Group: A non empty set under the operation $*$ is said to be a group if it satisfies following conditions.

(i) Closure property: $\forall a, b \in G$, $a*b \in G$.

(ii) Associative property:

$\forall a, b, c \in G$, $(a*b)*c = a*(b*c)$

(iii) Existence of an identity:

There exists an element $e \in G$ (to be found) such that $a*e = a = e*a$, $\forall a \in G$

(iv) Existence of an inverse:

$\forall a \in G$, there exists $b \in G$ (to be found) such that $a*b = e = b*a$.

Notation: $b = a^{-1}$. Also $b^{-1} = a$

Abelian group:

If $a*b = b*a$, $\forall a, b \in G$ then....

For example: $(Z, +)$, $(R, +)$, $(R-\{0\}, \times)$, $(Q-\{0\}, \times)$.

Note → 0 is identity element under addition. ($\because 0 + a = a$, $\forall a$)

→ 1 is identity element under multi. as $1 \times a = a$, $\forall a$.

Example:

1) Show that the set of square roots of unity form a group under multiplication

Proof: Here operation is multiplication which is given but set is not given.

We want to find x such that $x^2 = 1$

$\therefore x^2 - 1 = 0$

$\therefore (x-1)(x+1) = 0$ $\quad (\because a^2 - b^2 = (a-b)(a+b))$

$\therefore x - 1 = 0$ or $x + 1 = 0$

$\therefore x = 1$ or $x = -1$.

$\therefore$ Given set is $G = \{1, -1\}$

To show $(G, \times)$ is group,
we construct a composition table.

Composition table:

$\times$	1	-1
1	1	-1
-1	-1	1

(i) Closure property: Here all entries in composition table ~~is~~ belongs to G.
i.e. $\forall\, a, b \in G$, $a \times b \in G$.

(ii) Associative property:
We know that multiplication of integers is associative.
$\therefore \forall\, a, b, c \in G$, $(a \times b) \times c = a \times (b \times c)$

(iii) Existence of identity:
Here from composition table, it is clear that '1' is identity.
(as $1 \times 1 = 1$, $1 \times -1 = -1 \times 1 = -1$)

(iv) Existence of inverse:
From the composition table,
$(-1)^{-1} = -1$ as $\underset{a}{(-1)} \times \underset{b}{(-1)} = \underset{e}{1}$ and

$(1)^{-1} = 1$ as $\underset{a}{1} \times \underset{b}{1} = \underset{e}{1}$

Also from composition table,
$\forall\, a, b \in G$, $a \times b = b \times a$
Hence $(G, \times)$ is an abelian group.

Ex Show that the set of cube roots of unity form a group under multiplication

→ Let $x^3=1$. (we want to find x)

$\therefore x^3-1=0$

$\therefore (x-1)(x^2+x+1)=0$ $\quad (\because a^3-b^3=(a-b)(a^2+ab+b^2))$

$\therefore x-1=0$ or $x^2+x+1=0$

$\therefore x=1$ or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\quad$ (compare $x^2+x+1=0$ with $ax^2+bx+c=0$. Here $a=1, b=1, c=1$)

$x=\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\dfrac{-1\pm\sqrt{-3}}{2}$

$x=\dfrac{-1\pm\sqrt{3}i}{2}$ as $i=\sqrt{-1}$

$\therefore$ Cube roots of unity are $1, \omega$ and ω^2

where $\omega=\dfrac{-1+\sqrt{3}i}{2}$ and $\omega^2=\dfrac{(-1+\sqrt{3}i)^2}{4}=$

$\therefore \omega^2=\dfrac{1-2\sqrt{3}i+3i^2}{4}$ $\quad (\because (a+b)^2=a^2+2ab+b^2)$

$=\dfrac{1-2\sqrt{3}i-3}{4}=\dfrac{-2-2\sqrt{3}i}{4}$

$\omega^2=\dfrac{-1-\sqrt{3}i}{2}$

Let $G=\{1,\omega,\omega^2\}$. To prove that $(G,\times)$ is group.

composition table.

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3=1$
ω^2	ω^2	$\omega^3=1$	$\omega^4=\omega\cdot\omega^3=\omega$

(i) closure property: From table, it is clear that G is closed under $\times$.

(ii) Associative property: From composition table it is clear that $\forall a,b,c\in G$, $(a\times b)\times c=a\times(b\times c)$

(iii) Existence of identity: It is clear that 1 is identity

(iv) Existence of inverse:

$(1)^{-1}=1$ $\quad (\because \underset{a}{1}\times\underset{b}{1}=\underset{e}{1})$

$\omega^{-1}=\omega^2$ $\quad (\because \omega\times\omega^2=1)$

$(\omega^2)^{-1}=\omega$ $\quad (\because \omega^2\times\omega=1)$

Also from table it is clear that $a\times b=b\times a$

$\therefore (G,\times)$ is an abelian group.