8: Definition of Terms Relating to the Non-Ultimate Mechanical Properties of Polymers

CONTENTS

PREAMBLE

This document gives definitions of terms related to the non-ultimate mechanical behaviour or mechanical behaviour prior to failure of polymeric materials, in particular of bulk polymers and concentrated solutions and their elastic and viscoelastic properties.

The terms are arranged into sections dealing with basic definitions of stress and strain, deformations used experimentally, *stresses* observed experimentally, quantities relating stress and deformation, linear viscoelastic behaviour, and oscillatory deformations and *stresses* used experimentally for solids. The terms which have been selected are those met in the conventional mechanical characterization of polymeric materials.

To compile the definitions, a number of sources have been used. A number of the definitions were adapted from an International Organization for Standardization (ISO) manuscript on Plastics Vocabulary [1]. Where possible, the names for properties, their definitions and the symbols for linear viscoelastic properties were checked against past compilations of terminology [2-6]. Other documents consulted include ASTM publications [7-13].

The document does not deal with the properties of anisotropic materials. This is an extensive subject in its own right and the reader is referred to specialized texts [14,15] for information.

In the list of contents, main terms separated by / are alternative names, and terms in parentheses give those which are defined in the context of main terms, usually as notes to the definitions of main terms, with their names printed in bold type in the main text. Multicomponent quantities (vectors, tensors, matrices) are printed in bold type. Names

---

Originally prepared by a working group consisting of A. Kaye (UK), R. F. T. Stepto (UK), W. J. Work (USA), J. V. Alemán (Spain) and A. Ya. Malkin (Russia). Reprinted from *Pure Appl. Chem.* **70**, 701-754 (1998).

printed in italics are defined elsewhere in the document and their definitions can be found by reference to the alphabetical list of terms.

NOTE on tensor terminology and symbols*

The tensor quantities given in this chapter are all second rank, and are sometimes referred to as *matrices*, according to common usage, so that the two terms, tensor and matrix, are used interchangeably. In many cases, the components (or coefficients) of second-rank tensors are represented by 3 × 3 matrices. Symbols for tensors (matrices) are printed in bold italic type, while symbols for the components are printed in italic type. In general, the base tensors are those for a rectangular Cartesian coordinate system.

1 BASIC DEFINITIONS

In this section, quantities are expressed with respect to rectangular Cartesian co-ordinate axes, Ox_1, Ox_2, Ox_3, except where otherwise stated. The components of a vector $\boldsymbol{V}$ are denoted V_1, V_2 and V_3 with respect to these axes.

1.1 **traction**, $\boldsymbol{t}$, SI unit: Pa
stress vector

Vector force per unit area on an infinitesimal element of area that has a given normal and is at a given point in a body.

Note 1: The components of $\boldsymbol{t}$ are written as t_1, t_2, t_3.

Note 2: $\boldsymbol{t}$ is sometimes called true stress. The term traction (or stress vector) is preferred to avoid confusion with *stress tensor*.

1.2 **stress tensor**, $\boldsymbol{\sigma}$, SI unit: Pa
stress

Tensor with components σ_{ij} which are the components of the *traction* in the Ox_i direction on an element of area whose normal is in the Ox_j direction.

Note 1: A unit vector area with normal $\boldsymbol{n}$ can be resolved into three smaller areas equal to n_1, n_2 and n_3 with normals in the directions of the respective co-ordinate axes. Accordingly, each component of the *traction* on the original area can be considered as the sum of components in the same direction on the smaller areas to give

$$t_i = \sum_{j=1}^{3} \boldsymbol{\sigma}_{ij} n_j \quad , i = 1, 2, 3$$

Note 2: In usual circumstances, in the absence of body couples, $\boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_{ji}$.

Note 3: For a homogeneous stress $\boldsymbol{\sigma}$ is the same at all points in a body.

Note 4: For an inhomogeneous stress $\boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_{ij}(x_1, x_2, x_3)$.

Note 5: $\boldsymbol{\sigma}$ is a **true stress** because its components are forces per unit current area (c.f. Notes 3 and 4).

Note 6: If $\boldsymbol{\sigma}_{13}$ (= $\boldsymbol{\sigma}_{31}$) = $\boldsymbol{\sigma}_{23}$ (= $\boldsymbol{\sigma}_{32}$) = $\boldsymbol{\sigma}_{33}$ = 0 then the stress is called a **plane stress**. Plane *stresses* are associated with the deformation of a sheet of material in the plane of the sheet.

1.3 **deformation of an elastic solid**

Deformation of an elastic solid through which a mass point of the solid with co-ordinates X_1, X_2, X_3 in the undeformed state moves to a point with co-ordinates x_1, x_2, x_3 in the deformed state and the deformation is defined by

* This note has been added to this edition of the 1997 recommendations.

$x_i = x_i\,(X_1, X_2, X_3)$, $i = 1, 2, 3$

Note 1:A **homogeneous deformation** is one in which the relationships between the co-ordinates in the undeformed and deformed states reduce to

$$x_i = \sum_{j=1}^{3} f_{ij}\, X_j \quad i = 1, 2, 3$$

where the f_{ij} are constants.
Note 2:An **inhomogeneous deformation** is one in which the incremental changes in the undeformed and deformed co-ordinates are related by

$$\mathrm{d}x_i = \sum_{j=1}^{3} f_{ij}\, \mathrm{d}X_j \quad i = 1, 2, 3$$

where $f_{ij} = \partial x_i / \partial X_j \quad i, j = 1, 2, 3$, and where the f_{ij} are the functions of the coordinates x_j.
Note 3:The f_{ij} in notes 1 and 2 are **deformation gradients**.

1.4 deformation gradient tensor for an elastic solid, ***F***
Tensor whose components are deformation gradients in an elastic solid.
Note 1: The components of ***F*** are denoted f_{ij}.
Note 2: See Definition 1.3 for the definitions of f_{ij}.

1.5 deformation of a viscoelastic liquid or solid
Deformation of a viscoelastic liquid or solid through which a mass point of the viscoelastic liquid or solid with co-ordinates x_1', x_2', x_3' at time t' moves to a point with co-ordinates x_1, x_2, x_3 at time t such that there are functions g_i, $i = 1, 2, 3$, where

$$g_i\,(x_1', x_2', x_3'\; t') = g_i\,(x_1, x_2, x_3, t).$$

Note 1: t' often refers to some past time and t to the present time.
Note 2: The relationships between the total differentials of the functions g_i define how particles of the material move relative to each other. Thus, if two particles are at small distances $\mathrm{d}x_1', \mathrm{d}x_2', \mathrm{d}x_3'$ apart at time t' and $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ at time t then

$$\sum_{j=1}^{3} g_{ij}'\, \mathrm{d}x_j' = \sum_{j=1}^{3} g_{ij}\, \mathrm{d}x_j$$

where $g_{ij}'(x_1', x_2', x_3', t') = \dfrac{\partial g_i(x_1', x_2', x_3', t')}{\partial x_j'}$

and $g_{ij}(x_1, x_2, x_3, t) = \dfrac{\partial g_i(x_1, x_2, x_3, t)}{\partial x_j}$ where $i, j = 1, 2, 3$.

Note 3: The matrix with elements g_{ij} is denoted ***G*** and the matrix with elements g'_{ij} is denoted ***G***$'$.
Note 4: A **homogeneous deformation** is one in which the functions g_i are linear functions of the x_j, $i, j = 1, 2, 3$. As a result, the g_{ij} and ***G*** are functions of t only and the equations which define the deformation become

$$\sum_{j=1}^{3} g'_{ij}(t')x'_j = \sum_{j=1}^{3} g_{ij}(t)x_j$$

Note 5: *Homogeneous deformations* are commonly used or assumed in the methods employed for characterizing the mechanical properties of viscoelastic polymeric liquids and solids.

1.6 deformation gradients in a viscoelastic liquid or solid, f_{ij}

If two mass points of a liquid are at a small distance $\mathrm{d}x'_1, \mathrm{d}x'_2, \mathrm{d}x'_3$ apart at time t' then the deformation gradients are the rates of change of $\mathrm{d}x'_i$ with respect to $\mathrm{d}x_j$, $i, j = 1, 2, 3$.

Note: $f_{ij} = \partial x'_i / \partial x_j$, $i, j = 1, 2, 3$

1.7 deformation gradient tensor for a viscoelastic liquid or solid, $\boldsymbol{F}$

Tensor whose components are deformation gradients in a viscoelastic liquid or solid.

Note 1: The components of $\boldsymbol{F}$ are denoted f_{ij}.

Note 2: See Definition 1.6 for the definition of fij.

Note 3: By matrix multiplication, $\boldsymbol{F} = (\boldsymbol{G'})^{-1}\boldsymbol{G}$, where the matrices $\boldsymbol{G}$ and $\boldsymbol{G'}$ are those defined in Definition 1.5.

1.8 strain tensor

Symmetric tensor that results when a *deformation gradient tensor* is factorized into a rotation tensor followed or preceded by a symmetric tensor.

Note 1: A strain tensor is a measure of the relative displacement of the mass points of a body.

Note 2: The *deformation gradient tensor* $\boldsymbol{F}$ may be factorized as

$$\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{U} = \boldsymbol{V}\,\boldsymbol{R},$$

where $\boldsymbol{R}$ is an orthogonal matrix representing a rotation and $\boldsymbol{U}$ and $\boldsymbol{V}$ are strain tensors which are symmetric.

Note 3: Alternative strain tensors are often more useful, for example:

the **Cauchy** tensor, $\boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$
the **Green** tensor, $\boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$
the **Finger** tensor, $\boldsymbol{C}^{-1}$
the **Piola** tensor, $\boldsymbol{B}^{-1}$

where ‘T’ denotes transpose and ‘-1’ denotes inverse. $\boldsymbol{B}$ is most useful for solids and $\boldsymbol{C}$ and $\boldsymbol{C}^{-1}$ for viscoelastic liquids and solids.

Note 4: If the 1,3; 3,1; 2,3; 3,2; 3,3 elements of a strain tensor are equal to zero then the strain is termed **plane strain**.

1.9 Cauchy tensor, $\boldsymbol{C}$

Strain tensor for a viscoelastic liquid or solid, whose elements are

$$c_{ij} = \sum_{k=1}^{3} \frac{\partial x'_k}{\partial x_i} \cdot \frac{\partial x'_k}{\partial x_j} \;,$$

where x_i' and x_i are co-ordinates of a particle at times t' and t, respectively.

Note 1: See Definition 1.5 for the definition of x_i' and x_i.

Note 2: See Definition 1.8 for the definition of a *strain tensor*.

1.10 Green tensor, *B*

Strain tensor for an elastic solid, whose elements are

$$b_{ij} = \sum_{k=1}^{3} \frac{\partial x_i}{\partial X_k} \cdot \frac{\partial x_j}{\partial X_k} ,$$

where X_i and x_i are co-ordinates in the undeformed and deformed states, respectively.

Note 1: See Definition 1.3 for the definition of X_i and x_i.

Note 2: See Definition 1.8 for the definition of a *strain tensor*.

Note 3: For small strains, $\boldsymbol{B}$ may be expressed by the equation

$$\boldsymbol{B} = \boldsymbol{I} + 2\boldsymbol{\varepsilon} ,$$

where $\boldsymbol{I}$ is the unit matrix of order three and $\boldsymbol{\varepsilon}$ is the **small-strain tensor**. The components of $\boldsymbol{\varepsilon}$ are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

with $u_k = x_k - X_k$, $k = 1, 2, 3$, the displacements due to the deformation.

1.11 Finger tensor, $\boldsymbol{C}^{-1}$

Strain tensor, for a viscoelastic liquid or solid, whose elements are

$$c_{ij} = \sum_{k=1}^{3} \frac{\partial x_i}{\partial x_k'} \cdot \frac{\partial x_j}{\partial x_k'} ,$$

where x_i' and x_i are co-ordinates of a particle at times t' and t, respectively.

Note: See Definition 1.5 for the definition of x_i' and x_i.

1.12 rate-of-strain tensor, *D*, SI unit: s^{-1}

Time derivative of a *strain tensor* for a viscoelastic liquid or solid in *homogeneous deformation* at reference time, t.

Note 1: For an *inhomogeneous deformation*, the material derivative has to be used to find time derivatives of strain.

Note 2: $D = \lim_{t' \to t}\left(\frac{\partial U}{\partial t'} \right) = \lim_{t' \to t}\left(\frac{\partial V}{\partial t'} \right)$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are defined in Definition 1.8, note 2.

Note 3: The elements of $\boldsymbol{D}$ are

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) ,$$

where the v_k are the components of the velocity $\boldsymbol{v}$ at $\boldsymbol{x}$ and time, t.

1.13 vorticity tensor, ***W***, SI unit: s^{-1}
Derivative, for a viscoelastic liquid or solid in *homogeneous deformation*, of the rotational part of the deformation-gradient tensor at reference time, *t*.
Note 1: For an *inhomogeneous deformation* the material derivative has to be used.
Note 2: $W = \lim_{t' \to t} \left(\dfrac{\partial R}{\partial t'} \right)$, where ***R*** is defined in Definition 1.8, note 2.
Note 3: The elements of ***W*** are

$$w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) ,$$

where the v_k are the components of the velocity ***v*** at *x* and time *t*.

1.14 Rivlin-Ericksen tensors, $\boldsymbol{A}_n$, SI unit**:** s^{-n}
Rivlin-Ericksen tensor of order *n*, for a viscoelastic liquid or solid in *homogeneous deformation*, is the nth time derivative of the *Cauchy strain tensor* at reference time, *t*.
Note 1: For an *inhomogeneous deformation* the material derivatives have to be used.
Note 2: $A_n = \lim_{t' \to t} \left(\dfrac{\partial^n C}{\partial t'^n} \right)$, where ***C*** is defined in Definition 1.9.
Note 3: $\boldsymbol{A}_0 = \boldsymbol{I}$, where ***I*** is the unit matrix of order three.
Note 4: $A_1 = \dot{F}^{\mathrm{T}} + \dot{F} = 2D$ where ***F*** is the *deformation gradient tensor*, $\dot{F} = \lim_{t' \to t} \left(\dfrac{\partial F}{\partial t'} \right)$, where 'T' denotes transpose and ***D*** is the *rate-of-strain tensor*.
Note 5: In general, $A_{n+1} = \dot{A}_n + \dot{F}^{\mathrm{T}} A_n + A_n \dot{F}$, $n = 0, 1, 2, ...$

2 DEFORMATIONS USED EXPERIMENTALLY

All deformations used in conventional measurements of mechanical properties are interpreted in terms of *homogeneous deformations*.

2.1 general orthogonal homogeneous deformation of an elastic solid
Deformation, such that a mass point of the solid with co-ordinates X_1, X_2, X_3 in the undeformed state moves to a point with co-ordinates x_1, x_2, x_3 in the deformed state, with

$$x_i = \lambda_i X_i \text{ , } i = 1, 2, 3,$$

where the λ_i are constants.
Note 1: The relationships between the x_i and X_i for orthogonal *homogeneous deformation*s are a particular case of the general relationships given in Definition 1.3, provided the deformation does not include a rotation and the co-ordinate axes are chosen as the principal directions of the deformation.
Note 2: The λ_i are effectively **deformation gradients**, or, for finite deformations, the **deformation ratios** characterizing the deformation.
Note 3: For an incompressible material, $\lambda_1\lambda_2\lambda_3 = 1$.

Note 4: The λ_i are elements of the *deformation gradient tensor* ***F*** and the resulting *Cauchy* and *Green tensors* ***C*** and ***B*** are

$$C = B = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

2.2 uniaxial deformation of an elastic solid

Orthogonal, *homogeneous deformation* in which, say,

$$\lambda_1 = \lambda \quad \text{and} \quad \lambda_2 = \lambda_3.$$

Note 1: See Definition 2.1 for the definition of λ_i, $i = 1, 2, 3$.
Note 2: For an incompressible material, $\lambda_2 = \lambda_3 = 1/\lambda^{1/2}$.

2.3 uniaxial deformation ratio, λ
deformation ratio

Quotient of the length (l) of a sample under uniaxial tension or compression and its original length (l_0)

$$\lambda = l / l_0$$

Note 1: In tension λ (>1) may be termed the **extension ratio**.
Note 2: In compression λ (<1) may be termed the **compression ratio**.
Note 3: λ is equivalent to λ_1 in Definitions 2.1 and 2.2.

2.4 uniaxial strain, ε
engineering strain

Change in length of a sample in uniaxial tensile or compressive deformation divided by its initial length

$$\varepsilon = (l_1 - l_0) / l_0$$

where l_0 and l_1 are, respectively, the initial and final lengths.
Note 1: $\varepsilon = \lambda - 1$, where λ is the *uniaxial deformation ratio*.
Note 2: $\varepsilon > 0$ is referred to as **(uniaxial) tensile strain**.
Note 3: $\varepsilon < 0$ is referred to as **(uniaxial) compressive strain**.

2.5 Hencky strain, ε_H

Integral over the total change in length of a sample of the incremental strain in uniaxial tensile deformation

$$\varepsilon_H = \int_{l_0}^{l_1} \mathrm{d}l/l = \ln(l_1/l_0)$$

l_0 , l_1 and l are, respectively, the initial, final and instantaneous lengths.
Note 1: See *uniaxial strain* (Definition 2.4).

Note 2: The same equation can be used to define a quantity ε_H (< 0) in compression.

2.6 Poisson's ratio, μ

In a sample under small *uniaxial deformation*, the negative quotient of the lateral strain (ε_{lat}) and the longitudinal strain (ε_{long}) in the direction of the uniaxial force

$$\mu = -\left(\frac{\varepsilon_{lat}}{\varepsilon_{long}}\right)$$

Note 1: **Lateral strain** ε_{lat} is the strain normal to the *uniaxial deformation*.
Note 2: $\varepsilon_{lat} = \lambda_2 - 1 = \lambda_3 - 1$ (see Definitions 2.2 and 2.4).
Note 3: For an isotropic, incompressible material, μ = 0.5. It should be noted that, in materials referred to as incompressible, volume changes do in fact occur in deformation, but they may be neglected.
Note 4: For an anisotropic material, μ varies with the direction of the *uniaxial deformation*.
Note 5: Poisson's ratio is also sometimes called the **lateral contraction ratio** and is sometimes used in cases of non-linear deformation. The present definition will not apply in such cases.

2.7 pure shear of an elastic solid

Orthogonal, *homogeneous deformation* in which

$$\lambda_1 = \lambda$$
$$\lambda_2 = 1/\lambda$$
$$\lambda_3 = 1$$

Note: See Definition 2.1 for the definition of λ_i, i = 1, 2, 3.

2.8 simple shear of an elastic solid

Homogeneous deformation, such that a mass point of the solid with co-ordinates X_1, X_2, X_3 in the undeformed state moves to a point with co-ordinate x_1, x_2, x_3 in the deformed state, with

$$x_1 = X_1 + \gamma X_2$$
$$x_2 = X_2$$
$$x_3 = X_3$$

where γ is constant.
Note 1: The relationships between the x_i and X_i, i = 1, 2, 3, in simple shear are a particular case of the general relationships given in Definition 1.3.
Note 2: γ is known as the **shear** or **shear strain**.
Note 3: The *deformation gradient tensor* for the simple shear of an *elastic solid* is

$$\boldsymbol{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the *Cauchy* (***C***) and *Green* (***B***) *strain tensors* are

$$C = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.9 bulk compression, χ
Fractional decrease in volume (V) caused by a hydrostatic pressure

$$\chi = -\Delta V/V.$$

Note: Also referred to as **volume compression**, **isotropic compression** and **bulk compressive strain**.

2.10 general homogeneous deformation or flow of a viscoelastic liquid or solid
Flow or deformation such that a particle of the viscoelastic liquid or solid with co-ordinate vector $\boldsymbol{X'}$ at time t' moves to a point with co-ordinate vector $\boldsymbol{X}$ at time t with

$$\boldsymbol{G'X'} = \boldsymbol{GX}$$

where $\boldsymbol{G'}$ and $\boldsymbol{G}$ are tensors defining the type of deformation or flow and are functions of time only.
Note 1: The definition is equivalent to that given in Definition 1.5, note 4. Accordingly, the elements of $\boldsymbol{G'}$ and $\boldsymbol{G}$ are denoted $g'_{ij}(t')$ and $g_{ij}(t)$ and those of $\boldsymbol{X'}$ and $\boldsymbol{X}$, (x'_1, x'_2, x'_3) and (x_1, x_2, x_3).
Note 2: For an incompressible material

det $\boldsymbol{G}$ = 1, where det $\boldsymbol{G}$ is the determinant of $\boldsymbol{G}$.

Note 3: Deformations and flows used in conventional measurements of properties of viscoelastic liquids and solids are usually interpreted assuming incompressibility.

2.11 homogeneous orthogonal deformation or flow of an incompressible viscoelastic liquid or solid
Deformation or flow, as defined in Definition 2.10, such that

$$\boldsymbol{G} = \begin{pmatrix} g_{11}(t) & 0 & 0 \\ 0 & g_{22}(t) & 0 \\ 0 & 0 & g_{33}(t) \end{pmatrix}.$$

Note 1: The g_{ii} are defined in Definition 1.5, notes 2 to 4.
Note 2: If $g_{22} = g_{33} = 1/g_{11}^{1/2}$ the elongational deformation or flow is **uniaxial**.
Note 3: The *Finger strain tensor* for a homogeneous orthogonal deformation or flow of incompressible, viscoelastic liquid or solid is

$$\boldsymbol{C}^{-1} = \begin{pmatrix} \left(\dfrac{g'_{11}(t')}{g_{11}(t)}\right)^2 & 0 & 0 \\ 0 & \left(\dfrac{g'_{22}(t')}{g_{22}(t)}\right)^2 & 0 \\ 0 & 0 & \left(\dfrac{g'_{33}(t')}{g_{33}(t)}\right)^2 \end{pmatrix}.$$

2.12 steady uniaxial homogeneous elongational deformation or flow of an incompressible viscoelastic liquid or solid

Uniaxial homogeneous elongational flow in which

$$g_{11}(t) = \exp(-\dot{\gamma}_{\mathrm{E}} t)$$

where $\dot{\gamma}_{\mathrm{E}}$ is a constant, E denotes elongational and $g_{22} = g_{33} = 1/g_{11}^{1/2}$.

Note 1: $g_{11}(t)$, $g_{22}(t)$ and $g_{33}(t)$ are elements of the tensor $\boldsymbol{G}$ defined in Definition 1.5.

Note 2: From the definition of *general homogeneous flow* (Definition 1.5) ($\boldsymbol{G'X'} = \boldsymbol{GX}$ = constant) in the particular case of *steady uniaxial elongation flow*

$$x_1 g_{11}(t) = x_1 \exp(-\dot{\gamma}_{\mathrm{E}}\ t) = \text{constant}$$

and differentiation with respect to time gives

$$\dot{\gamma}_{\mathrm{E}} = (1/x_1)(\mathrm{d}x_1/\mathrm{d}t)\ .$$

Hence, $\dot{\gamma}_{\mathrm{E}}$ is the **elongational** or **extensional strain rate**.

Note 3: The *Finger strain tensor* for a steady uniaxial homogeneous elongation deformation or flow of an incompressible viscoelastic liquid or solid is

$$\boldsymbol{C}^{-1} = \begin{pmatrix} \exp(2\dot{\gamma}_{\mathrm{E}}(t - t')) & 0 & 0 \\ 0 & \exp(-\dot{\gamma}_{\mathrm{E}}(t - t')) & 0 \\ 0 & 0 & \exp(-\dot{\gamma}_{\mathrm{E}}(t - t')) \end{pmatrix}.$$

2.13 homogeneous simple shear deformation or flow of an incompressible viscoelastic liquid or solid

Flow or deformation such that

$$\boldsymbol{G} = \begin{pmatrix} 1 & -\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\gamma(t)$ is the **shear**.

Note 1: The general tensor $\boldsymbol{G}$ is defined in Definition 1.5.

Note 2: $\dot{\gamma} = \mathrm{d}\gamma(t)/\mathrm{d}t$ is the **shear rate**. The unit of $\dot{\gamma}$ is s^{-1}.

Note 3: If $\gamma(t) = \dot{\gamma} t$, where $\dot{\gamma}$ is a constant, then the flow has a constant shear rate and is known as **steady (simple) shear flow**.
Note 4: If $\gamma(t) = \gamma_0 \sin 2\pi\nu t$ then the flow is **oscillatory (simple) shear flow** of amplitude γ_0 and frequency ν expressed in Hz.
Note 5: The *Finger strain tensor* for *simple shear flow* is

$$\boldsymbol{C}^{-1} = \begin{pmatrix} 1+\left(\gamma(t)-\gamma(t')\right)^2 & \gamma(t)-\gamma(t') & 0 \\ \gamma(t)-\gamma(t') & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\gamma(t) - \gamma(t')$ is the amount of *shear* given to the liquid between the times t' and t. For *steady simple shear* flow $\boldsymbol{C}^{-1}$ becomes

$$\boldsymbol{C}^{-1} = \begin{pmatrix} 1+\dot{\gamma}^2(t-t') & \dot{\gamma}(t-t') & 0 \\ \dot{\gamma}(t-t') & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3 STRESSES OBSERVED EXPERIMENTALLY

For a given deformation or flow, the resulting *stress* depends on the material. However, the *stress tensor* does take particular general forms for experimentally used deformations (see **section 2**). The definitions apply to elastic solids, and viscoelastic liquids and solids.

3.1 stress tensor resulting from an orthogonal deformation or flow, $\boldsymbol{\sigma}$, SI unit: Pa
For an orthogonal deformation or flow the *stress tensor* is diagonal with

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}.$$

Note 1: See Definition 1.2 for the general definition of $\boldsymbol{\sigma}$.
Note 2: If the *strain tensor* is diagonal for all time then the *stress tensor* is diagonal for all time for isotropic materials.
Note 3: For a **uniaxial (orthogonal) deformation or flow** $\sigma_{22} = \sigma_{33}$.
Note 4: For a **pure shear deformation or flow** the *stresses* (σ_{11}, σ_{22}, σ_{33}) are usually all different from each other.
Note 5: The *stress tensor* resulting from a *pure shear deformation or flow* is called a **pure shear stress**.

3.2 tensile stress, σ, SI unit: Pa
Component σ_{11} of the *stress tensor* resulting from a tensile *uniaxial deformation*.
Note 1: The *stress tensor* for a *uniaxial deformation* is given in Definition 3.1.
Note 2: The Ox_1 direction is chosen as the direction of the *uniaxial deformation*.

3.3 compressive stress, σ, SI unit: Pa
Component σ_{11} of the *stress tensor* resulting from a compressive *uniaxial deformation*.
Note: See notes 1 and 2 of Definition 3.2.

3.4 nominal stress, σ, SI unit: Pa
engineering stress
Force resulting from an applied tensile or compressive *uniaxial deformation* divided by the initial cross-sectional area of the sample normal to the applied deformation.
Note: The term *engineering* or *nominal stress* is often used in circumstances when the deformation of the body is not infinitesimal and its cross-sectional area changes.

3.5 stress tensor resulting from a simple shear deformation or flow, $\boldsymbol{\sigma}$, SI unit: Pa
For a simple shear deformation or flow the *stress tensor* takes the form

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

where σ_{21} is numerically equal to σ_{12}.
Note 1: See Definition 1.2 for the general definition of σ.
Note 2: σ_{ii}, i = 1, 2, 3 are denoted **normal stresses**.
Note 3: σ_{12} is called the **shear stress**.

3.6 first normal-stress difference, N_1, SI unit: Pa
first normal-stress function
Difference between the first two normal *stresses* σ_{11} and σ_{22} in simple shear flow

$$N_1 = \sigma_{11} - \sigma_{22}.$$

Note 1: See Definition 3.5 for the definition of σ_{11} and σ_{22}.
Note 2: For *Newtonian liquids* $N_1 = 0$.

3.7 second normal-stress difference, N_2, SI unit: Pa
second normal-stress function
Difference between the second and third normal-*stresses* (σ_{22} - σ_{33}) in simple shear flow

$$N_2 = \sigma_{22} - \sigma_{33}.$$

Note 1: See Definition 3.5 for the definition of σ_{22} and σ_{33}
Note 2: For *Newtonian liquids*, $N_2 = 0$.

4 QUANTITIES RELATING STRESS AND DEFORMATION

4.1 constitutive equation for an elastic solid
Equation relating *stress* and strain in an elastic solid.
Note 1: For an elastic solid, the constitutive equation may be written

$$\sigma = \frac{2}{I_3^{1/2}}\left(\frac{\partial W}{\partial I_1}\boldsymbol{B} + \frac{\partial W}{\partial I_2}\left(I_1\boldsymbol{B} - \boldsymbol{B}^2\right) + I_3\frac{\partial W}{\partial I_3}\boldsymbol{I}\right),$$

where $\boldsymbol{B}$ is the *Green strain tensor*.
I_1, I_2, I_3 are invariants of $\boldsymbol{B}$, with
$I_1 = \mathrm{Tr}\,(\boldsymbol{B})$
$I_2 = 1/2\ \{(\mathrm{Tr}\,(\boldsymbol{B}))^2 - \mathrm{Tr}\,(\boldsymbol{B}^2)\}$
$I_3 = \det \boldsymbol{B}$,
where 'Tr' denotes trace and 'det' denotes determinant. The invariants are independent of the co-ordinate axes used and for symmetric tensors there are three independent invariants. W is a function of I_1, I_2 and I_3 and is known as the **stored energy function** and is the increase in energy (stored energy) per unit initial volume due to the deformation.
Note 2: For small deformations, the constitutive equation may be written

$$\boldsymbol{\sigma} = 2G\boldsymbol{\varepsilon} + l\boldsymbol{I}\mathrm{Tr}\,(\boldsymbol{\varepsilon}),$$

where G is the *shear modulus*, $\boldsymbol{\varepsilon}$ is the *small-strain tensor* and l is a **Lamé constant**.
Note 3: The Lamé constant, (l), is related to the *shear modulus* (G) and *Young's modulus* (E) by the equation

$$l = G(2G - E)/(E - 3G).$$

Note 4: For an incompressible elastic solid, the constitutive equation may be written

$$\sigma + P\boldsymbol{I} = 2\frac{\partial W}{\partial I_1}\boldsymbol{B} - 2\frac{\partial W}{\partial I_2}\boldsymbol{B}^{-1},$$

where P is the hydrostatic (or isotropic) pressure, $I_3 = 1$ and W is a function of I_1 and I_2, only.
Note 5: For small deformations of an incompressible, inelastic solid, the constitutive equation may be written

$$\boldsymbol{\sigma} + P\boldsymbol{I} = 2G\boldsymbol{\varepsilon}.$$

4.2 constitutive equation for an incompressible viscoelastic liquid or solid
Equation relating *stress* and deformation in an incompressible viscoelastic liquid or solid.
Note 1: A possible general form of constitutive equation when there is no dependence of *stress* on amount of strain is

$$\boldsymbol{\sigma} + P\boldsymbol{I} = f(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n),$$

where $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, ... are the *Rivlin-Ericksen tensors*.
Note 2: For a *non-Newtonian liquid* (see note 3), a form of the general constitutive equation which may be used is

$$\boldsymbol{\sigma} + P\boldsymbol{I} = \eta\boldsymbol{A}_1^2 + \alpha\boldsymbol{A}_1 + \beta\boldsymbol{A}_2,$$

where η is the *viscosity* and α and β are constants.

Note 3: A **Newtonian liquid** is a liquid for which the constitutive equation may be written

$$\boldsymbol{\sigma} + P\boldsymbol{I} = \eta \boldsymbol{A}_1 = \mathbf{2}\eta \boldsymbol{D},$$

where $\boldsymbol{D}$ is the *rate-of-strain tensor*. Liquids which do not obey this constitutive equation are termed **non-Newtonian liquids**.
Note 4: For cases where there is a dependence of *stress* on strain history the following constitutive equation may be used, namely

$$\boldsymbol{\sigma} + P\boldsymbol{I} = 2\int_{-\infty}^{t}\left(\frac{\partial \Omega}{\partial I_1}\boldsymbol{C}^{-1} - \frac{\partial \Omega}{\partial I_2}\boldsymbol{C}\right)\mathrm{d}t',$$

where $\boldsymbol{C}$ is the *Cauchy strain tensor* and Ω is a function of the invariants I_1, I_2 and I_3 of $\boldsymbol{C}^{-1}$ and the time interval $t\text{-}t'$. Ω is formally equivalent to the *stored-energy function*, W, of a solid.

4.3 modulus, in general M, SI unit: Pa
in bulk compressive deformation K, SI unit: Pa
in *uniaxial deformation* E, SI unit: Pa
in shear deformation G, SI unit: Pa
Quotient of *stress* and strain where the type of *stress* and strain is defined by the type of deformation employed.
Note 1: The detailed definitions of K, E and G are given in Definitions 4.5, 4.7 and 4.10.
Note 2: An **elastic modulus** or **modulus of elasticity** is a modulus of a body which obeys Hooke's law (*stress* $\propto$ strain).

4.4 compliance, in general C, SI unit: Pa
in bulk compressive deformation B, SI unit: Pa
in *uniaxial deformation* D, SI unit: Pa
in shear deformation J, SI unit: Pa
Quotient of strain and *stress* where the type of strain and *stress* is defined by the type of deformation employed.
Note 1: $C = 1/M$, where M is *modulus*.
Note 2: The detailed definitions of B, D and J are given in Definitions 4.6, 4.8 and 4.11.

4.5 bulk modulus, K, SI unit: Pa
Quotient of hydrostatic pressure (P) and *bulk compression* (χ)

$$K = P/\chi.$$

Note 1: Also known as **bulk compressive modulus**.
Note 2: For the definition of χ, see Definition 2.9.
Note 3: At small deformations, the *bulk modulus* is related to *Young's modulus* (E) by

$$K = \frac{E}{3(1-2\mu)}$$

where μ is *Poisson's ratio*.

4.6 bulk compliance, B, SI unit: Pa^{-1}

Quotient of *bulk compression* (χ) and hydrostatic pressure (P)

$$B = \chi / P.$$

Note 1: Also known as **bulk compressive compliance**.

Note 2: For the definition of χ, see Definition 2.9.

Note 3: $B = 1/K$, where K is the *bulk modulus*.

4.7 Young's modulus, E, SI unit: Pa

Quotient of uniaxial stress (σ) and strain (ε) in the limit of zero strain

$$E = \lim_{\varepsilon \to 0} (\sigma / \varepsilon) .$$

Note 1: The *stress* is a *true stress*, as in Definitions 3.2 and 3.3, and not a *nominal stress*, as in Definition 3.4.

Note 2: ε is defined in Definition 2.4.

Note 3: *Young's modulus* may be evaluated using *tensile* or *compressive uniaxial deformation*. If determined using tensile deformation it may be termed **tensile modulus**.

Note 4: For non-Hookean materials the *Young's modulus* is sometimes evaluated as:

(i) the **secant modulus** – the quotient of *stress* (σ) and strain at some nominal *strain* (ε) in which case

$$E = \sigma / \varepsilon$$

(ii) the **tangent modulus** – the slope of the *stress*-strain curve at some nominal *strain* (ε'), in which case

$$E = (\mathrm{d}\sigma / \mathrm{d}\varepsilon)_{\varepsilon = \varepsilon'}$$

4.8 uniaxial compliance, D, SI unit: Pa^{-1}

Quotient of *uniaxial strain* (ε) and uniaxial stress (σ) in the limit of zero strain

$$D = \lim_{\varepsilon \to 0} (\varepsilon / \sigma) .$$

Note 1: The *stress* is a *true stress* as in Definitions 3.2 and 3.3, and not a *nominal stress*, as in Definition 3.4.

Note 2: ε is defined in Definition 2.4.

Note 3: Uniaxial compliance may be evaluated using *tensile or compressive uniaxial deformation*. If determined using tensile deformation it may be termed **tensile compliance**.

Note 4: $D = 1/E$, where E is *Young's modulus*.

4.9 extensional viscosity, η_E, SI unit: Pa s
elongational viscosity

Quotient of the difference between the longitudinal *stress* (σ_{11}) and the lateral *stress* (σ_{22}) and the elongational strain rate ($\dot{\gamma}_E$) in steady uniaxial flow

$$\eta_E = (\sigma_{11} - \sigma_{22}) / \dot{\gamma}_E .$$

Note: See Definitions 3.1 and 2.12 for the definitions of σ_{11}, σ_{22} and $\dot{\gamma}_E$.

4.10 shear modulus, G, SI unit: Pa
Quotient of shear stress (σ_{12}) and shear strain (γ)

$$G = \sigma_{12}/\gamma.$$

Note 1: See Definition 2.8 for the definitions of γ for an elastic solid and Definition 3.5 for the definition of σ_{12}.
Note 2: The *shear modulus* is related to *Young's modulus* (E) by the equation

$$G = \frac{E}{2(1+\mu)}$$

where μ is *Poisson's ratio*.
Note 3: For elastomers, which are assumed incompressible, the *modulus* is often evaluated in *uniaxial tensile* or *compressive deformation* using $\lambda - \lambda^{-2}$ as the strain function (where λ is the *uniaxial deformation ratio*). In the limit of zero deformation the *shear modulus* is evaluated as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\lambda - \lambda^{-2})} = \frac{E}{3} = G \text{ (for } \mu = 0.5) ,$$

where σ is the *tensile* or *compressive stress*.

4.11 shear compliance, J, SI unit: Pa^{-1}
Quotient of shear strain (γ) and shear stress (σ_{12})

$$J = \gamma/\sigma_{12}.$$

Note 1: See Definition 2.8 for the definition of γ for an elastic solid and 3.5 for the definition of σ_{12}.
Note 2: $J = 1/G$, where G is the *shear modulus*.

4.12 shear viscosity, η, SI unit: Pa s
coefficient of viscosity
viscosity
Quotient of shear stress (σ_{12}) and shear rate ($\dot{\gamma}$) in steady, simple shear flow

$$\eta = \sigma_{12}/\dot{\gamma} .$$

Note 1: See Definitions 3.5 and 2.13 for the definitions of σ_{12} and $\dot{\gamma}$.
Note 2: For *Newtonian liquids*, σ_{12} is directly proportional to $\dot{\gamma}$ and η is constant.
Note 3: For *non-Newtonian liquids*, when σ_{12} is not directly proportional to $\dot{\gamma}$, η varies with $\dot{\gamma}$. The value of η evaluated at a given value of $\dot{\gamma}$ is termed the **non-Newtonian viscosity**.

Note 4: Some experimental methods, such as capillary flow and flow between parallel plates, employ a range of shear rates. The value of η evaluated at some nominal average value of $\dot{\gamma}$ is termed the **apparent viscosity** and given the symbol η_{app}. It should be noted that this is an imprecisely defined quantity.
Note 5: Extrapolation of η or η_{app} for *non-Newtonian liquids* to zero $\dot{\gamma}$ gives the **zero-shear viscosity**, which is given the symbol η_0.
Note 6: Extrapolation of η and η_{app} for *non-Newtonian liquids* to infinite $\dot{\gamma}$ gives the **infinite-shear viscosity**, which is given the symbol η_∞.

4.13 first normal-stress coefficient, ψ_1, SI unit: Pa s^2
Quotient of the first normal *stress* difference (N_1) and the square of the shear rate ($\dot{\gamma}$) in the limit of zero shear rate

$$\psi_1 = \lim_{\dot{\gamma}\to 0} (N_1/\dot{\gamma}^2)\,.$$

Note: See Definitions 3.6 and 2.13 for the definitions of N_1 and $\dot{\gamma}$.

4.14 second normal-stress coefficient, ψ_2, SI unit: Pa s^2
Quotient of the second normal *stress* difference (N_2) and the square of the shear rate ($\dot{\gamma}$) in the limit of zero shear rate

$$\psi_2 = \lim_{\dot{\gamma}\to 0} (N_2/\dot{\gamma}^2)\,.$$

Note: See Definitions 3.7 and 2.13 for the definitions of N_2 and $\dot{\gamma}$.

5 LINEAR VISCOELASTIC BEHAVIOUR

5.1 viscoelasticity
Time-dependent response of a liquid or solid subjected to *stress* or strain.
Note 1: Both viscous and elastic responses to *stress* or strain are required for the description of viscoelastic behaviour.
Note 2: Viscoelastic properties are usually measured as responses to an instantaneously applied or removed constant *stress* or strain or a **dynamic stress or strain**. The latter is defined as a sinusoidal *stress* or strain of small amplitude, which may or may not decrease with time.

5.2 linear viscoelastic behaviour
Interpretation of the viscoelastic behaviour of a liquid or solid in *simple shear* or *uniaxial deformation* such that

$$P(\mathrm{D})\sigma = Q(\mathrm{D})\gamma,$$

where σ is the shear stress or uniaxial stress, γ is the shear strain or *uniaxial strain*, and $P(\mathrm{D})$ and $Q(\mathrm{D})$ are polynomials in D, where D is the differential coefficient operator $\mathrm{d}/\mathrm{d}t$.

Note 1: In *linear viscoelastic behaviour*, *stress* and strain are assumed to be small so that the squares and higher powers of σ and γ may be neglected.
Note 2: See Definitions 3.5 and 2.13 for the definitions of σ and γ in *simple shear*.
Note 3: See Definitions 3.2 and 2.12 for definitions of σ and γ ($\equiv \gamma_E$) in *uniaxial deformations*.
Note 4: The polynomials $Q(\mathrm{D})$ and $P(\mathrm{D})$ have the forms:

$Q(\mathrm{D}) = a(\mathrm{D} + q_0) \ldots\ldots (\mathrm{D} + q_n)$
(a polynomial of degree $n + 1$)
$P(\mathrm{D}) = (\mathrm{D} + p_0)(\mathrm{D} + p_1) \ldots (\mathrm{D} + p_n)$
(a polynomial of degree $n + 1$)
$P(\mathrm{D}) = (\mathrm{D} + p_0)(\mathrm{D} + p_1) \ldots (\mathrm{D} + p_{n-1})$
(a polynomial of degree n)

where

(i) a is a constant
(ii) $q_0 \geq 0$, $p_0 > 0$ and $p_s, q_s > 0$, $s = 1, \ldots, n$.
(iii) $q_i < p_i < q_{i+1}$ and $q_n < p_n$ (if p_n exists) with p_i and q_i related to *relaxation* and *retardation times*, respectively.

Note 5: If $q_0 = 0$, the material is a liquid, otherwise it is a solid.
Note 6: Given that $Q(\mathrm{D})$ is a polynomial of degree $n + 1$; if $P(\mathrm{D})$ is also of degree $n + 1$ the material shows instantaneous elasticity; if $P(\mathrm{D})$ is of degree n, the material does not show instantaneous elasticity (i.e. elasticity immediately the deformation is applied.)
Note 7: There are definitions of linear viscoelasticity which use integral equations instead of the differential equation in Definition 5.2. (See, for example, [11].) Such definitions have certain advantages regarding their mathematical generality. However, the approach in the present document, in terms of differential equations, has the advantage that the definitions and descriptions of various viscoelastic properties can be made in terms of commonly used mechano-mathematical models (e.g. the *Maxwell* and *Voigt-Kelvin models*).

5.3 Maxwell model
Maxwell element

Model of the linear viscoelastic behaviour of a liquid in which

$$(\alpha D + \beta)\sigma = \mathrm{D}\gamma$$

where α and β are positive constants, D the differential coefficient operator $\mathrm{d}/\mathrm{d}t$, and σ and γ are the *stress* and strain in *simple shear* or *uniaxial deformation*.
Note 1: See Definition 5.2 for a discussion of σ and γ.
Note 2: The relationship defining the Maxwell model may be written

$$\mathrm{d}\sigma/\mathrm{d}t + (\beta/\alpha)\sigma = (1/\alpha)\,\mathrm{d}\gamma/\mathrm{d}t.$$

Note 3: Comparison with the general definition of *linear viscoelastic behaviour* shows that the polynomials $P(\mathrm{D})$ and $Q(\mathrm{D})$ are of order one, $q_0 = 0$, $p_0 = \beta/\alpha$ and $a = 1/\alpha$. Hence, a

material described by a Maxwell model is a *liquid* ($q_0 = 0$) having instantaneous elasticity (P(D) and Q(D) are of the same order).
Note 4: The Maxwell model may be represented by a spring and a dashpot filled with a *Newtonian liquid* in series, in which case $1/\alpha$ is the **spring constant** (force = $1/\alpha$)·extension) and $1/\beta$ is the **dashpot constant** (force = $(1/\beta)$·rate of extension).

5.4 Voigt-Kelvin model
Voigt-Kelvin element

Model of the *linear viscoelastic behaviour* of a solid in which

$$\sigma = (\alpha + \beta\mathrm{D})\gamma$$

where α and β are positive constants, D is the differential coefficient operator d/dt, and σ and γ are the *stresses* and strain in *simple shear* or *uniaxial deformation*.
Note 1: The Voigt-Kelvin model is also known as the **Voigt model** or **Voigt element**.
Note 2: See Definition 5.2 for a discussion of σ and γ.
Note 3: The relationship defining the Voigt-Kelvin model may be written

$$\sigma = \alpha\gamma + \alpha\beta\,\mathrm{d}\gamma/\mathrm{d}t.$$

Note 4: Comparison with the general definition *of linear viscoelastic behaviour* shows that the polynomial P(D) is of order zero, Q(D) is of order one, $aq_0 = \alpha$ and $a = \beta$. Hence, a material described by the Voigt-Kelvin model is a *solid* ($q_0 > 0$) without instantaneous elasticity (P(D) is a polynomial of order one less than Q(D)).
Note 5: The Voigt-Kelvin model may be represented by a spring and a dashpot filled with a Newtonian liquid in parallel, in which case α is the **spring constant** (force = α·extension) and β is the **dashpot constant** (force = β·rate of extension).

5.5 standard linear viscoelastic solid

Model of the *linear viscoelastic behaviour* of a solid in which

$$(\alpha_1 + \beta_1\mathrm{D})\sigma = (\alpha_2 + \beta_2\mathrm{D})\gamma$$

where α_1, β_1, α_2 and β_2 are positive constants, D is the differential coefficient operator d/dt, and σ and γ are the *stress* and strain in *simple shear* or *uniaxial deformation*.
Note 1: See Definition 5.2 for a discussion of σ and γ.
Note 2: The relationship defining *the standard linear viscoelastic solid* may be written

$$\alpha_1\sigma + \beta_1\,\mathrm{d}\sigma/\mathrm{d}t = \alpha_2\gamma + \beta_2\,\mathrm{d}\gamma/\mathrm{d}t$$

Note 3: Comparison with the general definition of a *linear viscoelastic behaviour* shows that the polynomial P(D) and Q(D) are of order one, $q_0 = \alpha_2/\beta_2$, $a = \beta_2/\beta_1$ and $p_0 = \alpha_1/\alpha_2$. Hence, the standard linear viscoelastic solid is a solid ($aq_0>0$) having instantaneous elasticity (P(D) and Q(D) are of the same order).
Note 4: The standard linear viscoelastic solid may be represented by:

(i) a *Maxwell model* (of spring constant h_2 and dashpot constant k_2) in parallel with a spring (of spring constant h_1) in which case $\alpha_1 = h_2$, $\beta_1 = k_2$, $\alpha_2 = h_1h_2$ and $\beta_2 = h_1k_2 + h_2k_2$.

(ii) a *Voigt-Kelvin model* (of spring constant h_2 and dashpot constant k_2) in series with a spring (of spring constant h_1) in which case $\alpha_1 = h_1 + h_2$, $\beta_1 = k_2$, $\alpha_2 = h_1 h_2$ and $\beta_2 = h_1 k_2$.

Note 5: The standard linear viscoelastic solid can be used to represent both *creep* and *stress relaxation* in materials in terms of single *retardation* and *relaxation times*, respectively.

5.6 relaxation time, τ, SI unit: s

Time characterizing the response of a viscoelastic liquid or solid to the instantaneous application of a constant strain.

Note 1: The response of a material to the instantaneous application of a constant strain is termed *stress relaxation*.

Note 2: The relaxation time of a *Maxwell element* is $\tau = 1/p_0 = \alpha/\beta$.

Note 3: The relaxation time of a *standard linear viscoelastic solid* is $\tau = 1/p_0 = \beta_1/\alpha_1$.

Note 4: Generally, a *linear viscoelastic material* has a spectrum of relaxation times, which are the reciprocals of p_i, $i = 0, 1, \ldots, n$ in the polynomial $P(\mathrm{D})$ (see Definition 5.2).

Note 5: The **relaxation spectrum** (**spectrum of relaxation times**) describing *stress relaxation* in polymers may be considered as arising from a group of *Maxwell elements* in parallel.

5.7 stress relaxation

Change in *stress* with time after the instantaneous application of a constant strain.

Note 1: The applied strain is of the form $\gamma = 0$ for $t < 0$ and $\gamma = \gamma_0$ for $t > 0$ and is usually a *uniaxial extension* or a *simple shear*.

Note 2: For *linear viscoelastic behaviour*, the *stress* takes the form

$$\sigma(t) = (c + \overline{\psi}(t))\gamma_0$$

c is a constant that is non-zero if the material has instantaneous elasticity and $\overline{\psi}(t)$ is the relaxation function.

Note 3: $\overline{\psi}(t)$ has the form

$$\overline{\psi}(t) = \sum_{i=0}^{n} \beta_i \mathrm{e}^{-p_i t}$$

where the β_i are functions of the p_i and q_i of the polynomials $P(\mathrm{D})$ and $Q(\mathrm{D})$ defining the *linear viscoelastic material*.

Note 4: The *relaxation times* of the material are $1/p_i$.

5.8 retardation time, τ, SI unit: s

Time characterizing the response of a viscoelastic material to the instantaneous application of a constant *stress*.

Note 1: The response of a material to the instantaneous application of a constant *stress* is termed *creep*.

Note 2: The *retardation time* of a *Voigt-Kelvin element* is $\tau = 1/q_0 = \beta/\alpha$ = (dashpot constant)/(spring constant).

Note 3: The *retardation time* of a *standard linear viscoelastic solid* is $\tau = 1/q_0 = \beta_2/\alpha_2$.

Note 4: Generally, a *linear viscoelastic material* has a spectrum of *retardation times*, which are reciprocals of q_i, $i = 0, 1, \ldots, n$ in the polynomial $Q(\mathrm{D})$.
Note 5: The **retardation spectrum** (spectrum of retardation times) describing *creep* in polymers may be considered as arising from a group of *Voigt-Kelvin elements* in series.

5.9 creep

Change in strain with time after the instantaneous application of a constant *stress*.
or
Time-dependent change of the dimensions of a material under a constant load.
Note 1: The applied *stress* is of the form $\sigma = 0$ for $t < 0$ and $\sigma = \sigma_0$ for $t > 0$ and is usually a *uniaxial stress* or a *simple shear stress*.
Note 2: For *linear viscoelastic behaviour*, the *strain* usually takes the form

$$\gamma(t) = (a + bt + \psi(t))\sigma_0$$

a is a constant that is non-zero if the material has instantaneous elasticity and b is a constant that is non-zero if the material is a liquid. $\psi(t)$ is the **creep function**. In addition,

$$J(t) = \gamma(t)/\sigma_0$$

is sometimes called the **creep compliance**.
Note 3: The creep function has the form

$$\psi(t) = \sum_i A_i \mathrm{e}^{-q_i t}$$

where the summation runs from $i = 0$ to n for a *solid* and 1 to n for a *liquid*. The A_i are functions of the p_i and q_i of the polynomials $P(\mathrm{D})$ and $Q(\mathrm{D})$ defining the *linear viscoelastic material* and the q_i are the q_i of the polynomial $Q(\mathrm{D})$ (see Definition 5.2).
Note 4: The *retardation times* of the material are $1/q_i$.
Note 5: Creep is sometimes described in terms of non-linear viscoelastic behaviour, leading, for example, to evaluation of recoverable shear and steady-state recoverable *shear compliance*. The definitions of such terms are outside the scope of this document.

5.10 forced oscillation

Deformation of a material by the application of a small sinusoidal strain (γ) such that

$$\gamma = \gamma_0 \cos \omega t$$

where γ_0 and ω are positive constants.
Note 1: γ may be in *simple shear* or *uniaxial deformation*.
Note 2: γ_0 is the **strain amplitude**.
Note 3: ω is the **angular velocity** of the circular motion equivalent to a sinusoidal frequency ν, with $\omega = 2\pi\nu$. The unit of ω is rad s^{-1}.
Note 4: For *linear viscoelastic behaviour*, a sinusoidal *stress* (σ) results from the sinusoidal strain with

$$\sigma = \sigma_0 \cos(\omega t + \delta) = \sigma_0 \cos\delta \cos\omega t - \sigma_0 \sin\delta \sin\omega t\,.$$

σ_0 is the **stress amplitude**. δ is the **phase angle** or **loss angle** between *stress* and strain.

Note 5: Alternative descriptions of the sinusoidal *stress* and strain in a viscoelastic material under forced oscillations are:

(i) $\gamma = \gamma_0 \sin \omega t$ $\quad \sigma = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \sin \delta \cos \omega t + \sigma_0 \cos \delta \sin \omega t$

(ii) $\gamma = \gamma_0 \cos(\omega t - \delta) = \gamma_0 \cos \delta \cos \omega t + \gamma_0 \sin \delta \sin \omega t$ $\quad \sigma = \sigma_0 \cos \omega t$

5.11 loss factor, $\tan \delta$
loss tangent

Tangent of the *phase angle* difference (δ) between *stress* and strain during forced oscillations.

Note 1: $\tan \delta$ is calculated using

$$\gamma = \gamma_0 \cos \omega t \text{ and } \sigma = \sigma_0 \cos(\omega t + \delta).$$

Note 2: $\tan \delta$ is also equal to the ratio of *loss* to *storage modulus*.

Note 3: A plot of $\tan \delta$ *vs*. temperature or frequency is known as a **loss curve**.

5.12 storage modulus in general M', SI unit: Pa
in simple shear deformation G', SI unit: Pa
in *uniaxial deformation* E', SI unit: Pa

Ratio of the amplitude of the *stress* in phase with the strain ($\sigma_0 \cos \delta$) to the amplitude of the strain (γ_0) in the *forced oscillation* of a material

$$M' = \sigma_0 \cos \delta / \gamma_0.$$

Note: See Definition 5.10 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos \omega t$ and $\sigma = \sigma_0 \cos(\omega t + \delta)$.

5.13 loss modulus in general M'', SI unit: Pa
in simple shear deformation G'', SI unit: Pa
in *uniaxial deformation* E'', SI unit: Pa

Ratio of the amplitude of the *stress* 90° out of phase with the strain ($\sigma_0 \sin \delta$) to the amplitude of the strain (γ_0) in the *forced oscillation* of a material

$$M'' = \sigma_0 \sin \delta / \gamma_0.$$

Note: See Definition 5.10 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos \omega t$ and $\sigma = \sigma_0 \cos(\omega t + \delta)$.

5.14 absolute modulus in general $|M^*|$, SI unit: Pa
in simple shear deformation $|G^*|$, SI unit: Pa
in *uniaxial deformation* $|E^*|$, SI unit: Pa

Ratio of the amplitude of the *stress* (σ_0) to the amplitude of the strain (γ_0) in the *forced oscillation* of a material

$|M^*| = \sigma_0 / \gamma_0$.

Note 1: See Definition 5.10 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos \omega t$ and $\sigma = \sigma_0 \cos(\omega t + \delta)$.
Note 2: The absolute modulus is related to the *storage modulus* and the *loss modulus* by the relationship

$$|M^*| = \left(\frac{\sigma_0^2 \cos^2 \delta}{\gamma_0^2} + \frac{\sigma_0^2 \sin^2 \delta}{\gamma_0^2} \right) = (M'^2 + M''^2)^{1/2}.$$

5.15 complex modulus in general M^*, SI unit: Pa
in simple shear deformation G^*, SI unit: Pa
in *uniaxial deformation* E^*, SI unit: Pa

Ratio of complex *stress* (σ^*) to complex strain (γ^*) in the *forced oscillation* of material

$M^* = \sigma^* / \gamma^*$.

Note 1: See Definition 5.10 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos \omega t$ and $\sigma = \sigma_0 \cos(\omega t + \delta)$.
Note 2: The **complex strain** $\gamma^* = \gamma_0 e^{i\omega t} = \gamma_0 (\cos \omega t + i \sin \omega t)$, where $i = \sqrt{-1}$, so that the real part of the complex strain is that actually applied to the material.
Note 3: SI unit: Pa The **complex stress** $\sigma^* = \sigma_0 e^{i(\omega t + \delta)} = \sigma_0 (\cos(\omega t + \delta) + i \sin(\omega t + \delta))$, so that the real part of the complex strain is that actually experienced by the material.
Note 4: The *complex modulus* is related to the *storage* and *loss moduli* through the relationships

$\mathrm{M}^* = \sigma^*/\gamma^* = \sigma_0 e^{i\delta}/\gamma_0 = (\sigma_0/\gamma_0)(\cos \delta + i \sin \delta) = M' + iM''$.

Note 5: For linear viscoelastic behaviour interpreted in terms of *complex stress* and *strain* (see notes 2 and 3)

$P(\mathrm{D})\sigma^* = Q(\mathrm{D})\gamma^*$

(see Definition 5.2). Further as $\mathrm{D}\sigma^* = \mathrm{d}\sigma^*/\mathrm{d}t = i\omega\sigma^*$ and $\mathrm{D}\gamma^* = i\omega\gamma^*$,

$M^* = \sigma^*/\gamma^* = Q(i\omega)/P(i\omega)$.

5.16 storage compliance in general C', SI unit: Pa^{-1}
in simple shear deformation J', SI unit: Pa^{-1}
in *uniaxial deformation* D', SI unit: Pa^{-1}

Ratio of the amplitude of the strain in phase with the *stress* ($\gamma_0 \cos \delta$) to the amplitude of the *stress* (σ_0) in the *forced oscillation* of a material

$C' = \gamma_0 \cos \delta / \sigma_0$.

Note: See Definition 5.10, note 5 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos(\omega t - \delta)$ and $\sigma = \sigma_0 \cos \omega t$.

5.17 loss compliance in general C", SI unit: Pa^{-1}
in simple shear deformation J'', SI unit: Pa^{-1}
in *uniaxial deformation* D'', SI unit: Pa^{-1}

Ratio of the amplitude of the strain 90° out of phase with the *stress* ($\gamma_0 \sin \delta$) to the amplitude of the *stress* (σ_0) in the *forced oscillation* of a material

$$C'' = \gamma_0 \sin \delta / \sigma_0 \cdot$$

Note: See Definition 5.10 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos(\omega t - \delta)$ and $\sigma = \sigma_0 \cos \omega t$.

5.18 absolute compliance in general $|C^*|$, SI unit: Pa^{-1}
in simple shear deformation $|J^*|$, SI unit: Pa^{-1}
in *uniaxial deformation* $|D^*|$, SI unit: Pa^{-1}

Ratio of the amplitude of the strain (γ_0) to the amplitude of the *stress* (σ_0) in the *forced oscillation* of a material

$$|C^*| = \gamma_0 / \sigma_0.$$

Note 1: See Definition 5.10, note 5 for the definition of a *forced oscillation* in which

$$\gamma = \gamma_0 \cos(\omega t - \delta) \text{ and } \sigma = \sigma_0 \cos \omega t.$$

Note 2: The absolute compliance is related to the *storage compliance* (5.16) and the *loss compliance* (Definition 5.17) by the relationship

$$|C^*| = \left(\frac{\gamma_0^2 \cos^2 \delta}{\sigma_0^2} + \frac{\gamma_0^2 \sin^2 \delta}{\sigma_0^2} \right)^{1/2} = (C'^2 + C''^2)^{1/2}.$$

Note 3: The absolute compliance is the reciprocal of the *absolute modulus*.

$$|C^*| = 1/|M^*|$$

5.19 complex compliance in general C^*, SI unit: Pa^{-1}
in simple shear deformation J^*, SI unit: Pa^{-1}
in *uniaxial deformation* D^*, SI unit: Pa^{-1}

Ratio of complex strain (γ^*) to complex *stress* (σ^*) in the *forced oscillation* of a material

$$C^* = \gamma^* / \sigma^*.$$

Note 1: See Definition 5.10 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos(\omega t - \delta)$ and $\sigma = \sigma_0 \cos \omega t$.
Note 2: The **complex strain** $\gamma^* = \gamma_0 e^{i(\omega t - \delta)} = \gamma_0 (\cos(\omega t - \delta) + i \sin(\omega t - \delta))$, where $i = \sqrt{-1}$, so that the real part of the complex strain is that actually experienced by the material.
Note 3: The **complex stress** $\sigma^* = \sigma_0 e^{i\omega t} = \sigma_0 (\cos \omega t + i \sin \omega t)$, so that the real part of the complex *stress* is that actually applied to the material.

Note 4: The complex compliance is related to the *storage* and *loss compliances* through the relationships

$$C^* = \gamma^*/\sigma^* = \gamma_0 e^{-i\delta}/\sigma_0 = (\gamma_0/\sigma_0)(\cos\delta - i\sin\delta) = C' - iC''$$

Note 5: The complex compliance is the reciprocal of the *complex modulus*.

$$C^* = 1/M^*$$

5.20 dynamic viscosity, η', SI unit: Pa s
Ratio of the *stress* in phase with the rate of strain ($\sigma_0 \sin\delta$) to the amplitude of the rate of strain ($\omega\gamma_0$) in the *forced oscillation* of a material

$$\eta' = \sigma_0 \sin\delta / \omega\gamma_0.$$

Note 1: See Definition 5.10, note 5 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \sin\omega t$ and $\sigma = \sigma_0 \sin(\omega t + \delta)$, so that $\dot{\gamma} = \omega\gamma_0 \cos\omega t$ and $\sigma = \sigma_0 \sin\delta\cos\omega t + \sigma_0 \cos\delta\sin\omega t$.
Note 2: See Definition 5.2, note 6: $\eta' = M''/\omega$ may be used for evaluating the *dynamic viscosity*. The same expression is often used to evaluate the *shear viscosity*. The latter use of this expression is not recommended.

5.21 out-of-phase viscosity, η'', SI unit: Pa s
Ratio of the *stress* 90° out of phase with the rate of strain ($\sigma_0 \cos\delta$) to the amplitude of the rate of strain ($\omega\gamma_0$) in the *forced oscillation* of a material

$$\eta'' = \sigma_0 \cos\delta / \omega\gamma_0 .$$

Note 1: See Definition 5.10, note 5 for the definition of a *forced oscillation* in which
$\gamma = \gamma_0 \sin\omega t$ and $\sigma = \sigma_0 \sin(\omega t + \delta)$, so that $\dot{\gamma} = \omega\gamma_0 \cos\omega t$ and
$\sigma = \sigma_0 \sin\delta\cos\omega t + \sigma_0 \cos\delta\sin\omega t$.
Note 2: See Definition 5.22, note 6: $\eta'' = M'/\omega$ may be used to evaluate the *out-of-phase viscosity*.

5.22 complex viscosity, η^*, SI unit: Pa s
Ratio of complex *stress* (σ^*) to complex rate of strain ($\dot{\gamma}^*$) in the *forced oscillation* of a material

$$\eta^* = \sigma^*/\dot{\gamma}^*.$$

Note 1: See Definition 5.10, note 5 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \sin\omega t$ and $\sigma = \sigma_0 \cos(\omega t + \delta)$ and the rate of strain $\dot{\gamma} = \omega_0 \cos\omega t$.
Note 2: The **complex rate of strain** $\dot{\gamma}^* = i\,\omega\gamma_0 e^{i\omega t} = i\,\omega\gamma_0 (\cos\omega t + i\sin\omega t)$, where $i = \sqrt{-1}$.
Note 3: The **complex stress** $\sigma^* = \sigma_0 e^{i(\omega t+\delta)} = \sigma_0 (\cos(\omega t+\delta) + i\sin(\omega t+\delta))$.
Note 4: The *complex viscosity* may alternatively be expressed as

$$\eta^* = \sigma^*/\dot{\gamma}^* = (\sigma_0 e^{i\delta})/(i\,\omega\gamma_0) = M^*/i\,\omega$$

where M^* is the *complex modulus*.
Note 5: The complex viscosity is related to the *dynamic* and *out-of-phase viscosities* through the relationships

$$\eta^* = \sigma^*/\dot{\gamma}^* = \sigma_0\,(\cos\delta + i\sin\delta)/(i\,\omega\gamma_0) = \eta' - i\,\eta''.$$

Note 6: The *dynamic* and *out-of-phase viscosities* are related to the *storage* and *loss moduli* by the relationships $\eta^* = \eta' - i\,\eta'' = M^*/i\,\omega = (M' + i\,M'')/i\,\omega$, so that $\eta' = M''/\omega$ and $\eta'' = M'/\omega$.

6 OSCILLATORY DEFORMATIONS AND STRESSES USED EXPERIMENTALLY

There are three modes of **free** and **forced** oscillatory deformations which are commonly used experimentally, **torsional oscillations**, **uniaxial extensional oscillations** and **flexural oscillations.**

The oscillatory deformations and *stresses* can be used for solids and liquids. However, the apparatuses employed to measure them are usually designed for solid materials. In principle, they can be modified for use with liquids.

Analyses of the results obtained depend on the shape of the specimen, whether or not the distribution of mass in the specimen is accounted for and the assumed model used to represent the linear viscoelastic properties of the material. The following terms relate to analyses which generally assume small deformations, specimens of uniform cross-section, non-distributed mass and a *Voigt-Kelvin* solid. These are the conventional assumptions.

6.1 free oscillation

Oscillatory deformation of a material specimen with the motion generated without the continuous application of an external force.
Note: For any real sample of material the resulting oscillatory deformation is one of decaying amplitude.

6.2 damping curve

Decreased deformation of a material specimen vs. time when the specimen is subjected to a *free oscillation*.
Note 1: See Definition 6.1 for the definition of a *free oscillation*.
Note 2: The term ‘damping curve’ is sometimes used to describe a *loss curve*.
Note 3: A damping curve is usually obtained using a **torsion pendulum**, involving the measurement of decrease in the axial, torsional displacement of a specimen of uniform cross-section of known shape, with the torsional displacement initiated using a torsion bar of known moment of inertia.
Note 4: Damping curves are conventionally analysed in terms of the *Voigt-Kelvin* solid giving a decaying amplitude and a single frequency.
Note 5: Given the properties of a *Voigt-Kelvin* solid, a damping curve is described by the equation

$$X = A\exp(-\beta t)\sin(\omega t - \phi),$$

where X is the displacement from equilibrium (for torsion $X = \theta$, the angular displacement), t is time, A is the amplitude, β is the *decay constant*, ω is the *angular velocity* corresponding to the *decay frequency* and ϕ is the *phase angle*.

6.3 decay constant, β, SI unit: s^{-1}

Exponential coefficient of the time-dependent decay of a *damping curve*, assuming Voigt-Kelvin behaviour.

Note 1: See *damping curve* and the equation therein (Definition 6.2, note 5).

Note 2: See *Voigt-Kelvin* solid.

Note 3: For small damping, β is related to the *loss modulus* (M''), through the equation

$$M'' = 2\beta\omega/H,$$

where ω is the *angular velocity* corresponding to the *decay frequency*. H is a parameter that depends on the cross-sectional shape of the specimen and the type of deformation. (For example, for the axial torsion of a circular rod of radius a and length l using a *torsion pendulum* with a torsion bar of moment of inertia I

$$H = \pi a^4/(2Il)$$

and $M'' \cong G''$, the *loss modulus* in simple shear).

6.4 decay frequency, ν, SI unit: Hz

Frequency of a *damping curve* assuming *Voigt-Kelvin* behaviour.

Note 1: See *damping curve* and the equation therein.

Note 2: See *Voigt-Kelvin* solid.

Note 3: $\nu = \omega/2\pi$, where ω is the *angular velocity* corresponding to ν.

Note 4: For small damping, the *storage modulus* (M') may be evaluated from ω through the equation

$$M' = \omega^2/H,$$

where H is as defined in Definition 6.3, note 3. Again, for torsion, $M' \cong G'$, the *storage modulus* in simple shear.

6.5 logarithmic decrement, Λ

Natural logarithm of the ratio of the displacement of a *damping curve* separated by one period of the displacement.

Note 1: *Voigt-Kelvin* behaviour is assumed so that the displacement decays with a single period T, where

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega},$$

with ν the frequency and ω is the *angular velocity* corresponding to ν.

Note 2: The *logarithmic decrement* can be used to evaluate the *decay constant*, β. From the equation for the *damping curve* of a *Voigt-Kelvin* solid.

$$\Lambda = \ln (X_n/X_{n+1}) = \beta(t_{n+1} - t_n) = \beta T,$$

where X_n and t_n are the displacement and time at a chosen point (usually near a maximum) in the n-th period of the decay, and X_{n+1} and t_{n+1} are the corresponding displacement and time one period later.
Note 3: Λ can also be defined using displacements k periods apart, with

$$\Lambda = (1/k)\ln (X_n/X_{n+k}).$$

Note 4: For small damping, Λ is related to the *loss tangent*, $\tan \delta$ by

$$\tan \delta = M''/M' = 2\beta/\omega = 2\Lambda/T\omega = \Lambda/\pi$$

(See Definitions 6.3 and 6.4 for expressions for M' and M'').

6.6 forced uniaxial extensional oscillations
Uniaxial extensional deformation of a material specimen of uniform cross-sectional area along its long axis by the continuous application of a sinusoidal force of constant amplitude.
Note 1: For a specimen of negligible mass, the linear-viscoelastic interpretation of the resulting deformation gives

$$(A/L)Q(\mathrm{D})l = P(\mathrm{D})f_0 \cos \omega t$$

where $P(\mathrm{D})$ and $Q(\mathrm{D})$ are the polynomials in D (=d/dt) characterizing the *linear-viscoelastic behaviour*, A is the cross-sectional area of the specimen, L its original length, l is here the change in length, f_0 the amplitude of the applied force of *angular velocity* ω and t the time.
Note 2: For a *Voigt-Kelvin* solid, with $P(\mathrm{D})=1$ and $Q(\mathrm{D})=\alpha+\beta\mathrm{D}$, where α is the spring constant and β the dashpot constant, the equation describing the deformation becomes

$$(A/L)\beta(\mathrm{d}l/\mathrm{d}t) + (A/L)\alpha l = f_0 \cos \omega t$$

or, in terms of *stress* and *strain*,

$$\alpha\varepsilon + \beta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \sigma_0 \cos \omega t$$

where $\varepsilon = l/L$ is the *uniaxial strain* and $\sigma_0 = f_0/A$ is the amplitude of the *stress*. The solution of the equation is

$$\varepsilon = \frac{\sigma_0}{\left(\alpha^2 + \beta^2\omega^2\right)^{1/2}}\cos(\omega t - \delta) = \varepsilon_0 \cos(\omega t - \delta)$$

where δ is the *phase angle* with $\tan\,\delta = \beta\omega/\alpha$.
Note 3: From Definition 5.14, the *absolute modulus* in *uniaxial deformation*

$$|E^*| = \sigma_0/\varepsilon_0 = (\alpha^2+\beta^2\omega^2)^{1/2}$$

where $\alpha = E'$, $\beta\omega = E''$ and $\tan\,\delta = E''/E'$ equal to the *loss tangent*.
Note 4: If one end of the specimen is fixed in position and a mass m is attached to the moving end, the *linear-viscoelastic* interpretation of the resulting deformation gives

$$mP(\mathrm{D})(\mathrm{d}^2l/\mathrm{d}t^2)+(A/L)Q(\mathrm{D})l = P(\mathrm{D})f_0 \cos\,\omega t$$

where the symbols have the same meaning as in note 1.
Note 5: For a *Voigt-Kelvin* solid (cf. note 2), the equation in note 4 describing the deformation becomes

$$m(\mathrm{d}^2l/\mathrm{d}t^2)+(A/L)\beta(\mathrm{d}l/\mathrm{d}t)+(A/L)\alpha\cdot l = f_0 \cos\,\omega t$$

with the solution

$$\varepsilon = \frac{\sigma_0 \cdot \left(A/(Lm)\right)}{\left(\left(\dfrac{A_\infty}{Lm} - \omega^2\right)^2 + \omega^2\left(\dfrac{A\beta}{Lm}\right)^2\right)^{1/2}} \cos(\omega t - \theta) = \varepsilon_0 \cos(\omega t - \theta)$$

where $\tan\,\theta = \dfrac{(A\beta/Lm)\omega}{(A\alpha/Lm) - \omega^2}$.

Note 6: The amplitude of the strain ε_0 is maximal when

$$\omega^2 = A\alpha/(Lm) = \omega_\mathrm{R}^2$$

giving the value of the *angular velocity* (ω_R) of the *resonance frequency* of the specimen in forced uniaxial extensional oscillation.
Note 7: Notes 2 and 5 show that application of a sinusoidal uniaxial force to a *Voigt-Kelvin* solid of negligible mass, with or without added mass, results in an out-of-phase sinusoidal uniaxial extensional oscillation of the same frequency.

6.7 forced flexural oscillation
Flexural deformation (bending) of a material specimen of uniform cross-sectional area perpendicular to its long axis by the continuous application of a sinusoidal force of constant amplitude.
Note 1: There are three modes of flexure in common use.
(i) Application of the flexural force at one end of the specimen with the other end clamped.
(ii) Application of the flexural force at the centre of the specimen with the two ends clamped (**three-point bending or flexure**).
(iii) Application of the flexural force at the centre of the specimen with the two ends resting freely on supports (also known as **three-point bending** or **flexure**).

Note 2: For specimens *without mass*, the linear-viscoelastic interpretation of the resulting deformations follows a differential equation of the same form as that for *a uniaxial extensional forced oscillation*, namely

$$(HJ/L^3)Q(\mathrm{D})y = P(\mathrm{D})f_0 \cos \omega t$$

where $P(\mathrm{D})$, $Q(\mathrm{D})$, f_0, ω and t have the same meaning as for a *forced uniaxial extensional oscillation* and H is a constant. The length of the specimen is $2L$. For mode of flexure (i) H=3, for (ii) H=24 and for (iii) H=6. J is the **second moment of area** of the specimen, defined by

$$J = \int_A q^2 \mathrm{d}A$$

where dA is an element of the cross-sectional area (A) of the specimen and q is the distance of that element from the **neutral axis or plane** of the specimen, lying centrally in the specimen and defined by points which experience neither compression nor extension during the flexure. For a specimen of circular cross-section $J=\pi r^2/4$, where r is the radius, and for one of rectangular cross-section $J=4ab^3/3$, where $2a$ and $2b$ are the lateral dimensions with flexure along the b dimension. Finally, y is the *flexural deflection* of the specimen at the point of application of the force, of either the end (mode of flexure (i)) or the middle (modes of flexure (ii) and (iii)).
Note 3: For a *Voigt-Kelvin* solid, the equation describing the deformation becomes

$$(HJ/L^3)\alpha \cdot y + (HJ/L^3)\beta(\mathrm{d}y/\mathrm{d}t) = f_0 \cos \omega t$$

with solution

$$y = \frac{f_0 L^3}{HJ\left(\alpha^2 + \beta^2\omega^2\right)^{1/2}} \cos(\omega t - \delta)$$

where δ is the *phase angle* with

$$\tan \delta = \beta\omega / \alpha$$

equal to the *loss tangent*.
Note 4: Unlike the strain in *forced uniaxial extensional oscillations*, those in *forced flexural deformations* are not homogeneous. In the latter modes of deformation, the strains vary from point-to-point in the specimen. Hence, the equation defining the displacement y in terms of the amplitude of applied force (f_0) cannot be converted into one defining strain in terms of amplitude of *stress*.
Note 5: If a mass m is attached to the specimen at the point of application of the force, the linear-viscoelastic interpretation of the resulting deformation gives

$$m \cdot P(\mathrm{D})\,(\mathrm{d}^2 y/\mathrm{d}t^2) + (HJ/L^3)Q(\mathrm{D})y = P(\mathrm{D})f_0 \cos \omega t$$

(cf. Definition 6.6, note 4).

Note 6: For a *Voigt-Kelvin solid* (cf. note 3 and Definition 6.6, note 5), the equation describing the deformation becomes

$$m(\mathrm{d}^2y/\mathrm{d}t^2)+(HJ/L^3)\beta(\mathrm{d}y/\mathrm{d}t)+(HJ/L^3)\alpha\cdot y = f_0\cos\,\omega t$$

with the solution

$$y = \frac{f_0/m}{\left(\left(\frac{HJ\alpha}{L^3 m}-\omega^2\right)^2+\omega^2\left(\frac{HJ\beta}{L^3 m}\right)^2\right)^{1/2}}\cos(\omega t - \delta)$$

where $\tan\,\delta = \dfrac{\left(HJ\beta/L^3 m\right)\omega}{\left(HJ\alpha/L^3 m\right)-\omega^2}$.

Note 7: The *flexural deflection y* is maximal when

$$\omega^2 = HJ\alpha/(L^3 m) = \omega_R^2$$

giving the value of the *angular velocity* (ω_R) of the *resonance frequency* of the specimen in forced flexural oscillations.
Note 8: Notes 3 and 6 show that the application of the defined sinusoidal flexural forces (i), (ii) and (iii) (note 1) to a *Voigt-Kelvin* solid of negligible mass, with or without added mass at the points of application of the forces, results in out-of-plane sinusoidal flexural oscillations of the same frequency.

6.8 flexural force, f_0, SI unit: N
Amplitude of the force applied to a material specimen to cause a *forced flexural oscillation*.
Note: A related quantity is the **flexural stress** which is somewhat arbitrarily defined as the amplitude of the *stress* in the convex, outer surface of a material specimen in *forced flexural oscillation*.

6.9 flexural deflection, y, SI unit: m
Deflection of a specimen subject to a *forced flexural oscillation* at the point of application of the *flexural force*.

6.10 flexural modulus, $|E^*|$, SI unit: Pa
Modulus measured using *forced flexural oscillations*.
Note 1: For a *Voigt-Kelvin solid* of negligible mass, the *absolute modulus* can be evaluated from the ratio of the *flexural force* (f_0) and the amplitude of the *flexural deflection* (y) with

$$f_0/Y_0 = (HJ/L^3)(\alpha^2 + \beta^2\omega^2)^{1/2}$$

where Y_0 is the amplitude of the *flexural deflection*,

$$|E^*| = (\alpha^2 + \beta^2\omega^2)^{1/2}$$

(see Definitions 5.14 and 6.6, note 3) and the remaining symbols are as defined in Definition 6.7, note 2.
Note 2: The ratio of the loss to the storage flexural modulus (E''/E') is derived from the *loss tangent* ($\tan\delta$) of the *forced flexural oscillation* with

$$\tan\delta = \beta\omega/\alpha = E''/E'$$

(see Definitions 5.11 and 6.7, note 3).
Note 3: The flexural modulus has been given the same symbol as the *absolute modulus* in *uniaxial deformation* as it becomes equal to that quantity in the limit of zero amplitudes of applied force and deformation. Under real experimental conditions it is often used as an approximation to $|E^*|$.

6.11 resonance curve, $A(\nu)$, SI unit: that of the amplitude A
Curve of the frequency dependence of the amplitude of the displacement of a material specimen subject to *forced oscillation*s in the region of a *resonance frequency*.
Note: See Definitions 6.6 and 6.7 for the description of modes of *forced oscillation* commonly used.

6.12 resonance frequency, ν_R, SI unit: Hz
Frequency at a maximum of a *resonance curve*.
Note 1: Material specimens subject to a *forced oscillation* in general have a spectrum of resonance frequencies.
Note 2: In cases of a single *resonance frequency*, the *resonance frequency* is proportional to the square root of the *storage modulus* (M') of the material.
Note 3: A material specimen which behaves as a *Voigt-Kelvin solid* under *forced oscillation*s with a mass added at the point of application of the applied oscillatory force has a single resonance frequency.
Note 4: Under a *forced uniaxial extensional oscillation* the resonance frequency

$$\nu_R = \omega_R/2\pi = \left(\frac{A\alpha}{Lm}\right)^{1/2} \Big/ 2\pi = \left(\frac{AE'}{Lm}\right)^{1/2} \Big/ 2\pi$$

(see Definition 6.6 for the origin of the equation and definitions of symbols). E' is *the storage modulus in uniaxial extension*.
Note 5: Under a *forced flexural oscillation* the resonance frequency

$$\nu_R = \omega_R/2\pi = \left(\frac{HJ\alpha}{L^3 m}\right)^{1/2} \Big/ 2\pi = \left(\frac{HJE'}{L^3 m}\right)^{1/2} \Big/ 2\pi$$

(see Definition 6.7 for the origin of the equation and the definition of symbols).

6.13 width of the resonance curve, $\Delta\nu$, SI unit: Hz
Magnitude of the difference in frequency between two points on a resonance curve on either side of ν_R which have amplitudes equal to $\left(1/\sqrt{2}\right)A(\nu_R)$.

Note 1: For a material specimen which behaves as a *Voigt-Kelvin solid* under *forced uniaxial extensional oscillation* with mass added at the point of application of the applied oscillatory force, $\Delta\nu$ is proportional to the *loss modulus* (E'').

$$2\pi\Delta\nu = \frac{A\beta}{Lm} = \frac{AE''}{Lm\omega_{\mathrm{R}}}$$

In addition the *storage modulus* (E')) may be evaluated from

$$\omega_{\mathrm{R}}^2 = \frac{A\alpha}{Lm} = \frac{AE'}{Lm}$$

(see Definition 6.6 for the definition of symbols).

Note 2: For a material specimen which behaves as *Voigt-Kelvin solid* under *forced flexural oscillations* with added mass at the point of application of the applied oscillatory force, $\Delta\nu$ is proportional to the *loss modulus* (E'')

$$2\pi\Delta\nu = \frac{HJ\beta}{L^3 m} = \frac{HJE''}{L^3 m\omega_{\mathrm{R}}}\ .$$

In addition, the *storage modulus* (E') may be evaluated from

$$\omega_{\mathrm{R}}^2 = \frac{HJ\alpha}{L^3 m} = \frac{HJE'}{L^3 m}$$

(see Definition 6.7 for the definition of symbols).

Note 3: For the *Voigt-Kelvin* behaviours specified in notes 1 and 2, the ratio of the *width of the resonance curve* ($\Delta\nu_{\mathrm{R}}$) to the *resonance frequency* (ν_{R}) is equal to the *loss tangent* ($\tan\delta$).

Under *forced uniaxial extensional oscillation*

$$\frac{\Delta\nu_{\mathrm{R}}}{\nu_{\mathrm{R}}} = \left(\frac{A}{Lm}\right)\beta\omega_{\mathrm{R}} \cdot \frac{Lm}{A\alpha} = \frac{\beta}{\alpha}\omega_{\mathrm{R}} = \frac{E''}{E'} = \tan\delta\ .$$

Under *forced flexural oscillation*

$$\frac{\Delta\nu_{\mathrm{R}}}{\nu_{\mathrm{R}}} = \left(\frac{HJ}{L^3 m}\right)\beta\omega_{\mathrm{R}} \cdot \frac{L^3 m}{HJ\alpha} = \frac{\beta}{\alpha}\omega_{\mathrm{R}} = \frac{E''}{E'} = \tan\delta\ .$$