

Maximum Likelihood and Bayesian Methods

Shyue Ping Ong

University of California, San Diego

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Overview

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- 2 Maximum Likelihood Inference
- 3 Bayesian Methods
- 4 Expectation Maximization

Preliminaries

- Thus far, we have discussed ML models in the context of minimizing a loss function, typically a sum of squares.
- These approaches have a probabilistic interpretation and are instances of the maximum likelihood (ML) approach to fitting.

Maximum Likelihood Inference

- Consider a probability distribution/mass function for the observations:

$$z_i \sim g_{\theta}(z)$$

- Also, known as a parametric model for Z , with θ being unknown parameter that govern the distribution.
- Let us assume that Z has a normal distribution ($\theta = (\mu, \sigma^2)$):

$$g_{\theta} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

- The likelihood of the observed data under model g_{θ} is then given by the *likelihood function*:

$$L(\theta; \mathbf{Z}) = \prod_{i=1}^N g_{\theta}(z_i)$$

Maximum Likelihood Inference, cont.

- Consider the log of L :

$$\ell(\theta; \mathbf{Z}) = \sum_{i=1}^N \log g_{\theta}(z_i)$$

- Score function $\dot{\ell}(\theta; \mathbf{Z}) = 0$ at maximum (dot means derivative).
- Information matrix:

$$\mathbf{I}(\theta) = - \sum_{i=1}^N \frac{\partial^2 \ell(\theta; z_i)}{\partial \theta \partial \theta^T}$$

- Observed information $= \mathbf{I}(\hat{\theta})$ ($\hat{\theta}$ is the estimated parameter)
- Fisher information $\mathbf{i}(\theta) = E_{\theta}[\mathbf{I}(\theta)]$, widely used in optimal experimental design.

Linear Regression Revisited as MLE

- $Y = \beta_0 + \beta_1 X + \varepsilon$
- $\varepsilon \sim N(0, \sigma^2)$, i.e., independent Gaussian noise.

$$\begin{aligned}
 L(\beta_0, \beta_1, \sigma; \mathbf{y}) &= \prod_{i=1}^N P(y_i | x_i; \hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}) \\
 &= \prod_{i=1}^N \frac{1}{\hat{\sigma} \sqrt{2\pi}} e^{-\frac{(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{2\hat{\sigma}^2}} \\
 \ell(\beta_0, \beta_1, \sigma; \mathbf{y}) &= \sum_{i=1}^N \log \frac{1}{\hat{\sigma} \sqrt{2\pi}} e^{-\frac{(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{2\hat{\sigma}^2}} \\
 &= -\frac{N}{2} \log \hat{\sigma}^2 2\pi - \sum_{i=1}^N \frac{(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{2\hat{\sigma}^2}
 \end{aligned}$$

- Clearly, MLE in this case is equivalent to minimizing least squares.

Bayesian Methods

- *Posterior* distribution given by:

$$P(\theta|\mathbf{Z}) = \frac{P(\mathbf{Z}|\theta)P(\theta)}{\int P(\mathbf{Z}|\theta)P(\theta)d\theta}$$

- Updated knowledge about θ after seeing data.

Expectation Maximization

- Popular approach for solving MLE problems.
- Algorithm (example for two-component Gaussian mixture):
 - 1 Start with initial guesses for parameters (e.g, two random y_i).
 - 2 Expectation step: Compute the *responsibilities*.

$$\hat{\gamma}_i = \frac{\hat{\pi} \phi_{\hat{\theta}_2}(y_i)}{(1 - \hat{\pi}) \phi_{\hat{\theta}_1}(y_i) + \hat{\pi} \phi_{\hat{\theta}_2}(y_i)}, i = 1, 2, \dots, N$$

- 3 Maximization step: Compute weighted means and variances.

$$\begin{aligned} \hat{\mu}_1 &= \frac{\sum_{i=1}^N (1 - \hat{\gamma}_i) y_i}{\sum_{i=1}^N (1 - \hat{\gamma}_i)}, \hat{\sigma}_1^2 = \frac{\sum_{i=1}^N (1 - \hat{\gamma}_i) (y_i - \hat{\mu}_1)^2}{\sum_{i=1}^N (1 - \hat{\gamma}_i)} \\ \hat{\mu}_2 &= \frac{\sum_{i=1}^N \hat{\gamma}_i y_i}{\sum_{i=1}^N \hat{\gamma}_i}, \hat{\sigma}_2^2 = \frac{\sum_{i=1}^N \hat{\gamma}_i (y_i - \hat{\mu}_2)^2}{\sum_{i=1}^N \hat{\gamma}_i} \end{aligned}$$

with mixing probability $\hat{\pi} = \sum_{i=1}^N \hat{\gamma}_i / N$.

- 4 Iterate until convergence.

Gaussian mixture using EM in scikit-learn

```
from sklearn.mixture import GaussianMixture

estimator = GaussianMixture(
    n_components=n_classes, covariance_type="spherical", max_iter=20, random_state=0
)
estimator.means_init = np.array(
    [x_train[y_train == i].mean(axis=0) for i in range(n_classes)]
)
estimator.fit(x_train)
```

The End