Zadatak 1.

a) Prijenosna funkcija otvorenog kruga glasi:

$$G_0(s) = G_R(s)G_p(s) = \frac{K_p}{T_I s\left(1 + T_p s\right)}$$

Na presječnoj frekvenciji amplituda u Bodeovom dijagramu iznosi 0, odnosno:

$$A = 20\log|G_0(j\omega_c)| = 0\ [\mathrm{dB}] \rightarrow |G_0(j\omega_c)| = 1$$

$$|G_0(j\omega_c)| = \frac{K_p}{T_I\omega_c\sqrt{1 + T_p^2\omega_c^2}} = \frac{8}{T_I\cdot 2\sqrt{1 + 1.2^2\cdot 2^2}} = 1 \rightarrow \boldsymbol{T_I} = \frac{8}{5.2} = \mathbf{1.53846153\ [s]}$$

Za fazno osiguranje zapišemo prijenosnu funkciju u obliku kompleksnog broja:

$$G_0(j\omega_c) = \frac{8}{j\frac{8}{5.2}\cdot 2(1 + j1.2\cdot 2)} = -0.923088 - j0.384615$$

Iz toga slijedi faza:

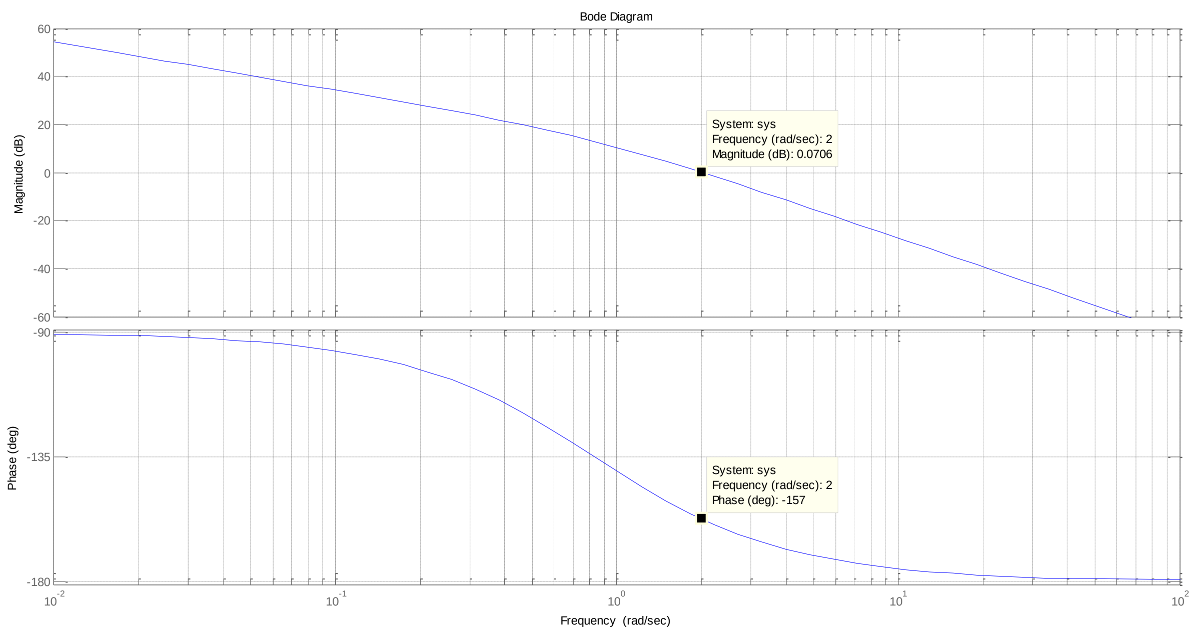
$$\varphi(\omega_c) = -157.38°$$

Fazno osiguranje je jednako:

$$\boldsymbol{\gamma_1} = 180° + \varphi(\omega_c) = 180° - 157.38° = \mathbf{22.62°}$$

Za Bodeov dijagram funkciju prikažemo u obliku:

$$G_0(j\omega) = \frac{8}{\frac{8}{5.2}j\omega(1 + 1.2j\omega)} = \frac{1}{j\frac{\omega}{5.2}\left(1 + j\frac{\omega}{\frac{1}{1.2}}\right)} = \frac{1}{j\frac{\omega}{5.2}\left(1 + j\frac{\omega}{0.8\dot{3}}\right)}$$



Slika 1. Bodeov dijagram

b) Kao i prethodne godine, na materijalima je 5. DZ.

c) Preporučeni raspon je:

$$T = (0.17 \div 0.34)\frac{1}{\omega_c} = 85\text{ ms } \div 170\text{ ms}$$

Odabire se vrijeme $\boldsymbol{T = 120\ [\mathrm{ms}]}$.

d) Dakle, imamo tri dijela:

I) Tustinova relacija:

$$\boldsymbol{G_R(z)} = G_R(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{1}{T_I\frac{2}{T}\frac{z-1}{z+1}} = \frac{z+1}{\frac{8\cdot 2}{5.2\cdot 0.12}(z-1)} = 0.039\frac{1+z^{-1}}{1-z^{-1}} = \boldsymbol{0.039\frac{z+1}{z-1}}$$

$$G_R(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \rightarrow b_0 = 0.039, b_1 = 0.039, a_1 = -1$$

$$u(k) = \sum_{i=0}^{m} b_i e(k-i) - \sum_{i=1}^{n} a_i u(k-i) = \sum_{i=0}^{1} b_i e(k-i) - \sum_{i=1}^{1} a_i u(k-i)$$

$$u(k) = b_0 e(k) + b_1 e(k-1) - a_1 u(k-1) \rightarrow \boldsymbol{u(k) = 0.039[e(k) + e(k-1)] + u(k-1)}$$

II) Eulerova unaprijedna diferencija:

$$\boldsymbol{G_R(z)} = G_R(s)\Big|_{s=\frac{z-1}{T}} = \frac{1}{T_I\frac{z-1}{T}} = \frac{1}{\frac{8}{5.2\cdot 0.12}(z-1)} = 0.078\frac{z^{-1}}{1-z^{-1}} = \boldsymbol{0.078\frac{1}{z-1}}$$

$$G_R(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \rightarrow b_0 = 0, b_1 = 0.078, a_1 = -1$$

$$u(k) = \sum_{i=0}^{m} b_i e(k-i) - \sum_{i=1}^{n} a_i u(k-i) = \sum_{i=0}^{1} b_i e(k-i) - \sum_{i=1}^{1} a_i u(k-i)$$

$$u(k) = b_0 e(k) + b_1 e(k-1) - a_1 u(k-1) \rightarrow \boldsymbol{u(k) = 0.078e(k-1) + u(k-1)}$$

II) Eulerova unazadna diferencija:

$$\boldsymbol{G_R(z)} = G_R(s)\Big|_{s=\frac{z-1}{Tz}} = \frac{1}{T_I\frac{z-1}{Tz}} = \frac{z}{\frac{8}{5.2\cdot 0.12}(z-1)} = 0.078\frac{1}{1-z^{-1}} = \boldsymbol{0.078\frac{z}{z-1}}$$

$$G_R(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \rightarrow b_0 = 1, b_1 = 0, a_1 = -1$$

$$u(k)=\sum_{i=0}^{m} b_i e(k-i)-\sum_{i=1}^{n} a_i u(k-i)=\sum_{i=0}^{1} b_i e(k-i)-\sum_{i=1}^{1} a_i u(k-i)$$

$$u(k)=b_0 e(k)+b_1 e(k-1)-a_1 u(k-1)\rightarrow \boldsymbol{u(k)=0.078e(k)+u(k-1)}$$

Zajedničko obilježje svih dobivenih diskretnih regulatora jest da svi imaju pol $z_p=1$ (integrator u kontinuiranoj domeni). Upravljački signal $u(k)$ svakog od regulatora ovisi o upravljačkom signalu iz prehodnog koraka $u(k-1)$.

e) ZOH diskretizacija:

$$G_p(z)=(1-z^{-1})\mathbf{Z}\left\{\frac{G_p(s)}{s}\right\}=(1-z^{-1})\mathbf{Z}\left\{\frac{K_p}{s(1+T_p s)}\right\}=(1-z^{-1})\mathbf{Z}\left\{\frac{K_p}{T_p s\left(\frac{1}{T_p}+s\right)}\right\}=$$

$$=(1-z^{-1})\mathbf{Z}\left\{K_p\frac{\frac{1}{T_p}}{s\left(\frac{1}{T_p}+s\right)}\right\}=(1-z^{-1})K_p\frac{\left(1-e^{-\frac{T}{T_p}}\right)z^{-1}}{(1-z^{-1})\left(1-e^{-\frac{T}{T_p}}z^{-1}\right)}=K_p\frac{\left(1-e^{-\frac{T}{T_p}}\right)z^{-1}}{1-e^{-\frac{T}{T_p}}z^{-1}}$$

$$\boldsymbol{G_p(z)}=8\frac{\left(1-e^{-\frac{0.12}{1.2}}\right)z^{-1}}{1-e^{-\frac{0.12}{1.2}}z^{-1}}=\frac{0.761301z^{-1}}{1-0.904837418036z^{-1}}=\boldsymbol{\frac{0.761301}{z-0.904837418036}}$$

Sada je prijenosna funkcija otvorenog kruga jednaka:

$$G_0(z)=G_R(z)G_p(z)=0.039\frac{z+1}{z-1}\frac{0.761301}{z-0.904837418036}=\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}$$

$$1+G_0(z)=1+\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}=0$$

pa je karakteristična jednadžba jednaka:

$$f(z)=z^2-1.90484z+0.904837+0.0296907(z+1)=z^2-1.8751493z+0.9345277$$

$$a_0=0.9345277,\quad a_1=-1.8751493,\quad a_2=1$$

Juryjev kriterij:

Uvjet A: $f(1)=1-1.8751493+0.9345277>0$
$(-1)^n f(-1)=(-1)^2(1+1.8751493+0.9345277)>0$

Uvjet B:

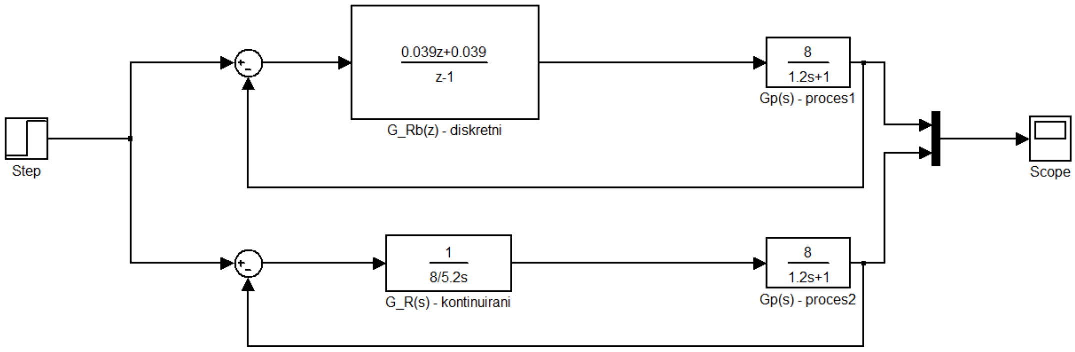
Tablica izgleda ovako:

Redak	z^0	z^1	z^2
1	0.9345277	−1.8751493	1

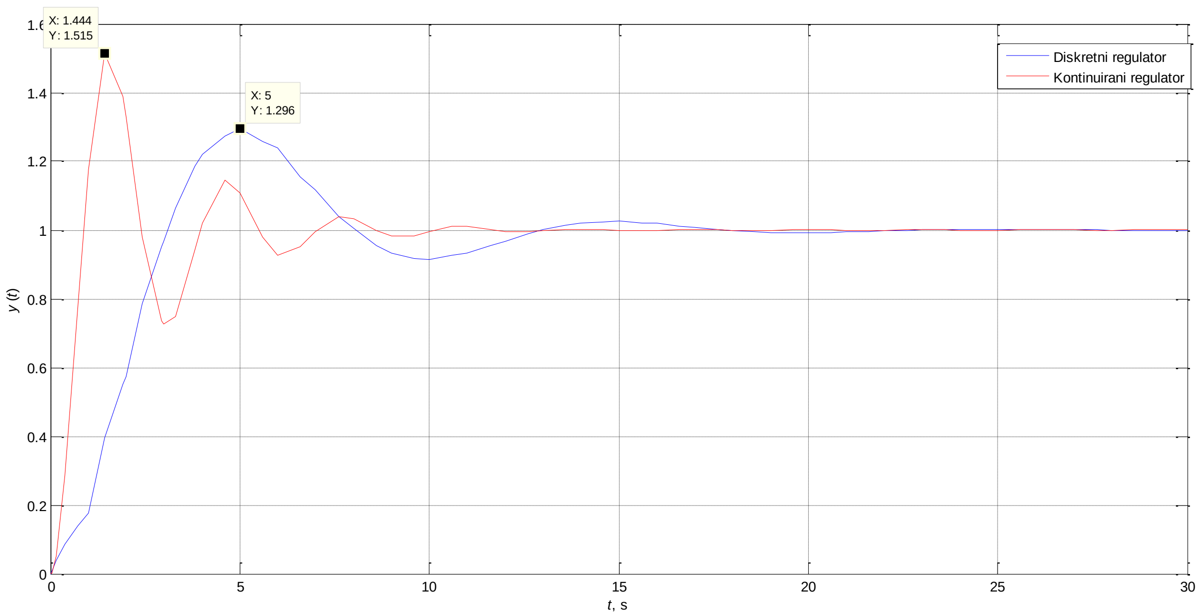
$$|a_0|<|a_n|\rightarrow|0.9345277|<|1|$$

Oba uvjeta su zadovoljena pa je sustav **stabilan**.

f)



Slika 2. Shema u *Simulinku*



Slika 3. Odziv na skokovitu promjenu

Iz očitanih vrijednosti vrijedi sljedeće:

1. diskretni regulator:

$$\boldsymbol{\sigma_m}[\%] = \frac{1.296 - 1}{1} \cdot 100[\%] = \mathbf{29.6}[\%]$$

$$\boldsymbol{t_m}[\mathbf{s}] = \mathbf{5}[\mathbf{s}]$$

2. kontinuirani regulator:

$$\boldsymbol{\sigma_m}[\%] = \frac{1.515 - 1}{1} \cdot 100[\%] = \mathbf{51.5}[\%]$$

$$\boldsymbol{t_m}[\mathbf{s}] = \mathbf{4.286}[\mathbf{s}]$$

g) Prijenosna funkcija zatvorenog kruga jednaka je:

$$G(z)=\frac{G_0(z)}{1+G_0(z)}=\frac{\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}}{1+\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}}=\frac{0.0296907(z+1)}{z^2-1.8751493z+0.9345277}$$

Statičko pojačanje je jednako:

$$\boldsymbol{K}=\lim_{z\to1}G(z)=\lim_{z\to1}\frac{0.0296907(z+1)}{z^2-1.8751493z+0.9345277}=\mathbf{1}$$

Regulacijsko odstupanje (izraz se može naći preko blokovske algebre npr.):

$$E(z)=\frac{R(z)}{1+G_0(z)}=\frac{\frac{z}{z-1}}{1+\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}}$$

$$\boldsymbol{e}(\infty)=\lim_{z\to1}\frac{z-1}{z}\frac{\frac{z}{z-1}}{1+\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}}=\lim_{z\to1}\frac{1}{1+\frac{0.0296907(z+1)}{z^2-1.90484z+0.904837}}=$$

$$=\lim_{z\to1}\frac{z^2-1.90484z+0.904837}{z^2-1.8751493z+0.9345277}=\mathbf{0}$$

Proces je diskretiziran ZOH diskretizacijom, pa je prema tome očuvana prijelazna funkcija, odnosno pojačanje sustava i regulacijsko odstupanje na skokovitu pobudu ostali su nepromijenjeni u odnosu na polazni kontinuirani sustav.

h)

$$\boldsymbol{G_0(\Omega)}=G_0(z)\bigg|_{z=\frac{1+\Omega\frac{T}{2}}{1-\Omega\frac{T}{2}}}=\frac{0.0296907\left(\frac{1+0.06\Omega}{1-0.06\Omega}+1\right)}{\left(\frac{1+0.06\Omega}{1-0.06\Omega}\right)^2-1.90484\frac{1+0.06\Omega}{1-0.06\Omega}+0.904837}$$

$$=\frac{\mathbf{-0.00356288\Omega+0.0593814}}{\mathbf{0.0137148\Omega^2+0.0114196\Omega}}$$

i) Uvrstimo $\Omega=j\omega^*$:

$$G_0(j\omega^*)=\frac{-0.00356288j\omega^*+0.0593814}{0.0137148(j\omega^*)^2+0.0114196j\omega^*}=\frac{1-j\frac{\omega^*}{16.67}}{j\frac{\omega^*}{5.2}\left(1+j\frac{\omega^*}{0.83}\right)}$$

Za presječnu frekvenciju:

$$|G_0(j\omega_c^*)|=1=\left|\frac{1-j\frac{\omega_c^*}{16.67}}{j\frac{\omega_c^*}{5.2}\left(1+j\frac{\omega_c^*}{0.83}\right)}\right|\to\boldsymbol{\omega_c^*}=\mathbf{2.0046\ [s^{-1}]}$$

Za dobivenu presječnu frekvenciju, prijenosna funkcija je:

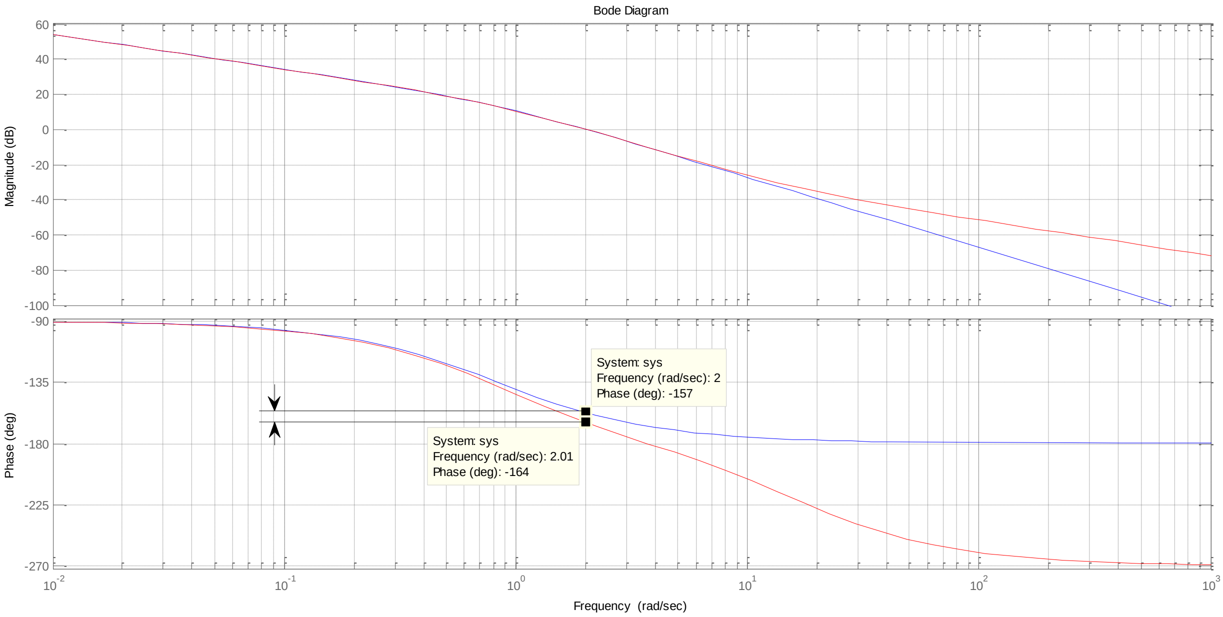
$$G_0(j\omega_c^*)=\frac{-0.00356288j\omega_c^*+0.0593814}{0.0137148(j\omega_c^*)^2+0.0114196j\omega_c^*}=-0.964836-j0.271169$$

Iz toga slijedi faza

$$\varphi(\omega_c^*) = -164.302°$$

Fazno osiguranje je jednako:

$$\boldsymbol{\gamma_2} = 180° + \varphi(\omega_c^*) = 180° - 164.302° = \mathbf{15.698°}$$



Slika 4. Bodeov dijagram

Primjećujemo kako je fazno osiguranje smanjeno u odnosu na polazni kontinuirani sustav. Zaključujemo kako digitalni regulatori narušavaju relativnu stabilnost sustava.

j) Potrebno je povećati iznos faznog osiguranja γ_2 na iznos γ_1. Samim time će se promijeniti presječna frekvencija, odnosno, amplitudni graf će se „spustiti“. Postupak je sljedeći:

I. Nađemo frekvenciju na kojoj je faza jednaka -157.38°.
II. Nađemo amplitudu na toj frekvenciji.
III. Nađemo novo pojačanje i novu konstantu.

I.

$$G_0(j\omega_{c,\text{novi}}^*) = \frac{-0.00356288 j\omega_{c,\text{novi}}^* + 0.0593814}{0.0137148\left(j\omega_{c,\text{novi}}^*\right)^2 + 0.0114196 j\omega_{c,\text{novi}}^*} =$$

$$= -\frac{6.577\omega_{c,\text{novi}}^*}{\omega_{c,\text{novi}}^*\left(1.451591232{\omega_{c,\text{novi}}^*}^2 + 1\right)} + j\frac{0.375828{\omega_{c,\text{novi}}^*}^2 - 5.2}{\omega_{c,\text{novi}}^*\left(1.451591232{\omega_{c,\text{novi}}^*}^2 + 1\right)}$$

$$\text{arctg}\frac{\text{Im}}{\text{Re}} = -157.38° \rightarrow \boldsymbol{\omega_{c,\text{novi}}^* = 1.566646\ [\mathrm{s}^{-1}]}$$

II.

$$G_0(j\omega_{c,\text{novi}}^*) = \frac{-0.00356288 j\omega_{c,\text{novi}}^* + 0.0593814}{0.0137148\left(j\omega_{c,\text{novi}}^*\right)^2 + 0.0114196 j\omega_{c,\text{novi}}^*} = -0.144425 - j0.601775$$

$$\left|G_0(j\omega_{c,\text{novi}}^*)\right| = 1.5646055479539214$$

III. Obzirom da amplituda na novoj presječnoj frekvenciji mora biti 0 dB (jer je to presječna frekvencij), mora se naći konstanta kojom ćemo pomnožiti amplitudu izračunatu pod II. tako da amplituda bude 0:

$$0 = 20\log\left(K'\left|G_0\left(j\omega^*_{c,\text{novi}}\right)\right|\right) \rightarrow K'\left|G_0\left(j\omega^*_{c,\text{novi}}\right)\right| = 1$$

$$K' = \frac{1}{\left|G_0\left(j\omega^*_{c,\text{novi}}\right)\right|} = \frac{1}{1.5646055479539214} = 0.6391387281655291$$

Uočimo da $G_0(j\omega^*)$ možemo napisati kao:

$$G_0(j\omega^*) = \frac{1 - j\frac{\omega^*}{16.67}}{j\frac{\omega^*}{5.2}\left(1 + j\frac{\omega^*}{0.83}\right)} = \frac{1 - 0.06j\omega^*}{j\frac{\omega^*}{5.2}(1 + 1.2j\omega^*)} = \frac{8(1 - 0.06j\omega^*)}{j\omega^*\frac{8}{5.2}(1 + 1.2j\omega^*)}$$

$$= \frac{8(1 - 0.06j\omega^*)}{j\omega^*\frac{8}{5.2}(1 + 1.2j\omega^*)} = \frac{K_p(1 - 0.06j\omega^*)}{T_I j\omega^*\left(1 + T_p j\omega^*\right)} = \frac{1}{T_I j\omega^*}\frac{K_p}{1 + T_p j\omega^*}(1 - 0.06j\omega^*)$$

Odnosno, s novom konstantnom će to izgledati ovako:

$$G_0(j\omega^*)_{\text{novo}} = K'\frac{1}{T_I j\omega^*}\frac{K_p}{1 + T_p j\omega^*}(1 - 0.06j\omega^*) = \frac{1}{\frac{T_I}{K'}j\omega^*}\frac{K_p}{1 + T_p j\omega^*}(1 - 0.06j\omega^*) =$$

$$= \frac{1}{T_{Ib}j\omega^*}\frac{K_p}{1 + T_p j\omega^*}(1 - 0.06j\omega^*)$$

Očito je sada:

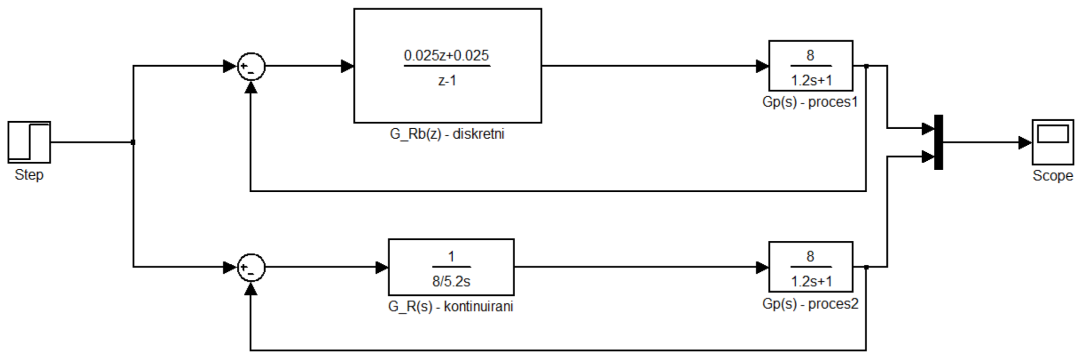
$$\boldsymbol{T_{Ib}} = \frac{T_I}{K'} = \frac{\frac{8}{5.2}}{0.6391387281655291} = \mathbf{2.407085458\ [s]}$$

Vrijeme uzorkovanja nalazi se unutar preporučenog raspona:

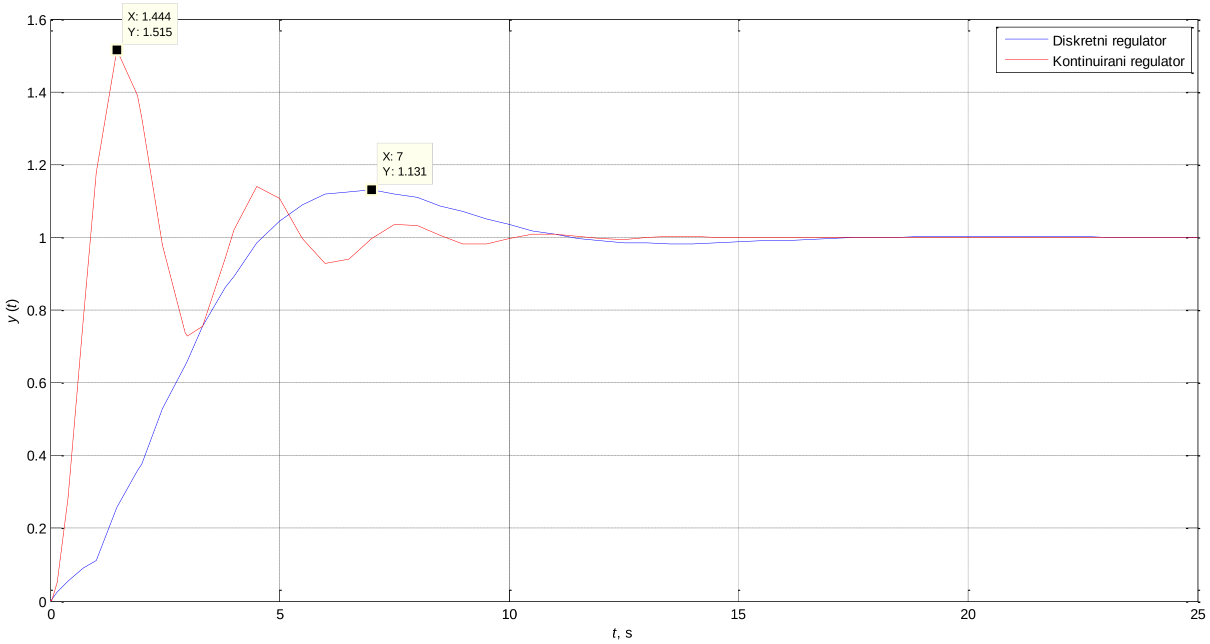
$$T = (0.17 \div 0.34)\frac{1}{\omega^*_{c,\text{novi}}} = 108.5 \text{ ms } \div 217 \text{ ms}$$

k)

$$\boldsymbol{G_R(z)} = G_R(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{1}{T_{Ib}\frac{2}{T}\frac{z-1}{z+1}} = \frac{z+1}{\frac{2\cdot 2.407085458}{0.12}(z-1)} = 0.025\frac{1 + z^{-1}}{1 - z^{-1}} = \mathbf{0.025\frac{z+1}{z-1}}$$



Slika 5. Shema u *Simulinku*



Slika 6. Odziv na skokovitu promjenu

Iz očitanih vrijednosti vrijedi sljedeće:

1. diskretni regulator:

$$\boldsymbol{\sigma_m}[\%] = \frac{1.13-1}{1}\cdot 100[\%] = \mathbf{13}[\%]$$

$$\boldsymbol{t_m}[\mathbf{s}] = \mathbf{7}[\mathbf{s}]$$

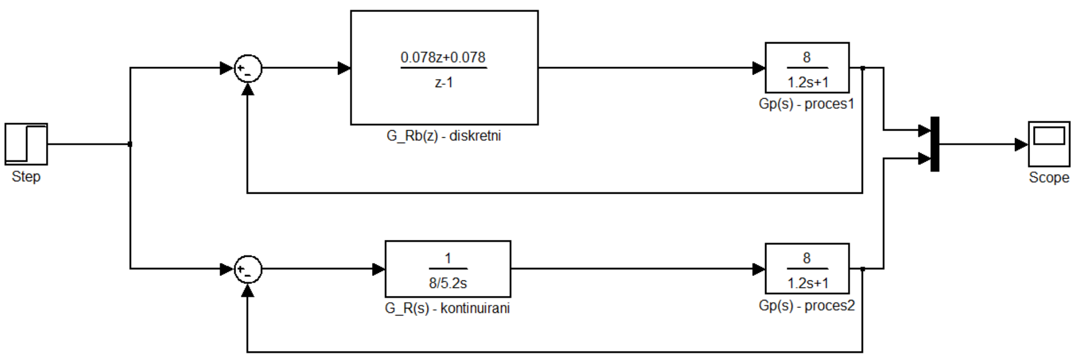
2. kontinuirani regulator:

$$\boldsymbol{\sigma_m}[\%] = \frac{1.515-1}{1}\cdot 100[\%] = \mathbf{51.5}[\%]$$

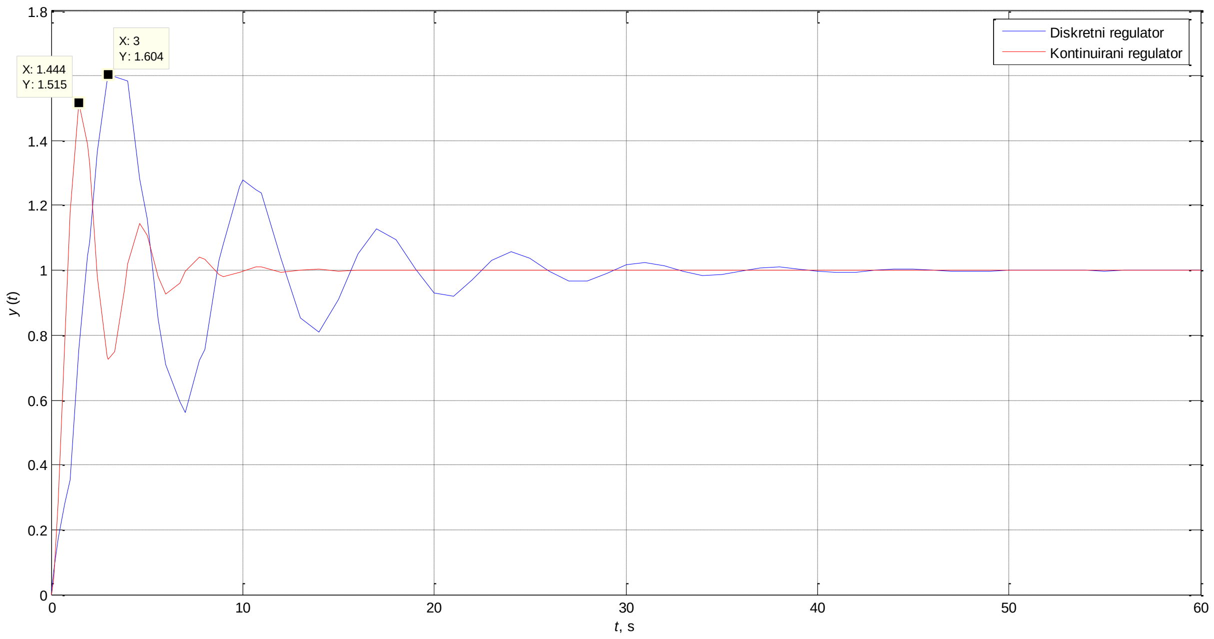
$$\boldsymbol{t_m}[\mathbf{s}] = \mathbf{1.444}[\mathbf{s}]$$

l)

$$\boldsymbol{G_R(z)} = G_R(s)\Big|_{s=\frac{2}{2T}\frac{z-1}{z+1}} = \frac{1}{T_I\frac{2}{2T}\frac{z-1}{z+1}} = \frac{z+1}{\frac{8}{5.2\cdot 0.12}(z-1)} = 0.078\frac{1+z^{-1}}{1-z^{-1}} = \mathbf{0.078}\frac{\boldsymbol{z+1}}{\boldsymbol{z-1}}$$



Slika 7. Shema u *Simulinku*



Slika 8. Odziv na skokovitu promjenu

Iz očitanih vrijednosti vrijedi sljedeće:

1. diskretni regulator:

$$\boldsymbol{\sigma_m}[\%] = \frac{1.604 - 1}{1} \cdot 100[\%] = \mathbf{60.4}[\%]$$

$$\boldsymbol{t_m}[\mathbf{s}] = \mathbf{3}[\mathbf{s}]$$

2. kontinuirani regulator:

$$\boldsymbol{\sigma_m}[\%] = \frac{1.515 - 1}{1} \cdot 100[\%] = \mathbf{51.5}[\%]$$

$$\boldsymbol{t_m}[\mathbf{s}] = \mathbf{1.444}[\mathbf{s}]$$