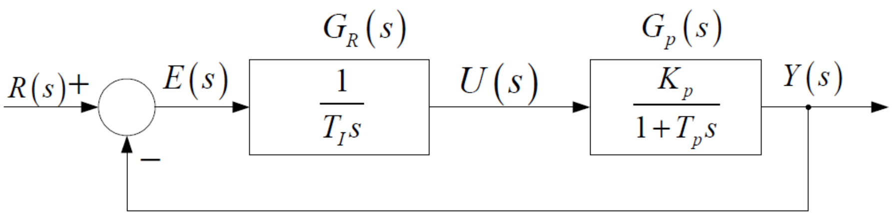
5. Domaća zadaća

Diskretni sustavi upravljanja

Zadatak 1



a) Najprije nađemo prijenosnu funkciju **otvorenog kruga**:

$$G_0(s) = G_R(s)G_p(s) = \frac{1}{T_I s}\frac{K_p}{1+T_p s} = \frac{K_p}{T_I s + T_I T_p s^2}$$

$$G_0(j\omega) = \frac{K_p}{T_I j\omega - T_I T_p \omega^2} = \frac{K_p}{T_I j\omega - T_I T_p \omega^2}\frac{T_I j\omega + T_I T_p \omega^2}{T_I j\omega + T_I T_p \omega^2} = -\frac{K_p T_I j\omega + K_p T_I T_p \omega^2}{T_I^2\omega^2 + T_I^2 T_p^2 \omega^4}$$

$$G_0(j\omega) = -\frac{K_p T_I T_p \omega^2}{T_I^2\omega^2 + T_I^2 T_p^2 \omega^4} + j\frac{-K_p T_I \omega}{T_I^2\omega^2 + T_I^2 T_p^2 \omega^4}$$

$$\varphi_0(\omega_c) = \mathrm{arctg}\frac{-\frac{K_p T_I \omega_c}{T_I^2\omega_c^2 + T_I^2 T_p^2 \omega_c^4}}{-\frac{K_p T_I T_p \omega_c^2}{T_I^2\omega_c^2 + T_I^2 T_p^2 \omega_c^4}} = \mathrm{arctg}\frac{K_p T_I \omega_c}{K_p T_I T_p \omega_c^2} = \mathrm{arctg}\frac{1}{T_p \omega_c}$$

Radi se o III. kvadrantu jer su i realni i imaginarni dio negativni.

Vrijedi izraz:

$$\gamma = \pi + \varphi_0(\omega_c) \rightarrow \frac{\pi}{4} = \pi + \mathrm{arctg}\frac{1}{T_p\omega_c} \rightarrow \mathrm{arctg}\frac{1}{T_p\omega_c} = -\frac{3\pi}{4}$$

$$\frac{1}{T_p\omega_c} = 1 \rightarrow \boldsymbol{\omega_c} = \frac{1}{T_p} = \frac{1}{0.1s} = \boldsymbol{10s^{-1}}$$

Sada, preko relacije

$$|G_0(j\omega_c)| = 1$$

nađemo T_I.

$$\sqrt{\left(-\frac{K_p T_I T_p \omega_c^2}{T_I^2\omega_c^2 + T_I^2 T_p^2 \omega_c^4}\right)^2 + \left(-\frac{K_p T_I \omega_c}{T_I^2\omega_c^2 + T_I^2 T_p^2 \omega_c^4}\right)^2} = 1$$

$$\frac{K_p^2}{T_I^2\omega_c^2 + T_I^2 T_p^2 \omega_c^4} = 1 \rightarrow T_I = \sqrt{\frac{K_p^2}{\omega_c^2 + T_p^2\omega_c^4}}$$

$$\boldsymbol{T_I} = \sqrt{\frac{176.89}{100 + 0.01 \cdot 10000}} = \mathbf{0.940452}\boldsymbol{s}$$

Sada nam je prijenosna funkcija otvorenoga kruga jednaka

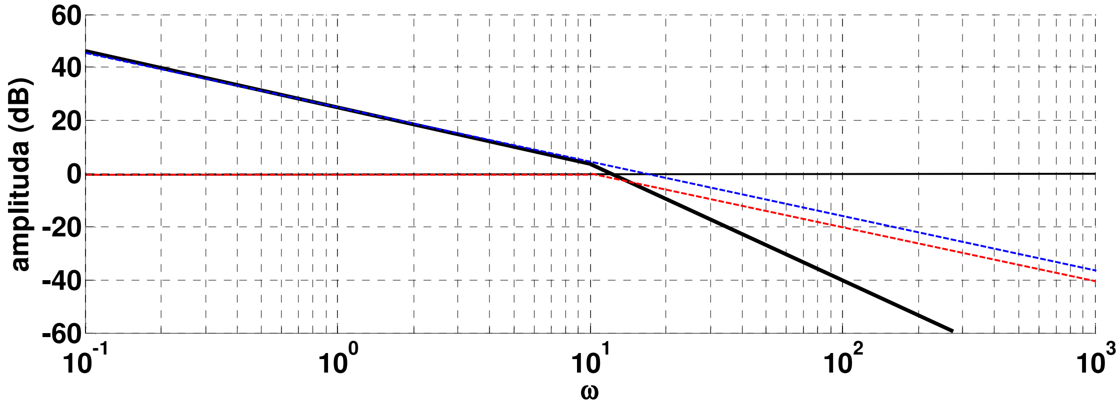
$$G_0(j\omega) = \frac{1}{T_I j\omega}\frac{K_p}{1 + T_p j\omega} = \frac{1}{j\frac{\omega}{\left(\frac{K_p}{T_I}\right)}}\frac{1}{1 + j\frac{\omega}{\left(\frac{1}{T_p}\right)}} = \frac{1}{j\frac{\omega}{14.142}}\frac{1}{1 + j\frac{\omega}{10}}$$

$G_{01}(j\omega) = \frac{1}{j\frac{\omega}{14.142}}$

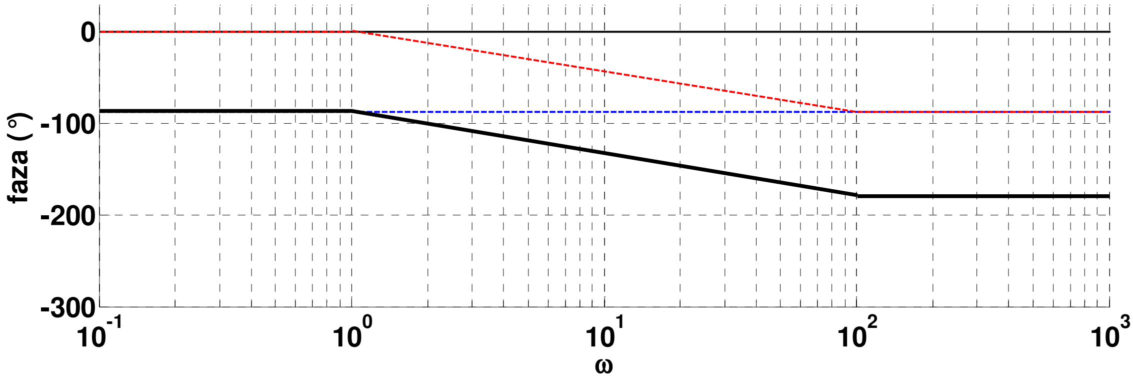
- amplituda je 0 u $\omega = 14.142 s^{-1}$; pravac nagiba -20 dB/dek.
- faza je -90°

$G_{02}(j\omega) = \frac{1}{1+j\frac{\omega}{10}}$

- amplituda je 0 do $\omega = 10 s^{-1}$, od tamo je pravac nagiba -20 dB/dek.
- faza je 0° za mali ω, -45° za $\omega = 10 s^{-1}$, -90° za veliki ω; crta se kroz dvije dekade
-

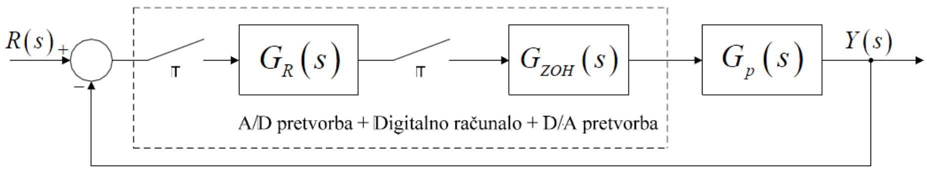


Sl. 1. Bodeov dijagram: Amplitudno-frekvencijska karakteristika



Sl. 2. Bodeov dijagram: Fazno-frekvencijska karakteristika

b)



c) (Predavanje 15 – slide 29.):

$$T = (0.17 : 0.34)\frac{1}{\omega_c}$$

Iz toga slijedi:

$$\frac{0.17}{\omega_c} \leq T \leq \frac{0.34}{\omega_c}$$

$$\frac{0.17}{10s^{-1}} \leq T \leq \frac{0.34}{10s^{-1}}$$

$$\mathbf{0.017s \leq T \leq 0.034s}$$

$$\mathbf{17ms \leq T \leq 34ms}$$

Odabiremo vrijeme $\boldsymbol{T = 20ms}$.

d) Dakle, imamo tri dijela:

- **d1)** Tustinova relacija (predavanje 16, slide 34.):

$$G_R(z) = G_R(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

$$G_R(z) = \frac{1}{T_I \frac{2}{T}\frac{z-1}{z+1}} = \frac{z+1}{\frac{2 \cdot 0.940452}{0.02}(z-1)} = 0.01063\frac{1+z^{-1}}{1-z^{-1}}$$

$$\boldsymbol{G_R(z) = \frac{0.01063 + 0.01063z^{-1}}{1-z^{-1}} = 0.01063\frac{z+1}{z-1}}$$

Za pripadni rekurzivni algoritam regulatora imamo relacije (predavanje 14, slide 39., 42.):

$$G_R(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

iz čega je $b_0 = 0.01063$, $b_1 = 0.01063$, $a_1 = -1$,

i

$$u(k) = \sum_{i=0}^{m} b_i e(k-i) - \sum_{i=1}^{n} a_i u(k-i) = \sum_{i=0}^{1} b_i e(k-i) - \sum_{i=1}^{1} a_i u(k-i)$$

$$u(k) = b_0 e(k) + b_1 e(k-1) - a_1 u(k-1) = 0.01063e(k) + 0.01063e(k-1) + u(k-1)$$

$$\boldsymbol{u(k) = 0.01063[e(k) + e(k-1)] + u(k-1)}$$

- **d2)** Eulerova unaprijedna diferencija (predavanje 16, slide 34.):

$$G_R(z) = G_R(s)\Big|_{s=\frac{z-1}{T}}$$

$$G_R(z) = \frac{1}{T_I \frac{z-1}{T}} = \frac{1}{\frac{0.940452}{0.02}(z-1)} = 0.02126\frac{z^{-1}}{1-z^{-1}}$$

$$\boldsymbol{G_R(z) = \frac{0.02126z^{-1}}{1-z^{-1}} = \frac{0.02126}{z-1}}$$

Pripadni rekurzivni algoritam:

$$G_R(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

iz čega je $b_0 = 0$, $b_1 = 0.02126$, $a_1 = -1$.

$$u(k) = \sum_{i=0}^{m} b_i e(k-i) - \sum_{i=1}^{n} a_i u(k-i) = \sum_{i=0}^{1} b_i e(k-i) - \sum_{i=1}^{1} a_i u(k-i)$$

$$u(k) = b_0 e(k) + b_1 e(k-1) - a_1 u(k-1) = 0.02126 e(k-1) + u(k-1)$$

$$\boldsymbol{u(k) = 0.02126 e(k-1) + u(k-1)}$$

- **d3)** Eulerova unazadna diferencija (predavanje 16, slide 34.):

$$G_R(z) = G_R(s)\Big|_{s=\frac{z-1}{Tz}}$$

$$G_R(z) = \frac{1}{T_I \frac{z-1}{Tz}} = \frac{z}{\frac{0.940452}{0.02}(z-1)} = 0.02126 \frac{1}{1 - z^{-1}}$$

$$\boldsymbol{G_R(z) = \frac{0.02126}{1 - z^{-1}} = 0.02126 \frac{z}{z-1}}$$

Pripadni rekurzivni algoritam:

$$G_R(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

iz čega je $b_0 = 0.02126$, $b_1 = 0$, $a_1 = -1$.

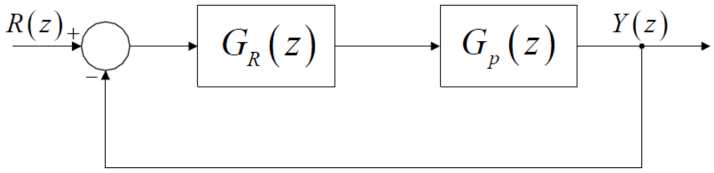
$$u(k) = \sum_{i=0}^{m} b_i e(k-i) - \sum_{i=1}^{n} a_i u(k-i) = \sum_{i=0}^{1} b_i e(k-i) - \sum_{i=1}^{1} a_i u(k-i)$$

$$u(k) = b_0 e(k) + b_1 e(k-1) - a_1 u(k-1) = 0.02126 e(k) + u(k-1)$$

$$\boldsymbol{u(k) = 0.02126 e(k) + u(k-1)}$$

Zajedničko obilježje svih dobivenih diskretnih regulatora jest da svi imaju pol $z_p = 1$ (integrator u kontinuiranoj domeni). Upravljački signal $u(k)$ svakog od regulatora ovisi o upravljačkom signalu iz prehodnog koraka $u(k-1)$.

e)



ZOH diskretizacija:

$$G_p(z) = (1 - z^{-1})\boldsymbol{Z}\left\{\frac{G_p(s)}{s}\right\}$$

Imamo

$$\frac{G_p(s)}{s} = \frac{K_p}{s(1 + T_p s)} = \frac{A}{s} + \frac{B}{1 + T_p s}$$

$$K_p = A + AT_p s + Bs = s(AT_p + B) + A$$

$$A = K_p = 13.3$$

$$AT_p + B = 0 \rightarrow B = -AT_p = -K_p T_p = -1.33$$

pa je

$$\frac{G_p(s)}{s} = \frac{13.3}{s} - \frac{1.33}{1 + 0.1s} = \frac{13.3}{s} - \frac{1.33}{1 + 0.1s} = 13.3\frac{1}{s} - 13.3\frac{1}{s + 10}$$

iz čega se dobije (pomoću tablice za $\boldsymbol{Z}$-transformaciju):

$$\boldsymbol{Z}\left\{\frac{G_p(s)}{s}\right\} = 13.3\frac{z}{z-1} - 13.3\frac{z}{z - e^{-10T}} = 13.3\frac{z}{z-1} - 13.3\frac{z}{z - e^{-10\cdot 0.02}}$$

$$\boldsymbol{Z}\left\{\frac{G_p(s)}{s}\right\} = 13.3\left[\frac{z}{z-1} - \frac{z}{z - 0.8187308}\right]$$

Konačno, dobijemo prijenosnu funkciju $G_p(z)$:

$$G_p(z) = (1 - z^{-1}) \cdot 13.3\left[\frac{z}{z-1} - \frac{z}{z - 0.8187308}\right] = \frac{z-1}{z} \cdot 13.3\left[\frac{z}{z-1} - \frac{z}{z - 0.8187308}\right]$$

$$G_p(z) = 13.3\left[1 - \frac{z-1}{z - 0.8187308}\right] = 13.3\frac{z - 0.8187308 - z + 1}{z - 0.8187308}$$

$$\boldsymbol{G_p(z) = \frac{2.41088036}{z - 0.8187308} = \frac{2.41088036z^{-1}}{1 - 0.8187308z^{-1}}}$$

f) Koristimo, dakle, prijenosnu funkciju regulatora

$$G_R(z) = \frac{0.01063 + 0.01063z^{-1}}{1 - z^{-1}} = 0.01063\frac{z+1}{z-1}$$

i prijenosnu funkciju procesa

$$G_p(z) = \frac{2.41088036}{z - 0.8187308} = \frac{2.41088036z^{-1}}{1 - 0.8187308z^{-1}}$$

pa je prijenosna funkcija **otvorenog kruga**

$$G_o(z) = G_R(z)G_p(z) = 0.01063\frac{z+1}{z-1}\frac{2.41088036}{z-0.8187308} = \frac{0.0256277z + 0.0256277}{z^2 - 1.8187308z + 0.8187308}$$

Karakteristična jednadžba je

$$f(z) = 1 + G_o(z)$$

$$f(z) = z^2 - 1.8187308z + 0.8187308 + 0.0256277z + 0.0256277$$

$$f(z) = z^2 - 1.7931031z + 0.8443585$$

Iz ovoga je $a_o = 0.8443585$, $a_1 = -1.7931031$ i $a_3 = 1$.

- Uvjet a):
  - $f(1) = 0.0512554 > 0$
  - $(-1)^n f(-1) = (-1)^4(1 + 1.7931031 + 0.8443585) = 3.6374616 > 0$

- Uvjet b):
  - tablica izgleda ovako (predavanje 18., slide 13.):

Redak	z^0	z^1	z^2
1	0.8443585	−1.7931031	1

  - pa nam mora biti samo $|a_o| < |a_n| \rightarrow |a_o| < |a_2| \rightarrow 0.8443585 < 1$

Oba uvjeta su zadovoljena pa je sustav **stabilan**.

g) Za statičko pojačanje treba nam prijenosna funkcija **zatvorenog kruga**:

$$G(z) = \frac{G_0(z)}{1 + G_0(z)} = \frac{\frac{0.0256277z + 0.0256277}{z^2 - 1.8187308z + 0.8187308}}{1 + \frac{0.0256277z + 0.0256277}{z^2 - 1.8187308z + 0.8187308}}$$

$$G(z) = \frac{0.0256277z + 0.0256277}{z^2 - 1.7931031z + 0.8443585}$$

Statičko pojačanje je jednako

$$\lim_{z \to 1} G(z) = \lim_{z \to 1} \frac{0.0256277z + 0.0256277}{z^2 - 1.7931031z + 0.8443585} = \frac{0.0256277 + 0.0256277}{1 - 1.7931031 + 0.8443585} = \frac{0.0512554}{0.0512554} = \mathbf{1}$$

Za odstupanje dobijemo:

$$E(z) = \frac{R(z)}{1 + G_0(z)} = \frac{\frac{z}{z-1}}{1 + \frac{0.0256277z + 0.0256277}{z^2 - 1.8187308z + 0.8187308}}$$

$$E(z) = \frac{z(z^2 - 1.8187308z + 0.8187308)}{(z-1)(z^2 - 1.7931031z + 0.8443585)}$$

$$\boldsymbol{e}_\infty = \lim_{z \to 1} \frac{z-1}{z} E(z) = \lim_{z \to 1} \frac{z^2 - 1.8187308z + 0.8187308}{z^2 - 1.7931031z + 0.8443585} = \frac{0}{0.0512554} = \mathbf{0}$$

Proces je diskretiziran ZOH diskretizacijom, pa je prema tome očuvana prijelazna funkcija, odnosno pojačanje sustava i regulacijsko odstupanje na skokovitu pobudu ostali su nepromijenjeni u odnosu na polazni kontinuirani sustav.

h) Imamo

$$G_0(\Omega) = G_0(z) \Bigg|_{z = \frac{1 + \Omega \frac{T}{2}}{1 - \Omega \frac{T}{2}}}$$

$$G_0(\Omega) = \frac{0.0256277 \frac{1 + \Omega \frac{T}{2}}{1 - \Omega \frac{T}{2}} + 0.0256277}{\left(\frac{1 + \Omega \frac{T}{2}}{1 - \Omega \frac{T}{2}} \right)^2 - 1.8187308 \frac{1 + \Omega \frac{T}{2}}{1 - \Omega \frac{T}{2}} + 0.8187308}$$

$$G_0(\Omega) = \frac{0.0256277\frac{1+0.01\Omega}{1-0.01\Omega} + 0.0256277}{\left(\frac{1+0.01\Omega}{1-0.01\Omega}\right)^2 - 1.8187308\frac{1+0.01\Omega}{1-0.01\Omega} + 0.8187308}$$

$$\boldsymbol{G_0(\Omega)} = \frac{0.0512554 - 0.000512554\Omega}{0.000363746\Omega^2 + 0.00362538\Omega} = \frac{\mathbf{14.1379386}}{\boldsymbol{\Omega}}\frac{\mathbf{1} - \frac{\boldsymbol{\Omega}}{\mathbf{100}}}{\mathbf{1} + \frac{\boldsymbol{\Omega}}{\mathbf{9.96679}}}$$

i) Uvrstimo $\Omega = j\omega^*$

$$G_0(j\omega^*) = \frac{14.1379386}{j\omega^*}\frac{1 - \frac{j\omega^*}{100}}{1 + \frac{j\omega^*}{9.96679}} = \frac{1}{j\frac{\omega^*}{14.1379386}}\frac{1 - j\frac{\omega^*}{100}}{1 + j\frac{\omega^*}{9.96679}}$$

ili zapisano na način $\mathrm{Re} + j\mathrm{Im}$:

$$G_0(j\omega^*) = -\frac{1.55988(\omega^*)^2}{0.0100668(\omega^*)^4 + 1.00000470492(\omega^*)^2} + j\frac{0.014185(\omega^*)^3 - 14.1379\omega^*}{0.0100668(\omega^*)^4 + 1.00000470492(\omega^*)^2}$$

pa dobijemo

$$|G_0(j\omega^*)| = \frac{14.1379386}{\omega^*}\frac{\sqrt{1 + \left(\frac{\omega^*}{100}\right)^2}}{\sqrt{1 + \left(\frac{\omega^*}{9.96679}\right)^2}}$$

odnosno, za $\omega^* = \omega_c^*$

$$|G_0(j\omega_c^*)| = \frac{14.1379386}{\omega_c^*}\frac{\sqrt{1 + \left(\frac{\omega_c^*}{100}\right)^2}}{\sqrt{1 + \left(\frac{\omega_c^*}{9.96679}\right)^2}} = 1$$

iz čega je

$$\boldsymbol{\omega_c^* = 10.0202 s^{-1}}$$

Za fazno osiguranje

$$\gamma = \pi + \varphi_0(\omega_c^*)$$

$$\gamma = \pi + \mathrm{arctg}\frac{\mathrm{Im}}{\mathrm{Re}}$$

Obzirom da su i imaginarni i realni dio negativni, nalazimo se u III. kvadrantu pa imamo

$$\mathrm{arctg}\frac{\mathrm{Im}}{\mathrm{Re}} = \mathrm{arctg}\frac{14.1379 - 0.014185(\omega_c^*)^2}{1.55988\omega_c^*} = 39.12° + 180°$$

39.12° izbaci kalkulator, a 180° dodamo kako bi došli u III. kvadrant.

Pa konačno dobijemo

$$\boldsymbol{\gamma} = \pi + 39.12° + 180° = 360° + 39.12° = \mathbf{39.12}°$$

Primjećujemo kako je fazno osiguranje smanjeno u odnosu na polazni kontinuirani sustav. Zaključujemo kako digitalni regulatori narušavaju relativnu stabilnost sustava.

Za Bodeov dijagram:

$G_{01}(j\omega^*) = \frac{1}{j\frac{\omega^*}{14.1379386}}$

- amplituda je 0 u $\omega^* = 14.1379886 s^{-1}$; pravac nagiba -20 dB/dek.
- faza je -90°

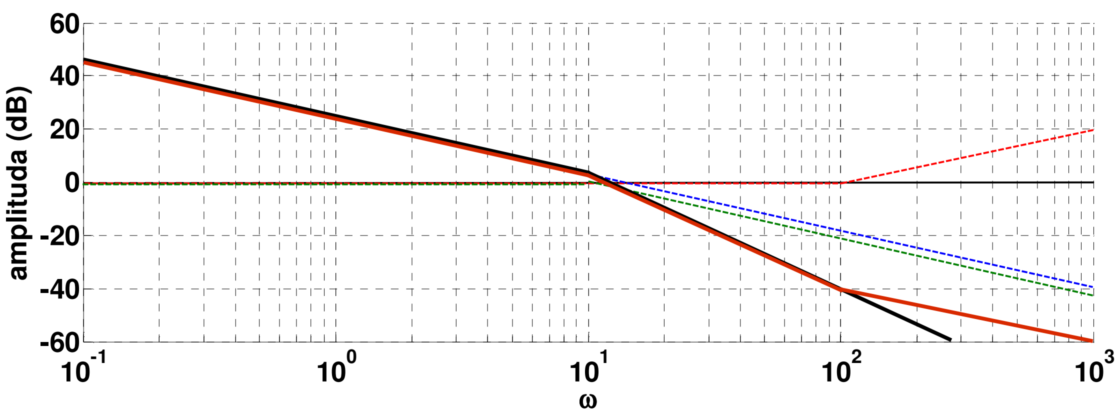
$G_{02}(j\omega^*) = 1 - j\frac{\omega^*}{100}$

- amplituda je 0 do $\omega^* = 100 s^{-1}$, od tamo je pravac nagiba 20 dB/dek.
- faza je 0° za mali ω^*, -45° za $\omega^* = 100 s^{-1}$, -90° za veliki ω^*; crta se kroz dvije dekade

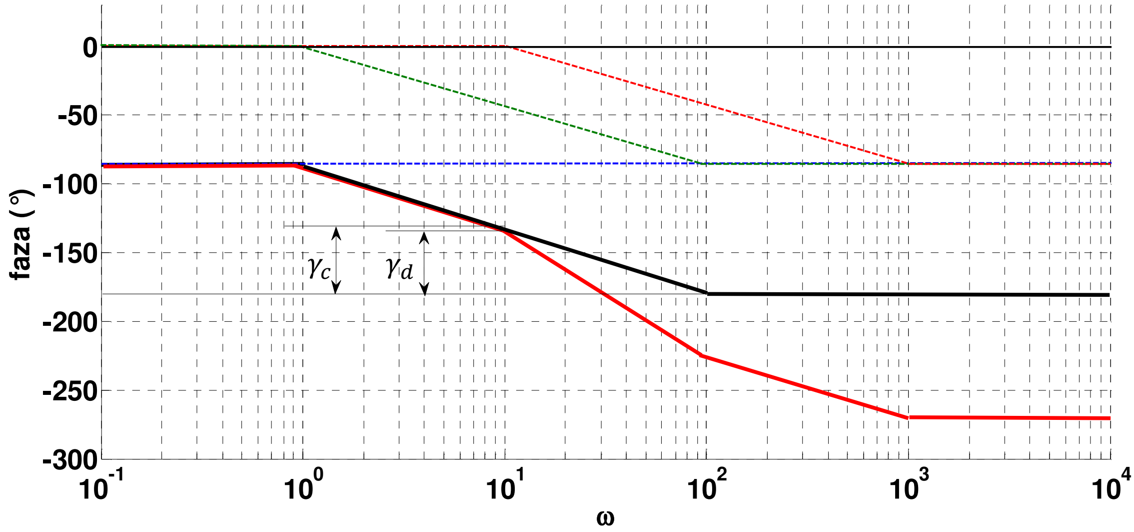
$G_{03}(j\omega^*) = \frac{1}{1+j\frac{\omega^*}{9.96679}}$

- amplituda je 0 do $\omega^* = 9.96679 s^{-1}$, od tamo je pravac nagiba -20 dB/dek.
- faza je 0° za mali ω^*, -45° za $\omega^* = 9.96679 s^{-1}$, -90° za veliki ω^*; crta se kroz dvije dekade

(Bodeov dijagram iz a) zadatka označen je crnom bojom, a iz ovog zadatka narančastom, jer je u zadaći navedeno da se oba crtaju na istom!)



Sl. 3. Bodeov dijagram: Amplitudno-frekvencijska karakteristika



Sl. 4. Bodeov dijagram: Fazno-frekvencijska karakteristika