(1.) $\left(d\left(m^2 - n^2\right)\right)^2 + \left(2\,d\,m\,n\right)^2 = \left(d\left(m^2 + n^2\right)\right)^2$

a) 15

$d = 1, 3, 5,$

$d = 1$

$m^2 + n^2 \neq 15$

$m^2 - n^2 = 15$

$(m - n)(m + n) = 15$

$m - n = 1$

$\underline{m + n = 15}$

$m = 8$

$n = 7$

$x = m^2 - n^2 = 15$

$y = 2mn = 112$

$z = m^2 + n^2 = 113$

$\left(15, 112, 113\right)$

$(m - n)(m + n) = 15$

$m - n = 3$

$\underline{m + n = 5}$

$m = 4$

$n = 1$

$x = 15$

$y = 8$

$z = 17$

$\left(8, 15, 17\right)$

$d = 3$

$m^2 + n^2 = 5$

$m = 2$

$n = 1$

$x = d\left(m^2 - n^2\right) = 9$

$y = d\,2mn = 12$

$z = d\left(m^2 + n^2\right) = 15$

$\left(9, 12, 15\right)$

$$m^2 - n^2 = 5$$
$$(m-n)(m+n) = 5$$
$$m - n = 5$$
$$m + n = 1$$
$$\overline{}$$
$$m = 3$$
$$n = -2$$

$$d = 5$$

$$m^2 + n^2 \neq 3$$
$$m^2 - n^2 = 3$$
$$(m-n)(m+n) = 3$$
$$m - n = 1$$
$$m + n = 3$$
$$\overline{}$$
$$m = 2$$
$$n = 1$$

a) 20

$$d = 1,5$$

$$d = 1$$
$$m^2 + n^2 = 20$$
$$m = 4$$
$$n = 2$$

$$2mn = 20$$
$$m = 10$$
$$n = 1$$

$$x = 15$$
$$y = 36$$
$$z = 39$$
$$(15, 36, 39)$$

$$x = 15$$
$$y = 20$$
$$z = 25$$
$$(15, 20, 25)$$

$$x = 12$$
$$y = 16$$
$$z = 20$$
$$(12, 16, 20)$$

$$x = 99$$
$$y = 20$$
$$z = 101$$
$$(20, 99, 101)$$

$2mn = 20$

$m = 5$

$n = 2$

$x = 21$

$y = 20$

$z = 29$

$$(20, 21, 29)$$

$m^2 - n^2 = 20$

$(m-n)(m+n) = 20$

$m - n = 1$ $\times$

$m + n = 20$

$x = 20$

$y = 48$

$z = 52$

$m - n = 2$

$m + n = 10$

$\overline{}$

$m = 6$

$n = 4$

$$(20, 48, 52)$$

$d = 5$

$m^2 + n^2 \neq 4$

$2mn = 4$

$m = 2$

$n = 1$

$x = 15$

$y = 20$

$z = 25$

$$(15, 20, 25)$$

$m^2 - n^2 = 4$

$(m-n)(m+n) = 4$

$m - n = 1$ $\times$

$m + n = 4$

c) 29

$d = 1,$

$m^2 + n^2 = 29$ $\sqrt{29} = 5,\sim$ $x = 21$

$m = 5$ $y = 20$

$n = 2$ $z = 29$

$$(21, 20, 29)$$

$m^2 - n^2 = 29$

$(m - n)(m + n) = 29$ $x = 29$

                               $y = 420$

$m - n = 1$ $z = 421$

$m + n = 29$

$$(29, 420, 421)$$

$m = 15$

$n = 14$

d) 38

$d = 1,$

$m^2 + n^2 \neq 38$

$2mn = 38$ $x = 360$

                        $y = 38$

$m = 19$ $z = 362$

$n = 1$

$$(38, 360, 362)$$

$m^2 - n^2 = 38$

$(m - n)(m + n) = 38$

$m - n = 1$ ✗

$m + n = 38$

② a) $\dfrac{51}{97}$

$$97 = \underline{1} \cdot 51 + 46$$
$$51 = \underline{1} \cdot 46 + 5$$
$$46 = \underline{9} \cdot 5 + 1$$
$$5 = \underline{5} \cdot 1$$

$$\dfrac{97}{51} = [1; 1, 9, 5]$$

$$\dfrac{51}{97} = [0; 1, 1, 9, 5)$$

b) $\dfrac{101}{31}$

$$101 = \underline{3} \cdot 31 + 8$$
$$31 = \underline{3} \cdot 8 + 7$$
$$8 = \underline{1} \cdot 7 + 1$$
$$7 = \underline{7} \cdot 1$$

$$\dfrac{101}{31} = [3; 3, 1, 7]$$

c) $\dfrac{58}{269}$

$$269 = \underline{4} \cdot 58 + 37$$
$$58 = \underline{1} \cdot 37 + 21$$
$$37 = \underline{1} \cdot 21 + 16$$
$$21 = \underline{1} \cdot 16 + 5$$
$$16 = \underline{3} \cdot 5 + 1$$
$$5 = \underline{5} \cdot 1$$

$$\dfrac{269}{58} = [4; 1, 1, 1, 3, 5]$$

$$\dfrac{58}{269} = [0; 4, 1, 1, 1, 3, 5]$$

③

$$a_i = \left\lfloor \frac{s_i + a_0}{t_i} \right\rfloor, \quad s_{i+1} = a_i t_i - s_i, \quad t_{i+1} = \frac{d - s_{i+1}^2}{t_i}$$

a) $\sqrt{23}$ $d = 23$

$a_0 = \lfloor \sqrt{23} \rfloor = 4$ $S_0 = 0$ $t_0 = 1$

$S_1 = 4$	$t_1 = 7$	$a_1 = 1$
$S_2 = 3$	$t_2 = 2$	$a_2 = 3$
$S_3 = 3$	$t_3 = 7$	$a_3 = 1$
$S_4 = 4$	$t_4 = 1$	$a_4 = 8$
$S_5 = 4$	$t_5 = 7$	

$$\left(s_1, t_1 \right) = \left(s_5, t_5 \right)$$

$$\sqrt{23} = \left[4 ; \overline{1, 3, 1, 8} \right]$$

b) $\sqrt{47}$ $d=47$

$a_o = 6$, $S_o = 0$, $t_o = 1$

$S_1 = 6$ $t_1 = 11$ $a_1 = 1$

$S_2 = 5$ $t_2 = 2$ $a_2 = 5$

$S_3 = 5$ $t_3 = 11$ $a_3 = 1$

$S_4 = 6$ $t_4 = 1$ $a_4 = 12$

$S_5 = 6$ $t_5 = 11$

$(S_1, t_1) = (S_5, t_5)$

$$\sqrt{47} = \left[6; \overline{1, 5, 1, 12}\right]$$

c) $\sqrt{57}$ $d = 57$

$a_o = 7$, $S_o = 0$, $t_o = 1$

$S_1 = 7$ $t_1 = 8$ $a_1 = 1$

$S_2 = 1$ $t_2 = 7$ $a_2 = 1$

$S_3 = 6$ $t_3 = 3$ $a_3 = 4$

$S_4 = 6$ $t_4 = 7$ $a_4 = 1$

$S_5 = 1$ $t_5 = 8$ $a_5 = 1$

$S_6 = 7$ $t_6 = 1$ $a_6 = 14$

$S_7 = 7$ $t_7 = 8$

$(S_1, t_1) = (S_8, t_8)$

$$\sqrt{57} = \left[7; \overline{1, 1, 4, 1, 1, 14}\right]$$

(4)

$$x^2 - 71y^2 = 1$$

$\sqrt{71}$ $\qquad$ $d = 71$

$a_0 = 8$, $S_0 = 0$, $t_0 = 1$

$S_1 = 8$	$t_1 = 7$	$a_1 = 2$
$S_2 = 6$	$t_2 = 5$	$a_2 = 2$
$S_3 = 4$	$t_3 = 11$	$a_3 = 1$
$S_4 = 7$	$t_4 = 2$	$a_4 = 7$
$S_5 = 7$	$t_5 = 11$	$a_5 = 1$
$S_6 = 4$	$t_6 = 5$	$a_6 = 2$
$S_7 = 6$	$t_7 = 7$	$a_7 = 2$
$S_8 = 8$	$t_8 = 1$	$a_8 = 16$
$S_9 = 8$	$t_9 = 7$	

$$\left(S_1, t_1\right) = \left(S_9, t_9\right) \qquad \sqrt{71} = \left[8; \overline{2,2,1,7,1,2,2,16}\right]$$

$$\ell = 8$$

$$\ell - 1 \Rightarrow \left(P_7, q_7\right)$$

$$P_n = P_{n-1} \cdot a_n + P_{n-2}$$

n	-1	0	1	2	3	4	5	6	7
a_n		8	2	2	1	7	1	2	2
P_n	1	8	17	42	59	455	514	1483	3480
q_n	0	1	2	5	7	54	61	176	413

$$\left(3480, 413\right)$$

⑤ $x^2 - 146 y^2 = 1$ $1 < x < 100\,000$

$\sqrt{146}$ $d = 146$

$a_0 = 12$ $S_0 = 0$ $t_0 = 1$

$S_1 = 12$ $t_1 = 2$ $a_1 = 12$
$S_2 = 12$ $t_2 = 1$ $a_2 = 24$
$S_3 = 12$ $t_3 = 2$

$(S_1, t_1) = (S_3, t_3)$

$\sqrt{146} = \left[\,12\,;\,\overline{12\,,\,24}\,\right]$

$\ell = 2$
$\ell - 1 = (p_1, q_1)$ $\ell + \ell - 1 = (p_3, q_3)$

n	-1	0	1	2	3	4
a_n		12	12	24	12	24
p_n	1	12	145	3492	42049	~~1012668~~
q_n	0	1	12	289	3480	

$\left(145, 12\right)$
$\left(42049, 3480\right)$