Numeričko rješavanje diferencijalnih jednadžbi

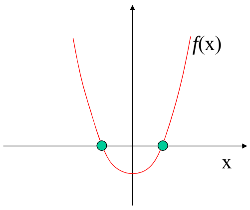
Diferencijalne jednadžbe

- Relacije između derivacija varijabli
- Broj nezavisnih varijabli
  - Diferencijalne jednadžbe (jedna nezavisna varijabla, derivacije) $\frac{d^2x}{dt^2} = x$
  - Parcijalne diferencijalne jednadžbe (dvije i više varijabli, parcijalne derivacije)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 1$$

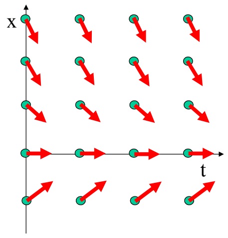
Algebarska jednadžba

$$f(x) = x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$$



Diferencijalna jednadžba

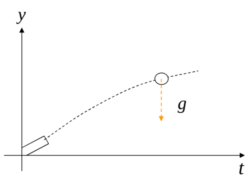
$$\frac{dx}{dt} = -x \quad \Rightarrow \quad x = Ce^{-t}$$



Potreban dodatni uvjet za određivanje konstante C

- Red diferencijalne jednadžbe: najveća derivacija u jednadžbi
- Za diferencijalnu jednadžbu n-tog reda potrebno je zadati n uvjeta za određivanje rješenja
  - Početni uvjeti: svi uvjeti su zadani u istoj (početnoj) točki
  - Rubni uvjeti (ostalo)

Primjer: Kosi hitac



$$\frac{d^2y}{dt^2} = -g = -10$$
$$\frac{dy}{dt} = -10t + c_1$$
$$y = -5t^2 + c_1 t + c_2$$

Početni uvjeti

$$y(0) = 0$$
$$\left.\frac{dy}{dt}\right|_{t=0} = 5$$
$$y = -5t^2 + 5t$$

Rubni uvjeti

$$y(0) = 0$$
$$y(10) = 100$$
$$y = -5t^2 + 60t$$

Linearne jednadžbe

- Nema produkta ili nelinearnih ovisnosti tražene funkcije i njenih derivacija
- Linearna diferencijalna jednadžba n-tog reda

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b$$

Numeričko rješavanje

- Početni uvjeti
- Kanonski oblik diferencijalne jednadžbe prvog reda

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x=0) = y_0$$

- Svaka diferencijalna jednadžba n-tog reda može se svesti na n jednadžbi prvog reda

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

$$\begin{aligned}
y_1(x) &\equiv y(x) \\
y_2(x) &\equiv \frac{dy_1}{dx} = \frac{dy}{dx} \\
y_3(x) &\equiv \frac{dy_2}{dx} = \frac{d^2 y}{dx^2} \\
&\vdots \\
y_n(x) &\equiv \frac{dy_{n-1}}{dx} = \frac{d^{n-1} y}{dx^{n-1}}
\end{aligned}
\quad
\left\{
\begin{aligned}
&\frac{dy_1}{dx} = y_2 \\
&\frac{dy_2}{dx} = y_3 \\
&\vdots \\
&\frac{dy_{n-1}}{dx} = y_n \\
&a_n(x)\frac{dy_n}{dx} + a_{n-1}(x)y_n + \cdots + a_1(x)y_2 + a_0(x)y_1 = b(x)
\end{aligned}
\right.$$

Primjer

$$\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}=5x$$

$$\begin{cases} y_1 = y \\ y_2 = \frac{dy_1}{dx} = \frac{dy}{dx} \\ y_3 = \frac{dy_2}{dx} = \frac{d^2y}{dx^2} \end{cases}$$

$$\begin{cases} \frac{dy_3}{dx} + 2y_3 - x^2 y_2 = 5x \\ \frac{dy_2}{dx} = y_3 \\ \frac{dy_1}{dx} = y_2 \end{cases}$$

$$\bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \bar{y}' = \vec{f}(\bar{y}) = \begin{pmatrix} y_2 \\ y_3 \\ x^2 y_2 - 2y_3 + 5x \end{pmatrix}$$

Kanonski problem

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f(x,y) \qquad \boxed{\Delta x \to 0, \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}}$$

$$\Delta y \approx f(x,y)\Delta x$$

$$y(x+\Delta x) - y(x) \equiv \Delta y \approx f(x,y)\Delta x$$

$$\Rightarrow y(x+\Delta x) \approx y(x) + f(x,y)\Delta x$$

To je Eulerova metoda

Primjer

$$\frac{dy}{dx} = -y, y(0) = 1$$

$f(x,y) = -y,\ f(0,1) = -1$

Odaberimo: $\Delta x = 0.1$

$y(0.1) = y(0) + (-1)\cdot 0.1 = 1 - 0.1 = 0.9$

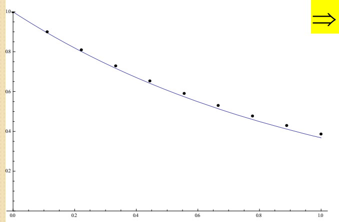
$y(0.2) = y(0.1) + (-0.9)\cdot 0.1 = 0.9 - 0.09 = 0.81$

$y(0.3) = y(0.2) + (-0.81)\cdot 0.1 = 0.81 - 0.081 = 0.729$

$\vdots$

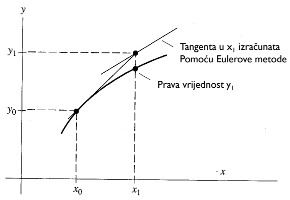
Usporedba s točnim rješenjem

$$\frac{dy}{y} = -dx$$
$$\ln y = -x + c$$
$$y = c'e^{-x} \qquad y(0) = c' = 1$$
$$\Rightarrow y = e^{-x}$$



Analiza greške

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$
$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f(x,y)$$
$$\Delta y \approx f(x,y)\Delta x$$
$$y_1 - y_0 \equiv \Delta y \approx f(x_0,y_0)\Delta x$$
$$\Rightarrow y_1 \approx y_0 + f(x_0,y_0)\Delta x$$



Taylorov red

$$y(x+\Delta x) = y(x) + y'(x)\Delta x + y''(x)\tfrac{\Delta x^2}{2} + y'''(x)\tfrac{\Delta x^3}{6} + \cdots$$
$$= \underbrace{y(x) + f(x,y)\Delta x}_{\text{Eulerova metoda}} + \underbrace{\tfrac{d}{dx} f(x,y)\tfrac{\Delta x^2}{2}}_{\text{Greška po koraku}} + \cdots$$

Metode Runge-Kutta

$y_{i+1} = y_i + \Phi(x_i, y_i, h)h$

gdje je

$y_i \equiv y(x_i), \quad y_{i+1} \equiv y(x_i + h)$

$\Phi(x_i, y_i, h)$: inkrementalna funkcija

(procjena nagiba od i do $i+1$)

$\Phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$

a : konstante

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$

$k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$

$\vdots$

$k_n = f(x_i + p_i h, y_i + q_{n-1,1} k_1 n + \cdots + q_{n-1,n-1} k_{n-1} n)$

p_i, q_{ij} : konstante

$\Rightarrow$ Konstante (a_i, p_i, q_{ij}) se biraju za uskladjivanje s Taylorovim redom

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Taylorov red

$$y(x_i + h) = y(x_i) + y'(x_i)h + \boxed{\tfrac{d}{dx} y'(x_i)}\tfrac{h^2}{2} + O(h^3)$$

$$= y(x_i) + f(x_i, y_i)h + \boxed{[\tfrac{\partial f}{\partial x}\Big|_i + \tfrac{\partial f}{\partial y}\Big|_i f_i]}\tfrac{h^2}{2} + O(h^3)$$

$$= y_i + f_i h + [\tfrac{\partial f}{\partial x}\Big|_i + \tfrac{\partial f}{\partial y}\Big|_i f_i]\tfrac{h^2}{2} + O(h^3)$$

$$\boxed{\begin{aligned}\frac{d}{dx} f(x,y) &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\end{aligned}}$$

Runge-Kutta prvog reda

$y(x_i+h) = y_i + f_i h + [\frac{\partial f}{\partial x}\big|_i + \frac{\partial f}{\partial y}\big|_i f_i]\frac{h^2}{2} + O(h^3)$

$n = 1$

$\Phi = a_1 k_1 = a_1 f(x_i, y_i)$

$y_{i+1} = y_i + a_1 f(x_i, y_i) h$

Usporedba s Taylorovim redom

$a_1 = 1$

$\Rightarrow y_{i+1} = y_i + f(x_i, y_i) h$ Eulerova metoda

Runge-Kutta drugog reda

$y(x_i+h) = y_i + f_i h + [\frac{\partial f}{\partial x}\big|_i + \frac{\partial f}{\partial y}\big|_i f_i]\frac{h^2}{2} + O(h^3)$

$n = 2 \quad \Phi = a_1 k_1 + a_2 k_2$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$

$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$

$$\boxed{\begin{array}{ll} f_i h: & a_1 + a_2 = 1 \\ \frac{\partial f}{\partial x} h^2: & a_2 p_1 = \frac{1}{2} \\ \frac{\partial f}{\partial y} f_i h^2: & a_2 q_{11} = \frac{1}{2} \end{array}}$$

Razvoj za k_2 :

$$k_2 = f(x_i, y_i) + \frac{\partial f}{\partial x}\bigg|_i (p_1 h) + \frac{\partial f}{\partial y}\bigg|_i (q_{11} k_1 h) + O(h^2)$$

$$y_{i+1} = y_i + a_1 k_1 h + a_2 [f_i + \tfrac{\partial f}{\partial x} p_1 h + \tfrac{\partial f}{\partial y} q_{11} k_1 h] h + O(h^3)$$

$$= y_i + a_1 f_i h + a_2 f_i h + a_2 \tfrac{\partial f}{\partial x} p_1 h^2 + a_2 \tfrac{\partial f}{\partial y} q_{11} f_i h^2 + O(h^3)$$

$$\left.\begin{array}{ll} f_i h: & a_1 + a_2 = 1 \\ \frac{\partial f}{\partial x} h^2: & a_2 p_1 = \frac{1}{2} \\ \frac{\partial f}{\partial y} f_i h^2: & a_2 q_{11} = \frac{1}{2} \end{array}\right\} \begin{array}{l} \text{3 jednadžbe,} \\ \text{4 nepoznanice} \end{array}$$

$a_2 = 1 \Rightarrow a_1 = 0,\ p_1 = \frac{1}{2}, q_{11} = \frac{1}{2}$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$$
$$= y_i + k_2 h = y_i + f(x_i + p_1 h, y_i + q_{11} k_1 h) h$$
$$y_{i+1} = y_i + f(x_i + \tfrac{1}{2} h, y_i + \tfrac{1}{2} f(x_i, y_i)) h$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h = y_i + \tfrac{1}{2}(k_1 + k_2) h$$
$$k_1 = f(x_i, y_i), k_2 = f(x_i + h, y_i + f_i h)$$

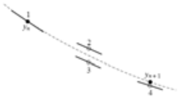
Runge-Kutta četvrtog reda

- Najraširenija

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

gdje je

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1 h)$$
$$k_3 = f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2 h)$$
$$k_4 = f(x_n + h, y_n + k_3 h)$$

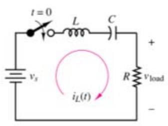


Sustavi diferencijalnih jednadžbi

- Sve metode se primjenjuju u vektorskom obliku

Odziv kruga drugog reda

Vs=12 V, R=200 Ω,
C=10 μF, L=0.5 H
$v_C(0)$=2 V; i_L=?



- Analitičko rješenje
  - Početni uvjeti (dva, $i_L(0)$=0)

$$V_s - v_C(0^+) - Ri_L(0^+) - v_L(0^+) = 0$$
$$V_s - v_C(0^+) - Ri_L(0^+) - L\frac{di_L(0^+)}{dt} = 0$$
$$\frac{di_L(0^+)}{dt} = \frac{V_S}{L} - \frac{v_C(0^+)}{L} = 20 \text{ A/s}$$

- Diferencijalna jednadžba

$$V_s - \frac{1}{C}\int_{-\infty}^{t} i_L(t)dt - Ri_L(t) - L\frac{di_L(t)}{dt} = 0 \quad ;t \geq 0 \quad \left| \frac{d}{dt} \right.$$

$$LC\frac{d^2 i_L(t)}{dt^2} + RC\frac{di_L(t)}{dt} + i_L(t) = C\frac{dV_S}{dt} \quad ;t \geq 0$$

- Standardni oblik

$$\frac{1}{\omega_n^2}\frac{d^2 x_N(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx_N(t)}{dt} + x_N(t) = 0$$

- Slijedi:

$$\omega_n = \sqrt{\frac{1}{LC}} = 447 \text{ rad/s}$$

$$\zeta = RC\frac{\omega_n}{2} = \frac{R}{2}\sqrt{\frac{C}{L}} = 0.447$$

- Rješenje ima oblik $i_L(t) = Ae^{st}$
- Uvrštavanje u diferencijalnu jednadžbu:

$$\frac{1}{\omega_n^2}s^2 Ae^{st} + \frac{2\zeta}{\omega_n}sAe^{st} + Ae^{st} = 0 \quad \Rightarrow \quad \frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1 = 0$$

- Rješenje ($i_L(\infty)=0$)

$$s_{1,2} = -\zeta\omega_n \pm \frac{1}{2}\sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\zeta\omega_n + j\omega_n\sqrt{\zeta^2-1})t} + A_2 e^{(-\zeta\omega_n - j\omega_n\sqrt{\zeta^2-1})t}$$

- Određivanje konstanti

$$i_L(0^+) = A_1 + A_2 = 0 \quad \Rightarrow A_1 = -A_2 = A$$

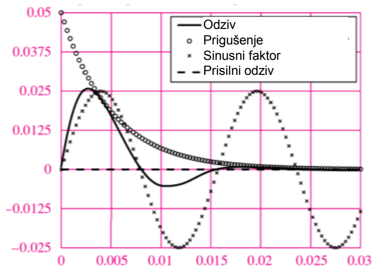
$$\frac{di_L(0^+)}{dt} = A_1(-\zeta\omega_n + j\omega_n\sqrt{\zeta^2-1}) + A_2(-\zeta\omega_n - j\omega_n\sqrt{\zeta^2-1}) = 2A(j\omega_n\sqrt{\zeta^2-1})$$

$$A = \frac{20}{2j\omega_n\sqrt{\zeta^2-1}} = -j0.025$$

- Rješenje

$$i_L(t) = -j0.025e^{(-200+j400)t} + j0.025e^{(-200-j400)t} = 0.025e^{-200t}\left(-je^{-j400t} + je^{-j400t}\right)$$

$$i_L(t) = 0.05e^{-200t}\sin(400t)$$



Numeričko rješenje

- Diferencijalna jednadžba

$$LC\frac{d^2 i_L(t)}{dt^2}+RC\frac{di_L(t)}{dt}+i_L(t)=0 \quad ;t\geq 0$$

- Svođenje na sustav

$$x_1=i_L(t), x_2=\frac{di_L(t)}{dt}$$

pa vrijedi

$$LC\frac{d^2 i_L(t)}{dt^2}+RC\frac{di_L(t)}{dt}+i_L(t)=0 \Rightarrow \begin{cases}\frac{dx_1}{dt}=x_2\\ \frac{dx_2}{dt}=\frac{1}{LC}(-RCx_2-x_1)\end{cases}$$

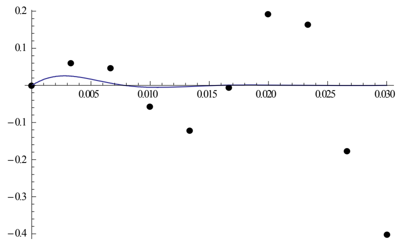
$$\text{pocetni uvjeti}: \begin{cases}x_1(0)=0\\ x_2(0)=20\end{cases}$$

Eulerova metoda

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

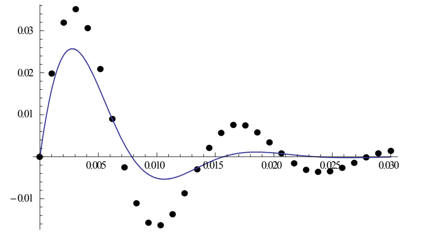
$$\begin{bmatrix}x_1\\x_2\end{bmatrix}_{t+\Delta t}=\begin{bmatrix}x_1\\x_2\end{bmatrix}_t+\begin{bmatrix}0 & \Delta t\\ -\frac{\Delta t}{LC} & -\frac{R\Delta t}{L}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}_t$$

Usporedba za Δt=0.003s



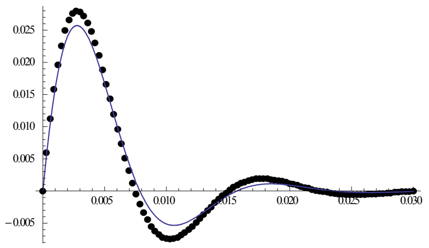
- Rješenje je nestabilno – divergira (10 koraka)

Usporedba za Δt=0.001s



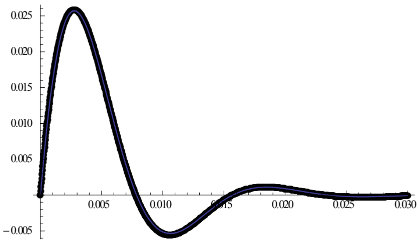
- Rješenje je stabilno, ali nedovoljno točno (30 koraka)

Usporedba za Δt=0.0003s



- Rješenje je stabilno, ali još uvijek nedovoljno točno (100 koraka)

Usporedba za Δt=0.00003s



- Rješenje je stabilno i točno (1000 koraka)

Runge-Kutta 4. reda

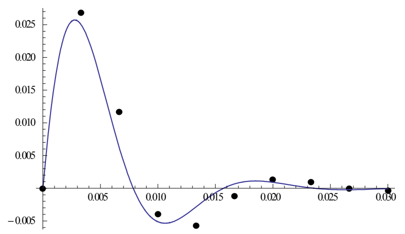
$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{LC}x_1(t) - \frac{R}{C}x_2(t) \end{bmatrix} \quad \Rightarrow \quad \vec{F}(x_1(t), x_2(t), t) = \begin{bmatrix} x_2(t) \\ -\frac{1}{LC}x_1(t) - \frac{R}{C}x_2(t) \end{bmatrix}$$

$$\vec{k}_1 = \vec{F}\left(x_1(t), x_2(t), t\right)\Delta t \quad ; \quad \vec{k}_2 = \vec{F}\left(x_1(t) + \frac{k_1(1)}{2}, x_2(t) + \frac{k_1(2)}{2}, t + \frac{\Delta t}{2}\right)\Delta t$$

$$\vec{k}_3 = \vec{F}\left(x_1(t) + \frac{k_2(1)}{2}, x_2(t) + \frac{k_2(2)}{2}, t + \frac{\Delta t}{2}\right)\Delta t \quad ; \quad \vec{k}_4 = \vec{F}\left(x_1(t) + k_3(1), x_2(t) + k_3(2), t + \Delta t\right)\Delta t$$

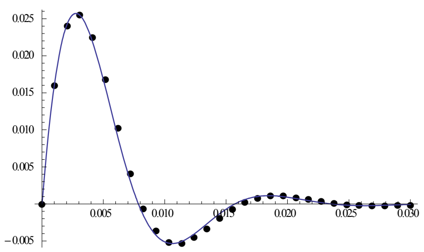
$$\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{1}{6}\left\{\begin{bmatrix} k_1(1) \\ k_1(2) \end{bmatrix} + 2\begin{bmatrix} k_2(1) \\ k_2(2) \end{bmatrix} + 2\begin{bmatrix} k_3(1) \\ k_3(2) \end{bmatrix} + \begin{bmatrix} k_4(1) \\ k_4(2) \end{bmatrix}\right\}$$

Usporedba za Δt=0.003s



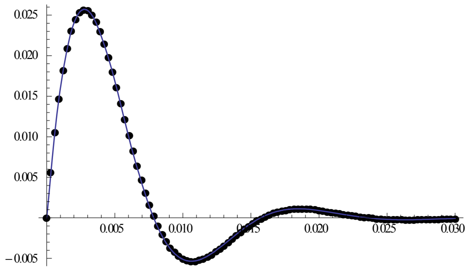
- Rješenje je stabilno ali nedovoljno točno (10 koraka)

Usporedba za Δt=0.001s



- Rješenje je stabilno i dovoljno (?) točno (30 koraka)

Usporedba za Δt=0.0003s



- Rješenje je stabilno i dovoljno točno (100 koraka)