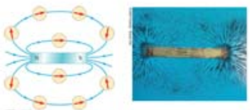
Magnetizam i elektrodinamika

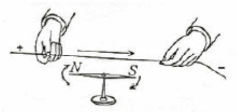
Osvježavanje znanja

Izvori magnetskog polja

- Permanentni magneti

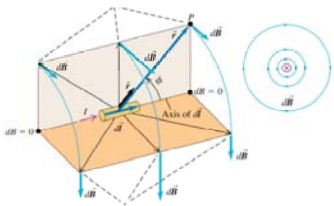


- Struje
  - H.C. Oersted, 1819



Biot-Savartov zakon

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\vec{l} \times \vec{R}}{|\vec{R}|^3} \quad ; \quad \vec{R} = \vec{r} - \vec{r}' \quad ; \quad \mu_0 = 4\pi \cdot 10^{-7} \ \text{(Vs/Am = H/m)}$$



Sila na ravni vodič

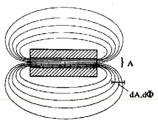
- Vodič duljine *l* kojim teče struja *I* postavljen je okomito na magnetsko polje indukcije *B*
- Ako u vremenu Δt brzinom *v* prođe *N* naboja *q* od jednog do drugog kraja vodiča, struja je: $I = \frac{\Delta Q}{\Delta t} = \frac{N \cdot q}{\Delta t}$
- Na svaki naboj *q* djeluje sila iznosa: $|\vec{f}| = q \cdot v \cdot B$
- Iznos ukupne sile na sve naboje (vodič) je:

$$|\vec{F}| = N \cdot q \cdot v \cdot B = N \cdot q \frac{l}{\Delta t} B = I \cdot l \cdot B$$

Gustoća magnetskog toka

- Gustoća magnetskog toka je:

$$B = \frac{\Phi}{A} \quad \left[\frac{Vs}{m^2}\right] = [Tesla]$$

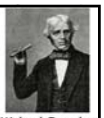


Ako je Φ konstantan na A

- Inače:

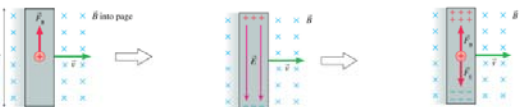
$$d\Phi = \vec{B} \cdot d\vec{A} \quad \Phi = \int \vec{B} \cdot d\vec{A}$$

Elektromagnetska indukcija



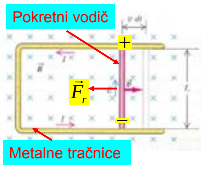
Michael Faraday (1791-1867)

- Faradayev zakon
- Gibanje vodiča u magnetskom polju



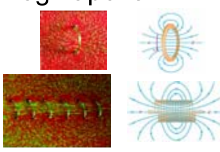
- Polje djeluje silom na naboje $F = q \cdot v \cdot B$
- Naboji se razdvajaju
- Rad koji obavi sila na elektrone jest $R = q \cdot v \cdot B \cdot l$

- Između krajeva štapa javlja se razlika potencijala $U_{AB}=\frac{R}{q}=v\cdot B\cdot l$
- Razmotrimo sada gibanje vodiča duž metalnih tračnica
  - Vodič je izvor napona
  - Zatvara se strujni krug
  - U Δt put vodiča je $\Delta s = v \cdot \Delta t$
  - Površina je: $\Delta S = \Delta s \cdot l = l \cdot v \cdot \Delta t$
  - Vrijedi:



$$|U_{ind}| = v\cdot B\cdot l = \frac{v\cdot B\cdot l\cdot \Delta t}{\Delta t} = \frac{B\Delta S}{\Delta t} = \frac{\Delta\Phi}{\Delta t}$$

- Polaritet induciranog napona: Lenzovo pravilo
  - Magnetski učinci induciranog napona (struje) se protive promjeni toka (uzroku koji ih je stvorio)
    - Smjer sile $\boldsymbol{F_r}$ na vodič je suprotan $\boldsymbol{v}$
  - Faradayev zakon: $U_{ind} = -\frac{\Delta\Phi}{\Delta t} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$
- Povećanje induciranog napona – *N* zavoja
- Zavojnice
  - Povećanje toka
  - *N* zavoja

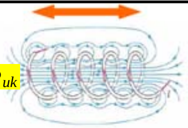


Induktiviteti

- Struja stvara oko sebe magnetsko polje (magnetski tok)
  - Struja i tok su povezani $\Phi = L\cdot I$
  - *L* je koeficijent samoindukcije, samoinduktivitet ili induktivitet $L=\frac{\Phi}{I}$
    - Jedinica za induktivitet je henri (1H)
  - Faradayev zakon:

$$U_{ind} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(LI) = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

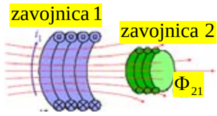
- Zavojnica s *N* zavoja
  - Φ_{uk} prolazi kroz sve zavoje
  - U svakom zavoju se inducir



$$U_{ind,1} = -\frac{d\Phi_{uk}}{dt}$$

  - Ukupni inducirani napon je $U_{ind} = N \cdot U_{ind,1} = -N\frac{d\Phi_{uk}}{dt}$
  - Uvodimo pojam obuhvaćenog (ulančenog) toka: $\psi = N \cdot \Phi_{uk}$
  - Induktivitet zavojnice je $L = \frac{\psi}{I} = \frac{N\Phi_{uk}}{I}$
  - Ukupni inducirani napon je: $U_{ind} = -\frac{d\psi}{dt} = -L\frac{dI}{dt}$

Međuinduktivitet



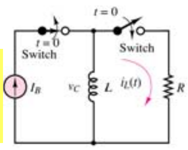
- Dvije bliske zavojnice
  - Zavojnica 1 stvara tok
  - Zavojnica 2 obuhvaća dio toka Φ_{21}
  - Međuinduktivitet je: $M = \frac{\psi_{21}}{I_1} = \frac{N_2\Phi_{21}}{I_1} = \frac{\psi_{12}}{I_2} = \frac{N_1\Phi_{12}}{I_2}$
  - Ako se tok mijenja u vremenu u zavojnici 2 inducira se napon:

$$U_{ind,2} = -\frac{d\psi_{21}}{dt} = -\frac{d}{dt}(MI_1) = -M\frac{dI_1}{dt}$$

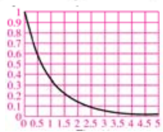
Energija pohranjena u induktivitetu

$$i_L(0^+) = i_L(0^-) = I_B$$

$$L\frac{di_L(t)}{dt} - Ri_L(t) = 0 \quad \Rightarrow \quad i_L(t) = I_B e^{-\frac{R}{L}t}$$



- Na R se u toplinu pretvori

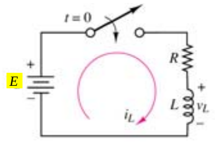


$$W_R = \int_0^\infty Ri_L^2(t)dt = \int_0^\infty RI_B^2 e^{-\frac{2R}{L}t}dt = I_B^2\frac{L}{2}$$

- Ta je energija bila pohranjena u induktivitetu

R-L krug

- Za induktivitet vrijedi $\lim_{t\to\infty} u_L(t) = 0$, $i_L(0^+) = i_L(0^-)$
- Krug prvog reda
  - $i_L(0^+)=0$ A
  - $i_L(t\to\infty)=E/R$
- Jednadžba

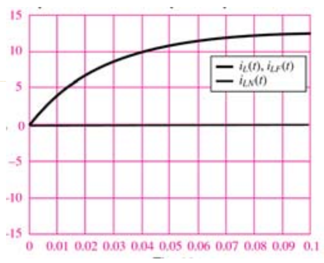


$$E - Ri_L(t) - L\frac{di_L(t)}{dt} = 0 \quad \Rightarrow \quad \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = \frac{E}{R}$$

$$\tau = \frac{L}{R} \quad ; \quad K_S = \frac{1}{R} \quad \Rightarrow \quad i_L(t) = i_L(t\to\infty) + [i_L(0) - i_L(t\to\infty)]e^{-\frac{R}{L}t}$$

$$i_L(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \Rightarrow \quad i_L(t) = i_{LN}(t) + i_{LF}(t)$$

$$i_{LN}(t) = 0 \quad ; \quad i_{LF}(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$



Krugovi drugog reda

- Opći oblik jednadžbe

$$\left.\begin{array}{l} a_2\dfrac{d^2x(t)}{dt^2} + a_1\dfrac{dx(t)}{dt} + a_0x(t) = b_0 f(t) \\ \dfrac{1}{\omega_n^2}\dfrac{d^2x(t)}{dt^2} + \dfrac{2\zeta}{\omega_n}\dfrac{dx(t)}{dt} + x(t) = K_S f(t) \end{array}\right\} \begin{array}{l} \omega_n = \sqrt{\dfrac{a_0}{a_2}} \\ \zeta = \dfrac{a_1}{2}\sqrt{\dfrac{1}{a_0a_2}} \\ K_S = \dfrac{b_0}{a_0} \end{array}$$

- ω_n – prirodna frekvencija
- ζ – prigušna konstanta

Prirodni odziv kruga drugog reda

$$\frac{1}{\omega_n^2}\frac{d^2x_N(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_N(t)}{dt}+x_N(t)=0$$

- Rješenje ovog sustava ima oblik $x_N(t)=Ae^{st}$
- Uvrštavanjem u jednadžbu dobivamo

$$\frac{1}{\omega_n^2}s^2Ae^{st}+\frac{2\zeta}{\omega_n}sAe^{st}+Ae^{st}=0 \quad\Rightarrow\quad \frac{1}{\omega_n^2}s^2+\frac{2\zeta}{\omega_n}s+1=0$$

- Karakteristični polinom – dva rješenja

$$s_{1,2}=-\zeta\omega_n\pm\frac{1}{2}\sqrt{(2\zeta\omega_n)^2-4\omega_n^2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

$$x_N(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

Korijeni karakterističnog polinoma

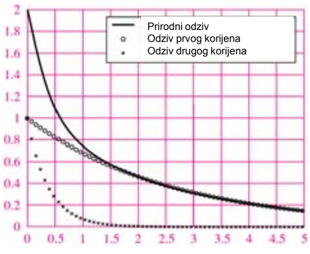
1. Realni i različiti ($\zeta>1$) – prigušeni odziv $s_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$
2. Realni i jednaki ($\zeta=1$) – kritično prigušeni odziv $s_{1,2}=-\zeta\omega_n$
3. Konjugirano kompleksni ($\zeta<1$) – pseudoperiodični odziv

   $s_{1,2}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$

Prigušeni odziv

$$x_N(t)=A_1e^{(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})t}+A_2e^{(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})t}=A_1e^{-\frac{t}{\tau_1}}+A_2e^{-\frac{t}{\tau_2}}$$

$$\tau_1=\frac{1}{\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}\quad;\quad\tau_2=\frac{1}{\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})}$$

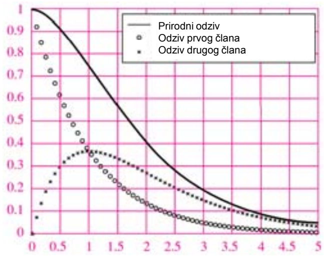


$A_1=A_2=1$
$\zeta=1.5$
$\omega_n=1$

Kritično prigušeni odziv

$$x_N(t) = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} = A_1 e^{-\frac{t}{\tau}} + A_2 t e^{-\frac{t}{\tau}}$$

$$\tau = \frac{1}{\omega_n}$$

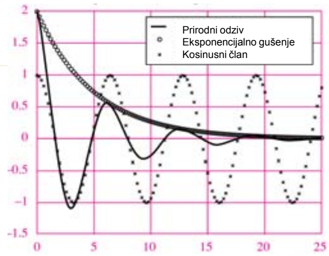


$$A_1 = A_2 = 1 \quad \zeta = 1 \quad \omega_n = 1$$

Pseudoperiodični odziv

$$x_N(t) = A_1 e^{(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})t} + A_2 e^{(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})t}$$

$$\text{za} \quad A_1 = A_2 = A \quad x_N(t) = 2Ae^{-\zeta\omega_n t}\cos\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]$$



$$A_1 = A_2 = 1 \quad \zeta = 0.2 \quad \omega_n = 1$$

Prisilni odziv

- Partikularno rješenje za istosmjernu pobudu f(t)=F je konstantno, pa je prisilni odziv $x_F(t) = K_S F = x_F(t \to \infty)$
- Ukupno rješenje je zbroj prirodnog i prisilnog

$$x(t) = x_N(t) + x_F(t) = A_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + A_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} + x(t \to \infty)$$

$$x(t) = x_N(t) + x_F(t) = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} + x(t \to \infty)$$

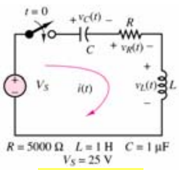
$$x(t) = x_N(t) + x_F(t) = A_1 e^{(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})t} + A_2 e^{(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})t} + x(t \to \infty)$$

Rješavanje krugova drugog reda sa istosmjernom pobudom

- Rješenje kruga $x(0^-)$ prije promjene stanja ($t=0^-$) i $x(t\to\infty)$ nakon završetka prijelazne pojave ($t\to\infty$)
- Odrediti početne uvjete korištenjem kontinuiranosti napona na kapacitetu i struje kroz induktivitet
- Napisati diferencijalnu jednadžbu za $t=0^+$. Nepoznata funkcija x će biti ili napon na C ili struja kroz L

- Reducirati jednadžbu na standardni oblik
- Odrediti parametre ω_n i ζ, te vrstu odziva
- Napisati ukupno rješenje
- Odrediti konstante A_1 i A_2 iz početnih uvjeta

Primjer



$$v_C(t\to\infty)=25\ \text{V} \quad ; \quad i_L(t\to\infty)=0\ \text{A}$$

Početni uvjeti – 2KZ u $t=0^+$

$$V_S - v_C(0^+) - Ri(0^+) + L\left.\frac{di}{dt}\right|_{t=0^+} = 0$$

$$i(0^-)=i(0^+)=0 \quad ; \quad v_C(0^-)=v_C(0^+)=5\ \text{V}$$

$$\left.\frac{di}{dt}\right|_{t=0^+} = \frac{V_S}{L} - \frac{v_C(0^+)}{L} = 25-5=20\ \text{V}$$

$$v_C(0^-)=5\ \text{V}$$

Diferencijalna jednadžba – 2KZ

$$V_S - v_C(t) - Ri(t) + L\frac{di(t)}{dt} = 0 \quad ; \quad v_C(t)=\frac{1}{C}\int_{-\infty}^{t} i(t')dt'$$

$$LC\frac{di(t)}{dt} + RCi(t) + \int_{-\infty}^{t} i(t')dt' = CV_S$$

Deriviranje obje strane po t rezultira s

$$LC\frac{d^2 i(t)}{dt^2} + RC\frac{di(t)}{dt} + i(t) = 0 \quad \Rightarrow$$

$$\omega_n = \sqrt{\frac{1}{LC}} = 1000 \quad \text{rad/s} \quad ; \quad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = 2.5$$

Rješenje je prigušeno:

$$i(t) = A_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + A_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} + i(t \to \infty) =$$
$$= A_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + A_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t}$$

Određivanje konstanti

$$i(0^+) = A_1 + A_2 = 0 \quad \Rightarrow \quad A_1 = -A_2 = A$$
$$\left.\frac{di}{dt}\right|_{t=0^+} = 2A\left(\omega_n\sqrt{\zeta^2-1}\right) = 20 \quad \Rightarrow \quad A = 0.00436$$

Rješenje $i(t) = 0.00436\left(e^{-208.7t} - e^{-4791.3t}\right)$