EMP – Matematički uvod – by Vedax

1.

$\vec{r} = 2\vec{a_x} + 2\vec{a_y} + 2\vec{a_z}$

$\vec{r}' = 4\vec{a_x} + 3\vec{a_y} + 3\vec{a_z}$

$\vec{R} = ?$

$\vec{a_R} = ?$ (u smjeru $\vec{R}$)

$$\vec{R} = \vec{r} - \vec{r}' = (2-4)\vec{a_x} + (2-3)\vec{a_y} + (2-3)\vec{a_z}$$

$$\boxed{\vec{R} = -2\vec{a_x} - \vec{a_y} - \vec{a_z}}$$

$$\vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{-2\vec{a_x} - \vec{a_y} - \vec{a_z}}{\sqrt{(-2)^2 + (-1)^2 + (-1)^2}} = \frac{-2\vec{a_x} - \vec{a_y} - \vec{a_z}}{\sqrt{6}}$$

$$\vec{a_R} = -\sqrt{\frac{4^2}{6_3}}\vec{a_x} - \frac{1}{\sqrt{6}}\vec{a_y} - \frac{1}{\sqrt{6}}\vec{a_z}$$

$$\boxed{\vec{a_R} = -\sqrt{\frac{2}{3}}\vec{a_x} - \frac{1}{\sqrt{6}}\vec{a_y} - \frac{1}{\sqrt{6}}\vec{a_z}}$$

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2. $T(2, 1, 3)$

a) $T_{cil} = ?$

b) $T_{sfe} = ?$

a) cilindrični k.s.

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \operatorname{tg}\varphi &= \frac{y}{x} \Rightarrow \varphi = \operatorname{arctg}\frac{y}{x} \end{aligned}$$

$T_{cil} = (r, \varphi, z)$

$r = \sqrt{4+1} = \sqrt{5} \approx 2.236$

$\varphi = \operatorname{arctg}\frac{1}{2} \approx 0.464$ (u radijanima)

$z = 3$

$$\boxed{T_{cil}\ (2.236, 0.464, 3)}$$

b) sferni k.s.

$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \frac{y}{z} &= \operatorname{tg}\theta\sin\varphi \end{aligned}$$

$r = \sqrt{4+1+9} = \sqrt{14} \approx 3.742$

$\varphi \approx 0.464$

$\frac{1}{3} = \operatorname{tg}\theta \sin 0.464$

$\theta \approx 0.641$

$T_{sfe} = (r, \theta, \varphi)$

$$\boxed{T_{sfe}\ (3.742, 0.641, 0.464)}$$

③ $\vec{A} = yz\vec{a_x} - 2x\vec{a_y} + \vec{a_z}$

$\vec{B} = \vec{a_x} + z\vec{a_y}$

$T(3,1,1)$

a) $(\vec{A}-\vec{B})_T = ?$

$$(\vec{A}-\vec{B})_T = \left((yz-1)\vec{a_x} + (-2x-z)\vec{a_y} + \vec{a_z}\right)_T =$$

$$= (1-1)\vec{a_x} + (-2\cdot 3 - 1)\vec{a_y} + \vec{a_z}$$

$$\boxed{(\vec{A}-\vec{B})_T = -7\vec{a_y} + \vec{a_z}}$$

b) $(\vec{A}+\vec{B})_T = ?$

$$(\vec{A}+\vec{B})_T = \left((yz+1)\vec{a_x} + (-2x+z)\vec{a_y} + \vec{a_z}\right)_T =$$

$$= (1+1)\vec{a_x} + (-6+1)\vec{a_y} + \vec{a_z}$$

$$\boxed{(\vec{A}+\vec{B})_T = 2\vec{a_x} - 5\vec{a_y} + \vec{a_z}}$$

c) $(\vec{A}\cdot\vec{B})_T = (yz - 2xz + 0)_T = 1\cdot 1 - 2\cdot 3\cdot 1 = -5$.

$$\boxed{(\vec{A}\cdot\vec{B})_T = -5}$$

d) $$(\vec{A}\times\vec{B})_T = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ yz & -2x & 1 \\ 1 & z & 0 \end{vmatrix} = \left(-z\vec{a_x} + \vec{a_y} + (yz^2+2x)\vec{a_z}\right)_T =$$

$$= -\vec{a_x} + \vec{a_y} + 7\vec{a_z}$$

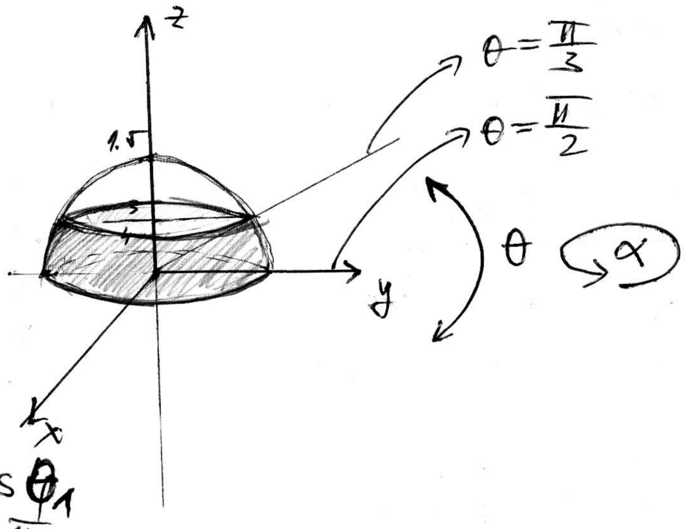
$$\boxed{(\vec{A}\times\vec{B})_T = -\vec{a_x} + \vec{a_y} + 7\vec{a_z}}$$

4. $r = 1.5\,m$

$0 \leq \alpha \leq 2\pi$

$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$

$P_s = ?$



sfera: $x^2+y^2+z^2 = 1.5^2\,m^2$

ograničena sa $0 \leq z \leq r\cos\theta_1$

odnosno sa $0 \leq z \leq 1.5\cos\frac{\pi}{3}$

$0 \leq z \leq \frac{3}{4}$

površina dijela plohe preko plošnog integrala!

↳ podintegralna f-ja = 1

$$P_s = \iint_S dS$$

$$x^2+y^2+z^2 = 1.5^2$$

$$z^2 = 1.5^2 - x^2 - y^2$$

$$z = \sqrt{2.25 - x^2 - y^2}$$

+ predznak

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$$

$$dS = \sqrt{1 + \frac{x^2}{2.25 - x^2 - y^2} + \frac{y^2}{2.25 - x^2 - y^2}}\,dxdy$$

$dxdy$ → projekcija na xy ravnu

$$dS = \sqrt{\frac{2.25 \cancel{- x^2 - y^2} \cancel{+ x^2 + y^2}}{2.25 - x^2 - y^2}}\,dxdy$$

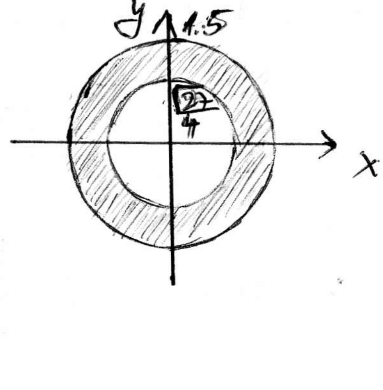
za $z = \frac{3}{4}$

⇓

$$x^2 + y^2 + \left(\frac{3}{4}\right)^2 = 1.5^2$$

$$x^2 + y^2 = \left(\frac{\sqrt{27}}{4}\right)^2$$

$$dS = \frac{1.5\,dxdy}{\sqrt{2.25 - x^2 - y^2}}$$



$$\boxed{P_s} = \iint \frac{1.5\,dxdy}{\sqrt{2.25 - x^2 - y^2}} =$$

$$= 1.5\int_0^{2\pi} d\varphi \int_{\frac{\sqrt{27}}{4}}^{\frac{3}{2}} \frac{r\,dr}{\sqrt{2.25 - r^2}} = \left[\begin{array}{c} 2.25 - r^2 = t \,/\,' \\ -2r\,dr = dt \\ r\,dr = -\frac{dt}{2} \end{array}\right] =$$

$$= -\frac{1.5}{\cancel{2}} \cdot \cancel{2}\pi \cdot \int_{0.5625}^{0} \frac{dt}{\sqrt{t}} = -1.5\pi \cdot \left.\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right|_{0.5625}^{0} = 2.25\pi\,m^2 =$$

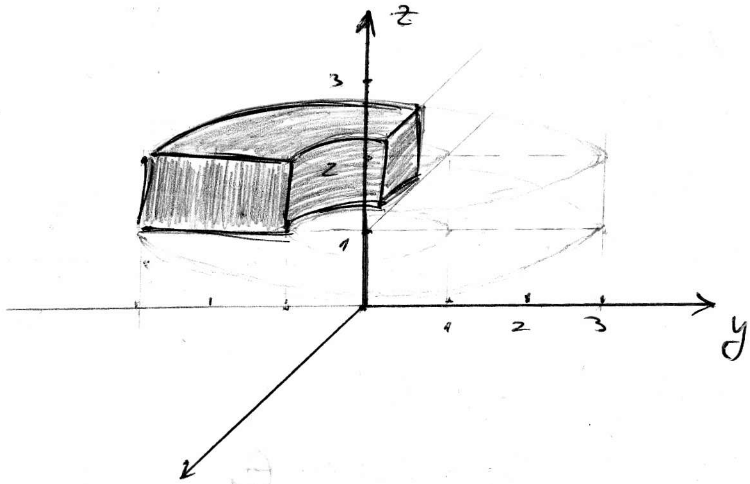
$$= \boxed{7.07\,m^2}$$

5. $1m \le r \le 3m$

$\pi \le \alpha \le \frac{3}{2}\pi$

$1m \le z \le 2m$

$V = ?$



$$V = \iiint_V dx\,dy\,dz =$$

$$= \int_{\pi}^{\frac{3\pi}{2}} d\varphi \int_1^3 r\,dr \int_1^2 dz = \left(\frac{3\pi}{2} - \pi\right) \cdot \frac{r^2}{2}\Big|_1^3 \cdot z\Big|_1^2 =$$

$$= \frac{\pi}{2} \cdot \frac{9-1}{2} \cdot (2-1) = \frac{\pi}{\cancel{2}_1} \cdot \cancel{4}^2 \cdot 1 = 2\pi$$

$$\boxed{V = 6.28 m^3}$$

6. $\vec{A} = 2\vec{a_x} - 3\vec{a_y} + 2\vec{a_z}$

$\vec{B} = \vec{a_x} + 2\vec{a_y} + \vec{a_z}$

a) vektorska projekcija vektora $\vec{B}$ na vektor $\vec{A}$:

$$\boxed{\vec{B}_{\vec{A}}} = \underbrace{|\vec{B}| \cos\varphi}_{} \cdot \vec{A_0} = \cancel{|\vec{B}|} \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\cancel{|\vec{B}|}} \cdot \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A} \cdot \vec{B} \cdot \vec{A}}{|\vec{A}|^2} = \frac{(2-6+2)\vec{A}}{4+9+4} =$$

$$= -\frac{2}{17}\vec{A} = \boxed{-\frac{2}{17}(2\vec{a_x} - 3\vec{a_y} + 2\vec{a_z})}$$

b) skalarna projekcija

$$\boxed{|\vec{B}| \cos\varphi} = \cancel{|\vec{B}|} \cdot \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\cancel{|\vec{B}|}} = \frac{-2}{\sqrt{4+9+4}} = \boxed{-0.485}$$

c) $$\cos\varphi = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \frac{-2}{\sqrt{17} \cdot \sqrt{6}} = -\frac{2}{\sqrt{102}} = -0.198029508\ldots$$

$$\boxed{\varphi = 101.42^\circ}$$

(7.) $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}||\sin\varphi|$

$$|\sin\varphi| = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ 2 & -3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \vec{a_x}(-3-4) - \vec{a_y}(2-2) + \vec{a_z}(4+3) =$$

$$= -7\vec{a_x} + 7\vec{a_z}$$

$$|\vec{A}\times\vec{B}| = \sqrt{49+49} = \sqrt{98}$$

$$|\sin\varphi| = \frac{\sqrt{98}}{\sqrt{102}} = 0.980196058\ldots$$

$$\sin\varphi = \pm 0.980196058$$

$\varphi_1 = 78°$, $\boxed{\varphi_2 = 101°}$ (točnije $\varphi = 101.42°$)

(8.) $\vec{A} = (y+2)\vec{a_x} + (x-1)\vec{a_y}$

$\vec{B} = 2\vec{a_x} - 3\vec{a_y} + 2\vec{a_z}$

$T(7,5,1)$

$$\vec{A}_T = (5+2)\vec{a_x} + (7-1)\vec{a_y} = 7\vec{a_x} + 6\vec{a_y}$$

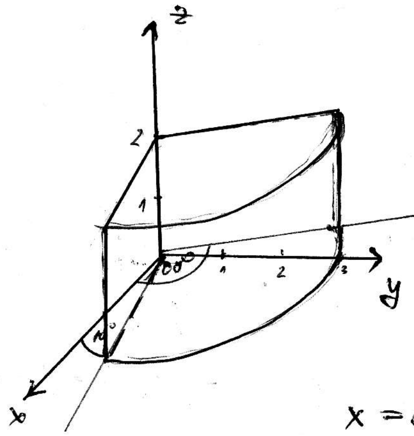
$$\vec{A}_{\vec{B}} = \underbrace{|\vec{A}|\cos\varphi}\,\vec{B_0}$$

$$\boxed{|\vec{A}|\cos\varphi} = \cancel{|\vec{A}|}\frac{\vec{A}\cdot\vec{B}}{\cancel{|\vec{A}|}|\vec{B}|} = \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = \frac{14-18}{\sqrt{4+9+4}} = \boxed{-\frac{4}{\sqrt{17}}}$$

(9.) $r = 3m$

$h = 2m$

$10^\circ \le \alpha \le 100^\circ$



$\varphi_1 = \pi \cdot \frac{10^\circ}{180^\circ} = \frac{\pi}{18}$

$\varphi_2 = \pi \cdot \frac{100^\circ}{180^\circ} = \frac{5\pi}{9}$

$x = r\cos\varphi$

$x_1 = 3\cos\frac{\pi}{18}$

$x_2 = 3\cos\frac{5\pi}{9}$

$x^2 + y^2 = 9 \rightarrow$ cilindar

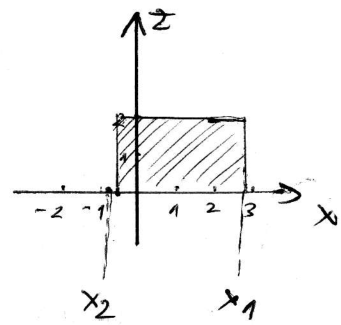
$y^2 = 9 - x^2$

$y = \sqrt{9-x^2}$

+ predznak

$dS = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\,dz$ → projekcija u xz ravninu

$dS = \sqrt{1 + \frac{x^2}{9-x^2}}\, dx\,dz = \frac{3\,dx\,dz}{\sqrt{9-x^2}}$



$S = 3\iint \frac{dx\,dz}{\sqrt{9-x^2}} = 3\int_{3\cos\frac{5\pi}{9}}^{3\cos\frac{\pi}{18}} \frac{dx}{\sqrt{9-x^2}} \int_0^2 dz =$

$= 3\arcsin\left(\frac{x}{3}\right)\Big|_{3\cos\frac{5\pi}{9}}^{3\cos\frac{\pi}{18}} \cdot 2 = 6\left[\arcsin\left(\cos\frac{\pi}{18}\right) - \arcsin\left(\cos\frac{5\pi}{9}\right)\right]$

$\boxed{S} = 6\left(\frac{4\pi}{9} - \left(-\frac{\pi}{18}\right)\right) = \cancel{6}^{1} \cdot \frac{8\pi + \pi}{\cancel{18}_3} = \frac{\cancel{9}^{3}\pi}{\cancel{3}_1} = \boxed{3\pi\ m^2}$

(10) $\vec{A} = 5\vec{a_x}$

$\vec{B} = 4\vec{a_x} + B_y\vec{a_y}$

$\varphi(\vec{A},\vec{B}) = 45^\circ$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos 45^\circ$$

$$(5\vec{a_x})\cdot(4\vec{a_x} + B_y\vec{a_y}) = \sqrt{5^2}\cdot\sqrt{4^2+B_y^2}\cdot\frac{\sqrt{2}}{2}$$

$$20 = 5\frac{\sqrt{2}}{2}\sqrt{16+B_y^2} \quad \Big/ \cdot\frac{2}{5\sqrt{2}}$$

$$\sqrt{16+B_y^2} = \frac{8}{\sqrt{2}} \quad \Big/^2$$

$$16+B_y^2 = 32$$

$$B_y^2 = 16 \Rightarrow \boxed{B_y = \pm 4}$$

(11.) $A(2,-5,-2)$

$B(14,-5,3)$

$$\vec{OA} = 2\vec{a_x} - 5\vec{a_y} - 2\vec{a_z}$$

$$\vec{OB} = 14\vec{a_x} - 5\vec{a_y} + 3\vec{a_z}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = 12\vec{a_x} + 5\vec{a_z}$$

$$\boxed{\vec{a}} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{12\vec{a_x} + 5\vec{a_z}}{\sqrt{144+25}} = \boxed{\frac{12}{13}\vec{a_x} + \frac{5}{13}\vec{a_z}}$$

(12.) $\vec{A} = \vec{a_r} + \pi\vec{a_\alpha} + 3\vec{a_z}$

$\vec{B} = \alpha\vec{a_r} + \beta\vec{a_\alpha} - 6\vec{a_z}$

paralelni! → kut između njih je 0°

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\cdot\sin 0 = |\vec{A}||\vec{B}|\cdot 0$$

$$|\vec{A}||\vec{B}| \neq 0 \Rightarrow |\vec{A}\times\vec{B}| = 0$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{a_r} & \vec{a_\alpha} & \vec{a_z} \\ 1 & \pi & 3 \\ \alpha & \beta & -6 \end{vmatrix} = (-6\pi - 3\beta)\vec{a_r} + (6+3\alpha)\vec{a_\alpha} + (\beta - \pi\alpha)\vec{a_z}$$

$$|\vec{A}\times\vec{B}| = \sqrt{(-6\pi-3\beta)^2 + (6+3\alpha)^2 + (\beta-\pi\alpha)^2} = 0$$

$$-6\pi - 3\beta = 0 \Rightarrow \boxed{\beta = -2\pi} \qquad 6+3\alpha = 0 \Rightarrow \boxed{\alpha = -2}$$

13.
$$\vec{A} = \vec{a_x} + 2\vec{a_y} - \vec{a_z}$$
$$\vec{B} = \alpha \vec{a_x} + \vec{a_y} + 3\vec{a_z}$$

$\varphi = 90^\circ$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos 90^\circ = |\vec{A}||\vec{B}|\cdot 0$$

$$|\vec{A}||\vec{B}| \neq 0 \Rightarrow \vec{A}\cdot\vec{B} = 0$$
$$\alpha + 2 - 3 = 0$$
$$\boxed{\alpha = 1}$$

14.
$$\varphi = 2xy - 5z$$
$T(1,2,3)$

$$\boxed{(\text{grad}\,\varphi)_T} = (2y\vec{a_x} + 2x\vec{a_y} - 5\vec{a_z})_T = \boxed{4\vec{a_x} + 2\vec{a_y} - 5\vec{a_z}}$$

15.
$$\vec{E} = \frac{50}{r}\vec{a_r} - 4\vec{a_z}$$
$(10, 20^\circ, 2) = (r, \varphi, z)$

$$\vec{E} = A_r\vec{a_r} + A_\alpha\vec{a_\alpha} + A_z\vec{a_z} \rightarrow \text{cil.}$$
$$\vec{E} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z} \rightarrow \text{kartez.}$$

$$A_r\vec{a_r} + A_\alpha\vec{a_\alpha} = A_x\vec{a_x} + A_y\vec{a_y}$$

vrijedi:
$$\vec{a_r} = \frac{x}{r}\vec{a_x} + \frac{y}{r}\vec{a_y}$$
$$\vec{a_\alpha} = -\frac{y}{r}\vec{a_x} + \frac{x}{r}\vec{a_y}$$

$$A_r = \frac{50}{r}, \quad A_\alpha = 0$$

$$\frac{50}{r}\left(\frac{x}{r}\vec{a_x} + \frac{y}{r}\vec{a_y}\right) = A_x\vec{a_x} + A_y\vec{a_y}$$

$$\frac{50x}{r^2}\vec{a_x} + \frac{50y}{r^2}\vec{a_y} = A_x\vec{a_x} + A_y\vec{a_y}$$

$$\frac{50\cos\varphi}{r}\vec{a_x} + \frac{50\sin\varphi}{r}\vec{a_y} = A_x\vec{a_x} + A_y\vec{a_y}$$

$$A_x = \frac{50\cos\varphi}{r} = \frac{5\cancel{0}\cos 20^\circ}{1\cancel{0}} = 5\cos 20^\circ = 4.698$$

$$A_y = 5\sin\varphi = 1.71$$

$$A_z = -4$$

$$\vec{E} = 4.698\vec{a_x} + 1.71\vec{a_y} - 4\vec{a_z}$$

$$\vec{a_E} = \frac{\vec{E}}{|\vec{E}|} = \frac{\vec{E}}{6.4} = 0.734\vec{a_x} + 0.267\vec{a_y} - 0.625\vec{a_z}$$

(16.) $|\vec{E}| = 10$

$$\vec{E} = \frac{50}{r}\vec{a_r} - 4\vec{a_z}$$

$$|\vec{E}| = \sqrt{\left(\frac{50}{r}\right)^2 + 16} = \sqrt{\frac{2500}{r^2} + 16} = 10 \;/^2$$

$$\frac{2500}{r^2} = 100 - 16$$

$$\frac{2500}{r^2} = 84$$

$$r^2 = \frac{2500}{84}$$

$$\boxed{r = 5.46}$$

(17.) $\vec{A} = 10\vec{a_r} - 3\vec{a_\theta} + 5\vec{a_\alpha}$

$\vec{B} = 2\vec{a_r} + 5\vec{a_\theta} + 3\vec{a_\alpha}$

najprije jedinični vektor u smjeru vektora $\vec{A}$:

$$\vec{A_0} = \frac{\vec{A}}{|\vec{A}|} = \frac{10\vec{a_r} - 3\vec{a_\theta} + 5\vec{a_\alpha}}{\sqrt{100 + 9 + 25}} = \frac{10}{\sqrt{134}}\vec{a_r} - \frac{3}{\sqrt{134}}\vec{a_\theta} + \frac{5}{\sqrt{134}}\vec{a_\alpha}$$

$$\vec{B} \cdot \vec{A_0} = \frac{20}{\sqrt{134}} - \cancel{\frac{15}{\sqrt{134}}} + \cancel{\frac{15}{\sqrt{134}}} = \frac{20}{\sqrt{134}}$$

$$\boxed{\vec{B} \cdot \vec{A_0} = 1.728}$$

(18.) $$\vec{n} = \vec{A} \times \vec{B} = \begin{vmatrix} \vec{a_r} & \vec{a_\theta} & \vec{a_\alpha} \\ 10 & -3 & 5 \\ 2 & 5 & 3 \end{vmatrix} = -34\vec{a_r} - 20\vec{a_\theta} + 56\vec{a_\alpha}$$

$$\boxed{\vec{n_0}} = \frac{\vec{n}}{|\vec{n}|} = \frac{-34\vec{a_r} - 20\vec{a_\theta} + 56\vec{a_\alpha}}{\sqrt{1156 + 400 + 3136}} = \boxed{-0.496\vec{a_r} - 0.292\vec{a_\theta} + 0.818\vec{a_\alpha}}$$

(oni su radili $\vec{B} \times \vec{A}$ što je jednako $-\vec{A} \times \vec{B}$ pa odtud drugačiji predznaci)

19. $\vec{A} = (x^2-y^2)\vec{a_y} + (x\cdot z)\vec{a_z}$

$P(r=6, \alpha=60^\circ, z=-4)$

$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$

$\vec{A} = A_r\vec{a_r} + A_\alpha\vec{a_\alpha} + A_z\vec{a_z}$

$$A_x\vec{a_x} + A_y\vec{a_y} = A_r\vec{a_r} + A_\alpha\vec{a_\alpha}$$

$$0\cdot\vec{a_x} + (x^2-y^2)\cdot\vec{a_y} = A_r\vec{a_r} + A_\alpha\vec{a_\alpha}$$

vrijedi: $\vec{a_r} = \frac{x}{r}\vec{a_x} + \frac{y}{r}\vec{a_y}$

$\vec{a_\alpha} = -\frac{y}{r}\vec{a_x} + \frac{x}{r}\vec{a_y}$

$$0\cdot\vec{a_x} + (x^2-y^2)\vec{a_y} = A_r\left(\frac{x}{r}\vec{a_x} + \frac{y}{r}\vec{a_y}\right) + A_\alpha\left(-\frac{y}{r}\vec{a_x} + \frac{x}{r}\vec{a_y}\right)$$

$$0\cdot\vec{a_x} + (x^2-y^2)\vec{a_y} = \vec{a_x}\left(A_r\frac{x}{r} - A_\alpha\frac{y}{r}\right) + \vec{a_y}\left(A_r\frac{y}{r} + A_\alpha\frac{x}{r}\right)$$

$$A_r\frac{x}{r} - A_\alpha\frac{y}{r} = 0 \quad /\cdot r$$

$$x^2-y^2 = A_r\frac{y}{r} + A_\alpha\frac{x}{r} \quad /\cdot r$$

$$A_r x - A_\alpha y = 0 \quad /\cdot x$$

$A_\alpha = \frac{x}{y}A_r$

$$A_r y + A_\alpha x = (x^2-y^2)r \quad /\cdot y$$

$$\left.\begin{aligned} A_r x^2 - \cancel{A_\alpha xy} &= 0 \\ A_r y^2 + \cancel{A_\alpha xy} &= (x^2-y^2)yr \end{aligned}\right\} +$$

$$A_r(x^2+y^2) = (x^2-y^2)yr$$

$$\underline{\underline{A_r}} = \frac{(x^2-y^2)yr}{x^2+y^2} = \frac{(9-27)\cdot 3\sqrt{3}\cdot 6}{36} = \underline{\underline{-9\sqrt{3}}}$$

$x = r\cos\alpha = 6\cos 60^\circ = 3$

$y = r\sin\alpha = 6\sin 60^\circ = 3\sqrt{3}$

$x^2+y^2 = r^2 = 36$

$$\underline{\underline{A_\alpha}} = \frac{x}{y}A_r = \frac{3}{3\sqrt{3}}\cdot(-9\sqrt{3}) = \underline{\underline{-9}}$$

$$\underline{\underline{A_z}} = xz = 3\cdot(-4) = \underline{\underline{-12}}$$

$$\boxed{\vec{A_r}\Big|_P = -9\sqrt{3}\,\vec{a_r} - 9\vec{a_\alpha} - 12\vec{a_z}}$$

20.

$R = 0.2\,m$

$\rho = \frac{1}{\sqrt{x^2+y^2}} \frac{C}{m^3}$ → volumna gustoća

$\rho = \frac{dQ}{dV}$

$Q = ?$

$Q = \int_V \rho\, dV$

$$Q = \iiint \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy\,dz$$

sferne koordinate:

$$\left.\begin{aligned} x &= R\sin\theta\cos\varphi \\ y &= R\sin\theta\sin\varphi \\ z &= R\cos\theta \end{aligned}\right\} \quad \begin{aligned} &\sqrt{x^2+y^2} = \sqrt{R^2\sin^2\theta} = R\sin\theta \\ &J = R^2\sin\theta \end{aligned}$$

$$Q = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{0.2} R^{\cancel{2}}\cancel{\sin\theta} \cdot \frac{1}{\cancel{R\sin\theta}}\, dR =$$

$$= \cancel{2}\pi \cdot \pi \cdot \frac{R^2}{\cancel{2}}\Big|_0^{0.2} = \pi^2 \cdot 0.2^2 = 0.04\pi^2$$

$$\boxed{Q = 0.394\, C}$$

21.

$f(x,y,z) = 5x + 10xz - xy + 6$

$\text{grad} f = ?$

$$\boxed{\text{grad } f} = \frac{\partial f}{\partial x}\vec{a_x} + \frac{\partial f}{\partial y}\vec{a_y} + \frac{\partial f}{\partial z}\vec{a_z} = \boxed{(5 + 10z - y)\vec{a_x} + (-x)\vec{a_y} + 10x\vec{a_z}}$$

22.

$f(r, \alpha, z) = 2\sin\alpha - rz + 4$

$\text{grad} f = ?$

$$\boxed{\text{grad } f} = \frac{\partial f}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial f}{\partial \alpha}\vec{a_\alpha} + \frac{\partial f}{\partial z}\vec{a_z} = \boxed{(-z)\vec{a_r} + \frac{2\cos\alpha}{r}\vec{a_\alpha} - r\vec{a_z}}$$

23.

$f(r, \theta, \alpha) = 2r\cos\theta - 5\alpha + 2$

$\text{grad} f = ?$

$$\text{grad } f = \frac{\partial f}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \alpha}\vec{a_\alpha}$$

$$\boxed{\text{grad } f = 2\cos\theta\,\vec{a_r} - 2\sin\theta\,\vec{a_\theta} - \frac{5}{r\sin\theta}\vec{a_\alpha}}$$

(24.) $f(x,y,z) = xy + 2z^2$

$T(1,1,1)$

$\vec{a} = \vec{a_x} - 2\vec{a_y} + \vec{a_z}$

$$\boxed{(\text{grad} f)_T \cdot \vec{a_0}} = (\text{grad} f)_T \cdot \frac{\vec{a}}{|\vec{a}|} =$$

$$= (y\vec{a_x} + x\vec{a_y} + 4z\vec{a_z})_T \cdot \frac{\vec{a_x} - 2\vec{a_y} + \vec{a_z}}{\sqrt{6}} =$$

$$= (\vec{a_x} + \vec{a_y} + 4\vec{a_z}) \cdot \frac{\vec{a_x} - 2\vec{a_y} + \vec{a_z}}{\sqrt{6}} = \boxed{\frac{3}{\sqrt{6}}}$$

(25.) $\vec{F}(x,y,z) = (x+y)\vec{a_x} - x\vec{a_y} + z\vec{a_z}$

$A(0,1,2) \leftrightarrow B(1,0,2)$

$$I = \int_A^B \vec{F} d\vec{r} = \int_A^B (x+y)\,dx - x\,dy + z\,dz$$

parametrizacija dužine $\overline{AB}$:

$x = t \longrightarrow dx = dt$

$y = 1-t \longrightarrow dy = -dt$

$z = 2, \quad t \in [0,1] \longrightarrow dz = 0$

$$\boxed{I} = \int_0^1 (\cancel{t} + 1 - \cancel{t})\,dt - t\cdot(-dt) + 2\cdot 0 =$$

$$= \int_0^1 (1+t)\,dt = t\Big|_0^1 + \frac{t^2}{2}\Big|_0^1 = 1 + \frac{1}{2} = \frac{3}{2} = \boxed{1.5}$$

(26.) $\vec{F}(x,y,z) = \vec{a_x} + 2\vec{a_y} + \vec{a_z}$

$A(1,0,1) \Leftrightarrow B(0,1,1) \rightarrow$ kružni luk $\rightarrow$ $x = \cos t$, $y = \sin t$, $z = 1$, $t \in [0, \frac{\pi}{2}]$

$dx = -\sin t\,dt$

$dy = \cos t\,dt$

$dz = 0$

$$\boxed{I} = \int_A^B dx + 2dy + dz = \int_0^{\frac{\pi}{2}} -\sin t\,dt + 2\cos t\,dt = (\cos t + 2\sin t)\Big|_0^{\frac{\pi}{2}} = \boxed{1}$$

(27.) $\vec{F}(x,y,z) = 2x^2y\,\vec{a_x} + z\vec{a_y} + y\,\vec{a_z}$

$0 \leq x, y, z \leq 1$

tok vektorskog polja?

$$\oint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dx\,dy\,dz = \iiint_V (4xy + 0 + 0)\,dx\,dy\,dz =$$

$$= 4\iiint_V xy\,dx\,dy\,dz = 4\int_0^1 x\,dx \int_0^1 y\,dy \int_0^1 dz = 4\left(\frac{x^2}{2}\cdot\frac{y^2}{2}\cdot z\right)\Big|_0^1 = \boxed{1}$$

(28.) $\vec{F}(x,y,z) = e^{-\alpha y}\left(\cos(\beta x)\vec{a_x} - \sin(\beta x)\vec{a_y}\right)$

$\operatorname{div}\vec{F} = ?$

$$\vec{F} = e^{-\alpha y}\cos(\beta x)\vec{a_x} - e^{-\alpha y}\sin(\beta x)\vec{a_y}$$

$$\boxed{\operatorname{div}\vec{F}} = \frac{\partial}{\partial x}\left(e^{-\alpha y}\cos(\beta x)\right) - \frac{\partial}{\partial y}\left(e^{-\alpha y}\sin(\beta x)\right) =$$

$$= -\beta e^{-\alpha y}\sin(\beta x) + \alpha e^{-\alpha y}\sin(\beta x) =$$

$$= \boxed{(\alpha - \beta)\,e^{-\alpha y}\sin(\beta x)}$$

(29.) $\vec{F}(x,y,z) = 5x^2\sin(\pi x)\,\vec{a_x}$

$x = 0.5$

$$\left(\operatorname{div}\vec{F}\right)_{x=0.5} = \left(10x\sin(\pi x) + 5\pi x^2\cos(\pi x)\right)_{x=0.5} =$$

$$= \underbrace{5\sin\left(\frac{\pi}{2}\right)}_{1} + \underbrace{5\pi\cdot 0.25\,\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}}_{0} = \boxed{5}$$

(30.) $\vec{F}(x,y,z) = xy\,\vec{a_x} + 2yz\,\vec{a_y} - \vec{a_z}$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & -1 \end{vmatrix} = -2y\vec{a_x} + 0\cdot\vec{a_y} - x\vec{a_z} = \boxed{-2y\,\vec{a_x} - x\,\vec{a_z}}$$

(31.) $\vec{F}(r,\alpha,z) = 2\vec{a_r} + \sin\alpha\,\vec{a_\alpha} - z\vec{a_z}$

$\text{rot}\,\vec{F} = ?$

$A_r = 2$
$A_\alpha = \sin\alpha$
$A_z = -z$

$$\text{rot}\,\vec{F} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}\right)\vec{a_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{a_\alpha} +$$

$$+ \frac{1}{r}\left(\frac{\partial (r A_\alpha)}{\partial r} - \frac{\partial A_r}{\partial \alpha}\right)\vec{a_z} =$$

$$= \left(\frac{1}{r}\cdot 0 - 0\right)\vec{a_r} + (0-0)\,\vec{a_\alpha} + \frac{1}{r}(\sin\alpha - 0)\vec{a_z} =$$

$$= \boxed{\frac{\sin\alpha}{r}\vec{a_z}}$$

(32.) $\vec{F}(x,y,z) = xy\,\vec{a_x} + 2\vec{a_y}$

$(0,0,0) \rightarrow (1,0,0) \rightarrow (0,1,0)$

$A \rightarrow B \rightarrow C$

$\overline{AB}$: $x = t$, $t \in [0,1]$ $\Rightarrow dx = dt$
$y = 0$ $\Rightarrow dy = 0$
$z = 0$ $\Rightarrow dz = 0$

$\overline{BC}$: $x = -t+1$, $t \in [0,1]$ $\Rightarrow dx = -dt$
$y = t$ $\Rightarrow dy = dt$
$z = 0$ $\Rightarrow dz = 0$

$\overline{CA}$: $x = 0$, $t \in [1,0]$ $\Rightarrow dx = 0$
$y = t$ $\Rightarrow dy = dt$
$z = 0$ $\Rightarrow dz = 0$

$$\boxed{I} = \int \vec{F}\,d\vec{r} = \int xy\,dx + 2dy = \int_0^1 t\cdot 0\cdot dt + 2\cdot 0 +$$

$$+ \int_0^1 (-t+1)\cdot t\,(-dt) + 2dt + \int_1^0 0\cdot t\cdot 0 + 2\cdot dt =$$

$$= \int_0^1 (1-t)t\,dt + 2dt + 2\int_1^0 dt = \left(\frac{t^2}{2} - \frac{t^3}{3} + 2t\right)\Big|_0^1 + 2t\Big|_1^0 =$$

$$= \frac{1}{2} - \frac{1}{3} + 2 - 2 = \boxed{\frac{1}{6}}$$