Sudari

a) Savršeno elastičan sudar

Očuvana je količina gibanja

$$\vec{P} = \underbrace{\sum_i \vec{p_i}}_{\text{prije}} = \underbrace{\sum_i \vec{p_i}'}_{\text{poslije sudara}}$$

Očuvana je kinetička energija

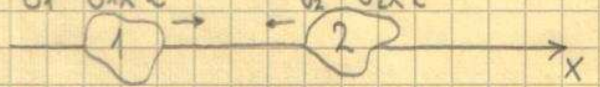
$$E_{kin.} = \underbrace{\sum_i E_{kin,i}}_{\text{prije}} = \underbrace{\sum_i E_{kin,i}'}_{\text{poslije sudara}}$$

Razmotrimo slučaj 2 čestice:

Količina gibanja: $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$

Pojednostavljenje: 1D gibanje

$\vec{v}_1 = \vec{v}_{1x}\vec{i}$ $\quad$ $\vec{v}_2 = \vec{v}_{2x}\vec{i}$



Sustav: $m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}' + m_2 v_{2x}'$

$$m_1 v_{1x}^2 + m_2 v_{2x}^2 = m_1 v_{1x}'^2 + m_2 v_{2x}'^2$$

Iz kvadratne jednadžbe dobivamo:

$$v_{1x}' = \begin{cases} v_{1x} \text{ (sudar se nije dogodio)} \\ \frac{m_1 - m_2}{m_1 + m_2} v_{1x} + \frac{2m_2}{m_1 + m_2} v_{2x} \end{cases}$$

$$v_{2x}' = \begin{cases} v_{2x} \text{ (sudar se nije dogodio)} \\ \frac{m_2 - m_1}{m_1 + m_2} v_{2x} + \frac{2m_1}{m_1 + m_2} v_{1x} \end{cases}$$

b) Savršeno neelastičan sudar

Očuvana je količina gibanja

$E_{kin.} = E_{kin.}' + Q$

Najveći dopušteni dio $E_{kin.}$ se pretvara u toplinu.

"Pokazuje se" da je Q najveći kada je $\vec{v}_1' = \vec{v}_2' = \underset{\text{brzina težišta}}{\underbrace{\vec{v}_{CM}}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

$$\vec{v}_{CM} = \dot{\vec{r}}_{CM} = \frac{\sum_i m_i \dot{\vec{r}}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i}$$

c) Općenit neelastičan sudar

Gubitak $E_{kin.}$ manji je od najvećeg dopuštenog.

Sustav N čestica

Centar mase (težište) $\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$

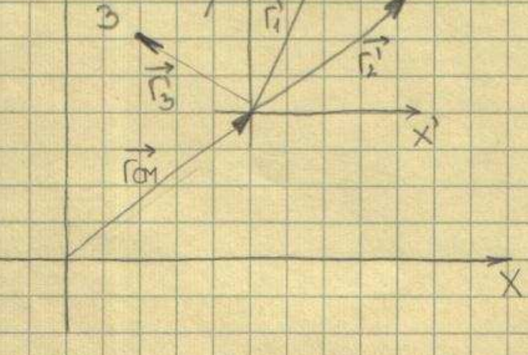
Brzina centra mase $\vec{v}_{CM} = \dot{\vec{r}}_{CM}$

Ukupna količina gibanja $\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i = \left(\sum_i m_i\right) \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} = \underset{\text{ukupna masa}}{\underbrace{M}} \cdot \vec{v}_{CM}$

Jednadžba gibanja za centar mase $\frac{d}{dt}\vec{P} = \sum_i \vec{F}_{vanjska,i} = \vec{F}_{vanjska}$

Ako na sustav čestica ne djeluju vanjske sile, ukupna kol. gib. je očuvana.

Kutna količina gibanja



$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_{CM} + \vec{r}_i') \times \vec{p}_i = \sum_i \vec{r}_{CM} \times \vec{p}_i + \sum_i \vec{r}_i' \times \vec{p}_i$$

$$= \vec{r}_{CM} \times \sum_i \vec{p}_i + \sum_i \vec{r}_i' \times \vec{p}_i$$

$$= \underbrace{\vec{r}_{CM} \times \vec{P}}_{\vec{L}_{CM}} + \underbrace{\sum_i \vec{r}_i' \times \vec{p}_i}_{\vec{L}_0}$$

$\vec{L}_{CM}$ → kutna kol. gibanja za CM

$\vec{L}_0$ → kutna kol. gibanja oko CM

Kinetička energija

$$E_{kin} = \sum_i E_{kin,i} = \sum_i \frac{m_i v_i^2}{2}$$

$$\vec{v}_i = \vec{v}_{CM} + \vec{v}_i' \ , \quad v_i^2 = v_{CM}^2 + 2\vec{v}_{CM}\cdot\vec{v}_i' + v_i'^2$$

$$E_{kin} = \sum_i \frac{1}{2} m_i \left(v_{CM}^2 + 2\vec{v}_{CM}\cdot\vec{v}_i' + v_i'^2\right)$$

$$= \underbrace{\frac{1}{2} M v_{CM}^2}_{E_{kin,CM}} + \underbrace{\vec{v}_{CM}\cdot\sum_i m_i \vec{v}_i'}_{0} + \underbrace{\sum_i \frac{1}{2} m_i v_i'^2}_{E_{kin,0} \rightarrow \text{ oko CM}}$$

Položaj centra mase tijela (računanje)

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\int \vec{r}\, dm}{\int dm}$$

1. $\rho = \frac{dm}{dV}$, $dm = \rho dV$ $\quad \vec{r}_{CM} = \frac{\int \vec{r} \rho dV}{\int \rho dV}$

   ↓ volumna gustoća mase

2. $\sigma = \frac{dm}{dA}$, $dm = \sigma dA$ $\quad \vec{r}_{CM} = \frac{\int \vec{r} \sigma dA}{\int \sigma dA}$

   ↓ površinska gustoća mase

3. $\lambda = \frac{dm}{dl}$, $dm = \lambda dl$ $\quad \vec{r}_{CM} = \frac{\int \vec{r} \lambda dl}{\int \lambda dl}$

   ↓ linijska gustoća mase

Statika krutog tijela

1. uvjet statike: Zbroj vanjskih sila mora biti jednak nuli. $\sum_i \vec{F}_{ext,i} = 0$

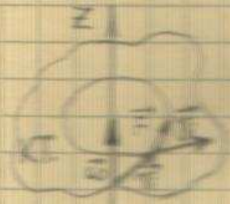
2. uvjet statike: Zbroj momenata vanjskih sila oko bilo koje osi mora biti jednak nuli. $\sum_i \vec{M}_{ext,i} = 0$, $\vec{M}_{ext,i} = \vec{r}_i \times \vec{F}_{ext,i}$

Dokaz 2. uvjeta: $\vec{M} = \sum_i \vec{M}_{ext,i} = \sum_i \vec{r}_i \times \vec{F}_{ext,i}$ udaljenost od osi

$$= \sum_i (\overset{\text{pomak}}{\vec{a}} + \vec{r}_i') \times \vec{F}_{ext,i} = \vec{a} \times \underbrace{\sum_i \vec{F}_{ext,i}}_{0 \text{ zbog 1. uvjeta}} + \boxed{\sum_i \vec{r}_i' \times \vec{F}_{ext,i}}$$

→ 2. uvjet zadovoljen za bilo koji položaj

Rotacija krutog tijela (oko čvrste osi)



Kutna količina gibanja (zamah, zakretni impuls, angularni moment)

obodna brzina: $\vec{v}_i = \omega \times \vec{r}_i$

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i = \vec{r}_i \times (m_i\vec{v}_i) = \vec{r}_i \times m_i(\omega\times\vec{r}_i) = m_i \cdot \vec{r}_i(\omega \times \vec{r}_i)$$

$$= m_i\left[(\vec r_i\cdot\vec r_i)\omega - (\vec r_i \omega)\vec r_i\right]$$

$$= m_i(r_i^2\omega - |r_i||\omega|\cdot\cos\alpha\cdot\vec r_i)$$

$$= m_i r_i^2\omega(\hat\omega - \cos\alpha\, \hat{r}_i) \quad /\cdot\hat\omega$$

$$= m_i r_i^2 \omega(1-\cos^2\alpha)$$

$$= m_i(r_i\sin\alpha)^2\omega = m_i r_i'^2\omega = \vec l_{iz}$$

$$\vec L_z = \sum_i \vec l_{iz} = \sum_i m_i r_i'^2 \omega = \left(\sum_i m_i r_i'^2\right)\omega = I_z\omega$$

↳ moment tromosti u odnosu na z os

Jednadžba gibanja

$$\frac{d}{dt}\vec L = \frac{d}{dt}\sum_i \vec l_i = \frac{d}{dt}\sum_i \vec r_i\times\vec p_i = \sum_i(\vec v_i\times\vec p_i + \vec r_i\times\vec F_{ext,i}) = \sum_i(\vec r_i\times \vec F_{ext,i}) = \sum_i M_{ext.i} = \vec M$$

analogija iz $\frac{d\vec p}{dt} = \vec F$

Kinetička energija

$$E_{kin.} = \frac12 m v_{CM}^2 + \sum_i \frac12 m_i v_i'^2$$

$$\sum_i \frac12 m_i v_i'^2 = \sum_i \frac12 m_i (r_i'\omega)^2 = \frac12\sum_i m_i r_i'^2\omega^2 = \frac12 I_{CM}\omega^2$$

Rad, energija i snaga

$$dW_i = \vec F_i\cdot ds_i = \vec F_i\cdot\vec v_i dt = \vec F_i\cdot(\vec\omega\times\vec r_i)dt = \vec\omega(\vec r_i\times\vec F_i)dt = \vec M_i\omega dt = \vec M_i d\vec\varphi$$

$$dW = \sum_i dW_i = \sum_i \vec M_i d\vec\varphi = d\vec\varphi\cdot\vec M$$

$$\vec M = I_z\cdot\vec\alpha = I_z\frac{d\vec\omega}{dt}$$

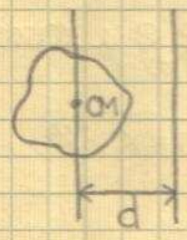
$$dW = d\vec\varphi\cdot I_z\cdot\frac{d\vec\omega}{dt} = \frac{\overset{\vec\omega dt}{d\vec\varphi}}{dt}\cdot I_z \cdot d\vec\omega = I_z\vec\omega\, d\vec\omega \quad / \int_p^k$$

$$W_k - W_p = \frac{I_z\omega_k^2}{2} - \frac{I_z\omega_p^2}{2}$$

$\Delta W = \Delta E_{kin.}$ → Teorem o radu i Ekin. kod rotacije.

Računanje momenta tromosti $I_z = \sum_i r_i^2 m_i$

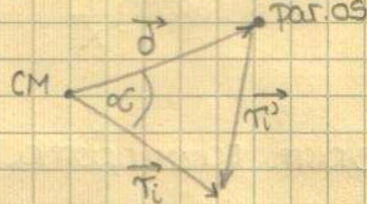
Teorem o paralelnim osima (Steinerov teorem) (za 3D tijelo)



$I = I_{CM} + md^2$

↳ moment tromosti kroz paralelnu os

Dokaz: $I = \sum_i m_i r_i'^2 = \sum_i m_i (\vec{r_i} - \vec{d})^2$

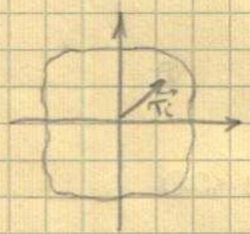


$= \sum_i m_i (r_i^2 - \underbrace{2|r_i||d|\cos\alpha}_{\text{nestaje}} + d^2)$

$= \sum_i m_i r_i^2 + \sum_i m_i d^2$

$= I_{CM} + md^2$

Teorem o okomitim osima (za plohe u ravnini)



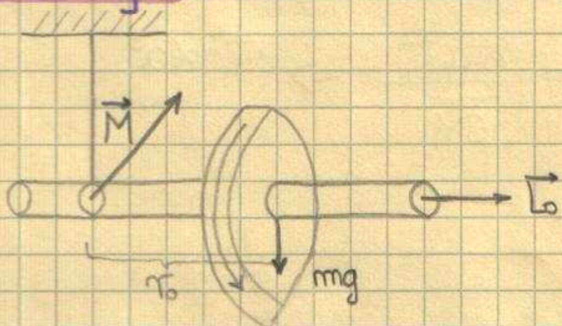
$I_z = I_x + I_y$

Dokaz: $I_z = \int r^2 \sigma dA = \int (x^2 + y^2) \sigma dx dy = \int x^2 \overbrace{\sigma dx dy}^{dm} + \int y^2 \overbrace{\sigma dx dy}^{dm}$

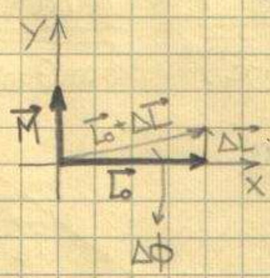
$= I_x + I_y$

Moment tromosti za os kroz centar štapa: $\frac{1}{12} m a^2$ → a duljina štapa

Precesija



- Pogled odozgo:



$\vec{L_0} \to \vec{L_0} + \Delta\vec{L}$

- mijenja se u skladu s jednadžbom gibanja

$\Delta\vec{L} = \vec{M} \cdot \Delta t$

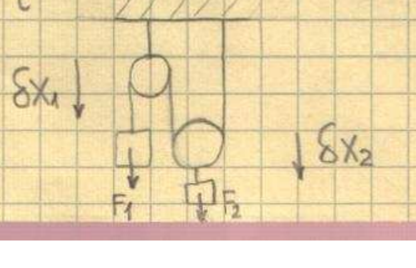
$\Delta\vec{L} = \vec{L_0} \sin\Delta\phi$, $\Delta\vec{L} = \vec{L_0} \Delta\phi$ (zanemarimo mali)

$\frac{\Delta\phi}{\Delta t} = \frac{\vec{M}}{\vec{L_0}} = \frac{mgr_0}{I_0\omega} = \Omega$ → kutna brzina precesije

Princip virtualnog rada

$\sum_i \delta W_i = \sum_i (\vec{F_i} \cdot \delta\vec{r_i} + \vec{M_i} \cdot \delta\vec{\vartheta_i}) = 0$

Primjer:



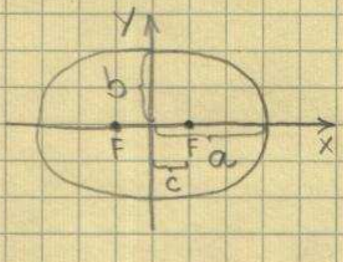
$\sum_i \delta W_i = F_1 \delta x_1 + F_2 \delta x_2 = 0$, $\delta x_1 = -2\delta x_2$

$2F_1 = F_2$

GRAVITACIJA

Keplerovi zakoni

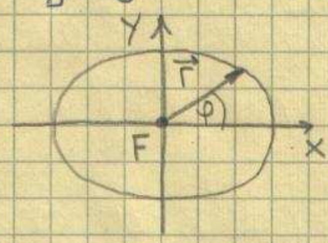
1. (1609) Planeti se oko Sunca gibaju po elipsama, a Sunce je u žarištu.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$c=\sqrt{a^2-b^2}$$

$$\varepsilon=\frac{c}{a}$$

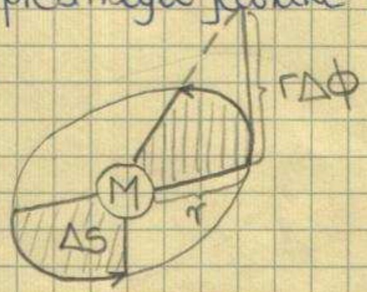


$$r=\frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\varphi} \qquad r=\sqrt{x^2+y^2}$$

$$x=r\cos\varphi \qquad \varphi=\mathrm{arctg}\,\frac{y}{x}$$

$$y=r\sin\varphi$$

2. (1609) Radijus vektori planeta u jednakim vremenskim razmacima prebrisuju jednake površine.



$$\Delta S=\frac{1}{2}r\cdot r\Delta\phi=\frac{1}{2}r^2\dot{\phi}\Delta t$$

$$\frac{\Delta S}{\Delta t}=\frac{1}{2}r^2\dot{\phi}$$

3. (1619) Kvadrati ophodnih vremena dvaju planeta odnose se kao kubovi njihovih srednjih udaljenosti od Sunca.

površina elipse

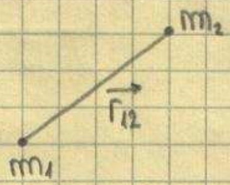
$$S=ab\pi=\frac{L}{2m}T\ , \quad b=\sqrt{1-\varepsilon^2} \qquad L=\sqrt{mka(1-\varepsilon^2)}$$

$$T=\frac{2mab\pi}{L}=\frac{2ma^2\sqrt{1-\varepsilon^2}\,T}{\sqrt{mka(1-\varepsilon^2)}}=\frac{2ma^2\pi}{\sqrt{mGmMa}}=\boxed{\frac{2\pi a^{3/2}}{\sqrt{GM}}} \Rightarrow T^2=\alpha a^3$$

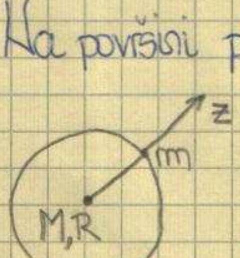
Opći zakon gravitacije

$$\vec{F}_{12}=G\frac{m_1m_2}{|\vec{r}_{12}|^2}\hat{r}_{12}=-\vec{F}_{21}$$

$$G \rightarrow 6.673\cdot10^{-11}\frac{Nm^2}{kg^2}$$



Na površini planeta

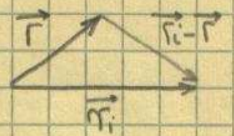


$$\vec{F}=G\frac{mM}{R^2}\vec{k}=mg$$

$$\vec{g}=-\frac{GM}{R^2}\vec{k}$$

Gravitacijsko polje

$$\vec g(\vec r) = G\sum_i \frac{m_i}{|\vec r_i-\vec r|^3}(\vec r_i - \vec r)$$

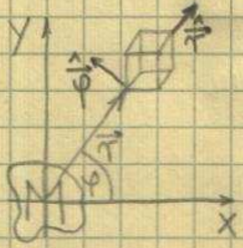


Gravitacijski potencijal

$$U(\vec r) = -G\sum_i \frac{m}{|\vec r_i-\vec r|} \qquad E_{pot.}=mU$$

$$\vec g = -\nabla U$$

Kinematika i dinamika Keplerovih zakona



$\vec r$ ... položaj

$\vec v = \dot{\vec r} = \dot r\hat r + r\dot\varphi\hat\varphi$ ... brzina

Efektivni potencijal

- energija = kin. energija + potencijalna energija = konstanta

$$E = \frac12 m v^2 - G\frac{Mm}{r}, \qquad k = GMm \text{ (pokrata)}$$

$$= \frac12 m(\dot r\hat r + r\dot\varphi\hat\varphi)^2 - \frac kr$$

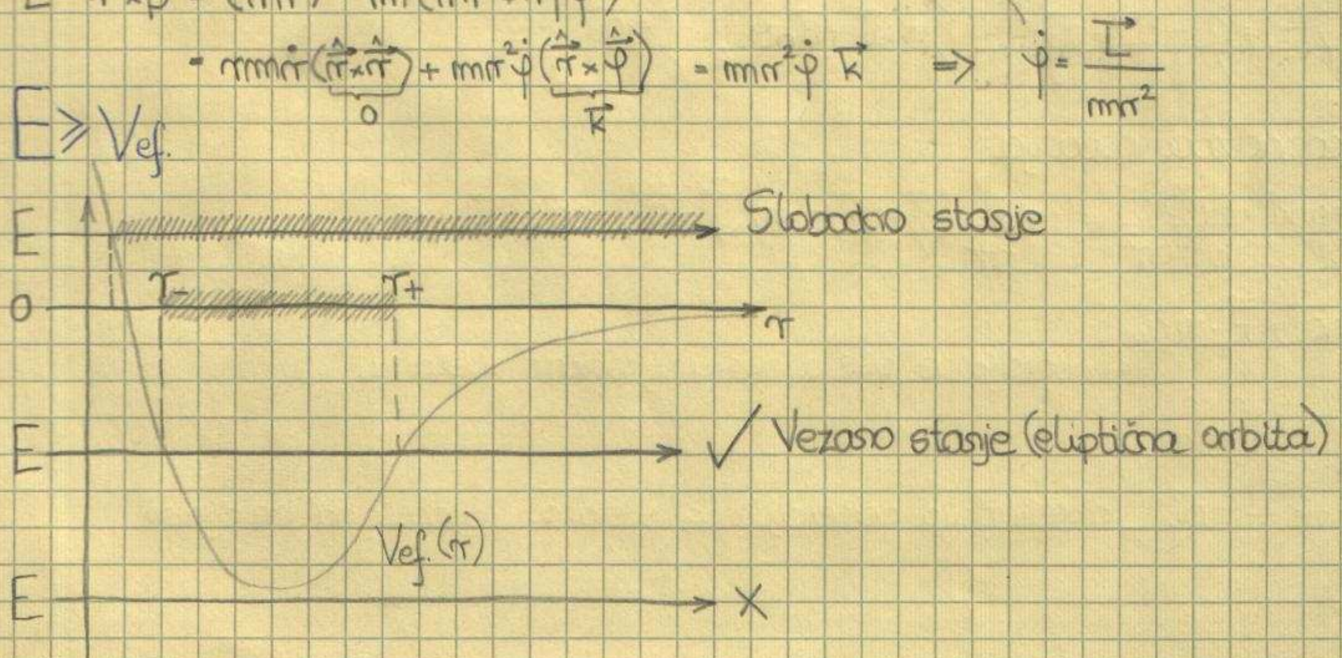
$$= \underbrace{\frac12 m\dot r^2}_{\geq 0} + \underbrace{\frac{L^2}{2mr^2} - \frac kr}_{\equiv V_{ef}(r)\text{ potencijal}}$$

- kutna količina gibanja = konstanta

$$\vec L = \vec r\times\vec p = (r\hat r)\times m(\dot r\hat r + r\dot\varphi\hat\varphi)$$

$$= mr\dot r\underbrace{(\hat r\times\hat r)}_{0} + mr^2\dot\varphi\underbrace{(\hat r\times\hat\varphi)}_{\vec k} = mr^2\dot\varphi\,\vec k \Rightarrow \dot\varphi = \frac{L}{mr^2}$$

$$E \geq V_{ef.}$$



$$E = V_{ef} = \frac{L^2}{2mr^2} - \frac{k}{r} \quad / \cdot \frac{r^2}{k}$$

$$\frac{Er^2}{k} - \frac{L^2}{2mr^2} + r = 0$$

$$r_\pm = \frac{-k}{2E}\left[1 \pm \sqrt{1+\frac{2EL^2}{mk^2}}\right] \qquad L = \sqrt{mka(1-\varepsilon^2)}$$

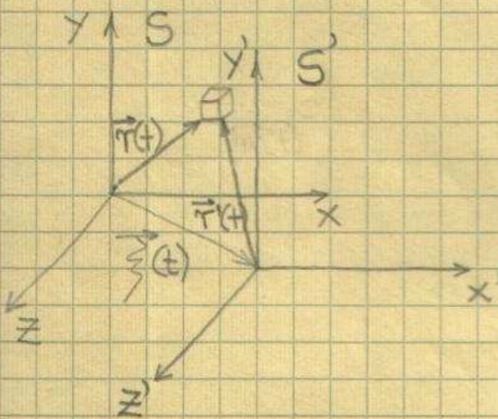
$$= a\left[1 \pm \varepsilon\right]$$

INERCIJALNI I NEINERCIJALNI SUSTAVI

vrijedi 1. Newtonov aksiom onako kako ga je iskazao

ne vrijedi 1. Newtonov aksiom

D'Alembertov princip



Sustav S je inercijalan: $\frac{d\vec{p}}{dt} = m\vec{a} = \vec{F}$

Veza sa sustavom S':

$$\vec{r}(t) = \vec{r}'(t) + \vec{\xi}(t)$$

$$\dot{\vec{r}} = \dot{\vec{r}}' + \dot{\vec{\xi}}$$

$$\underbrace{\ddot{\vec{r}}}_{\vec{a}} = \underbrace{\ddot{\vec{r}}'}_{\vec{a}'} + \ddot{\vec{\xi}}$$

$$\vec{\xi}(t) = \underbrace{\vec{\xi}(t_0)}_{\vec{\xi}_0} + \underbrace{\dot{\vec{\xi}}(t_0)}_{\vec{u}_0}(t-t_0) + \frac{1}{2}\underbrace{\ddot{\vec{\xi}}(t_0)}_{\vec{a}_0}(t-t_0)^2 + \dots \qquad t_0 = 0$$

$$\vec{\xi}(t) = \vec{\xi}_0 + \vec{u}_0 t + \frac{\vec{a}_0 t^2}{2}, \quad \dot{\vec{\xi}} = \vec{u}_0 + \vec{a}_0 t, \quad \ddot{\vec{\xi}} = \vec{a}_0$$

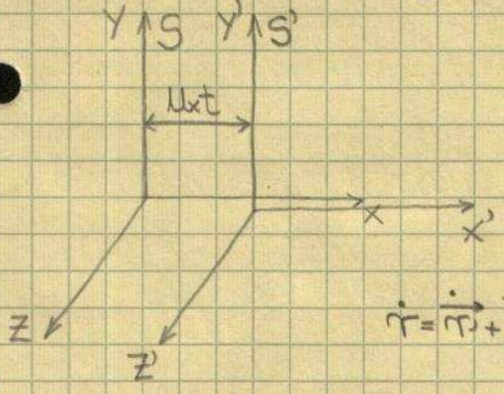
Jednadžba gibanja:

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d^2}{dt^2}(\vec{\xi} + \vec{r}') = m(\ddot{\vec{\xi}} + \ddot{\vec{r}}') = m(\vec{a}_0 + \vec{a}')$$

$$m\vec{a}' = \underbrace{\vec{F}}_{\text{stvarne sile}} \underbrace{- m\vec{a}_0}_{\text{inercijalne sile u neinercijalnom sustavu}}$$

$\vec{F}_{inerc.} = -m\vec{a}_0$, $\vec{a}_0$ → akceleracija neinercijalnog sustava u odnosu na inercijalni

Galilejeve transformacije



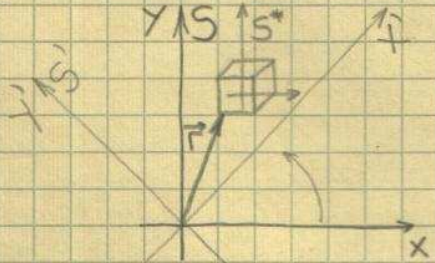
$$\vec{\xi} = u_x t\vec{i}$$

$$\vec{r} = \vec{r}' + \vec{\xi} \Rightarrow \begin{cases} x = u_x t + x' \\ y = y' \\ z = z' \end{cases}$$

$$\dot{\vec{r}} = \dot{\vec{r}}' + \dot{\vec{\xi}} \Rightarrow \begin{cases} v_x = u_x + v_x' \\ v_y = v_y' \\ v_z = v_z' \end{cases} \qquad \ddot{\vec{r}} = \ddot{\vec{r}}' + \ddot{\vec{\xi}} \Rightarrow \begin{cases} a_x = a_x' \\ a_y = a_y' \\ a_z = a_z' \end{cases}$$

ovo je preuredno

Rotirajući sustav



S-inercijalni sustav, S'-rotirajući sustav

S*-sustav vezan uz česticu, miruje u S'

S' i S* skupa rotiraju

$$\vec{a_0} = -\omega^2\vec{r}$$

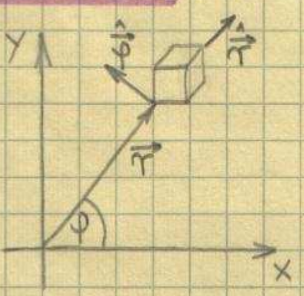
$$\vec{F} - m\vec{a_0} = m\vec{a}^* = m\vec{a}'$$

$$m\vec{a}' = \vec{F} - m\vec{a_0} = \vec{F} + \underbrace{m\omega^2\vec{r}}$$

centrifugalna sila (djeluje na van pa nam treba i centripetalna)

Coriolisova sila (Koriolisova)

Općenito:



$$\dot{\hat{\vec{r}}} = \dot{\varphi}\hat{\vec{\varphi}}$$

$$\dot{\hat{\vec{\varphi}}} = -\dot{\varphi}\hat{\vec{r}}$$

Položaj: $\vec{r} = r\hat{\vec{r}}$

Brzina: $\vec{v} = \dot{\vec{r}} = \dot{r}\hat{\vec{r}} + r\dot{\hat{\vec{r}}} = \dot{r}\hat{\vec{r}} + r\dot{\varphi}\hat{\vec{\varphi}}$

Akceleracija: $\vec{a} = \dot{\vec{v}} = \ddot{r}\hat{\vec{r}} + \dot{r}\dot{\hat{\vec{r}}} + \dot{r}\dot{\varphi}\hat{\vec{\varphi}} + r\ddot{\varphi}\hat{\vec{\varphi}} + r\dot{\varphi}\dot{\hat{\vec{\varphi}}}$

$$= \ddot{r}\hat{\vec{r}} + \dot{r}\dot{\varphi}\hat{\vec{\varphi}} + \dot{r}\dot{\varphi}\hat{\vec{\varphi}} + r\ddot{\varphi}\hat{\vec{\varphi}} - r\dot{\varphi}^2\hat{\vec{r}}$$

$$= \hat{\vec{r}}(\ddot{r} - r\dot{\varphi}^2) + \hat{\vec{\varphi}}(2\dot{r}\dot{\varphi} + r\ddot{\varphi})$$

Poseban slučaj:

$$\dot{\varphi} = \omega = konst. \Rightarrow \ddot{\varphi} = 0$$

$$\dot{r} = v_r = konst. \Rightarrow \ddot{r} = 0$$

$$\vec{a} = -r\omega^2 \hat{r} + 2 v_r \omega \hat{\varphi}$$

$$\vec{F}_{inerc} = -m\vec{a} = \underbrace{m r \omega^2}_{\text{centrifugalna}} \hat{r} - \underbrace{2 m v_r \omega}_{\text{Coriolisova}} \hat{\varphi}$$

$$\vec{F}_{coriolis} = 2m\vec{v}' \times \vec{\omega}$$

brzina kojom se platforma vrti