Formule za arkus i area funkcije

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Popis formula:

•Arc sin

$$w = -i \cdot Ln(iz + \sqrt{1 - z^2}). \tag{1}$$

•Izvod.

$$w = Arcsin\ z,$$

$$\sin w = z,$$

$$\frac{1}{2i}\left(e^{iw} - e^{-iw}\right) = z,$$

$$e^{iw} - e^{-iw} = 2iz.$$

Množimo obje strane s e^{iw}:

$$e^{2iw} - 1 = 2ize^{iw},$$

$$e^{2iw} - 2ize^{iw} - 1 = 0.$$

Rješavamo kvadratnu jednadžbu.

$$e^{iw} = iz \pm \sqrt{1 - z^2}.$$

Logaritmiramo obje strane.

$$iw = Ln\left(iz \pm \sqrt{1 - z^2}\right),$$

$$w = -iLn\left(iz \pm \sqrt{1 - z^2}\right).$$

Kako imamo korijen kompleksnog broja možemo maknuti $\pm$, pa je konačna formula:

$$w = -i \cdot Ln\left(iz + \sqrt{1 - z^2}\right).$$

•Arc cos

$$w = -i \cdot Ln\left(z + \sqrt{z^2 - 1}\right). \tag{2}$$

•Izvod.

$$w = Arccos\ z,$$

$$\cos w = \frac{1}{2}\left(e^{iw} + e^{-iw}\right),$$

$$e^{iw} + e^{-iw} = 2z.$$

Množimo obje strane s e^{iw} :

$$e^{2iw} - 2ze^{iw} + 1 = 0.$$

Rješavamo kvadratnu jednadžbu:

$$e^{iw} = z \pm \sqrt{z^2 - 1}.$$

Logaritmiramo obje strane, slijedi (analogno prethodnom primjeru sređujemo i uklanjamo $\pm$):

$$w = -i \cdot Ln\left(z + \sqrt{z^2 - 1}\right).$$

•Arc tg

$$w = -\frac{i}{2} Ln \left(\frac{1+iz}{1-iz} \right). \tag{3}$$

•Izvod.

$$w = Arctg\ z,$$

$$tg\ w = z,$$

$$\frac{-i(e^{iw} - e^{-iw})}{e^{iw} + e^{-iw}} = z,$$

$$-i \left(e^{iw} - e^{-iw} \right) = z \left(e^{iw} + e^{-iw} \right).$$

Množimo obje strane s e^{iw}:

$$-i \left(e^{2iw} - 1 \right) = z \left(e^{2iw} + 1 \right),$$

$$e^{2iw}(-i - z) = z - i,$$

$$e^{2iw} = \frac{z - i}{-i - z}.$$

Logaritmiramo obje strane:

$$w = -\frac{i}{2} Ln \left(\frac{z - i}{-i - z} \right).$$

Možemo dodatno srediti koristeći:

$$\frac{z - i}{-i - z} \cdot \frac{i}{i} = \frac{1 + iz}{1 - iz},$$

pa je krajnji izraz:

$$w = -\frac{i}{2} Ln \left(\frac{1+iz}{1-iz} \right).$$

•Arc ctg

$$w = \frac{i}{2} Ln\left(\frac{z-i}{z+i}\right). \tag{4}$$

•Izvod.

$$w = Arcctg\ z,$$

$$ctg\ w = z,$$

$$\frac{e^{iw} + e^{-iw}}{-i(e^{iw} - e^{-iw})} = z,$$

$$e^{iw} + e^{-iw} = -zi\left(e^{iw} - e^{-iw}\right).$$

Množimo obje strane s e^{iw}:

$$e^{2iw} + 1 = -zi\left(e^{2iw} - 1\right),$$

$$e^{2iw}\left(1 + zi\right) = -1 + zi,$$

$$e^{2iw} = \frac{-1 + zi}{1 + zi},$$

$$w = -\frac{i}{2} Ln\left(\frac{-1 + zi}{1 + zi}\right).$$

Ako uzmemo u obzir svojstvo logaritma i izraz:

$$\frac{-1 + zi}{1 + zi} \cdot \frac{i}{i} = \frac{-z - i}{-z + 1} = \frac{z + i}{z - i},$$

dobivamo:

$$w = \frac{i}{2} Ln\left(\frac{z-i}{z+i}\right).$$

•Ar sh

$$w = Ln\left(z + \sqrt{z^2+1}\right). \tag{5}$$

•Izvod.

$$w = Arsh\ z,$$

$$sh\ w = z,$$

$$\frac{1}{2}\left(e^w - e^{-w}\right) = z,$$

$$e^w - e^{-w} = 2z.$$

Množimo obje strane s e^w:

$$e^{2w} - 2ze^w - 1 = 0.$$

Rješavanjem kvadratne jednadžbe dobivamo:

$$e^w = z \pm \sqrt{z^2+1},$$

odnosno:

$$w = Ln\left(z + \sqrt{z^2+1}\right).$$

•Ar ch

$$w = Ln\left(z + \sqrt{z^2 - 1}\right). \tag{6}$$

•Izvod.

$$w = Arch\ z,$$

$$chw = z,$$

$$\frac{1}{2}\left(e^w + e^{-w}\right) = z,$$

$$e^w + e^{-w} = 2z.$$

Množimo obje strane s e^w:

$$e^{2w} - 2ze^w + 1 = 0.$$

Rješavanjem kvadratne jednadžbe dobivamo:

$$e^w = z \pm \sqrt{z^2 - 1},$$

$$w = Ln\left(z + \sqrt{z^2 - 1}\right).$$

•Ar th

$$w = \frac{1}{2} Ln\left(\frac{1+z}{1-z}\right). \tag{7}$$

•Izvod.

$$w = Arth\ z,$$

$$th\ w = z,$$

$$\frac{e^{w} - e^{-w}}{e^{w} + e^{-w}} = z,$$

$$e^{w} - e^{-w} = z\left(e^{w} + e^{-w}\right).$$

Množimo obje strane s e^{w}:

$$e^{2w} - 1 = z\left(e^{2w} + 1\right),$$

$$e^{2w}\left(1 - z\right) = z + 1,$$

$$e^{2w} = \frac{1+z}{1-z}.$$

Slijedi krajnji izraz:

$$w = \frac{1}{2} Ln\left(\frac{1+z}{1-z}\right).$$

•Ar cth

$$w = \frac{1}{2} Ln \left(\frac{z+1}{z-1} \right). \tag{8}$$

•Izvod.

$$w = Arcth\ z,$$

$$cth\ w = z,$$

$$e^{w} + e^{-w} = \left(e^{w} - e^{-w}\right) z.$$

Množimo obje strane s e^{w}:

$$e^{2w} + 1 = \left(e^{2w} - 1\right) z,$$

$$e^{2w} (1 - z) = -z - 1,$$

$$e^{2w} = \frac{-z-1}{1-z}.$$

Imamo krajnje rješenje:

$$w = \frac{1}{2} Ln \left(\frac{z+1}{z-1} \right).$$

•Dodatak

$$-Ln(z) = Ln\left(\frac{1}{z}\right). \tag{9}$$

•Izvod.

$$Ln(z) = \ln|z| + i\left(arg(z) + 2k\pi\right),$$

$$\frac{1}{z} = \frac{x - yi}{|z|^2},$$

$$\left|\frac{1}{z}\right| = \frac{1}{|z|},$$

$$arg\left(\frac{1}{z}\right) = -arg(z).$$

Iz navedenih izraza slijedi (9).