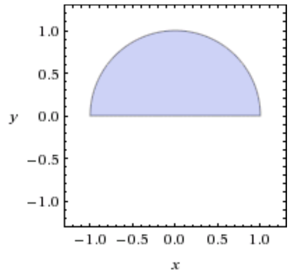
KOMPLEKSNA ANALIZA

Zadaci za vježbu

Konformna preslikavanja – I. dio

$G^* = \{\operatorname{Im} w > 0\}$

31. A) $G = \{|z| < 1, \operatorname{Im} z > 0\}$



Korak (1):

$$z_1 = 1 \to w_1 = 0$$

$$z_2 = i \to w_2 = 1$$

$$z_3 = -1 \to w_3 = \infty$$

$$S_1(z) = \frac{az + b}{z - z_3} = \frac{az + b}{z + 1}$$

$$S_1(1) = \frac{a + b}{2} = 0$$

$$S_1(i) = \frac{ai + b}{1 + i} = 1$$

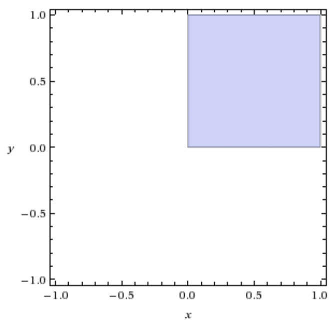
$$a = -i, \qquad b = i$$

$$S_1(z) = -i\frac{z - 1}{z + 1}$$

Korak (2):

$$z_4 = 0 \to S_1(0) = i = w_4$$

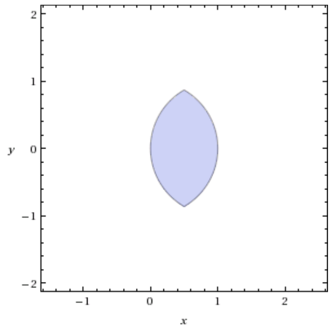
Nacrtate te točke u w-ravnini i dobije se prvi kvadrant.



Prvi kvadrant proširujemo na gornju poluravninu sa $S_2(z) = z^2$. Konačno je:

$$S(z) = S_2\big(S_1(z)\big) = \left(\frac{z - 1}{z + 1}\right)^2$$

B) $G = \{|z| < 1, |z-1| < 1\}$



Korak (1):

$$z_1 = \frac{1+i\sqrt{3}}{2} \to w_1 = 0$$

$$z_2 = 0 \to w_2 = 1$$

$$z_3 = \frac{1-i\sqrt{3}}{2} \to w_3 = \infty$$

$$S_1(z) = \frac{az+b}{z-z_3} = \frac{2az+b}{2z-1+i\sqrt{3}}$$

$$S_1(0) = 1 \to b = \frac{-1+i\sqrt{3}}{2}$$

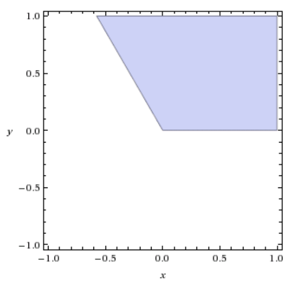
$$S_1\left(\frac{1+i\sqrt{3}}{2}\right) = 0 \to a = \frac{-1-i\sqrt{3}}{2}$$

$$S_1(z) = \frac{\left(-1-i\sqrt{3}\right)z - 1 + i\sqrt{3}}{2z-1+i\sqrt{3}}$$

Korak (2):

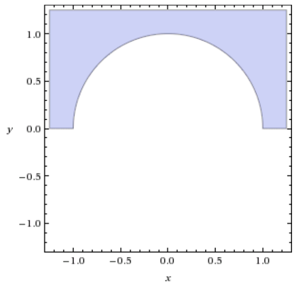
$$z_4 = 1 \to S_1(1) = \frac{-1+i\sqrt{3}}{2} = w_4$$

Nacrtate te točke u *w*-ravnini i dobije područje u prvom i drugom kvadrantu do kuta $2\pi/3$.



To proširujemo na gornju poluravninu sa $S_2(z) = z^{\frac{3}{2}}$. Konačno je: $S(z) = S_2\big(S_1(z)\big) = \left(\frac{\left(-1-i\sqrt{3}\right)z-1+i\sqrt{3}}{2z-1+i\sqrt{3}}\right)^{\frac{3}{2}}$.

32. A) $G = \{|z| > 1, \mathrm{Im}\, z > 0\}$



Korak (1):

$$z_1 = -1 \to w_1 = \infty$$

$$z_2 = i \to w_2 = i$$

$$z_3 = 1 \to w_3 = 0$$

$$S_1(z) = \frac{az+b}{z-z_1} = \frac{az+b}{z+1}$$

$$S_1(i) = i \to \frac{a(1+i)+b(1-i)}{2} = i$$

$$S_1(1) = 0 \to \frac{a+b}{2} = 0$$

$$a = 1, \qquad b = -1$$

$$S_1(z) = \frac{z-1}{z+1}$$

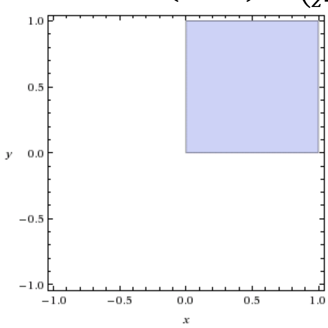
Korak (2):

$$z_{4,5} = -2 \to S_1(-2) = 3 = w_4$$

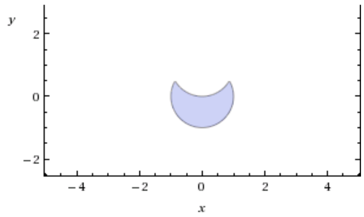
$$z_5 = 2 \to S_1(2) = \frac{1}{3} = w_5$$

$$z_6 = \infty \to S_1(\infty) = 1$$

Nacrtate te točke u *w*-ravnini i dobije se prvi kvadrant. Prvi kvadrant proširujemo na gornju poluravninu sa $S_2(z) = z^2$. Konačno je: $S(z) = S_2\big(S_1(z)\big) = \left(\frac{z-1}{z+1}\right)^2$



B) $G = \{|z| < 1, |z - i| > 1\}$



Korak (1):

$$z_1 = \frac{\sqrt{3} + i}{2} \to w_1 = 0$$

$$z_2 = 0 \to w_2 = 1$$

$$z_3 = \frac{-\sqrt{3} + i}{2} \to w_3 = \infty$$

$$S_1(z) = \frac{az + b}{z - z_3} = \frac{az + b}{z + \frac{\sqrt{3} - i}{2}}$$

$$S_1(0) = 1 \to b = \frac{\sqrt{3} - i}{2}$$

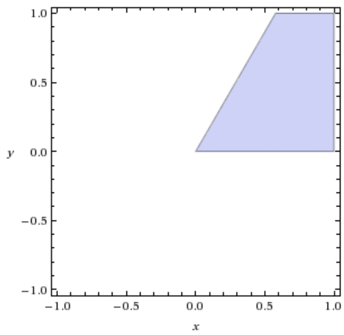
$$S_1\left(\frac{\sqrt{3} + i}{2}\right) = 0 \to a = \frac{-1 + i\sqrt{3}}{2}$$

$$S_1(z) = \frac{(-1 + i\sqrt{3})z + \sqrt{3} - i}{2z + \sqrt{3} - i}$$

Korak (2):

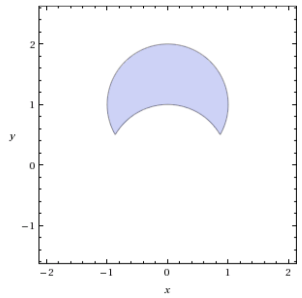
$$z_4 = -i \to S_1(-i) = \frac{1 + i\sqrt{3}}{4} = w_4$$

Nacrtate te točke u *w*-ravnini i dobije područje u prvom i drugom kvadrantu do kuta $\pi/3$.



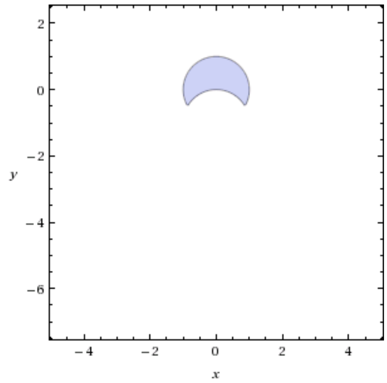
To proširujemo na gornju poluravninu sa $S_2(z) = z^3$. Konačno je: $S(z) = S_2\big(S_1(z)\big) = \left(\frac{(-1+i\sqrt{3})z+\sqrt{3}-i}{2z+\sqrt{3}-i}\right)^3$.

33. A) $G = \{|z| > 1, |z - i| < 1\}$



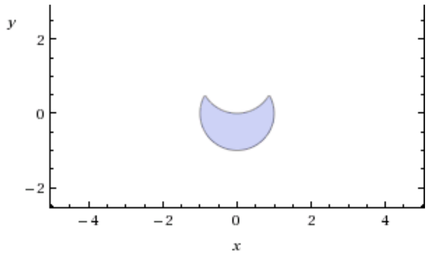
Korak (1):

$$S_1(z) = z - i$$



Korak (2):

$$S_2(z) = e^{i\pi} z = -z$$



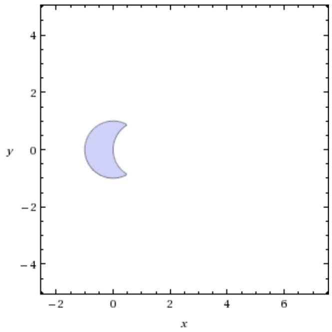
Ovo je sada kao 32. B) pa imamo:

$$S_3(z) = \left(\frac{\left(-1 + i\sqrt{3}\right)z + \sqrt{3} - i}{2z + \sqrt{3} - i} \right)^3$$

Konačno je:

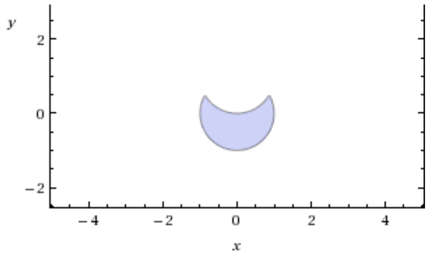
$$S(z) = S_3\left(S_2\left(S_1(z)\right)\right) = \left[\frac{\left(1 - i\sqrt{3}\right)z - 2i}{-2z + \sqrt{3} + i} \right]^3$$

B) $G = \{|z| < 1, |z-1| > 1\}$



Korak (1):

$$S_1(z) = e^{\frac{i\pi}{2}} z = iz$$



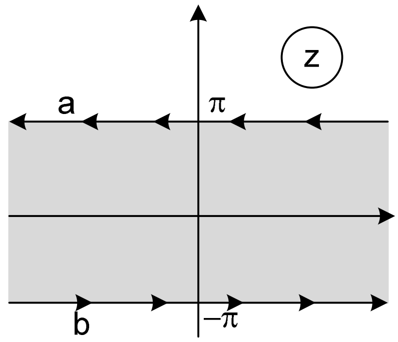
Ovo je sada kao 32. B) pa imamo:

$$S_2(z) = \left(\frac{(-1 + i\sqrt{3})z + \sqrt{3} - i}{2z + \sqrt{3} - i} \right)^3$$

Konačno je:

$$S(z) = S_2(iz) = \left[\frac{(-1 + i\sqrt{3})iz + \sqrt{3} - i}{2iz + \sqrt{3} - i} \right]^3$$

34. A) $G = \{|\mathrm{Im}\, z| < \pi\}, w = e^z$



$$w = e^z = e^x(\cos y + i \sin y) \to u = e^x \cos y\,, \qquad v = e^x \sin y$$

a ... $y = \pi, \qquad x \in (\overleftarrow{-\infty, \infty})$

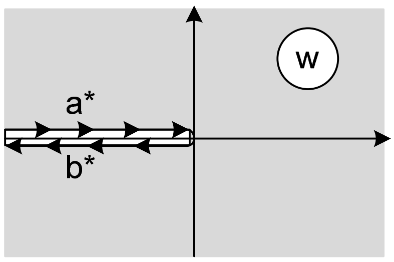
a* ... $u \in \left(\overrightarrow{-\infty, 0}\right), \qquad v = 0$

b ... $y = \pi, \qquad x \in (\overrightarrow{-\infty, \infty})$

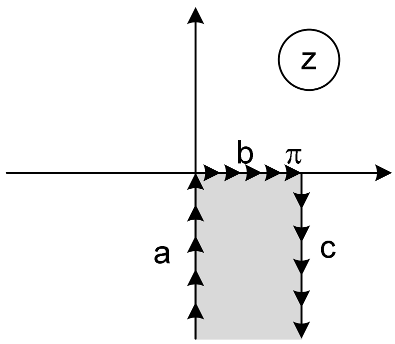
b* ... $u \in \left(\overleftarrow{-\infty, 0}\right), \qquad v = 0$

Obzirom da je definiranim obilaskom područje uvijek s lijeve strane, i u *w*-ravnini mora biti s lijeve strane.

Rješenje je očito cijela *w*-ravnina s prorezom duž $(-\infty, 0)$.



B) $G = \{0 < \operatorname{Re} z < \pi, \operatorname{Im} z < 0\}$, $w = \cos z$



$$w = \cos z = \cos x \operatorname{ch} y - i \sin x \operatorname{sh} y \to u = \cos x \operatorname{ch} y\,, \qquad v = -\sin x \operatorname{sh} y$$

a ... $x = 0, \qquad y \in (\overrightarrow{-\infty, 0})$

a* ... $u = \operatorname{ch} y \to u \in (\overleftarrow{1, \infty}), \qquad v = 0$

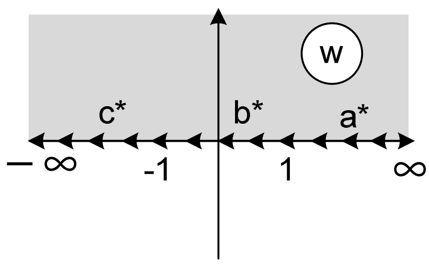
b ... $y = 0, \qquad x \in (\overrightarrow{0, \pi})$

b* ... $u = \cos x \to u \in (\overleftarrow{-1,1}), \qquad v = 0$

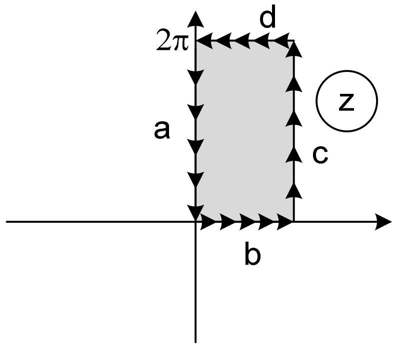
c ... $x = \pi, \qquad y \in (\overleftarrow{-\infty, 0})$

c* ... $u = -\operatorname{ch} y \to u \in (\overleftarrow{-\infty, -1}), \qquad v = 0$

Zadano područje je, definiranim obilaskom, s desne strane. Rješenje je gornja poluravnina.



35. A) $G = \{0 < \operatorname{Im} z < 2\pi, 0 < \operatorname{Re} z < 1\}$, $w = e^z$



$$w = e^z = e^x(\cos y + i\sin y) \rightarrow u = e^x \cos y\,, \qquad v = e^x \sin y$$

a ... $x = 0, \qquad y \in \left(\overleftarrow{0{,}2\pi}\right)$

a* ... $u = \cos y\,, v = \sin y \rightarrow kružnica\; u^2 + y^2 = 1$

b ... $y = 0, \qquad x \in \left(\overrightarrow{0{,}1}\right)$

b* ... $u = e^x \rightarrow u \in \left(\overrightarrow{1, e}\right), \qquad v = 0$

c ... $x = 1, \qquad y \in \left(\overrightarrow{0{,}2\pi}\right)$

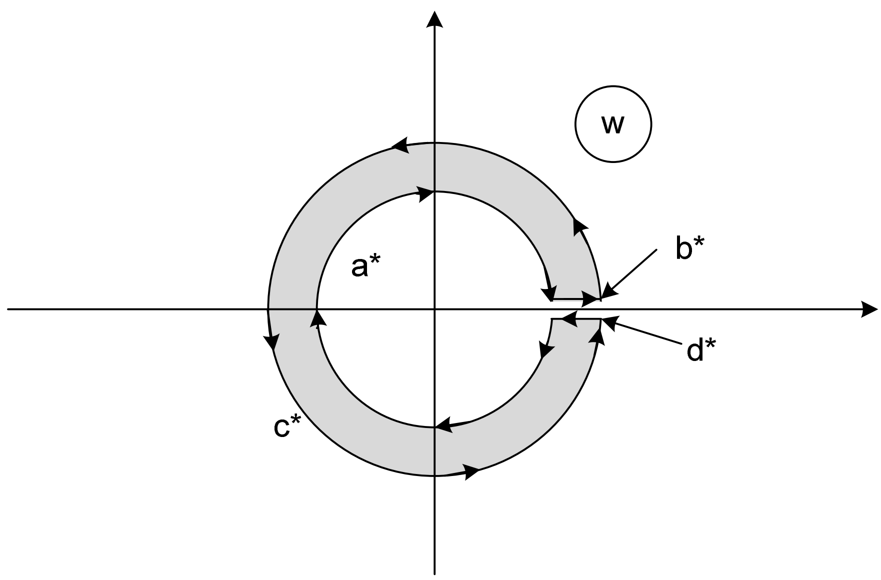
c* ... $u = e\cos y\,, v = e\sin y \rightarrow kružnica\; u^2 + y^2 = e^2$

d ... $y = 2\pi, \qquad x \in \left(\overleftarrow{0{,}1}\right)$

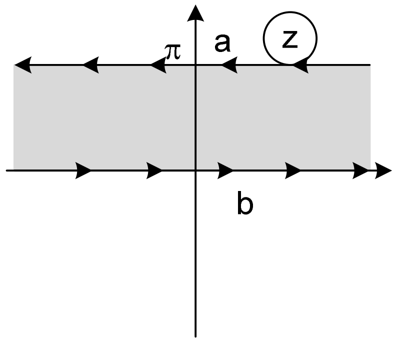
d* ... $u = e^x \rightarrow u \in \left(\overleftarrow{1, e}\right), \qquad v = 0$

Obzirom da je definiranim obilaskom područje uvijek s lijeve strane, i u *w*-ravnini mora biti s lijeve strane.

Rješenje je prsten $1 < |w| < e$ s prorezom duž $(1, e)$.



B) $G = \{0 < \operatorname{Im} z < \pi\}$, $w = \operatorname{ch} z$



$$w = \operatorname{ch} z = \operatorname{ch} x \cos y + i \operatorname{sh} x \sin y \to u = \operatorname{ch} x \cos y\,, \qquad v = \operatorname{sh} x \sin y$$

a ... $y = \pi, \qquad x \in (\overleftarrow{-\infty, \infty})$

a* ... $u = -\operatorname{ch} x \to u \in \boxed{(\overrightarrow{-\infty, -1})}, \qquad v = 0$

b ... $y = 0, \qquad x \in (\overrightarrow{-\infty, \infty})$

b* ... $u = \operatorname{ch} x \to u \in \boxed{(\overleftarrow{1, \infty})}, \qquad v = 0$

Rješenje je očito cijela *w*-ravnina s prorezima duž $(-\infty, -1)$ i $(1, \infty)$.