22.12.2009.

BURIĆ - PRED

# ORTOGONALNI SUSTAVI

$I = (a, b)$ , $a, b \in \overline{\mathbb{R}}$ — $\infty$ ULAZI

$\rho : I \to \mathbb{R}_+$ TEŽINSKA FJA

$$\int_a^b |f(x)|^2 \cdot \rho(x) dx < \infty$$

$L^2(a, b, \rho)$ = PROSTOR KVADRATNO INTEGRABILNIH FJA

- **def** SKALARNI PRODUKT:

$$(f|g) = \int_a^b f(x) g(x) \rho(x) dx$$

- **def** NORMA

$$\|f\| = \sqrt{(f|f)} \qquad \|f\| = \sqrt{\int_a^b f^2(x) \rho(x) dx}$$

- UDALJENOST $f$ i $g$:

$$\|f-g\| = \sqrt{\int_a^b (f(x) - g(x))^2 \rho(x) dx}$$

- **def** $f$ i $g$ SU ORTOGONALNE FJE AKO JE $(f|g) = 0$

- **def** SUSTAV FJA $w_0, w_1 \ldots$ JE ORTOGONALAN S TEŽINSKOM FJOM $\rho$ NA $(a, b)$ AKO JE:

$$(w_n|w_m) = \begin{cases} > 0, & n = m \\ 0, & n \neq m \end{cases}$$

DOKAŽI DA JE SKUP ORTOG MOGUĆE U Z1

1MI-09-MAT3 DOKAŽI DA JE SUSTAV FJA $\cos\pi x, \sin\pi x, \ldots \cos m\pi x, \sin m\pi x$

4.b.1

ORTOGONALAN $\left( \rho(x) = 1, \ (a, b) = [-1, 1] \right)$

$$(\cos m\pi x | \cos m\pi x) = \int_{-1}^{1} \cos m\pi x \cdot \cos m\pi x \, dx = \left( \frac{1}{2}x + \frac{1}{4m\pi} \sin 2m\pi x \right) \Big|_{-1}^{1} = 1 > 0$$

$$\int_{-1}^{1} \sin^2 m\pi x \, dx = 1$$

$m \neq n$

$$\int_{-1}^{1} \cos(n\pi x) \sin(m\pi x) \, dx = \emptyset$$ NEPARNA FJA NA SIMET. INTERV.

$m \neq n$

$$\int_{-1}^{1} \cos(n\pi x) \cos(m\pi x) \, dx =$$

$$= \frac{1}{2\pi} \left( \frac{\sin \pi x (m-n)}{m-n} + \frac{\sin \pi x (m+n)}{m+n} \right) \Big|_{-1}^{1} = \emptyset$$

$n \neq m$

$$\int_{-1}^{1} \sin(n\pi x) \sin(m\pi x) \, dx = \ldots \ldots \emptyset$$

---

$1, x, x^2, x^3 \ldots x^m$ HTJELI BI DA BUDE ORTOGONALAN

$$(x | x^3) = \int_a^b x \cdot x^3 \rho(x) \, dx \overset{\text{UVIJEK}}{>} 0$$ NISU ORTOGONALNI

## GRAMM-SCHMIDTOV POSTUPAK ORTOGONALIZACIJE

$f_0, f_1, f_2 \ldots$ NISU ORTOGON. $\longrightarrow$ $w_0, w_1, w_2 \ldots$ ORTOGONALNE FJE

$$w_0 = f_0$$

$$w_1 = f_1 - \frac{(f_1 | w_0)}{\|w_0\|^2} \cdot w_0$$

$\vdots$

$$w_n = f_n - \frac{(f_n | w_0)}{\|w_0\|^2} w_0 - \frac{(f_n | w_1)}{\|w_1\|^2} w_1 - \ldots - \frac{(f_n | w_{n-1})}{\|w_{n-1}\|^2} w_{n-1}$$

ZAD - GRAM. ORTOGONALIZIRAJ:

$$1, x, x^2 \quad \text{NA } [0,1] \text{ UZ } \rho(x) = x$$

$$w_0 = \boxed{1}$$

$$w_1 = x - \frac{(x|1)}{\|1\|^2}\cdot 1 = x - \frac{\frac{1}{3}}{\frac{1}{2}}\cdot 1 = \boxed{x - \frac{2}{3}}$$

$$(x|1) = \int_0^1 x\cdot 1\cdot x\,dx = \frac{1}{3}, \qquad \|1\|^2 = \int_0^1 1^2 x\,dx = \frac{1}{2}$$

$$w_2 = x^2 - \frac{(x^2|1)}{\|1\|^2}\cdot 1 - \frac{(x^2|x-\frac{2}{3})}{\|x-\frac{2}{3}\|^2}\cdot(x-\tfrac{2}{3}) = \boxed{x^2 - \frac{6}{5}x + \frac{3}{10}}$$

$$(x^2|1) = \int_0^1 x^2\cdot 1\cdot x\,dx = \frac{1}{4}, \qquad \|x^2|x-\tfrac{2}{3}\| = \frac{1}{36} = \int_0^1 (x-\tfrac{2}{3})^2 x\,dx = \frac{1}{36}$$

$$(x^2|x-\tfrac{2}{3}) = \int_0^1 x^2(x-\tfrac{2}{3})x\,dx = \frac{1}{30}$$

---

ODABIR PRIGODNE TEŽINE?

$\rho(x)$ = ZADOVOLJAVA DIF. JEDN. $\dfrac{\rho'(x)}{\rho(x)} = \dfrac{\alpha(x)}{B(x)}$

UZ UVJET $\rho(a)\cdot B(a) = \rho(b)B(b) = 0$

$\alpha(x) = \alpha_0 + \alpha_1 x, \quad B(x) = B_0 + B_1 x + B_2 x^2$

$\rho$-UVJETI

BILO NA ZI def

PRIM

1) LEGENDREOVI POLINOMI, $P_n$

$\rho(x) = 1, \quad [-1, 1]$

$$\frac{0}{1} = \frac{\alpha(x)}{B(x)} \Rightarrow \alpha(x) = 0$$
$$B(x) = 1 - x^2$$

2) ČEBIŠEVLJEVI POLINOMI, $T_n$

$\rho(x) = \dfrac{1}{\sqrt{1-x^2}} \quad [-1, 1]$

$$\frac{\rho'(x)}{\rho(x)} = \frac{x}{1-x^2} \Rightarrow \alpha(x) = x$$
$$B(x) = 1 - x^2$$

3) HERMITEOVI POLINOMI, $H_n$

$\rho(x) = e^{-x^2}$ $\quad [-\infty, \infty]$

$$\frac{\rho'(x)}{\rho(x)} = \frac{-2x}{1} \Rightarrow \begin{array}{l} \alpha(x) = -2x \\ \beta(x) = 1 \end{array}$$

4) LAGUERROVI POLINOMI, $L_n$

$\rho(x) = e^{-x}$ $\quad [0, \infty]$

$$\frac{\rho'(x)}{\rho(x)} = \frac{-1}{1} \qquad \begin{array}{l} \alpha(x) = -x \\ \beta(x) = x \end{array}$$

---

TM: NEKA JE $(Q_n)$ SISTEM POLINOMA, ORTOGONALAN S TEŽINOM $\rho(x)$ KOJA ZADOVOLJAVA $\rho$-UVJETE
TADA JE $Q_n$ RJEŠENJE DIF. JEDN. (LIN. DIF. 2-REDA)

$$\beta(x)\, y'' + (\alpha(x) + \beta'(x))\, y' - n[\alpha_1 + (n+1)\beta_2]\, y = 0$$

DOKAZ NE TREBA

PR.

LEGEN... $\beta(x) = 1-x^2$ $\quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$
$\alpha(x) = 0$

TM: RODRIGUESOVA FORMULA

NEKA JE $(Q_n)$ SISTEM POLINOMA, ORTOGONALAN S TEŽINOM $\rho(x)$ KOJA ZADOVOLJAVA $\rho$-UVJETE TADA:

$$Q_n(x) = \frac{A_n}{\rho(x)} \cdot \frac{d^n}{dx^n}\left[\rho(x)\, \beta(x)^n\right]$$

$A_n$ → PO VOLJI ODABRANA KONSTANTA

PR. LEGENDREOVI $\quad P_n(x) = \frac{A_n}{1} \frac{d^n}{dx^n}\left[(1-x^2)^n\right]$

$P_0 = 1 \quad (A_0 = 1)$

$P_1 = A_1(-2x) = x \quad (A_1 = -\frac{1}{2})$

$P_2 = \frac{3}{2}x^2 - \frac{1}{2} \quad (A_2 = \frac{1}{8})$

$\boxed{P_n(1) = 1} \rightarrow \boxed{A_n = \frac{(-1)^n}{2^n n!}}$

- def FJA IZVODNICA ORTOGONALNOG SUSTAVA $(Q_n)$ JE:

$$\Psi(x, w) = \sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} w^n$$

TM: FJA IZVODNICA SUSTAVA ORTOGONALNIH POLINOMA $\widetilde{Q_n}(x)$ JE:

$$\Psi(x, w) = \frac{1}{\rho(x)} \frac{\rho(U(w))}{1 - wB'(U(w))}$$

GDJE JE:

$U$ RJEŠENJE JEDNADŽBE $U - x - wB(U) = 0$

$$\widetilde{Q}_n = \frac{1}{\rho(x)} \cdot \frac{d^n}{dx^n}\left[\rho(x) B(x)^n\right] \quad (\text{RF. BEZ } A_n)$$

RODRIGUES

MOŽDA U ZI

PR FJA IZVODNICA ZA $P_n$ (LA.)

$U - x - w \cdot (1 - U^2) = 0, \; U = ?$

$wU^2 + U - x - w = 0$

$$U_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4w \cdot (-x - w)}}{2w}$$ UZIMA SE ONAJ U KOJI JE BLIŽI X

$$\Rightarrow \quad U(x, w) = \frac{-1 + \sqrt{1 + 4xw + 4w^2}}{2w}$$

$$\Psi(x, w) = \frac{1}{1} \cdot \frac{1}{\sqrt{1 + 4wx + 4w^2}} = \sum_{n=0}^{\infty} \frac{\widetilde{P}_n}{n!} w^n$$

UMJESTO $w$ UVRSTIMO $-\frac{w}{2}$

$$\sum_{n=0} \frac{\widetilde{P}_n}{n!}\left(-\frac{w}{2}\right)^n = \sum \frac{(-1)^n}{2^n n!} \widetilde{P}_n \, w^n = \sum P_n \cdot w^n$$

$$= \boxed{\frac{1}{\sqrt{1 - 2wx + w^2}} = \sum P_n(x) \cdot w^n}$$

$$\sum \boxed{P_n(1)} w^n = \frac{1}{1 - w} = \sum w^n$$

$P_n(1) = 1$ ✓ PRIMIJETI