15. 12. 2009.

BURIĆ – PREDAVANJE NASTAVAK

def **PSI** ILI **DIGAMA** FJA

$$\psi(x) = (\ln\Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}$$

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{d}{dz}\left[\ln\left(z^{-1}e^{-\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1}e^{\frac{z}{k}}\right)\right]$$

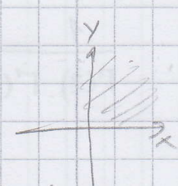
$$= \frac{d}{dz}\left[-\ln z - \gamma z + \sum_{k=1}^{\infty}\left(\frac{z}{k} - \ln\left(1+\frac{z}{k}\right)\right)\right]$$

$$\boxed{\psi(z) = -\frac{1}{z} - \gamma + \sum_{k=1}^{\infty}\frac{z}{k(z+k)}}$$

BETA FJA

$\alpha, \beta > 0$

$$\Gamma(\alpha)\,\Gamma(\beta) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt \cdot \int_0^{\infty} s^{\beta-1}e^{-s}ds = \left|\begin{array}{ll} t = x^2, & s = y^2 \\ dt = 2x\,dx, & ds = 2y\,dy \end{array}\right| =$$



$$= \int_0^{\infty} 2x^{2\alpha-1}e^{-x^2}dx \int_0^{\infty} 2y^{2\beta-1}e^{-y^2}dy = 4\int_0^{\infty}\int_0^{\infty} x^{2\alpha-1}y^{2\beta-1}e^{-(x^2+y^2)}dx\,dy$$

$$= \left|\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \end{array}\right| = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty}(r\cos\varphi)^{2\alpha-1}(r\sin\varphi)^{2\beta-1}e^{-r^2}r\,dr\,d\varphi =$$

$$= 2\int_0^{\frac{\pi}{2}}(\cos\varphi)^{2\alpha-1}(\sin\varphi)^{2\beta-1}d\varphi\int_0^{\infty} r^{2\alpha+2\beta-2}e^{-r^2}2r\,dr = \left|\begin{array}{l} u = r^2 \\ du = 2r\,dr \end{array}\right| =$$

$$\Gamma(\alpha)\cdot\Gamma(\beta) = 2\int_0^{\frac{\pi}{2}}\cos^{2\alpha-1}\varphi\,\sin^{2\beta-1}\varphi\,d\varphi\underbrace{\int_0^{\infty}u^{\alpha+\beta-1}e^{-u}du}_{\Gamma(\alpha+\beta)}$$

" $\alpha = 2\alpha - 1$ "

$$\underbrace{\int_0^{\frac{\pi}{2}}\cos^{2\alpha-1}\varphi\,\sin^{2\beta-1}\varphi\,d\varphi = \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}} \Rightarrow \boxed{\int_0^{\frac{\pi}{2}}\cos^{\alpha}\varphi\,\sin^{\beta}\varphi\,d\varphi = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{2\Gamma\left(\frac{\alpha+\beta}{2}+1\right)}}$$

BILO NA PZI 08 — INTEGRAL POTENCIJA TRIG. FJA

$\ast$

ZAD) PZI

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{tg\,x}} = \int_0^{\frac{\pi}{2}} (\cos x)^{\frac{1}{2}} (\sin x)^{-\frac{1}{2}} dx = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{2\underbrace{\Gamma(1)}_{1}} =$$

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$

$$= \frac{\frac{\pi}{\sin\frac{\pi}{4}}}{2\cdot 1} = \frac{\pi}{\sqrt{2}}$$

Pr

$$\int_0^1 \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array} \right| = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \ldots$$

$\alpha = 2,\ \beta = 0$

$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{\pi}{4}$$

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$$\left| \begin{array}{l} x = \cos^2\varphi \\ dx = -2\cos\varphi\sin\varphi\,d\varphi \end{array} \right| = \frac{1}{2}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

def BETA FJA. ILI EULEROV INTEGRAL I-VRSTE JE

$$\boxed{B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx}$$

$$\cancel{\frac{1}{2}} B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\cancel{2}\Gamma(\alpha+\beta)} \Rightarrow \boxed{B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

ZAD. PRIK. PREKO B I Γ FJE

$$\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \left| \begin{array}{l} x^3 = t \\ 3x^2dx = dt \\ dx = \frac{dt}{3x^2} \end{array} \right| = \frac{1}{3}\int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{1}{3}}dt = \frac{1}{3}B\left(\frac{1}{3},\frac{2}{3}\right) =$$

$x = t^{\frac{1}{3}}$

$$= \frac{1}{3}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\underbrace{\Gamma(1)}_{1}} = \frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{3\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}}$$

ZAD/ $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \left| \begin{array}{ll} 1+x^4 = \frac{1}{1-t} & \to x = \sqrt[4]{\frac{1}{1-t}-1} = \sqrt[4]{\frac{t}{1-t}} \\ 4x^3 dx = \frac{1}{(1-t)^2}dt & t = 1-\frac{1}{1+x^4} \end{array} \right| =$

$$= \frac{1}{4}\int_0^1 \frac{dt}{x^3(1-t)} = \frac{1}{4}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{4}}\,dt = \frac{1}{4}B\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4}\cdot\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(1)} = \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

ZAD/ DOKAZATI : $B(x, y+1) = \frac{y}{x+y}\cdot B(x,y)$

$$B(x, y+1) = \int_0^1 t^{x-1}(1-t)^y\,dt = \int_0^1 t^{x-1}(1-t)(1-t)^{y-1}\,dt =$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt - \int_0^1 t^x(1-t)^{y-1}\,dt = B(x,y) - B(x+1,y)$$

$$B(x,y+1) = \left| \begin{array}{ll} u = (1-t)^y & dv = t^{x-1}dt \\ du = -y(1-t)^{y-1}dt & v = \frac{t^x}{x} \end{array} \right| = \frac{t}{x}(1-t)^y\Big|_0^1 + \frac{y}{x}\int_0^1 t^x(1-t)^{y-1}\,dt$$

$B(x,y+1) = \frac{y}{x}B(x+1,y)$ UVRSTI , GOTOVO

2. FJE IZVODNICE, BERNOULLIJEVI POLINOMI

$$f(x) \to \sum_{n=0}^{\infty} c_n x^n$$

- def FJA $g(x)$ JE FJA IZVODNICA $\boxed{\text{NIZA}}$ (c_n) AKO VRIJEDI:

$c_n \to g(x)$

$$\boxed{g(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n}$$

Pr) $c_n = 1$ $\quad g(x) = \sum \frac{x^n}{n!} = e^x$

$(1, 1, 1, \dots)$

Pr) $c_n = n!$ $\quad g(x) = \sum \frac{n!}{n!} x^n = \frac{1}{1-x}$

$(0!, 1!, 2!, 3!, \dots)$

- def FJA $g(x, s)$ JE FJA IZVODNICA $\boxed{\text{NIZA}}$ $\boxed{\text{FJA}}$ $(h_n(s))$

$$g(x,s) = \sum_{n=0}^{\infty} \frac{h_n(s)}{n!} x^n$$

def BEORNULLIJEVI BROJEVI (B_n) SE DEFINIRAJU FJOM IZVODNICOM:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Pr.) $B_n = g^{(n)}(0)$

$$B_0 = \left. \frac{x}{e^x - 1} \right|_{x=0} \overset{L'H}{=} \lim_{x \to 0} \frac{1}{e^x} = 1$$

$$B_1 \overset{DER}{=} \left. \frac{e^x - 1 - xe^x}{(e^x-1)^2} \right|_{x=0} \overset{L'H}{=} \lim_{x \to 0} \dots = -\frac{1}{2}$$

TM (REKURZIVNA FORMULA ZA ZA B_n)

$$\begin{cases} B_0 = 1 \\ \sum_{n=0}^{m} \binom{n+1}{k} B_k = 0 \quad (n = 1) \end{cases}$$

DOKAZ →

Dokaz

$$x = (e^x - 1)\sum_{k=0}^{\infty} \frac{B_k}{k!}x^k = \sum_{j=1}^{\infty}\frac{x^j}{j!}\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k - \sum_{j=0}^{\infty}\frac{x^{j+1}}{(j+1)!}\sum \frac{B_k}{k!}x^k$$

$$x = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_k}{(j+1)!\,k!}x^{k+j+1} = \sum_{n=0}^{\infty}\Big(\sum_{\substack{j+k=n \\ j=n-k}}\frac{B_k}{(j+1)!\,k!}\Big)x^{n+1}\,\frac{(n+1)!}{(n+1)!}$$

$$x = \sum_{n=0}^{\infty}\underbrace{\Big(\sum_{k=0}^{n}\binom{n+1}{k}B_k\Big)}_{=0}\frac{x^{n+1}}{(n+1)!} \underset{=}{\overset{}{=}} x \quad \checkmark$$

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def BERNOULLIJEVI POLINOMI $B_n(s)$ DEFINIRANI SU:

$$\frac{x\,e^{xs}}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n(s)}{n!}x^n$$

Pr)
$B_0(s) = 1$
$B_1(s) = -\frac{1}{2} + s$
$B_2(s) = \frac{1}{6} - s + s^2$
$B_3(s) = \frac{1}{2}s - \frac{3}{2}s^2 + s^3$

TM (SVOJSTVA $B_n(s)$) NA ZI JEDNO OD SVOJS.

1.) $B_n(0) = B_n$ (BERN. BROJ)

2.) $B_n(1-s) = (-1)^n B_n(s)$

3.) $B_n'(s) = n\,B_{n-1}(s)$

4.) $B_n(s+1) - B_n(s) = n\cdot s^{n-1}$, $n \geq 1$

5.) $B_n(s+h) = \sum_{k=0}^{n}\binom{n}{k}B_k(s)\,h^{n-k}$

DOKAZI

1.) $s=0: \quad \dfrac{x}{e^x-1} = \sum \dfrac{B_n(0)}{n!}x^n$

$\to B_n(0) = B_n$

2.) $\displaystyle\sum_{n=0}^{\infty} \frac{B_n(1-s)}{n!}x^n = \frac{x e^{x(1-s)}}{e^x-1} = \frac{x e^x e^{-xs}}{e^x-1} = \frac{-x e^{-xs}}{-1+e^{-x}} = \frac{(-x)\, e^{(-x)s}}{e^{-x}-1} = \sum_{n=0}^{\infty} (-1)^n B_n(s) \frac{x^n}{n!}$

3.) $\displaystyle\frac{x e^{xs}}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n(s)}{n!}x^n \Big/ \frac{d}{ds}$

$\displaystyle\frac{x^2 e^{xs}}{e^x-1} = \sum_{n=1}^{\infty} \frac{B_n'(s)}{n!}x^n$

$\displaystyle x\frac{x e^{xs}}{e^x-1} = x\sum_{n=0}^{\infty} \frac{B_n(s)}{n!}x^n = \sum_{n=0}^{\infty} \frac{B_n(s)}{n!}x^{n+1}\cdot\frac{n+1}{n+1} = \sum_{n=0}^{\infty} (n+1)B_n(s)\frac{x^{n+1}}{(n+1)!} =$

$\displaystyle = \sum_{n=1}^{\infty} n B_{n-1}(s)\frac{x^n}{n!}$

4.) $\displaystyle\sum_{n=1}^{\infty} B_n(s+1)-B_n(s)\frac{x^n}{n!} = \frac{x e^{x(s+1)}}{e^x-1} - \frac{x e^{xs}}{e^x-1} = \frac{x e^{xs}\cancel{(e^x-1)}}{\cancel{e^x-1}} =$

$\displaystyle = x\sum_{n=0}^{\infty} \frac{x^n s^n}{n!} = \sum_{n=0}^{\infty}\frac{s^n}{n!}x^{n+1} = \sum_{n=0}^{\infty}(n+1)s^n\cdot\frac{x^{n+1}}{(n+1)!} =$

$\displaystyle = \sum_{n=1}^{\infty} n s^{n-1}\frac{x^n}{n!}$

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5.) NE TREBA ZNATI...

PRIMJENA: 1.) $1+2+\ldots+n = \dfrac{n(n+1)}{2}$

$1^2+2^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$1^k+2^k+\ldots+n^k = \dfrac{1}{k+1}\left[B_{k+1}(n+1) - B_{k+1}\right]$

2.) $\displaystyle\int_0^n f(x)\,dx =$

3.) APROKSIMACIJA FAKTORIJELA I GAMA FJE