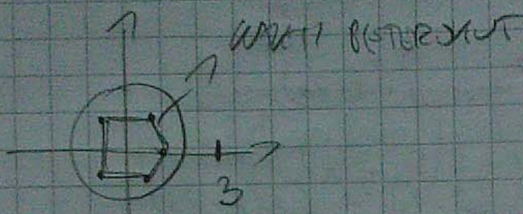
$I = \oint \frac{dz}{(z-3)(z^5-1)}$



Reziduum (ostatak = reziduum)

$Res(f,z_0) = C_{-1}$ LAURENTOV RED OKO TOČKE z_0

$Res(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z)$ ako je z_0 pol 1. reda (IZVOD!)

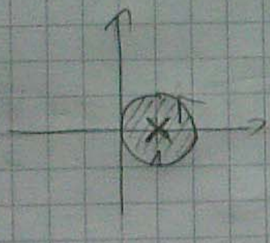
$Res(f,z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$ z_0 pol m-tog reda

AKO JE DRUGI SINGULARITET PO DEFINICIJI!

TEOREM: $\int_\Gamma f(z)\,dz = 2\pi i \sum_{z_k\in G} Res(f,z_k)$ DOKAZ!! (3 REDA) OSNOVNI TM O REZIDUUMU (str. 62)

Pr. $\int_{|z-1|=1}\frac{\sin \pi z}{(z-1)^3}\,dz$ $\begin{matrix}\to 1\\ \to 3\end{matrix}\Big\}\, 3-1=2$

$z_0=1$ SINGULARITET (POL JER JE NULTOČKA NAZIVNIKA, DRUGOG REDA) pol 2. reda



$= 2\pi i\, Res(f,1)$

$Res(f,1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{\sin\pi z}{(z-1)^{3}}\right] =$

$\lim_{z\to 1}\frac{\pi\cos\pi z(z-1)-\sin\pi z}{(z-1)^2} = \left(\frac{0}{0}\right) =$ L'HOSPITAL $= \lim_{z\to 1}\frac{-\pi^2\sin\pi z\cancel{(z-1)}+\pi\cos\pi z-\pi\cos\pi z}{2\cancel{(z-1)}\cdot 1} = 0$

$= 2\pi i\cdot 0 = 0$

2. MI 2008-4) $\displaystyle\oint_{|z|=2} z^2 e^{\frac{1}{z-1}}\,dz$

SIN, EKSP → BITNI SING ($9z_0$)

$z_0 = 1$ BITNI SINGULARITET

NIJE POL, NIJE U NAZIVNIKU CIJELE FJE. e^{∞} NIJE DEF. U KOMPL.

$= 2\pi i \operatorname{Res}(f,1)$

$f(z) = z^2 e^{\frac{1}{z-1}}$ U LAURENTOV RED OKO $z_0=1$

$$= (z-1+1)^2 e^{\frac{1}{z-1}} =$$

$$(z-1)^2 e^{\frac{1}{z-1}} + 2(z-1)e^{\frac{1}{z-1}} + e^{\frac{1}{z-1}}$$

$$= \left[(z-1)^2 + 2(z-1) + 1\right] \sum_{m=0}^{\infty} \frac{\frac{1}{(z-1)^m}}{m!}$$

$$= \left[(z-1)^2 + 2(z-1) + 1\right]\left[1 + \frac{1}{z-1} + \frac{1}{2!(z-1)^2} + \frac{1}{3!(z-1)^3} + \dots\right]$$

$$\operatorname{Res}(f,1) = C_{-1} = \frac{1}{3!} + \frac{2}{2!} + 1 = \frac{1}{6} + 1 + 1 = \frac{13}{6}$$

$$= 2\pi i \cdot \frac{13}{6} = \frac{13}{3}\pi i$$

12. Primjena reziduuma

1. TIP $\displaystyle\int_0^{2\pi} R(\cos t, \sin t)\,dt$

$$= \left|\begin{array}{l} z = e^{it} \\ \sin t = \frac{1}{2i}\left(z - \frac{1}{z}\right) \\ \cos t = \frac{1}{2}\left(z + \frac{1}{z}\right) \end{array}\right| \qquad dt = \frac{dz}{iz}$$

$$= \oint_{|z|=1} f(z)\,dz = 2\pi i \sum_{z_k \in G} \operatorname{Res}(f, z_k)$$

ZAD $\displaystyle\int_0^{2\pi} \frac{dt}{2+\sin t} = \oint_{|z|=1} \frac{\frac{dz}{iz}}{2 + \frac{1}{2i}\left(z - \frac{1}{z}\right)} = \int \frac{2\,dz}{4iz + z^2 - 1}$

$$z^2 + 4iz - 1 = 0$$

$$z_{1,2} = \frac{-4i \pm \sqrt{16i^2 - 4\cdot(-1)}}{2} = \frac{-4i \pm 2i\sqrt{3}}{2} = -2i \pm i\sqrt{3}$$

✓ $z_1 = (-2+\sqrt{3})i$ POLOVI

✗ $z_2 = (-2-\sqrt{3})i$ PRVOG REDA

$2\pi i \,\mathrm{Res}(f, (-2+\sqrt{3})i)$

$$\mathrm{Res}(f, (-2+\sqrt{3})i) = \lim_{z \to (-2+\sqrt{3})i} \cancel{(z-(-2+\sqrt{3})i)} \cdot \frac{2}{\cancel{(z-(-2+\sqrt{3})i)}(z-(-2-\sqrt{3})i)} =$$

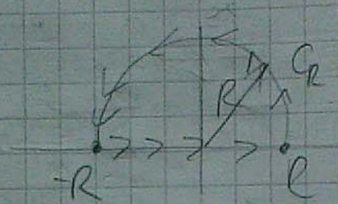
$$\frac{2}{(-2+\sqrt{3})i - (-2-\sqrt{3})i} = \frac{2}{2\sqrt{3}i} = \frac{1}{\sqrt{3}i}$$

$$\int = 2\pi i \cdot \frac{1}{\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}$$

2. TIP $\int_{-\infty}^{+\infty} f(x)\,dx$

RAC. FJA (RACIONALNA)

NE SMIJE IMATI REALNIH SINGULARITETA



$$\oint f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \underbrace{\int_{C_R} f(z)\,dz}_{\text{JORDANOVA LEMA}} \Big/ \lim_{R\to\infty}$$

$$= 2\pi i \sum \mathrm{Res}(f, z_k)$$
$$\mathrm{Im}\, z_k > 0$$

TM: JORDANOVA LEMA

$$M(R) = \max_{z \in C_R} |f(z)|$$

AKO $R \cdot M(R) \to 0$ kad $R \to \infty$

$$\int_{C_R} f(z)\,dz = 0$$

ZAD. $I = \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$

PROVJERA JORDANA:

$$|f(z)| = \left|\frac{1}{(x^2+1)^2}\right| = \left|\frac{1}{(z^2+1)^2}\right| = \frac{1}{|z^2+1|^2} \leq \frac{1}{\big||z|^2 - |-1|\big|^2}$$
$$|z^2-(-1)|$$

$$\begin{cases} |z_1+z_2| \leq |z_1| + |z_2| \\ |z_1 - z_2| \geq \big||z_1| - |z_2|\big| \end{cases}$$

$$* \quad \frac{1}{|z_1-z_2|} \leq \left|\frac{1}{|z_1|-|z_2|}\right|$$

$(|z|=R) \leq \frac{1}{|R^2-1|^2} = M(R)$

$\lim_{R\to\infty} R\cdot M(R) = \lim_{R\to\infty} R\cdot \frac{1}{|R^2-1|^2} = 0$ ✓

$f(z) = \frac{1}{(z^2+1)^2}$

$z_{1,2} = \pm i$ POLOVI DRUGOG REDA (RSI. 2)

UZIMAMO SAMO i (U GORNJOJ POLURAVNINI JE)

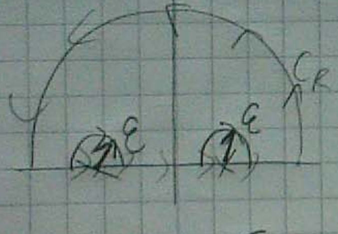
$= \frac{1}{2} \cdot 2\pi i \cdot Res(f,i)$

$Res(f,i) = \frac{1}{1!} \lim_{z\to i} \left[\cancel{(z-i)^2} \frac{1}{\cancel{(z-i)^2}(z+i)^2} \right]' =$

$\lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{2^3 i^3} = \frac{-2}{8(-i)} = \underline{\frac{1}{4i}}$

$= \frac{1}{2} \cdot 2\pi i \cdot \frac{1}{4i} = \pi i \cdot \frac{1}{4i} = \frac{\pi}{4}$

ZAOBILAŽENJE SINGULARITETA NA REALNOJ OSI



$I = 2\pi i \sum_{Im z_k > 0} Res(f, z_k) + \pi i \sum_{Im z_k = 0} Res(f, z_k)$ FORMULA!

3. TIP INTEGRALA (KOMB. 1. i 2.)

a) $\int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = Re\left[2\pi i \sum_{Im z_k > 0} Res(f(z) e^{i\alpha t}, z_k) + \pi i \sum_{Im z_k = 0} Res(f(z) e^{i\alpha t}, z_k) \right]$

b) $\int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx = Im\left[\quad -11- \quad \right]$

JORDANOVA LEMA : $\alpha > 0$

ako $M(R) \to 0$, kada $R\to\infty$

tada $\int f(z) e^{i\alpha t} dz \to 0$

ZAD. $I = \int_{-\infty}^{\infty} \frac{\cos x}{x^4-1} dx$

JORDAN : $\left| \frac{1}{z^4-1} \right| = \frac{1}{|z^4-1|} \leq \frac{1}{||z|^4-1|} \leq \frac{1}{|R^4-1|} = M(R)$

$\lim_{R\to\infty} M(R) = \lim_{R\to\infty} \frac{1}{\infty} = 0$

$z^4 - 1 = 0$

$z = \sqrt[4]{1}$

$(z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i) = 0$

$z = 1$, $z = -1$, $z = i$, $\cancel{z = -i}$

POLOVI PRVOG REDA

$= Re\left[2\pi i \, Res(f e^{iz}, i) + \pi i \, Res(f e^{iz}, 1) + Res(f e^{iz}, -1)\right]$

$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) \to$ pol 1. reda

$\hookrightarrow \ldots = \dfrac{BROJNIK(z_0)}{NAZIVNIK'(z_0)}$

$Res(f e^{iz}, i) = \dfrac{e^{i \cdot i}}{4 i^3} = \dfrac{e^{-1}}{-4i} = \dfrac{-1}{4ei}$

$\dfrac{B}{N} = \dfrac{e^{iz}}{z^4 - 1}$

$Res(f e^{iz}, 1) = \dfrac{e^{i}}{4 \cdot 1^3} = \dfrac{e^{i}}{4}$

$Res(f e^{iz}, -1) = \dfrac{e^{-i}}{-4}$

$I = Re\left[2\pi \cancel{i} \dfrac{e^{-1}}{-4\cancel{i}} + \pi i \left(\dfrac{e^{i}}{4} + \dfrac{e^{-i}}{-4}\right)\right] =$

$\sin 1 = \dfrac{1}{2i}\left(e^{i} - e^{-i}\right)$

$= Re\left[-\dfrac{\pi e^{-1}}{2} - \dfrac{\pi}{2} \sin 1\right] = -\dfrac{\pi e^{-1}}{2} - \dfrac{\pi}{2} \sin 1$