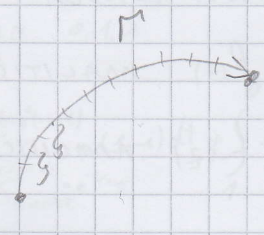
27.10.2009

BURIĆ – PRED.

INTEGRALI FJA KOMPL. VAR.



– DEF $\displaystyle\int_\Gamma f(z)\,dz := \lim_{n\to\infty}\sum_{k=0}^{n-1} f(\zeta)(z_{k+1}-z_k)$

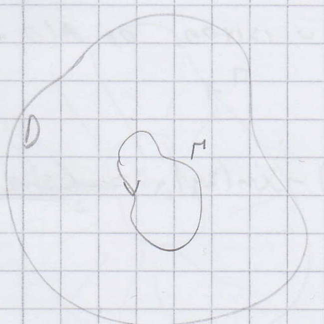
– RAČUNANJE: I $\displaystyle\int_\Gamma f(z)\,dz = \int (u(x,y)+i\cdot v(x,y))(dx+i\,dy) = \int_\Gamma u(x,y)\,dx - v(x,y)\,dy + i\int_\Gamma v(x,y)\,dx + \ldots$

$z = x+iy \,/\, d$

$dz = dx + i\,dy$

$= \ldots$

II PARAMETRIZACIJA $z=\varphi(t)$, $dz = \varphi'(t)\,dt$



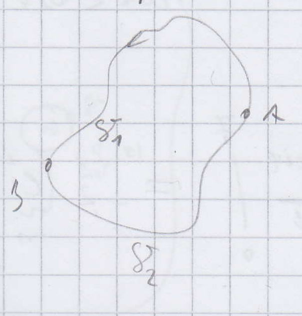
GREEN

$\displaystyle \Rightarrow = \iint_G \underbrace{\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)}_{C.R} dx\,dy + i\iint_G \underbrace{\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)}_{C.R} dx\,dy$

TM: CAUCHYJEV TEOREM

UZMI

POSLJEDICA:



$\displaystyle 0 \overset{\text{CAUCHY}}{=} \int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\underline{\gamma_2}} f(z)\,dz$

$\displaystyle \int_{\gamma_1} f(z)\,dz = -\int_{\underline{\gamma_2}} f(z)\,dz$

$\displaystyle \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$

ZAKLJUČAK:

1. INTEG. PO OTVOR. KRIVULJI Γ — KAO OBIČ. INTEGR.

   a) AKO JE $f(z)$ ANALITIČKA $\int_\Gamma f(z)\,dz = F(z) = F(B)-F(A)$

   b) AKO NIJE $f(z)$ ANALITIČKA $\rightarrow$ PARAMETRIZACIJA $|z|$ $\bar{z}$

2. INT. PO ZATV. KRIVULJI Γ

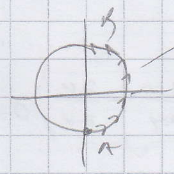
   a) ANALITIČKA $I = 0$

   b) AKO NIJE ANALIT. $\rightarrow$ PARAMETRIZACIJA

3. $f(z)$ ANALITIČKA OSIM U NEKIM TOČKAMA (KONAČNO MNOGO TOČAKA) $\rightarrow$ CAUCHYJEVA FORMULA ILI REZI...

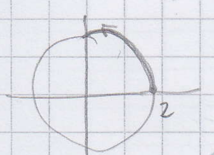
PRIMJER 1

$\int_\Gamma \cos(z-i)\,dz$, Γ - KRUŽNI LUK JED. KRUŽ. U ISHOD. OD $A(0,-1)$ DO B(... $Re\, z > 0$



$$= \sin(z-i)\Big|_A^B = \sin(0) - \sin(2i) = \underline{i\,sh\,2}$$

PRIM 2 $\int_\Gamma \frac{1}{\bar{z}}\,dz$ .. $\Gamma = |z| = 2$ 1. KVADR.

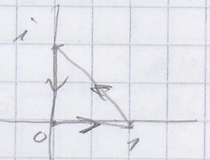


$$z = re^{i\varphi} = 2e^{i\varphi} \Rightarrow \bar{z} = 2e^{-i\varphi}, \quad dz = i2e^{i\varphi}\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cancel{2}e^{-i\varphi}} \cdot \cancel{2}e^{i\varphi} i\,d\varphi = i\int_0^{\frac{\pi}{2}} e^{2i\varphi}\,d\varphi = \cancel{i}\,\frac{e^{2i\varphi}}{2\cancel{i}}\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}(\underbrace{e^{i\pi}}_{-1}-1) = \underline{\underline{-1}}$$

ZMI - 2008

① $\int_\Gamma e^{\bar z}\,dz$ $\quad \Gamma \to$ poz. orient. trójkąt. o wierz. $0, 1, i$



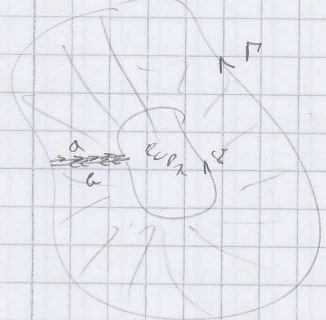
$$= \int_a e^{\bar z}dz + \int_b e^{\bar z}dz + \int_c e^{\bar z}dz = \int_0^1 e^t dt + \int_1^0 \underbrace{e^{t-i(1-t)}}_{e^{t-i+it} = e^{t(1+i)}\cdot \text{konst.}} \cdot (1-i)\,dt + \int_1^0 e^{-it}\, i\,dt =$$

a: $y=0$, $x=t$, $t\in\langle 0,1\rangle$
$z = x+iy = t$
$\bar z = t$
$dz = dt$

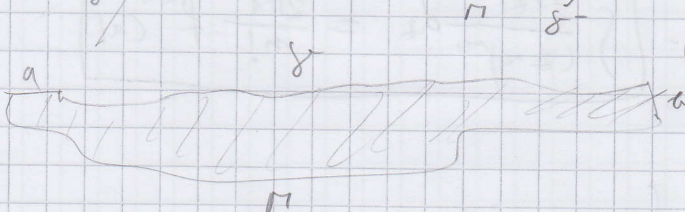
b: $y = 1-x$
$x = t$, $y = 1-t$
$z = t + i(1-t)$
$dz = (1-i)dt$

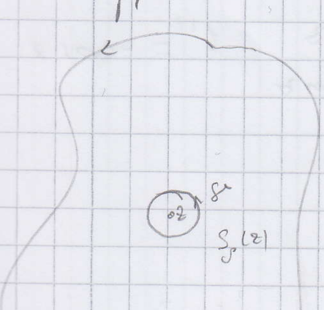
c: $x=0$, $y=t$
$z = it$, $\bar z = -it$
$dz = i\,dt$

$$= e - 1 + \underbrace{(1-i)e^{-i}\frac{1}{i+1}}_{-i} e^{t(i+1)}\Big|_1^0 + i\frac{1}{-i}e^{-it}\Big|_1^0 = e-1-ie^{-i}(1-e^{1+i}) - (1-e^{-i})$$



$$\int_\Gamma + \int_a + \int_{\gamma^-} + \int_b = \emptyset \qquad \int_\Gamma = -\int_{\gamma^-} \Rightarrow \boxed{\int_\Gamma f(z)dz = \int_{\gamma^-} \ldots}$$





$$\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi = \int_\gamma \frac{f(\xi)}{\xi - z}d\xi \ \Big/ \lim_{\rho\to 0}$$

$\gamma \ldots \ |\xi - z| = \rho$

ξ postaje z jak ρ postaje ∅

$$\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi = \int_\gamma \frac{f(z)}{\xi - z}d\xi$$

$$= f(z)\cdot \int_\gamma \frac{d\xi}{\xi - z}$$

PR(1)

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{r e^{i\varphi} i\,d\varphi}{r e^{i\varphi}} = i\varphi\Big|_0^{2\pi} = 2\pi i$$

$z - a = r e^{i\varphi}$

TM: CAUCHYJEVA INT. FORMULA

$$\boxed{f(z) = \frac{1}{2\pi i}\int \frac{f(\xi)}{\xi - z}\,d\xi}$$

DER. ANAL. FJE

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{h}\int_\Gamma\left(\frac{f(\xi)}{\xi - z - h} - \frac{f(\xi)}{\xi - z}\right)d\xi =$$

$$= \frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{\cancel{h}}\int\frac{\cancel{h}\cdot f(\xi)}{(\xi - z - h)(\xi - z)}\,d\xi =$$

$$f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^2}\,d\xi$$

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi}$$

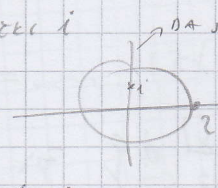
• PRIMJENA: $\boxed{\int \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}f^{(n)}(a)}$

ZAD $\int \frac{\cos z}{(z-i)^3}\,dz$

NIJE ANAL. U TOČKI i

$\Gamma \ldots \quad |z| = 2$

DA JE TO TOČKA ONDA JE $\emptyset$

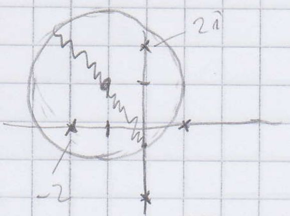


$f(z) = \cos z \qquad f'' = -\cos z$

$f' = -\sin z$

$$= \frac{2\pi i}{2!}f''(i) = \frac{2\pi i}{\cancel{2}}\cdot(-\cos i) = -\pi i \operatorname{ch} 1$$

Pr.) $\displaystyle\int_\Gamma \frac{z^3}{z^4-16}\,dz$, $\Gamma \ldots |z+1-i| = 2$



NIJE ANAL. ZA $z^4-16=0$

$z^4 = 16$

$(z^2-4)(z^2+4)=0$

$(z-2)(z+2)(z-2i)(z+2i)=0$

$$\int \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}$$

$$= \int_{\mathcal{S}_1} \frac{\overbrace{\frac{z^3}{(z-2)(z+2i)(z-2i)}}^{f(z)}}{z+2}\,dz + \int_{\mathcal{S}_2} \frac{\overbrace{\frac{z^3}{(z-2)(z+2)(z+2i)}}^{f(z)}}{z-2i}\,dz = \frac{2\pi i}{0!}\cdot\frac{-8}{-4\cdot 8} + \frac{2\pi i}{0!}\cdot\frac{+8\cdot(-i)}{-8\cdot 4i}$$

$a=-2$, $n=0$ $\qquad$ $a=2i$, $n=0$

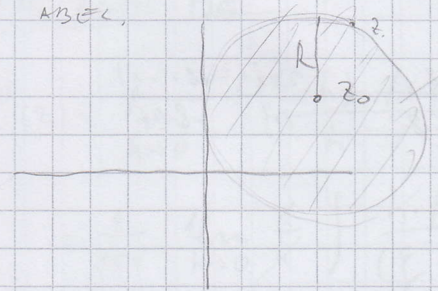
$$= \frac{\pi}{2}i + \frac{\pi}{2}i = \pi i$$

2 REDOVI

$\sum_{n=0}^{\infty} c_n \cdot (z-z_0)^n$ RED POTENCIJA OKO TOČKE z_0

↳ TOČKA OKO KOJE SE RAZVIJA

ABEL



UNUTAR, ZA RUB NE ZNAMO

$$\sum z^n = \frac{1}{1-z} \quad |z|<1$$

R – POLUMJ. KONVERG.

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$$

$$\boxed{\begin{aligned} \frac{1}{R} &= \lim_{n\to\infty} \sqrt[n]{|c_n|} \\ \frac{1}{R} &= \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \end{aligned}}$$

PRIM 1

a) $\sum_{n=0}^{\infty} \left(\frac{z}{z-i}\right)^n$ $\quad c_n = \left(\frac{1}{z-i}\right)^n$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\sqrt[n]{\frac{1}{(z-i)^n}}\right| = \frac{1}{|z-i|} = \frac{1}{\sqrt{5}} \qquad \boxed{R = \sqrt{5}}$$

PRIM 2

b.) $\sum_{n=0}^{\infty} \underbrace{\frac{n!}{n^n}}_{c_n} z^n$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$R = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \boxed{e}$$

PRIM 3

c) $\sum \underbrace{(\mathrm{ch}\, in)^n}_{c_n} z^n$

NAJVEĆE OD SVIH GOMILIŠTA

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|\mathrm{ch}\, in|^n} = \limsup_{n\to\infty} |\cos n| = 1 \Rightarrow \boxed{R = 1}$$

0 1