5. Domaća zadaća

DOMAĆA ZADAĆA

1) A) $\int_0^\infty e^{-x^3}dx = \left| \begin{array}{l} x^3=t \to x=t^{1/3} \\ dt=3x^2dx \\ dx=\frac{dt}{3x^2}=\frac{1}{3}t^{-2/3}dt \end{array} \right| = \int_0^\infty \frac{1}{3}t^{-\frac{2}{3}}e^{-t}dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)=\Gamma\left(\frac{4}{3}\right)$ ✓

B) $\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx = \int_{-1}^{1}(1+x)^{1/2}(1-x)^{-1/2}dx = \left| \begin{array}{l} 1+x=2t \to x=2t-1 \\ 1-x=2-2t \\ dx=2dt \end{array} \right| =$

$= \int_0^1 \cancel{2}^{1/2}\cdot t^{1/2}\cdot\cancel{2}^{-1/2}\cdot(1-t)^{-1/2}\cdot 2\cdot dt = 2\int_0^1 t^{1/2}(1-t)^{-1/2}dt = \left| \begin{array}{l} \alpha=\frac{3}{2} \\ \beta=\frac{1}{2} \end{array} \right| =$

$= 2\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = 2\cdot\frac{\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+1)} = 2\cdot\frac{\frac{1}{2}\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}\cdot\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}}{\underbrace{\Gamma(1)}_{1}} = \pi$ ✓

2) A) $\int_0^\infty e^{-x^6}dx = \left| \begin{array}{l} x^6=t \to x=t^{1/6} \\ dt=6x^5dx \\ dx=\frac{1}{6}\frac{dt}{x^5}=\frac{1}{6}t^{-5/6}dt \end{array} \right| = \int_0^\infty \frac{1}{6}t^{-\frac{5}{6}}e^{-t}dt = \frac{1}{6}\Gamma\left(\frac{1}{6}\right)=\Gamma\left(\frac{7}{6}\right)$ ✓

B) $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \int_0^1 (1-x^3)^{-1/3}dx = \left| \begin{array}{l} 1-x^3=t \to x=(1-t)^{1/3} \\ dt=-3x^2dx \\ dx=-\frac{1}{3}\frac{dt}{x^2}=-\frac{1}{3}(1-t)^{-\frac{2}{3}}dt \end{array} \right| =$

$= \int_1^0 -\frac{1}{3}t^{-1/3}(1-t)^{-2/3}dt = \int_0^1 \frac{1}{3}t^{-1/3}(1-t)^{-2/3}dt = \left| \begin{array}{l} \alpha=\frac{2}{3} \\ \beta=\frac{1}{3} \end{array} \right| =$

$= \frac{1}{3}\cdot\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(1)} = \frac{1}{3}\,\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)} = \left| \begin{array}{l} \text{svojstvo} \\ \text{simetrije} \\ \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \end{array} \right| =$

$= \frac{1}{3}\,\frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{3}\,\frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{3\sqrt{3}}$ ✓

③ A) $$\int_0^\infty x^2 e^{-x^2} dx = \left| \begin{array}{l} x^2 = t \to x = t^{1/2} \\ dt = 2x\,dx \\ dx = \frac{1}{2} t^{-1/2} dt \end{array} \right| = \int_0^\infty t \cdot e^{-t} \cdot \frac{1}{2} t^{-1/2} dt = \int_0^\infty \frac{1}{2} t^{1/2} e^{-t} dt =$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\pi}$$

B) $$\int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} = \int_0^{\pi/2} (\cos x)^{-1/2} dx = \left| \begin{array}{l} \alpha = -\frac{1}{2} \\ \beta = 0 \end{array} \right| = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

④ A) $$\int_0^\infty x \cdot e^{-x^3} dx = \left| \begin{array}{l} x^3 = t \to x = t^{1/3} \\ dt = 3x^2 dx \\ dx = \frac{1}{3} t^{-2/3} dt \end{array} \right| = \int_0^\infty \frac{1}{3} t^{-\frac{1}{3}} e^{-t} dt = \frac{1}{3}\Gamma\left(\frac{2}{3}\right)$$

B) $$\int_0^{\pi/2} \sqrt{\operatorname{tg} x}\, dx = \int_0^{\pi/2} \sqrt{\frac{\sin x}{\cos x}}\, dx = \int_0^{\pi/2} (\sin x)^{1/2} \cdot (\cos x)^{-1/2} dx = \left| \begin{array}{l} \alpha = -\frac{1}{2} \\ \beta = \frac{1}{2} \end{array} \right|$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{2} = \left| \begin{array}{l} \text{svojstvo} \\ \text{simetrije} \end{array} \right| = \frac{\pi}{2\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

⑤ A) $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 (1-x^4)^{-1/2} dx = \left| \begin{array}{l} 1-x^4 = t \to x = (1-t)^{1/4} \\ dt = -4x^3 dx \\ dx = -\frac{1}{4}(1-t)^{-3/4} dt \end{array} \right| = \int_1^0 -\frac{1}{4} t^{-1/2}(1-t)^{-3/4} dt =$$

$$= \int_0^1 \frac{1}{4} t^{-1/2}(1-t)^{-3/4} dt = \left| \begin{array}{l} \alpha = \frac{1}{2} \\ \beta = \frac{1}{4} \end{array} \right| = \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

⑤ 3.)

$$\int_0^{\infty} \frac{x^2}{1+x^4}\,dx = \left| \begin{matrix} x^2 = tg\varphi \to x = \sqrt{tg\varphi} \\ 2x\,dx = \frac{1}{\cos^2\varphi}\,d\varphi \\ dx = \frac{1}{2\cos^2\varphi}\cdot(tg\varphi)^{-1/2}\,d\varphi \end{matrix} \right| \quad \begin{matrix} \text{granice:} \\ tg\varphi = 0 \to \varphi = 0 \\ tg\varphi = \infty \to \varphi = \pi/2 \end{matrix}$$

$$= \int_0^{\pi/2} \frac{tg\varphi}{1+tg^2\varphi}\,\frac{1}{2\cos^2\varphi}\cdot(tg\varphi)^{-1/2}\,d\varphi = \int_0^{\pi/2} \frac{tg^{1/2}\varphi}{1+tg^2\varphi}\,\frac{1}{2\cos^2\varphi}\,d\varphi =$$

$$= \int_0^{\pi/2} \frac{\frac{\sin^{1/2}\varphi}{\cos^{1/2}\varphi}}{1+\frac{\sin^2\varphi}{\cos^2\varphi}}\,\frac{1}{2\cos^2\varphi}\,d\varphi = \frac{1}{2}\int_0^{\pi/2} \frac{\frac{\sin^{1/2}\varphi}{\cos^{1/2}\varphi}}{\frac{\cos^2\varphi+\sin^2\varphi}{\cancel{\cos^2\varphi}}}\,\frac{1}{\cancel{\cos^2\varphi}}\,d\varphi =$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{1/2}\varphi}{\cos^{1/2}\varphi}\,d\varphi = \frac{1}{2}\int_0^{\pi/2} \cos^{-1/2}\varphi\cdot\sin^{1/2}\varphi\,d\varphi = \left| \begin{matrix} \alpha = -\frac{1}{2} \\ \beta = \frac{1}{2} \end{matrix} \right| = \frac{1}{2}\,\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{2\Gamma(1)} =$$

$$= \frac{1}{2}\,\frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{2\Gamma(1)} = \frac{1}{4}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4}\,\frac{\pi}{\frac{\sqrt{2}}{2}} = \frac{\pi}{2\sqrt{2}}$$

⑥ $$B(\alpha,\alpha) = \frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha-1}\,\Gamma(\alpha+\frac{1}{2})}$$

$$B(\alpha,\alpha) = \int_0^1 x^{\alpha-1}(1-x)^{\alpha-1}\,dx = \left| \begin{matrix} x = \sin^2 x \\ dx = 2\sin x\cos x\,dx \end{matrix} \right| = 2\int_0^{\frac{\pi}{2}} \sin^{2\alpha-2}x\,\cos^{2\alpha-2}x\,\sin x\cos x\,dx =$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1}x\,\cos^{2\alpha-1}x\,dx = 2\int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2}\sin 2x\right)^{2\alpha-1}dx = \frac{2}{2^{2\alpha-1}}\int_0^{\frac{\pi}{2}} (\sin 2x)^{2\alpha-1}\,dx = \left| \begin{matrix} u = 2x \\ du = 2dx \end{matrix} \right| =$$

$$= \frac{\cancel{2}}{2^{2\alpha-1}}\int_0^{\pi} (\sin u)^{2\alpha-1}\,\frac{du}{\cancel{2}} = \frac{2}{2^{2\alpha-1}}\int_0^{\frac{\pi}{2}} (\sin u)^{2\alpha-1}\,du = \frac{\cancel{2}}{2^{2\alpha-1}}\;\frac{\Gamma(\frac{0+1}{2})\Gamma(\frac{(2\alpha-1)+1}{2})}{\cancel{2}\,\Gamma(\frac{0+(2\alpha-1)}{2}+1)} =$$

$$= \frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha-1}\,\Gamma(\alpha+\frac{1}{2})}$$

⑦ $xB(x,y+1) = yB(x+1,y)$

$$x\int_0^1 t^{x-1}(1-t)^y\,dt = y\int_0^1 t^x (1-t)^{y-1}\,dt$$

$$= \left| \begin{matrix} u = t^x & dv = (1-t)^{y-1}\,dt \\ du = x\cdot t^{x-1}\,dt & v = -\frac{1}{y}(1-t)^y \end{matrix} \right|$$

$$= y\left[\cancel{-t^x\cdot\frac{1}{y}(1-t)^y\Big|_0^1} + \int_0^1 \frac{x}{y} t^{x-1}(1-t)^y\,dt\right]$$

$$x\int_0^1 t^{x-1}(1-t)^y\,dt = x\int_0^1 t^{x-1}(1-t)^y\,dt$$

✓✓

⑧ $B(x,y) = B(x+1,y) + B(x,y+1)$

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^1 t^x(1-t)^{y-1}\,dt + \int_0^1 t^{x-1}(1-t)^y\,dt$$

$$= \int_0^1 t^x(1-t)^{y-1}\,dt + \int_0^1 t^{x-1}(1-t)(1-t)^{y-1}\,dt$$

$$= \cancel{\int_0^1 t^x(1-t)^{y-1}\,dt} + \int_0^1 t^{x-1}(1-t)^{y-1}\,dt - \cancel{\int_0^1 t^x(1-t)^{y-1}\,dt}$$

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

✓

⑨ $(x+y)B(x,y+1) = yB(x,y)$

$$(x+y)\int_0^1 t^{x-1}(1-t)^{y} = y\cdot\int_0^1 t^{x-1}(1-t)^{y-1}$$

$$(x+y)\int_0^1 t^{x-1}(1-t)(1-t)^{y-1} = y\cdot B(x,y)$$

$$(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}dt - (x+y)\int_0^1 t^{x}(1-t)^{y-1}dt = yB(x,y)$$

$$(x+y)B(x,y) - (x+y)\cdot B(x+1,y) = yB(x,y)$$

$$B(x,y+1) = \int_0^1 t^{x-1}(1-t)^{y}dt = \begin{vmatrix} u=(1-t)^y & dv = t^{x-1}dt \\ du = -y(1-t)^{y-1}dt & v=\frac{1}{x}t^x \end{vmatrix}$$

$$= \cancel{\frac{t^x}{x}(1-t)^y}\Big|_0^1 + \int_0^1 \frac{y}{x}t^x(1-t)^{y-1}dt = \frac{y}{x}B(x+1,y)$$

$$B(x+1,y) = \frac{x}{y}B(x,y+1)$$

$$(x+y)B(x,y) - (x+y)\cdot\frac{x}{y}B(x,y+1) = yB(x,y)$$

$$-(x+y)\frac{x}{y}B(x,y+1) = -xB(x,y)$$

$$(x+y)B(x,y+1) = yB(x,y) \quad ✓$$

GRAMM - SCHMIDT

(11) $f_0(x) = 1, \quad f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = x^3$

$[0,1], \quad \rho(x) = x^2$

$w_0(x) = f_0(x) = 1$ ✓

$$w_1(x) = x - \frac{(x|1)}{\|1\|^2}\cdot 1 = x - \frac{\int_0^1 x^3\,dx}{\int_0^1 x^2\,dx} = x - \frac{\left.\frac{x^4}{4}\right|_0^1}{\left.\frac{x^3}{3}\right|_0^1} = x - \frac{\frac{1}{4}}{\frac{1}{3}} = x - \frac{3}{4}$$ ✓

$$w_2(x) = f_2 - \frac{(f_2|w_0)}{\|w_0\|^2}w_0 - \frac{(f_2|w_1)}{\|w_1\|^2}w_1$$

$$= x^2 - \frac{(x^2|1)}{\|1\|^2} - \frac{(x^2|x-\frac{3}{4})}{\|x-\frac{3}{4}\|^2}\left(x-\frac{3}{4}\right)$$

$$= x^2 - \frac{\int_0^1 x^4\,dx}{\int_0^1 x^2\,dx} - \frac{\int_0^1 x^4(x-\frac{3}{4})\,dx}{\int_0^1 x^2(x-\frac{3}{4})^2\,dx}\left(x-\frac{3}{4}\right)$$

$$= x^2 - \frac{3}{5} - \frac{\int_0^1 x^5\,dx - \frac{3}{4}\int_0^1 x^4\,dx}{\int_0^1 x^4\,dx - \int_0^1 \frac{3}{2}x^3\,dx + \frac{9}{16}\int_0^1 x^2\,dx}\left(x-\frac{3}{4}\right)$$

$$= x^2 - \frac{3}{5} - \frac{\frac{1}{6}-\frac{3}{20}}{\frac{1}{5}-\frac{3}{2}\cdot\frac{1}{4}+\frac{9}{16}\cdot\frac{1}{3}}\left(x-\frac{3}{4}\right) = x^2 - \frac{3}{5} - \frac{\frac{1}{60}}{\frac{1}{80}}\left(x-\frac{3}{4}\right) =$$

$$= x^2 - \frac{3}{5} - \frac{4}{3}\left(x-\frac{3}{4}\right) = x^2 - \frac{4}{3}x + \frac{2}{5}$$ ✓

$$w_3(x) = f_3 - \frac{(f_3|w_0)}{\|w_0\|^2}\cdot w_0 - \frac{(f_3|w_1)}{\|w_1\|^2}w_1 - \frac{(f_3|w_2)}{\|w_2\|^2}w_2$$

$$= x^3 - \frac{(x^3|1)}{\|1\|^2} - \frac{(x^3|x-\frac{3}{4})}{\|x-\frac{3}{4}\|^2}\left(x-\frac{3}{4}\right) - \frac{(x^3|(x^2-\frac{4}{3}x+\frac{2}{5}))}{\|x^2-\frac{4}{3}x+\frac{2}{5}\|^2}\left(x^2-\frac{4}{3}x+\frac{2}{5}\right)$$

$$\frac{(x^3|1)}{\|1\|^2} = \frac{\int_0 x^5\,dx}{\int_0 x^2\,dx} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

$$\frac{(x^3|x-\frac{3}{4})}{\|x-\frac{3}{4}\|^2} = \frac{\int_0^1 x^5(x-\frac{3}{4})\,dx}{\int_0^1 x^2(x-\frac{3}{4})^2\,dx} = \frac{\frac{1}{7}-\frac{3}{4}\cdot\frac{1}{\cancel{6}2}}{\frac{1}{80}} = \frac{\frac{1}{7}-\frac{1}{8}}{\frac{1}{80}} = \frac{10}{7}$$

$$\frac{(x^3|x^2-\frac{4}{3}x+\frac{2}{5})}{\|x^2-\frac{4}{3}x+\frac{2}{5}\|^2} = \frac{\int_0^1 x^5(x^2-\frac{4}{3}x+\frac{2}{5})\,dx}{\int_0^1 x^2(x^2-\frac{4}{3}x+\frac{2}{5})^2\,dx} = \frac{\frac{1}{8}-\frac{4}{3}\cdot\frac{1}{7}+\frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{6}3}}{\frac{1}{7}-\frac{\cancel{8}}{3}\cdot\frac{1}{\cancel{6}_3}-\frac{16}{45}\cdot\frac{1}{5}-\frac{16}{15}\frac{1}{4}+\frac{4}{25}\cdot\frac{1}{3}} =$$

$$x^2\left(x^4+\frac{16}{9}x^2+\frac{4}{25}-\frac{8}{3}x^3+\frac{4}{5}x^2-\frac{16}{15}x\right)$$

$$= \frac{\frac{1}{8}-\frac{4}{21}+\frac{1}{15}}{\frac{1}{7}-\frac{4}{9}-\frac{116}{225}-\frac{4}{15}+\frac{4}{15}} = \frac{\frac{1}{8}-\frac{13}{105}}{-\frac{164}{225}-\frac{10}{63}} = \frac{\frac{1}{840}}{\frac{-14.602}{14.175}} = \frac{1}{\frac{1.\cancel{[illegible]}\,544}{15\cancel{[illegible]}\;105}} = \frac{\frac{1}{\cancel{8}\cancel{[illegible]}168}}{\frac{544}{\cancel{[illegible]}\,105}} = \frac{105}{168.544}$$

$$\frac{10}{7}\left(x-\frac{3}{4}\right) = \frac{10}{7}x - \frac{30}{28}$$

$$\frac{105}{168.544}\left(x^2-\frac{4}{3}x+\frac{2}{5}\right) = \frac{105}{168.544}x^2 - \frac{35}{42.544}x + \frac{1}{4.541}$$

12. $f_0(x) = e^{-x}$ $f_1(x) = e^{-2x}$ $f_2(x) = e^{-3x}$ $f_3(x) = e^{-\dots}$

$[0, \infty]$, $\rho(x) = 1$

$w_0(x) = f_0(x) = e^{-x}$

$$w_1(x) = f_1(x) - \frac{(f_1 | w_0)}{\|w_0\|^2} w_0 = e^{-2x} - \frac{(e^{-2x} | e^{-x})}{\|e^{-x}\|^2} e^{-x} =$$

$$= e^{-2x} - \frac{\int_0^\infty e^{-3x}\,dx}{\int_0^\infty e^{-2x}\,dx} e^{-x} = e^{-2x} - \frac{+\frac{1}{3} e^{-3x}\Big|_0^\infty}{+\frac{1}{2} e^{-2x}\Big|_0^\infty} \cdot e^{-x} =$$

$$= e^{-2x} - \frac{2}{3} \cdot e^{-x}$$

$$w_2(x) = f_2(x) - \frac{(f_2 | w_0)}{\|w_0\|^2} w_0 - \frac{(f_2 | w_1)}{\|w_1\|^2} w_1$$

$$= e^{-3x} - \frac{(e^{-3x} | e^{-x})}{\|e^{-x}\|^2} e^{-x} - \frac{(e^{-3x} | e^{-2x} - \frac{2}{3} e^{-x})}{\|e^{-2x} - \frac{2}{3} e^{-x}\|^2} (e^{-2x} - \frac{2}{3} e^{-x})$$

$$= e^{-3x} - \frac{\int_0^\infty e^{-4x}\,dx}{\int_0^\infty e^{-2x}\,dx} e^{-x} - \frac{\int_0^\infty e^{-3x}(e^{-2x} - \frac{2}{3} e^{-x})\,dx}{\int_0^\infty (e^{-2x} - \frac{2}{3} e^{-x})^2\,dx} (e^{-2x} - \frac{2}{3} e^{-x})$$

$$= e^{-3x} - \frac{1}{2} \cdot e^{-x} - \frac{\frac{1}{5} - \frac{\cancel{2}}{3} \frac{1}{\cancel{4}2}}{\frac{1}{4} - \frac{\cancel{4}}{3} \cdot \frac{1}{3} + \frac{4}{9} \frac{1}{2}} (e^{-2x} - \frac{2}{3} e^{-x})$$

$$\frac{\frac{1}{5} - \frac{1}{6}}{\frac{1}{4} - \frac{4}{9} + \frac{4}{18}} = \frac{\frac{1}{30}}{\frac{1}{36}} = \frac{6}{5}$$

$$= e^{-3x} - \frac{6}{5} e^{-2x} + \frac{3}{10} e^{-x}$$

$$\frac{6}{5} \cdot \frac{2}{3}$$

$$\frac{4}{5} - \frac{1}{2} = \frac{8-5}{10}$$

$$w_3(x) = f_3(x) - \frac{(f_3|w_0)}{\|w_0\|^2} w_0 - \frac{(f_3|w_1)}{\|w_1\|^2} w_1 - \frac{(f_3|w_2)}{\|w_2\|^2} w_2$$

$$\frac{(f_3|w_0)}{\|w_0\|^2} w_0 = \frac{(e^{-4x}|e^{-x})}{\|e^{-x}\|^2} e^{-x} = \frac{\int_0^\infty e^{-5x}\,dx}{\int_0^\infty e^{-2x}\,dx} e^{-x} = \frac{+\frac15 e^{-5x}\Big|_0^\infty}{+\frac12 e^{-2x}\Big|_0^\infty} e^{-x} = \frac25 e^{-x}$$

$$\frac{(f_3|w_1)}{\|w_1\|^2} = \frac{(e^{-4x}|e^{-2x}-\frac23 e^{-x})}{\|e^{-2x}-\frac23 e^{-x}\|^2} = \frac{\int_0^\infty e^{-4x}(e^{-2x}-\frac23 e^{-x})\,dx}{\int_0^\infty (e^{-2x}-\frac23 e^{-x})^2\,dx} = \frac{\frac16 - \frac23\cdot\frac15}{\frac14 - \frac43\cdot\frac13 + \frac49\cdot\frac12} =$$

$$= \frac{\frac16 - \frac2{15}}{\frac14 - \frac29} = \frac{\frac1{30}}{\frac1{36}} = \frac65$$

$$\frac{(f_3|w_2)}{\|w_2\|^2} = \frac{(e^{-4x}|e^{-3x}-\frac65 e^{-2x}+\frac3{10}e^{-x})}{\|e^{-3x}-\frac65 e^{-2x}+\frac3{10}e^{-x}\|^2} = \frac{\int_0^\infty e^{-4x}(e^{-3x}-\frac65 e^{-2x}+\frac3{10}e^{-x})\,dx}{\int_0^\infty (e^{-3x}-\frac65 e^{-2x}+\frac3{10}e^{-x})^2\,dx} =$$

$$= \frac{\frac17 - \frac65\cdot\frac16 + \frac3{10}\cdot\frac15}{\int_0^\infty (e^{-6x} + \frac{36}{25}e^{-4x} + \frac9{100}e^{-2x} - \frac{12}5 e^{-5x} + \frac35 e^{-4x} - \frac{18}{25}e^{-3x})\,dx} =$$

$$= \frac{\frac17 - \frac15 + \frac3{50}}{\frac16 - \frac{12}5\cdot\frac15 + \frac{51}{25}\cdot\frac14 - \frac{18}{25}\cdot\frac13 + \frac9{100}\cdot\frac12} = \frac{-\frac2{35}+\frac3{50}}{\frac16 - \frac{12}{25} + \frac{51}{100} - \frac6{25} + \frac9{200}} =$$

$$= \frac{\frac1{350}}{-\frac{611}{600}} = \frac{-\cancel{600}^{12}}{\cancel{350}_{7}\cdot 611} = \frac{-12}{7\cdot 611}$$

(13) $f_0(x)=1$, $f_1(x)=\sin x$, $f_2(x)=\cos x$

$[-\pi,\pi]$ $\qquad \underline{\rho(x)=1-x}$

$w_0(x)=f_0(x)=\underline{1}$

$$w_1(x)=f_1(x)-\frac{(f_1|w_0)}{\|w_0\|^2}w_0=\sin x-\frac{\int_{-\pi}^{\pi}\sin x(1-x)\,dx}{\int_{-\pi}^{\pi}(1-x)\,dx}=$$

$u=x \quad dv=\sin x\,dx$
$du=dx \quad v=-\cos x$

$$=\sin x-\frac{\int_{-\pi}^{\pi}\sin x\,dx-\int_{-\pi}^{\pi}x\cdot\sin x\,dx}{\int_{-\pi}^{\pi}dx-\int_{-\pi}^{\pi}x\,dx}=\sin x-\frac{-\cos x\Big|_{-\pi}^{\pi}-\left[-x\cos x\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\cos x\,dx\right]}{2\pi-\frac{x^2}{2}\Big|_{-\pi}^{\pi}}$$

$$=\sin x-\frac{-(\cos\pi-\cos(-\pi))-\left[(+\pi\cos\pi+\pi\cos\pi)+\cancel{\sin x}\Big|_{-\pi}^{\pi}\right]}{2\pi-\frac{1}{2}(\pi^2-\pi^2)}=\sin x+1$$

$$w_2(x)=f_2(x)-\frac{(f_2|w_0)}{\|w_0\|^2}w_0-\frac{(f_2|w_1)}{\|w_1\|^2}w_1=\cos x-\frac{(\cos x|1)}{\|1\|^2}-\frac{(\cos x|\sin x+1)}{\|\sin x+1\|^2}(\sin x+1)$$

$$=\cos x-\frac{\int_{-\pi}^{\pi}\cos x(1-x)\,dx}{\int_{-\pi}^{\pi}(1-x)\,dx}-\frac{\int_{-\pi}^{\pi}\cos x(\sin x+1)(1-x)\,dx}{\int_{-\pi}^{\pi}(1-x)(\sin x+1)^2\,dx}(\sin x+1)$$

$$=\cos x-\frac{\cancel{\sin x}\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}x\cdot\cos x\,dx}{2\pi}-\frac{\int_{-\pi}^{\pi}\cos x\sin x\,dx-\int_{-\pi}^{\pi}x\cos x\sin x\,dx+\int_{-\pi}^{\pi}\cancel{\cos x\,dx}-\int_{-\pi}^{\pi}x\cos x\,dx}{\int_{-\pi}^{\pi}(1-x)(\sin^2x+2\sin x+1)\,dx}(\sin x+1)$$

$u=x \quad dv=\cos x\,dx$
$du=dx \quad v=\sin x$

$u=x \quad dv=\sin 2x\,dx$
$du=dx \quad v=-\frac{1}{2}\cos 2x$

$$=\cos x-\frac{-\int_{-\pi}^{\pi}\cancel{\sin x\,dx}}{2\pi}-\frac{\int_{-\pi}^{\pi}\frac{1}{2}\cancel{\sin 2x\,dx}-\int_{-\pi}^{\pi}\frac{x}{2}\sin 2x\,dx+\int_{-\pi}^{\pi}\sin x\,dx}{\int_{-\pi}^{\pi}\frac{1-\cos 2x}{2}\,dx+2\int_{-\pi}^{\pi}\cancel{\sin x\,dx}+2\pi-\int_{-\pi}^{\pi}x\cdot\frac{1-\cos 2x}{2}\,dx-\underbrace{\int 2x\sin x}_{4\pi}-\int_{-\pi}^{\pi}x\,dx}$$

$$=\cos x-\frac{-\frac{1}{2}\left[\overbrace{-\frac{x}{2}\cos 2x\Big|_{-\pi}^{\pi}}^{-\pi}+\int\frac{1}{2}\cancel{\cos 2x\,dx}\right]}{\underbrace{2\pi-4\pi+\pi}_{-\pi}}(\sin x+1)=\cos x+\frac{1}{2}(\sin x+1)$$