12. 01. 2010

BURIĆ – PRED. ZADNJE

FOURIEROV RED PO ORTOGONALNIM SUSTAVIMA

cilj: $f = \sum_{k=1}^{\infty} c_k \varphi_k \,/\cdot(\varphi_i)$ (φ_k) ORTOG. SUSTAV

$$(f|\varphi_i) = c_i \underbrace{(\varphi_i|\varphi_i)}_{\|\varphi_i\|^2}$$

$$c_i = \frac{(f|\varphi_i)}{\|f_i\|^2}$$

– def RED OBLIKA $f = \sum_{j=1}^{\infty} \frac{(f|\varphi_j)}{\|\varphi_j\|^2} \varphi_j$ SE NAZIVA FOURIEROV RED PO ORTOG SUSTAVU

n-TI FOURIEROV POLINOM (APROKSIMACIJA) JE $f = \sum_{j=1}^{n} c_j \varphi_j$

Prim | ORTOG SUSTAV. $\frac{1}{2}$, $\sin x$, $\cos x$ .... $\sin(nx)$, $\cos(nx)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx\, f(x)\, dx, \quad b_n = \dots$$

TM | FOURIEROV RED IMA SVOJSTVO DA NAJBOLJE APROKSIMIRA FJU f, TJ ZA SVAKI n, UDALJENOST $\|f - \sum_{k=1}^{n} a_n \varphi_n\|$ JE NAJMANJA UKOLIKO JE $a_k = c_k = \frac{(f|\varphi_k)}{\|\varphi_k\|^2}$

NAZI – DOKAZ

BESSELOVA NEJEDNAKOST $\sum_{k=1}^{\infty} c_k^2 \|\varphi_k\|^2 \leq \|f\|^2$

DOKAZ: $d(f, \sum_{k=1}^{n} a_n \varphi_k)^2 = \|f - \sum_{k=1}^{n} a_k \varphi_k\|^2 = (f - \sum_{k=1}^{n} a_k \varphi_k \mid f - \sum_{k=1}^{n} a_k \varphi_k)$

$$= (f|f) - \left(\sum a_k \varphi_k \middle| f\right) - \left(f \middle| \sum a_k \varphi_k\right) + \left(\sum a_k \varphi_k \middle| \sum a_j \varphi_j\right) =$$

$$= \|f\|^2 - 2\left(\sum a_k \varphi_k \middle| f\right) + \sum_{k=1}^{n} a_k^2 (\varphi_k|\varphi_k) =$$

$$= \|f\|^2 - 2\sum_{k=1}^{n} a_k (\varphi_k|f) + \sum_{k=1}^{n} a_k^2 \|\varphi_k\|^2$$

$$\sum a_k^2 \|\varphi_k\|^2$$

$$= \|f\|^2 - 2\sum_{k=1}^{n} a_k (\varphi_k|f) + \sum_{k=1}^{n} a_k^2 \|\varphi_k\|^2 = \|f\|^2 - 2\sum_{k=1}^{n} a_k \cdot c_k \|\varphi_k\|^2 +$$

$$= \|f\|^2 - \sum_{k=1}^{n} c_k^2 \|\varphi_k\|^2 + \sum_{k=1}^{n} (a_k - c_k)^2 \|\varphi_k\|^2$$ $\Rightarrow d$ JE NAJMANJI ZA $a_k = c_k$ ✓

$$\geq \|f\|^2 - \sum_{k=1}^{n} c_k^2 \|\varphi_k\|^2 \geq 0$$

$$\sum_{k=1}^{n} c_k^2 \|\varphi_k\|^2 \leq \|f\|^2 \quad \text{TM}$$

-**def** - KAŽEMO DA JE ORTOG. SUS (φ_n) BAZA U X, AKO SE SVAKA $f \in X$ MOŽE NAPISATI KAO $f = \sum_{k=1}^{n} a_k \varphi_k$ I VRIJEDI $\|f - \sum_{k=1}^{n} c_k \varphi_k\| \to 0$, KADA $n \to \infty$

-**def** (φ_n) JE ZATVOREN AKO VRIJEDI PARSEVALOVA JEDNAKOST

$$\sum c_k^2 \|\varphi_k\|^2 = \|f\|^2$$

-**def** (φ_n) JE POTPUN AKO JE SVAKA FJA KOJA JE OKOMITA NA SVE (φ_n) ~~NUL~~ IDENTIČKI JEDNAKA NULI

TM BAZA $\Leftrightarrow$ ZATVOREN $\Leftrightarrow$ POTPUN

5.07 –16) $w_n = \sqrt{x}\sin(nx)$, $n \geq 1$ NA $[0, 2\pi]$ S $\rho(x) = 1$

ORT: $(w_n|w_n) = \int_0^{2\pi} \sqrt{x}\sin(nx)\sqrt{x}\sin(nx)\,1\,dx = \ldots = \emptyset$

RAZVIJ U FOUR(.) FJU $f(x) = \sqrt{x^3}$

$$f(x) = \sum_{n=1}^{\infty} C_n \cdot w_n, \quad C_n = \frac{(f|w_n)}{\|w_n\|^2}$$

$$(f|w_n) = \int_0^{2\pi} \underbrace{\sqrt{x^3}\sqrt{x}\sin(nx)}_{x^2 \sin(nx)}\,1 \cdot dx = \ldots = \frac{-4\pi^2}{n}$$

$$\|w_n\|^2 = (w_n|w_n) = \int_0^{2\pi} x \cdot \underbrace{\sin^2(nx)}_{\frac{1-\cos 2nx}{2}}\,dx =$$

$$f(x) = \sum_{n=1}^{\infty} \frac{\frac{-4\cancel{\pi^2}}{n}}{\cancel{\pi^2}} \cdot \sqrt{x}\sin(nx) = -4\sqrt{x}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

FOURIEROV RED PO LEGENDREOVIM POLINOMIMA

– PODSJETNIK $[-1, 1]$, $\rho(x) = 1$, $\beta(x) = 1 - x^2$, $\alpha(x) = \emptyset$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \ldots$$

$$P_n(x) = A_n \frac{d^n}{dx^n}\left[(1-x^2)^n\right], \quad P_n(1) = 1$$

FOUR. RED PO LEG.

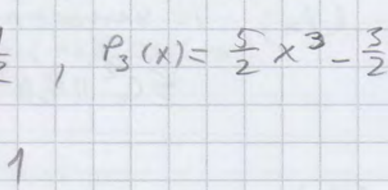
$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x), \quad C_n = \frac{(f|P_n)}{\|P_n\|^2}, \quad \|P_n\|^2 = \frac{2}{2n+1}$$

ZAD) IZRAČUNAJ PRVIH NEKOLIKO (NPR DO 3) KOEFICIJENATA U RAZVOJU FJE f U F. RED PO LEG. POLINOMIMA

$$f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 1, & 0 < x \leq 1 \end{cases}$$

$$\boxed{C_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\,dx}$$

$$C_0 = \frac{1}{2}\left[\int_{-1}^{0} -1 \cdot \underset{P_0(x)}{1}\,dx + \int_0^1 1 \cdot 1\,dx\right] = \emptyset$$



→ f JE NEPARNA

→ $C_{2n} = 0$

$C_0 = C_2 = C_4 = \emptyset$

$$C_1 = \frac{3}{2}\left[\int_{-1}^{0} -1 \cdot x\,dx + \int_0^1 1 \cdot x\,dx\right] = \frac{3}{2}$$

$$c_3 = \frac{7}{2}\left[2\int_0^1 1\cdot\left(\frac{5}{2}x^2-\frac{3}{2}x\right)dx\right. = -\frac{7}{8}$$

$$c_4 = 0$$

$$c_5 = \frac{11}{2}\left[2\int_0^1 1\cdot \underbrace{\frac{1}{8}\left(63x^5-70x^3+15x\right)}_{P_5(x)}dx\right. = \frac{11}{16}$$

$$f(x) = c_0P_0(x)+c_1P_1(x)+c_2P_2(x)+\ldots$$

$$= \frac{3}{2}P_1(x)-\frac{7}{8}P_3(x)+\frac{11}{16}P_5(x)+\ldots$$

Rješ

NE TREBA IH UVRSTITI AKO JE ZADATAK TAKO ZADAN

Pr. IZRAČUNAJ TREĆI FOURIEROV POLINOM FJE f PO LEG. POLINOMIMA

$$S_3(f) = c_0P_0(x)+c_1P_1(x)+c_2P_2(x)+c_3P_3(x)$$

I POLINOME UVRSTITI

$$= \frac{3}{2}x-\frac{7}{8}\left(\frac{5}{2}x^3-\frac{3}{2}x\right) = -\frac{35}{16}x^3+\frac{45}{16}x$$

FOURIEROV RED PO ČEBIŠEVLJEVIM POLINOMIMA

– PODSJETNIK $[-1,1]$, $\rho(x)=\frac{1}{\sqrt{1-x^2}}$

$T_0(x)=1$

$T_1(x)=x$

$T_2(x)=2x^2-1$

$T_3(x)=4x^3-3x$

$$T_n(x) = \cos[n\cdot\arccos x]$$

$$T_{n+1}(x) = 2xT_n(x)-T_{n-1}(x)$$

$$f(x)=\sum_{n=0}^{\infty}c_nT_n(x), \quad c_n=\frac{(f|T_n)}{\|T_n\|^2}$$

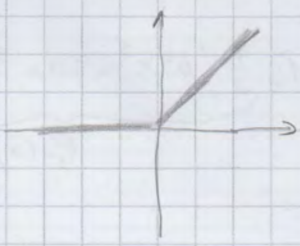
$$\boxed{\|T_0\|^2=\pi, \quad \|T_n\|^2=\frac{\pi}{2}, \ n\geq 1}$$

$$(f|T_n) = \int_{-1}^{1}\frac{f(x)\cdot T_n(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1}\frac{f(x)\cdot\cos[n\arccos x]}{\sqrt{1-x^2}}dx = \left|\begin{matrix}x=\cos t\\ dx=-\sin t\,dt\end{matrix}\right| =$$

$$= \int_0^{\pi} f(\cos t)\cos(nt)\,dt \Rightarrow \boxed{\begin{aligned} c_0 &= \frac{1}{\pi}\int_0^{\pi} f(\cos t)\,dt \\ c_n &= \frac{2}{\pi}\int_0^{\pi} f(\cos t)\cos nt\,dt\end{aligned}}$$

ZAD] RAZVIJ U FOURIJ. RED PO ČEBIŠ. POL. FJU

$$f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x, & 0 \le x < 1 \end{cases}$$



1 korak ?

$$f(\cos t) = \begin{cases} 0, & \frac{\pi}{2} \le t \le \pi \\ \cos t, & 0 \le t \le \frac{\pi}{2} \end{cases}$$

$$c_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos t \, dt = \frac{1}{\pi}$$

$$c_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos t \cos nt \, dt = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (\cos(n+1)t + \cos(n-1)t)\, dt =$$

$$= \frac{1}{\pi}\left(\frac{\sin\frac{\pi}{2}(n+1)}{n+1} + \frac{\sin\frac{\pi}{2}(n-1)}{n-1} \right) = -\frac{2}{\pi}\cdot\frac{\cos\frac{n\pi}{2}}{n^2-1} \quad \text{ZA } \boxed{n \neq 1}$$

$$c_1 = -\frac{2}{\pi}\lim_{n\to 1}\frac{\cos\frac{n\pi}{2}}{n^2-1} = -\frac{2}{\pi}\lim_{n\to 1}\frac{-\frac{\pi}{2}\sin n\frac{\pi}{2}}{2n} = \frac{1}{2}$$

$$c_{2n} = -\frac{2}{\pi}\frac{\cos(n\pi)}{4n^2+1} = -\frac{2}{\pi}\frac{(-1)^n}{4n^2-1}$$

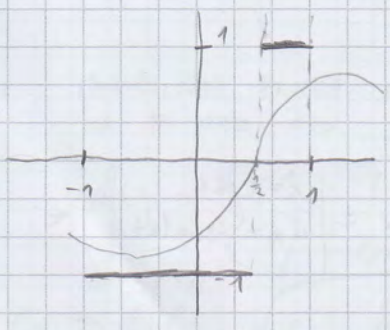
$$c_{2n+1} = \emptyset$$

$$\Rightarrow \boxed{f(x) = \frac{1}{\pi}T_0(x) + \frac{1}{2}T_1(x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}T_{2n}(x)}$$

5.02
20.) RAZVIJ U F. RED PO ČEB. POL. FUNKCIJU

$f(x) = \operatorname{sgn}(\sin(x - \frac{1}{2}))$, POMOĆU DOBIVENOG RAZVOJA IZRAČ. SUMU

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \dots$$



$$f(\cos t) = \begin{cases} -1, & \frac{\pi}{3} \le t \le \pi \\ 1, & 0 \le t \le \frac{\pi}{3} \end{cases}$$

$$c_0 = \frac{1}{\pi}\left[\int_0^{\frac{\pi}{3}} 1\, dt + \int_{\frac{\pi}{3}}^{\pi} (-1)\, dt\right] = -\frac{1}{3}$$

$$c_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{3}} 1\cdot\cos nt \, dt + \int_{\frac{\pi}{3}}^{\pi} -1\cos nt\, dt\right] = +\frac{4}{\pi}\frac{\sin\frac{n\pi}{3}}{n}$$

$$f(x) = -\frac{1}{3}T_0(x) + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n}T_n(x)$$

$\sin \frac{n\pi}{3} \Rightarrow \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, \dots$

$T_n(x) = \cos nt \quad , \quad t = \arccos x$

Ex $\boxed{x = \frac{1}{2}}$

$\cos \frac{n\pi}{3} \Rightarrow \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, 1 \dots$

$+ \quad - \quad \cancel{+} \quad + \quad - \quad + \quad - \quad +$

$f\left(\frac{1}{2}\right) = -\frac{1}{3} T_0\left(\frac{1}{2}\right) + \frac{4}{\pi} \cdot \frac{\sqrt{3}}{\cancel{2}} \frac{1}{\cancel{2}} \cdot S$

$0 = -\frac{1}{3} + \frac{\sqrt{3}}{\pi} \cdot S \quad \Rightarrow \quad \boxed{S = \frac{\pi}{3\sqrt{3}}}$

Ry.