3 Elementarne funkcije

16. **A)** Opća eksponencijalna funkcija definira se kao

$$a^z := e^{z \operatorname{Ln} a}$$

pa imamo:

$$z = 3^{2-i} = e^{(2-i)\operatorname{Ln} 3}$$

$$\operatorname{Ln} 3 = \ln|3| + i(0 + 2k\pi) = \ln 3 + i2k\pi, k \in \mathbb{Z}$$

$$3^{2-i} = e^{(2-i)(\ln 3 + i2k\pi)} = e^{2\ln 3 + 2k\pi + i(4k\pi - \ln 3)} = e^{\ln 9} e^{2k\pi} e^{i(4k\pi - \ln 3)} = 9e^{2k\pi} e^{i(4k\pi - \ln 3)}, k \in \mathbb{Z}$$

$$|z| = 9e^{2k\pi}, k \in \mathbb{Z}$$

$$\arg z = 4k\pi - \ln 3 = -\ln 3$$

B)

$$z = \operatorname{ch}^2(i\ln 3) = [\operatorname{ch}(i\ln 3)]^2 = [\cos(\ln 3)]^2 = \cos^2(\ln 3)$$

$$|z| = \cos^2(\ln 3)$$

$$\arg z = 0$$

C)

$$z = \operatorname{th}(\pi i) = \frac{\operatorname{sh}(\pi i)}{\operatorname{ch}(\pi i)} = \frac{i\sin\pi}{\cos\pi} = 0$$

$$|z| = 0$$

$$\arg z = \infty$$

17. **A)**

$$\operatorname{Ln}(-i) = \ln|-i| + i\left(\frac{3\pi}{2} + 2k\pi\right) = \ln 1 + i\left(\frac{3\pi}{2} + 2k\pi\right) = i\left(\frac{3\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$$

B)

$$z = i^{\operatorname{Ln} i}$$

$$\operatorname{Ln} i = \ln|i| + i\left(\frac{\pi}{2} + 2k\pi\right) = \ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right) = i\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$$

$$i^{\operatorname{Ln} i} = i^{i\left(\frac{\pi}{2} + 2k\pi\right)} = e^{i\left(\frac{\pi}{2} + 2k\pi\right)\operatorname{Ln} i} = e^{i\left(\frac{\pi}{2} + 2k\pi\right)\cdot i\left(\frac{\pi}{2} + 2l\pi\right)} = e^{-\left(\frac{\pi}{2} + 2k\pi\right)\left(\frac{\pi}{2} + 2l\pi\right)}, k, l \in \mathbb{Z}$$

C)

$$\operatorname{Arctg}\left(\frac{i}{3}\right) = -\frac{i}{2}\operatorname{Ln}\left(\frac{1 + i\frac{i}{3}}{1 - i\frac{i}{3}}\right) = -\frac{i}{2}\operatorname{Ln}\left(\frac{1}{2}\right)$$

$$\operatorname{Ln}\left(\frac{1}{2}\right) = \ln\left|\frac{1}{2}\right| + i(0 + 2k\pi) = \ln 2^{-1} + i2k\pi = -\ln 2 + i2k\pi, k \in \mathbb{Z}$$

$$\operatorname{Arctg}\left(\frac{i}{3}\right) = -\frac{i}{2}\operatorname{Ln}\left(\frac{1}{2}\right) = -\frac{i}{2}(-\ln 2 + i2k\pi) = \frac{i}{2}\ln 2 + k\pi = k\pi + i\ln\sqrt{2}, k \in \mathbb{Z}$$

18. **A)**

$$\operatorname{Ln} e = \ln|e| + i(0 + 2k\pi) = \ln e + i2k\pi = 1 + i2k\pi, k \in \mathbb{Z}$$

B)

$$z = 1^{\frac{1}{i}}$$

$$1^{\frac{1}{i}} = e^{\frac{1}{i}\operatorname{Ln} 1}$$

$$\operatorname{Ln} 1 = \ln|1| + i(0 + 2k\pi) = \ln 1 + i2k\pi = i2k\pi, k \in \mathbb{Z}$$

$$1^{\frac{1}{i}} = e^{\frac{1}{i}i2k\pi} = e^{2k\pi}, k \in \mathbb{Z}$$

C)

$$\operatorname{sh}\left(\frac{\pi i}{2}\right) = i\sin\left(\frac{\pi}{2}\right) = i$$

19. A)

$$e^z + i = 0 \to e^z = -i$$

$$z = \mathrm{Ln}\,(-i) = i\left(\frac{3\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$$

Vidi 17. A).

B)

$$i + \mathrm{sh}\,(iz) = 0$$

$$i + i\sin z = 0$$

$$i\,(1 + \sin z) = 0$$

$$1 + \sin z = 0$$

$$\sin z = -1 \to z = \mathrm{Arcsin}\,(-1)$$

$$z = -i\,\mathrm{Ln}\left(-i \pm \sqrt{1-1}\right) = -i\,\mathrm{Ln}\,(-i) = -i \cdot i\left(\frac{3\pi}{2} + 2k\pi\right) = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}$$

20. A) Obzirom da je "mali" ln, radi se o glavnoj logaritamskoj grani. Neka je $z = x + iy$. Imamo:

$$\ln\,(i - z) = \ln\,(-x + i\,(1-y)) = \frac{1}{2}\ln\left(x^2 + 1 - 2y + y^2\right) + i\,\mathrm{arctg}\left(\frac{1-y}{-x}\right) = 1$$

Da bi ovo vrijedilo, mora realni dio biti jednak realnom, a imaginarni dio jednak imaginarnome:

$$\frac{1}{2}\ln\left(x^2 + 1 - 2y + y^2\right) = 1$$

$$\mathrm{arctg}\left(\frac{1-y}{-x}\right) = 0$$

Obratite pažnju na sljedeću stvar: da bi arctg bio jednak 0, to znači da se broj $-x + i\,(1-y)$ mora nalaziti na pozitivnom dijelu realne osi, a to vrijedi ako je $1 - y = 0 \to y = 1$ i $-x > 0 \to x < 0$!!! Kada ubacimo $y = 1$ u prvu jednadžbu, dobijemo:

$$\frac{1}{2}\ln\left(x^2 + 1 - 2 + 1\right) = 1$$

$$\ln\left(x^2\right) = 2$$

$$x^2 = e^2 \to (x - e)(x + e) = 0 \to x_1 = e, x_2 = -e$$

Zbog $x < 0$ odbacujemo $x_1 = e$ i slijedi da je konačno rješenje:

$$z = x + iy = -e + i$$

B)

$$\sin z = \pi i$$

$$z = \mathrm{Arcsin}\,(\pi i) = -i\,\mathrm{Ln}\left(i \cdot \pi i \pm \sqrt{1 - (\pi i)^2}\right) = -i\,\mathrm{Ln}\left(-\pi \pm \sqrt{1 + \pi^2}\right)$$

$$z_1 = -i\,\mathrm{Ln}\left(-\pi + \sqrt{1 + \pi^2}\right)$$

$$\mathrm{Ln}\left(-\pi + \sqrt{1+\pi^2}\right) = \ln\left|-\pi + \sqrt{1+\pi^2}\right| + i\,(0 + 2k\pi) = \ln\left(-\pi + \sqrt{1+\pi^2}\right) + i2k\pi, k \in \mathbb{Z}$$

$$z_1 = -i\left(\ln\left(-\pi + \sqrt{1+\pi^2}\right) + i2k\pi\right) = 2k\pi - i\ln\left(-\pi + \sqrt{1+\pi^2}\right), k \in \mathbb{Z}$$

$$z_2 = -i\,\mathrm{Ln}\left(-\pi - \sqrt{1+\pi^2}\right)$$

$$\mathrm{Ln}\left(-\pi - \sqrt{1+\pi^2}\right) = \ln\left|-\pi - \sqrt{1+\pi^2}\right| + i\,(\pi + 2k\pi) = \ln\left(\pi + \sqrt{1+\pi^2}\right) + i\,(\pi + 2k\pi), k \in \mathbb{Z}$$

$$z_2 = -i\left(\ln\left(\pi + \sqrt{1+\pi^2}\right) + i\,(\pi + 2k\pi)\right) = \pi + 2k\pi - i\ln\left(\pi + \sqrt{1+\pi^2}\right), k \in \mathbb{Z}$$