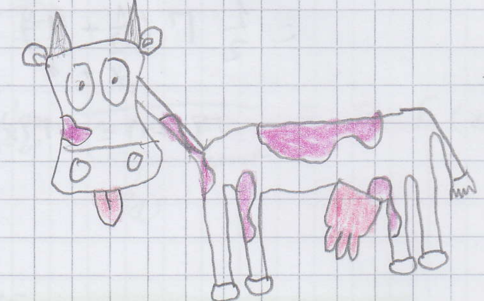
08. 12. 2009.

SPECIJALNE FJE

BURIĆ

FALA ŽELJKI NA BOMBONAMA



GAMA FJA

- def $\quad \Gamma(x) = \int_0^\infty t^{x+1} e^{-t}\,dt \quad , x>0$

Prim $\quad \Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$

$$\Gamma(x+1) = \int_0^t t^x e^{-t}\,dt = \left| \begin{matrix} u = t^x & dv = e^{-t}dt \\ du = x t^{x-1}dt & v = -e^{-t} \end{matrix} \right| = \underbrace{-t^x e^{-t}\Big|_0^\infty}_{\emptyset} + x \underbrace{\int_0^\infty t^{x-1} e^{-t}\,dt}_{\Gamma(x)}$$

→ OSNOVNO SVOJSTVO GAMA FJE $\quad$ (BILO NA ZI) $\quad$ DEF I DOKAZ

$$\boxed{\Gamma(x+1) = x \cdot \Gamma(x)}$$

$x = n \in \mathbb{N}$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) \dots = n!$$

$$\boxed{\begin{matrix} \Gamma(n+1) = n! \\ \Gamma(n) = (n-1)! \end{matrix}}$$

ZD. PRIKAŽI INTEGRAL e^{-x^3} PREKO GAMA FJE

$$\int_0^\infty e^{-x^3}dx = \left| \begin{matrix} x^3 = t \\ 3x^2 dx = dt \\ dx = \frac{dt}{3x^2} \end{matrix} \right| = \frac{1}{3}\int_0^\infty t^{-\frac{2}{3}} e^{-t}\,dt = \frac{1}{3}\Gamma\left(\frac{4}{3}\right)$$

PROŠLE GOD. ZA 3B

$^{\frac{1}{3}-1}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt = \left| \begin{matrix} t = u^2 \\ dt = 2u\,du \end{matrix} \right| = 2\int_0^\infty \frac{1}{\cancel{u}} e^{-u^2} \cancel{u}\,du = 2\int_0^\infty e^{-u^2}du = \sqrt{\pi}$$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

FJA GUSTOĆE

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = \left| \begin{matrix} x = u\sqrt{2} \\ dx = du\sqrt{2} \end{matrix} \right| = \frac{1}{\sqrt{\pi}}\, 2\int_0^\infty e^{-u^2}du = 1$$

IZVOD ZA PRIM IZNAD

ZAD.) PRIKAŽI PREKO Γ I IZRAČUNAJ DO KRAJA... INTEGRAL

$$\int_0^\infty x^2 e^{-x^2} dx = \left| \begin{matrix} t = x^2 \\ 2x\,dx = dt \end{matrix} \right| = \frac{1}{2} \int_0^\infty t^{\frac{1}{2}} e^{-t} dt = \frac{1}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2}\Gamma\left(\underbrace{\frac{1}{2}+1}_{x+1}\right) = \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}\sqrt{\pi}}$$

$\Gamma(x+1) = x\Gamma(x)$

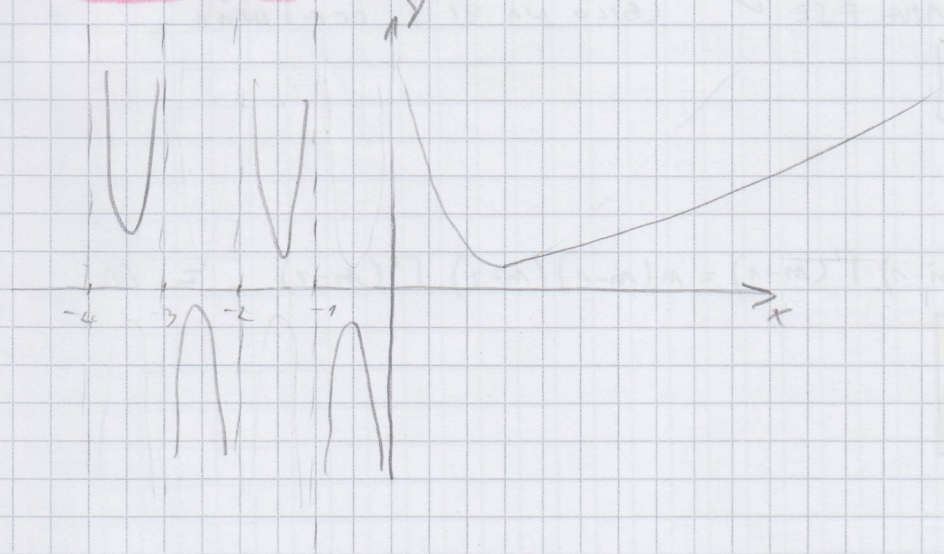
PROŠIRENJE NA NEGATIVNU OS

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad ZA \quad x \in (-1, 0)$$

$\rightarrow$ ITERATIVNO SE PROŠIRI NA SVE $x<0$, OSIM ZA $0, -1, -2, -3, -4, \ldots$

SINGULARITETI

GRAF Γ FJE



PROŠ. NA $\mathbb{C}$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad \operatorname{Re} z > 0$$

def

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2 \cdots n}{z(z+1)\cdots(z+n)} \cdot n^z$$

$$\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2 \cdots n}{(z+1)(z+2)\cdots(z+1+n)} n^{z+1} \cdot \frac{z}{z}$$

$$= z \cdot \lim_{n\to\infty} \frac{1\cdot 2 \cdots n}{z(z+1)\cdots(z+n)} n^z \cdot \lim_{n\to\infty} \underbrace{\frac{n}{n+1+\underset{\text{KONST.}}{z}}}_{1} = z \cdot \Gamma(z)$$

TM: OBJE DEF GAME SU EKVIVALENTNE

BILO NA ZI

DOKAZ

$$f(z,n) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \left| \begin{array}{l} s = \frac{t}{n} \\ ds = \frac{1}{n}dt \end{array} \right| = \int_0^1 (1-s)^n n^{z-1} s^{z-1} \cdot n\,ds =$$

$$= n^z \int_0^1 \underbrace{(1-s)^n}_{u} \underbrace{s^{z-1}}_{dv} ds = \left| \begin{array}{l} u = (1-s)^n \\ du = n(1-s)^{n-1} \\ dv = s^{z-1}ds \\ v = \frac{s^z}{z} \end{array} \right| = n^z \cdot \left(\frac{s^z}{z}(1-s)^n \Big|_0^1 + \int_0^1 \right.$$

$$\left. + \frac{n}{z}\int_0^1 (1-s)^{n-1} \cdot s^z\,dz \right) = \left| \begin{array}{l} \text{PONAVLJAJUĆI} \\ \text{POSTUPAK} \end{array} \right| =$$

$$= n^z \cdot \frac{n(n-1)(n-2)\cdots 1}{z(z+1)(z+2)\cdots(z+n-1)} \int_0^1 s^{z+n-1}\,ds$$

$$f(z,n) = n^z \cdot \frac{1\cdot 2 \cdots n}{z(z+1)\cdots(z+n)} \quad / \lim_{n\to\infty}$$

$$\lim_{n\to\infty} f(z,n) = \lim_{n\to\infty} \int_0^n \left(1-\frac{z}{n}\right)^n t^{z-1}\,dt = \int_0^\infty e^{-t} t^{z-1}\,dt = \Gamma(z) =$$

$$= \lim_{n\to\infty} \frac{1\cdot 2 \cdots n}{z(z+1)\cdots(z+n)} \cdot n^z$$

$$\Gamma(n+1) = n! \qquad \Pi(z) = \Gamma(z+1)$$

$$\Pi(n) = n! \qquad \frac{1}{\Pi(z)} = \frac{1}{\Gamma(z+1)} = \frac{1}{z\Gamma(z)} \rightarrow \boxed{\Gamma(z)} \ \boxed{\text{NEMA NULTOČKU}}$$

TM (WEIERSTRASSOV BESKONAČAN PRODUKT)

$$\frac{1}{\Gamma(z)} = z \cdot e^{\gamma z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$ PRI ČEMU JE γ EULER-MASCHERONIJEVA KONSTANTA

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n\right) = 0.577\dots$$

DOKAZ
$$\Gamma(z) = \lim_{n\to\infty}\frac{1\cdot 2\cdots n}{z(z+1)\cdots(z+n)} = \frac{1}{z}\lim n^{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}$$

$$\frac{z+1}{1} = \frac{z}{1}+1, \quad \frac{z+n}{n} = \frac{z}{n}+1 \qquad n^{z} = e^{z\ln n}$$

$$\frac{1}{\Gamma(z)} = z\cdot\lim_{n\to\infty} e^{-z\ln n}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}+\frac{z}{k}}$$

$$= z\lim_{n\to\infty} e^{z\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n\right)}\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}$$

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} \;//$$

$$1+\frac{z}{k} = 0 \Rightarrow \boxed{z=-k},\; k\in\mathbb{N} \rightarrow$$ POLOVI 1. REDA

NULTOČKE OD $\frac{1}{\Gamma(z)}$ $(0,-1,-2,-3,\dots)$

TM:
$$\Gamma(z) = \int_1^{\infty} t^{z-1}e^{-t}\,dt + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\cdot\frac{1}{z+k} \;//$$

TOČKE $z=-k$ $(k=0,1,2,3,\dots)$ SU POLOVI 1. REDA I VRIJEDI

$$\operatorname{Res}(\Gamma(z),-k) = \frac{(-1)^k}{k!}$$

DOKAZ:
$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt = \int_0^1\dots + \int_1^{\infty}\dots$$

$$\int_0^1 t^{z-1}e^{-t}\,dt = \int_0^1\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}t^{z-1}\,dt \underset{\text{PMF}}{=} \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^1 t^{k+z-1}\,dt = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\cdot\frac{1}{z+k}$$

$$P(z) = (z-c_1)(z-c_2)\dots(z-c_n)$$

$$f(z) = (z-c)^n \varphi(z)$$

$$\varphi(0) \neq 0$$

$$\sin z = z\underbrace{(z-\pi)(z+\pi)}_{(z^2-\pi^2)}\underbrace{(z+2\pi)(z-2\pi)}_{(z^2-(2\pi)^2)}\dots\dots = \lim_{n\to\infty} z(z^2-\pi^2)\dots(z^2-n^2\pi^2)$$

$$= z\lim_{n\to\infty} c\prod_{k=1}^{n}\left(1-\frac{z^2}{k^2\pi^2}\right.$$

$$\boxed{\sin z = z\cdot\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right)}$$

SVOJSTVO SIMETRIJE

$$\frac{1}{\Gamma(z)\,\Gamma(-z)} = z\,e^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\cdot(-z)\,e^{-\gamma z}\prod_{k=1}^{\infty}\left(1-\frac{z}{k}\right)e^{\frac{z}{k}}$$

$$= z^2\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) = -\frac{z}{\pi}\cdot\pi z\cdot\prod_{k=1}^{\infty}\left(1-\frac{(z\pi)^2}{\pi^2k^2}\right)$$

$$= -\frac{z}{\pi}\sin(\pi z) \to \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

$$\Gamma(1-z) = -z\,\Gamma(-z)$$
$$\to \Gamma(-z) = \frac{-\Gamma(1-z)}{z}$$

Π