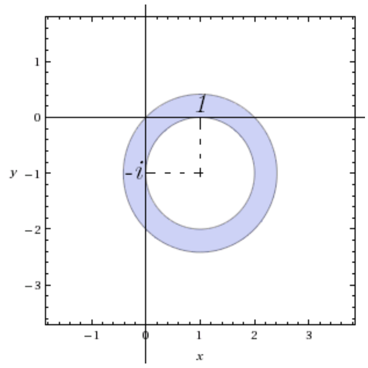
1 Kompleksna ravnina

1. A)

$$1 < |z - (1 - i)| < \sqrt{2}$$

Kružni vijenac polumjera 1 i $\sqrt{2}$ sa središtem u S(1, -i). Rubovi nisu uključeni.



B)

$$z = x + yi$$

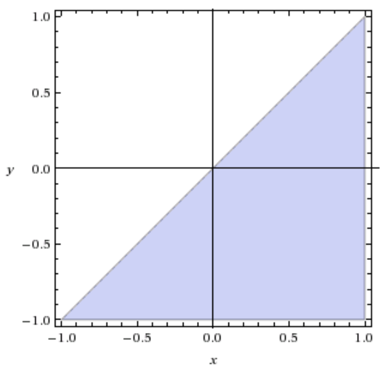
$$|x - 1 + yi| < |x + (y - 1)i|$$

$$\sqrt{(x-1)^2 + y^2} < \sqrt{x^2 + (y-1)^2}$$

$$x^2 - 2x + 1 + y^2 < x^2 + y^2 - 2y + 1$$

$$-2x < -2y \rightarrow y < x$$

Poluravnina $y < x$. Rub nije uključen.



C)

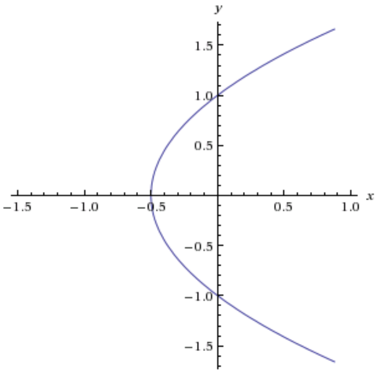
$$z = x + yi$$

$$x + 1 = \sqrt{x^2 + y^2}$$

$$x^2 + 2x + 1 = x^2 + y^2$$

$$x = \frac{y^2 - 1}{2}$$

Parabola.
Uvjet: $x + 1 \geq 0 \rightarrow x \geq -1$



2. A)

$$z = x + yi$$

$$|x - 1 + (y + 2)i| = |x + 3 + (y - 1)i|$$

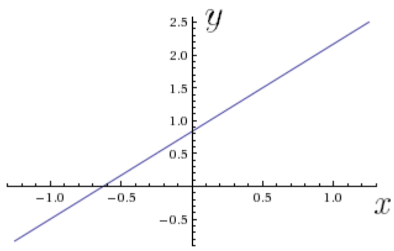
$$\sqrt{(x - 1)^2 + (y + 2)^2} = \sqrt{(x + 3)^2 + (y - 1)^2}$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = x^2 + 6x + 9 + y^2 - 2y + 1$$

$$-8x + 6y - 5 = 0$$

$$y = \frac{4}{3}x + \frac{5}{6}$$

Pravac.



B)

$$z = x + yi$$

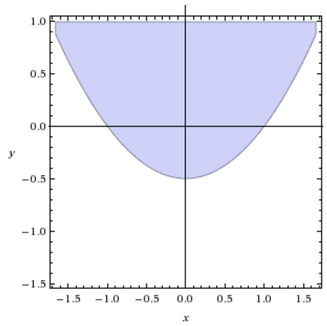
$$\sqrt{x^2 + y^2} < y + 1$$

$$x^2 + y^2 < y^2 + 2y + 1$$

$$y > \frac{x^2 - 1}{2}$$

Uvjet: $y + 1 \geq 0 \rightarrow y \geq -1$

Dio ravnine iznad parabole (parabola nije uključena).



C)

$$z = x + yi$$

$$x^2 + y^2 = |1 + x^2 - y^2 + 2xyi| = \sqrt{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

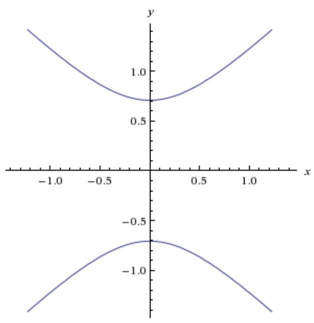
$$x^4 + 2x^2y^2 + y^4 = x^4 + y^4 + 1 - 2x^2y^2 + 2x^2 - 2y^2 + 4x^2y^2$$

$$0 = 1 + 2x^2 - 2y^2$$

$$2y^2 - 2x^2 = 1$$

$$\frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

Hiperbola. $a = b = \frac{1}{\sqrt{2}}$. Tjemena su: $T_{1,2} = \left(\pm\frac{1}{\sqrt{2}}, 0\right)$.



3. A)

$$z = x + yi$$

$$\frac{z-1}{z-i} = \frac{x - 1 + yi}{x + (y-1)i} \cdot \frac{x - (y-1)i}{x - (y-1)i} = \frac{x^2 - x + y^2 - y + i[x + y - 1]}{x^2 + (y-1)^2}$$

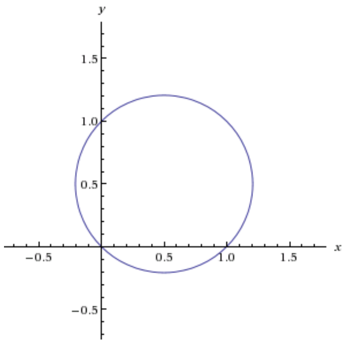
$$\mathrm{Re}\left(\frac{z-1}{z-i}\right) = \frac{x^2 - x + y^2 - y}{x^2 + (y-1)^2} = 0$$

$$x^2 - x + y^2 - y = 0$$

$$x^2 - 2x \cdot \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + y^2 - 2y \cdot \frac{1}{2} + \frac{1}{4} - \frac{1}{4} = 0$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{1}{\sqrt{2}}\right)^2$$

Kružnica sa središtem u $S\left(\frac{1}{2},\frac{1}{2}i\right)$ polumjera $\frac{1}{\sqrt{2}}$.



B)

$$z=x+yi$$

$$|x^2+(y-3)i|+|x^2+(y+3)i|=8$$

$$\sqrt{x^2+(y-3)^2}+\sqrt{x^2+(y+3)^2}=8$$

$$\sqrt{x^2+(y-3)^2}=8-\sqrt{x^2+(y+3)^2}$$

$$x^2+y^2-6y+9=64-16\sqrt{x^2+y^2+6y+9}+x^2+y^2+6y+9$$

$$12y+64=16\sqrt{x^2+y^2+6y+9}$$

$$3y+16=4\sqrt{x^2+y^2+6y+9}$$

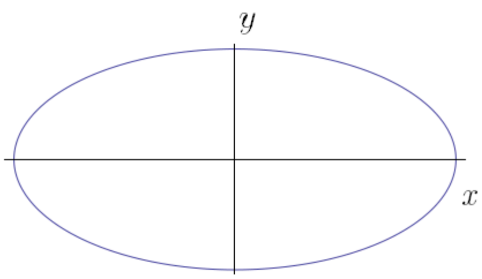
$$9y^2+96y+256=16x^2+16y^2+96y+144$$

$$16x^2+7y^2=112$$

$$\frac{x^2}{7}+\frac{y^2}{16}=1$$

Elipsa.

Uvjet: $3y+16\geq 0\rightarrow y\geq -\frac{16}{3}$



C)

$$z = x + yi$$

$$|x - 3 + yi| - |x + 3 + yi| > 4$$

$$\sqrt{(x-3)^2 + y^2} - \sqrt{(x+3)^2 + y^2} > 4$$

$$\sqrt{(x-3)^2 + y^2} > 4 + \sqrt{(x+3)^2 + y^2}$$

$$x^2 - 6x + 9 + y^2 > 16 + 8\sqrt{x^2 + 6x + 9 + y^2} + x^2 + 6x + 9 + y^2$$

$$-12x - 16 > 8\sqrt{x^2 + 6x + 9 + y^2}$$

$$-3x - 4 > 2\sqrt{x^2 + 6x + 9 + y^2}$$

$$9x^2 + 24x + 16 > 4x^2 + 24x + 36 + 4y^2$$

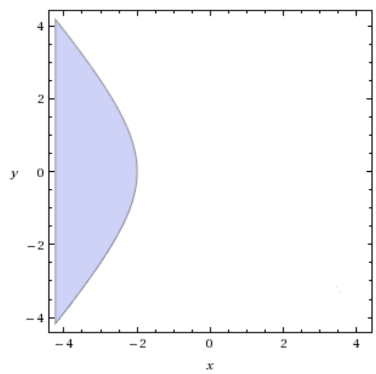
$$5x^2 - 4y^2 > 20$$

$$\frac{x^2}{4} - \frac{y^2}{5} > 1$$

Hiperbola.

Uvjet: $-3x - 4 \geq 0 \rightarrow x \leq -\frac{4}{3}$

Dio ravnine lijevo od lijeve grane hiperbole.



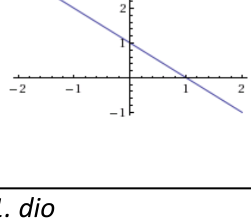
4. A) (Zad. 3. A))

$$\mathrm{Im}\left(\frac{z-1}{z-i}\right) = \frac{x + y - 1}{x^2 + (y-1)^2} = 0$$

$$x + y - 1 = 0$$

$$y = 1 - x$$

Pravac.



B)

$$z = x + yi$$

$$|x^2 + (y-1)i| - |x^2 + (y+1)i| = 1$$

$$\sqrt{x^2 + (y-1)^2} - \sqrt{x^2 + (y+1)^2} = 1$$

$$\sqrt{x^2 + (y-1)^2} = 1 + \sqrt{x^2 + (y+1)^2}$$

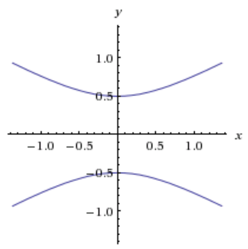
$$x^2 + y^2 - 2y + 1 = 1 + 2\sqrt{x^2 + y^2 + 2y + 1} + x^2 + y^2 + 2y + 1$$

$$-4y - 1 = 2\sqrt{x^2 + y^2 + 2y + 1}$$

$$16y^2 + 8y + 1 = 4x^2 + 4y^2 + 8y + 4$$

$$12y^2 - 4x^2 = 3$$

Hiperbola.



C)

$$z = x + yi$$

$$|z - 1| + |z + 1| = \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2}$$

(1)

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} > 4$$

$$\sqrt{(x-1)^2 + y^2} > 4 - \sqrt{(x+1)^2 + y^2}$$

$$x^2 - 2x + 1 + y^2 > 16 - 8\sqrt{x^2 + 2x + 1 + y^2} + x^2 + 2x + 1 + y^2$$

$$-4x - 16 > -8\sqrt{x^2 + 2x + 1 + y^2}$$

$$x + 4 < 2\sqrt{x^2 + 2x + 1 + y^2}$$

$$x^2 + 8x + 16 < 4x^2 + 8x + 4 + 4y^2$$

$$3x^2 + 4y^2 > 12$$

$$\frac{x^2}{4} + \frac{y^2}{3} > 1$$

(2)

$$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2}<8$$

$$\sqrt{(x-1)^2+y^2}<8-\sqrt{(x+1)^2+y^2}$$

$$x^2-2x+1+y^2<64-16\sqrt{x^2+2x+1+y^2}+x^2+2x+1+y^2$$

$$-4x-64<-16\sqrt{x^2+2x+1+y^2}$$

$$x+16>4\sqrt{x^2+2x+1+y^2}$$

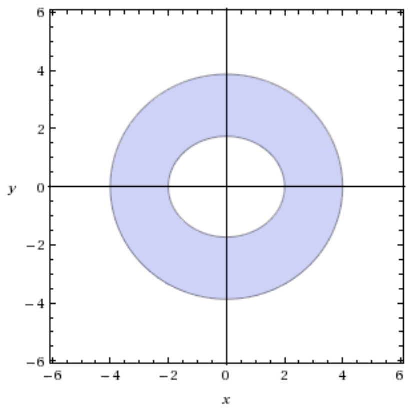
$$x^2+32x+256>16x^2+32x+16+16y^2$$

$$15x^2+16y^2<240$$

$$\frac{x^2}{16}+\frac{y^2}{15}<1$$

Uvjet: $x+16\geq 0 \to x\geq -16$

Eliptički vijenac između elipsa (1) i (2).



5. A)

$$z(t)=2e^{-it},\ \ t\in[0,2\pi]$$

Jednadžbom $z(t)=Re^{it}, t\in[0,2\pi]$, zadana je kružnica polumjera R, kretanjem u smjeru obrnutom od smjera kazaljke na satu. Zbog $-it$, smjer kretanja će biti u smjeru kazaljke na satu (JEDAN obilazak). Odredimo neke karakteristične točke:

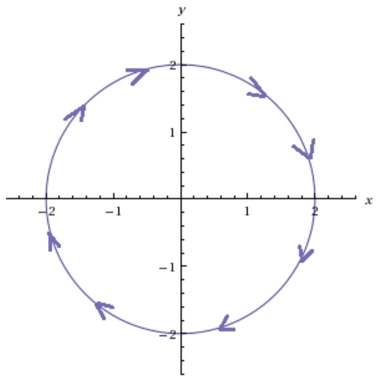
$$z(0)=2$$

$$z\left(\frac{\pi}{2}\right)=-2i$$

$$z(\pi)=-2$$

$$z\left(\frac{3\pi}{2}\right)=2i$$

$$z(2\pi)=2$$



B)

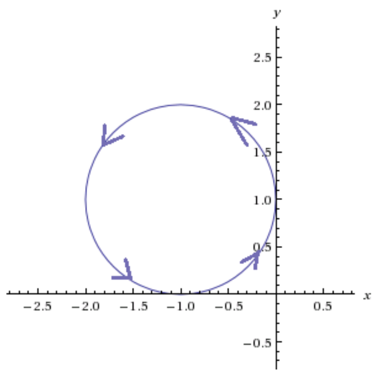
$$z(t) = e^{2it} - 1 + i, \quad t \in [0,2\pi]$$

Jednadžbom $z(t) = z_0 + Re^{2it}, t \in [0,\pi]$, zadan je jedan obilazak po kružnici polumjera R sa središtem u z_0, kretanjem u smjeru obrnutom od smjera kazaljke na satu.

Jednadžbom $z(t) = z_0 + Re^{2it}, t \in [0,2\pi]$, zadana su dva obilaska po kružnici polumjera R, kretanjem u smjeru obrnutom od smjera kazaljke na satu.

$$z(t) = e^{2it} + (-1 + i), \quad t \in [0,2\pi]$$

Znači, dva obilaska po kružnici sa središtem u $z_0 = (-1, i)$ polumjera 1 u smjeru obrnutom od smjera kazaljke na satu.



C)

$$z(t) = 1 + i\cos t, \quad t \in [0,2\pi]$$

Realni dio je konstantan i jednak je 1. Imaginarni dio mijenja vrijednosti od 1 prema -1, i od -1 prema 1. Ovo je segment od (1, i) do (1, -i). Obzirom da ćemo na početku krenuti od točke (1, i) prema točci (1, -i), i vratiti se u točku (1, i), to znači da ćemo segment proći dva puta.