II DOMAĆA ZADAĆA

1.) $p \geq 1$ Naći sve $a \in \mathbb{R}$ t.d.

(a) $x^a \in L^p(\langle 0,1\rangle)$

(b) $x^a \in L^p(\langle 1,+\infty\rangle)$

(c) $x^a \in L^p(\langle 0,+\infty\rangle)$

(a) Treba odrediti za koje $a \in \mathbb{R}$ vrijedi $\int_0^1 (x^a)^p dx < \infty$

1° $ap = -1$ tj. $a = -\frac{1}{p}$ tada je

$$\int_0^1 x^{ap} dx = \int_0^1 \frac{1}{x} dx = \ln 1 - \lim_{b \to 0} \ln b = +\infty$$

2° $ap \neq -1$ tada je

$$\int_0^1 x^{ap} dx = \frac{1}{ap+1} - \lim_{b \to 0} \frac{x^{ap+1}}{ap+1} = \frac{1}{ap+1} - \frac{1}{ap+1} \lim_{b \to 0} x^{ap+1}$$

Ako je $ap+1 > 0$, tada je $\lim\limits_{b \to 0} x^{ap+1} = 0$

Ako je $ap+1 < 0$, tada je $\lim\limits_{b \to 0} x^{ap+1} = +\infty$

Zaključujemo da je $x^a \in L^p(\langle 0,1\rangle) \iff \boxed{a > -\frac{1}{p}}$

(b) Za koje $a \in \mathbb{R}$ vrijedi $\int_1^{+\infty} (x^a)^p dx < \infty$?

1° $ap = -1$ tj. $a = -\frac{1}{p}$

$$\int_1^{+\infty} x^{ap} dx = \int_1^{+\infty} \frac{1}{x} dx = \lim_{b \to +\infty} \ln b - \ln 1 = +\infty$$

2° $ap \neq -1$, tada je

$$\int_1^{+\infty} x^{ap} dx = \frac{1}{ap+1} \lim_{b \to +\infty} x^{ap+1} - \frac{1}{ap+1}$$

Ako je $ap+1 > 0 \Rightarrow \lim\limits_{b \to +\infty} x^{ap+1} = +\infty$

Ako je $ap+1 < 0 \Rightarrow \lim\limits_{b \to +\infty} x^{ap+1} = 0$

Dakle, $x^a \in L^p(\langle 1,+\infty\rangle) \iff \boxed{a < -\frac{1}{p}}$

(c) Koristeći (a) i (b) lako se vidi da $x^a \notin L^r(\langle 0, +\infty\rangle)$ niti za jedan $a \in \mathbb{R}$.

2.) Pokažite da je $f(x) = \dfrac{x^{-2} + x^{-3} + x^{-5}}{1 + x^{-1}} \in L^1(\langle 1, +\infty\rangle)$

$$\frac{f(x)}{x^{-2}} = \frac{1 + x^{-1} + x^{-3}}{1 + x^{-1}} = 1 + \frac{x^{-3}}{1 + x^{-1}} = 1 + \frac{1}{\underbrace{x^3 + x^2}_{\leq 1 \text{ za } x \geq 1}} \leq 2$$

$\Rightarrow f(x) \leq 2x^{-2}$ za $x \geq 1$

$$\int_1^{+\infty} |f(x)|\,dx \leq \int_1^{+\infty} 2x^{-2}\,dx = -2\frac{1}{x}\Big|_1^{+\infty} = 2$$

$\Rightarrow f \in L^1(\langle 1, +\infty\rangle)$