ZZV 13

Zad 1

1) $y'' + y = x \cdot \sin x$

y_H: $r^2+1=0$ $r=\pm i$

$$\boxed{y_H = C_1\cos x + C_2\sin x}$$

$$\boxed{y = y_H + y_p}$$

$y_p = (Ax+B)\cdot x\,[h_1\cos x + h_2 \sin x]$

$= A h_1 x^2\cos x + A h_2 x^2 \sin x + B h_1 x \cos x + B h_2 x \sin x$

$y_p' = 2A h_1 x\cos x - A h_1 x^2 \sin x + 2A h_2 x \sin x + A h_2 x^2 \cos x$

$+ B h_1 \cos x - B h_1 x \sin x + B h_2 \sin x + B h_2 x \cos x$

$= (2Ah_1 + Bh_2) x\cos x + (2Ah_2 - Bh_1)x\sin x$

$+ Ah_2 x^2\cos x - Ah_1 x^2 \sin x + Bh_1\cos x + Bh_2 \sin x$

$y_p'' = (2Ah_1 + Bh_2)\cos x - (2Ah_1+Bh_2)x\sin x + (2Ah_2 - Bh_1)\sin x$

$+ (2Ah_2 - Bh_1)x\cos x + 2Ah_2 x\cos x - Ah_2 x^2 \sin x$

$- 2Ah_1 x \sin x - Ah_1 x^2\cos x - Bh_1\sin x + Bh_2\cos x$

$= (2Ah_1 + 2Bh_2)\cos x + (2Ah_2 - 2Bh_1)\sin x$

$- (4Ah_1 + Bh_2)x\sin x + (4Ah_2 - Bh_1)x\cos x$

$- Ah_2 x^2 \sin x - Ah_1 x^2 \cos x$

$y_p'' + y_p = x\cdot\sin x$

$(Ah_1 - \cancel{Ah_1})x^2\cos x + (Ah_2 - \cancel{Ah_2})x^2\sin x + (\cancel{Bh_1} + 4Ah_2 - \cancel{Bh_1})x\cos x$

$+ (\cancel{Bh_2} - 4Ah_1 - \cancel{Bh_2})x\sin x + (2Ah_1 + 2Bh_2)\cos x + (2Ah_2 - 2Bh_1)\sin x$

$= x\cdot \sin x$

$4Ah_2 x\cos x - 4Ah_1 x\sin x + 2(Ah_1 + Bh_2)\cos x + 2(Ah_2 - Bh_1)\sin x$

$= 1x \cdot \sin x$

$-4Ah_1 = 1$ $\quad Ah_1 + Bh_2 = 0$ $\quad 4Ah_2 = 0$ $\quad Ah_2 - Bh_1 = 0$

$A h_1 = -\frac{1}{4}$ $\quad Bh_2 = \frac{1}{4}$ $\quad A h_2 = 0$ $\quad Bh_1 = 0$

$$\boxed{y_p = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x}$$

2) $y'' + 2y' + 5y = e^{-x}\cos 2x$

y_H $\quad r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$$\boxed{y_H = C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x}$$

$y_p = x \cdot e^{-x}\left[A \cdot \cos 2x + B \cdot \sin 2x\right] = Axe^{-x}\cos 2x + Bxe^{-x}\sin 2x$

$y_p' = Ae^{-x}\cos 2x - Axe^{-x}\cos 2x - 2Axe^{-x}\sin 2x$

$+ Be^{-x}\sin 2x - Bxe^{-x}\sin 2x + 2Bxe^{-x}\cos 2x$

$= (2B - A)xe^{-x}\cos 2x + (-2A - B)xe^{-x}\sin 2x$

$+ Ae^{-x}\cos 2x + Be^{-x}\sin 2x$

$y_p'' = (2B - A)e^{-x}\cos 2x + (A - 2B)xe^{-x}\cos 2x + (2A - 4B)xe^{-x}\sin 2x$

$+ (-2A - B)e^{-x}\sin 2x + (2A + B)xe^{-x}\sin 2x + (-4A - 2B)xe^{-x}\cos 2x$

$- Ae^{-x}\cos 2x - 2Ae^{-x}\sin 2x - Be^{-x}\sin 2x + 2Be^{-x}\cos 2x$

$= (2B - A - A + 2B)e^{-x}\cos 2x + (-2A - B - 2A - B)e^{-x}\sin 2x$

$+ (A - 2B - 4A - 2B)xe^{-x}\cos 2x + (2A - 4B + 2A + B)xe^{-x}\sin 2x$

$= (-2A + 4B)e^{-x}\cos 2x + (-4A - 2B)e^{-x}\sin 2x$

$+ (-3A - 4B)xe^{-x}\cos 2x + (4A - 3B)xe^{-x}\sin 2x$

$y_p'' + 2y_p' + 5y_p = e^{-x}\cos 2x$

$(-\cancel{2A} + 4B + \cancel{2A})e^{-x}\cos 2x + (-4A - \cancel{2B} + \cancel{2B})e^{-x}\sin 2x$

$\cancel{+ (-3A - 4B + 4B - 2A + 5A)xe^{-x}\cos 2x}$

$\cancel{+ (4A - 3B - 4A - 2B + 5B)xe^{-x}\sin 2x} = e^{-x}\cos 2x$

$4B = 1 \qquad -4A = 0$

$B = \frac{1}{4} \qquad A = 0$

$$\boxed{y_p = \frac{1}{4}xe^{-x}\sin 2x}$$

$y = y_H + y_p = e^{-x}(c_1\cos 2x + c_2\sin 2x)$

$+ \frac{1}{4}xe^{-x}\sin 2x$

3) $y'' + y = \sin(x+\frac{\pi}{3}) = \sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3} = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$

y_H - $r^2 + 1 = 0$
$r = \pm i$ $\Rightarrow \boxed{y_H = C_1 \cos x + C_2 \sin x}$

$y_p = x\,[A\cos x + B\sin x] = Ax\cos x + Bx\sin x$

$y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x$

$y_p'' = -A\sin x - A\sin x - Ax\cos x + B\cos x + B\cos x - Bx\sin x$

$= -2A\sin x + 2B\cos x - Ax\cos x - Bx\sin x$

$y_p'' + y_p = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$

$-2A\sin x + 2B\cos x + \cancel{(A-A)x\cos x} + \cancel{(B-B)x\sin x} = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$

$-2A = \frac{1}{2}$ $\quad 2B = \frac{\sqrt{3}}{2}$

$A = -\frac{1}{4}$ $\quad B = \frac{\sqrt{3}}{4}$ $\Rightarrow \boxed{y_p = -\frac{1}{4}x\cos x + \frac{\sqrt{3}}{4}x\sin x}$

4) $y'' + y' - 2y = 3e^{-2x}$

y_H - $r^2 + r - 2 = 0 \Rightarrow r_1 = 1,\ r_2 = 2$

$\boxed{y_H = c_1 e^{x} + c_2 e^{-2x}}$

$y_p = k\cdot x e^{-2x}$ $\quad y_p' = ke^{-2x} - 2kxe^{-2x}$

$y_p'' = -2ke^{-2x} - 2ke^{-2x} + 4kxe^{-2x}$

$y_p'' + y_p' - 2y_p = 3e^{-2x}$

$-4ke^{-2x} + \cancel{4kxe^{-2x}} + ke^{-2x} \cancel{-2kxe^{-2x}} \cancel{-2kxe^{-2x}} = 3e^{-2x}$

$-3ke^{-2x} = 3e^{-2x} \Rightarrow k = -1$

$\boxed{y_p = -xe^{-2x}}$

5.) $y'' + 2y' + 2y = e^{2x} + 4\sin x\cos x = e^{2x} + 2\sin 2x$

y_H: $r^2 + 2r + 2 = 0 \quad r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$$y_H = \boxed{C_1 e^{-x}\cos x + C_2 e^{-x}\sin x}$$

$y_{P1} = k e^{2x}$, $y_{P1}' = 2k e^{2x}$, $y_{P1}'' = 4k e^{2x}$

$$y_{P1}'' + 2y_{P1}' + 2y_{P1} = e^{2x}$$

$$4k e^{2x} + 4k e^{2x} + 2k e^{2x} = e^{2x}$$

$$10k \cancel{e^{2x}} = \cancel{e^{2x}}$$

$$\boxed{k = \frac{1}{10}} \Rightarrow y_{P1} = \frac{1}{10} e^{2x}$$

$y_{P2} = A\cos 2x + B\sin 2x \quad y_{P2}' = -2A\sin 2x + 2B\cos 2x$

$$y_{P2}'' = -4A\cos 2x - 4B\sin 2x$$

$$y_{P2}'' + 2y_{P2}' + 2y_{P2} = 2\sin 2x$$

$$(-4A + 4B + 2A)\cos 2x + (-4B - 4A + 2B)\sin 2x = 2\sin 2x$$

$4B - 2A = 0$ $\qquad$ $-2B - 4A = 2$

$2B = A$ $\qquad$ $-A - 4A = 2$

$\boxed{B = -\frac{1}{5}}$ $\qquad$ $\boxed{A = -\frac{2}{5}}$

$$\Rightarrow y_{P2} = -\frac{2}{5}\cos 2x - \frac{1}{5}\sin 2x$$

$$y_P = y_{P1} + y_{P2} \Rightarrow \boxed{y_P = \frac{1}{10}e^{2x} - \frac{2}{5}\cos 2x - \frac{1}{5}\sin 2x}$$

6.) $y'' - y = 4x e^{x} + 2\sin x$

$y_H \quad r^2 - 1 = 0 \qquad \boxed{y_H = C_1 e^{x} + C_2 e^{-x}}$

$r = \pm 1$

$y_1 = A\cos x + B\sin x \qquad y_1' = -A\sin x + B\cos x \qquad y_1'' = -A\cos x - B\sin x$

$y_1'' - y_1 = 2\sin x \quad \Rightarrow \quad -2A\cos x - 2B\sin x = 2\sin x$

$-2B = 2 \qquad A = 0$

$B = -1$

$\boxed{y_1 = -\sin x}$

$y_2 = x\cdot(Ax + B)e^{x} = Ax^2 e^{x} + Bx e^{x}$

$y_2' = 2Axe^{x} + Ax^2e^{x} + Be^{x} + Bxe^{x}$

$= (2A+B)xe^{x} + Ax^2e^{x} + Be^{x}$

$y_2'' = (2A+B)e^{x} + (2A+B)xe^{x} + 2Axe^{x} + Ax^2e^{x} + Be^{x}$

$= (2A+2B)e^{x} + (4A+B)xe^{x} + Ax^2e^{x}$

$y_2'' - y = 4xe^{x}$

$(2A+2B)e^{x} + (4A + \cancel{B} - B)xe^{x} + \cancel{(A - A)x^2e^{x}} = 4xe^{x}$

$2A + 2B = 0 \qquad 4A = 4$

$\boxed{B = -1} \qquad \boxed{A = 1}$

$\boxed{y_2 = x^2e^{x} - xe^{x}} \qquad \boxed{y_p = y_1 + y_2 = -\sin x + x^2e^{x} - xe^{x}}$

Zad 2.

1) $y''' - y = 4e^x$ $\quad y_H$: $r^3 - 1 = 0 \quad (r-1)(r^2+r+1)=0$

$r_1 = 1 \quad r_{2,3} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$\boxed{y_H = C_1 e^x + C_2 e^{-\frac{x}{2}} \cos \frac{\sqrt{3}}{2} x + C_3 e^{-\frac{x}{2}} \sin \frac{\sqrt{3}}{2} x}$$

$y_p = kxe^x \quad y_p' = ke^x + kxe^x \quad y_p'' = ke^x + ke^x + kxe^x = 2ke^x + kxe^x$

$y_p''' = 2ke^x + ke^x + kxe^x = 3ke^x + kxe^x$

$y_p''' - y_p = 4e^x \quad 3ke^x + \cancel{kxe^x} - \cancel{kxe^x} = 4e^x$

$\boxed{k = \frac{4}{3}} \Rightarrow \boxed{y_p = \frac{4}{3} xe^x}$

2)

$y''' + y'' - 2y' = x - e^x$

$y_H \quad r^3 + r^2 - 2r = 0 \quad r(r^2+r-2)=0 \quad r(r-1)(r+2)=0$

$y_H = C_1 + C_2 e^x + C_3 e^{-2x}$

$y_1 = x\cdot(Ax+B) = Ax^2 + Bx \quad y_1' = 2Ax + B \quad y_1'' = 2A \quad y_1''' = 0$

$y_1''' + y_1'' - 2y_1' = x \Rightarrow 0 + 2A - 4Ax - 2B = x$

$-4A = 1 \quad 2A - 2B = 0$

$A = -\frac{1}{4} \quad B = A = -\frac{1}{4}$

$$\boxed{y_1 = -\frac{1}{4}x^2 - \frac{1}{4}x}$$

$y_2 = kxe^x \quad y_2' = ke^x + kxe^x \quad y_2'' = 2ke^x + kxe^x \quad y_2''' = 3ke^x + kxe^x$

$y_2''' + y_2'' - 2y_2' = -e^x$

$3ke^x + \cancel{kxe^x} + 2ke^x + \cancel{kxe^x} - 2ke^x - \cancel{2kxe^x} = -e^x$

$k = -\frac{1}{3} \Rightarrow \boxed{y_2 = -\frac{1}{3}xe^x}$

$$\boxed{y_p = y_1 + y_2 = -\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{3}xe^x}$$

3) $y''' + y'' = xe^{-x}$

y_H .. $v^3 + v^2 = 0$

$v^2(v+1) = 0$

$v_{1,2} = 0 \quad v_{1,3} = -1$

$$\boxed{y_H = C_1 + C_2 x + C_3 e^{-x}}$$

$y_p = x(Ax+B)e^{-x} = Ax^2 e^{-x} + Bxe^{-x}$

$y_p' = 2Axe^{-x} - Ax^2 e^{-x} + Be^{-x} - Bxe^{-x}$

$= (2A-B)xe^{-x} - Ax^2 e^{-x} + Be^{-x}$

$y_p'' = (2A-B)e^{-x} + (-2A+B)xe^{-x} - 2Axe^{-x} + Ax^2e^{-x} - Be^{-x}$

$= (2A-2B)e^{-x} + (-4A+B)xe^{-x} + Ax^2e^{-x}$

$y_p''' = (-2A+2B)e^{-x} + (-4A+B)e^{-x} + (4A-B)xe^{-x} + 2Axe^{-x}$

$- Ax^2e^{-x}$

$= (-6A+3B)e^{-x} + (6A-B)xe^{-x} - Ax^2e^{-x}$

$y_p''' + y_p'' = xe^{-x}$

$(-6A+3B+2A-2B)e^{-x} + (6A-\cancel{B}-4A+\cancel{B})xe^{-x} + (\cancel{-A+A})x^2e^{-x}$

$= xe^{-x}$

$-4A + B = 0$

$\boxed{B=2}$

$2A = 1$

$\boxed{A = \frac{1}{2}} \Rightarrow \boxed{y_p = \frac{1}{2}x^2e^{-x} + 2xe^{-x}}$

4) $y''' - y'' = 4\sinh x = 2e^{x} - 2e^{-x}$

y_H .. $v^3 - v^2 = 0 \quad v^2(v-1) = 0$

$$\boxed{y_H = C_1 + C_2 x + C_3 e^{x}}$$

$y_1 = kxe^{x} \quad y_1' = ke^{x} + kxe^{x}$

$y_1'' = 2ke^{x} + kxe^{x}$

$y_1''' = 3ke^{x} + kxe^{x}$

$3ke^{x} + \cancel{kxe^{x}} - 2ke^{x} - \cancel{kxe^{x}} = 2e^{x}$

$\boxed{k=2} \quad \boxed{y_1 = 2xe^{x}}$

$y_2 = ke^{-x} \quad y_2' = -ke^{-x}$

$y_2'' = ke^{-x} \quad y_2''' = -ke^{-x}$

$-2ke^{-x} = -2e^{-x}$

$k = 1 \quad \boxed{y_2 = e^{-x}}$

$y_p = y_1 + y_2 \Rightarrow \boxed{y_p = 2xe^{x} + e^{-x}}$

Zad 3

1) $y'' - y' = \frac{e^x}{1+e^x}$ $\quad y_H: \ r^2 - r = 0 \quad r(r-1) = 0$

$$y_H = C_1 + C_2 e^x$$

$$y = C_1(x) + C_2(x) e^x$$

$$\begin{cases} C_1' \cdot 1 + C_2' e^x = 0 \\ C_1' \cdot 0 + C_2' \cancel{e^x} = \frac{\cancel{e^x}}{1+e^x} \end{cases} \quad C_2' = \frac{1}{1+e^x} \quad C_1' = \frac{-e^x}{1+e^x}$$

$$C_2(x) = \int \frac{dx}{1+e^x} = \left| \begin{array}{l} t = e^x + 1 \\ dt = e^x dx \\ = (t-1)dx \end{array} \right| = \int \frac{dt}{(t-1)t} = -\int \frac{dt}{t} + \int \frac{dt}{t-1}$$

$$= -\ln(e^x+1) + \ln(e^x) =$$

$$= \boxed{-\ln(e^x+1) + x + C_2}$$

$$C_1(x) = -\int \frac{e^x}{1+e^x} dx = -\int \frac{d(1+e^x)}{1+e^x} = \boxed{-\ln|e^x+1| + C_1}$$

$$\boxed{y = C_1 + e^x C_2 - \ln(e^x+1) + e^x \left[x - \ln(e^x+1) \right]}$$

2) $y'' + 4y = \frac{1}{\cos 2x}$ $\quad y_H: \ r^2 + 4 = 0 \quad r = \pm 2i$

$$y_H = C_1 \cos 2x + C_2 \sin(2x)$$

$$\begin{array}{l} C_1' \cos 2x + C_2' \sin 2x = 0 \\ -2C_1' \sin 2x + 2C_2' \cos 2x = \frac{1}{\cos 2x} \end{array} \Rightarrow C_2' = \frac{1}{2\cos^2 2x} + C_1' \frac{\sin 2x}{\cos 2x}$$

$$C_1' \left[\cos 2x + \frac{\sin^2 2x}{\cos 2x} \right] + \frac{\sin 2x}{2\cos^2 2x} = 0 \qquad C_2' = \frac{1}{2\cos^2 2x} \left[1 - \sin^2 2x \right]$$

$$C_1' \left[\frac{\cos^2 2x \cancel{+ \sin^2 2x}}{\cos 2x} \right] = -\frac{\sin 2x}{2\cos^{\cancel{2}} 2x} \qquad = \frac{\cos^2 2x}{2\cos^2 2x} = \frac{1}{2}$$

$$C_1' = -\frac{1}{2} \frac{\sin 2x}{\cos 2x} \qquad C_2(x) = \frac{1}{2} \int dx = \boxed{\frac{1}{2} x + C_2}$$

$$C_1(x) = \frac{1}{4} \int \frac{-2\sin 2x}{\cos 2x} dx = \frac{1}{4} \int \frac{d(\cos 2x)}{\cos 2x} = \frac{1}{4} \ln|\cos 2x| + C_1$$

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{2} x \cdot \sin 2x + \frac{1}{4} \cos 2x \cdot \ln|\cos 2x|}$$

3)

$$y'' - 4y' + 5y = \frac{e^{2x}}{\cos x} \qquad y_H \quad r^2 - 4r + 5 = 0$$

$$r = 2 \pm i$$

$$y_H = C_1 e^{2x}\cos x + C_2 e^{2x}\sin x$$

1) $C_1' \cancel{e^{2x}} \cos x + C_2' \cancel{e^{2x}} \sin x = 0 \Rightarrow C_1' \cos x + C_2' \sin x = 0$

2) $C_1'\left[2\cancel{e^{2x}}\cos x - \cancel{e^{2x}}\sin x\right] + C_2'\left[2\cancel{e^{2x}}\sin x + \cancel{e^{2x}}\cos x\right] = \frac{\cancel{e^{2x}}}{\cos x}$

$$C_1'(2\cos x - \sin x) + C_2'(2\sin x + \cos x) = \frac{1}{\cos x}$$

$$\cancel{2\cdot\left[C_1'\cos x + C_2'\sin x\right]} - C_1'\sin x + C_2'\cos x = \frac{1}{\cos x}$$

3.) $C_2' = \frac{1}{\cos^2 x} + C_1' \frac{\sin x}{\cos x} \Rightarrow 1$

$$C_1'\left[\cos x + \frac{\sin^2 x}{\cos x}\right] + \frac{\sin x}{\cos^2 x} = 0$$

$$C_1' \frac{\cos^2 x + \sin^2 x}{\cos x} = -\frac{\sin x}{\cos^2 x}$$

$$C_1' = -\frac{\sin x}{\cos x} \Rightarrow 2. \quad C_2' = \frac{1}{\cos^2 x}\left[1 - \sin^2 x\right] = 1$$

$$C_1 = \int \frac{d(\cos x)}{\cos x} = \ln|\cos x| + C_1 \qquad C_2 = \int dx = x + C_2$$

$$\boxed{y = C_1 e^{2x}\cos x + C_2 e^{2x}\sin x + e^{2x}\cos x \ln|\cos x| + e^{2x}\sin x \cdot x}$$

4.)

$$y'' - 2y' + y = \frac{e^x}{x^2+1} \qquad y_H \quad r^2 - 2r + 1 = 0 \quad (r-1)^2 = 0$$

$$y_H = C_1 e^x + C_2 x e^x$$

$C_1' \cancel{e^x} + C_2' x \cancel{e^x} = 0 \Rightarrow C_1' + x C_2' = 0$

$C_1' \cancel{e^x} + C_2'(\cancel{e^x} + x\cancel{e^x}) = \frac{\cancel{e^x}}{x^2+1}$

$\underbrace{\left[C_1' + C_2' x\right]}_{0} + C_2' = \frac{1}{x^2+1} \Rightarrow C_2(x) = \arctan x + C_2$

$$C_1(x) = \int -\frac{x}{x^2+1} = -\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = -\frac{1}{2}\ln(x^2+1) + C_1$$

$$\boxed{y = C_1 e^x + C_2 x e^x - \frac{1}{2} e^x \ln(x^2+1) + x e^x \arctan x}$$

8) $y'' + 4y = \cot 2x$ $\qquad y_H$: $r^2 + 4 = 0$ $\quad r_{1,2} = \pm 2i$

$$y_H = C_1 \cos 2x + C_2 \sin 2x$$

$$C_1' \cos 2x + C_2' \sin 2x = 0$$

$$-2C_1' \sin 2x + 2C_2' \cos 2x = \frac{\cos 2x}{\sin 2x}$$

$$C_2' = \frac{1}{2}\frac{1}{\sin 2x} + C_1' \frac{\sin 2x}{\cos 2x}$$

$$C_1'\left[\cos 2x + \frac{\sin^2 2x}{\cos 2x}\right] + \frac{1}{2} = 0$$

$$\boxed{C_1' = -\frac{1}{2}\cos 2x}$$

$$C_2' = \frac{1}{2}\frac{1}{\sin 2x} - \frac{1}{2}\sin 2x$$

$$= \frac{1}{2}\left[\frac{1-\sin^2 2x}{\sin 2x}\right] = \frac{1}{2}\frac{\cos^2 2x}{\sin 2x}$$

$$C_1(x) = -\frac{1}{2}\int \cos 2x \, dx = -\frac{1}{4}\sin 2x + C_1$$

$$C_2(x) = \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}dx = \left|\begin{matrix} t = 2x \\ dt = 2dx \end{matrix}\right| = \frac{1}{4}\int \frac{\cos^2 t}{\sin t}dt$$

$$= \frac{1}{4}\int \frac{1-\sin^2 t}{\sin t}dt = \frac{1}{4}\left[\underbrace{\int \frac{dt}{\sin t}}_{A} - \underbrace{\int \sin t \, dt}_{-\cos t}\right]$$

$$A = \int \frac{dt}{\sin t} = \ln \tan \frac{t}{2}$$

$$C_2(x) = \frac{1}{4}\left(\ln|\tan x| + \cos 2x\right) + C_2$$

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cancel{\cos 2x \sin 2x} + \frac{1}{4}\cancel{\cos 2x \sin 2x}$$

$$+ \frac{1}{4}\sin 2x \ln\left|\tan\frac{x}{2}\right|$$

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4}\sin 2x \ln\left|\tan \frac{x}{2}\right|}$$

6) $y'' + y = \frac{2}{\sin^3 x}$ $\quad y_H \quad r^2+1=0 \Rightarrow r=\pm i$

$y_H = C_1\cos x + C_2 \sin x$

$C_1'\cos x + C_2'\sin x = 0$

$-C_1'\sin x + C_2'\cos x = \frac{2}{\sin^3 x}$

$C_2' = \frac{2}{\cos x \sin^3 x} + C_1'\frac{\sin x}{\cos x}$

$C_1'\left[\cos x + \frac{\sin^2 x}{\cos x}\right] + \frac{2}{\cos x \sin^3 x} = 0$

$C_1' \cdot \frac{1}{\cancel{\cos x}} = -\frac{2}{\cancel{\cos x}\cdot \sin^3 x}$

$\boxed{C_1' = -\frac{2}{\sin^3 x}}$

$C_2' = \frac{2}{\cos x \cdot \sin^3 x} - 2\frac{1}{\sin x \cos x} = \frac{2}{\sin x \cos x}\left[\frac{1}{\sin^2 x} - 1\right] =$

$= 2 \cdot \frac{1}{\sin x \cancel{\cos x}} \cdot \frac{\cos^{\cancel{2}} x}{\sin^2 x} = \boxed{\frac{2\cos x}{\sin^3 x}}$

$C_1(x) = -2\int \frac{dx}{\sin^2 x} = 2\cot x + C_1$

$C_2(x) = 2\int \frac{\cos x}{\sin^3 x} = 2\int \overset{I}{\cot x \frac{dx}{\sin^2 x}} = \left| \begin{matrix} u = \cot x & du = -\frac{1}{\sin^2 x}dx \\ dv = \frac{dx}{\sin^2 x} & v = -\cot x \end{matrix} \right.$

$= 2\left[-\cot^2 x - \int \overset{I}{\cot x \frac{1}{\sin^2 x}}\right]$

$2 \quad I = -\cot^2 x - I \Rightarrow I = -\frac{1}{2}\cot^2 x$

$C_2(x) = -\cot^2 x + C_2$

$y = C_1\cos x + C_2 \sin x + \underbrace{2\cot x \cos x - \sin x \cot^2 x}_{A}$

$A = 2\cdot\frac{\cos^2 x}{\sin x} - \cancel{\sin x}\frac{\cos^2 x}{\sin^{\cancel{2}} x} = \frac{\cos^2 x}{\sin x} = \cot \cos x$

$$\boxed{y = C_1\cos x + C_2 \sin x + \cot x \cdot \cos x}$$

Krivo je rješenje u knjizi, tj. greška je, ovako kako je rješenje zad 1.6.

7) $y'' + 2y' + y = 3e^{-x}\sqrt{x+1} \Rightarrow r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$C_1' \cancel{e^{-x}} + C_2' x \cancel{e^{-x}} = 0$ $\qquad y_H = C_1 e^{-x} + C_2 x e^{-x}$

$\underbrace{-C_1' e^{-x} - C_2' x e^{-x}}_{0} + C_2' \cancel{e^{-x}} = 3\cancel{e^{-x}}\sqrt{x+1}$

$\boxed{C_2' = 3\sqrt{x+1}}$ $\qquad \boxed{C_1' = -3x\sqrt{x+1}}$

$C_1(x) = -3\int x\sqrt{x+1}\,dx = \left|\begin{matrix} t^2 = x+1 \\ 2t\,dt = dx \end{matrix}\right|$

$= -3\int (t^2-1)\cdot t \cdot 2t\,dt = -6\int t^2(t^2-1)\,dt$

$= -6\int \left[t^4 - t^2\right] dt = -6\left[\frac{1}{5}t^5 - \frac{1}{3}t^3\right] + C_1$

$C_2(x) = 3\int \sqrt{x+1}\,dx = \left|\begin{matrix} t^2 = x+1 \\ 2t\,dt = dx \end{matrix}\right| = 6\int t^2\,dt = 6\cdot\frac{1}{3}t^3 + C_2$

$p = e^{-x}$

$y = C_1 p + C_2 x \cdot p - 6p\cdot\frac{1}{5}t^5 \underbrace{+ \cancel{6p\cdot\frac{1}{3}t^3}} + \underbrace{p\cdot(t^2-1)\cdot 6\cdot\frac{1}{3}t^3}$

$y = C_1 \cdot p + C_2 x\, p - 6p\cdot\frac{1}{5}t^5 + 2p\,t^5$

$y = C_1 \cdot p + C_2 \cdot x \cdot p + \frac{4}{5} p \cdot t^5$

vratimo oznake

$$\boxed{y = C_1 e^{-x} + C_2 x e^{-x} + \frac{4}{5} e^{-x} (x+1)^{\frac{5}{2}}}$$

3)

$y'' + y = \frac{1}{\cos^3 x}$ $\qquad y_H: \quad r^2+1=0 \quad r = \pm i$

$$\boxed{y_H = C_1 \cos x + C_2 \sin x}$$

$C_1' \cos x + C_2' \sin x = 0$

$-C_1' \sin x + C_2' \cos x = \frac{1}{\cos^3 x}$

$C_2' = \frac{1}{\cos^4 x} + C_1' \frac{\sin x}{\cos x}$

$C_1' \left(\frac{\cos^2 x + \sin^2 x}{\cos x} \right) + \frac{\sin x}{\cos^4 x} = 0$

$C_1' \frac{1}{\cancel{\cos} x} = -\frac{\sin x}{\cos^{\cancel{4}3} x}$

$$\boxed{C_1' = -\frac{\sin x}{\cos^3 x}}$$

$C_2' = \frac{1}{\cos^4 x} - \frac{\sin^2 x}{\cos^4 x} = \frac{1}{\cos^4 x}\left[1 - \sin^2 x\right] = \frac{\cos^2 x}{\cos^4 x}$

$$\boxed{C_2' = \frac{1}{\cos^2 x}}$$

$C_1(x) = \int \frac{-\sin x}{\cos^3 x}\,dx = \int \frac{d(\cos x)}{\cos^3 x} = \boxed{-\frac{1}{-2} \frac{1}{\cos^2 x} + C_1}$

$C_2(x) = \int \frac{dx}{\cos^2 x} = \boxed{\tan x + C_2}$

$y = C_1 \cos x + C_2 \sin x \underbrace{- \frac{1}{2} \frac{\cancel{\cos x}}{\cos^{\cancel{2}} x} + \frac{\sin x \cdot \sin x}{\cos x}}_{A}$

$A = \frac{2\sin^2 x - 1}{2\cos x} = -\frac{\cos 2x}{2\cos x}$ $\qquad \cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x = 2\cos^2 x - 1$

$$\boxed{y = C_1 \cos x + C_2 \sin x - \frac{\cos 2x}{2\cos x}}$$

Zad 14.

1) $y'' - y = \operatorname{th} x$

y_H: $r^2 - 1 = 0$, $r = \pm 1$

$y_H = C_1 e^x + C_2 e^{-x}$

$C_1' e^x + C_2' e^{-x} = 0 \quad / \cdot e^x \quad C_1' e^{2x} + C_2' = 0 \Rightarrow -C_2' = C_1' e^{2x}$

$C_1' e^x - C_2' e^{-x} = \operatorname{th} x \qquad C_1' e^x + C_1' e^x = \operatorname{th} x$

$\boxed{C_1' = \frac{\operatorname{th} x}{2e^x}} \qquad C_2' = -C_1' e^{2x} = \boxed{-\frac{1}{2} e^x \operatorname{th} x}$

$C_1(x) = \int \frac{\operatorname{th} x}{2e^x} dx = \frac{1}{2} \int \frac{e^x - e^{-x}}{e^x(e^x + e^{-x})} dx = \left| \begin{array}{l} e^x = t \\ t\,dx = dt \end{array} \right| =$

$= \frac{1}{2} \int \frac{t - \frac{1}{t}}{t(t + \frac{1}{t})} \cdot \frac{dt}{t} = \frac{1}{2} \int \frac{t^2 - 1}{t^2(t^2+1)} dt$

$\frac{t^2-1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct + D}{t^2+1} \quad / \cdot t^2(t^2+1)$

$t^2 - 1 = At(t^2+1) + B(t^2+1) + t^2(Ct + D)$

$t^2 - 1 = t^3(A + C) + t^2(B + D) + At + B$

$\boxed{B = -1} \quad \boxed{A = 0} \quad B + D = 1 \quad \boxed{D = 2} \quad A + C = 0 \quad C = 0$

$= \frac{1}{2} \int \left[-\frac{1}{t^2} + \frac{2}{t^2+1} \right] dt = \boxed{\frac{1}{2} \cdot \frac{1}{e^x} + \operatorname{arctg} e^x + C_1}$

$C_2(x) = -\frac{1}{2} \int e^x \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \left| \begin{array}{l} t = e^x \\ dt = t\,dx \end{array} \right| = -\frac{1}{2} \int \cancel{t} \frac{t - \frac{1}{t}}{t + \frac{1}{t}} \frac{dt}{\cancel{t}}$

$= -\frac{1}{2} \int \frac{t^2 - 1}{t^2+1} dt = -\frac{1}{2} \int \left(1 - \frac{2}{t^2+1}\right) dt = \boxed{-\frac{1}{2} e^x + \operatorname{arctg} e^x + C_2}$

$t^2 - 1 : t^2 + 1 = 1$

$t^2 + 1$

-2

$y = C_1 e^x + C_2 e^{-x} + \cancel{\frac{1}{2}} + e^x \operatorname{arctg} e^x$

$\cancel{-\frac{1}{2}} + e^{-x} \operatorname{arctg} e^x$

$\boxed{y = C_1 e^x + C_2 e^{-x} + 2 \operatorname{ch} x \operatorname{arctg} e^x}$

2) $y'' + y = \frac{1}{1+\cos 2x}$ $\quad y_H \quad r^2+1=0 \quad r=\pm i$

$$\boxed{y_H = C_1\cos x + C_2\sin x}$$

$$C_1'\cos x + C_2'\sin x = 0$$

$$-C_1'\sin x + C_2'\cos x = \frac{1}{1+\cos 2x} = \frac{1}{2\cdot\frac{1+\cos 2x}{2}} = \frac{1}{2\cos^2 x}$$

$$C_2' = \frac{1}{2\cos^3 x} + C_1'\frac{\sin x}{\cos x}$$

$$C_1'\left[\cos x + \frac{\sin^2 x}{\cos x}\right] + \frac{1}{2}\frac{\sin x}{\cos^3 x} = 0$$

$$C_1'\cdot\frac{1}{\cancel{\cos x}} = -\frac{1}{2}\frac{\sin x}{\cos^3 x} \Rightarrow C_1' = \boxed{-\frac{1}{2}\frac{\sin x}{\cos^2 x}}$$

$$C_2' = \frac{1}{2\cos^3 x} - \frac{1}{2}\frac{\sin^2 x}{\cos^3 x} = \frac{1}{2\cos^3 x}\left[1-\sin^2 x\right] = \frac{1}{2}\frac{\cos^2 x}{\cos^3 x}$$

$$\boxed{C_2' = \frac{1}{2}\frac{1}{\cos x}}$$

$$C_1(x) = -\frac{1}{2}\int\frac{\sin x}{\cos^2 x}dx = \frac{1}{2}\int\frac{d(\cos x)}{\cos^2 x} = \boxed{-\frac{1}{2}\cdot\frac{1}{\cos x} + C_1}$$

$$C_2(x) = \frac{1}{2}\int\frac{dx}{\cos x} = \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C_2$$

$$\boxed{y = C_1\cdot\cos x + C_2\sin x - \frac{1}{2} + \frac{1}{2}\sin x\cdot\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|}$$

3) $y'' - 2y' + y = x^{-2}e^x$ $\quad y_H \quad r^2-2r+1=0 \quad (r-1)^2=0$

$$\boxed{y_H = C_1e^x + C_2\,xe^x}$$

$$C_1'\cancel{e^x} + C_2'x\cancel{e^x} = 0$$

$$\underbrace{C_1'e^x + C_2'xe^x}_{0} + C_2'\cancel{e^x} = x^{-2}\cancel{e^x} \Rightarrow \boxed{C_2' = \frac{1}{x^2}} \Rightarrow \boxed{C_1' = -\frac{1}{x}}$$

$$C_1(x) = -\int\frac{1}{x}dx = -\ln x + C_1$$

$$C_2(x) = \int\frac{dx}{x^2} = -\frac{1}{x} + C_2$$

$$\boxed{y = C_1e^x + C_2xe^x - e^x\ln x - e^x}$$

4.) $y'' + 2y' + y = \frac{1}{xe^x}$ $\quad y_H: \; r^2 + 2r + 1 = 0 \quad (r+1)^2 = 0$

$$\boxed{y_H = c_1 e^{-x} + c_2 x e^{-x}}$$

$c_1' \cancel{e^{-x}} + c_2' x \cancel{e^{-x}} = 0$

$-c_1' e^{-x} - c_2' x e^{-x} + c_2' \cancel{e^{-x}} = \frac{1}{x} \cancel{e^{-x}}$

$0 \quad \boxed{c_2' = \frac{1}{x}} \quad \boxed{c_1' = -1}$

$\ldots \; c_2(x) = \ln x + c_2'$

$c_1(x) = -x + c_1$

$y = c_1 e^{-x} + c_2 x e^{-x} - x e^{-x} + x e^{-x} \ln x$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + x e^{-x} [\ln x - 1]}$$

5.) $y'' + y = \tan x \quad y_H: \; r^2 + 1 = 0 \quad r = \pm i$

$y_H = c_1 \cos x + c_2 \sin x$

$c_1' \cos x + c_2' \sin x = 0$

$-c_1' \sin x + c_2' \cos x = \frac{\sin x}{\cos x}$

$c_2' = \frac{\sin x}{\cos^2 x} + c_1' \frac{\sin x}{\cos x}$

$c_1' \cdot \frac{1}{\cancel{\cos x}} = -\frac{\sin^2 x}{\cos^{\cancel{2}} x}$

$\boxed{c_1' = -\frac{\sin^2 x}{\cos x}}$

$\cos^2 x$

$c_2' = \frac{\sin x}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \cdot \sin^2 x = \frac{\sin x}{\cancel{\cos^2 x}} [\cancel{1 - \sin^2 x}]$

$\boxed{c_2' = \sin x}$

$c_1(x) = -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx = \int (\cos x - \frac{1}{\cos x}) dx$

$= \sin x - \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + c_1$

$c_2(x) = \int \sin x \, dx = -\cos x + c_2$

$y = c_1 \cdot \cos x + c_2 \cdot \sin x + \cancel{\cos x \sin x} - \cos x \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| - \cancel{\sin x \cos x}$

$$\boxed{y = c_1 \cdot \cos x + c_2 \cdot \sin x - \cos x \ln|\tan(\frac{x}{2} + \frac{\pi}{4})|}$$

6) $y'' - y' = e^{2x}\cos(e^x)$ $\qquad y_H \quad r^2 - r = 0$

$r(r-1) = 0$

$y_H = C_1 + C_2 e^x$

$C_1' + C_2' e^x = 0$

$0 + C_2' \cancel{e^x} = e^{2x}\cos(e^x) \Rightarrow \boxed{C_2' = e^x \cos(e^x)}$

$\boxed{C_1' = -e^{2x}\cos(e^x)}$

$C_1(x) = -\int e^{2x}\cos e^x \left| \begin{matrix} t = e^x \\ dt = t\,dx \end{matrix} \right. = -\int t^{\cancel{2}} \cos t \cdot \frac{dt}{\cancel{t}}$

$= -\int t \cdot \cos t\,dt = \left| \begin{matrix} u = t & du = dt \\ dv = \cos t\,dt & v = \sin t \end{matrix} \right.$

$= -\left[t\cdot \sin t - \int \sin t\,dt \right] = -t\sin t - \cos t$

$\boxed{C_1 = -e^x \sin e^x - \cos e^x + C_1}$

$C_2(x) = \int e^x \cos e^x = \left| \begin{matrix} t = e^x \\ dt = e^x dx \end{matrix} \right. = \int \cos t\,dt = \sin t$

$\boxed{C_2 = \sin e^x + C_2}$

$y = C_1 + C_2 e^x - \cancel{e^x \sin e^x} - \cos e^x + \cancel{e^x \sin e^x}$

$\boxed{y = C_1 + C_2 e^x - \cos e^x}$

7) $y'' - 2y' + y = \frac{e^x}{x} \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

$y_H = C_1 e^x + C_2 x e^x$

$C_1' \cancel{e^x} + C_2' x \cancel{e^x} = 0$

$\underbrace{C_1' e^x + C_2' x e^x}_{0} + C_2' \cancel{e^x} = \frac{\cancel{e^x}}{x} \Rightarrow \boxed{C_2' = \frac{1}{x}} \quad \boxed{C_1' = -1}$

$\boxed{C_1(x) = -x + C_1}$

$\boxed{C_2(x) = \ln x + C_2}$

$y = C_1 e^x + C_2 x e^x - x e^x + x e^x \ln x$

$\boxed{y = C_1 e^x + C_2 x e^x + x e^x [\ln x - 1]}$

8.) $y'' + 4y' + 4y = e^{-2x} x^{-3} \Rightarrow r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$$\boxed{y_H = C_1 e^{-2x} + C_2 x e^{-2x}}$$

$C_1' \cancel{e^{-2x}} + C_2' x \cancel{e^{-2x}} = 0$

$\underbrace{-2C_1' e^{-2x} - 2C_2' x e^{-2x}}_{0} + C_2' \cancel{e^{-2x}} = \cancel{e^{-2x}} \frac{1}{x^3}$

$$\boxed{C_2' = \frac{1}{x^3}} \qquad \boxed{C_1' = -\frac{1}{x^2}}$$

$$\boxed{C_1(x) = \frac{1}{x} + C_1}$$

$$\boxed{C_2(x) = -\frac{1}{2} \cdot \frac{1}{x^2} + C_2}$$

$y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{x} e^{-2x} - \frac{1}{2} \cdot \frac{1}{x} e^{-2x}$

$$\boxed{y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{2x} e^{-2x}}$$

Zad 5.

$y''' + y' = \sin x \qquad y(0) = y'(0) = 0, \; y''(0) = 1$

$y_p = x(A\cos x + B\sin x) \qquad r^3 + r = 0 \quad r(r^2+1) = 0 \Rightarrow r_1 = 0, \; r_{2,3} = \pm i$

$= Ax\cos x + Bx\sin x \qquad y_H = C_1 + C_2\cos x + C_3 \sin x$

$y_p' = A\cos x + B\sin x - Ax\sin x + Bx\cos x$

$y_p'' = -A\sin x - A\sin x + B\cos x + B\cos x - Ax\cos x - Bx\sin x$

$= -2A\sin x + 2B\cos x - Ax\cos x - Bx\sin x$

$y_p''' = -2A\cos x - A\cos x - 2B\sin x - B\sin x + Ax\sin x - Bx\cos x$

$= -3A\cos x - 3B\sin x + Ax\sin x - Bx\cos x$

$y_p''' + y_p' = \sin x \Rightarrow (-3A + A)\cos x + (-3B + B)\sin x$

$+ (A - \cancel{A}) x\sin x + (\cancel{-B} + B) x\cos x = \sin x$

$A = 0 \qquad -2B = 1 \quad B = -\frac{1}{2}$

$y_p = -\frac{1}{2} x\sin x$

$y(0) = 0$

$y = C_1 + C_2\cos x + C_3\sin x - \frac{1}{2} x\sin x \Rightarrow 0 = C_1 + C_2 + 0 + 0$

$y'(0) = 0$

$y' = -C_2\sin x + C_3\cos x - \frac{1}{2}\sin x - \frac{1}{2} x\cos x \Rightarrow 0 = 0 + C_3 + 0 + 0 \qquad \boxed{C_3 = 0}$

$y'' = -C_2\cos x - C_3\sin x - \frac{1}{2}\cos x - \frac{1}{2}\cos x + \frac{1}{2} x\sin x$

$y''(0) = 1 \Rightarrow 1 = -C_2 + 0 - \frac{1}{2} - \frac{1}{2} + 0$

$\boxed{C_2 = -2} \Rightarrow \boxed{C_1 = 2}$

$$\boxed{y = 2 - 2\cos x - \frac{1}{2} x\sin x}$$

Zad 6. Eulerova jednadžba

$$x^2y'' - 3xy' + 4y = 0 \qquad y = x^\alpha$$

$$\alpha(\alpha-1)\cancel{x^\alpha} - 3\alpha\cancel{x^\alpha} + 4\cancel{x^\alpha} = 0$$

$$\alpha^2 - \alpha - 3\alpha + 4 = 0$$

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0 \quad \alpha_{1,2} = 2 \Rightarrow \boxed{y = C_1x^2 + C_2x^2\ln x}$$

Zad 7.

$$(x^2-1)y'' + (x - 8\sqrt{x^2-1})y' + 15y = x$$

$$\boxed{x = \operatorname{ch} t}$$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{\dot{y}}{\operatorname{sh} t}} \qquad y'' = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{\dot{y}}{\operatorname{sh} t}\right]}{\operatorname{sh} t} = \boxed{\frac{\ddot{y}\operatorname{sh} t - \dot{y}\operatorname{ch} t}{\operatorname{sh}^3 t}}$$

$$\boxed{\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1}$$

$$\underbrace{(\operatorname{ch}^2 t - 1)}_{\operatorname{sh}^2 t} \cdot \frac{\ddot{y}\operatorname{sh} t - \dot{y}\operatorname{ch} t}{\operatorname{sh}^3 t} + (\operatorname{ch} t - 8\operatorname{sh} t)\frac{\dot{y}}{\operatorname{sh} t} + 15y = \operatorname{ch} t$$

$$\ddot{y} - \cancel{\dot{y}\operatorname{cth} t} + \cancel{\dot{y}\operatorname{cth} t} - 8\dot{y} + 15y = \operatorname{ch} t$$

$$\ddot{y} - 8\dot{y} + 15y = \operatorname{ch} t = \frac{1}{2}e^t + \frac{1}{2}e^{-t} \Rightarrow r^2 - 8r + 15 = 0$$

$$r_1 = 5 \quad r_2 = 3$$

$$\boxed{y_H = C_1e^{5t} + C_2e^{3t}}$$

$$y_P = Ae^t + Be^{-t}$$

$$\dot{y}_P = Ae^t - Be^{-t}$$

$$\ddot{y}_P = Ae^t + Be^{-t}$$

$$e^t(A - 8A + 15A) + e^{-t}(B + 8B + 15B) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$$

$$8A = \frac{1}{2} \qquad 24B = \frac{1}{2}$$

$$\boxed{A = \frac{1}{16}} \qquad \boxed{B = \frac{1}{48}}$$

$$\boxed{y = C_1e^{5t} + C_2e^{3t} + \frac{1}{16}e^t + \frac{1}{48}e^{-t}}$$

Ovo je točno! valjda ☺

Zad 8

Nesto kao 7

Zad 9

$$(1-x^2)\,y'' + \left(\sqrt{1-x^2} - x\right) y' - 2y = 2x^2$$

$x = \sin t$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot y}{\cos t}$$

$$y'' = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\dot y}{\cos t}\right)}{\cos t} = \frac{\frac{\ddot y \cos t + \dot y \sin t}{\cos^2 t}}{\cos t} = \frac{\ddot y \cos t + \dot y \sin t}{\cos^3 t}$$

$$\frac{(1-\sin^2 t)}{\cancel{\cos^2 t}} \cdot \left[\frac{\ddot y \cos t + \dot y \sin t}{\cos^3 t}\right] + \left(\overset{\cos t}{\sqrt{1-\sin^2 t}} - \sin t\right) \cdot \frac{\dot y}{\cos t} - 2y = 2\sin^2 t$$

$$\ddot y + \dot y \cdot \cancel{\tan t} + \dot y - \dot y \cancel{\tan t} - 2y = 2\sin^2 t$$

$$\ddot y + \dot y - 2y = 1 - \cos 2t$$

$r^2 + r - 2 = 0$

$(r-1)(r+2) = 0 \quad r_1 = 1 \quad r_2 = -2$

$$\boxed{y_H = C_1 e^{-2t} + C_2 e^{t}}$$

$y_p = At + B + k_1 \cos 2t + k_2 \sin 2t$

$\dot y_p = A - 2k_1 \sin 2t + 2k_2 \cos 2t$

$\ddot y_p = 0 - 4k_1 \cos 2t - 4k_2 \sin 2t$

$$\ddot y_p + \dot y_p - 2y_p = \cos 2t\,(-4k_1 + 2k_2 - 2k_1) + \sin 2t\,(-4k_2 - 2k_1 - 2k_2) + A - 2B - 2At = 1 - \cos 2t$$

$A - 2B = 1 \qquad \boxed{2A = 0}$

$\boxed{B = -\frac{1}{2}}$

$-6k_1 + 2k_2 = -1 \qquad -6k_2 - 2k_1 = 0$

$-2k_1 = 6k_2$

$18k_2 + 2k_2 = -1$

$\boxed{k_2 = -\frac{1}{20}} \qquad k_1 = -3k_2 = \boxed{\frac{3}{20}}$

$$y = C_1 e^{-2t} + C_2 e^{t} - \frac{1}{2} - \frac{1}{20}\sin 2t + \frac{3}{20}\cos 2t$$