Dvostruki integral

(1. tjedan)

by

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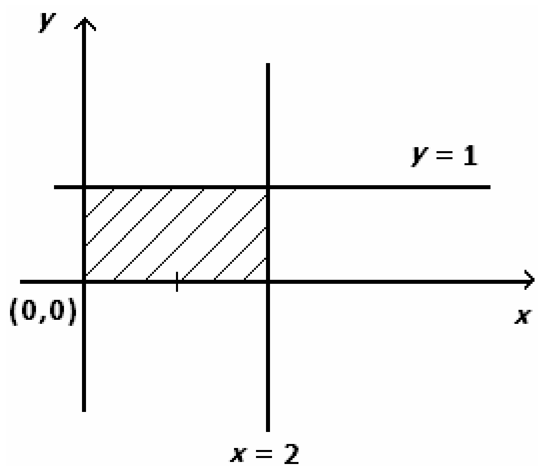
Zadatak 1. Izračunajte dvostruke integrale:

A. $\iint_D (x^4 + x^2y^2 + y^4)dxdy$ ako je područje integracije D omeđeno koordinatnim osima i pravcima $x = 2$, $y = 1$



$$\iint_D (x^4 + x^2y^2 + y^4)dxdy = \iint_D x^4 dxdy + \iint_D x^2y^2 dxdy + \iint_D y^4 dxdy =$$

$$= \int_0^2 x^4 \left(\int_0^1 dy \right) dx + \int_0^2 x^2 \left(\int_0^1 y^2 dy \right) dx + \int_0^2 \left(\int_0^1 y^4 dy \right) dx =$$

$$= \int_0^2 x^4 dx + \int_0^2 x^2 \left(\frac{y^3}{3} \Big|_0^1 \right) dx + \int_0^2 \left(\frac{y^5}{5} \Big|_0^1 \right) dx =$$

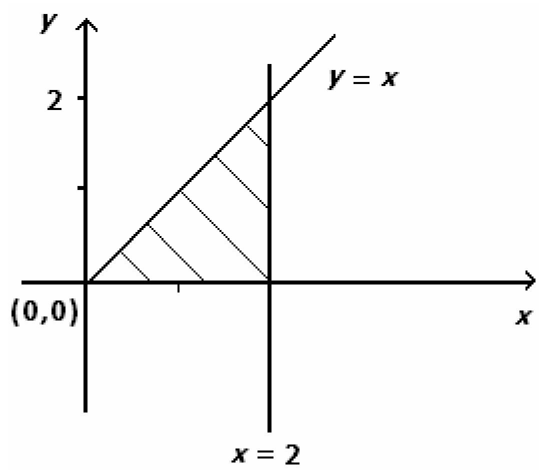
$$= \frac{x^5}{5} \Big|_0^2 + \frac{1}{3} \int_0^2 x^2 dx + \frac{1}{5} \int_0^2 dx =$$

$$= \frac{32}{5} + \frac{1}{3} \frac{x^3}{3} \Big|_0^2 + \frac{1}{5} x \Big|_0^2 =$$

$$= \frac{32}{5} + \frac{8}{9} + \frac{2}{5} =$$

$$= \frac{346}{45}$$

B. $\iint_D x^3 y e^{xy}\,dxdy$ ako je područje integracije D omeđeno koordinatnom osi x i pravcima $x = 2$, $y = x$



$$\iint_D x^3 y e^{xy}\,dxdy = \int_0^2 x^3\left(\int_0^x y e^{xy}\,dy\right)dx = \begin{vmatrix} xy = t & xdy = dt \\ y = \dfrac{t}{x} & dy = \dfrac{dt}{x}\end{vmatrix} =$$

$$= \int_0^2 x^3\left(\int_0^{x^2} \frac{t}{x} e^t \frac{dt}{x}\right)dx = \int_0^2 x^3\left(\frac{1}{x^2}\int_0^{x^2} te^t\,dt\right)dx = \begin{vmatrix} u = t & dv = e^t dt \\ du = dt & v = e^t\end{vmatrix} =$$

$$= \int_0^2 x^3\left(\frac{1}{x^2}\left(te^t\Big|_0^{x^2} - \int_0^{x^2} e^t\,dt\right)\right)dx = \int_0^2 x^3\left(\frac{1}{x^2}\left(x^2e^{x^2} - e^t\Big|_0^{x^2}\right)\right)dx =$$

$$= \int_0^2 x^3\left(\frac{1}{x^2}\left(x^2e^{x^2} - e^{x^2} + 1\right)\right)dx = \int_0^2 x\left(x^2e^{x^2} - e^{x^2} + 1\right)dx = \int_0^2 x^3e^{x^2}dx - \int_0^2 xe^{x^2}dx + \int_0^2 xdx$$

$$\int_0^2 x^3e^{x^2}dx = \frac{1}{2}\int_0^2 x^2e^{x^2}\cdot 2xdx \begin{vmatrix} x^2 = t \\ 2xdx = dt\end{vmatrix} = \frac{1}{2}\int_0^4 te^t dt = \begin{vmatrix} u = t & dv = e^t dt \\ du = dt & v = e^t\end{vmatrix} =$$

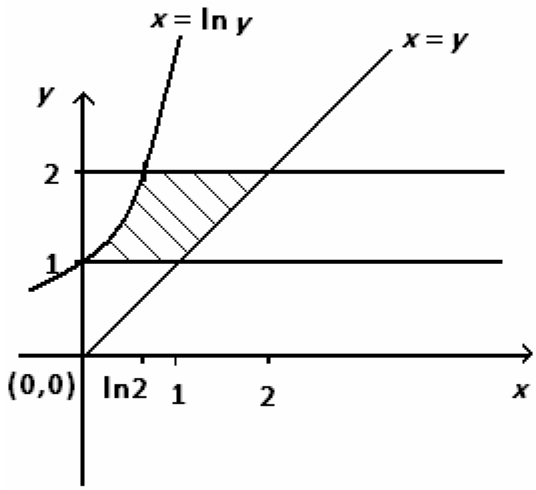
$$= \frac{1}{2}\left(te^t\Big|_0^4 - \int_0^4 e^t dt\right) = \frac{1}{2}\left(te^t\Big|_0^4 - e^t\Big|_0^4\right) = \frac{1}{2}\left(4e^4 - e^4 + 1\right) = 2e^4 - \frac{1}{2}e^4 + \frac{1}{2}$$

$$\int_0^2 xe^{x^2}dx = \begin{vmatrix} x^2 = t \\ 2xdx = dt\end{vmatrix} = \frac{1}{2}\int_0^4 e^t dt = \frac{1}{2}e^t\Big|_0^4 = \frac{1}{2}e^4 - \frac{1}{2}$$

$$\iint_D x^3 y e^{xy}\,dxdy = 2e^4 - \frac{1}{2}e^4 + \frac{1}{2} - \left(\frac{1}{2}e^4 - \frac{1}{2}\right) + 2 = 2e^4 - \frac{1}{2}e^4 + \frac{1}{2} - \frac{1}{2}e^4 + \frac{1}{2} + 2 = e^4 + 3$$

Zadatak 2. Promijenite poredak integracije u integralima:

A. $\int_{1}^{2} dy \int_{\ln y}^{y} f(x,y)dx$



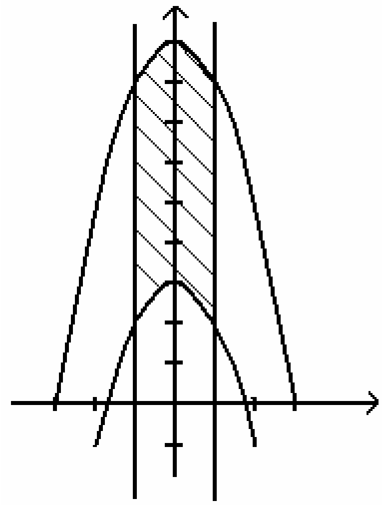
U početnom integralu su fiksne granice po *y*. Znači, mi najprije trebamo odrediti fiksne granice od *x*, i onda gledati koje nam krivulje zatvaraju površinu u tim granicama. Imat ćemo tri integrala sa sljedećim granicama:

1. $0 \to \ln 2$
2. $\ln 2 \to 1$
3. $1 \to 2$

Za granicu od 0 do ln2, dio površine omeđen nam je sa funkcijama 1 i e^x. Za granicu od ln2 do 1, dio površine omeđen nam je sa 1 i 2, a za granicu od 1 do 2, dio površine nam je omeđen sa funkcijama 2 i *y*. Pa pišemo integral:

$$\int_{0}^{\ln 2} dx \int_{1}^{e^x} f(x,y)dy + \int_{\ln 2}^{1} dx \int_{1}^{2} f(x,y)dy + \int_{1}^{2} dx \int_{2}^{x} f(x,y)dy$$

B. $\int_{-1}^{1} dx \int_{3-x^2}^{9-x^2} f(x,y)dy$



U početnom integralu su fiksne granice po *x*. Znači, mi najprije trebamo odrediti fiksne granice od *y*, i onda gledati koje nam krivulje zatvaraju površinu u tim granicama. Imat ćemo četiri integrala sa sljedećim granicama:

1.1. $2 \to 3$

1.2. $2 \to 3$

2. $3 \to 8$

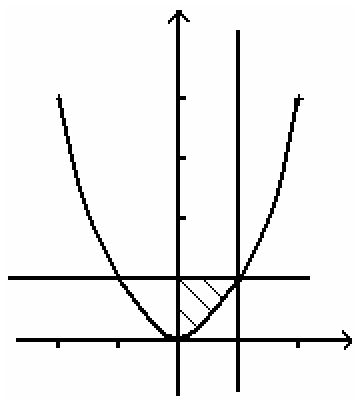
3. $8 \to 9$

Za granicu od 2 do 3, dio površine omeđen nam je sa funkcijama -1 i $-\sqrt{3-y}$ (lijevi dio), odnosno sa $\sqrt{3-y}$ i 1 (desni dio). Za granicu od 3 do 8, dio površine omeđen nam je sa -1 i 1, a za granicu od 8 do 9, dio površine nam je omeđen sa funkcijama $-\sqrt{9-y}$ (lijevi dio) i $\sqrt{9-y}$ (desni dio). Pa pišemo integral:

$$\int_{2}^{3} dy \int_{-1}^{-\sqrt{3-y}} f(x,y)dx + \int_{2}^{3} dy \int_{\sqrt{3-y}}^{1} f(x,y)dx + \int_{3}^{8} dy \int_{-1}^{1} f(x,y)dx + \int_{8}^{9} dy \int_{-\sqrt{9-y}}^{\sqrt{9-y}} f(x,y)dx$$

Zadatak 3. Promjenom poretka integracije izračunajte integrale:

A. $\int\limits_0^1 x^5 dx \int\limits_{x^2}^1 e^{y^2} dy$



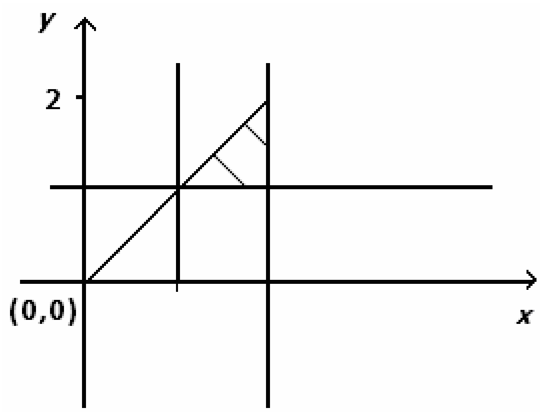
Imamo fiksne granice po x, što znači da ih mijenjamo na y i onda gledamo koje krivulje prekrivaju određeni dio površine D (osjenčane na crtežu).

$$\int\limits_0^1 e^{y^2} dy \int\limits_0^{\sqrt{y}} x^5 dx = \int\limits_0^1 e^{y^2} \left(\frac{x^6}{6} \Bigg|_0^{\sqrt{y}} \right) dy = \frac{1}{6} \int\limits_0^1 e^{y^2} y^3 dy = \frac{1}{12} \int\limits_0^1 e^{y^2} y^2 \cdot 2y dy = \begin{vmatrix} y^2 = t \\ 2ydy = dt \end{vmatrix} =$$

$$= \frac{1}{12} \int\limits_0^1 t e^t dt = \begin{vmatrix} u = t & dv = e^t dt \\ du = dt & v = e^t \end{vmatrix} = \frac{1}{12} \left(t e^t \Big|_0^1 - \int\limits_0^1 e^t dt \right) = \frac{1}{12} \left(e - e^t \Big|_0^1 \right) = \frac{1}{12} (e - e + 1) =$$

$$= \frac{1}{12}$$

B. $\int_1^2 x dx \int_1^x \sqrt{x^2 - y^2}\, dy$



$$\int_1^2 dy \int_y^2 x\sqrt{x^2 - y^2}\, dx = \left| \begin{matrix} x^2 - y^2 = t \\ 2x dx = dt \end{matrix} \right| = \int_1^2 dy \int_0^{4-y^2} \sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int_1^2 \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_0^{4-y^2} \right) dy =$$

$$= \frac{1}{3}\int_1^2 \left(\left(4 - y^2\right)^{\frac{3}{2}} \right) dy = \left| \begin{matrix} y = 2\sin t \\ dy = 2\cos t dt \end{matrix} \right| = \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\left(4 - 4\sin^2 t\right)^{\frac{3}{2}} \right) \cos t dt =$$

$$= \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\left[4\left(1 - \sin^2 t\right)\right]^{\frac{3}{2}} \right) \cos t dt = \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\left[4\cos^2 t\right]^{\frac{3}{2}} \right) \cos t dt = \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\left[2^2 \cos^2 t\right]^{\frac{3}{2}} \right) \cos t dt =$$

$$= \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^3 t \cos t dt = \frac{16}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^4 t dt = \frac{16}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\cos^2 t\right)^2 dt = \frac{16}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2t}{2} \right)^2 dt =$$

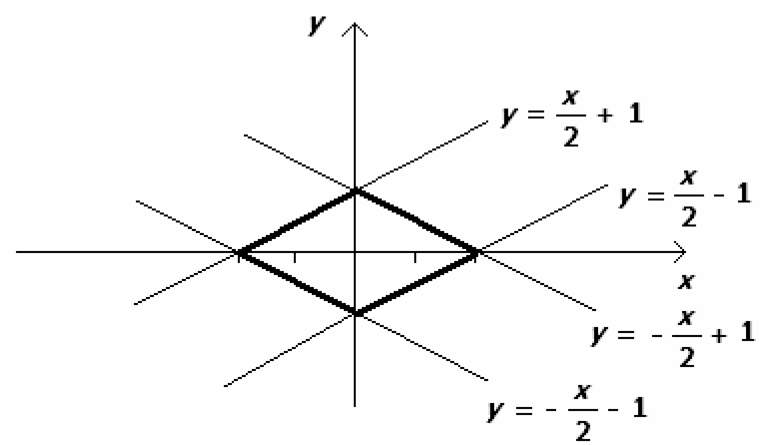
$$= \frac{4}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(1 + 2\cos 2t + \cos^2 2t\right) dt = \frac{4}{3}\left(\frac{\pi}{2} - \frac{\pi}{6} \right) + \frac{8}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2t dt + \frac{4}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1 + \cos 4t}{2} dt =$$

$$= \frac{4}{3}\frac{\pi}{3} + \frac{8}{3}\frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{3} \right) + \frac{2}{3}\left(\frac{\pi}{2} - \frac{\pi}{6} \right) + \frac{2}{3}\left(\sin 2\pi - \sin\frac{2\pi}{3} \right) =$$

$$= \frac{4\pi}{9} - \frac{4}{3}\frac{\sqrt{3}}{2} + \frac{2\pi}{9} - \frac{2}{3}\frac{\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{4\sqrt{3}}{3}$$

Zadatak 4. Neka je D četverokut s vrhovima $A(2,0)$, $B(0,1)$, $C(-2,0)$ i $D(0,-1)$.

A. Postavite granice integracije u integralu $\iint_D f(x,y)dxdy$.



$$\int_{-2}^{0} dx \int_{-\frac{x}{2}-1}^{\frac{x}{2}+1} f(x,y)dy + \int_{0}^{2} dx \int_{\frac{x}{2}-1}^{-\frac{x}{2}+1} f(x,y)dy$$

Izračunajte integral za

B. $f(x,y) = e^{x+y}$

$$\int_{-2}^{0} dx \int_{-\frac{x}{2}-1}^{\frac{x}{2}+1} e^x e^y dy + \int_{0}^{2} dx \int_{\frac{x}{2}-1}^{-\frac{x}{2}+1} e^x e^y dy = \int_{-2}^{0} e^x \left(e^{\frac{x}{2}+1} - e^{-\frac{x}{2}-1} \right) dx + \int_{0}^{2} e^x \left(e^{-\frac{x}{2}+1} - e^{\frac{x}{2}-1} \right) dx =$$

$$= e\int_{-2}^{0} e^{\frac{3}{2}x} dx - \frac{1}{e}\int_{-2}^{0} e^{\frac{1}{2}x} dx + e\int_{0}^{2} e^{\frac{1}{2}x} dx - \frac{1}{e}\int_{0}^{2} e^{\frac{3}{2}x} dx =$$

$$= \frac{2}{3} e e^{\frac{3}{2}x} \Big|_{-2}^{0} - \frac{2}{e} e^{\frac{1}{2}x} \Big|_{-2}^{0} + 2e e^{\frac{1}{2}x} \Big|_{0}^{2} - \frac{2}{3e} e^{\frac{3}{2}x} \Big|_{0}^{2} =$$

$$= \frac{2}{3}e - \frac{2}{3}ee^{-3} - \frac{2}{e} + \frac{2}{e}e^{-1} + 2ee - 2e - \frac{2}{3e}e^3 + \frac{2}{3e} =$$

$$= \frac{4}{3}\left(e^2 - e - \frac{1}{e} + \frac{1}{e^2} \right)$$

C. $f(x,y)=x$

$$\int_{-2}^{0}dx\int_{-\frac{x}{2}-1}^{\frac{x}{2}+1}xdy+\int_{0}^{2}dx\int_{\frac{x}{2}-1}^{-\frac{x}{2}+1}xdy=\int_{-2}^{0}x\left(\frac{x}{2}+1+\frac{x}{2}+1\right)dx+\int_{0}^{2}x\left(-\frac{x}{2}+1-\frac{x}{2}+1\right)dx=$$

$$=\int_{-2}^{0}x(x+2)dx+\int_{0}^{2}x(-x+2)dx=$$

$$=\int_{-2}^{0}x^{2}dx+2\int_{-2}^{0}xdx-\int_{0}^{2}x^{2}dx+2\int_{0}^{2}xdx=$$

$$=\frac{x^{3}}{3}\bigg|_{-2}^{0}+x^{2}\bigg|_{-2}^{0}-\frac{x^{3}}{3}\bigg|_{0}^{2}+x^{2}\bigg|_{0}^{2}=$$

$$=\frac{8}{3}-4-\frac{8}{3}+4=$$

$$=0$$

D. $f(x,y)=y$

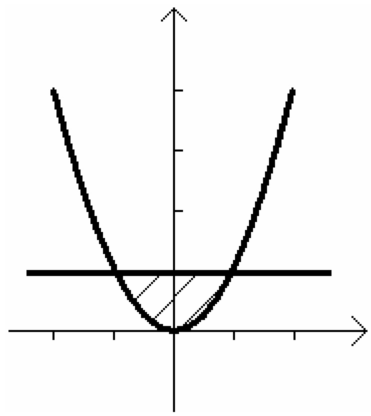
$$\int_{-2}^{0}dx\int_{-\frac{x}{2}-1}^{\frac{x}{2}+1}ydy+\int_{0}^{2}dx\int_{\frac{x}{2}-1}^{-\frac{x}{2}+1}xdy=$$

$$=\frac{1}{2}\int_{-2}^{0}\left(\frac{x^{2}}{4}+x+1-\frac{x^{2}}{4}-x-1\right)dx+\int_{0}^{2}\left(\frac{x^{2}}{4}-x+1-\frac{x^{2}}{4}+x-1\right)dx=$$

$$=0$$

Zadatak 5. Neka je D lik omeđen krivuljama $y = x^2$ i $y = 1$.

A. Izračunajte površinu lika D.



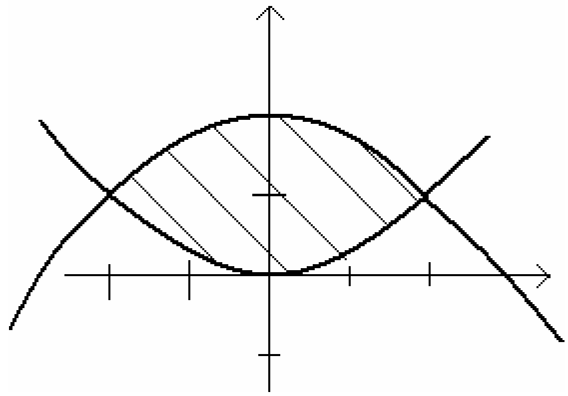
$$\int_{-1}^{1} dx \int_{x^2}^{1} dy = \int_{-1}^{1}\left(y\Big|_{x^2}^{1}\right)dx = \int_{-1}^{1}(1 - x^2)dx = x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1} = 2 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}$$

B. Izračunajte $\iint_D (x+1)dxdy$.

$$\int_{-1}^{1} dx \int_{x^2}^{1} (x+1)dy = \int_{-1}^{1}\left(xy\Big|_{x^2}^{1} + y\Big|_{x^2}^{1}\right)dx = \int_{-1}^{1}(x - x^3 + 1 - x^2)dx =$$

$$= \frac{x^2}{2}\Big|_{-1}^{1} - \frac{x^4}{4}\Big|_{-1}^{1} + x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4} + 1 + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}$$

Zadatak 6. Izračunajte površinu lika omeđenog krivuljama $y=\frac{x^2}{4}$ i $y=\frac{8}{x^2+4}$. Nacrtajte sliku!



$$\int_{-2}^{2}dx\int_{\frac{x^2}{4}}^{\frac{8}{x^2+4}}dy=\int_{-2}^{2}\left(\frac{8}{x^2+4}-\frac{x^2}{4}\right)dx=\int_{-2}^{2}\frac{32-x^4-4x^2}{4\left(x^2+4\right)}dx=$$

$$=\int_{-2}^{2}\frac{8}{x^2+4}dx-\frac{1}{4}\int_{-2}^{2}\frac{x^4}{x^2+4}dx-\int_{-2}^{2}\frac{x^2}{x^2+4}dx=$$

$$=8\cdot\frac{1}{2}arctg\frac{x}{2}\Big|_{-2}^{2}-\frac{1}{4}\int_{-2}^{2}\frac{\left(x^2+4\right)\left(x^2-4\right)+16}{x^2+4}dx-\int_{-2}^{2}\frac{x^2+4-4}{x^2+4}dx=$$

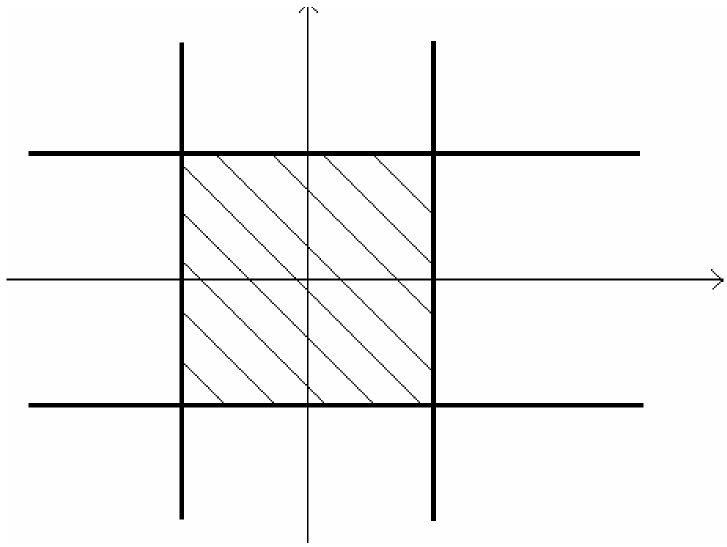
$$=4(arctg\,1-arctg\,(-1))-\frac{1}{4}\int_{-2}^{2}\left(x^2-4\right)dx-4\int_{-2}^{2}\frac{1}{x^2+4}dx-\int_{-2}^{2}dx+4\int_{-2}^{2}\frac{1}{x^2+4}dx=$$

$$=4\left(\frac{\pi}{4}+\frac{\pi}{4}\right)-\frac{1}{4}\frac{x^3}{3}\Big|_{-2}^{2}+x\Big|_{-2}^{2}-4\cdot\frac{1}{2}arctg\frac{x}{2}\Big|_{-2}^{2}-x\Big|_{-2}^{2}+4\cdot\frac{1}{2}arctg\frac{x}{2}\Big|_{-2}^{2}=$$

$$=2\pi-\frac{2}{3}-\frac{2}{3}=$$

$$=2\pi-\frac{4}{3}$$

Zadatak 7. Izračunajte $\iint_D e^{-|x|-|y|}dxdy$ pri čemu je $D = \{(x,y) \in R : \max\{|x|,|y| = 1\}\}$



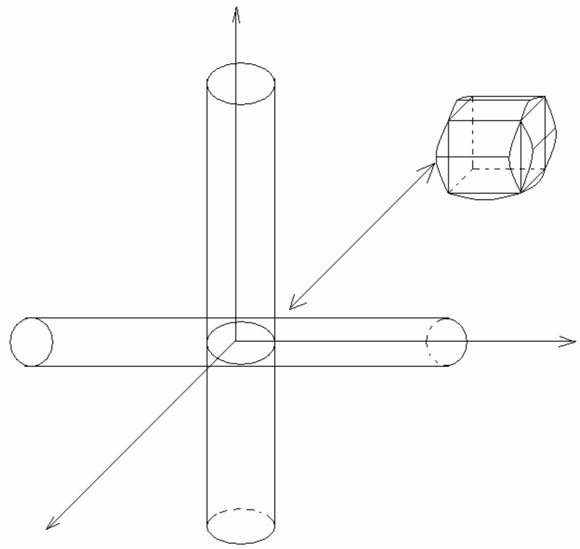
Ovaj zadatak možemo riješiti ili na način da izračunamo površinu jednog malog kvadrata unutar osjenčane površine, pa ukupnu površinu dobijemo samo tako da pomnožimo sa 4, ili da za svaki dio površine pišemo posebno integral. Mislim da je prvi način lakši i brži.

Uzet ćemo fiksne granice od *x*, i to za gornji desni kvadrat. Tada su one od 0 do 1. Na tim granicama, funkcija unutar modula je pozitivna, pa kad mičemo modul ne trebamo stavljati još jedan minus ispred toga. Na fiksnim granicama od *x*, granice za *y* su nam također od 0 do 1. Analogno za modul od *x*, vrijedit će i za modul od *y*,

$$P = 4\int_0^1 dx\int_0^1 e^{-x-y}dy = 4\int_0^1 e^{-x}\left(-e^{-y}\Big|_0^1\right)dx = 4\int_0^1 e^{-x}\left(-e^{-1}+1\right)dx = 4\left(-e^{-1}+1\right)\int_0^1 e^{-x}dx =$$

$$= 4\left(-e^{-1}+1\right)\left(-e^{-x}\Big|_0^1\right) = 4\left(-e^{-1}+1\right)\left(-e^{-1}+1\right) = 4\left(1-\frac{1}{e}\right)^2$$

Zadatak 8. Izračunajte volumen tijela omeđenog plohama $x^2 + y^2 = 1$ i $x^2 + z^2 = 1$. Nacrtajte sliku!

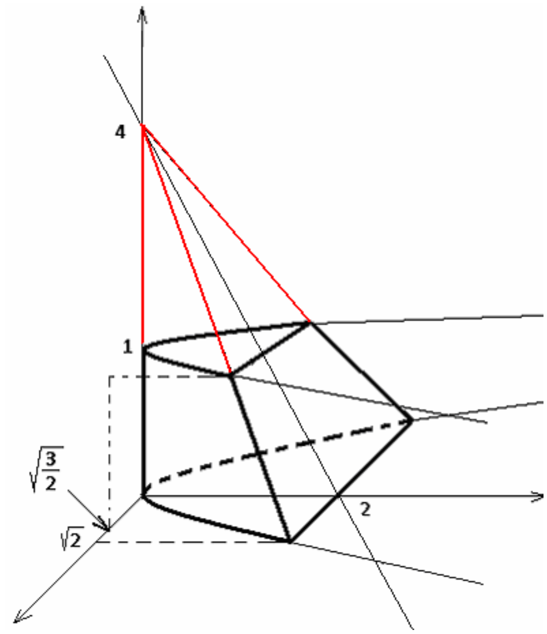


Presjek ovih dvaju cilindara je kocka sa zaobljenim plohama. (Čisto da si možete predočiti, iako nije ni bitno ako znamo granice integracije. A znamo.)

$$V = \iint_D z\,dxdy = 2\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy = 2\int_{-1}^{1} \sqrt{1-x^2}\left(\sqrt{1-x^2} + \sqrt{1-x^2}\right)dx =$$

$$= 2\int_{-1}^{1} \left(1 - x^2 + 1 - x^2\right)dx = 4x\Big|_{-1}^{1} - 4\frac{x^3}{3}\Big|_{-1}^{1} = 8 - \frac{8}{3} = \frac{16}{3}$$

Zadatak 9. Izračunajte volumen tijela određenog nejednadžbama $y \geq x^2$, $z \leq 1$, $z \leq 4-2y$ i $z \geq 0$. Nacrtajte sliku!



Za veliko tijelo:

$$V_1 = \iint_D z\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{x^2}^{2} (4-2y)\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left(4y \Big|_{x^2}^{2} - y^2 \Big|_{x^2}^{2} \right) dx = \int_{-\sqrt{2}}^{\sqrt{2}} \left(8 - 4x^2 - 4 + x^4\right) dx =$$

$$= 4x \Big|_{-\sqrt{2}}^{\sqrt{2}} - \frac{4}{3}x^3 \Big|_{-\sqrt{2}}^{\sqrt{2}} + \frac{1}{5}x^5 \Big|_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\sqrt{2}}{15}$$

Za malo tijelo:

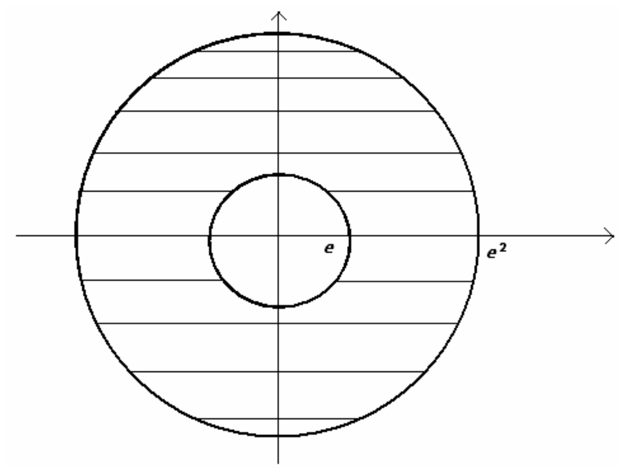
$$V_2 = \iint_D z\,dx\,dy = \int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} dx \int_{x^2}^{\frac{3}{2}} (4-2y)\,dy = \int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} \left(4y \Big|_{x^2}^{\frac{3}{2}} - y^2 \Big|_{x^2}^{\frac{3}{2}} \right) dx = \int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} \left(6 - 4x^2 - \frac{9}{4} + x^4 \right) dx =$$

$$= \frac{15}{4} x \Big|_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} - \frac{4}{3}x^3 \Big|_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} + \frac{1}{5}x^5 \Big|_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} = \frac{22}{5}\sqrt{\frac{3}{2}}$$

Pa nam je konačni volumen: $V = V_1 - V_2 = \dfrac{64\sqrt{2}}{15} - \dfrac{22}{5}\sqrt{\dfrac{3}{2}}$

Zadatak 10. Izračunajte $\iint_D \ln(x^2+y^2)dxdy$, pri čemu je D kružni vijenac $e^2 \le x^2+y^2 \le e^4$, gdje je e baza prirodnog logaritma.

Naša površina se nalazi između kružnica $x^2+y^2=e^2$ i $x^2+y^2=e^4$. Kao u prethodnom primjeru, izračunat ćemo samo jedan komad vjenčića, a ukupnu površinu dobit tako da pomnožimo sa 4.



Također, zbog toga što imamo u funkciji x^2+y^2, bit će nam lakše pređemo li na polarne koordinate: $x=r\cos\varphi$ i $y=r\sin\varphi$.

Kada prelazimo na polarne koordinate vrijedi:

$$\iint_D f(x,y)dxdy = \iint_{D'} r\cdot f(r\cos\varphi, r\sin\varphi)drd\varphi$$

Za ovaj zadatak ćemo uzeti fiksne granice od φ.

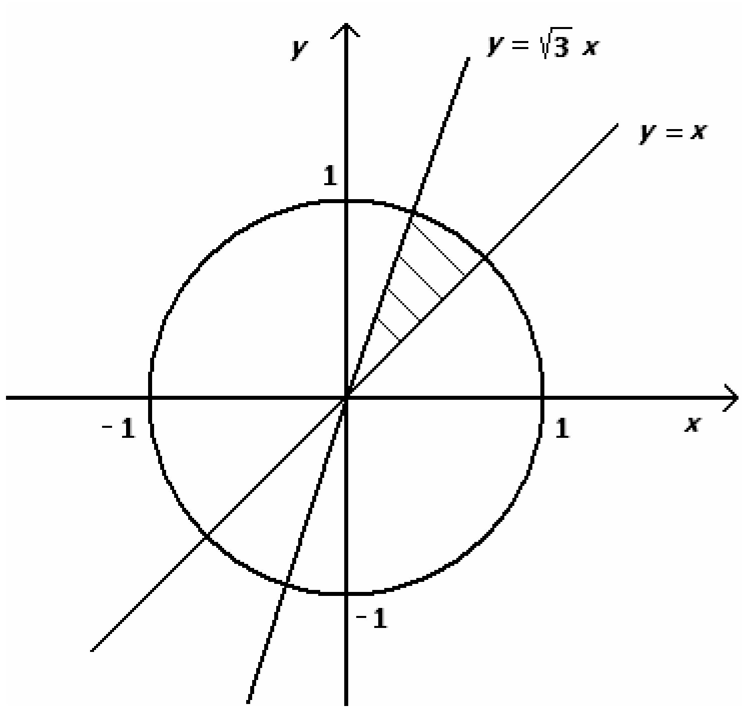
$$P = 4\int_0^{\frac{\pi}{2}} d\varphi \int_e^{e^2} r\ln r^2 dr = \begin{vmatrix} r^2 = t \\ 2rdr = dt \end{vmatrix} = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi \int_{e^2}^{e^4} \ln t dt = \begin{vmatrix} u = \ln t & dv = dt \\ du = \frac{1}{t}dt & v = t \end{vmatrix} =$$

$$= 2\int_0^{\frac{\pi}{2}} \left(t\ln t \Big|_{e^2}^{e^4} - \int_{e^2}^{e^4} dt \right) d\varphi = 2\int_0^{\frac{\pi}{2}} \left(e^4\ln e^4 - e^2\ln e^2 - e^4 + e^2\right)d\varphi = 2\left(3e^4 - e^2\right)\int_0^{\frac{\pi}{2}} d\varphi =$$

$$= 2\left(3e^4 - e^2\right)\varphi\Big|_0^{\frac{\pi}{2}} = \pi e^2\left(3e^2 - 1\right)$$

Zadatak 11. Izračunajte $\iint_D \sqrt{1-x^2-y^2}\,dxdy$, pri čemu je D kružni isječak određen nejednadžbama $x^2+y^2\leq 1$, $y\geq x$, $y\leq\sqrt{3}x$, $y\geq 0$.

Ovaj zadatak ćemo također riješiti preko polarnih koordinata. Nacrtajmo sliku.



Nagib pravca $y=x$ je 45°, odnosno $\frac{\pi}{4}$, a pravca $y=\sqrt{3}x$ je 60°, odnosno $\frac{\pi}{3}$. To su nam granice za φ. Za r su nam granice od 0 do 1.

$$P=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^1 r\sqrt{1-r^2}\,dr = \begin{vmatrix} 1-r^2=t \\ -2rdr=dt \end{vmatrix} = -\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_1^0 \sqrt{t}\,dt = -\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\left. \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^0 \right) d\varphi =$$

$$= -\frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (-1)\,d\varphi = \frac{1}{3}\varphi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{1}{3}\frac{\pi}{12} = \frac{\pi}{36}$$