Matematika 3

Masovne instrukcije 28.10.2011.

Laplaceova transformacija

Tablica transformacija

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$f(t) \circ\!\!-\!\!\!-\!\!\bullet F(s)$$

$$\alpha f(t) \circ\!\!-\!\!\!-\!\!\bullet \alpha F(s)$$

$$u(t) \circ\!\!-\!\!\!-\!\!\bullet \frac{1}{s}$$

$$e^{at} \circ\!\!-\!\!\!-\!\!\bullet \frac{1}{s-a}$$

$$\sin(\omega t) \circ\!\!-\!\!\!-\!\!\bullet \frac{\omega}{s^2+\omega^2}$$

$$\cos(\omega t) \circ\!\!-\!\!\!-\!\!\bullet \frac{s}{s^2+\omega^2}$$

$$\mathrm{sh}(\omega t) \circ\!\!-\!\!\!-\!\!\bullet \frac{\omega}{s^2-\omega^2}$$

$$\mathrm{ch}(\omega t) \circ\!\!-\!\!\!-\!\!\bullet \frac{s}{s^2-\omega^2}$$

$$t^n \circ\!\!-\!\!\!-\!\!\bullet \frac{n!}{s^{n+1}}$$

$$e^{-at} f(t) \circ\!\!-\!\!\!-\!\!\bullet F(s+a)$$

$$g_{[a,b]}(t) = u(t-a) - u(t-b) \circ\!\!-\!\!\!-\!\!\bullet \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

$$f^{(n)}(t) \circ\!\!-\!\!\!-\!\!\bullet s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

$$t^n f(t) \circ\!\!-\!\!\!-\!\!\bullet (-1)^n F^{(n)}(s)$$

$$\int_0^t f(\tau)\, d\tau \circ\!\!-\!\!\!-\!\!\bullet \frac{F(s)}{s}$$

$$\frac{f(t)}{t} \circ\!\!-\!\!\!-\!\!\bullet \int_s^\infty F(s)\, ds$$

$$f(t+T) = f(t) \circ\!\!-\!\!\!-\!\!\bullet \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, dt \circ\!\!-\!\!\!-\!\!\bullet F_1(s) F_2(s)$$

$$R \circ\!\!-\!\!\!-\!\!\bullet R \qquad L \circ\!\!-\!\!\!-\!\!\bullet sL \qquad C \circ\!\!-\!\!\!-\!\!\bullet \frac{1}{sC}$$

Zadaci

1. Odredite Laplaceov transformat sljedećih funkcija:

(a)

$$f(t) = \sin\left(t - \frac{\pi}{4}\right)\cos\left(t - \frac{\pi}{4}\right)$$

Rješenje: $F(s) = -\frac{1}{2}\frac{s}{s^2+4}$

(b)

$$f(t) = (t-2)^3 e^{-t} u(t-2)$$

Rješenje: $F(s) = \frac{6e^{-2(s+1)}}{(s+1)^4}$

(c)

$$f(t) = \frac{(2t-3)^{n+1} e^{-2nt}}{(n+1)!} u\left(t - \frac{3}{2}\right)$$

Rješenje: $F(s) = \frac{2^{n+1} e^{-\frac{3}{2}(s+2n)}}{(s+2n)^{n+2}}$

(d)

$$f(t) = \frac{1-t^7}{1-t}$$

Rješenje: $F(s) = \frac{6!}{s^7} + \frac{5!}{s^6} + \frac{4!}{s^5} + \frac{3!}{s^4} + \frac{2!}{s^3} + \frac{1!}{s^2} + \frac{0!}{s^1}$

(e)

$$f(t) = e^{-t} u(t-1) e^{-2t} u(t-2) \cdots e^{-100t} u(t-100)$$

Rješenje: $F(s) = \frac{e^{-100(s+5050)}}{s+5050}$

(f)

$$f(t) = t^n \cdot 3^{\frac{t-1}{2}}$$

Rješenje: $F(s) = \frac{n!}{\sqrt{3}\left(s - \ln\sqrt{3}\right)^{n+1}}$

(g)

$$f(t) = t \sin\left(\frac{t}{3}\right)$$

Rješenje: $F(s) = \frac{2}{3}\frac{s}{\left(s^2 + \frac{1}{9}\right)^2}$

(h)

$$f(t) = t^2 \operatorname{ch}(3t)$$

Rješenje: $F(s) = \frac{2s\left(s^2+27\right)}{(s^2-9)^3}$

2. Izračunajte sljedeće integrale:

(a)

$$\int_0^\infty e^{-4t} \operatorname{sh} t \operatorname{ch} t dt$$

Rješenje: $I = \frac{1}{12}$

(b)

$$\int_0^\infty \frac{1}{x} e^{-x} \sin x dx$$

Rješenje: $I = \frac{\pi}{4}$

(c)

$$\int_0^\infty e^{-2t} t^2 \sin t dt$$

Rješenje: $I = \frac{22}{125}$

(d)

$$\int_0^\infty e^{-2t}\cos(4t)dt$$

Rješenje: $I = \frac{1}{10}$

3. Odredite Laplaceov transformat sljedećih funkcija:

(a)

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases}$$

$$T = 2\pi$$

Rješenje: $F(s) = \frac{1}{1-e^{-2\pi s}}\frac{1+e^{-\pi s}}{s^2+1}$

(b)

$$f(t) = \operatorname{sgn}(\cos t)$$

Rješenje: $F(s) = \frac{1}{1-e^{-s}}\frac{1-e^{-s}(s+1)}{s^2}$

(c)

$$f(t) = (t - \lfloor t \rfloor)\, u(t)$$

Rješenje: $F(s) = \frac{1}{1-e^{-2\pi s}}\frac{e^{-2\pi s}\left(e^{\frac{\pi}{2}s}-1\right)^3\left(e^{\frac{\pi}{2}s}+1\right)}{s}$

4. Odredite original funkcije:

(a)

$$F(s) = \frac{se^{-2s}}{s^2+24s+169}$$

Rješenje: $f(t) = e^{-12(t-2)}\left[\cos(5(t-2)) - \frac{12}{5}\sin(5(t-2))\right]u(t-2)$

(b)

$$F(s) = \frac{1}{s^3-6s^2+8s}$$

Rješenje: $f(t) = \frac{1}{8}\left(1-2e^{2t}+e^{4t}\right)u(t)$

(c)

$$F(s) = \frac{s}{s^2-16s+5}$$

Rješenje: $f(t) = e^{8t}\left[\operatorname{ch}\left(\sqrt{59}t\right) + \frac{4}{\sqrt{59}}\operatorname{sh}\left(\sqrt{59}t\right)\right]u(t)$

(d)

$$F(s) = \frac{3e^{-3s}}{s(s^2+1)}$$

Rješenje: $f(t) = [1-\cos(t-3)]\,u(t-3)$

(e)

$$F(s) = \frac{s}{\left(s^2+1\right)^2}$$

Rješenje: $f(t) = \frac{1}{2}t\sin(t)u(t)$

(f)

$$F(s) = \frac{se^{-4s}}{\left(s^2+1\right)^2}$$

Rješenje: $f(t) = \frac{1}{2}(t-4)\sin(t-4)u(t-4)$

5. Riješite sljedeće jednadžbe:

(a)

$$y'(t) + \int_0^t y(\tau)\, d\tau = 0$$

$$y(0) = 1$$

Rješenje: $y(t) = \cos(t) u(t)$

(b)

$$2x''(t) - 3x'(t) = e^t$$

$$x(0) = x'(0) = 0$$

Rješenje: $x(t) = \frac{1}{3}\left(1 - 3e^t + 2e^{\frac{3}{2}t}\right) u(t)$

(c)

$$y(t) = t + 4\int_0^t (t-\tau)\, y(\tau)\, d\tau$$

Rješenje: $y(t) = \frac{1}{2}\operatorname{sh}(2t) u(t)$

(d)

$$y'(t) = y''(t) + \int_0^t e^{\tau} \sin \tau d\tau$$

$$y(0) = y'(0) = 0$$

Rješenje: $f(t) = y(t) = \frac{1}{2}\left[1 - e^t\left(2 - \sin(t) - \cos(t)\right)\right] u(t)$

(e)

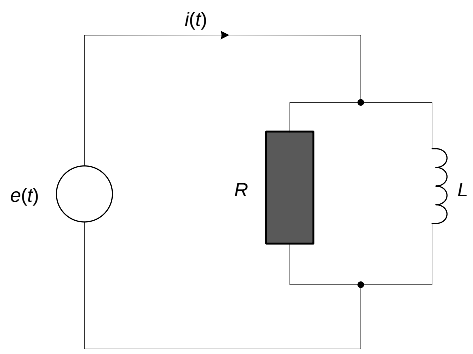
$$y'(t) \cdot e^{-t} = 1 + \int_0^t e^{-\tau} y(\tau)\, d\tau$$

$$y(0) = 0$$

Rješenje: $y(t) = \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) u(t)$

6. Odredite struju $i(t)$ ako je zadan sljedeći električni krugi:

$$e(t) = e^{-t} u(t-1)$$
$$R = L = 1$$



Slika 1: Shema električnog kruga

Rješenje: $E(s) = \frac{e^{-s-1}}{s+1}$, $Z(s) = \frac{s}{s+1}$, $I(s) = \frac{e^{-s-1}}{s}$, $i(t) = \frac{1}{e} u(t-1)$