(15.) $\iiint_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$

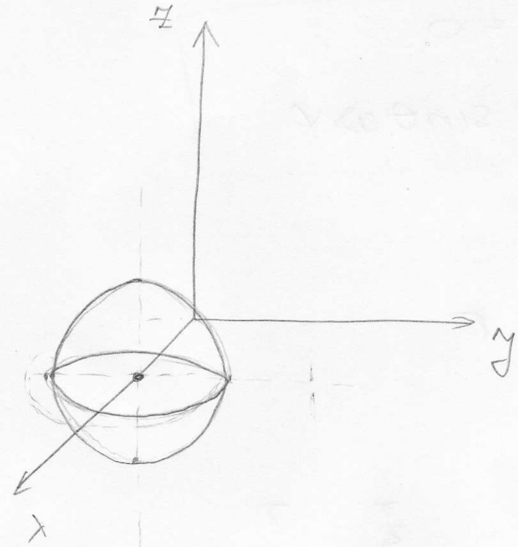
$V \rightarrow x^2+y^2+z^2 \leq x$

$x^2+y^2+z^2=x$

$x^2-x+y^2+z^2=0$

$x^2-x+\frac{1}{4}-\frac{1}{4}+y^2+z^2=0$

$\left(x-\frac{1}{2}\right)^2+y^2+z^2=\frac{1}{2}$ $\longrightarrow$ kugla polumjera $\frac{1}{2}$ sa središtem u $\left(\frac{1}{2},0,0\right)$

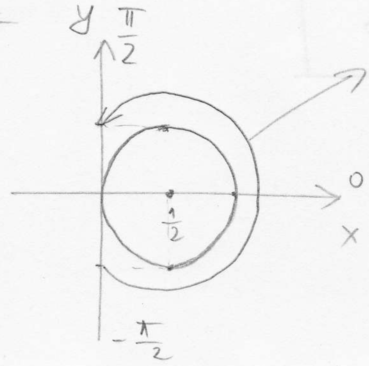


Uvodimo sferne koordinate:

$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}\right\} J = r^2\sin\theta$$

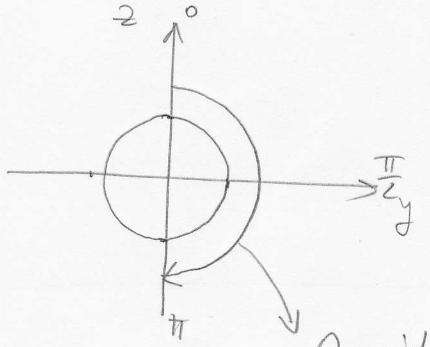
Projekcija na:

a) xy



$\rightarrow \varphi$ ide od $-\frac{\pi}{2}$ do $\frac{\pi}{2}$

b) $\underline{zy}$



θ ide od 0 do π

r prati jednadžbu sfere:

$$r^2 = r\sin\theta\cos\varphi$$

$$r(r - \sin\theta\cos\varphi) = 0$$

$$\boxed{\begin{array}{l} r_{DOLJE} = 0 \\ r_{GORE} = \sin\theta\cos\varphi \end{array}}$$

Podintegralna f-ja:

$$\sqrt{x^2+y^2+z^2} = \sqrt{r^2} = r$$

Pa imamo:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\sin\theta\cos\varphi} r^3 dr = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\pi} \sin^5\theta\cos^4\varphi\, d\theta =$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi\, d\varphi \int_0^{\pi} \sin^5\theta\, d\theta = \frac{1}{4}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{16}{15}\cdot\cos^4\varphi\, d\varphi =$$

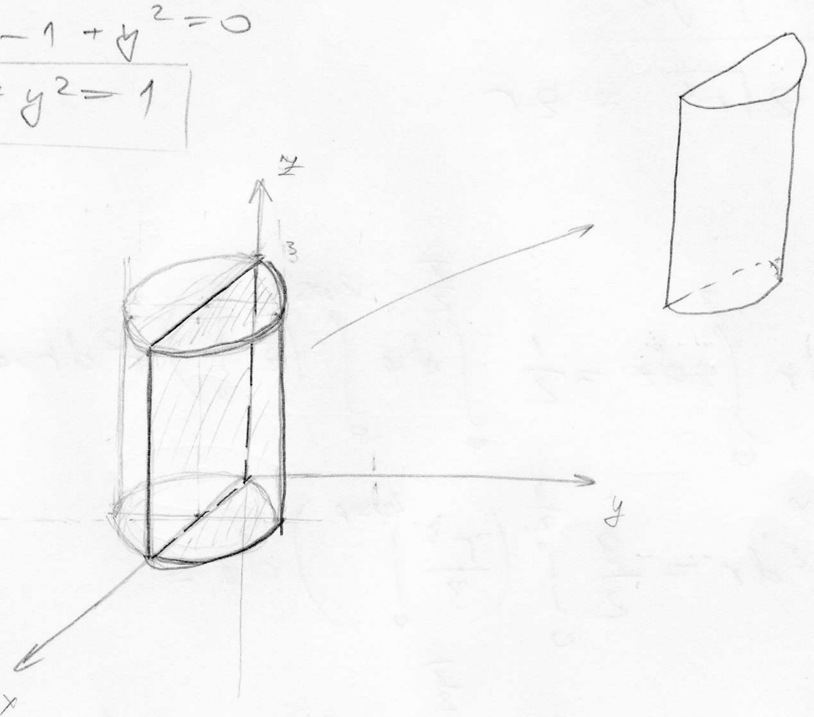
$$= \frac{4}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi\, d\varphi = \frac{\cancel{4}^{1}}{\cancel{15}_{5}}\cdot\frac{\cancel{3}^{1}\pi}{\cancel{8}_{2}} = \boxed{\frac{\pi}{10}}$$

16.

$$\iiint_V z\sqrt{x^2+y^2}\,dx\,dy\,dz$$

$$V \to \begin{array}{l} x^2+y^2=2x \\ y=0 \\ z=0 \\ z=3 \\ y\geq 0 \end{array}$$

$$x^2+y^2=2x$$
$$x^2-2x+y^2=0$$
$$x^2-2x+1-1+y^2=0$$
$$\boxed{(x-1)^2+y^2=1}$$

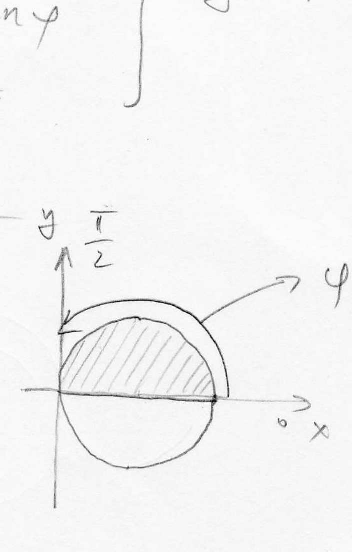


Uvodimo cilindrične koordinate:

$$\left.\begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \\ z=z \end{array}\right\} J=r$$

Projekcija na:

xy



$\to \varphi$ ide od 0 do $\frac{\pi}{2}$

r prati jednadžbu valjkaste plohe:

$$r^2 = 2r\cos\varphi$$

$$r(r-2\cos\varphi)=0$$

$$\boxed{\begin{aligned} r_{DONJE} &= 0 \\ r_{GORE} &= 2\cos\varphi \end{aligned}}$$

$z \Rightarrow$ donja granica je $\boxed{z=0}$, a gornja $\boxed{z=3}$

podintegralna f-ja:

$$z\sqrt{x^2+y^2} = z\sqrt{r^2} = zr$$

Pa imamo:

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 dr \int_0^3 z dz = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \left(z^2 \Big|_0^3\right) r^2 dr =$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 dr = \frac{9}{2}\int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3}\Big|_0^{2\cos\varphi}\right) d\varphi =$$

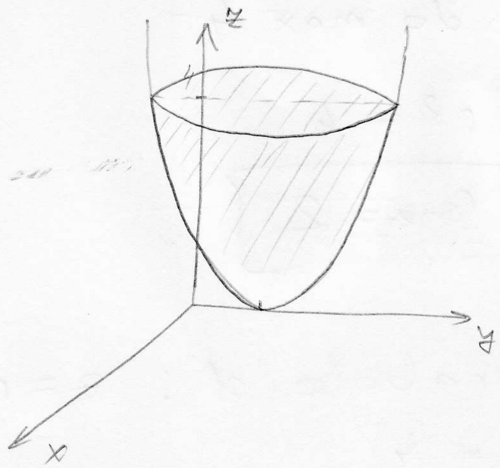
$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} 8\cos^3\varphi\, d\varphi = 12\int_0^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi = \cancel{12}^{4} \cdot \frac{2}{\cancel{3}_1} = \boxed{8}$$

17.

$$\iiint_V yz\,dx\,dy\,dz$$

$V \rightarrow \quad z = x^2 + (y-1)^2$

$z = 4$



Presjecište:

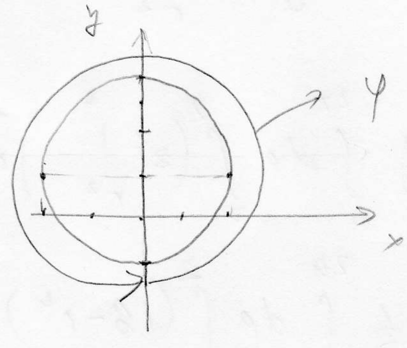
$z = x^2 + (y-1)^2$

$z = 4$

$4 = x^2 + (y-1)^2 \Rightarrow$ kružnica sa središtem u $(0,1)$ polumjera 2

Projekcija na:

xy



φ ide od 0 do 2π

Prelazimo na cilindrične koordinate (pomaknute):

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi + 1 \\ z &= z \end{aligned}\right\} J = r$$

time nam se mijenja i položaj tijela, ali kut φ ostaje isti!!!

r prati jednadžbu rotacijskog paraboloida

$$z = x^2 + (y-1)^2$$

$$z = r^2$$

$\downarrow$

z ide od 0 do max 4

$r^2 = 0$ $\qquad$ $r^2 = 4$

$\boxed{r_{donje} = 0}$ $\qquad$ $\boxed{r_{gornje} = 2}$

$z \rightarrow$ donja granica paraboloid : $z = r^2$

gornja granica : $z = 4$

podintegralna f-ja :

$$yz = (r\sin\varphi + 1)z = zr\sin\varphi + z$$

Pa imamo :

$$\int_0^{2\pi} \sin\varphi\, d\varphi \int_0^2 r^2 dr \int_{r^2}^4 z\, dz + \int_0^{2\pi} d\varphi \int_0^2 r\, dr \int_{r^2}^4 z\, dz =$$

$$= \frac{1}{2}\int_0^{2\pi} \sin\varphi\, d\varphi \int_0^2 \left(z^2 \Big|_{r^2}^4\right) r^2 dr + \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^2 \left(z^2 \Big|_{r^2}^4\right) r\, dr =$$

$$= \frac{1}{2}\int_0^{2\pi} \sin\varphi\, d\varphi \int_0^2 (16 - r^4) r^2 dr + \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^2 (16 - r^4) r\, dr =$$

$$= 8\int_0^{2\pi} \sin\varphi\, d\varphi \int_0^2 r^2 dr - \frac{1}{2}\int_0^{2\pi} \sin\varphi\, d\varphi \int_0^2 r^6 dr + 8\int_0^{2\pi} d\varphi \int_0^2 r\, dr -$$

$$- \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^2 r^5 dr = \underbrace{\frac{64}{3}\int_0^{2\pi} \sin\varphi\, d\varphi}_{0} - \underbrace{\frac{64}{7}\int_0^{2\pi} \sin\varphi\, d\varphi}_{0} + 16 \cdot 2\pi -$$

$$- \frac{1}{\cancel{2}_1} \frac{\cancel{32}^{16}}{3} \cdot 2\pi = 32\pi - \frac{32\pi}{3} = \boxed{\frac{64\pi}{3}}$$

18.

$$\iiint_V x^2\,dx\,dy\,dz$$

$$V \rightarrow \quad z = 3 - \sqrt{x^2+y^2}$$

$$z = 1$$

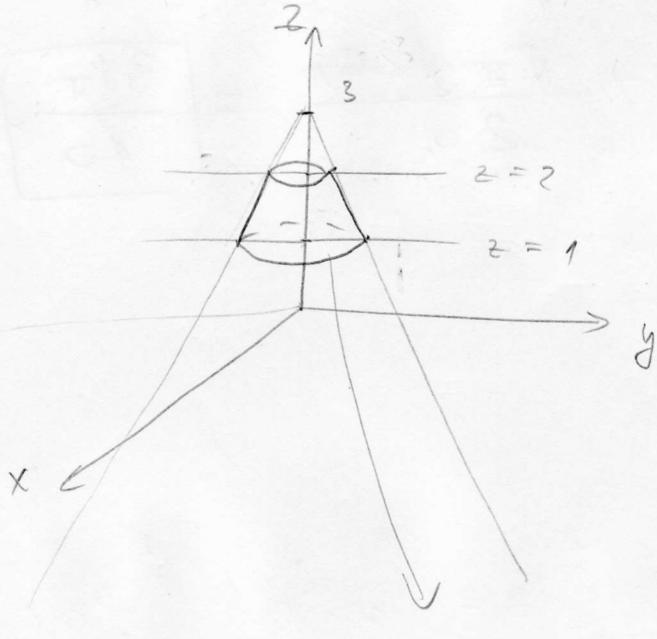
$$z = 2$$

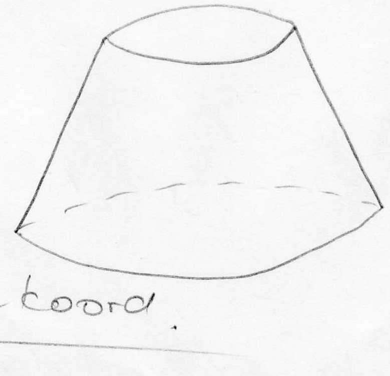
$$z = 3 - \sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = 3 - z \quad /^2$$

$$x^2+y^2 = (3-z)^2$$

$$x^2+y^2 = (z-3)^2 \rightarrow \text{stožac sa vrhom u } (3,0,0) \text{ okrenut prema dolje}$$



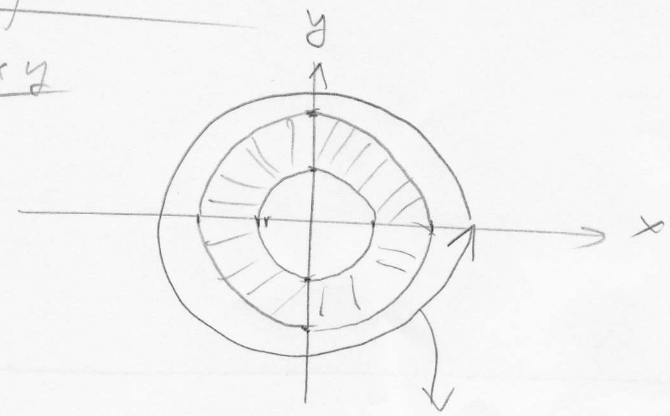


Uvest ćemo cilindrične koord.

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} \; J = r$$

Projekcija na:

xy



φ ide od 0 do 2π

presjecišta:

$z=1 \Rightarrow 1 = 3-\sqrt{x^2+y^2}$

$-2 = -\sqrt{x^2+y^2}$

$$\boxed{x^2+y^2=4}$$

$z=2 \Rightarrow 2 = 3-\sqrt{x^2+y^2}$

$-1 = -\sqrt{x^2+y^2}$

$$\boxed{x^2+y^2=1}$$

Moramo podijeliti na dva integrala:

$$\int_0^{2\pi}\cos^2\varphi\, d\varphi\int_0^1 r^3 dr\int_1^2 dz + \int_0^{2\pi}\cos^2\varphi\, d\varphi\int_1^2 r^3 dr\int_1^{3-r} dz =$$

$$= \frac{\pi}{4} + \frac{13\pi}{10} = \frac{5\pi+26\pi}{20} = \boxed{\frac{31\pi}{10}}$$

(20.)

$$\int_{-\sqrt{3}}^{\sqrt{3}} dx \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} dy \int_{1}^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz$$

A. cilindrične

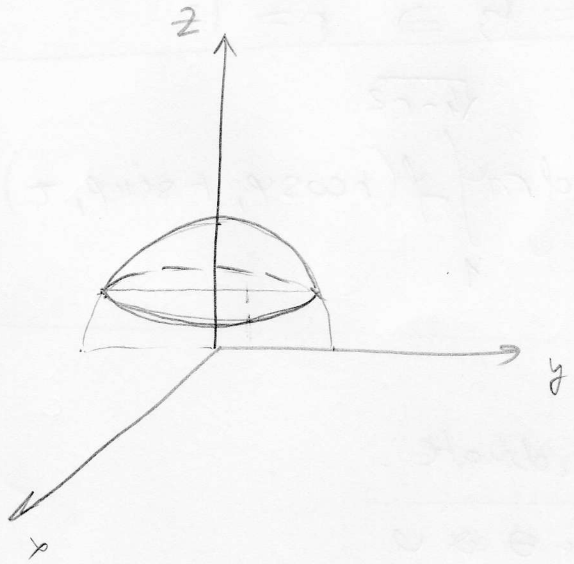
B. sferne

$$\boxed{\begin{gathered} x=-\sqrt{3} \\ x=\sqrt{3} \\ x^2+y^2=3 \\ z=1 \\ x^2+y^2+z^2=4 \end{gathered}}$$

$z=1 \rightarrow x^2+y^2+1=4$

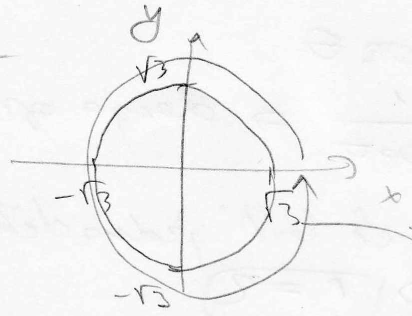
$x^2+y^2=3$

$x^2+y^2=(\sqrt{3})^2$



Projekcija na:

xy



$\rightarrow$ φ ide od 0 do 2π

r ide od 0 do jednadžbe sfere:

z ide od 1 do sfere

A. cilindrične koord.

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$

$$x^2 + y^2 + z^2 = 4$$

$$r^2 + z^2 = 4$$

$$z^2 = 4 - r^2$$

$$\boxed{z = \sqrt{4 - r^2}} \Rightarrow \text{jed. sfere u cilindričnim koord.}$$

$$r^2 + 1 = 4$$

$$r^2 = \sqrt{3} \Rightarrow r = \sqrt{3}$$

$$\boxed{\int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} r\,dr \int_1^{\sqrt{4-r^2}} f(r\cos\varphi, r\sin\varphi, z)\,dz}$$

B. Sferne koordinate:

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

~~za~~ $z = 1 \Rightarrow 1 = r\cos\theta$

$r = \frac{1}{\cos\theta} \Rightarrow$ donja granica za r

gornja granica za r će biti jednadžba sfere:

$$r^2 = 4 \Rightarrow \boxed{r = 2}$$

Za sjecište pravca $x = \sqrt{3}$ i sfere imamo da je kut $\theta = \frac{\pi}{3} \Rightarrow \theta$ ide od 0 do $\frac{\pi}{3}$

$$\boxed{\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{3}} \sin\theta\,d\theta \int_{\frac{1}{\cos\theta}}^{2} r^2 f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\,dr}$$

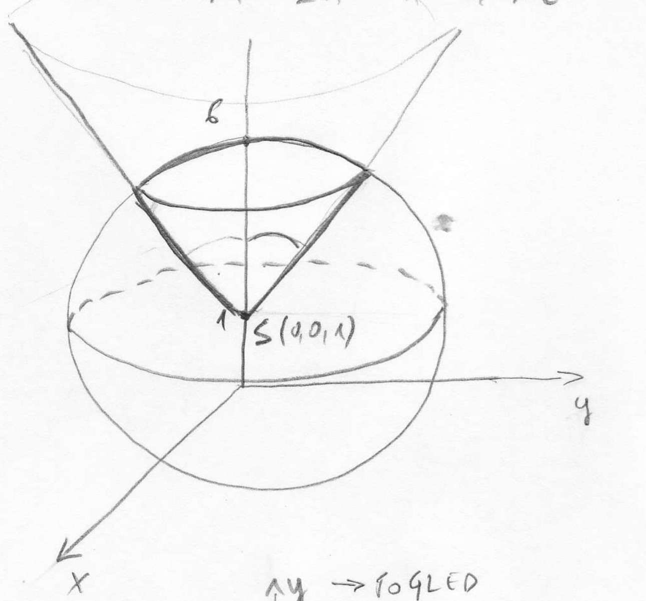
$\boxed{240}$ $\iiint_V (x^2+y^2)\,dx\,dy\,dz$ $\quad V = \{(x,y,z) : x^2+y^2+(z-1)^2 \le 25$ ← *

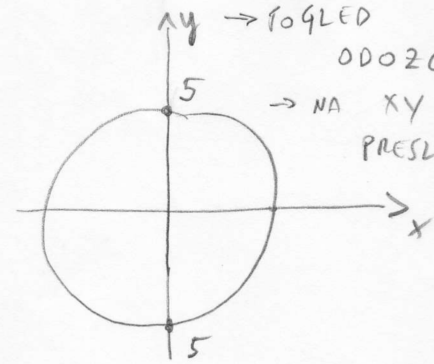
$: z \ge \sqrt{x^2+y^2} + 1$ ←

$z^2 \ge \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1$

* POMAKNUTA SFERA SA R=5

STOŽAC, SAMO GORNJI DIO, TRANSLATIRA ZA 1 GORE





→ POGLED ODOZGO → NA XY RAVNINU PRESLIKAMO

$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} d\theta \int_0^5 r^2 \sin^2\theta \cdot r^2 \sin\theta \, dr$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} \sin^3\theta \, d\theta \int_0^5 r^4 \, dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} \sin^3 \frac{5^5}{5} \, d\theta =$$

$$= 5^4 \cdot \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} (1-\cos^2\theta) \cdot \sin\theta \, d\theta \quad \left| \begin{array}{l} \cos\theta = t \\ dt = -\sin\theta \, d\theta \\ D: \cos 0 = 1 \\ G: \cos\frac{\pi}{4} = \end{array} \right|$$

$$= -5^4 \int_0^{2\pi} d\varphi \int_1^{\frac{\sqrt{2}}{2}} (1-t^2) \, dt =$$

$$= -5^4 \cdot \int_0^{2\pi} d\varphi \left(\frac{\sqrt{2}}{2} - 1 - \frac{\frac{2\sqrt{2}}{8}}{3} + 1 \right)$$

$$= -5^4 \int_0^{2\pi} \frac{10\sqrt{2}}{24} = -5^4 \cdot \frac{10\sqrt{2}}{24} \cdot 2\pi$$

- vjerojatno sam negdje falio ali manje bitno -.-

→ STOŽAC U POMAKNUTIM SFERNIMA

$$\sqrt{x^2+y^2} + 1 \le z$$

$$\sqrt{r^2 \sin^2\theta} + 1 = \cancel{r} \cos\theta$$

$$|\sin\theta| = \cos\theta$$

$\Downarrow$

vidimo da je $\theta = \frac{\pi}{4}$

USTVARI BI TREBAO RASPISIVATI NA SLUČAJEVE JER COS MIJENJA PREDZNAK

$[0, \frac{\pi}{2}]$ $\cos\theta \ge 0$ $\qquad$ $[\frac{\pi}{2}, \pi]$ $\cos\theta \le 0$

$\operatorname{tg}\theta = 1$ $\qquad$ ooo

POMAKNUTE SFERNE:

$$\left.\begin{array}{l} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta + 1 \end{array}\right\} dx\,dy\,dz = r^2 \sin\theta \, dr\, d\varphi\, d\theta$$

-1

$$x^2+y^2+(z-1)^2 = 25$$

$\Downarrow$

$$r = 5$$

FUNKCIJA: (x^2+y^2)

$$r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi$$

$$= r^2\sin^2\theta$$

ZAD 22.

$$\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 4r^2(1+2\cos^2\theta)\cdot 8\sqrt{3}r^2\sin\theta\,dr$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} (1+2\cos^2\theta)\sin\theta \int_0^1 8\sqrt{3}r^2\cdot 4r^2$$

$$\left/\begin{array}{l}\cos\theta = t\\ -\sin\theta\,d\theta = dt\end{array}\right/$$

$$= \int_0^{2\pi} d\varphi \int_{\cos 0 = 1}^{\cos\pi = -1} \frac{32\sqrt{3}}{5}\cdot(1+2t^2)\,dt$$

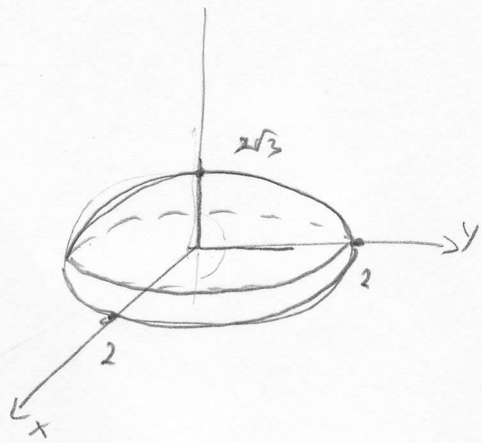
$$= \frac{32\sqrt{3}}{5}\int_0^{2\pi} d\varphi \int_1^{-1} (1+2t^2)\,dt$$

$$= \frac{32\sqrt{3}}{5}\cdot 2\pi\cdot\left(2+2\cdot\frac{2}{3}\right) =$$

$$= \frac{128\pi\sqrt{3}}{3}$$

V: elipsoid $3x^2+3y^2+z^2=12 \;/:12$

$$\frac{x^2}{2^2}+\frac{y^2}{2^2}+\frac{z^2}{(2\sqrt{3})^2}=1$$



$x = 2r\sin\theta\cos\varphi$

$y = 2r\sin\theta\sin\varphi$

$z = 2\sqrt{3}\,r\cos\theta$

$dx\,dy\,dz = 2\cdot 2\cdot 2\sqrt{3}\,r^2\sin\theta$

funkcja:

$$(2r\sin\theta\cos\varphi)^2+(2r\sin\theta\sin\varphi)^2+(2\sqrt{3}r\cos\theta)^2$$

$$= 4r^2\sin^2\theta\underbrace{\cos^2\varphi + 4r^2\sin^2\theta\sin^2\varphi}_{=1} + 12r^2\cos^2\theta \quad (8+4)r^2\cos^2\theta$$

$$= 4r^2\sin^2\theta + 4r^2\cos^2\theta + 8r^2\cos^2\theta$$

$$= 4r^2 + 8r^2\cos^2\theta$$

$$= 4r^2(1+2\cos^2\theta)$$

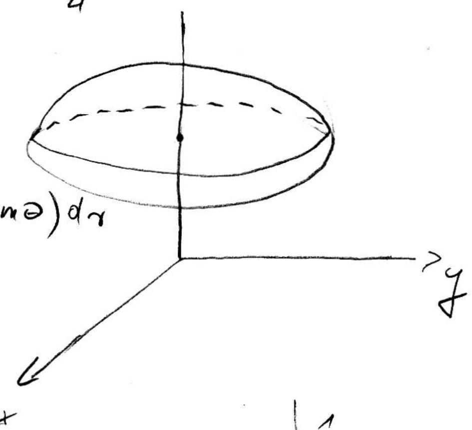
(23.) $\iiint z^2\,dx\,dy\,dz$ V: elipsoid $x^2+4y^2+(z-5)^2=4$

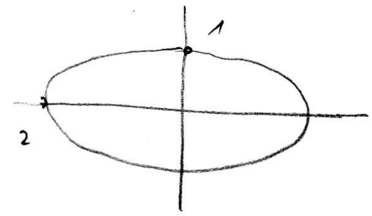
$$\frac{x^2}{4}+\frac{y^2}{1}+\frac{(z-5)^2}{4}=1$$

$$=\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^1 \left(4r^2\cos^2\theta+20r\cos\theta+25\right)4r^2\sin\theta\,dr$$

$$=\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \left(16r^4\cos^2\theta\sin\theta+80r^3\cos\theta\sin\theta+100r^2\sin\theta\right)dr$$

$$=\int_0^{2\pi} d\varphi \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta \int_0^1 16r^4\,dr$$
$$+\int_0^{2\pi} d\varphi \int_0^{\pi}\cos\theta\sin\theta\,d\theta \int_0^1 80r^3\,dr$$
$$+\int_0^{2\pi} d\varphi \int_0^{\pi}\sin\theta\,d\theta \int_0^1 100r^2\,dr$$





POOPĆENE SFERNE:

$x = 2r\sin\theta\cos\varphi$

$y = 1r\sin\theta\sin\varphi$

$z-5 = 2r\cos\theta$

$|J| = 2\cdot 1\cdot 2\,r^2\sin\theta$

elipsoid $\rightarrow$ 1

funkcija $(2r\cos\theta+5)^2$

$= 4r^2\cos^2\theta+20r\cos\theta+25$

$$=2\pi\cdot\int_0^{\pi}\cos^2\theta\sin\theta\cdot\frac{16}{5}\,d\theta \quad \left|\begin{array}{ll}\cos\theta = t & \\ dt=-\sin\theta & \\ D: 1 & G: -1\end{array}\right|$$
$$+2\pi\int_0^{\pi}\cos\theta\sin\theta\cdot\frac{80}{4}\,d\theta \quad \left|\begin{array}{ll}\cos\theta = t & \\ dt=-\sin\theta & \\ D: 1 & G: -1\end{array}\right|$$
$$+2\pi\int_0^{\pi}\sin\theta\cdot\frac{100}{3}\,d\theta$$

$$=2\pi\cdot\left[-\frac{16}{5}\int_1^{-1}t^2\,dt-20\int_1^{-1}t\,dt-\frac{100}{3}\cos\theta\Big|_0^{\pi}\right]$$

$$=2\pi\cdot\left[\frac{16}{5}\,\frac{t^3}{3}\Big|_{-1}^{1}+20\cdot\frac{t^2}{2}\Big|_{-1}^{1}-\frac{100}{3}(-1-1)\right]$$

$$=2\pi\cdot\left[\frac{16}{5}\cdot\frac{2}{3}+0+\frac{100}{3}\cdot 2\right]=$$

$$=2\pi\cdot\left[\frac{32}{15}+\frac{200\cdot 5}{15}\right]$$

$$=\frac{688}{5}\pi \quad //$$