Laplaceova transformacija

(2. dio)

by Vedax

Konvolucija

95. str.

1. Izračunaj konvolucije sljedećih funkcija:

A. $1 * t$

$$1 * t = \int_0^t 1 \cdot (t-\tau) d\tau = t\int_0^t d\tau - \int_0^t \tau d\tau = t \cdot \tau\Big|_0^t - \frac{\tau^2}{2}\Big|_0^t = t \cdot \tau - \frac{\tau^2}{2}$$

B. $t * t^3$

$$t * t^3 = \int_0^t \tau(t-\tau)^3 d\tau = \int_0^t \left(\tau \cdot t^3 - 3t^2 \cdot \tau^2 + 3t \cdot \tau^3 - \tau^4\right) d\tau =$$

$$= t^3 \frac{\tau^2}{2}\Big|_0^t - 3t^2 \frac{\tau^3}{3}\Big|_0^t + 3t\frac{\tau^4}{4}\Big|_0^t - \frac{\tau^5}{5}\Big|_0^t = \frac{t^5}{2} - t^5 + \frac{3t^5}{4} - \frac{t^5}{5} =$$

$$= \frac{t^5}{20}$$

C. $t * t^3$

$$t^2 * t^2 = \int_0^t \tau^2 (t-\tau)^2 d\tau = t^2 \frac{\tau^3}{3}\Big|_0^t - 2t\frac{\tau^4}{4}\Big|_0^t + \frac{\tau^5}{5}\Big|_0^t = \frac{t^5}{30}$$

D. $e^t * e^t$

$$e^t * e^t = \int_0^t e^\tau e^{t-\tau} d\tau = e^t \int_0^t d\tau = e^t \tau\Big|_0^t = te^t$$

2. Koristeći teorem o konvoluciji, izračunaj original funkcija:

A. $\dfrac{1}{s(s+3)}$

$$\frac{1}{s(s+3)}=\frac{1}{s}\cdot\frac{1}{s+3}$$

$$\frac{1}{s}\bullet\!\!-\!\!\circ\, u(t)$$

$$\frac{1}{s+3}\bullet\!\!-\!\!\circ\, e^{-3t}$$

$$\frac{1}{s(s+3)}\bullet\!\!-\!\!\circ\, e^{-3t} * u(t)=\int_0^t e^{-3\tau}u(t-\tau)d\tau=\int_0^t e^{-3\tau}d\tau=-\frac{e^{-3\tau}}{3}\Bigg|_0^t=\frac{1}{3}-\frac{1}{3}e^{-3t}$$

B. $\dfrac{1}{s\left(s^2-4s+5\right)}$

$$\frac{1}{s\left(s^2-4s+5\right)}=\frac{1}{s}\cdot\frac{1}{s^2-4s+5}=\frac{1}{s}\cdot\frac{1}{(s-2)^2+1}$$

$$\frac{1}{s}\bullet\!\!-\!\!\circ\, u(t)$$

$$\frac{1}{(s-2)^2+1}\bullet\!\!-\!\!\circ\, e^{2t}\sin t$$

$$\frac{1}{s\left(s^2-4s+5\right)}\bullet\!\!-\!\!\circ\, e^{2t}\sin t * u(t)=\int_0^t e^{2\tau}\sin\tau\cdot u(t-\tau)d\tau=\int_0^t e^{2\tau}\sin\tau d\tau$$

$$\int e^{2t}\sin t dt = \left| \begin{matrix} u = e^{2t} & dv = \sin t dt \\ du = 2e^{2t} dt & v = -\cos t \end{matrix} \right| = -e^{2t}\cos t + 2\int e^{2t}\cos t dt =$$

$$= \left| \begin{matrix} u = e^{2t} & dv = \cos t dt \\ du = 2e^{2t} dt & v = \sin t \end{matrix} \right| = -e^{2t}\cos t + 2e^{2t}\sin t - 4\int e^{2t}\sin t dt$$

$$\int e^{2t}\sin t dt = -e^{2t}\cos t + 2e^{2t}\sin t - 4\int e^{2t}\sin t dt$$

$$5\int e^{2t}\sin t dt = -e^{2t}\cos t + 2e^{2t}\sin t$$

$$\int e^{2t}\cos t dt = -\frac{1}{5}e^{2t}\cos t + \frac{2}{5}e^{2t}\sin t$$

$$\frac{1}{s\left(s^2 - 4s + 5\right)} \bullet\!\!-\!\!\circ -\frac{1}{5}e^{2\tau}\cos\tau\Big|_0^t + \frac{2}{5}e^{2\tau}\sin\tau\Big|_0^t = \frac{1}{5} - \frac{1}{5}e^{2t}\cos t + \frac{2}{5}e^{2t}\sin t$$

Rješavanje diferencijalnih i integralnih jednadžbi

102. str.

1. Primjenom Laplaceove transformacije riješi sljedeće diferencijalne jednadžbe:

A. $y'' + y' - 2y = 2t$, $y(0) = 0$, $y'(0) = 1$

$$y''(t) \circ\!\!-\!\!\bullet\, s^2Y(s) - sy(0) - y'(0)$$

$$y'(t) \circ\!\!-\!\!\bullet\, sY(s) - y(0)$$

$$-2y(t) \circ\!\!-\!\!\bullet\, -2Y(s)$$

$$2t \circ\!\!-\!\!\bullet\, \frac{2}{s^2}$$

Pa početna jednadžba prelazi u:

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = \frac{2}{s^2}$$

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = \frac{2}{s^2}$$

$$Y(s)\left(s^2 + s - 2\right) = \frac{s^2 + 2}{s^2} \Leftrightarrow Y(s) = \frac{s^2 + 2}{s^2(s-1)(s+2)} = \frac{As + B}{s^2} + \frac{C}{s-1} + \frac{D}{s+2}$$

$$s^2 + 2 = (As + B)\left(s^2 + s - 2\right) + Cs^2(s+2) + Ds^2(s-1)$$
$$s^2 + 2 = As^3 + As^2 - 2As + Bs^2 + Bs - 2B + Cs^3 + 2Cs^2 + Ds^3 - Ds^2$$
$$s^2 + 2 = s^3(A + C + D) + s^2(A + B + 2C - D) + s(-2A + B) - 2B$$

$$A+C+D=0$$
$$A+B+2C-D=1$$
$$-2A+B=0 \Rightarrow A=-\frac{1}{2}$$
$$-2B=2 \Rightarrow B=-1$$

$$C+D=\frac{1}{2}$$
$$2C-D=\frac{5}{2}$$

$$C=1$$
$$D=-\frac{1}{2}$$

Pa imamo:

$$Y(s)=\frac{-\frac{1}{2}s-1}{s^2}+\frac{1}{s-1}+\frac{-\frac{1}{2}}{s+2}$$
$$Y(s)=-\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s-1}-\frac{1}{2}\cdot\frac{1}{s+2}$$

Pa je rješenje:

$$y(t)=-\frac{1}{2}-t+e^t-\frac{1}{2}e^{-2t}$$

5. Primjenom Laplaceove transformacije riješi integralnu jednažbu:

A. $y(t) = at^2 + \int_0^t \sin\tau \cdot y(t-\tau)d\tau$

$$\int_0^t \sin\tau \cdot y(t-\tau)d\tau = \sin t * y(t)$$

$$\sin t \circ\!\!-\!\!\bullet \frac{1}{s^2+1}$$

$$y(t) \circ\!\!-\!\!\bullet Y(s)$$

Gornja jednadžba se preslikava u:

$$Y(s) = a\frac{2}{s^3} + \frac{1}{s^2+1}Y(s)$$

$$Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{2a}{s^3}$$

$$Y(s) \cdot \frac{s^2}{s^2+1} = \frac{2a}{s^3} \Leftrightarrow Y(s) = \frac{2a\left(s^2+1\right)}{s^5} = \frac{2a}{s^3} + \frac{1}{12}\frac{24a}{s^5}$$

Pa je rješenje:

$$y(t) = at^2 + \frac{t^4}{12}$$

7. Primjenom Laplaceove transformacije riješi integralno-diferencijalnu jednadžbu:

A. $y'(t)=\int_0^t y(u)\cos(t-u)du\,,\ y(0)=1$

$$y'(t) \circ\!\!-\!\!\bullet\, sY(s)-y(0)=sY(s)-1$$

$$\int_0^t y(u)\cos(t-u)du = y(t) * \cos t$$

$$y(t) \circ\!\!-\!\!\bullet\, Y(s)$$

$$\cos t \circ\!\!-\!\!\bullet\, \frac{s}{s^2+1}$$

$$\int_0^t y(u)\cos(t-u)du \circ\!\!-\!\!\bullet\, Y(s)\cdot\frac{s}{s^2+1}$$

Pa se početna jednadžba preslikava u:

$$sY(s)-1=Y(s)\cdot\frac{s}{s^2+1}$$

$$Y(s)\left(s-\frac{s}{s^2+1}\right)=1 \Leftrightarrow Y(s)=\frac{s^2+1}{s^3}=\frac{1}{s}+\frac{1}{s^3}$$

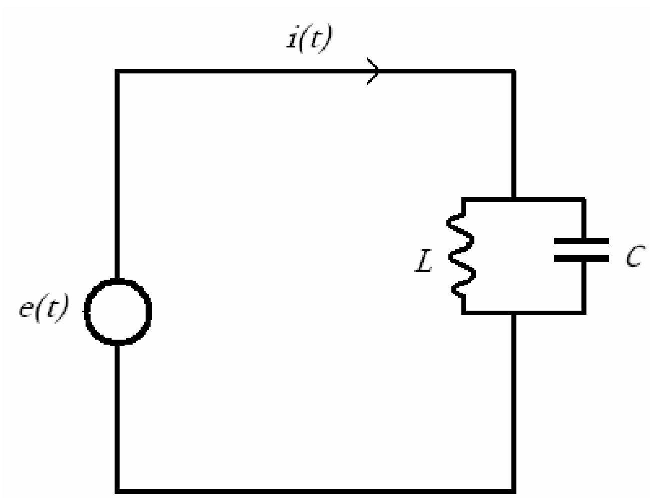
Pa je rješenje:

$$y(t)=1+\frac{t^2}{2}$$

Primjene

Ovo sam riješio zadatke sa prošlih međuispita.

2007./08.



Zadan je $e(t) = 1 + \cos 2t$, $L = 1$ i $C = 1$ i treba naći struju $i(t)$.

Najprije napon preslikamo u Laplaceovu domenu:

$$e(t) = 1 + \cos 2t \circ\!\!-\!\!\bullet\, E(s) = \frac{1}{s} + \frac{s}{s^2 + 4}$$

Zatim izračunamo imedanciju:

$$\frac{1}{Z(s)} = \frac{1}{\frac{1}{sC}} + \frac{1}{sL} \Rightarrow Z(s) = \frac{\frac{L}{C}}{\frac{1}{sC} + sL} = \frac{1}{\frac{1}{s} + s} = \frac{s}{s^2 + 1}$$

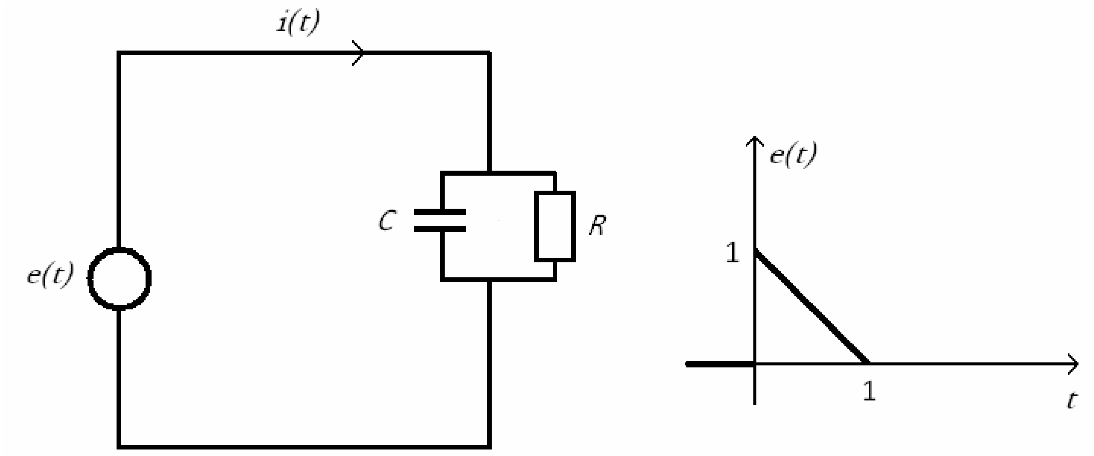
I onda nam je struja prema Ohmovom zakonu:

$$I(s)=\frac{E(s)}{Z(s)}=\frac{s^2+1}{s}\cdot\left(\frac{1}{s}+\frac{s}{s^2+4}\right)=\frac{s^2+1}{s^2}+\frac{s^2+1}{s^2+4}=1+\frac{1}{s^2}+1-\frac{3}{s^2+4}$$

Pa nam je struja jednaka:

$$i(t)=\left(2\delta(t)+t-\frac{3}{2}\sin 2t\right)u(t)$$

2008./09.



Zadana nam je napon. Moramo najprije očitati iz slike koliko on iznosi. Samo uvrstite točke (0,1) i (1,0) u jednadžbu pravca kroz dvije točke, i dobijete da je napon jednak:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Leftrightarrow e(t) - 1 = \frac{0-1}{1-0}(t-0) \Leftrightarrow e(t) = -t + 1$$

Međutim, takav napon je definiran na granici samo od 0 do 1, pa da biste dobili pravu vrijednost napon, pomnožiti $-t+1$ sa gate funkcijom:

$$e(t) = (-t+1)g_{[0,1]}(t) = (-t+1)(u(t) - u(t-1)) = -tu(t) + u(t) + (t-1)u(t-1)$$

Najprije odredimo sliku napona u Laplaceovoj domeni:

$$e(t) \circ\!\!-\!\!\bullet\, E(s) = -\frac{1}{s^2} + \frac{1}{s} + \frac{e^{-s}}{s^2}$$

Zatim izračunamo impedanciju:

$$Z(s) = \frac{\frac{1}{sC} \cdot R}{\frac{1}{sC} + R} = \text{(uvrstimo vrijednosti } C = 1 \text{ i } R = 1\text{)} = \frac{\frac{1}{s}}{\frac{1}{s} + 1} = \frac{1}{s+1}$$

I prema Ohmovom zakonu nam je:

$$I(s)=\frac{E(s)}{Z(s)}=(s+1)\left(-\frac{1}{s^2}+\frac{1}{s}+\frac{e^{-s}}{s^2}\right)=-\frac{1}{s}+1+\frac{e^{-s}}{s}-\frac{1}{s^2}+\frac{1}{s}+\frac{e^{-s}}{s^2}$$

$$I(s)=1+\frac{e^{-s}}{s}-\frac{1}{s^2}+\frac{e^{-s}}{s^2}$$

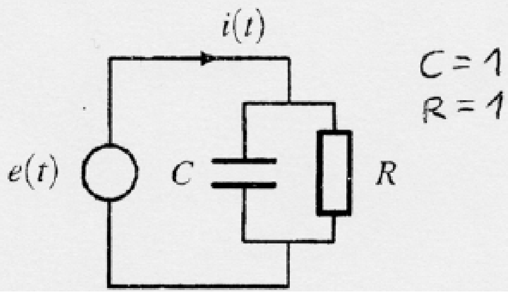
Pa nam je rješenje:

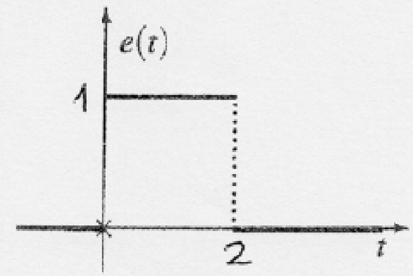
$$i(t)=\delta(t)-tu(t)+u(t-1)+(t-1)u(t-1)=$$
$$=\delta(t)-tu(t)+u(t-1)+tu(t-1)-u(t-1)$$

$$i(t)=\delta(t)-tu(t)+tu(t-1)=\delta(t)-t[u(t)-u(t-1)]$$

2008./09. – ponovljeni

8. (4 boda)
Pomoću Laplaceove transformacije izračunajte i skicirajte struju $i(t)$ strujnog kruga zadanog slikom 1 uz priključeni napon $e(t)$ zadan slikom 2.





Najprije odredimo napon. Iz slike vidimo da je napon jednak 1 samo na intervalu od 0 do 2. Znači, zapisat ćemo ga preko gate funkcije.

$$e(t) = 1 \cdot g_{[0,2]}(t) = u(t) - u(t-2)$$

Zatim preslikamo napon u Laplaceovu domenu:

$$e(t) \circ\!\!-\!\!\bullet \frac{1}{s} - \frac{e^{-2s}}{s}$$

Onda izračunamo impedanciju:

$$Z(s) = \frac{\frac{1}{sC} \cdot R}{\frac{1}{sC} + R} = \text{(uvrstimo vrijednosti } C = 1 \text{ i } R = 1\text{)} = \frac{\frac{1}{s}}{\frac{1}{s} + 1} = \frac{1}{s+1}$$

Pa je prema Ohmovom zakonu struja jednaka:

$$I(s) = \frac{E(s)}{Z(s)} = (s+1)\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) = 1 - e^{-s} + \frac{1}{s} - \frac{e^{-s}}{s}$$

Pa nam je rješenje:

$$i(t) = \delta(t) - \delta(t-1) + u(t) - u(t-1)$$