ZI - 30.1.2007.

(1) $f(x,y,z) = \ln \frac{yz}{x} + 2$

$(\text{grad} f)_T = 2\vec{i} - \vec{j} + 3\vec{k}$

a) $\text{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} =$

$= \frac{x}{yz}\cdot\left(-\frac{yz}{x^2}\right)\vec{i} + \frac{x}{yz}\cdot\frac{z}{x}\vec{j} + \frac{x}{yz}\cdot\frac{y}{x}\vec{k} =$

$= -\frac{1}{x}\vec{i} + \frac{1}{y}\vec{j} + \frac{1}{z}\vec{k}$

b) $-\frac{1}{x} = 2 \Rightarrow x = -\frac{1}{2}$

$\frac{1}{y} = -1 \Rightarrow y = -1$ $\quad T\left(-\frac{1}{2}, -1, \frac{1}{3}\right)$

$\frac{1}{z} = 3 \Rightarrow z = \frac{1}{3}$

(2) $f(x,y,z) = x - y^2 z$

$A(1,-1,3)$

$B(4,1,3)$

$\frac{\partial f}{\partial \vec{AB}} = ?$

$\frac{\partial f}{\partial \vec{AB}} = \vec{AB}_0 \cdot \text{grad} f$

$\vec{AB} = 3\vec{i} + 2\vec{j} + 0\cdot\vec{k}$

$\vec{AB}_0 = \frac{\vec{AB}}{|\vec{AB}|} = \frac{3\vec{i} + 2\vec{j}}{\sqrt{9+4}} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j}$

$\text{grad} f = \vec{i} - 2yz\vec{j} - y^2\vec{k}$

$\frac{\partial f}{\partial \vec{AB}} = \left(\frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j}\right)\cdot\left(\vec{i} - 2yz\vec{j} - y^2\vec{k}\right) = \frac{3}{\sqrt{13}} - \frac{4yz}{\sqrt{13}} = \frac{1}{\sqrt{13}}(3 - 4yz)$

③ $\vec{v}(x,y,z) = xz\vec{i} - 3\vec{j} - y\sqrt{z}\,\vec{k}$

$\vec{s} = 2\vec{i} - 2\vec{j} - \vec{k}$

$T(0,-4,1)$

$\left.\dfrac{\partial \vec{v}}{\partial \vec{s}}\right|_T = ?$

$$\frac{\partial \vec{v}}{\partial \vec{s}} = \left(s_{01}\frac{\partial}{\partial x} + s_{02}\frac{\partial}{\partial y} + s_{03}\frac{\partial}{\partial z}\right)\cdot \vec{v}$$

$$\vec{s}_0 = \frac{\vec{s}}{|\vec{s}|} = \frac{2\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{4+4+1}} = \underbrace{\frac{2}{3}}_{s_{01}}\vec{i} - \underbrace{\frac{2}{3}}_{s_{02}}\vec{j} - \underbrace{\frac{1}{3}}_{s_{03}}\vec{k}$$

$$\frac{\partial}{\partial x}\vec{v} = z\vec{i}$$

$$\frac{\partial}{\partial y}\vec{v} = -\sqrt{z}\,\vec{k}$$

$$\frac{\partial}{\partial z}\vec{v} = x\vec{i} - \frac{y}{2\sqrt{z}}\vec{k}$$

$$\frac{\partial \vec{v}}{\partial \vec{s}} = \frac{2}{3}\cdot z\vec{i} + \frac{2}{3}\sqrt{z}\,\vec{k} - \frac{1}{3}x\vec{i} + \frac{y}{6\sqrt{z}}\vec{k} = \left(-\frac{1}{3}x + \frac{2}{3}z\right)\vec{i} + \left(\frac{2}{3}\sqrt{z} + \frac{y}{6\sqrt{z}}\right)\vec{k}$$

$$\left.\frac{\partial \vec{v}}{\partial \vec{s}}\right|_T = \left(-\frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 1\right)\vec{i} + \left(\frac{2}{3}\sqrt{1} + \frac{-4}{6\sqrt{1}}\right)\vec{k} = \frac{2}{3}\vec{i}$$

④ $$\nabla\Big[\underbrace{(\vec{a}\cdot\vec{r})}_{\text{skalar}}\ \underbrace{\nabla\left(\tfrac{1}{r}\right)}_{\text{vektor}}\Big]$$

$$\Downarrow \qquad \Downarrow$$

$$f \qquad \vec{g}$$

$$\nabla(f\vec{g}) = (\nabla f)\vec{g} + f(\nabla\vec{g})$$

$$(\nabla f)\vec{g} \Leftrightarrow \left[\nabla(\vec{a}\cdot\vec{r})\right]\left[\nabla\left(\frac{1}{r}\right)\right] = \left[\nabla(a_1x + a_2y + a_3z)\right]\left[\left(\frac{1}{r}\right)'\cdot\frac{\vec{r}}{r}\right] =$$

$$= \underbrace{\left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right)}_{\vec{a}}\cdot\left(-\frac{\vec{r}}{r^3}\right) = -\frac{\vec{a}\cdot\vec{r}}{r^3}$$

$$f(\nabla\vec{g}) \Leftrightarrow (\vec{a}\cdot\vec{r})\left(\nabla\left(\nabla\left(\frac{1}{r}\right)\right)\right) = (\vec{a}\cdot\vec{r})\cdot\left(\nabla\left(-\frac{\vec{r}}{r^3}\right)\right) =$$

$$= (\vec{a}\cdot\vec{r})\underbrace{\left(\nabla\left(-\frac{1}{r^3}\cdot\vec{r}\right)\right)}_{*} = 0$$

$$* \quad \left[\nabla\left(-\frac{1}{r^3}\right)\right]\cdot\vec{r} - \frac{1}{r^3}\cdot(\nabla\vec{r}) = \left(-\frac{1}{r^3}\right)'\cdot\frac{\vec{r}}{r}\cdot\vec{r} - \frac{1}{r^3}\cdot 3 =$$

$$* \left[\nabla\left(-\frac{1}{r^3}\right)\right]\cdot\vec r - \frac{1}{r^3}\cdot(\nabla\vec r) = \left(-\frac{1}{r^3}\right)'\cdot\frac{\vec r}{r}\cdot\vec r - \frac{1}{r^3}\cdot 3 =$$

$$= \frac{3}{r^{4}}\cdot\frac{r^2}{r} - \frac{3}{r^3} = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

$$\nabla\left[(\vec a\cdot\vec r)\,\nabla\left(\frac{1}{r}\right)\right] = -\frac{\vec a\cdot\vec r}{r^3}$$

$$\left[\begin{aligned}&\vec a = a_1\vec i + a_2\vec j + a_3\vec k\\ &\vec r = x\vec i + y\vec j + z\vec k\\ &r = |\vec r| = \sqrt{x^2+y^2+z^2}\end{aligned}\right]$$

5.) $\vec a = z\vec i + x\vec j$

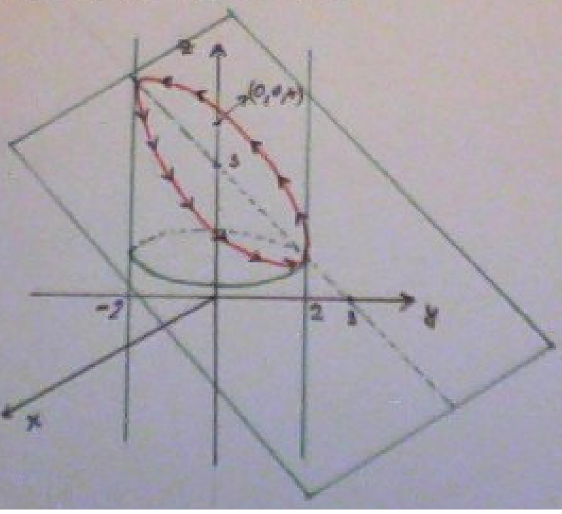
$K \ldots \quad x^2+y^2=4$

$\qquad\quad y+z=3$

poz. orijentirana gledajući iz $(0,0,4)$

$$\oint_K \vec a\, d\vec r = ?$$

→ KRIVULJNI INTEGRAL II. VRSTE



$$I = \oint_k \vec{a}\, d\vec{r} = \int_k z\,dx + x\,dy$$

$$x = 2\cos t$$
$$y = 2\sin t$$
$$z = 3 - y = 3 - 2\sin t, \quad t \in [0, 2\pi]$$

$$dx = -2\sin t\,dt$$
$$dy = 2\cos t\,dt$$

$$I = \int_k (3 - 2\sin t)(-2\sin t\,dt) + 2\cos t \cdot 2\cos t\,dt =$$

$$= \int_k (-6\sin t + 4\sin^2 t + 4\cos^2 t)\,dt = \int_k (4 - 6\sin t)\,dt =$$

$$= \int_0^{2\pi} 4\,dt - \int_0^{2\pi} 6\sin t\,dt = 4 \cdot 2\pi + 6\cos t \Big|_0^{2\pi} = 8\pi$$

6

$$\operatorname{rot} \vec{a} = \vec{0}$$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x^2} & z & y \end{vmatrix} = \vec{i}\,(1-1) - \vec{j}\,(0-0) + \vec{k}\,(0-0) = \vec{0}$$

$$p(x,y,z) = \int_{x_0}^{x} \frac{1}{x^2}\,dx + \int_{y_0}^{y} z\,dy + \int_{z_0}^{z} y_0\,dz =$$

$$= \int_1^x \frac{dx}{x^2} + \int_0^y z\,dy + \int_0^z 0\,dz =$$

$$= -\frac{1}{x}\Big|_1^x + zy\Big|_0^y = -\frac{1}{x} + 1 + zy + C =$$

$$= -\frac{1}{x} + yz + K$$

(7.) $\iint_S dS = ?$, $S \ldots \; x^2 + z^2 = 1$

$y \geq 0$

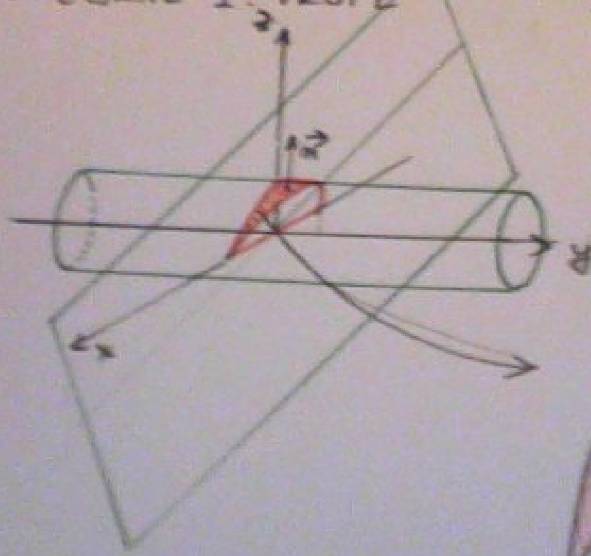
$y \leq z$

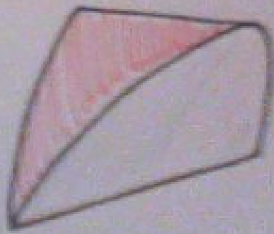
* nije potrebna normala xD slučajno nacrtana

PLOŠNI INTEGRAL I. VRSTE

-Iz slike $\Rightarrow z \geq 0$

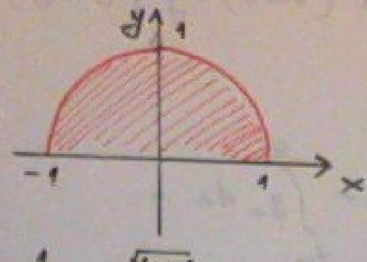
$z = \sqrt{1 - x^2}$





$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx\,dy$$

$$dS = \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx\,dy = \frac{dx\,dy}{\sqrt{1 - x^2}}$$



$$\iint_S dS = \int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}} \, dy =$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} \, dx =$$

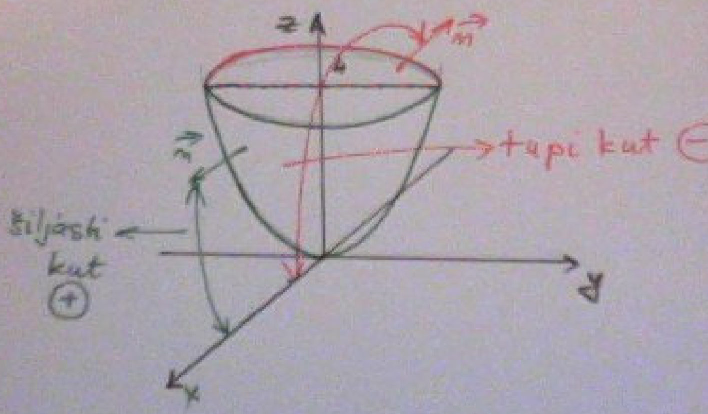
$$= \int_{-1}^{1} dx = 2$$

(8) S ... $z = x^2 + y^2$

$0 \le z \le 4$

$$\iint_S \sqrt{z}\, dy\, dz = ?$$

PLOŠNI INTEGRAL II. VRSTE



→ u integralu $dydz$ → tražimo x → $x = \pm\sqrt{z - y^2}$

x je pozitivan, predznak integrala pozitivan

x je negativan, predznak integrala negativan

$$\iint_{S,\, x>0} \sqrt{z}\, dy\, dz - \iint_{S,\, x<0} \sqrt{z}\, dy\, dz = 0$$