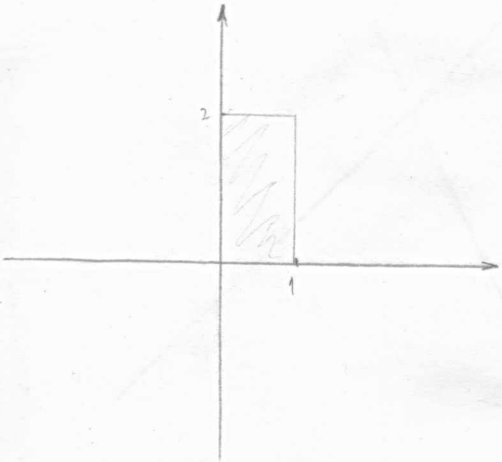
4. DOMAĆA ZADAĆA IZ MATEMATIKE 3

1. $\iint_P x\,dx\,dy$ $\quad X=1 \quad X=0 \quad Y=2 \quad Y=0$

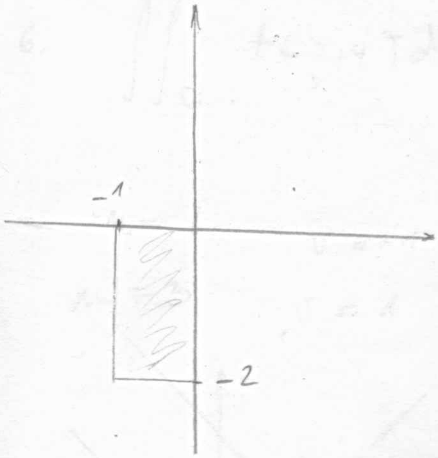


$$\int_0^1 x\,dx \int_0^2 dy =$$

$$\int_0^1 x \cdot y \Big|_0^2 dx = 2\int_0^1 x\,dx$$

$$= 2\,\frac{x^2}{2}\Big|_0^1 = 1$$

2. $\iint_P y\,dx\,dy$ $\quad X=-1 \quad X=0 \quad Y=-2 \quad Y=0$

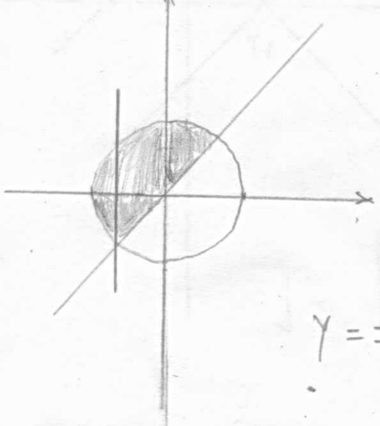


$$\int_{-1}^0 dx \int_{-2}^0 y\,dy = \int_{-1}^0 \frac{y^2}{2}\Big|_{-2}^0 dx =$$

$$\int_{-1}^0 (0-2)\,dx = -2\int_{-1}^0 dx = -2\,x\Big|_{-1}^0$$

$$= -2$$

3. $\iint_D f(x,y)\,dx\,dy$ $\quad Y \geq X \quad x^2+y^2 \leq 1$



$$\int_{-1}^{-\frac{\sqrt{2}}{2}} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy +$$

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} dx \int_x^{\sqrt{1-x^2}} f(x,y)\,dy$$

$\cos 45 = \frac{x}{1}$

$x = \cos 45$

$=$

$Y = \pm\sqrt{1-x^2}$

$Y = x$

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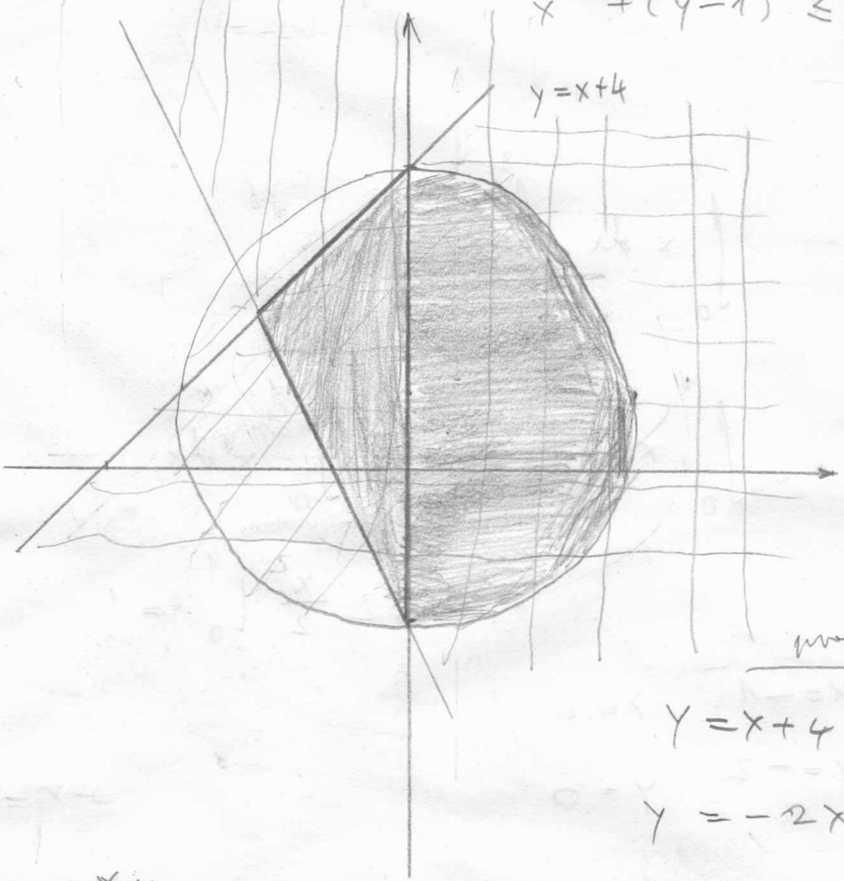
4. $\iint_D f(x,y)\,dx\,dy$

$\overset{0}{y} \le \overset{4}{x+4}$

$\overset{0}{y} \ge -2x-2$ $\quad 0 > -2$

$x^2+(y-1)^2 \le 9$



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$y = x+4$

$y = -2x-2$

$x+4 = -2x-2$

$3x = -6$

$x = 2$

$$\int_{-2}^{0} dx \int_{-2x-2}^{x+4} f(x,y)\,dy$$

$$+\int_{0}^{3} dx \int_{-\sqrt{9-x^2}+1}^{\sqrt{9-x^2}+1} f(x,y)\,dy$$

$(y-1)^2 = 9-x^2$

$y-1 = \pm\sqrt{9-x^2}$

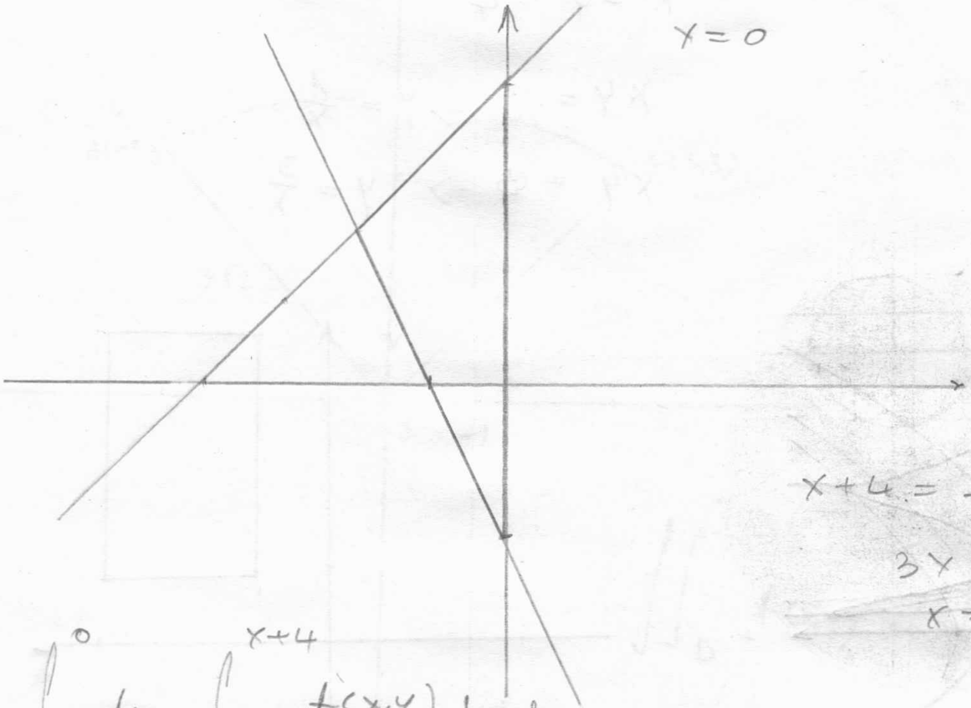
$y = \pm\sqrt{9-x^2}+1$

5. $\iint_D f(x,y)\,dx\,dy$

$y = x+4$

$y = -2x-2$

$x = 0$



$x+4 = -2x-2$

$3x = -6$

$x = -2$

$\int_{-2}^{0} dx \int_{-2x-2}^{x+4} f(x,y)\,dx\,dy$

6. $\iint_D f(x,y)\,dx\,dy$

$x+y=1 \qquad x-y=-1 \qquad y = -x+1$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad +3$

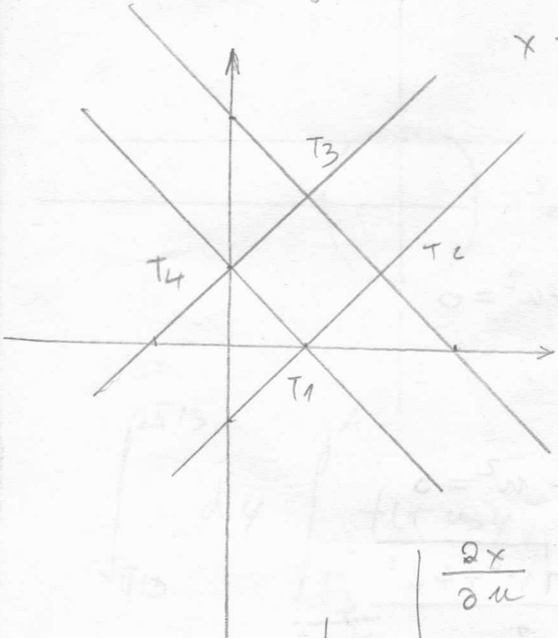
$x+y=3 \qquad x-y=1 \qquad y = x+1$

$y = x-1$

$u = 1 \qquad v = -1 \qquad u = x+y \qquad v = x-y \Rightarrow x = v+y$

$u = 3 \qquad v = 1 \qquad x = 1-y$

$x = 3-y \qquad x = u-y \qquad v = u-y-y$



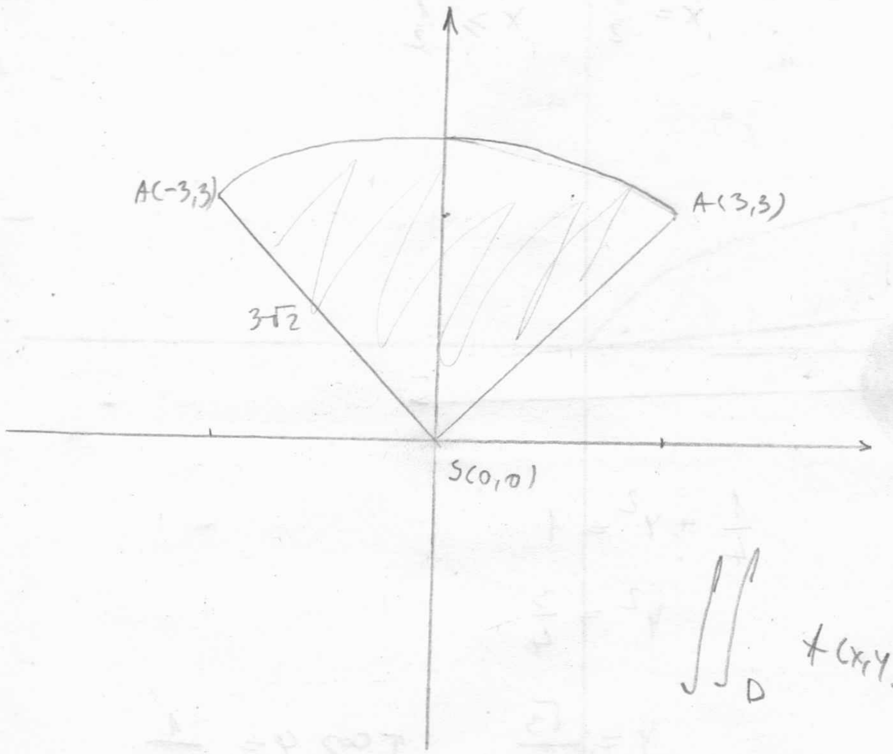
$u = v+2y \qquad y = \dfrac{u-v}{2}$

$x = \dfrac{u+v}{2}$

$I = \int_1^3 du \int_{-1}^{1} \frac{1}{2}\, f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) dv$

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{2} \quad -\frac{1}{2}$

8. $\iint_D f(x,y)\,dx\,dy$



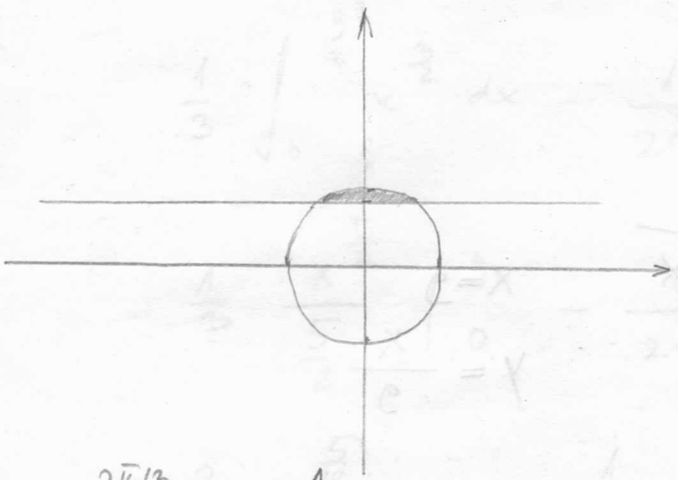
$x^2+y^2=r^2$

$\iint_D f(x,y)\,dx\,dy$

$$\int_{\pi/4}^{3\pi/4} d\varphi \int_0^{3\sqrt{2}} f(r\cos\varphi, r\sin\varphi)\, r\,dr$$

9. $\iint_D f(x,y)\,dx\,dy$ $\quad x^2+y^2=1$

$y=\frac{\sqrt{3}}{2} \quad y \geqslant \frac{\sqrt{3}}{2}$



$x^2+\left(\frac{\sqrt{3}}{2}\right)^2=1$

$x^2=-\frac{3}{4}+1=\frac{1}{4}$

$x=\pm\frac{1}{2}$

$$\int_{\pi/3}^{2\pi/3} d\varphi \int_{\frac{\sqrt{3}}{2\sin\varphi}}^{1} f(r\cos\varphi, r\sin\varphi)\, r\,dr$$

$\varphi=\arccos\frac{1}{2}$

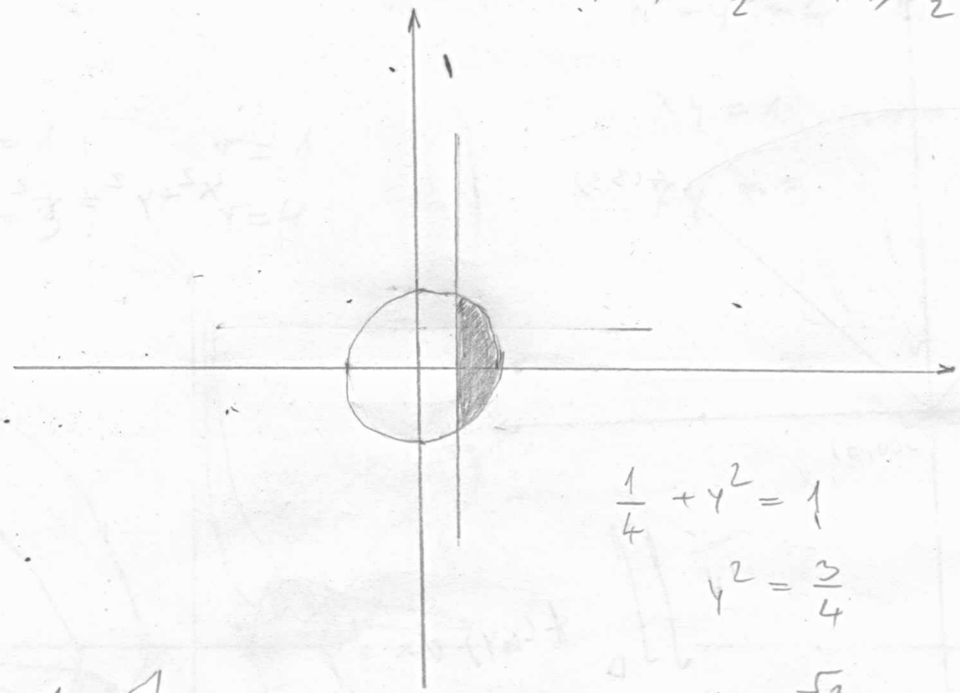
$\varphi=\frac{\pi}{3}$

$y=\frac{\sqrt{3}}{2} \qquad r\sin\varphi=\frac{\sqrt{3}}{2}$

$r=\frac{\sqrt{3}}{2\sin\varphi}$

10. $\iint_D f(x,y)\,dx\,dy$: $x^2+y^2=1$

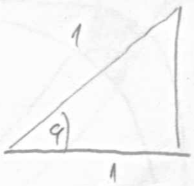
$x=\frac{1}{2}$ $\quad x \geq \frac{1}{2}$



$\frac{1}{4}+y^2=1$

$y^2=\frac{3}{4}$

$y=\frac{\sqrt{3}}{2}$ $\qquad r\cos\varphi=\frac{1}{2}$



$r=\frac{1}{2\cos\varphi}$

$\varphi = \arccos\frac{1}{2}$

$=60^\circ = \pm\frac{\pi}{3}$ rad

$$I=\int_{-\pi/3}^{\pi/3} d\varphi \int_{\frac{1}{2\cos\varphi}}^{1} f(r\cos\varphi, r\sin\varphi)\, r\,dr$$

11. $\iint_D y^2\,dx\,dy$ $\qquad y=\sqrt{x}$ $\qquad x=9$

$y=\frac{x}{9}$

$$\int_0^9 dx \int_{\frac{x}{9}}^{\sqrt{x}} y^2\,dy$$

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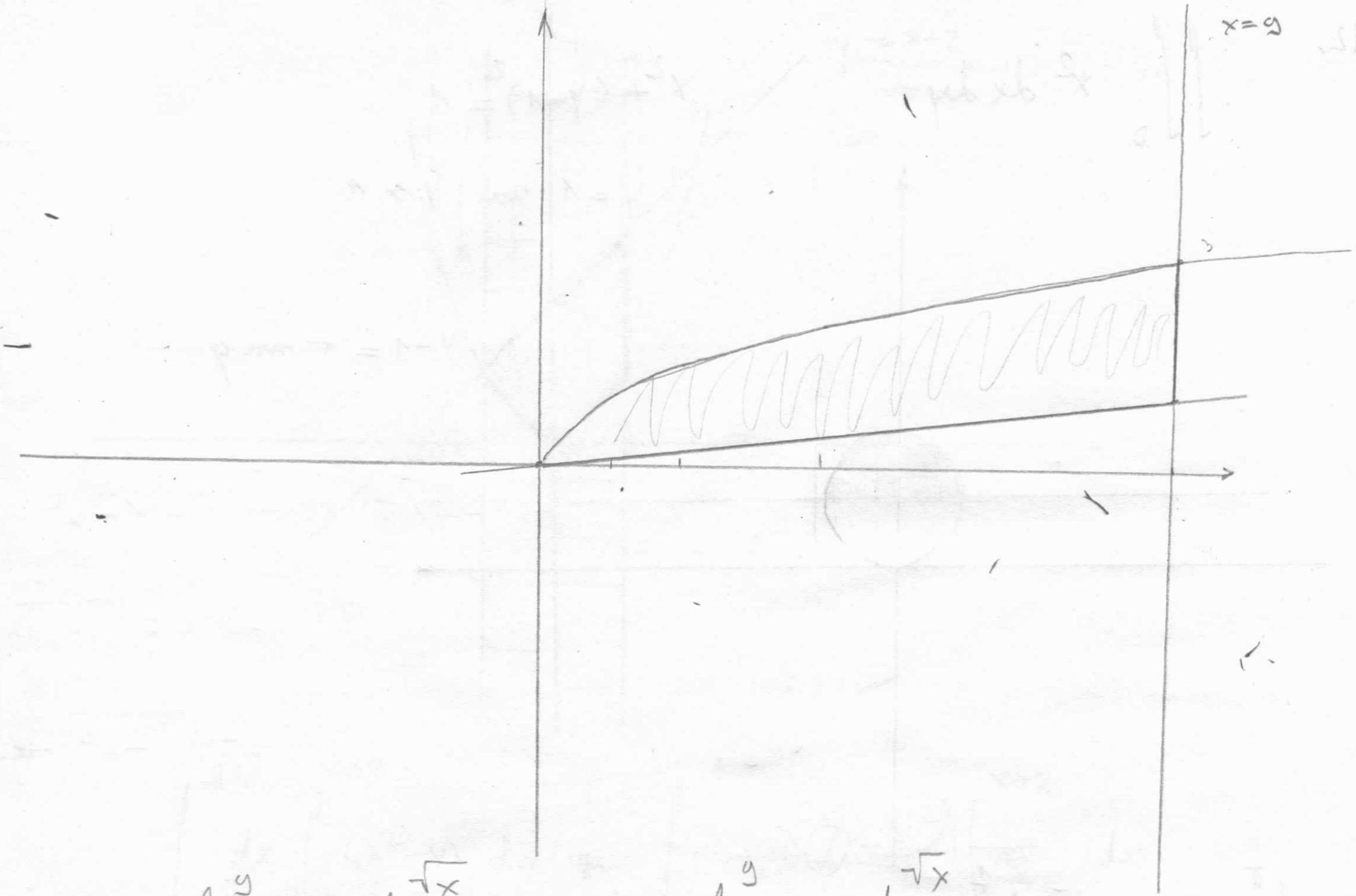


$$I = \int_0^9 dx \int_{\frac{x}{9}}^{\sqrt{x}} y^2\, dy = \int_0^9 \frac{y^3}{3} \Big|_{\frac{x}{9}}^{\sqrt{x}} dx =$$

$$\int_0^9 \frac{\sqrt{x^3}}{3} - \frac{x^3}{3 \cdot 9^3} \, dx =$$

$$\frac{1}{3} \int_0^9 x^{\frac{3}{2}} dx - \frac{1}{2187} \int_0^9 x^3 \, dx =$$

$$\frac{1}{3} \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \Big|_0^9 - \frac{1}{2187} \frac{x^4}{4} \Big|_0^9 =$$

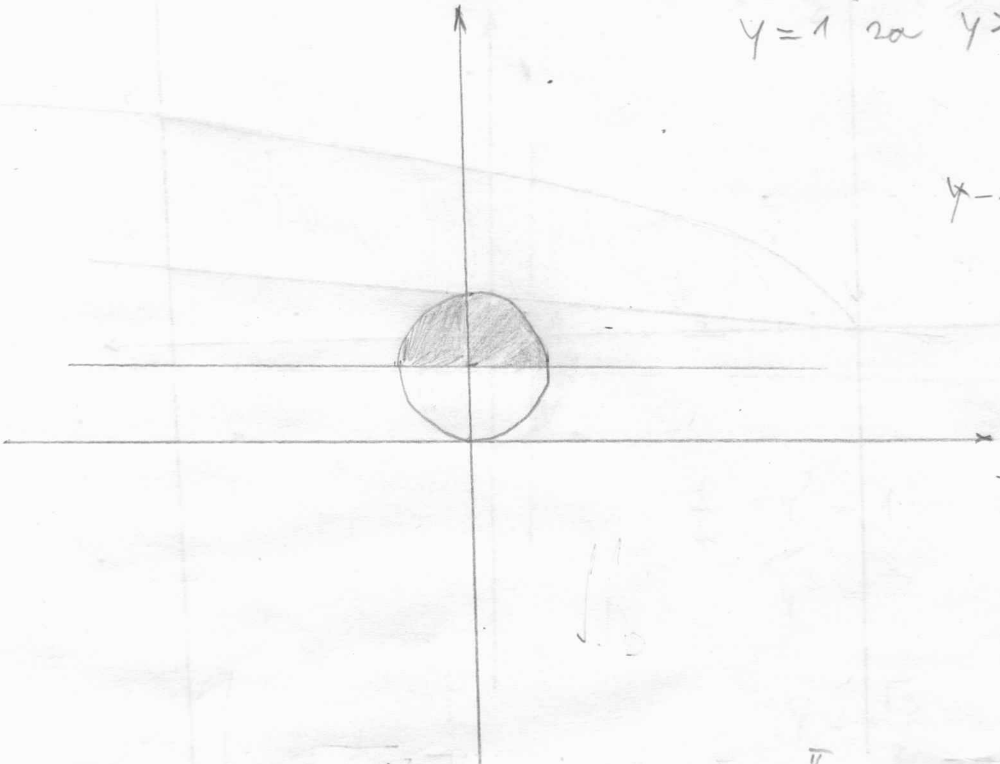
$$\frac{2}{15} 9^{\frac{5}{2}} - \frac{1}{8748} 9^4.$$

$$32.4 - 0.75 = 31.65$$

12. $\iint_D x^2\,dx\,dy$ $\qquad x^2+(y-1)^2=1$

$y=1$ za $y\geq 1$

$y-1 \leq r\sin\varphi$



$$\int_0^{\pi}\cos^2\varphi\,d\varphi\int_0^{1} r^3\,dr = \frac{1}{4}\int_0^{\pi}\cos^2\varphi\,d\varphi =$$

$$\frac{1}{4}\int_0^{\pi}\frac{1+\cos 2\varphi}{2}\,d\varphi =$$

$$\frac{1}{8}\varphi\Big|_0^{\pi} + \frac{1}{4}\cancel{\sin 2\varphi}\Big|_0^{\pi} = \frac{\pi}{8}$$

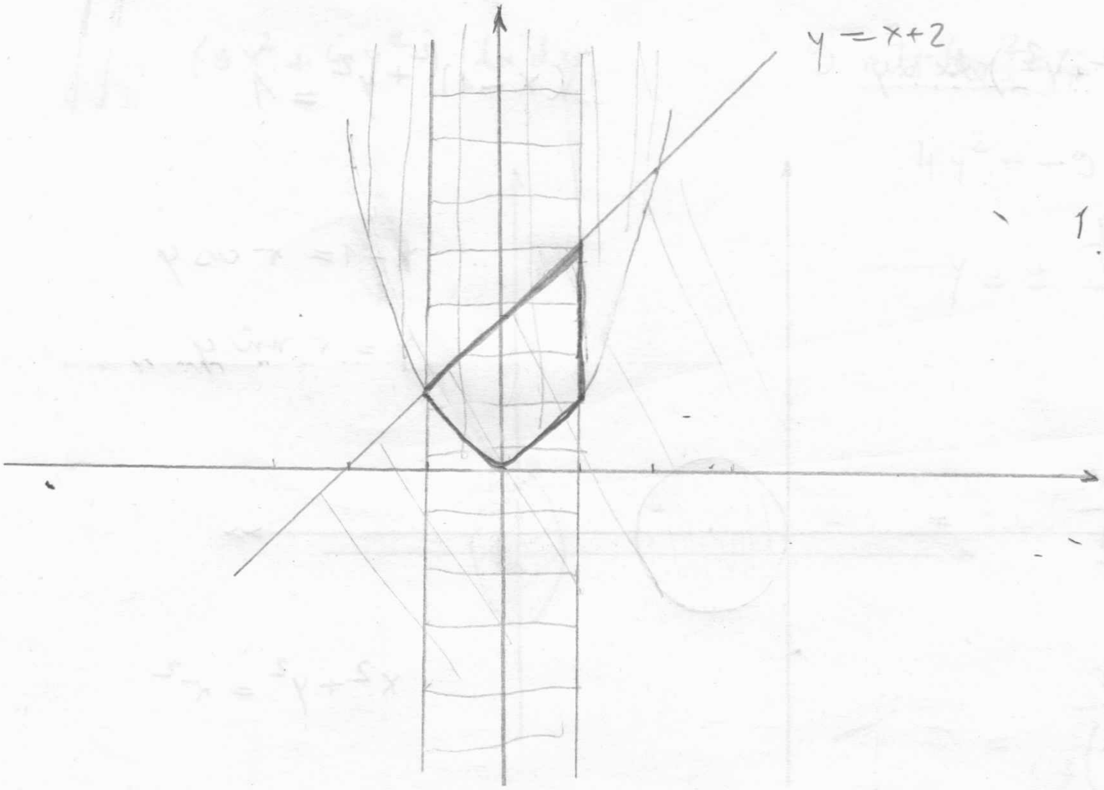
13. $\iint_D (x^2-y)\,dx\,dy$

$-1\leq x\leq 1$

$x^2\leq y\leq x+2$

$y\geq x^2$

$y\leq x+2$



$$\int_{-1}^{1} dx \int_{x^2}^{x+2} (x^2-y)\,dy = \int_{-1}^{1} \left(x^2 y - \frac{y^2}{2}\right)\Big|_{x^2}^{x+2} dx$$

$$\int_{-1}^{1}\left[x^2(x+2-x^2) - \frac{1}{2}(x^2+4x+4-x^4)\right]dx =$$

$$\int_{-1}^{1}\left[x^3+2x^2-x^4-\frac{1}{2}x^2-2x-2+\frac{x^4}{2}\right]dx =$$

$$\int_{-1}^{1}\left(-\frac{1}{2}x^4+x^3+\frac{3}{2}x^2-2x-2\right)dx =$$

$$-\frac{1}{2}\frac{x^5}{5}\Big|_{-1}^{1} + \frac{x^4}{4}\Big|_{-1}^{1} + \frac{3}{2}\frac{x^3}{3}\Big|_{-1}^{1} - 2\frac{x^2}{2}\Big|_{-1}^{1} - 2x\Big|_{-1}^{1} =$$

$$-\frac{1}{10}(2) + 0 + 1 - 0 - 4 = -3.2$$

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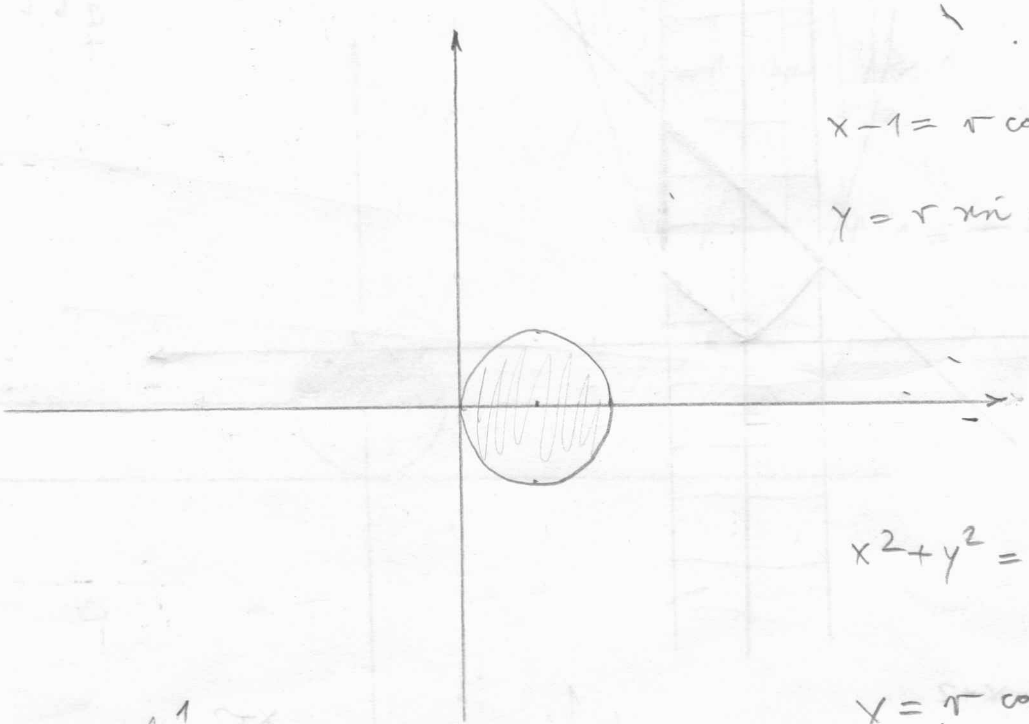
14. $\iint_D (x^2+y^2)\,dx\,dy$

$(x-1)^2+y^2=1$



$x-1 = r\cos\varphi$

$y = r\sin\varphi$

$x^2+y^2=r^2$

$x = r\cos\varphi+1$

$x^2 = r^2\cos^2\varphi + 2r\cos$

$+1$

$= x^2+y^2 = r^2\cos^2\varphi + 2r$

$+1 + r^2\sin^2\varphi$

$= r^2 + 2r\cos\varphi +$

$$\int_0^{2\pi} d\varphi \int_0^1 (r^2+2r\cos\varphi+1)\,r\,dr =$$

$$\int_0^{2\pi} d\varphi \int_0^1 (r^3+2r^2\cos\varphi+r)\,dr =$$

$$\int_0^{2\pi}\left[\frac{r^4}{4}\Big|_0^1 + 2\,\frac{r^3}{3}\cos\varphi\Big|_0^1 + \frac{r^2}{2}\right]d\varphi =$$

$$\int_0^{2\pi}\left[\frac{1}{4}+\frac{2}{3}\cos\varphi+\frac{1}{2}\right]d\varphi =$$

$$\frac{1}{4}\varphi\Big|_0^{2\pi} + \frac{2}{3}\cancel{\sin\varphi}\Big|_0^{2\pi} + \frac{1}{2}\varphi\Big|_0^{2\pi} = \frac{2\pi}{4}+\frac{2\pi}{2} = \frac{3\pi}{2}$$

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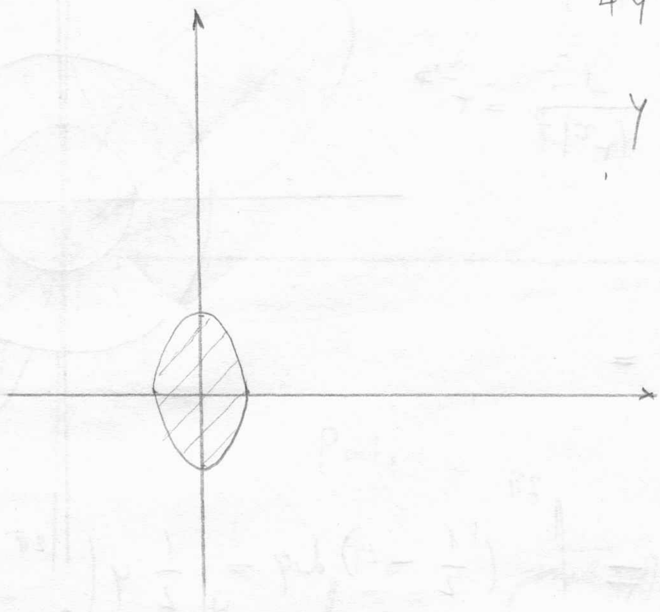
15. $\iint_D (9x^2+4y^2)\,dx\,dy$

$9x^2+4y^2=4 \quad |:4$

$4y^2=-9x^2+4$

$y=\pm\frac{\sqrt{-9x^2+4}}{4}$



$\frac{x^2}{\frac{4}{9}}+\frac{y^2}{1}=1$

$\frac{x^2}{\left(\frac{2}{3}\right)^2}+\frac{y^2}{1}=1$

$\frac{3}{2}x=r\cos\varphi$

$x=\frac{2}{3}r\cos\varphi$

$y=r\sin\varphi$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}=$$

$\frac{x^2}{1}+\frac{y^2}{1}=r^2$

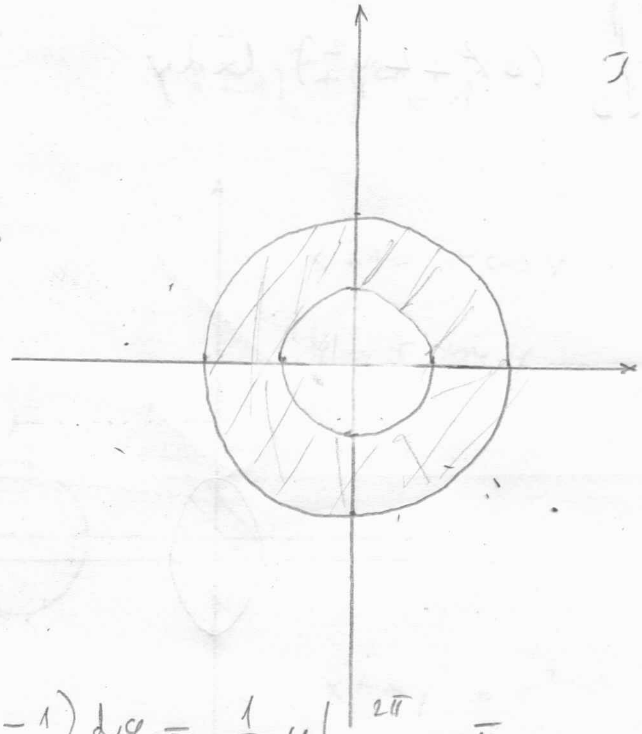
$x\frac{2}{3}r\cos\varphi$

$$\begin{vmatrix} \frac{2}{3}\cos\varphi & -\frac{2}{3}r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}=\frac{2}{3}r\cos^2\varphi+\frac{2}{3}r\sin^2\varphi=\frac{2}{3}r$$

$9\cdot\left(\frac{2}{3}r\cos\varphi\right)^2+4r^2\sin^2\varphi=4r^2$

$$\int_0^{2\pi}d\varphi\int_0^1\frac{8}{3}r^3\,dr=\int_0^{2\pi}\frac{8}{3}\frac{r^4}{4}\,d\varphi=\frac{2}{3}\int_0^{2\pi}d\varphi=\frac{4\pi}{3}$$

16. $\iint_D \dfrac{dx\,dy}{(x^2+y^2)^{\frac{3}{2}}}$



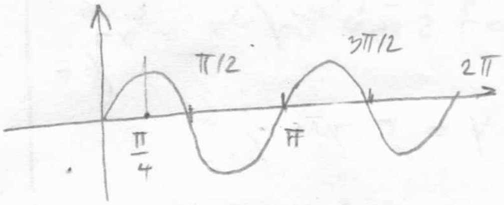
$x^2+y^2 \le 4$ $\qquad \dfrac{1}{(r^2)^{\frac{3}{2}}} = r^{-3}$

$x^2+y^2 \ge 1$

$\int_0^{2\pi} d\varphi \int_1^2 r^{-2}\,dr =$

$-\int_0^{2\pi} \dfrac{1}{r}\Big|_1^2 \,d\varphi = -\int_0^{2\pi} \left(\dfrac{1}{2}-1\right) d\varphi = \dfrac{1}{2}\varphi\Big|_0^{2\pi} = \pi$

17. $r = \sqrt{\sin 2\varphi}$ $\qquad \sin 2\varphi \ge 0$



$\varphi \in [0, \frac{\pi}{2}]$

ili $[\pi, \frac{3\pi}{2}]$

1. ili III. kvadrant

$\sin 2\varphi_{max} = 1$ | . arc sin

$2\varphi_{max} = 90^\circ$ $\qquad$ ili 2φ $\qquad \sin 2\varphi_{max} = +1$

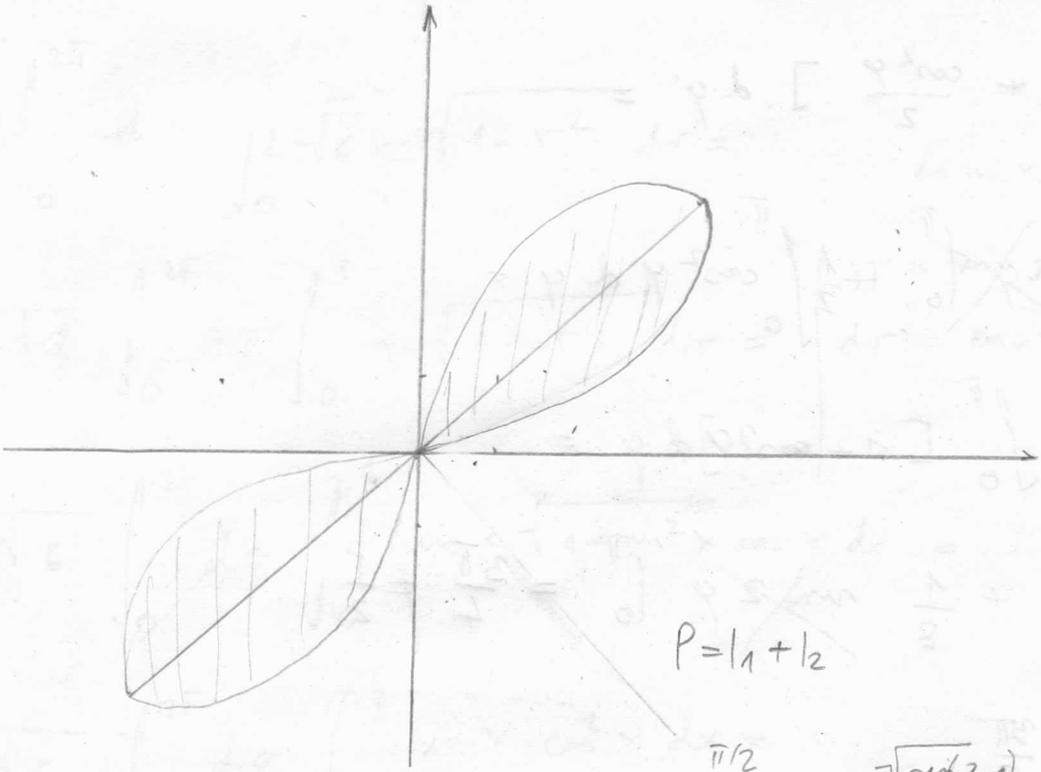
$\varphi_{max} = 45^\circ$ $\qquad 2\varphi_{max} = 225^\circ$

$\varphi =$

$\sin 2\varphi_{min} = 0$ $\qquad \sin 2\varphi_{min} = 0$

$2\varphi_{min} = 0$ $\qquad 2\varphi_{min} = 180$

$\varphi_{min} = 0$ $\qquad \varphi_{min} = 90^\circ$



$P = P_1 + P_2$

zbog simetrije situacije

$$\frac{P}{2} = \int_0^{\pi/2} d\varphi \int_0^{\sqrt{\sin(2\varphi)}} r\, dr =$$

$$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{\sqrt{\sin 2\varphi}} d\varphi = \int_0^{\pi/2} \frac{\sin 2\varphi}{2} d\varphi =$$

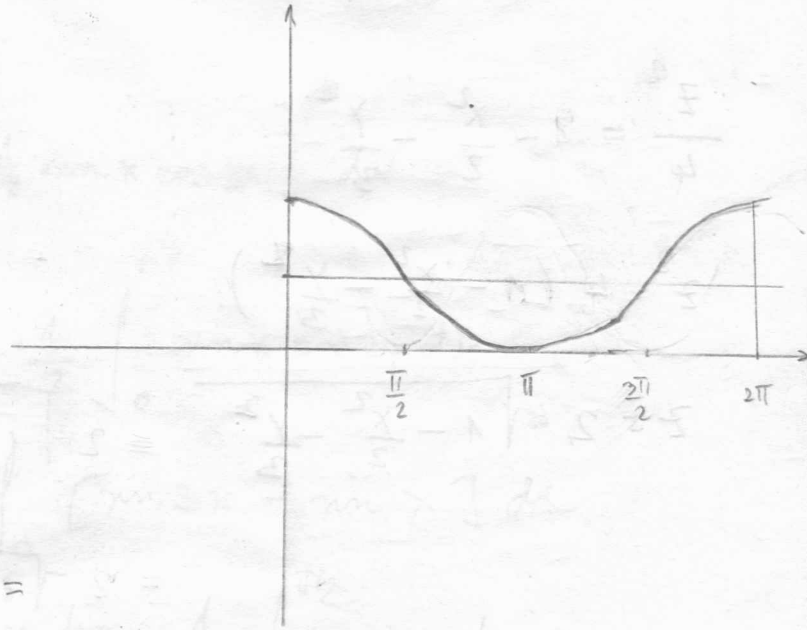
$P = 1$

$$= -\frac{1}{4} \cos 2\varphi \Big|_0^{\pi/2} =$$

$$-\frac{1}{4}(-2) = \frac{1}{2}$$

18. $r = 1 + \cos\varphi$

$$\frac{P}{2} = \int_0^{\pi} d\varphi \int_0^{1+\cos\varphi} r\, dr =$$



$$\int_0^{\pi} \frac{r^2}{2}\Big|_0^{1+\cos\varphi} d\varphi =$$

$$\int_0^{\pi} \frac{1 + 2\cos\varphi + \cos^2\varphi}{2} d\varphi =$$

$$\int_0^{\pi} \left[\frac{1}{2} + \cos\varphi + \frac{\cos^2\varphi}{2} \right] d\varphi =$$

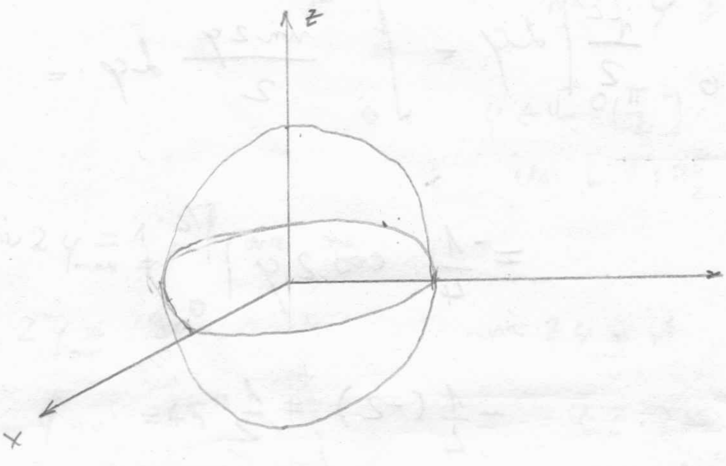
$$\frac{1}{2}\varphi \Big|_0^{\pi} + \cancel{\sin\varphi \Big|_0^{\pi}} + \frac{1}{2}\int_0^{\pi} \cos^2\varphi \, d\varphi =$$

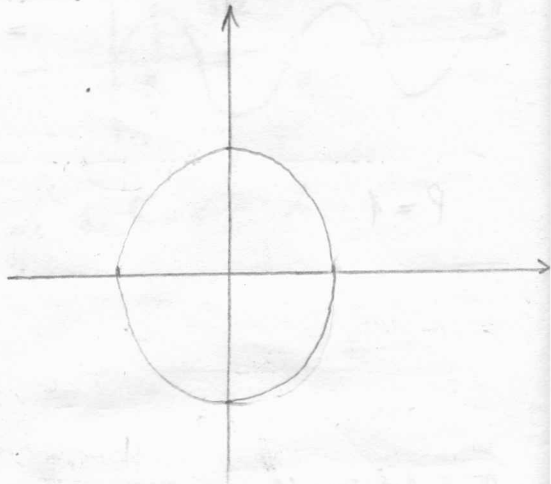
$$\frac{\pi}{2} + \frac{1}{4}\int_0^{\pi} [1 + \cos 2\varphi] \, d\varphi =$$

$$\frac{\pi}{2} + \frac{\pi}{4} + \frac{1}{8} \cancel{\sin 2\varphi} \Big|_0^{\pi} = \frac{3\pi}{4} = \frac{P}{2}$$

$$P = \frac{3\pi}{2}$$

19. $\frac{x^2}{2} + \frac{y^3}{3} + \frac{z^2}{4} = 1$ $\qquad J = ab\,r = \sqrt{2}\cdot\sqrt{3}\,r = \sqrt{6}\,r$





$$\frac{z^2}{4} = 1 - \frac{x^2}{2} - \frac{y^2}{3}$$

$$z^2 = 4\left(1 - \frac{x^2}{2} - \frac{y^2}{3}\right)$$

$$z = 2\sqrt{1 - \frac{x^2}{2} - \frac{y^2}{3}} = 2\sqrt{1 - \left(\frac{x^2}{2} + \frac{y^2}{3}\right)}$$

$$= 2\sqrt{1 - r^2}$$

$$V=2\int_0^{2\pi} d\varphi \int_0^1 2\sqrt{6}\, r\sqrt{1-r^2}\, dr = \qquad 1=\sin x$$

$$4\sqrt{6}\int_0^{2\pi} d\varphi \int_0^1 r\sqrt{1-r^2}\, dr = \left| \begin{matrix} r=\sin x \\ dr=\cos x\, dx \end{matrix} \right| =$$

$$4\sqrt{6}\int_0^{2\pi} d\varphi \int_0^1 \sin x\sqrt{1-\sin^2 x}\cos x\, dx =$$

$$4\sqrt{6}\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin x\cdot\cos^2 x\, dx =$$

$$\underline{I=\int_0^{\pi/2}\sin x\cdot\cos^2 x\, dx} = \int_0^{\pi/2}\sin x\,(1-\sin^2 x)\, dx =$$

$$\int_0^{\pi/2}\left[\sin x-\sin^3 x\right]dx = \int_0^{\pi/2}\sin x\, dx - \int_0^{\pi/2}\sin^3 x\, dx$$

$$= -\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin^3 x\, dx =$$

$$= 1-\int_0^{\pi/2}\frac{1-\cos 2x}{2}\sin x\, dx =$$

$$= 1-\int_0^{\pi/2}\left[\frac{1}{2}\sin x-\frac{1}{2}\sin x\cos 2x\right]dx =$$

$$1-\frac{1}{2}\int_0^{\pi/2}\sin x\, dx+\frac{1}{2}\int_0^{\pi/2}\sin x\cos 2x\, dx =$$

$$1+\frac{1}{2}\cos x\Big|_0^{\pi/2}+\frac{1}{4}\int_0^{\pi/2}\left[\sin 3x-\sin x\right]dx$$

$$\frac{1}{2}+\frac{1}{4}\int_0^{\pi/2}\sin 3x\, dx-\frac{1}{4}\int_0^{\pi/2}\sin x\, dx =$$

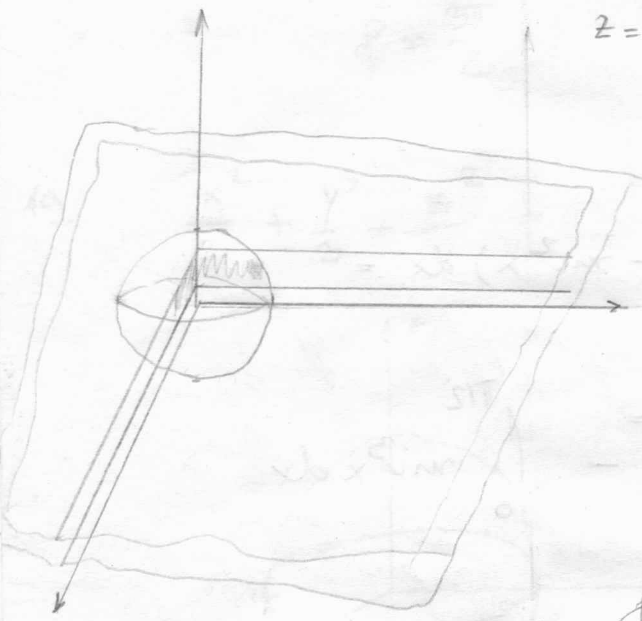
$$\frac{1}{2} - \frac{1}{12}\cos 3x \Big|_0^{\pi/2} + \frac{1}{4}\cos x \Big|_0^{\pi/2} =$$

$$\frac{1}{2} + \frac{1}{12} - \frac{1}{4} = \frac{1}{3}$$

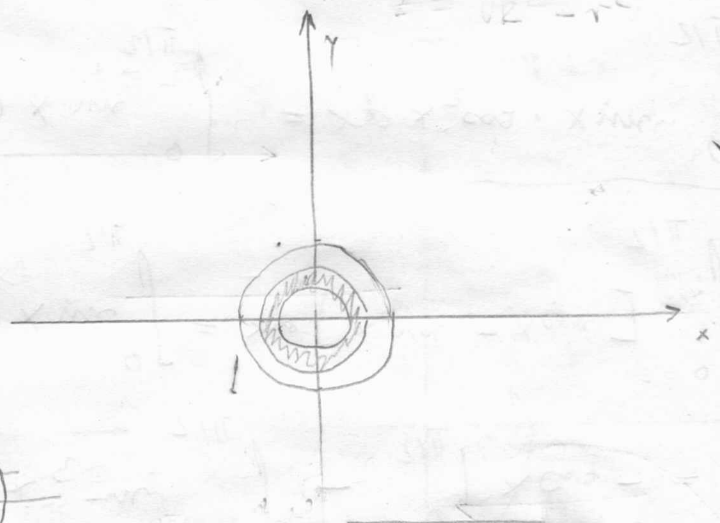
$$V = \frac{2}{3}\sqrt{6}\int_0^{2\pi} d\varphi = \frac{4}{3}\sqrt{6}\cdot 2\pi = \frac{8\sqrt{6}}{3}\pi$$

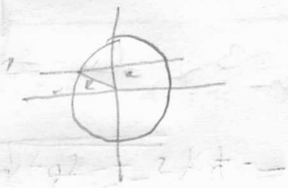
20. $x^2+y^2+z^2=R^2$ $\quad z=a$ $\quad z=b$ $\quad z^2=R^2-x^2-y^2$

$z=\pm\sqrt{R^2-x^2-y^2}$



$z=\sqrt{R^2-r^2}$





$z=\sqrt{R^2-r^2} - a$

$\qquad\qquad\quad - b$

$$V=\int_0^{2\pi} d\varphi \int_0^{\sqrt{R^2-a^2}} r\left(\sqrt{R^2-r^2}-a\right) dr -$$

$$\int_0^{2\pi} d\varphi \int_0^{\sqrt{R^2-b^2}} r\left(\sqrt{R^2-r^2}-b\right) dr$$

$$\int \left(r\sqrt{R^2-r^2}-ar\right) dr = \int r\sqrt{R^2-r^2}\,dr - a\int r\,dr =$$

$u = R^2$

$$\left| \begin{array}{l} u = R^2 - r^2 \\ du = -2r\,dr \\ dr = \dfrac{du}{-2r} \end{array} \right| = \int \frac{\cancel{r}}{-2\cancel{r}} \sqrt{u}\, du - a\frac{r^2}{2} =$$

$$-\frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{a}{2} r^2 = -\frac{1}{3}\sqrt{(R^2-r^2)^3} - \frac{a}{2} r^2$$

$$\int_0^{2\pi} \left[-\frac{1}{3}\sqrt{(R^2-r^2)^3} \Big|_0^{\sqrt{R^2-a^2}} - \frac{a}{2} r^2 \Big|_0^{\sqrt{R^2-a^2}} \right] d\varphi$$

$$-\int_0^{2\pi} \left[-\frac{1}{3}\sqrt{(R^2-r^2)^3} \Big|_0^{\sqrt{R^2-b^2}} - \frac{b}{2} r^2 \Big|_0^{\sqrt{R^2-b^2}} \right] d\varphi =$$

$$\int_0^{2\pi} \left[-\frac{1}{3}\sqrt{(\cancel{R^2}-\cancel{R^2}+a^2)^3} + \frac{1}{3}\sqrt{R^6} - \frac{a}{2}(R^2-a^2) \right] d\varphi$$

$$-\int_0^{2\pi} \left[-\frac{1}{3}\sqrt{(\cancel{R^2}-\cancel{R^2}+b^2)^3} + \frac{1}{3}\sqrt{R^6} - \frac{b}{2}(R^2-b^2) \right] d\varphi$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} a^3 + \frac{1}{3} R^3 - \frac{a}{2}(R^2-a^2) \right] d\varphi$$

$$-\int_0^{2\pi} \left[-\frac{1}{3} b^3 + \frac{1}{3} R^3 - \frac{b}{2}(R^2-b^2) \right] d\varphi =$$

$$\int_0^{2\pi} \left[-\frac{1}{3} a^3 + \frac{1}{3} b^3 - \frac{a}{2}(R^2-a^2) + \frac{b}{2}(R^2-b^2) \right] d\varphi$$

$$\int_0^{2\pi} \left[-\frac{a^3}{3} + \frac{b^3}{3} - \frac{aR^2}{2} + \frac{a^3}{2} + \frac{b}{2} R^2 - \frac{b^3}{2} \right] d\varphi$$

$$\int_0^{2\pi}\left[\frac{a^3}{6}-\frac{b^3}{6}+\frac{R^2}{2}(b-a)\right]d\varphi =$$

$$2\pi\left[\frac{a^3}{6}-\frac{b^3}{6}+\frac{R^2}{2}(b-a)\right] =$$

$$\frac{\pi}{3}(a^3-b^3)+R^2\pi(b-a)$$