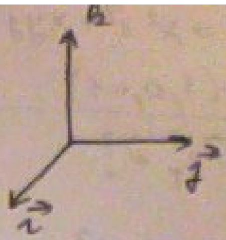
PZI - 2007



1) a) $\mathrm{rot}\,\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

b) $\vec{a} = x^2\vec{i} + xyz\vec{j} + z^2\vec{k}$

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\vec{i}\times\vec{i} = 0 \qquad \vec{j}\times\vec{i} = -\vec{k} \qquad \vec{k}\times\vec{i} = \vec{j}$

$\vec{i}\times\vec{j} = \vec{k} \qquad \vec{j}\times\vec{j} = 0 \qquad \vec{k}\times\vec{j} = -\vec{i}$

$\vec{i}\times\vec{k} = -\vec{j} \qquad \vec{j}\times\vec{k} = \vec{i} \qquad \vec{k}\times\vec{k} = 0$

$$\nabla\times\vec{a} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(x^2\vec{i} + xyz\vec{j} + z^2\vec{k}\right) =$$

$$= \frac{\partial}{\partial x}(xyz)\vec{k} - \frac{\partial}{\partial x}(z^2)\vec{j} - \frac{\partial}{\partial y}(x^2)\vec{k} + \frac{\partial}{\partial y}(z^2)\vec{i} + \frac{\partial}{\partial z}(x^2)\vec{j} - \frac{\partial}{\partial z}(xyz)\vec{i} =$$

$$= yz\vec{k} - xy\vec{i}$$

2) $\vec{a} = (x^2 - y)\vec{i} + yz\vec{k}$

$\vec{s} = 4\vec{i} - 3\vec{k}$

$\vec{s}_0 = \frac{\vec{s}}{|\vec{s}|} = \frac{4\vec{i} - 3\vec{k}}{\sqrt{25}} = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{k}$

$\frac{\partial\vec{a}}{\partial\vec{s}} = \vec{s}$

$$\frac{\partial\vec{a}}{\partial\vec{s}} = \left(\frac{4}{5}\frac{\partial}{\partial x} - \frac{3}{5}\frac{\partial}{\partial z}\right)\left((x^2 - y)\vec{i} + yz\vec{k}\right) =$$

$$= \frac{4}{5}2x\vec{i} - \frac{3}{5}y\vec{k}$$

$$\frac{4}{5}2x\vec{i} - \frac{3}{5}y\vec{k} = 4\vec{i} - 3\vec{k}$$

$\frac{8x}{5} = 4 \Rightarrow x = \frac{5}{2} \qquad z = z \qquad T\left(\frac{5}{2}, 5, z\right), z\in\mathbb{R}$

$-\frac{3}{5}y = -3 \Rightarrow y = 5$

3. $\vec{v} = x^2\vec{i} + xy\vec{j} + xz^3\vec{k}$

$T(1,0,-1)$

$\Delta\vec{v}|_T = ?$

$$\Delta\vec{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x^2\vec{i} + xy\vec{j} + xz^3\vec{k}\right) =$$

$$= 2\vec{i} + 0 + 6xz\vec{k} = 2\vec{i} + 6xz\vec{k}$$

$$\Delta\vec{v}|_T = 2\vec{i} + 6\cdot 1\cdot(-1)\vec{k} = 2\vec{i} - 6\vec{k}$$

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4. $\nabla[r\cdot\nabla(r\vec{r})] = ?$

$$\nabla(r\vec{r}) = (\nabla r)\vec{r} + r(\nabla\vec{r}) = \frac{\vec{r}}{r}\cdot\vec{r} + r\cdot 3 = \frac{r^2}{r} + 3r = 4r$$

$$\nabla(r\cdot 4r) = \nabla(4r^2) = (4r^2)'\cdot\frac{\vec{r}}{r} = 8r\cdot\frac{\vec{r}}{r} = 8\vec{r}$$

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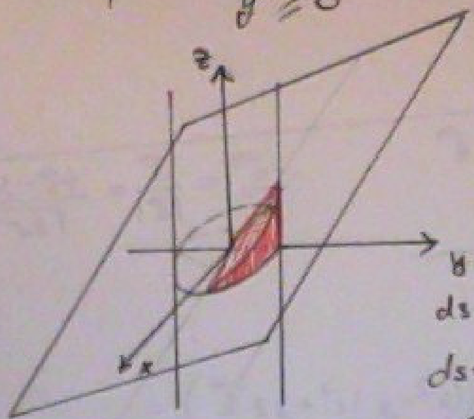
5. $\int_K (x^2+y^2)\,ds$, $K \ldots\ x^2+2y^2=4$, $z=y$, $y \geq 0$



$$ds = \sqrt{(x')^2+(y')^2+(z')^2}\,dt$$

$$ds = \sqrt{4\sin^2 t + 2\cos^2 t + 2\cos^2 t}\,dt$$

$$ds = 2\,dt$$

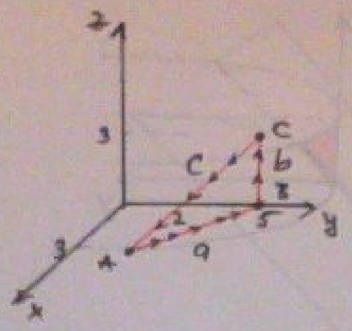
$$\frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow \begin{array}{l} x = 2\cos t \\ y = \sqrt{2}\sin t \\ z = \sqrt{2}\sin t \end{array}$$

$$I = 2\int_0^\pi (4\cos^2 t + 2\sin^2 t)\,dt = 2\int_0^\pi (2\cos^2 t + 2)\,dt = 4\int_0^\pi (\cos^2 t + 1)\,dt =$$

$$= 4\int_0^\pi \frac{1+\cos 2t}{2}\,dt + 4\int_0^\pi dt = 2\pi + 4\pi = 6\pi$$

⑥ $\int_K yz\,dx + 4xz\,dz$, K ... $A(3,2,0), B(0,5,0), C(0,5,3)$

→ neg. orij. ie isk.



$$t = \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

a: $\frac{x-3}{0-3} = \frac{y-2}{5-2} = \frac{z-0}{0-0}$

$\frac{x-3}{-3} = \frac{y-2}{3} = \frac{z}{0} = t$

$x = 3-3t \Leftrightarrow dx = -3dt$
$y = 2+3t$
$z = 0 \Leftrightarrow dz = 0$
$t \in [0,1]$

$$\int_0^1 (2+3t)\cdot 0\cdot(-3dt) + 4(3-3t)\cdot 0 = 0$$

b: $\frac{x-0}{0-0} = \frac{y-5}{5-5} = \frac{z-0}{3-0}$

$\frac{x}{0} = \frac{y-5}{0} = \frac{z}{3} = t$

$x = 0 \Leftrightarrow dx = 0$
$y = 5$
$z = 3t \Leftrightarrow dz = 3dt$
$t \in [0,1]$

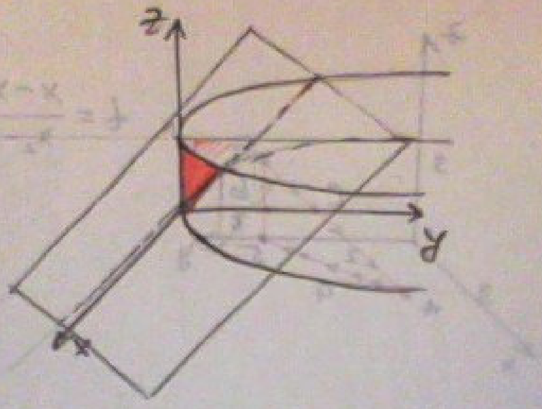
$$\int_0^1 5\cdot 3t\cdot 0 + 4\cdot 0\cdot 3dt = 0$$

c: $\frac{x-0}{3-0} = \frac{y-5}{2-5} = \frac{z-3}{0-3} = t$

$x = 3t \Leftrightarrow dx = 3dt$
$y = 5-3t$
$z = 3-3t \Leftrightarrow dz = -3dt$
$t \in [0,1]$

$$\int_0^1 (5-3t)(3-3t)3dt + 4\cdot 3t\cdot(-3dt) =$$
$$= \int_0^1 (15-15t-9t+9t^2)\cdot 3dt - 36t\,dt =$$
$$= \int_0^1 (45-72t+27t^2-36t)\,dt =$$
$$= \dots = 0$$

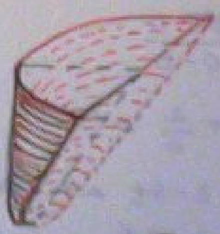
① $\iint_S x\,dS$, S: $y=x^2$, $z\le 1$, $z\ge y$, $x\ge 0$



$y=x^2$

$dS=\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$

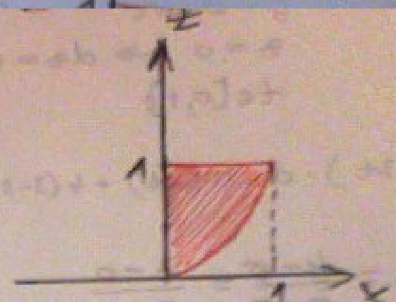
$dS=\sqrt{1+4x^2}\,dx\,dz$



$I=\iint x\sqrt{1+4x^2}\,dx\,dz=$

$=\int_0^1 dx\int x\sqrt{1+4x^2}\,d$

$I=\iint x\sqrt{1+4x^2}\,dx\,dz=$



$=\int_0^1 dx\int_{x^2}^1 x\sqrt{1+4x^2}\,dz=$

$=\int_0^1 (1-x^2)\,x\sqrt{1+4x^2}\,dx=$

$=\int_0^1 x\sqrt{1+4x^2}\,dx-\int_0^1 x^2\sqrt{1+4x^2}\cdot x\,dx=$

$=\left[\begin{array}{c}1+4x^2=t \;\big|\; ' \\ x\,dx=\frac{dt}{8} \\ x^2=\frac{t-1}{4}\end{array}\right]=\frac{1}{8}\int_1^5\sqrt{t}\,dt-\frac{1}{8}\int_1^5\frac{t-1}{4}\sqrt{t}\,dt=$

$=\frac{1}{8}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^5-\frac{1}{32}\,\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\Big|_1^5+\frac{1}{32}\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^5=\ldots=\frac{1}{24}(5$