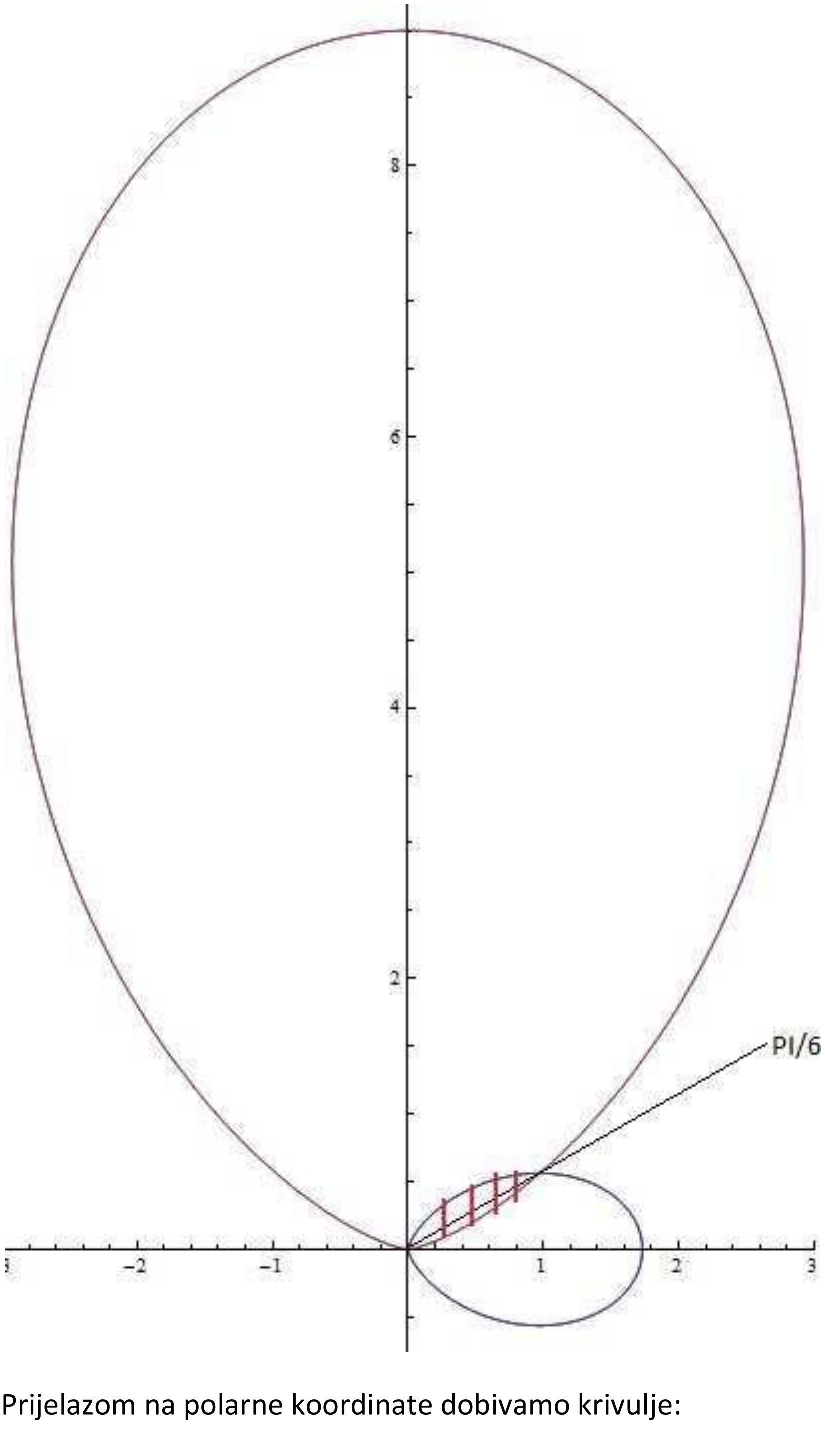
Dvostruki Integral (polarne koordinate)

(2. tjedan)

By

Vedax & Ninjo & Wolfman & kate89 & barby,

14. Izračunajte $\iint_d \frac{dxdy}{\sqrt{x^2+y^2}}$pri čemu je područje integracije određeno nejednadžbama (x^2+y^2)^2<=Sqrt(3)x^3 i (x^2+y^2)^2<=9y^3



Prijelazom na polarne koordinate dobivamo krivulje:

Pišem f umjesto φ

$r = \sqrt{3}\cos^3 \quad r = 9\sin^3 f$

Izjednačavanjem dobivamo zraku fi na kojoj leži točka u kojoj se krivulje sijeku:

$$tg^3 f = \frac{\sqrt{3}}{9} => f = \frac{\pi}{3}$$

Funkcija f(x,y)= $\frac{1}{\sqrt{x^2+y^2}}$ je u polarnima $f(r,f) = \frac{1}{r}$

Dakle, integral je:

$$I = \int_0^{\pi/6} df \int_0^{9\sin^3 f} \frac{1}{r} r\,dr + \int_{\pi/6}^{\pi/2} df \int_0^{\sqrt{3}\cos^3 f} \frac{1}{r} r\,dr = 9\int_0^{\pi/6} \sin^3 f\,df + \sqrt{3}\int_{\pi/6}^{\pi/2} \cos^3 f\,df = \cdots = 6 - \frac{19\sqrt{3}}{6}$$

$$\sin^3 f\ rastaviti\ na\ sinf * \sin^2 f, pri\ čemu\ je \sin^2 f = 1 - \cos^2 f. Sada\ imamo\ dva\ integrala, \int sinf\ i\ \int sinf \cos^2 f$$

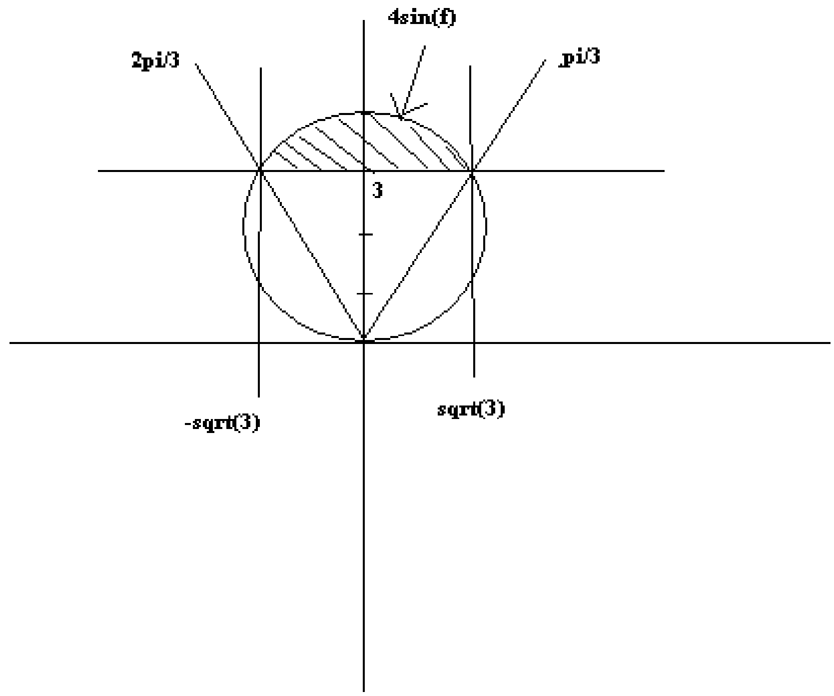
$$\int sinf \cos^2 f \left| \begin{matrix} cosf = t \\ dt = -sinf \end{matrix} \right| nakon\ ove\ supstitucije\ imamo - \int t^2 dt\ sto\ se\ rjesava\ bez\ frke\ :)$$

Analogno radimo i sa cos.

15.Prijalzom na polarne koord izracunajte:

A

$\int_{-\sqrt{3}}^{\sqrt{3}} dx \ \int_{3}^{2+\sqrt{4-x^2}} \frac{dy}{(x^{\vee}+y^2)^{\frac{3}{2}}}$



$$P = \int_{\pi/3}^{2\pi/3} df \int_{3/sinf}^{4sinf} r * \frac{dr}{r^3} = \int_{\pi/3}^{2\pi/3} \frac{r^{-1}}{-1} \Big|_{\frac{3}{sinf}}^{4sinf} df = -\int_{\pi/3}^{2\pi/3} \left(\frac{1}{4sinf} - \frac{sinf}{3}\right) df$$

$$= -\frac{1}{4} \int_{\pi/3}^{2\pi/3} \frac{df}{sinf} + \int_{\pi/3}^{2\pi/3} sinf df$$

-ovo je tablični integral pa imamo:

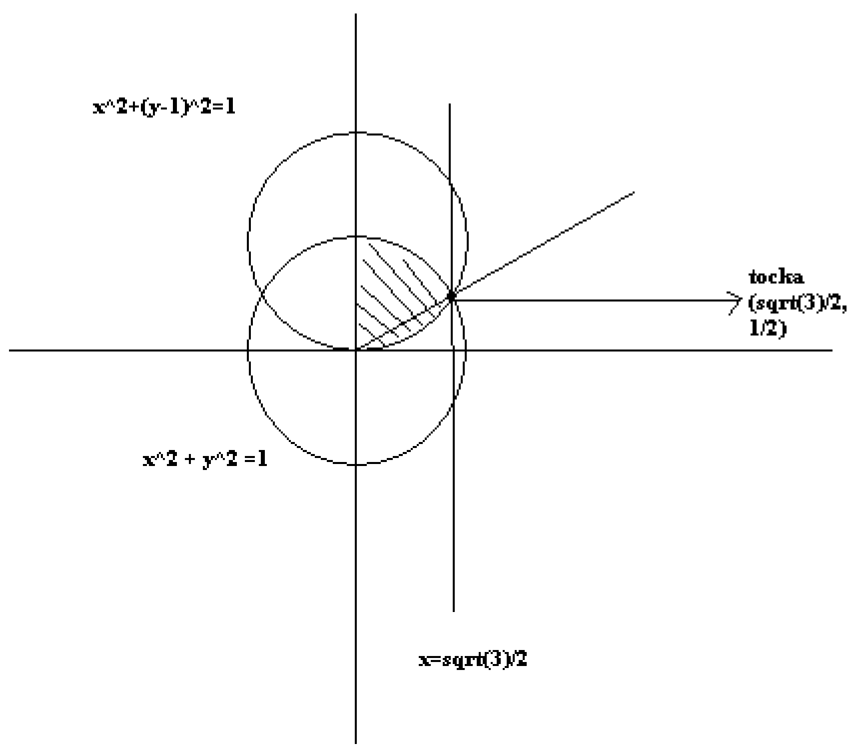
$$-\frac{1}{4}(\ln\left|tg\frac{f}{2}\right| \left|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right) + \frac{1}{3}\left(-cosf \left|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right.\right) = -\frac{1}{4}(\ln\left|tg\frac{\pi}{3}\right| - \ln\left|tg\frac{\pi}{6}\right| - \frac{1}{3}(\cos\frac{2\pi}{3} - \cos\frac{\pi}{3})$$

$$= -\frac{1}{4}\left(ln\sqrt{3} - \ln(\frac{\sqrt{3}}{3}\right) + \frac{1}{3} = -\frac{1}{4}\ln\frac{\sqrt{3}}{\frac{\sqrt{3}}{3}} + \frac{1}{3} = \frac{1}{2}\ln\frac{1}{\sqrt{3}} + \frac{1}{3}$$

B $\int_0^{\frac{\sqrt{3}}{2}} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{y^2 - x^2}\, dy$

$$\int_0^{\frac{\sqrt{3}}{2}} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{y^2 - x^2}\, dy = \int_0^{\frac{\pi}{6}} df \int_0^{2sinf} r^2 dr + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} df \int_0^1 r^2 dr$$

Kako smo došli do ovoga:



Iz granica se očita da imamo dvije kružnice. Moramo naći sjecišta tih kružnica

$$x^2 + y^2 = 1$$

$$x^2 - (y-1)^2 = 1$$

Dobiva se da su to tocke za $x1 = 0\ i\ x2 = \mp\frac{\sqrt{3}}{2}\ y1 = 0\ i\ y2 = \frac{1}{2}$

$nadalje\ moramo\ izračunati\ donju\ granicu\ integracije\ i\ radij\ vektor\ na\ kojem\ se\ nalazi\ točka\ sjecišta$

$$x^2 - (y-1)^2 = 1\ \ uz\ x = rcosf\ i\ y = rsinf \rightarrow r^2\cos^2 f + r^2 \sin^2 f - 2rsinf + 1 = 1$$

$$r^2 - 2rsinf = 0 \ \rightarrow r = 2sinf\ (ovo\ nam\ je\ donja\ granica\ integracija, ova\ pomaknuta\ kruznica)$$

$$za\ tocku\ x = \frac{\sqrt{3}}{2}\ i\ y = \frac{1}{2}$$

$$x = \frac{\sqrt{3}}{2} = rcosf$$

$$y = \frac{1}{2} = rsinf\ \ \rightarrow dijeljenjem\ ove\ dvije\ jednadbe\ dobivamo:$$

$$\frac{\frac{\sqrt{3}}{2}}{0.5} = ctgf \rightarrow f = \frac{\pi}{6}$$

Dakle

$$P = \int_0^{\frac{\pi}{6}} df \int_0^{2sinf} r^2 dr + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} df \int_0^1 r^2 dr = \int_0^{\frac{\pi}{6}} r^3 \Big|_0^{2sinf} df + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} r^3 \Big|_0^1 df$$

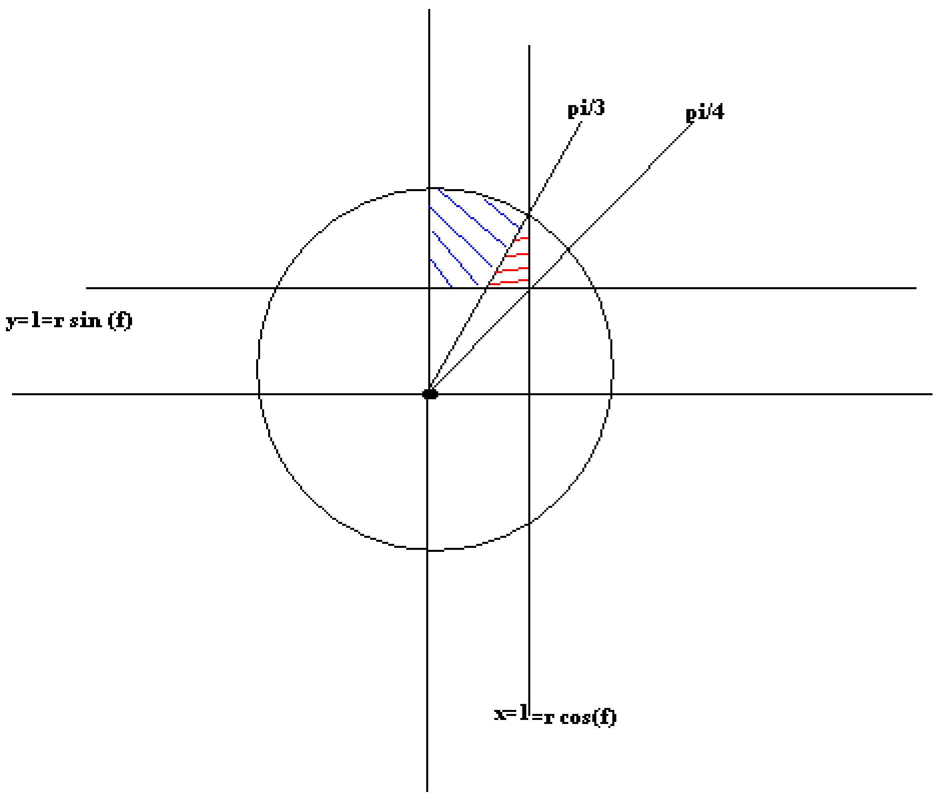
$$\frac{8}{3}\int_0^{\frac{\pi}{6}} \sin^3 f\ df + \frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} df = \frac{8}{3}\int_0^{\frac{\pi}{6}} sinf\ (1-\cos^2 f)\ df + \frac{1}{3}(\frac{\pi}{2} - \frac{\pi}{6}) =$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{6}} sinfdf + \int_0^{\frac{\pi}{6}} -sinfcos^2 f\ df \left| \begin{matrix} cosf = t & donja: cos0 = 1 \\ -sinfdf = dt & gornja: cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \end{matrix} \right| + \frac{\pi}{9}$$

$$= \frac{8}{3}\left(-cosf \Big|_0^{\frac{\pi}{6}}\right) + \frac{8}{3}\int_1^{\frac{\sqrt{3}}{2}} t^2 dt + \frac{\pi}{9} = samo\ par\ osnovnih\ integrala\ \ :) = -\sqrt{3} + \frac{16}{9} + \frac{\pi}{3}$$

C

$$\int_0^1 dx \int_1^{\sqrt{4-x^2}} \frac{dy}{(x^2+y^2)^{\frac{5}{2}}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} df \int_{\frac{1}{sinf}}^{\frac{1}{cosf}} \frac{dr}{r^4} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} df \int_{\frac{1}{sinf}}^{2} \frac{dr}{r^4}$$



Prva idemo vidjeti koja nam funkcija ostane:

$(x^2 + \ y^2)^{\frac{5}{2}} = (r^2)^{5/2} = r^5 \rightarrow dakle\ kada\ mnozimo\ sa\ Jakobijanom\ (r)\ imamo \frac{r}{r^5} = r^4$

$$sjecište\ pravca\ x = 1\ i\ kružnice\ x^2 + y^2 = 4\ \ je\ u\ tocki(1, \sqrt{3}\ )$$

$$radij\ vektori\ kutevi\ za\ tg = 1\ i\ tg = \sqrt{3}\ \rightarrow f = \frac{\pi}{4}\ i\ f = \frac{\pi}{3}$$

Dakle:

$$P = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} df \int_{\frac{1}{sinf}}^{\frac{1}{cosf}} \frac{dr}{r^4} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} df \int_{\frac{1}{sinf}}^{2} \frac{dr}{r^4} \;\; (crveni + plavi)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{r^{-3}}{-3} | \frac{\frac{1}{cosf}}{\frac{1}{sinf}} df + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{r^{-3}}{-3} | \frac{2}{\frac{1}{sinf}} df =$$

$$-\frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^3 f \, df - \frac{1}{24} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} df + \frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 f \, df$$

-ovdje su dvije iste podintegralne funkcije ubačene u jedne granice

$\frac{1}{3} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^3 f \, df + \frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3 f \, df = \frac{1}{3} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^3 f \, df$

$$= -\frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cosf \, (1 - \sin^2 f) df - \frac{1}{24} \left(\frac{\pi}{2} - \frac{\pi}{3}\right) + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} sinf \, (1 - \cos^2 f) \, df$$

$$= -\frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cosf \;\; df$$

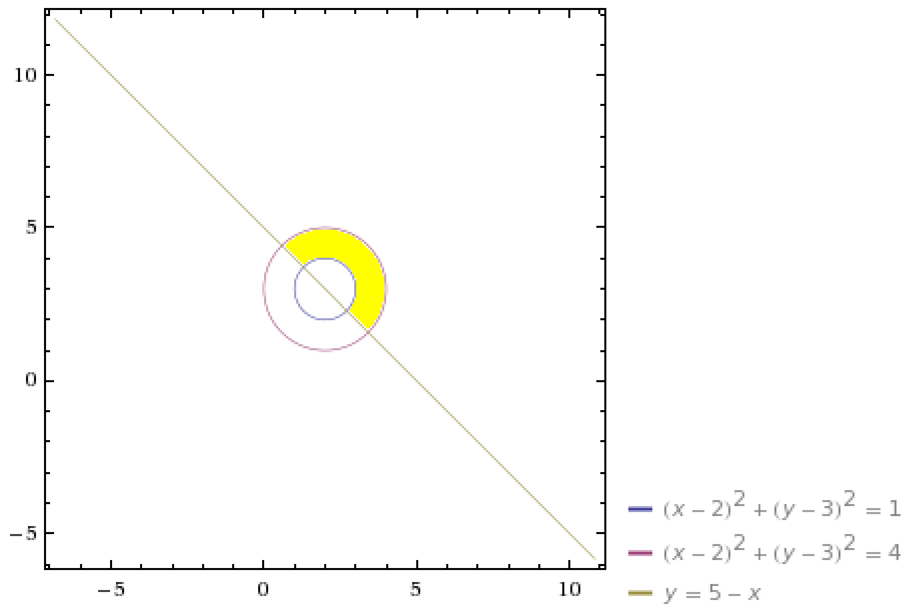
$$+ -\frac{1}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cosf \sin^2 f \, df \left| \begin{matrix} sinf = t & D: \frac{\sqrt{2}}{2} \\ cosfdf = dt & G: \frac{\sqrt{3}}{2} \end{matrix} \right| - \frac{1}{24} \left(\frac{\pi}{6}\right)$$

$$+ \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} sinf \, df - \frac{1}{3} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} sinf \;\cos^2 f \, df \left| \begin{matrix} cosf = u & D: \frac{\sqrt{2}}{2} \\ -sinfdf = du & G: 0 \end{matrix} \right|$$

$$= -\frac{1}{3} sinf \, | \frac{\frac{\pi}{3}}{\frac{\pi}{4}} + \frac{1}{3} \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} t^2 dt - \frac{\pi}{144} - \frac{1}{3} \cos | \frac{\frac{\pi}{2}}{\frac{\pi}{4}} + \frac{1}{3} \int_{\frac{\sqrt{2}}{2}}^{0} u^2 du = malo \; lakog \; računa =$$

$$-\frac{\pi}{144} - \frac{\sqrt{3}}{8} + 5\frac{\sqrt{2}}{18}$$

16.



Koristit ćemo pomaknute polarne koordinate:

$x-2=rcos\varphi$
$y-3=rsin\varphi$

Jednadžbe pravca i kružnica su nam onda:

$(rcos\varphi)^2+(rsin\varphi)^2=1\leftrightarrow r^2=1\leftrightarrow r=1$

$(rcos\varphi)^2+(rsin\varphi)^2=4\leftrightarrow r^2=4\leftrightarrow r=2$

$rsin\varphi+3=-rcos\varphi-2+5\leftrightarrow r(sin\varphi+cos\varphi)=0\leftrightarrow sin\varphi+cos\varphi=0$

Iz ovoga zadnjeg možemo dobiti kuteve:

$sin\varphi+cos\varphi=0\leftrightarrow sin\varphi=-cos\varphi$

Ovo su II. i IV. kvadrant pa su kutevi od $-\frac{\pi}{4}$ do $\frac{3\pi}{4}$.

Pa nam je rješenje:

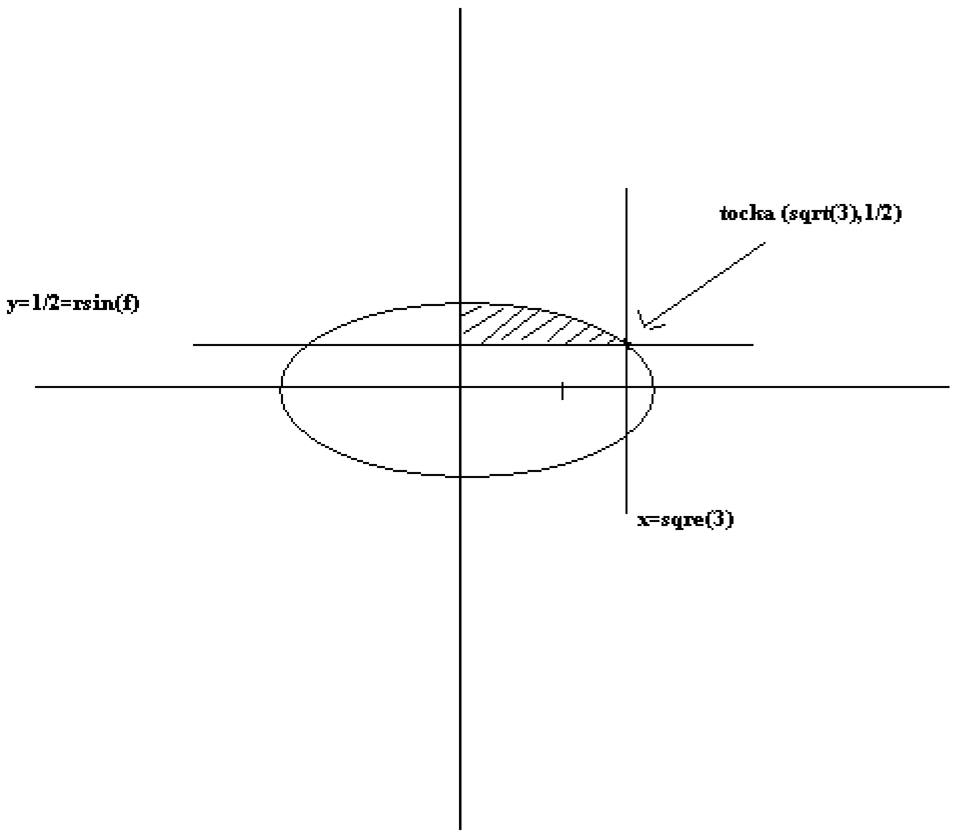
$$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_{1}^{2} r(rcos\varphi + 2)dr = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\int_{1}^{2} (r^2 cos\varphi + 2r)dr \right) d\varphi = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} cos\varphi \left(\frac{r^3}{3} \Big|_1^2 \right) d\varphi + \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(r^2 \Big|_1^2 \right) d\varphi =$$

$$= \frac{7}{3} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} cos\varphi d\varphi + 3 \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi = \frac{7}{3} sin\varphi \Bigg|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} + 3\varphi \Bigg|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} = 3\pi + \frac{7\sqrt{2}}{3}$$

17.Prijelazom na eliptičke koord izračunaj:

$$\int_{0}^{\sqrt{3}} dx \int_{1/2}^{\sqrt{1-\frac{x^2}{4}}} \frac{dy}{\left(y^2+\frac{x^2}{4}\right)^{\frac{3}{2}}}$$

$$imamo\ elipsu \frac{x^2}{2^2}+\frac{y^2}{1^2}=1\ i\ uvodimo\ eliptičke\ koord\ x=2rcosf\ i\ y=rsinf$$



$$y=\frac{1}{2}=rsinf \rightarrow r=\frac{1}{2sinf} \rightarrow donja\ granica\ integracije\ po\ r$$

$$\sqrt{1-\frac{x^2}{4}}=y \rightarrow 1-\frac{x^2}{4}=y^2=\frac{4r^2\cos^2 f}{4}+r^2\sin^2 f=1 \rightarrow r=1\ gornja\ granica\ za\ r$$

$$sjeciste\ elipse\ i\ pravca\ x=\sqrt{3}\ u\ tocki\ \left(\sqrt{3},\frac{1}{2}\right)$$

$$\sqrt{3}=2rcosf\quad i\quad \frac{1}{2}=rsinf \rightarrow slijedi\ djeljenjem \frac{\sqrt{3}}{0.5}=2ctgf \rightarrow f=\frac{\pi}{6}\ \ (kut\ radij\ vektora\ tocke)$$

Pa imamo(uz mnozenje Jakobijanom koja iznosi 2r)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} df \int_{1/2sinf}^{1} 2r\frac{dr}{r^3} = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{r^{-1}}{-1} \,\Big|_{\frac{1}{2sinf}}^{1} \; df = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (-1+2sinf)df =$$

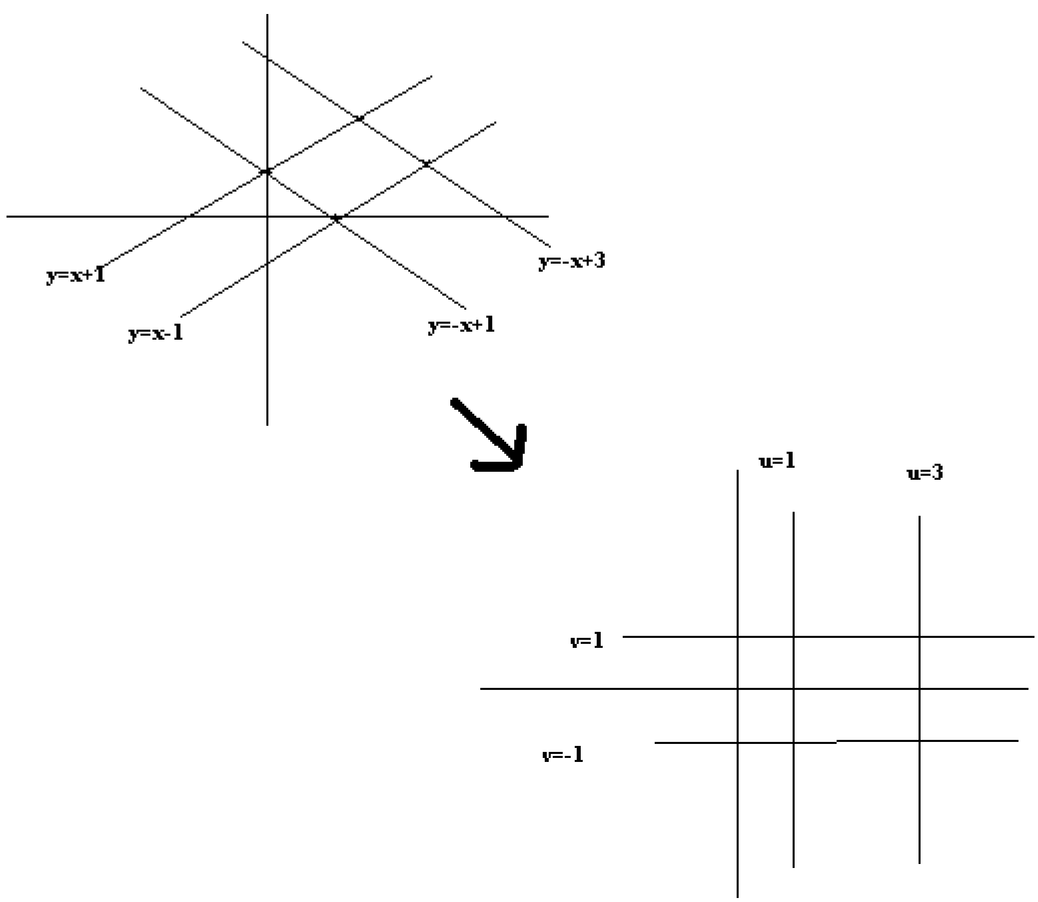
$$4(-cosf\,\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}) - 2f\,\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \cdots = 2\sqrt{3} - \; 2\frac{\pi}{3}$$

18.

$$\iint_D (x+y)^2(x-y)^2 dx\, dy$$

D:

x+y= 1 x+y=3 x-y=-1 x-y=1 → uz u=x+y i v= x-y imamo u=1 u=3 v=1 v=-1



da bi nasli Jakobianu moramo izraziti x i y preko u i v →$iz\ sustava\ u = x + y\ i\ v = x - y\ \ imamo$

$$x = \frac{u+v}{2} \; i \; y = \frac{u-v}{2}$$

J=-0.5

$$I = \int_{1}^{3} du \int_{-1}^{1} u^3 v^2 * |J| dv = \frac{1}{2} \int_{1}^{3} u^3 du \int_{-1}^{1} v^2 dv = malo\ lakog\ racuna : D = \frac{20}{3}$$

19.

Vidite i sami da zadatak ako zadatak nije toliko kompliciran, no kod računa Jakobijana javljaju se korijeni korijena i tako neke ružne stvari.