Rješenja 2. domaće zadaće MAT3E/R

verzija: 1.0

1. **Zadanu funkciju prikaži u obliku Fourierovog integrala:** $f(x) = sgn(x-1) - sgn(x-3)$
Rješenje:

$$f(x) = sgn(x-1) - sgn(x-3) = \frac{x-1}{|x-1|} - \frac{x-3}{|x-3|}$$

$$sgn(x-1) = \begin{cases} -1 & \text{ako } x \in \langle -\infty, 1\rangle \\ 1 & \text{ako } x \in \langle 1, \infty\rangle \end{cases}$$

$$sgn(x-3) = \begin{cases} -1 & \text{ako } x \in \langle -\infty, 3] \\ 1 & \text{ako } x \in \langle 3, \infty] \end{cases}$$

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{1}(-1)\cos\lambda\xi d\xi + \frac{1}{\pi}\int_{1}^{\infty} 1\cos\lambda\xi d\xi - \frac{1}{\pi}\int_{-\infty}^{3}(-1)\cos\lambda\xi d\xi - \frac{1}{\pi}\int_{3}^{\infty} 1\cos\lambda\xi d\xi$$

$$A(\lambda) = -\frac{\sin\lambda}{\pi\lambda} - \frac{\sin\lambda}{\pi\lambda} + \frac{\sin 3\lambda}{\pi\lambda} + \frac{\sin 3\lambda}{\pi\lambda} = \frac{2}{\pi\lambda}(\sin 3\lambda - \sin\lambda)$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{1}(-1)\sin\lambda\xi d\xi + \frac{1}{\pi}\int_{1}^{\infty} 1\sin\lambda\xi d\xi - \frac{1}{\pi}\int_{-\infty}^{3}(-1)\sin\lambda\xi d\xi - \frac{1}{\pi}\int_{3}^{\infty} 1\sin\lambda\xi d\xi$$

$$B(\lambda) = \frac{\cos\lambda}{\pi\lambda} + \frac{\cos\lambda}{\pi\lambda} - \frac{\cos 3\lambda}{\pi\lambda} - \frac{\cos 3\lambda}{\pi\lambda} = \frac{2}{\pi\lambda}(\cos\lambda - \cos 3\lambda)$$

$$f(x) = \int_0^\infty \frac{2}{\pi\lambda}(\sin 3\lambda - \sin\lambda) + \frac{2}{\pi\lambda}(\cos\lambda - \cos 3\lambda) d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda(3-x) - \sin\lambda(1-x)}{\lambda} d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda(x-1) - \sin\lambda(x-3)}{\lambda} d\lambda$$

2. **Zadanu funkciju prikaži u obliku Fourierovog integrala:** $f(x) = \sin x, 0 \le x \le \pi$
Rješenje:

$$A(\lambda) = \frac{1}{\pi}\int_0^\pi \sin x \cdot \cos\lambda x \, dx = \frac{1}{2\pi}\left(\int_0^\pi \sin(x+\lambda x)dx + \int_0^\pi \sin(x-\lambda x)dx\right)$$

$$= \frac{1}{2\pi}\left(-\frac{1}{1+\lambda}\cos(x+\lambda x)\Big|_0^\pi - \frac{1}{1-\lambda}\cos(x-\lambda x)\Big|_0^\pi\right) = -\frac{1}{2\pi}\left(\frac{\cos(\lambda\pi)+\pi}{1+\lambda} + \frac{\cos(\lambda\pi)+\pi}{1-\lambda}\right)$$

$$= \frac{2(\cos(\lambda\pi)+1}{2\pi(1-\lambda^2)} = \frac{\cos(\lambda\pi)+1}{\pi(1-\lambda^2)}$$

$$B(\lambda) = \frac{1}{\pi}\int_0^\pi \sin x \cdot \sin\lambda x \, dx = \frac{1}{2\pi}\left(\int_0^\pi \cos(x-\lambda x)dx - \int_0^\pi \cos(x+\lambda x)dx\right)$$

$$\frac{1}{2\pi}\left(\frac{1}{1-\lambda}\sin(x-\lambda x)\Big|_0^\pi - \frac{1}{1+\lambda}\sin(x+\lambda x)\Big|_0^\pi\right) = \frac{1}{2\pi}\left(\frac{\sin(\pi-\lambda\pi)}{1-\lambda} - \frac{\sin(\lambda\pi+\pi)}{1+\lambda}\right)$$

3. **Zadanu funkciju prikaži u obliku Fourierovog integrala:**

$$f(x) = \begin{cases} x & \textbf{ako } x \in [-1,1] \\ 0 & \textbf{ako } x \notin [-1,1] \end{cases}$$

Rješenje:

$$A(\lambda) = 0$$

$$B(\lambda) = \frac{1}{\pi}\int_{-1}^{1} x\sin(\lambda x)\ dx = \begin{vmatrix} u = x & dv = \sin(\lambda x)dx \\ du = dx & v = -\frac{1}{\lambda}\cos(\lambda x) \end{vmatrix} = \frac{1}{\pi}\left[-\frac{x}{\lambda}\cos(\lambda x)\Big|_{-1}^{1} + \frac{1}{\lambda}\int_{-1}^{1}\cos(\lambda x)\ dx\right]$$

$$= \frac{1}{\pi}\left[-2\cdot\frac{\cos\lambda}{\lambda} + 2\cdot\frac{\sin\lambda}{\lambda^2}\right] = \frac{2}{\pi}\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^2}$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^2}\sin(\lambda x)\ dx$$

4. **Zadanu funkciju prikaži u obliku Fourierovog integrala:**

$$f(x) = \begin{cases} 1-x & \textbf{ako } x \in (0,1) \\ 1+x & \textbf{ako } x \in (-1,0) \\ 0 & \textbf{inače} \end{cases}$$

Rješenje:
Funkcija je parna. Probajte nacrtati. $B(\lambda) = 0$

$$A(\lambda) = \frac{2}{\pi}\int_{0}^{1}(1-x)\cos(\lambda x)\ dx = \frac{2}{\pi}\left(\int_{0}^{1}\cos(\lambda x)\ dx - \int_{0}^{1} x\cdot(\lambda x)\ dx\right) = \begin{vmatrix} u = x & dv = \cos(\lambda x)dx \\ du = dx & v = \frac{1}{\lambda}\sin(\lambda x) \end{vmatrix}$$

$$= \frac{2}{\pi}\left(\frac{1}{\lambda}\sin(\lambda x)\Big|_{0}^{1} - \frac{x}{\lambda}\sin(\lambda x)\Big|_{0}^{1} + \frac{1}{\lambda}\int_{0}^{1}\sin(\lambda x)\ dx\right) = -\frac{2}{\pi}\frac{\cos(\lambda x)}{\lambda^2}\Big|_{0}^{1} = \frac{2}{\pi}\left(\frac{1-\cos\lambda}{\lambda^2}\right)$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{1-\cos\lambda}{\lambda^2}\cdot\cos(\lambda x)\ d\lambda$$

5. **Odredi sinusni i kosinusni spektar funkcije**

$$f(x) = \begin{cases} \frac{A}{T}x & \textbf{ako } 0 \le x \le T \\ 0 & \textbf{ako } x < 0 \textbf{ ili } x > T \end{cases}$$

Rješenje:

$$A(\lambda) = \frac{A}{\pi T}\int_{0}^{T} x\cdot\cos(\lambda x)\ dx = \begin{vmatrix} u = x & dv = \cos(\lambda x)dx \\ du = dx & v = \frac{1}{\lambda}\sin(\lambda x) \end{vmatrix} = \frac{A}{\pi T}\left[\frac{x}{\lambda}\sin(\lambda x)\Big|_{0}^{T} - \frac{1}{\lambda}\int_{0}^{T}\sin(\lambda x)\ dx\right]$$

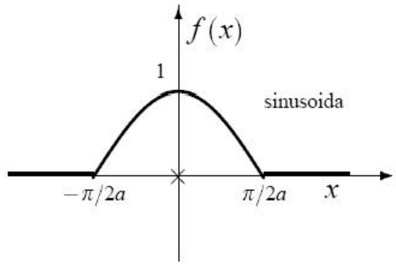
$$= \frac{A}{\pi T}\left[\frac{T}{\lambda}\sin(\lambda T) + \frac{1}{\lambda^2}\cos(\lambda x)\Big|_{0}^{T}\right] = \frac{A}{\pi T}\left[\frac{T}{\lambda}\sin(\lambda T) + \frac{1}{\lambda^2}(\cos(\lambda T) - 1)\right]$$

$$B(\lambda) = \frac{A}{\pi T}\int_{0}^{T} x\cdot\sin(\lambda x)\ dx = \begin{vmatrix} u = x & dv = \sin(\lambda x)dx \\ du = dx & v = -\frac{1}{\lambda}\cos(\lambda x) \end{vmatrix} = \frac{A}{\pi T}\left[-\frac{x}{\lambda}\cos(\lambda x)\Big|_{0}^{T} - \frac{1}{\lambda^2}\sin(\lambda x)\Big|_{0}^{T}\right]$$

$$= \frac{A}{\pi T}\left[\frac{1}{\lambda^2}\sin(\lambda T) - \frac{T}{\lambda}\cos(\lambda T)\right]$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

6. **Funkciju zadanu slikom razvij u Fourierov integral, a zatim pomoću tog prikaza odredi vrijednost integrala** $\int_0^\infty \frac{\cos(\frac{t\pi}{2a})}{a^2-t^2}dt$



Rješenje:

$$f(x) = \cos(ax)$$

$$A(\lambda) = \frac{2}{\pi}\int_0^{\frac{\pi}{2a}} \cos(a\xi)\cdot\cos(\lambda\xi) = \frac{1}{\pi}\int_0^{\frac{\pi}{2a}} \cos((a+\lambda)\xi)d\xi + \int_0^{\frac{\pi}{2a}} \cos((a-\lambda)\xi)d\xi$$

$$= \frac{1}{\pi}\left(\frac{\sin(\frac{\pi}{2a}+\lambda\frac{\pi}{2a})}{a+\lambda} + \frac{\sin(\frac{\pi}{2a}-\lambda\frac{\pi}{2a})}{a-\lambda}\right) = \frac{1}{\pi}\left(\frac{\sin\frac{\pi}{2}\cdot\cos\frac{\lambda\pi}{2a}+\sin\frac{\lambda\pi}{2a}\cdot\cos\frac{\pi}{2}}{a+\lambda} + \frac{\sin\frac{\pi}{2}\cdot\cos\frac{\lambda\pi}{2a}-\sin\frac{\lambda\pi}{2a}\cdot\cos\frac{\pi}{2}}{a-\lambda}\right)$$

$$= \frac{1}{\pi}\left(\frac{\cos\frac{\lambda\pi}{2a}}{a+\lambda} + \frac{\cos\frac{\lambda\pi}{2a}}{a-\lambda}\right) = \frac{2a}{\pi}\frac{\cos\frac{\lambda\pi}{2a}}{a^2-\lambda^2}$$

$$F(x) = \frac{2a}{\pi}\int_0^\infty \frac{\cos\frac{\lambda\pi}{2a}}{a^2-\lambda^2}\cdot\cos(\lambda x)\ d\lambda$$

$$\text{za } x=0,\ f(x)=1,\ \cos(\lambda x)=1$$

$$1 = \frac{2a}{\pi}\int_0^\infty \frac{\cos\frac{\lambda\pi}{2a}}{a^2-\lambda^2}d\lambda \Rightarrow \int_0^\infty \frac{\cos\frac{\lambda\pi}{2a}}{a^2-\lambda^2} = \frac{\pi}{2a}$$

7. **Računajući preko definicije Laplaceovog transformata, odredi sliku sljedeće funkcije. Za svaki transformat naznači njegovo područje definicije**

$$A.\ 2t+1 \qquad B.\ te^{-t} \qquad C.\ e^t\sin t$$

Rješenje:

A.

$$F(s) = \int_0^\infty e^{-st}\cdot(2t+1)\ dt = \begin{vmatrix} u=2t+1 & dv=e^{-st}\,dt \\ du=2dt & v=-\frac{e^{st}}{s}\end{vmatrix}$$

$$= -(2t+1)\cdot\frac{e^{-st}}{s}\Big|_0^\infty + 2\int_0^\infty \frac{e^{st}}{s}$$

$$= -(2t+1)\cdot\frac{e^{-st}}{s}\Big|_0^\infty - \frac{2}{s^2}e^{-st}\Big|_0^\infty$$

$$= \frac{2}{s^2}+\frac{1}{s}$$

B.

$$F(s) = \int_0^\infty e^{-st} \cdot e^{-t} \cdot t\ dt = \int_0^\infty e^{-t(s+1)} \cdot t\ dt = \begin{vmatrix} u = t & dv = e^{-t(s+1)}\ dt \\ du = dt & v = -\frac{e^{-t(s+1)}}{s+1} \end{vmatrix}$$

$$= -\frac{-t \cdot e^{-t(s+1)}}{s+1}\Bigg|_0^\infty + \frac{1}{s+1}\int_0^\infty e^{-t(s+1)}\ dt$$

$$= -\frac{-t \cdot e^{-t(s+1)}}{s+1}\Bigg|_0^\infty - \frac{e^{-t(s+1)}}{(s+1)^2}\Bigg|_0^\infty$$

$$= 0 - 0 - \left(0 - \frac{1}{(s+1)^2}\right) = \frac{1}{(s+1)^2}$$

C.

$$F(s) = \int_0^\infty e^{-st+t} \sin t\ dt = \begin{vmatrix} u = \sin t & dv = e^{t(1-s)}\ dt \\ du = \cos t\ dt & v = \frac{e^{t(1-s)}}{1-s} \end{vmatrix}$$

$$= \frac{\sin t \cdot e^{t(1-s)}}{1-s}\Bigg|_0^\infty - \frac{1}{1-s}\int_0^\infty e^{t(1-s)} \cos t\ dt = \begin{vmatrix} u = \cos t & dv = e^{t(1-s)}\ dt \\ du = -\sin t\ dt & v = \frac{e^{t(1-s)}}{1-s} \end{vmatrix}$$

$$= \frac{\sin t \cdot e^{t(1-s)}}{1-s}\Bigg|_0^\infty - \frac{1}{1-s}\left[\frac{\cos t \cdot e^{t(1-s)}}{1-s}\Bigg|_0^\infty + \frac{1}{1-s}\int_0^\infty e^{t(1-s)} \sin t\ dt\right]$$

$s > 1$ da bi limes postojao.

$$-\frac{1}{(1-s)^2}\cos t \cdot e^{t(1-s)}\Bigg|_0^\infty - \frac{1}{(1-s)^2} I = I$$

$$-\frac{1}{(1-s)^2} \cdot (0-1) = I + \frac{I}{(1-s)^2}$$

$$\frac{1}{(1-s)^2} = I\left(1 + \frac{1}{(1-s)^2}\right)$$

$$I = \frac{\frac{1}{(1-s)^2}}{\frac{(1-s)^2+1}{(1-s)^2}}$$

$$I = \frac{1}{s^2 - 2s + 2}$$

8. **Računajući preko definicije Laplaceovog transformata, odredi sliku sljedeće funkcije. Za svaki transformat naznači njegovo područje definicije**

$$A.\ f(t) = \begin{cases} 1 & \text{ako } t \geq T \\ 0 & \text{ako } t < T \end{cases} \qquad B.\ f(t) = \begin{cases} 1 & \text{ako } t \leq T \\ 0 & \text{ako } t > T \end{cases}$$

Rješenje:

A.

$$F(s) = \int_T^\infty e^{-st} \cdot 1\ dt = -\frac{e^{-st}}{s}\Bigg|_T^\infty = \frac{e^{-sT}}{s}$$

B.

$$F(s) = \int_0^T e^{-st} \cdot 1\ dt = -\frac{e^{-st}}{s}\Bigg|_0^T = \frac{1}{s} - \frac{e^{-sT}}{s}$$

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9. **Računajući preko definicije Laplaceovog transformata, odredi sliku sljedeće funkcije. Za svaki transformat naznači njegovo područje definicije**

$$A.\ f(t) = \begin{cases} t & \text{ako } 0 \le t \le 1 \\ 1 & \text{ako } t > 1 \end{cases} \qquad B.f(t) = \begin{cases} 0 & \text{ako } 0 \le t \le 1 \\ 1 & \text{ako } 1 \le t \le 2 \\ 0 & \text{ako } t > 2 \end{cases}$$

Rješenje:

A.

$$F(s) = \int_0^1 e^{-st} \cdot t\,dt + \int_0^\infty e^{-st}\,dt = \begin{vmatrix} u = t & dv = e^{-st}\,dt \\ du = dt & v = -\frac{e^{-st}}{s} \end{vmatrix} = -\frac{e^{-st}}{s} \cdot t + \frac{1}{s}\int e^{-st}\,dt - \frac{e^{-st}}{s}$$

$$= -\frac{e^{-st}}{s} \cdot t\Big|_0^1 - \frac{e^{-st}}{s^2}\Big|_0^1 - \frac{e^{-st}}{s}\Big|_1^\infty = \frac{1}{s^2} - \frac{e^{-s}}{s^2} = \frac{1}{s^2}(1 - e^{-s})$$

B.

$$F(s) = \int_1^2 e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_1^2 = -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s} = \frac{1}{s}(e^{-s} - e^{-2s})$$

10. **Provjeri jesu li ove funkcije originali ili nisu. Ako jesu, odredi eksponent rasta.**

$$A.\ ch\ 3t \qquad B.\ \sin(t^2) \qquad C.\ e^{-2t}\sin t \qquad D.\ \frac{1}{t}$$

Rješenje:

A.

$$f(t) = ch(3t) = \frac{1}{2} \cdot (e^{3t} + e^{-3t})$$

$$\lim_{t\to\infty} \frac{1}{2} \cdot (e^{3t} + e^{-3t}) \cdot e^{-at} = \lim_{t\to\infty} \frac{1}{2}\frac{e^{3t} + e^{-3t}}{e^{at}} = \frac{1}{2}\lim_{t\to\infty} \frac{e^{3t}}{e^{at}} + \frac{e^{-3t}}{e^{at}}$$

Izraz $\frac{e^{-3t}}{e^{at}}$ gledamo kao 0 i zanemarujemo ga.

$$\frac{1}{2}\lim_{t\to\infty} \frac{e^{3t}}{e^{at}}$$

Izraz pod limesom mora davati neku konstantu. Ako izraz pod limesom preuredimo $e^{t(3-a)}$ te ako je $e^0 = 1$, u slučaju da uvrstimo $a = 3$ dobili bi limes 1, što je konstanta.

Eksponent rasta je $a_0 = 3$.

B.

$$f(t) = \sin(t^2)$$

$$\lim_{t\to\infty} \frac{\sin(t^2)}{e^{at}}$$

Sinus uvijek ima vrijednosti u intervalu $[-1, 1]$, a to znači da za bilo koju vrijednost argumenta on će poprimiti neku vrijednost iz tog intervala $[-1, 1]$, tj. uvijek će bit konstantan pa limes pišemo:

$$\lim_{t\to\infty}\frac{\sin(t^2)}{e^{at}}=\lim_{t\to\infty}\frac{C}{e^{at}}$$

Ako za a uvrstimo 0 tada je limes konstantan.

$$\lim_{t\to\infty}\frac{C}{e^{at}}=\lim_{t\to\infty}\frac{C}{1}=\lim_{t\to\infty}C=C$$

Eksponent rasta je $a_0=0$.

C.

$$f(t)=e^{-2t}\sin t$$

$$\lim_{t\to\infty}\frac{e^{-2t}\cdot\sin(t)}{e^{at}}=\lim_{t\to\infty}\frac{e^{-2t}\cdot C}{e^{at}}=\lim_{t\to\infty}e^{-2t-at}\cdot C=\lim_{t\to\infty}e^{t(-2-a)}\cdot C$$

Ako je $a=-2$ limes je konstanta pa je eksponent rasta

$$a_0=-2$$

D.

$$f(t)=\frac{1}{t}$$

$$\lim_{t\to\infty}\frac{1}{t\cdot e^{at}}$$

Uvjet da je funkcija original (str. 67 knjižica): "f je na svakom konačnom intervalu po dijelovima neprekinuta" te "Laplaceova transformacija je definirana na intervalu $[0,\infty>$". Znači uključujući 0.

Ako uvrstimo 0 u izraz, ne dobijemo da je vrijednost limesa konstanta, što znači da uvjet neprekinutosti nije ispunjen, pa ova funkcija nije original.

11. **Koristeći tablicu Laplaceovih transformata, odredi originale sljedećih funkcija:**

$$A.\ \frac{2s-3}{s^2-6}\qquad B.\ \frac{4s+1}{s^2+5}\qquad C.\ \frac{1}{s-1}-\frac{2}{s+2}$$

Rješenje:

A.

$$\frac{2s-3}{s^2-6}=2\frac{s}{s^2-(\sqrt{6})^2}-\frac{3}{\sqrt{6}}\frac{\sqrt{6}}{s^2-(\sqrt{6})^2}\ \circ\!\!-\!\!\bullet\ 2ch(\sqrt{6}t)-\frac{3}{\sqrt{6}}sh(\sqrt{6}t)$$

B.

$$\frac{4s+1}{s^2+5}=4\frac{s}{s^2+(\sqrt{5})^2}+\frac{1}{s^2+(\sqrt{5})^2}\ \circ\!\!-\!\!\bullet\ 4\cos(\sqrt{5}t)+\frac{1}{\sqrt{5}}\sin(\sqrt{5}t)$$

C.

$$\frac{1}{s-1}-\frac{2}{s+2}\ \circ\!\!-\!\!\bullet\ e^{t}-2e^{-2t}$$

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12. **Koristeći tablicu Laplaceovih transformata, odredi originale sljedećih funkcija:**

$$A.\ \frac{2}{s-2}+\frac{4}{s^2} \qquad B.\ \frac{1}{s^3}-\frac{2}{s^2} \qquad C.\ \frac{1}{s^4}\left(2+\frac{1}{s}-s^2\right)$$

Rješenje:

A.

$$\frac{2}{s-2}+\frac{4}{s^2} \circ\!\!-\!\!\!-\!\!\bullet\ 2e^{2t}+4t$$

B.

$$\frac{1}{s^3}-\frac{2}{s^2} \circ\!\!-\!\!\!-\!\!\bullet\ \frac{1}{2}t^2-2t$$

C.

$$\frac{1}{s^4}\left(2+\frac{1}{s}-s^2\right)=2\frac{1}{s^4}+\frac{1}{s^5}-\frac{1}{s^2} \circ\!\!-\!\!\!-\!\!\bullet\ \frac{1}{3}t^3+\frac{t^4}{4!}-t$$

13. **Odredi slike sljedećih funkcija:**

$$A.\ t^2(t-2) \qquad B.\ sh^3(2t) \qquad C.\ f(t)=\int_0^t u\sin u\ du \qquad D.\ f(t)=\begin{cases}1 & \text{ako } 0<t<3\\ 2 & \text{ako } t\geq 3\end{cases}$$

Rješenje:

A.

$$t^2\cdot u(t-2)=(t-2)^2\cdot u(t-2)+4t\cdot u(t-2)-4\cdot u(t-2)=(t-2)^2\cdot u(t-2)+4(t-2)\cdot u(t-2)-4\cdot u(t-2)+8\cdot u(t-2)$$

$$=(t-2)^2\cdot u(t-2)+4(t-2)\cdot u(t-2)+4\cdot u(t-2)$$

$$(t-2)^2\cdot u(t-2)+4(t-2)\cdot u(t-2)+4\cdot u(t-2) \circ\!\!-\!\!\!-\!\!\bullet\ \frac{2}{s^3}e^{-2s}+\frac{4}{s^2}e^{-2s}+\frac{4}{s}e^{-2s}=e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

B.

$$sh^3(2t)=\left(\frac{1}{2}e^{2t}-\frac{1}{2}e^{-2t}\right)^3=\frac{1}{8}e^{6t}-\frac{3}{8}e^{2t}e^{-4t}+\frac{3}{8}e^{2t}e^{-4t}-\frac{1}{8}e^{-6t}=\frac{1}{8}e^{6t}-\frac{3}{8}e^{2t}+\frac{3}{8}e^{-2t}-\frac{1}{8}e^{-6t}$$

$$\frac{1}{8}e^{6t}-\frac{3}{8}e^{2t}+\frac{3}{8}e^{-2t}-\frac{1}{8}e^{-6t} \circ\!\!-\!\!\!-\!\!\bullet\ \frac{1}{8}\frac{1}{s-6}-\frac{3}{8}\frac{1}{s-2}+\frac{3}{8}\frac{1}{s+2}-\frac{1}{8}\frac{1}{s+6}$$

$$=\frac{1}{8}\left(\frac{s+6-s+6}{s^2-36}+\frac{-3s-6+3s-6}{s^2-4}\right)=\frac{3}{2}\left(\frac{1}{s^2-36}-\frac{1}{s^2-4}\right)$$

C.

$$f(t)=\int_0^t u\sin u\ du=\begin{vmatrix}u=m & dv=\sin u\ du\\ du=dm & v=-\cos u\end{vmatrix}=...=-t\cos t+\sin t$$

Teorem o deriviranju slike (str.77 "Fourierov red i Laplaceova transformacija")

$$-t\cos t=F'(s)$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

$$\cos t \circ\!\!-\!\!\bullet \frac{s}{s^2+1}$$

$$F'(s) = \frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{-s^2+1}{(s^2+1)^2}$$

$$-t\cos t + \sin t \circ\!\!-\!\!\bullet -\frac{s^2+1}{(s^2+1)^2} + \frac{1}{s^2+1} = \frac{-s^2+1+s^2+1}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$$

D.

$$f(t) = 1\cdot g_{[0,3]} + 2\cdot g_{[3,\infty]} = u(t) - u(t-3) + 2\cdot u(t-3) = u(t) + u(t-3)$$

$$u(t) + u(t-3) \circ\!\!-\!\!\bullet \frac{1}{s} + \frac{1}{s}e^{-3s} = \frac{1}{s}(1+e^{-3s})$$

14. **Odredi slike sljedećih funkcija:**

$A.\ 5\cdot u(t-2) - 2\cdot u(t-3) \qquad B.\ t^3e^{-2t} + t^2 \qquad C.\ f(t) = \begin{cases} 1 & \text{ako } 0 \le t < 1 \\ t & \text{ako } t \le t < 2 \\ 2 & \text{ako } t \ge 2 \end{cases} \qquad D.\ f(t) = \int_0^t u^2e^u\, du$

Rješenje:

A.

$$5\cdot u(t-2) - 2\cdot u(t-3) \circ\!\!-\!\!\bullet \frac{5}{s}e^{-2s} - \frac{2}{s}e^{-3s}$$

B.

$$t^3e^{-2t} + t^2 \circ\!\!-\!\!\bullet \frac{6}{(s+2)^4} + \frac{2}{s^3}$$

C.

$$f(t) = g_{[0,1]} + t\cdot g_{[1,2]} + 2\cdot g_{[2,\infty]} = u(t) - u(t-1) + t\cdot u(t-1) - t\cdot u(t-2) + 2\cdot u(t-2)$$

$$= u(t) - u(t-1) + (t-1)\cdot u(t-1) + u(t-1) - (t-2)\cdot u(t-2) - 2\cdot u(t-2) + 2\cdot u(t-2) = u(t) + (t-1)\cdot u(t-1) - (t-2)\cdot u(t-2)$$

$$u(t) + (t-1)\cdot u(t-1) - (t-2)\cdot u(t-2) \circ\!\!-\!\!\bullet \frac{1}{s} + \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} = \frac{1}{s}\left(1 + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right)$$

D.

Teorem o integriranju originala

$$u^2e^u \circ\!\!-\!\!\bullet \frac{2}{(s-1)^3}$$

$$\int_0^t u^2e^u\, du \circ\!\!-\!\!\bullet \frac{2}{s(s-1)^3}$$

15. **Odredi slike sljedećih funkcija:**

$$A.\ 1-t^2e^{-2t}\cdot u(t-3) \qquad B.\ \frac{sh\ t}{t} \qquad C.\ f(t)=\int_0^t (u^3+1)e^{-u}\ du \qquad D.\ f(t)=\begin{cases}0 & \text{ako } t<2\\ t+1 & \text{ako } t\geq 2\end{cases}$$

Rješenje:

A.

$$\begin{aligned}1-t^2e^{-2t}\cdot u(t-3) &= 1-(t-3)^3e^{-2t}\cdot u(t-3)-6te^{-2t}\cdot u(t-3)+9e^{-2t}\cdot u(t-3)\\ &= 1-(t-3)^3e^{-2t}\cdot u(t-3)-6te^{-2t}\cdot u(t-3)-9e^{-2t}\cdot u(t-3)\end{aligned}$$

$$1-(t-3)^3e^{-2t}\cdot u(t-3)-6te^{-2t}\cdot u(t-3)+9e^{-2t}\cdot u(t-3) \circ\!\!-\!\!\bullet\ \frac{1}{s}-\frac{2}{(s+2)^3}e^{-3(s+2)}-\frac{6}{(s+2)^2}e^{-3(s+2)}-\frac{9}{s+2}e^{-3(s+2)}$$

B.

Teorem o integriranju slike (str.79)

$$sh\ t \circ\!\!-\!\!\bullet\ \frac{1}{s^2-1}$$

$$\frac{sh\ t}{t} \circ\!\!-\!\!\bullet\ \int_s^\infty \frac{1}{s^2-1}\ ds = \frac{1}{2}\ln\left|\frac{s-1}{s+1}\right|\Bigg|_s^\infty = \frac{1}{2}\ln\left(\lim_{s\to\infty}\frac{s-1}{s+1}\right)-\frac{1}{2}\ln\left|\frac{s-1}{s+1}\right| = \frac{1}{2}\ln\left|\frac{s+1}{s-1}\right|$$

C.

Teorem o integriranju originala str.81

$$(u^3+1)e^{-u} = u^3\cdot e^{-u}+e^{-u} \circ\!\!-\!\!\bullet\ \frac{6}{(s+1)^4}+\frac{1}{s+1}$$

$$\int_0^t (u^3+1)e^{-u}\ du \circ\!\!-\!\!\bullet\ \frac{1}{s}\left(\frac{6}{(s+1)^4}+\frac{1}{s+1}\right)$$

D.

$$f(t)=(t+1)\cdot g_{[2,\infty]} = (t+1)\cdot u(t-2) = (t-2)\cdot u(t-2)+3\cdot u(t-2)$$

$$(t-2)\cdot u(t-2)+3\cdot u(t-2) \circ\!\!-\!\!\bullet\ \frac{1}{s^2}e^{-2s}+\frac{3}{s}e^{-2s}$$

16. **Odredi sliku periodičke funkcije perioda $T=4$, a koja je zadana formulama**

$$f(t)=\begin{cases}3t & \text{ako } 0<t<2\\ 6 & \text{ako } 2<t<4\end{cases}$$

Rješenje:

Teorem 11 - Slika periodične funkcije (str.82)

$$f(t)=3t\cdot g_{[0,2]}+6\cdot g_{[2,4]} = 3t\cdot u(t)-3t\cdot u(t-2)+6\cdot u(t-2)-6\cdot u(t-4) = 3t\cdot u(t)-3(t-2)\cdot u(t-2)-6\cdot u(t-2)+6\cdot u(t-2)-6\cdot u(t-4)$$

$$3t\cdot u(t)-3(t-2)\cdot u(t-2)-6\cdot u(t-4) \circ\!\!-\!\!\bullet\ \frac{3}{s^2}-\frac{3}{s^2}e^{-2s}-\frac{6}{s}e^{-4s}$$

$$F(s)=\frac{3}{s^2(1-e^{-4s})}(1-e^{-2s}-2se^{-4s})$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

17. **Primjenom Laplaceove transformacije izračunaj integral**

$$A.\ \int_0^\infty e^{-ax}\cdot\frac{\sin x}{x}\,dx \qquad B.\ \int_0^\infty e^{-2t}\cdot t\cdot\cos t\ dt \qquad C.\ \int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\,dt$$

Rješenje:

A.

Integral od Laplace $\frac{\sin x}{x}$ uz a kao donju granicu.

Laplace $\sin x$.

$$\sin x \circ\!\!-\!\!\bullet \frac{1}{1+a^2}$$

$$\frac{\sin x}{x} = \int_a^\infty \frac{1}{1+a^2}\,da = arctg\ a\Big|_a^\infty = \frac{\pi}{2} - arctg\ a$$

$$\int_0^\infty e^{-ax}\cdot\frac{\sin x}{x}\,dx = \frac{\pi}{2} - arctg\ a$$

B.

Laplace $\cos t\cdot t$, $s=2$.

$$\cos t \circ\!\!-\!\!\bullet \frac{s}{s^2+1}$$

deriviranje

$$-t\cdot\cos t \circ\!\!-\!\!\bullet \frac{1-s^2}{(s^2+1)^2}$$

$$t\cdot\cos t \circ\!\!-\!\!\bullet \frac{s^2-1}{(s^2+1)^2}$$

uvrstimo $s=2$ u zadnjeg Laplacea.

$$\int_0^\infty e^{-2t}\cdot t\cdot\cos t\ dt = \frac{3}{25}$$

C.

$$\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\,dt = \int_0^\infty \frac{e^{-2t}}{t}\,dt - \int_0^\infty \frac{e^{-4t}}{t}\,dt$$

$$\int_0^\infty \frac{e^{-2t}}{t}\,dt = |s=2| = \int_2^\infty \frac{1}{s}\,ds = \ln s\Big|_2^\infty = \ln(\infty) - \ln 2$$

$$\int_0^\infty \frac{e^{-4t}}{t}\,dt = |s=4| = \int_4^\infty \frac{1}{s}\,ds = \ln s\Big|_4^\infty = \ln(\infty) - \ln 4 = \ln(\infty) - 2\ln 2$$

$$\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\,dt = \int_0^\infty \frac{e^{-2t}}{t}\,dt - \int_0^\infty \frac{e^{-4t}}{t}\,dt = \ln(\infty) - \ln 2 - \ln(\infty) + 2\ln 2 = \ln 2$$

$$\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\,dt = \ln 2$$

18. **Odredi original funkcije**

$$A.\ \frac{1}{s^2+4s+3} \qquad B.\ \frac{s^3e^{-2s}}{(s^2+4)(s-1)} \qquad C.\ \frac{(s+2)e^{-4s}}{s^2+2s+5}$$

Rješenje:

A.

$$\frac{1}{s^2+4s+3} = \frac{1}{(s+2)^2-1} \bullet\!\!-\!\!\circ\ sh\ t\cdot e^{-2t}$$

B.

$$\frac{s^3e^{-2s}}{(s^2+4)(s-1)} = \frac{s^3-s^2+4s-4+s^2-4s+4}{s^3-s^2+4s-4} = 1+\frac{s^2-4s+4}{(s^2+1)(s-1)} = 1+\frac{As+B}{s^2+4}+\frac{C}{s-1}$$

$$(As+B)(s-1)+C(s^2+4) = s^2-4s+4$$

$$za\ s=1\ \Rightarrow 5C=1\Rightarrow C=\frac{1}{5}$$

$$za\ s=0\ \Rightarrow -B+\frac{4}{5}=4\Rightarrow B=-\frac{16}{5}$$

$$za\ s=2\ \Rightarrow 2A-\frac{16}{5}+\frac{8}{5}=0\Rightarrow A=\frac{4}{5}$$

$$e^{-2s}+\frac{4}{5}\frac{s}{s^2+4}e^{-2s}-\frac{8}{5}\frac{2}{s^2+4}e^{-2s}+\frac{1}{5}\frac{1}{s-1}e^{-2s} \bullet\!\!-\!\!\circ \left(\sigma(t-2)+\frac{4}{5}\cos 2(t-2)-\frac{8}{5}\sin 2(t-2)+\frac{1}{5}e^{t-2}\right)\cdot u(t-2)$$

$$f(t) = \left(\sigma(t-2)+\frac{4}{5}\cos 2(t-2)-\frac{8}{5}\sin 2(t-2)+\frac{1}{5}e^{t-2}\right)\cdot u(t-2)$$

C.

$$\frac{(s+2)e^{-4s}}{s^2+2s+5} = e^{-4s}\frac{s+1}{(s+1)^2+4}+e^{-4s}\frac{1}{(s+1)^2+4} = e^{-4s}\frac{s+1}{(s+1)^2+4}+\frac{1}{2}e^{-4s}\frac{2}{(s+1)^2+4}$$

$$\frac{s+1}{(s+1)^2+4} \bullet\!\!-\!\!\circ \cos(2t)\cdot e^{-t}$$

$$\frac{2}{(s+1)^2+4} \bullet\!\!-\!\!\circ \sin(2t)\cdot e^{-t}$$

$$\frac{s+1}{(s+1)^2+4}+\frac{1}{2}\frac{2}{(s+1)^2+4} \bullet\!\!-\!\!\circ \cos(2t)\cdot e^{-t}+\frac{1}{2}\sin(2t)\cdot e^{-t}$$

$$e^{-4s}\left(\frac{s+1}{(s+1)^2+4}+\frac{1}{2}\frac{2}{(s+1)^2+4}\right) \bullet\!\!-\!\!\circ [\cos 2(t-4)\cdot e^{-(t-4)}+\frac{1}{2}\sin 2(t-4)\cdot e^{-(t-4)}]u(t-4)$$

19. **Odredi original funkcije**

$$A.\ \frac{s+1}{s^2(s-1)(s+2)} \qquad B.\ \frac{1}{(s-1)^2(s-2)^2}$$

Rješenje:

A.

$$\frac{As+B}{s^2} + \frac{C}{s-1} + \frac{D}{s+2} = \frac{s+1}{s^2(s-1)(s+2)}$$

$$(As+B)(s-1)(s+2) + C \cdot s^2(s+2) + D \cdot s^2(s-1) = s+1$$

$$As^3 + As^2 - 2As + Bs^2 + Bs - 2B + Cs^3 + 2Cs^2 + Ds^3 - Ds^2 = s+1$$

$$A+C+D=0$$

$$A+B+2C-D=0$$

$$-2A+B=1 \Rightarrow A = \frac{B-1}{2} \Rightarrow A = -\frac{3}{4}$$

$$-2B=1 \Rightarrow B=-\frac{1}{2}$$

Iz prve dvije jednadžbe dobijemo C i D.

$$C=\frac{2}{3} \quad D=\frac{1}{12}$$

$$\frac{s+1}{s^2(s-1)(s+2)} = -\frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{2}{3}\frac{1}{s-1} + \frac{1}{12}\frac{1}{s+2} \circ\!\!-\!\!\bullet\ -\frac{3}{4} - \frac{1}{2}t + \frac{2}{3}e^t + \frac{1}{12}e^{-2t}$$

B.

$$\frac{1}{(s-1)^2(s-2)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$A(s-1)(s-2)^2 + B(s-2)^2 + C(s-2)(s-1)^2 + D(s-1)^2 = 1$$

$$As^3 - 5As^2 + 8As - 4A + Bs^2 - 4Bs + 4B + Cs^3 - 4Cs^2 + 5Cs - 2C + Ds^2 - 2Ds + D = 1$$

$$A+C=0$$

$$-5A+B-4C+D=0$$

$$8A-4B+5C-2D=0$$

$$-4A+4B-2C+D=1$$

Nakon rješavanja sustava jednadžbi:

$$A=2,\ B=1,\ C=-2,\ D=1$$

$$\frac{1}{(s-1)^2(s-2)^2} = \frac{2}{s-1} + \frac{1}{(s-1)^2} - \frac{2}{s-2} + \frac{1}{(s-2)^2} \circ\!\!-\!\!\bullet\ 2e^t + te^t - 2e^{2t} + te^{2t}$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

20. **Koristeći teorem o konvoluciji, izračunaj original funkcije**

$$A.\ \frac{1}{s(s+3)} \qquad B.\ \frac{1}{s(s^2-4s+5)} \qquad C.\ \frac{1}{s^3(s+1)^3}$$

Rješenje:

A.

$$\frac{1}{s(s+3)}$$

$$\frac{1}{s} \bullet\!\!-\!\!\circ\ u(t)$$

$$\frac{1}{s+3} \bullet\!\!-\!\!\circ\ e^{-3t}$$

$$\frac{1}{s(s+3)} = u(t) * e^{-3t} = \int_0^t e^{-3\tau} u(t-\tau)\ d\tau = -\frac{1}{3} e^{-3\tau}\Big|_0^t = \frac{1}{3} - \frac{1}{3} e^{-3t}$$

B.

$$\frac{1}{s(s^2-4s+5)} = \frac{1}{s((s-2)^2+1)} \bullet\!\!-\!\!\circ\ \sin t \cdot e^{2t} * u(t)$$

$$\frac{1}{s} \bullet\!\!-\!\!\circ\ u(t)$$

$$\frac{1}{(s-2)^2+1} \bullet\!\!-\!\!\circ\ \sin t \cdot e^{2t}$$

$$\sin t \cdot e^{2t} * u(t) = \int_0^t \sin\tau e^{2\tau}\ d\tau = \begin{vmatrix} u = e^{2\tau} & dv = \sin\tau d\tau \\ du = 2e^{2\tau} d\tau & v = -\cos\tau \end{vmatrix} =$$

$$= -e^{2t}\cos t + 1 + 2\int_0^t \cos\tau e^{2\tau}\ d\tau = \begin{vmatrix} u = e^{2\tau} & dv = \cos\tau d\tau \\ du = 2e^{2\tau} d\tau & v = \sin\tau \end{vmatrix}$$

$$\int_0^t \sin\tau e^{2\tau} d\tau = -e^{2t}\cos t + 1 + 2e^{2t}\sin t - 4\int_0^t \sin\tau e^{2\tau}\ d\tau$$

$$\int_0^t \sin\tau e^{2\tau}\ d\tau = \frac{1}{5}(-e^{2t}\cos t + 1 + 2e^{2t}\sin t)$$

C.

$$F(s) = \frac{1}{s^3(s+1)^3}$$

$$F_1(s) = \frac{1}{s^2} \Rightarrow f_1(t) = t$$

$$F_2(s) = \frac{1}{(s+1)^3} \Rightarrow f_2(t) = \frac{1}{2} t^2 e^{-t}$$

Konvolucija:

$$f(t) = \frac{1}{2}\int_0^t e^{-\tau}\tau^2(t-\tau) d\tau = \frac{1}{2} t \int_0^t e^{-\tau}\tau^2 d\tau - \frac{1}{2}\int_0^t e^{-\tau}\tau^3 d\tau$$

$$f(t) = \frac{1}{2} t \cdot I_1 - \frac{1}{2} \cdot I_2$$

$$I_1 = \int_0^t e^{-\tau}\tau^2 d\tau = ... = e^{-t} \cdot (-t^2 - 2t - 2) + 2$$

$$I_2 = \int_0^t e^{-\tau}\tau^3 d\tau = ... = e^{-t} \cdot (-t^3 - 3t^2 - 6t - 6) + 6$$

$$f(t) = t - 3 + \frac{1}{2}e^{-t}(t^2 + 4t + 6)$$