MASS by Burić

Plošni integrali

I vrste

$$\iint_S f(x,y,z)\,dS$$

$\hookrightarrow z = z(x,y) \rightarrow dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

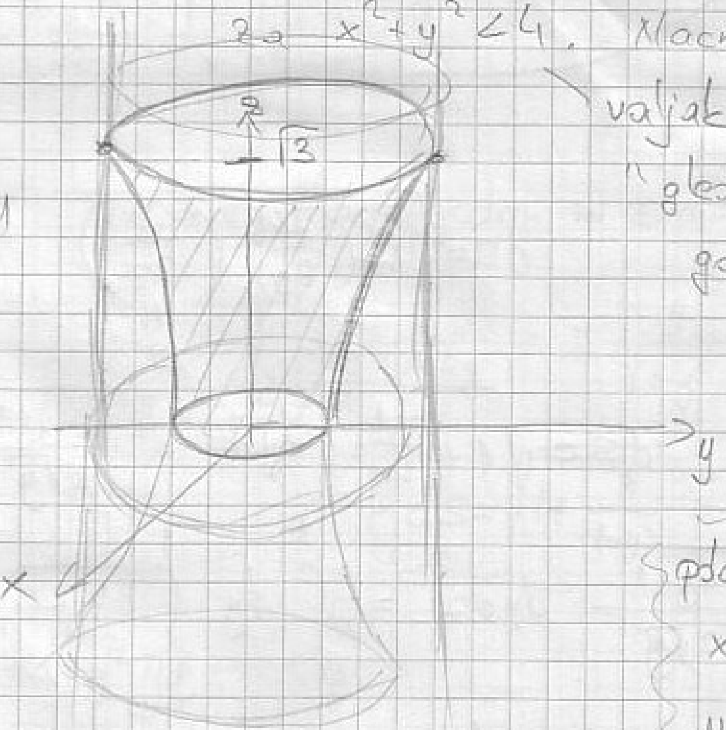
površina plohe: $P = \iint_S dS$

7.DZ - 135)

$\iint_S z\,dS$, gdje je s dio plohe $z=\sqrt{x^2+y^2-1}$ za $x^2+y^2<4$. Nacrtaj sliku.

$z^2 = x^2+y^2-1$

$x^2+y^2-z^2=1$



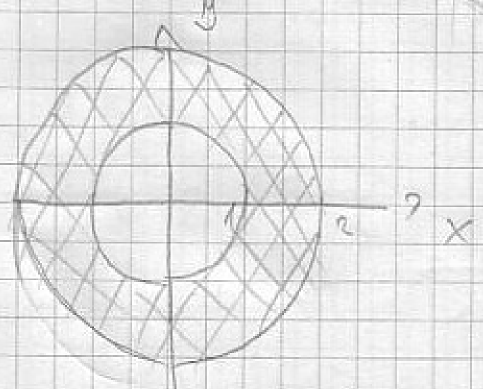
valjak

"gledamo samo gornji dio"

polarne koo.

$x = r\cos\varphi$

$y = r\sin\varphi$



$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2-1}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2-1}}$$

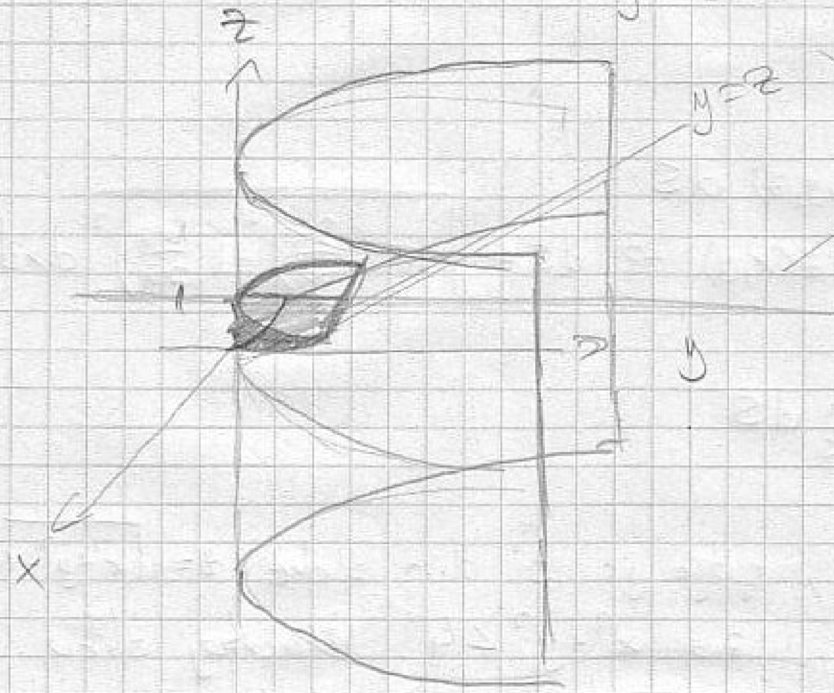
$$dS = \sqrt{\frac{x^2+y^2-1+x^2+y^2}{x^2+y^2-1}}\,dx\,dy$$

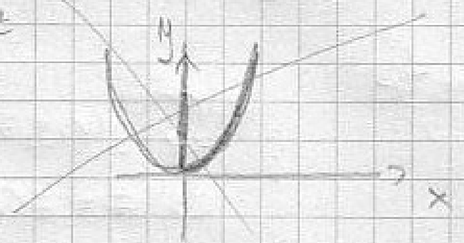
$$I = \iint \cancel{\sqrt{x^2+y^2-1}}\sqrt{\frac{2x^2+2y^2-1}{\cancel{x^2+y^2-1}}}\,dx\,dy = \int_0^{2\pi} d\varphi \int_1^2 \sqrt{2r^2-1}\,r\,dr = \ldots = \frac{\pi}{3}(7\sqrt{7}-1)$$

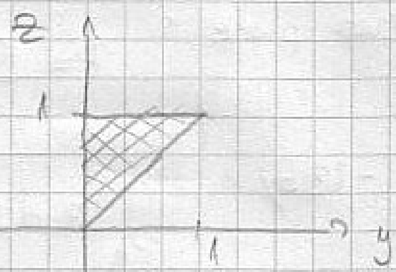
PZI-07-7)

$\iint_S x\,dS$, ... S dio plohe $y=x^2$ za koji je

$y \le z \le 1$, $x \ge 0$







$x = f(y,z)$

$x = \pm\sqrt{y}$

$x = \sqrt{y}$

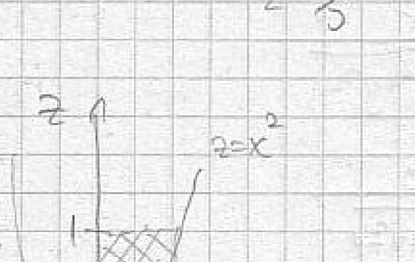
$$dS = \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2 \underbrace{\left(\frac{\partial x}{\partial z}\right)}_{0}}\; dy\,dz$$

$$dS = \sqrt{1+\frac{1}{4y}}\; dy\,dz = \frac{\sqrt{4y+1}}{2\sqrt{y}}\; dy\,dz$$

$$I = \iint_{\Omega_{yz}} \cancel{\sqrt{y}} \cdot \frac{\sqrt{4y+1}}{2\cancel{\sqrt{y}}}\; dy\,dz = \frac{1}{2}\int_0^1 dy \int_y^1 \sqrt{4y+1}\, dz = \dots =$$

$$= \frac{1}{2}\int_0^1 (1-y)\sqrt{4y+1}\, dy = \dots$$

ILI :



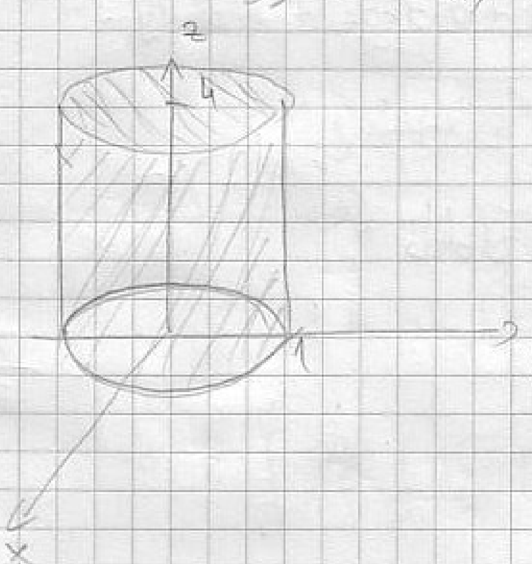
$$y = f(x,z) = x^2$$

$$\frac{\partial y}{\partial x} = 2x \qquad \frac{\partial y}{\partial z} = 0$$

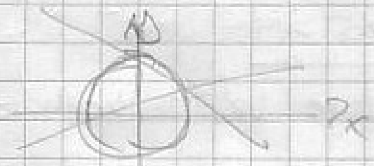
$$dS = \sqrt{1+4x^2}\, dx\,dz$$

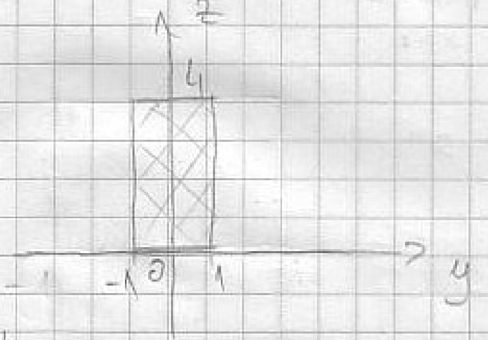
$$I = \iint x\sqrt{1+4x^2}\, dx\,dz = \int_0^1 x\sqrt{1+4x^2}\, dx \int_{x^2}^1 dz = \dots = \frac{1}{24}\left(5\sqrt{5} - \frac{11}{5}\right)$$

7DZ. 13 a) $\iint (x^2+xy)\,dS$, $S \ldots x^2+y^2=1$, $0 \le z \le 4$



x-y ravnina





$$dS = \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

$x = \pm\sqrt{1-y^2}$ - 2 integrala (jedan za ⊕ dio, drugi za ⊖ dio)

$$\frac{\partial x}{\partial y} = \mp \frac{y}{\sqrt{1-y^2}}$$

$$dS = \sqrt{1+\frac{y^2}{1-y^2}} = \frac{1}{\sqrt{1-y^2}}\,dy\,dz$$

$$I = \iint\limits_{\substack{\Omega_{yz}^{+} \\ x>0}} \left(\left(+\sqrt{1-y^2}\right)^2 + \cancel{\sqrt{1-y^2}\cdot y}\right)\cdot\frac{dy\,dz}{\sqrt{1-y^2}} +$$

$$+ \iint\limits_{\substack{\Omega_{yz}^{-} \\ x<0}} \left(\left(-\sqrt{1-y^2}\right)^2 - \cancel{\sqrt{1-y^2}\,y}\right)\frac{dy\,dz}{\sqrt{1-y^2}}$$

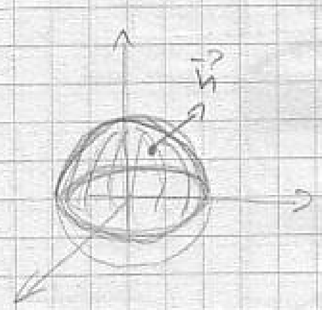
$$= 2\iint\limits_{\Omega_{yz}} (1-y^2)\frac{dy\,dz}{\sqrt{1-y^2}} = 2\int_{-1}^{1}\sqrt{1-y^2}\,dy\int_0^4 dz = 4\pi$$

II vrste

$$\iint_{S^+} \vec{a}\cdot d\vec{S} = \iint_{S^+} a_1\, dy\, dz + a_2\, dx\, dz + a_3\, dx\, dy$$

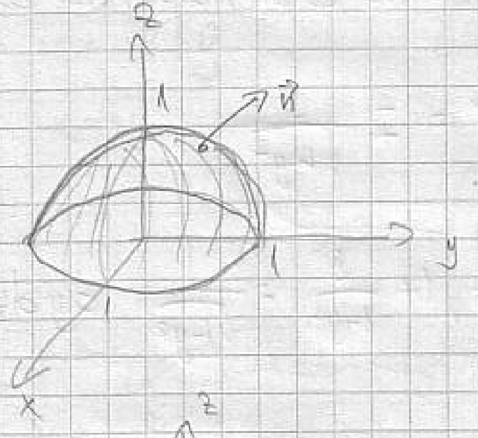
Strana orijentacija

orijentaciju određuje vektor normale na plohu



PZI-07-8) $$\iint_S x^2\, dy\, dz + y\, dx\, dz + (z^2+1)\, dx\, dy$$

S ... vanjska strana plohe $x^2+y^2+z^2=1$ za $z \geq 0$

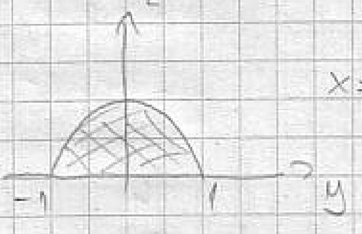


1) $\iint x^2\, dy\, dz = \oplus \iint \left(+\sqrt{1-y^2-z^2}\right)^2 dy\, dz \ominus \iint \left(-\sqrt{1-y^2-z^2}\right)^2 dy\, dz$

$\alpha < 90°$, $\cos\alpha > 0$

$\alpha > 90°$, $\cos\alpha < 0$

$= 0$



$x = \pm\sqrt{1-y^2-z^2}$

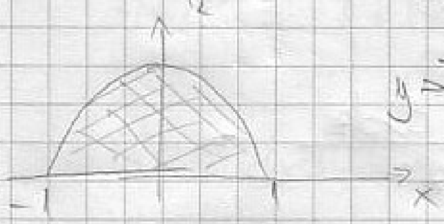
gledamo kut između poz. strane x-osi i vektora normale

možemo gledati i parnost dijela, x^2 je parna, integriramo po ⊕ i ⊖ dijelu te je int. $= 0$

y je neparna, integriramo po ⊕ i ⊖ dijelu pa je int $= 2\iint \ldots$

z je parna ali integriramo samo po ⊕ dijelu pa imamo int $= \iint \ldots$

2) $\iint y\,dx\,dz = +\iint\left(+\sqrt{1-x^2-z^2}\right)dx\,dz - \iint\left(-\sqrt{1-x^2-z^2}\right)dx\,dz$



$y = \pm\sqrt{1-x^2-z^2}$

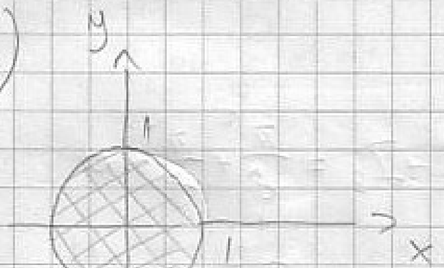
$\beta < 90$, $\cos\beta > 0$

$\beta > 90^\circ$, $\cos\beta < 0$

$$= 2\iint \sqrt{1-x^2-z^2}\,dx\,dz = \begin{vmatrix} x = r\cos\varphi \\ z = r\sin\varphi \end{vmatrix} =$$

$$= 2\int_0^{\pi} d\varphi \int_0^1 \sqrt{1-r^2}\, r\,dr = \frac{2\pi}{3}$$

3)



$$+\iint (1-x^2-y^2+1)\,dx\,dy = \begin{vmatrix} x = r\cos\varphi \\ y = r\sin\varphi \end{vmatrix} =$$

$\gamma < 90^\circ$, $\cos\gamma > 0$

$z = \pm\sqrt{1-x^2-y^2}$

$$= \int_0^{2\pi} d\varphi \int_0^1 (2-r^2)\,r\,dr =$$

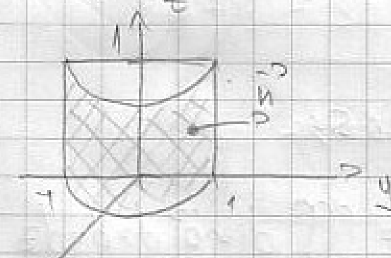
$$= \frac{3\pi}{2}$$

$$I = I_1 + I_2 + I_3 = 0 + \frac{2\pi}{3} + \frac{3\pi}{2} = \frac{13\pi}{6}$$

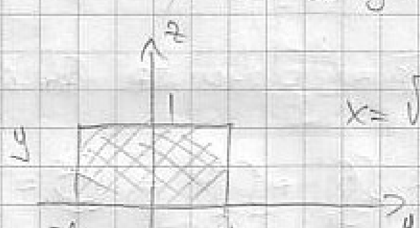
ZZV) $I = \iint_S \dfrac{x\,dy\,dz + y\,dx\,dz + z\,dx\,dy}{x^2+y^2+z^2}$

S je dio plohe $x = \sqrt{1-y^2}$ za $0 \le z \le 1$ orijentiran tako da normala na plohu zatvara s osi x šiljasti kut

$x^2 + y^2 = 1$



$$I_1 = \iint \frac{x\,dy\,dz}{x^2+y^2+z^2} = +\iint \frac{\sqrt{1-y^2}\,dy\,dz}{\cancel{1-y^2}+\cancel{y^2}+z^2} =$$



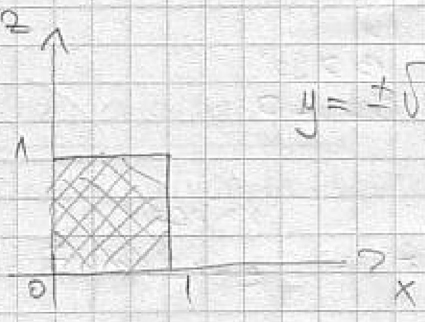
$x = \sqrt{1-y^2}$

$$= \underbrace{\int_{-1}^1 \sqrt{1-y^2}\,dy}_{\frac{\pi}{2}} \underbrace{\int_0^1 \frac{dz}{1+z^2}}_{\frac{\pi}{4}} =$$

supst. $y = \sin t$

$$= \frac{\pi^2}{8}$$

I_2) $\iint \frac{y\,dx\,dz}{x^2+y^2+z^2} = \oplus \iint\limits_{\beta<90^\circ} \frac{+\sqrt{1-x^2}\,dx\,dz}{x^2+1-x^2+z^2} \ominus \iint\limits_{\beta>90^\circ} \frac{-\sqrt{1-x^2}\,dx\,dz}{x^2+1-x^2+z^2}$

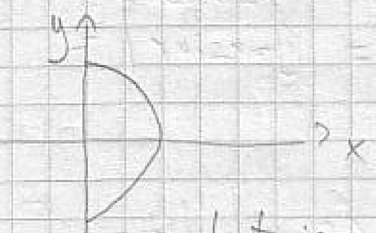


$y=\pm\sqrt{1-x^2}$

$$= 2\iint \frac{\sqrt{1-x^2}\,dx\,dz}{1+z^2} =$$

$$= 2\underbrace{\int_0^1 \sqrt{1-x^2}\,dx}_{\frac{\pi}{4}} \underbrace{\int_0^1 \frac{dz}{1+z^2}}_{\frac{\pi}{4}} = \frac{\pi^2}{8}$$

I_3) $\iint \frac{z\,dx\,dy}{x^2+y^2+z^2} = \emptyset$



kut je uvijek 90°

$\cos 90^\circ = 0$

$$I = I_1 + I_2 + I_3 = \frac{\pi^2}{8} + \frac{\pi^2}{8} + 0$$

$$= \frac{\pi^2}{4}$$

ZI-09-6) a) Iskaži i dokaži teorem o divergenciji

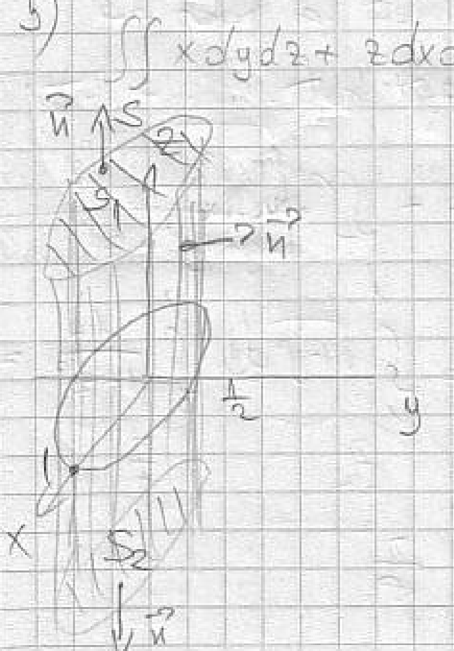
$$\oiint\limits_{S^+} \vec{a}\,d\vec{s} = \iiint\limits_V \operatorname{div}\vec{a}\,dx\,dy\,dz =$$

$$= \iiint\limits_V \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) dx\,dy\,dz$$

b) $\iint x\,dy\,dz + z\,dx\,dy$ po vanjskoj strani plohe $x^2+4y^2=1$

za $-1 \le z \le 1$

$$x^2 + \frac{y^2}{(\frac{1}{2})^2} = 1$$



zatvaram plohu sa S_1 i S_2

$$\underbrace{\iint\limits_S}_{=?} + \iint\limits_{S_1} + \iint\limits_{S_2} = \oiint = \iiint \operatorname{div}\vec{a}\,dx\,dy\,dz$$

$$\iint\limits_S = \iiint - \iint\limits_{S_1} - \iint\limits_{S_2}$$

$\vec{a} = x\vec{i} + z\vec{k}$ — eliptične koordinate

$\operatorname{div}\vec{a} = 1+1 = 2$

$x = r\cos\varphi$
$y = \frac{1}{2} r\sin\varphi$
$dxdy = \frac{1}{2} r\,dr\,d\varphi$

$$\iiint_V 2\,dxdydz = 2\int_0^{2\pi} d\varphi \int_0^1 \frac{r}{2}\,dr \int_{-1}^{1} dz$$

$$= 2\pi$$

S_1

$$\iint_{S_1} \underbrace{x\,dydz}_{=0} + z\,dxdy = \oplus \iint 1\,dxdy = \frac{\pi}{2}$$

$\gamma = 0^\circ$

jer je $\alpha = 90^\circ$

S_2

$$\iint_{S_2} \underbrace{x\,dydz}_{=0} + z\,dxdy = \ominus \iint -1\,dxdy = \frac{\pi}{2}$$

$\gamma = 180^\circ$

$\alpha = 90^\circ$

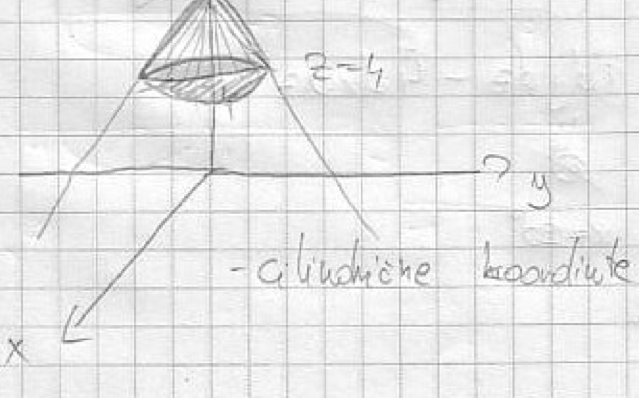
$$\iint_S = 2\pi - \frac{\pi}{2} - \frac{\pi}{2} = \pi$$

PZI -09-6) a) isto ko i prijašnji zad

b) Izračunaj tok vektorskog polja

$a = x^2\vec{i} + y^2\vec{j} + z\vec{k}$ kroz vanjski dio plohe

koja je rub $V = \left\{ (z-8)^2 \geq x^2+y^2,\ z \geq 4,\ z \leq 8,\ x \geq 0 \right\}$



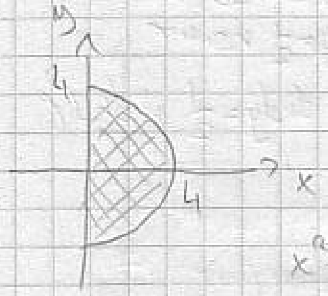
- zatvorena ploha - netreba zatvarat

$$\oiint \vec{a}\,d\vec{s} = \iiint_V \operatorname{div}\vec{a}\,dxdydz$$

$\operatorname{div}\vec{a} = 2x + 2y + 1$

- cilindrične koordinate

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^4 r\,dr \int_4^{8-r} (2r\cos\varphi + 2r\sin(\varphi+\pi))\,dz = \dots$$



$$= \frac{32}{3}(8+\pi)$$

$x^2+y^2=16$

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$

$z-8 = \pm\sqrt{x^2+y^2}$

$z = 8-\sqrt{x^2+y^2}$

$z = 8-r$

7DZ - 20) a)

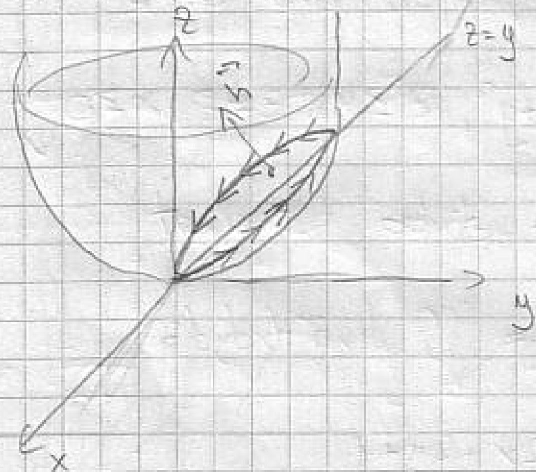
$$\oint_{\Gamma} a_1 dx + a_2 dy + a_3 dz =$$

$$= \iint_{S+} \operatorname{rot}\vec{a}\, d\vec{S}$$

b) $\oint_C -4z\,dx + 3dy + y\,dz$ C presječnica ploha

$z = x^2+y^2$ i $z = y$

Orjentiran tako da je njena projekcija u x-y ravnini pozitivno orjentirana

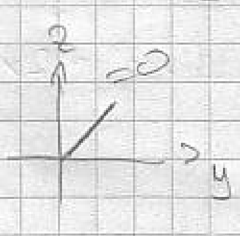


$$\operatorname{rot}\vec{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4z & 3 & y \end{vmatrix}$$

$$= \vec{i} - 4\vec{j} + 0\vec{k} = 1\vec{i} - 4\vec{j}$$

$$I = \iint_S 1\,dy\,dz - 4\,dx\,dz$$

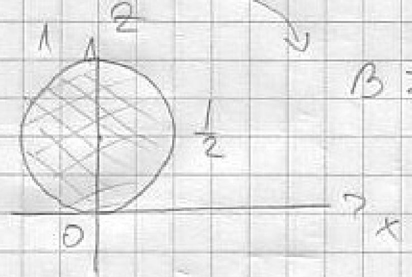
$\parallel$
$\emptyset$



$\alpha = 90^\circ$

$$I = -4(-)\iint dx\,dz = 4\underbrace{\iint dx\,dz}_{r^2\pi}$$

ili polarne koordinate



$\beta > 90^\circ$

$$I = 4 \cdot \frac{1}{4}\pi = \pi$$

$$z = x^2 + z^2$$

$$x^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$r = \frac{1}{2}$$