# MAT3E: Višestruki integrali – Dvostruki integrali – 1.5. Zadaci za vježbu

**~Wolfman**

19. Izračunajte $\iint (x^2 + y\text{^}2)dxdy$, pri čemu je područje integracije lik u prvom kvadrantu omeđen krivuljama x^2-y^2=1, x^2-y^2=4, xy=1 i xy=3. Naputak: uvedite nove varijable u=x^2-y^2 i v=xy!

$$u = x^2 - y^2$$
$$v = xy$$
$$y^2 = x^2 - u => y = \sqrt{x^2 - u}$$
$$x^2 = y^2 + u => x = \sqrt{y^2 + u}$$

$$v = x\sqrt{x^2 - u}$$
$$v^2 = x^4 - x^2u$$
$$x^2 = \frac{u \pm \sqrt{u^2 + 4v^2}}{2} = \frac{u + \sqrt{u^2 + 4v^2}}{2}$$

U gornjem redu smo zanemarili rješenje sa minusom u sredini, jer je sqrt(u^2+4v^2) sigurno veće od u, a x^2 ne može biti negativan ☺

$$x = \pm\frac{\sqrt{2}}{2}\sqrt{\sqrt{u^2 + 4v^2} + u}$$

Slično se za y dobije:

$$y = \pm\frac{\sqrt{2}}{2}\sqrt{\sqrt{u^2 + 4v^2} - u}$$

Jacobijan je:

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = A - B$$

$$\frac{\partial x}{\partial u} = \pm\frac{\sqrt{2}}{2}\frac{1}{2\sqrt{\sqrt{u^2 + 4v^2} + u}}\left(1 + \frac{2u}{2\sqrt{u^2 + 4v^2}}\right) = \pm\frac{\sqrt{2}}{4}\frac{\sqrt{u^2 + 4v^2} + u}{\sqrt{\sqrt{u^2 + 4v^2} + u}\cdot\sqrt{u^2 + 4v^2}}$$
$$\frac{\partial x}{\partial v} = \pm\frac{\sqrt{2}}{2}\frac{1}{2\sqrt{\sqrt{u^2 + 4v^2} + u}}\frac{8v}{2\sqrt{u^2 + 4v^2}} = \pm\sqrt{2}\frac{v}{\sqrt{\sqrt{u^2 + 4v^2} + u}\cdot\sqrt{u^2 + 4v^2}}$$
$$\frac{\partial y}{\partial v} = \pm\frac{\sqrt{2}}{2}\frac{1}{2\sqrt{\sqrt{u^2 + 4v^2} - u}}\frac{8v}{2\sqrt{u^2 + 4v^2}} == \pm\sqrt{2}\frac{v}{\sqrt{\sqrt{u^2 + 4v^2} - u}\cdot\sqrt{u^2 + 4v^2}}$$
$$\frac{\partial y}{\partial u} = \pm\frac{\sqrt{2}}{2}\frac{1}{2\sqrt{\sqrt{u^2 + 4v^2} - u}}\left(-1 + \frac{2u}{2\sqrt{u^2 + 4v^2}}\right) = \pm\frac{\sqrt{2}}{4}\frac{u - \sqrt{u^2 + 4v^2}}{\sqrt{\sqrt{u^2 + 4v^2} - u}\cdot\sqrt{u^2 + 4v^2}}$$

$$A = \pm\frac{\sqrt{2}}{4}\frac{\sqrt{u^2+4v^2}+u}{\sqrt{\sqrt{u^2+4v^2}+u}\cdot\sqrt{u^2+4v^2}}\cdot\sqrt{2}\frac{v}{\sqrt{\sqrt{u^2+4v^2}-u}\cdot\sqrt{u^2+4v^2}}$$
$$= \pm\frac{1}{2}\frac{v(\sqrt{u^2+4v^2}+u)}{(u^2+4v^2)\sqrt{\left(\sqrt{u^2+4v^2}+u\right)(\sqrt{u^2+4v^2}-u)}}$$
$$= \pm\frac{1}{2}\frac{v(\sqrt{u^2+4v^2}+u)}{(u^2+4v^2)\sqrt{u^2+4v^2-u^2}} = \pm\frac{1}{2}\frac{v(\sqrt{u^2+4v^2}+u)}{(u^2+4v^2)2v} = \pm\frac{1}{4}\frac{\sqrt{u^2+4v^2}+u}{u^2+4v^2}$$

$$B = \pm\sqrt{2}\frac{v}{\sqrt{\sqrt{u^2+4v^2}+u}\cdot\sqrt{u^2+4v^2}}\frac{\sqrt{2}}{4}\frac{u-\sqrt{u^2+4v^2}}{\sqrt{\sqrt{u^2+4v^2}-u}\cdot\sqrt{u^2+4v^2}} = \pm\frac{1}{4}\frac{u-\sqrt{u^2+4v^2}}{u^2+4v^2}$$
$$J = A - B = \pm\frac{1}{4}\left(\frac{\sqrt{u^2+4v^2}+u}{u^2+4v^2} - \frac{u-\sqrt{u^2+4v^2}}{u^2+4v^2}\right) = \pm\frac{1}{4}\frac{2\sqrt{u^2+4v^2}}{u^2+4v^2} = \pm\frac{1}{2}\frac{1}{\sqrt{u^2+4v^2}}$$
$$|J| = \frac{1}{2}\frac{1}{\sqrt{u^2+4v^2}}$$

$$f(x,y) = x^2 + y^2$$

$$f(u,v) = \frac{u+\sqrt{u^2+4v^2}}{2} + \frac{-u+\sqrt{u^2+4v^2}}{2} = \frac{2\sqrt{u^2+4v^2}}{2} = \sqrt{u^2+4v^2}$$

Granice integriranja u novim koordinatama dobijemo tako da uvrstimo nove koordinate u zadane granice:

$$u = x^2 - y^2 = 1$$
$$u = x^2 - y^2 = 4$$
$$v = xy = 1$$
$$v = xy = 3$$

Dakle koordinata u ide od 1 do 4, a v od 1 do 3. ☺

Sada možemo napisati traženi integral:

$$\iint (x^2+y^2)dxdy = \int_1^4 du\int_1^3 \sqrt{u^2+4v^2}\cdot|J|dv = \int_1^4 du\int_1^3\sqrt{u^2+4v^2}\cdot\frac{1}{2}\frac{1}{\sqrt{u^2+4v^2}}dv$$
$$= \frac{1}{2}\int_1^4 du\int_1^3 dv = \frac{1}{2}(4-1)(3-1) = \frac{1}{2}\cdot 3\cdot 2 = 3$$

Sada ja imam samo jedno pitanje... Zar sva ova zajebancija samo zbog 3?! TRI?! Jebote, čovjek bi očekivao da će nakon svega ovoga dobiti neki korijen ili bar $\frac{\pi}{6}$ ili tako nešto. Ali neeeeeeeee... Tri. Fuckin' 3. What's so special about 3? Od danas mrzim taj broj. 3. Puj. ☺