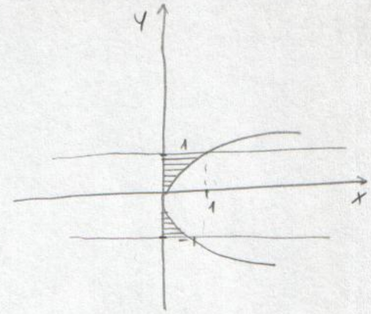
DVOSTRUKI INTEGRALI

1. $\iint\limits_D f(x,y)\,dx\,dy$

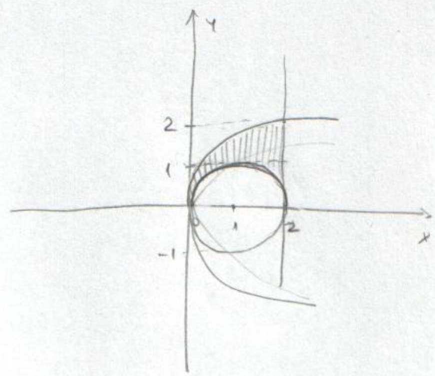
$D = \{ (x,y) : -1 \le y \le 1,\ 0 \le x \le y^2 \}$



$$\int_{-1}^{1} dy \int_{0}^{y^2} f(x,y)\,dx$$

$$\int_{0}^{1} dx \int_{-1}^{-\sqrt{x}} f(x,y)\,dy + \int_{0}^{1} dx \int_{\sqrt{x}}^{1} f(x,y)\,dy$$

2. $\int_{0}^{2} dx \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)\,dy$



$0 \le x \le 2$

$\sqrt{2x-x^2} \le y \le \sqrt{2x}$

$\sqrt{2x-x^2} = y \quad /()^2$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

$x = 1 + \sqrt{1-y^2}$

$y = \sqrt{2x} \quad /()^2$

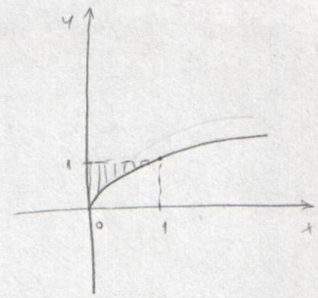
$y^2 = 2x \to x = \frac{y^2}{2}$

$\left.\begin{array}{l} y^2 = 2x - x^2 \\ y^2 = 2x \end{array}\right\} \quad x = 0$

$x = 2 \to y = \sqrt{4} = 2$

$$\int_{0}^{1} dy \int_{\frac{y^2}{2}}^{1-\sqrt{1-y^2}} f(x,y)\,dx + \int_{0}^{1} dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)\,dx + \int_{1}^{2} dy \int_{\frac{y^2}{2}}^{2} f(x,y)\,dx$$

3. $\int_0^1 dx \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy$



$$\int_0^1 dy \int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx =$$

$$= \int_0^1 \sin\left(\frac{y^3+1}{2}\right) dy \int_0^{y^2} dx$$

$$= \int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy$$

$$= \left| \begin{array}{l} y^3+1 = t \\ 3y^2 dy = dt \rightarrow y^2 dy = \frac{dt}{3} \end{array} \right|$$

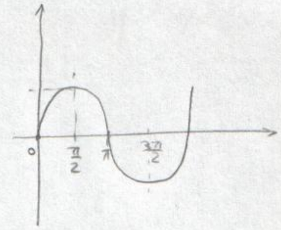
$$= \frac{1}{3} \int_1^2 \sin\left(\frac{t}{2}\right) dt$$

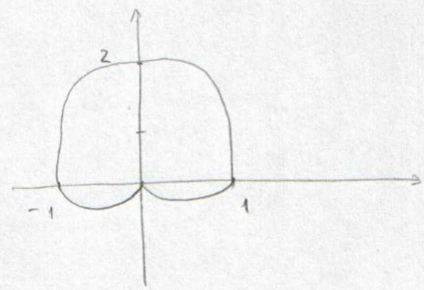
$$= -\frac{2}{3} \cos\left(\frac{t}{2}\right)\Big|_1^2$$

$$= -\frac{2}{3}\left(\cos(1) - \cos\left(\frac{1}{2}\right)\right)$$

$$= \frac{2}{3}\left(\cos\left(\frac{1}{2}\right) - \cos(1)\right)$$

4. $r = 1 + \sin\varphi$





$$P = \int_0^{2\pi} d\varphi \int_0^{1+\sin\varphi} r\,dr$$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{1+\sin\varphi} d\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + 2\sin\varphi + \sin^2\varphi)\,d\varphi$$

$$= \frac{1}{2}\varphi\Big|_0^{2\pi} - \cos\varphi\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \sin^2\varphi\,d\varphi$$

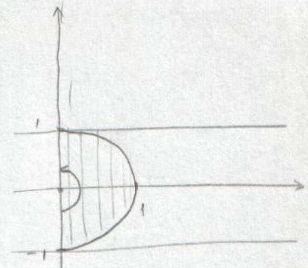
$$= \pi - 0 + \frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\varphi}{2}\,d\varphi$$

$$= \pi + \frac{1}{4}\varphi\Big|_0^{2\pi} - \frac{1}{8}\sin 2\varphi\Big|_0^{2\pi}$$

$$= \pi + \frac{\pi}{2}$$

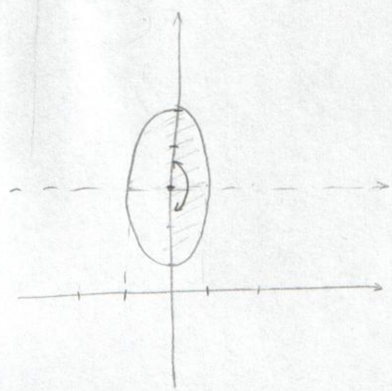
$$= \frac{3}{2}\pi$$

5. $\int\limits_{-1}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx$



$$\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{1} r^2 dr = \frac{1}{3} \cdot \varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{3}$$

6. $\iint\limits_{D} \sqrt{4x^2+y^2-6y+17}\; dx\, dy$



$x^2 + \frac{(y-3)^2}{4} \leq 1$

$x \geq 0$

$x = r\cos\varphi$

$y - 3 = 2r\sin\varphi$

$J = 2r$

$$4x^2+y^2-6y+17 = 4r^2\cos^2\varphi + 4r^2\sin^2\varphi + \cancel{12r\sin\varphi} + 9 \cancel{-12r\sin\varphi} - 18 + 17$$
$$= 4r^2 + 8$$
$$= 4(r^2+2)$$

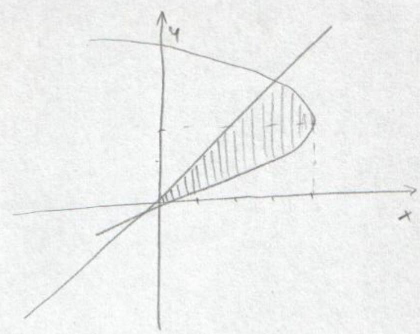
$$\sqrt{4(r^2+2)} = 2\sqrt{r^2+2}$$

$$4\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{0}^{1} r\sqrt{r^2+2}\, dr = \left| \begin{matrix} r^2+2 = t \\ 2r\,dr = dt \end{matrix} \right| = 2\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_{2}^{3} \sqrt{t}\, dt = 2\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \Big|_{2}^{3} d\varphi =$$

$$= \frac{4}{3}\pi\left(\sqrt{27} - \sqrt{8}\right) = \frac{4}{3}\pi\left(3\sqrt{3} - 2\sqrt{2}\right)$$

7. $x = 4y - y^2$

$y = x$



$x = -(y^2 - 4y)$

$x = -\left[(y-2)^2 - 4\right]$

$x - 4 = (y-2)^2$

$$P = \int_0^3 dy \int_y^{4y-y^2} dx = \int_0^3 (3y - y^2)\, dy = \frac{3}{2} y^2 \Big|_0^3 - \frac{1}{3} y^3 \Big|_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}$$

8. a) $x = x(u,v)$

$y = y(u,v)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

b) $x = \sqrt{\frac{x+v}{2}}$

$y = \sqrt{\frac{u-v}{2}}$

$$\frac{\partial x}{\partial u} = \frac{1}{\sqrt{\frac{u+v}{2}}} \cdot \frac{1}{2} = \frac{\partial x}{\partial v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{\sqrt{\frac{x-v}{2}}} \cdot \frac{1}{2} \qquad \frac{\partial y}{\partial v} = \frac{1}{\sqrt{\frac{u-v}{2}}} \cdot \left(-\frac{1}{2}\right)$$

$$J = \begin{vmatrix} \frac{1}{2} \frac{\sqrt{2}}{\sqrt{u+v}} & \frac{1}{2} \frac{\sqrt{2}}{\sqrt{u+v}} \\ \frac{1}{2} \frac{\sqrt{2}}{\sqrt{u-v}} & -\frac{1}{2} \frac{\sqrt{2}}{\sqrt{u-v}} \end{vmatrix} = -\frac{1}{4} \cdot \frac{1}{\sqrt{u^2 - v^2}}$$

8. c) $\iint_D xy \, dx\, dy$

$x^2+y^2=4$

$x^2+y^2=9$

$x^2-y^2=1$

$x^2-y^2=4$

$u=x^2+y^2 \qquad u\in[4,9]$

$v=x^2-y^2 \qquad v\in[1,4]$

$$x=\sqrt{\frac{u+v}{2}}$$

$$y=\sqrt{\frac{u-v}{2}}$$

$$\int_4^9 du \int_1^4 \sqrt{\frac{u+v}{2}}\cdot\sqrt{\frac{u-v}{2}}\cdot\frac{1}{4\sqrt{u^2-v^2}}\, dv =$$

$$=\frac{1}{8}\int_4^9 du\int_1^4 dv = \frac{15}{8}$$

9. b) $x = u + v^2$

$y = uv - u^2$

$\frac{\partial x}{\partial u} = 1$ $\quad$ $\frac{\partial x}{\partial v} = 2v$

$\frac{\partial y}{\partial u} = v - 2u$ $\quad$ $\frac{\partial y}{\partial v} = u$

$$J = \begin{vmatrix} 1 & 2v \\ v-2u & u \end{vmatrix} = u - 2v^2 + 4uv$$

c) $\iint_D f(x,y)\,dx\,dy$

$x + y = 1$

$x + y = 3$

$x - y = -1$

$x - y = 1$

$u = x + y$

$v = x - y$

$$\frac{1}{2}\int_1^3 du \int_{-1}^{1} f(x(u,v), y(u,v))\,dv$$

$x = \frac{u+v}{2}$

$y = \frac{u-v}{2}$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

TROSTRUKI INTEGRALI

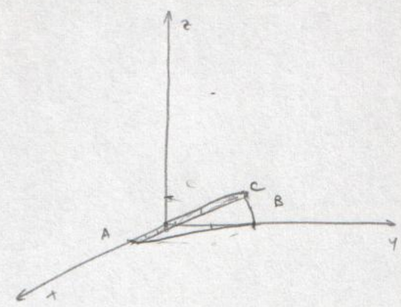
1. $\iiint\limits_V x\,dx\,dy\,dz$

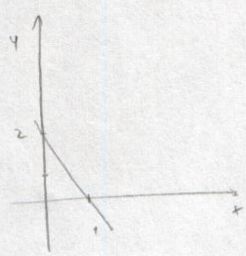
O(0, 0, 0)

A(1, 0, 0)

B(0, 2, 0)

C(0, 2, 1)





$$\int_0^1 x\,dx \int_0^{2-2x} dy \int_0^{\frac{y}{2}} dz = \int_0^1 x\,dx \int_0^{2-2x} \frac{y}{2}\,dy = \frac{1}{4}\int_0^1 y^2\Big|_0^{2-2x} x\,dx = \frac{1}{4}\int_0^1 (4-8x+4x^2)\,x\,dx =$$

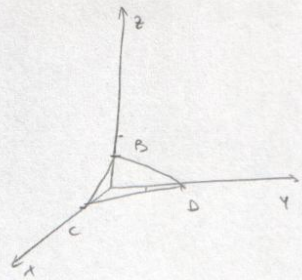
$$= \frac{1}{4}\left(2x^2\Big|_0^1 - \frac{8}{3}x^3\Big|_0^1 + x^4\Big|_0^1 \right) = \frac{1}{4}\left(2 - \frac{8}{3} + 1\right) = \frac{1}{12}$$

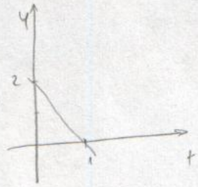
$$\vec{n} = \vec{CA} \times \vec{CO} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 0 & -2 & -1 \end{vmatrix} = \vec{i}\cdot 0 - \vec{j}\cdot(-1) + \vec{k}\cdot(-2) = \vec{j} - 2\vec{k}$$

$$y \cancel{-2} - 2z + 2 = 0 \quad \to \quad z = \frac{y}{2}$$

2. $\iiint_V x \, dV$

$A(0,0,0)$
$B(0,0,1)$
$C(1,0,0)$
$D(0,2,0)$





$$\vec{n} = \vec{BC} \times \vec{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix} = \vec{i} \cdot 2 - \vec{j} \cdot (-1) + \vec{k} \cdot 2 = 2\vec{i} + \vec{j} + 2\vec{k}$$

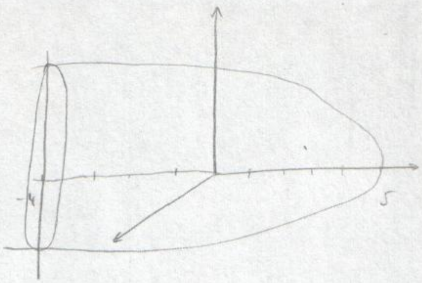
$$2(x-0) + 1(y-0) + 2(z-1) = 0$$
$$2x + y + 2z - 2 = 0 \rightarrow z = -x - \frac{y}{2} + 1$$

$$\int_0^1 x \, dx \int_0^{2-2x} dy \int_0^{1-x-\frac{y}{2}} dz = \int_0^1 x \, dx \int_0^{2-2x} \left(1 - x - \frac{y}{2}\right) dy = \int_0^1 x \left(y\Big|_0^{2-2x} - xy\Big|_0^{2-2x} - \frac{y^2}{4}\Big|_0^{2-2x} \right) dx =$$

$$= \int_0^1 x \left(2 - 2x \cancel{-2x} + 2x^2 - 1 \cancel{+2x} - x^2 \right) dx = \int_0^1 (x^3 - 2x^2 + x) \, dx =$$

$$= \frac{x^4}{4}\Big|_0^1 - \frac{2}{3}x^3\Big|_0^1 + \frac{x^2}{2}\Big|_0^1 = \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{3 - 8 + 6}{12} = \frac{1}{12}$$

3. $\iiint\limits_V y \, dx\,dy\,dz$



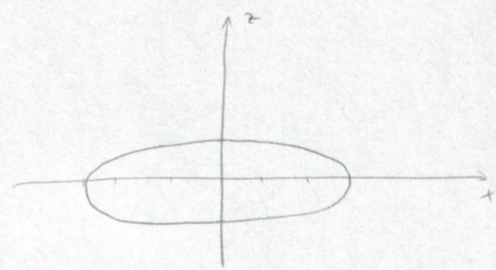
$y = -x^2 - 3z^2 + 5$

$y = -4$

$-4 = -x^2 - 3z^2 + 5$

$x^2 + 3z^2 = 9 \;\big|\; :9$

$\frac{x^2}{9} + \frac{z^2}{3} = 1$



$x = 3r\cos\varphi$

$z = \sqrt{3}\, r \sin\varphi$

$y = y$

$0 \leq \varphi \leq 2\pi$

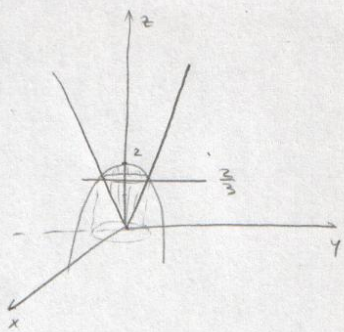
$$\int_0^{2\pi} d\varphi \int_0^1 3\sqrt{3}\, r \int_{-4}^{5-9r^2} y \, dy$$

$9r^2\cos^2\varphi + 9r^2\sin^2\varphi = 9$

$r^2 = 1$

4. $z=\sqrt{x^2+y^2}$

$z = 2-3(x^2+y^2)$



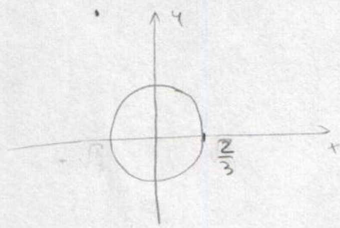
$z-2=-3(x^2+y^2)$

$z^2=x^2+y^2$

$z-2=-3z^2$

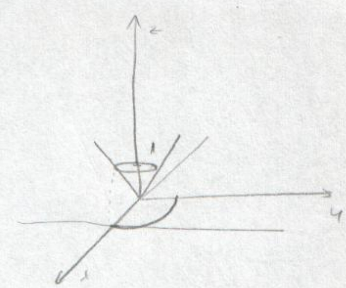
$3z^2+z-2=0$

$z_{1,2}=\frac{-1\pm 5}{6}=\frac{2}{3}$



$$\int_0^{2\pi} d\varphi \int_0^{\frac{2}{3}} r\,dr \int_r^{2-3r^2} dz$$

5. $\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz$



$0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq r \leq 1$

$\int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r\,dr \int_r^1 f(r\cos\varphi, r\sin\varphi, z)\,dz$

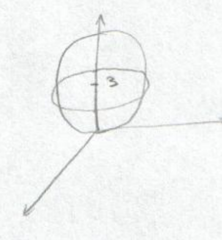
6. a) $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

b) $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$= r^2\sin\theta$



c) $\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$

$x^2+y^2+(z-3)^2 = 9$

$x^2+y^2+z^2-6z+9 = 9$

$r^2 - 6r\cos\theta = 0$

$r(r-6\cos\theta) = 0 \rightarrow r_D = 0$

$r_U = 6\cos\theta$

$\int_0^{2\pi} d\varphi \int_0^{6\cos\theta} r^4\,dr \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta$

VEKTORI

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$y - y_0 = f'(x_0)(x - x_0)$

$y - y_0 = y'(x_0)(x - x_0)$

$x(t) \rightarrow x - x_0 = x'(t_0)(t - t_0) = x'(t_0)\cdot u$

$y(t) \rightarrow y - y_0 = y'(t_0)(t - t_0) = y'(t_0)\cdot u$

$z(t) \rightarrow z - z_0 = z'(t_0)(t - t_0) = z'(t_0)\cdot u$

$x = x_0 + x'(t_0)\cdot u$

$y = y_0 + y'(t_0)\cdot u$

$z = z_0 + z'(t_0)\cdot u$

$\vec{r}(u) = (x_0 + x'(t_0)\cdot u)\vec{i} + (y_0 + y'(t_0)u)\vec{j} + (z_0 + z'(t_0)u)\vec{k}$

$= x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + (x'(t_0)\vec{i} + y'(t_0)\vec{j} + z'(t_0)\vec{k})\cdot u$

$\vec{r}(t) = (2\cos t)\vec{i} + (3\sin t)\vec{j} + t\vec{k}$

$t = \pi$

$x_0 = 2\cos\pi = -2$

$y_0 = 3\sin\pi = 0$

$z_0 = \pi$

$x_0' = -2\sin t_0 = -2\sin\pi = 0$

$y_0' = 3\cos t_0 = 3\cos\pi = -3$

$z_0' = 1$

$\vec{r}(t) = -2\vec{i} + \pi\vec{k} + (-3\vec{j} + \vec{k})\cdot u$

c) $z = x^2 + y^2$

$2x + z = 0$

$-2x = x^2 + y^2$

$(x+1)^2 + y^2 = 1$

$x + 1 = \cos t$

$y = \sin t$

$z = -2x = -2(\cos t - 1) = 2 - 2\cos t$

$x = \cos t - 1$

$y = \sin t$

$z = 2 - 2\cos t$

$t \in [0, 2\pi]$

d) $\vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + t^2\vec{k}$

$\vec{r}'(t) = (-2\sin t)\,\vec{i} + (2\cos t)\vec{j} + 2t\vec{k}$

$|\vec{r}'(t)| = \sqrt{5}$

$\sqrt{5} = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2} = \sqrt{4 + 4t^2} \quad |()^2$

$5 = 4 + 4t^2 \to t^2 = \frac{1}{4}$

$t = \pm\frac{1}{2}$

2. $\vec{r}(t) = \frac{\cos \pi t}{t} \vec{i} + \frac{\sin \pi t^2}{\pi} \vec{j} + t^3 \vec{k}$

$t = 1$

$x_0(t) = -1$

$y_0(t) = 0$

$z_0(t) = 1$

$\vec{r}\,'(t) = \frac{-\sin \pi t \cdot \pi - \cos \pi t}{t^2} \vec{i} + \frac{\cos \pi t^2 \cdot 2\pi}{\pi} \vec{j} + 3t^2 \vec{k}$

$x'(t) = 1$

$y'(t) = -2$

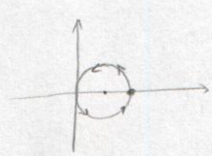
$z'(t) = 3$

$r'(t) = \vec{i} - 2\vec{j} + 3\vec{k}$ → vektor smjera tangente

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \varphi$

$\cos \varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(\vec{i} - 2\vec{j} + 3\vec{k}) \cdot \vec{i}}{\sqrt{1+4+9} \cdot 1} = \frac{1}{\sqrt{14}}$

$(x-1)^2 + y^2 = 1$



$t = 0 \rightarrow x = 2$
$y = 0$

$t = \frac{\pi}{2} \rightarrow x = 1$
$y = 1$

$x - 1 = \cos t \rightarrow x = 1 + \cos t$

$y = \sin t$

$t \in [0, 2\pi]$