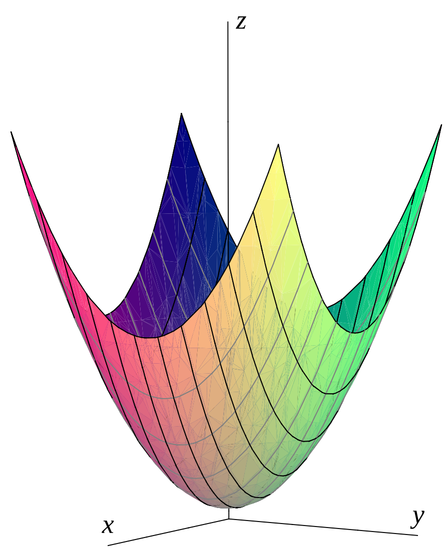
Parametric Surfaces

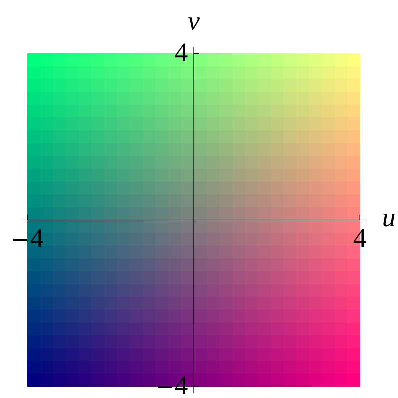
1. (a) *Parameterize the elliptic paraboloid $z = x^2 + y^2 + 1$. Sketch the grid curves defined by your parameterization.*

Solution. There are several ways to parameterize this. Here are a few.

i. One way to think of parameterizing is simply that we want to describe the surface using 2 variables. This amounts to describing x, y, and z using just 2 variables. In this case, z is already written in terms of x, y, and z, so we can describe the surface just using x and y. That is, we can use the parameterization $\vec{r}(x, y) = \langle x, y, x^2 + y^2 + 1\rangle$. We often use the variables u and v as the parameters (just as we usually used t for the parameter when parameterizing curves), so we could also write this as $\boxed{\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1\rangle}$. (It is certainly not necessary to use u and v though.)

Here is a picture of this parameterization:

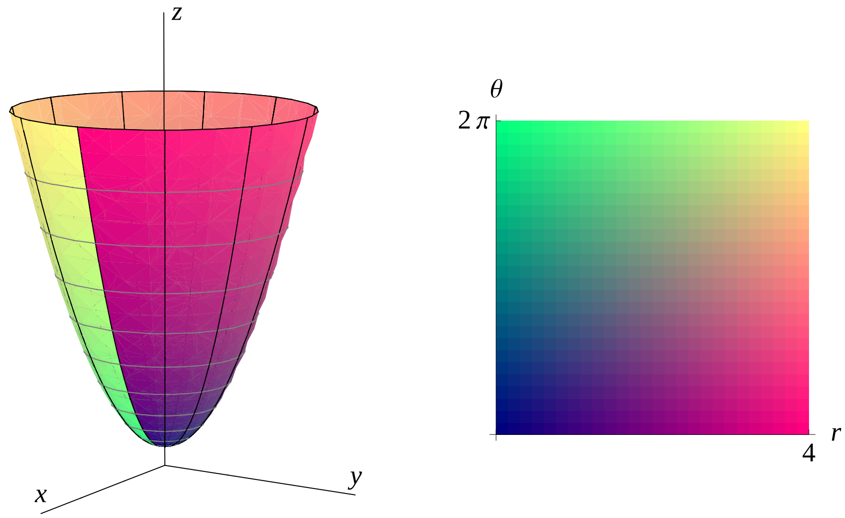




The gray curves are where u is constant, and the black curves are where v is constant. The right picture shows a way of coloring the uv-plane; the paraboloid is colored according to the corresponding u and v value at each point. For instance, from the right picture, we see that $u = 4$, $v = -4$ is colored pink. Therefore, the point $\vec{r}(4, -4) = \langle 4, -4, 17\rangle$ on the paraboloid is colored pink.

ii. Another possibility is to use cylindrical coordinates to rewrite the surface as $z = r^2 + 1$. Then, every point can be described in terms of r and θ since $z = r^2 + 1$. Converting back to Cartesian coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $z = r^2 + 1$, so we have the parameterization $\boxed{\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 + 1\rangle}$.

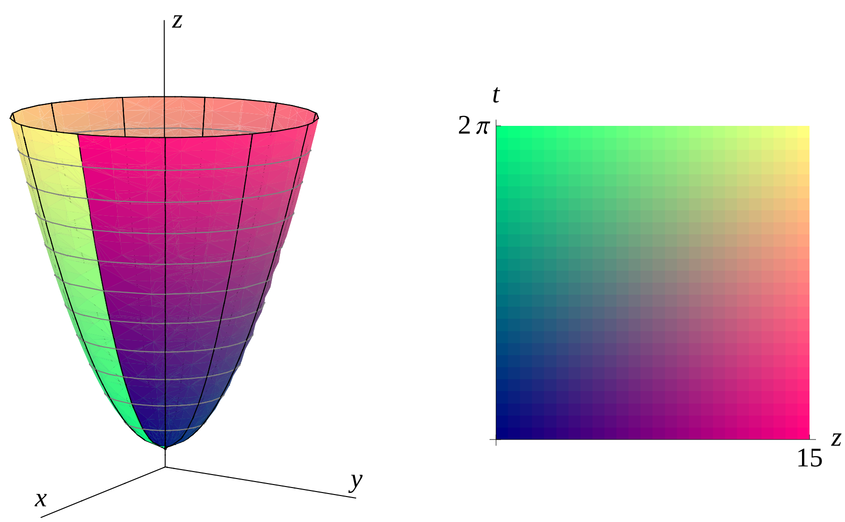
Here is a picture of this parameterization:



The gray curves are where r is constant, and the black curves are where θ is constant.

iii. Yet another possibility is to think of slicing (or taking cross-sections or traces). This approach is slightly more difficult than the previous one, but it's also more flexible. Looking at the surface, we know that taking traces in $z = k$ gives us circles, and circles are curves that we know how to parameterize. If we imagine slicing at a particular z-value, then the slice is the circle $x^2 + y^2 = z - 1$, which is a circle centered at $(x, y) = (0, 0)$ with radius $\sqrt{z-1}$. Therefore, we know that x and y can be described by $x = \sqrt{z-1}\cos t, y = \sqrt{z-1}\sin t$. This gives the parameterization $\boxed{\vec{r}(z,t) = \langle \sqrt{z-1}\cos t, \sqrt{z-1}\sin t, z\rangle}$.

Here is a picture of this parameterization:



The gray curves are where z is constant, and the black curves are where t is constant.

(b) *If we only want to parameterize the part of the elliptic paraboloid under the plane $z = 10$, what restrictions would you place on the parameters you used in (a)?*

Solution.

i. For the parameterization $\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1\rangle$, we need to restrict u and v. Since we want z (the last component) to be less than 10, we need $\boxed{u^2 + v^2 < 9}$.

ii. For the parameterization $\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 + 1\rangle$, we need to restrict r and θ. Since the paraboloid was written as $z = r^2 + 1$ in cylindrical coordinates and we want $z < 10$, we need $r < 3$. We know that θ can be anything, so our restrictions are $\boxed{0 \le r < 3,\ 0 \le \theta < 2\pi}$.

iii. For the parameterization $\vec{r}(z, t) = \langle \sqrt{z-1}\cos t, \sqrt{z-1}\sin t, z\rangle$, we need to restrict z and t. We already know that we want $z < 10$. Looking at the paraboloid, we also want $z \ge 1$.[(1)] Looking back, we used t to parameterize a circle, and the parameterization we chose means $0 \le t < 2\pi$ is a good restriction. So, for this parameterization, we have $\boxed{1 \le z < 10,\ 0 \le t < 2\pi}$.

2. *Parameterize the plane that contains the 3 points $P(1, 0, 1)$, $Q(2, -2, 2)$, and $R(3, 2, 4)$.*

Solution. One way to parameterize the plane is to let the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ define our grid. We can think of P as an "origin" for the plane and the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as a set of "axes" for the plane. That is, we can reach any point in the plane by starting at P, going in the direction of $\overrightarrow{PQ}$ for a while, and then going in the direction of $\overrightarrow{PR}$ for a while.

In this case, $\overrightarrow{PQ} = \langle 1, -2, 1\rangle$ and $\overrightarrow{PR} = \langle 2, 2, 3\rangle$, so our parameterization is

$$\boxed{\vec{r}(u, v) = \langle 1, 0, 1\rangle + u\langle 1, -2, 1\rangle + v\langle 2, 2, 3\rangle = \langle 1 + u + 2v, -2u + 2v, 1 + u + 3v\rangle}$$

(You can think of this as saying: start at $P(1, 0, 1)$, go off in the direction of $\overrightarrow{PQ} = \langle 1, -2, 1\rangle$ for a bit — how long is determined by u — and then go off in the direction of $\overrightarrow{PR} = \langle 2, 2, 3\rangle$ for a bit.)

Alternatively, you could find the equation of the plane (see the worksheet "Lines and Planes") — it is $8x + y - 6z = 2$. Then, we can write any one of the variables in terms of the other two and use those other two as parameters. For instance, $y = 2 - 8x + 6z$ expresses y in terms of x and z. If we want to use u and v as our parameters, then we can just have $x = u$, $z = v$, and $y = 2 - 8u + 6v$, which gives the parameterization $\boxed{\vec{r}(u, v) = \langle u, 2 - 8u + 6v, v\rangle.}$

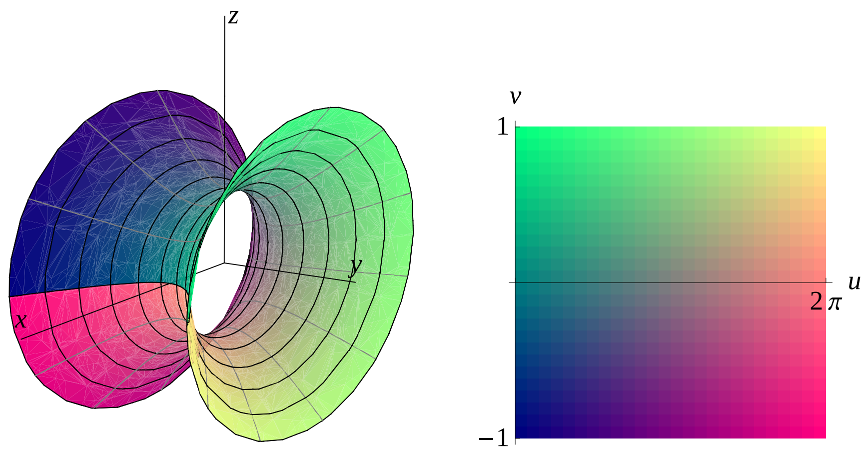
Of course, there are many other parameterizations. One way to check whether your parameterization is reasonable is to remember that you are supposed to be parameterizing $8x+y-6z = 2$. So, however you parameterize, this relationship should be satisfied. For instance, in our first parameterization $\vec{r}(u, v) = \langle 1+u+2v, -2u+2v, 1+u+3v\rangle$, you can easily check that $8(1+u+2v)+(-2u+2v)-6(1+u+3v) = 2$.

3. *Parameterize the hyperboloid $x^2 - 4y^2 + z^2 = 1$.*

Solution. The traces in $y = k$ of this surface will be circles. In particular, since $x^2 + z^2 = 1 + 4y^2$, the trace in $y = k$ will be a circle centered at $(x, z) = (0, 0)$ with radius $\sqrt{1 + 4y^2}$. We can parameterize this by taking $x = \sqrt{1 + 4y^2}\cos u$, $z = \sqrt{1 + 4y^2}\cos u$ with $0 \le u < 2\pi$. Our other parameter is just y; if we rename it v, then we have the parameterization $\boxed{\vec{r}(u, v) = \langle \sqrt{1 + 4v^2}\cos u, v, \sqrt{1 + 4v^2}\sin u\rangle}$ with $\boxed{0 \le u < 2\pi}$ (v can be anything).

[(1)] Notice that this is implied in our parameterization since $\sqrt{z-1}$ is not defined if $z < 1$.

Here is a picture of the parameterization.

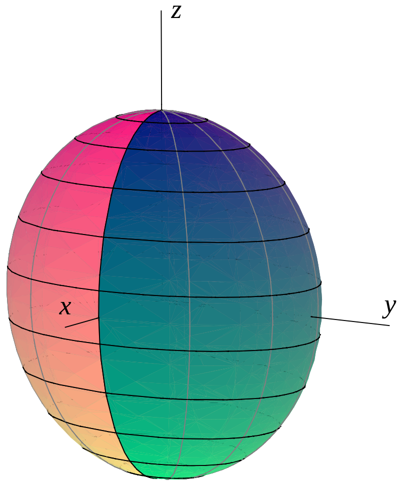


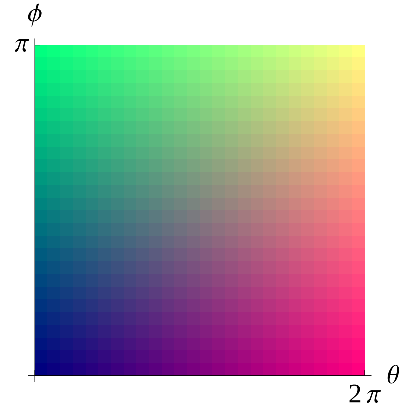
The gray curves are where u is constant, and the black curves are where v is constant.

4. *Parameterize the ellipsoid* $9x^2 + 4y^2 + z^2 = 36$.

   Solution. There is not much work to do here if we take a clever approach. Let's rewrite the given equation as $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{6}\right)^2 = 1$. We know that $X^2 + Y^2 + Z^2 = 1$ can be parameterized as $X = \sin\phi\cos\theta$, $Y = \sin\phi\sin\theta$, and $Z = \cos\phi$, and if we think of X as being $\frac{x}{2}$, Y as being $\frac{y}{3}$, and Z as being $\frac{z}{6}$, then we get $x = 2\sin\phi\cos\theta$, $y = 3\sin\phi\sin\theta$, $z = 6\cos\phi$. That is, our parameterization is $\boxed{\vec{r}(\theta, \phi) = \langle 2\sin\phi\cos\theta, 3\sin\phi\sin\theta, 6\cos\phi\rangle}$ with $\boxed{0 \le \theta < 2\pi,\ 0 \le \phi \le \pi}$.

   Here is a picture of the parameterization.



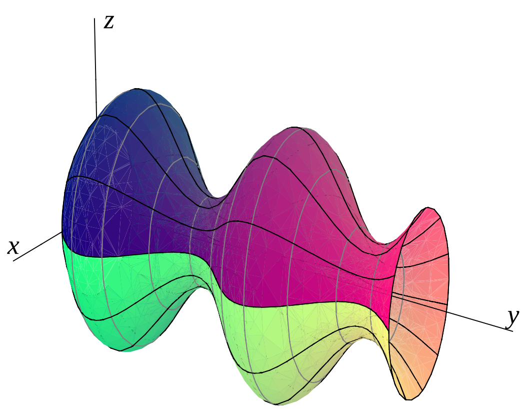


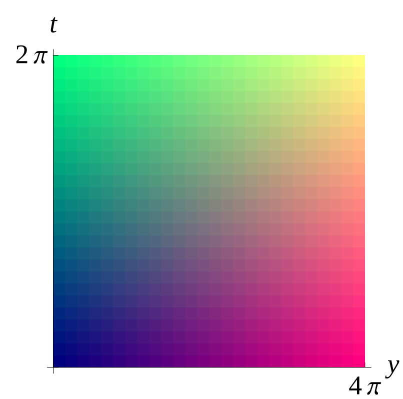
The gray curves are where θ is constant, and the black curves are where ϕ is constant.

5. *Consider the curve* $z = 2 + \sin y$, $0 \leq y \leq 4\pi$ *in the* yz*-plane. Let* S *be the surface obtained by rotating this curve about the* y*-axis. Find a parameterization of* S*.*

   Solution. The traces in $y = k$ of this surface will be circles. In particular, if we look at the trace in $y = k$, we see a circle centered at $(x, z) = (0, 0)$ with radius $2 + \sin y$. We can parameterize this by taking $x = (2 + \sin y)\cos t$, $z = (2 + \sin y)\sin t$ with $0 \leq t < 2\pi$. Thus, a parameterization of the surface is $\boxed{\vec{r}(y, t) = \langle (2 + \sin y)\cos t, y, (2 + \sin y)\sin t \rangle}$ with $\boxed{0 \leq y \leq 4\pi,\ 0 \leq t < 2\pi}$.

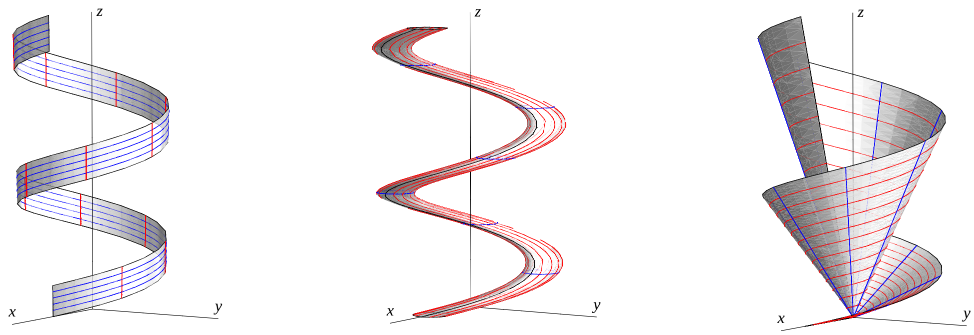
   Here is a picture of the parameterization.





The gray curves are where y is constant, and the black curves are where t is constant.

6. Here are three surfaces.



(I) (II) (III)

Match each function with the surface it parameterizes. Which curves are where u is constant and which curves are where v is constant?

(a) $\vec{r}(u,v) = \left\langle \dfrac{\cos u}{4} + \cos v, \dfrac{\sin u}{4} + \sin v, v \right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

Solution. If u is a constant, then $\vec{r}(u,v)$ has the form $\langle C_1 + \cos v, C_2 + \sin v, v\rangle$ where C_1 and C_2 are constants. You should recognize this as a helix (remember $\langle \cos t, \sin t, t\rangle$), shifted.

On the other hand, if v is a constant, then $\vec{r}(u,v)$ has the form $\left\langle \frac{1}{4}\cos u + C_1, \frac{1}{4}\sin u + C_2, C_3 \right\rangle$ where C_1, C_2, and C_3 are constants. You should recognize this as parameterizing a circle (which is parallel to the xy-plane since the z-component does not vary with u.

The surface which has helices as one set of grid curves and circles as the other is (II). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

(b) $\vec{r}(u,v) = \left\langle \cos u, \sin u, u + \dfrac{v}{4} \right\rangle$, $0 \le u \le 4\pi$, $0 \le v \le 2\pi$.

Solution. If u is a constant, then $\vec{r}(u,v)$ has the form $\left\langle C_1, C_2, C_3 + \frac{v}{4} \right\rangle$ where C_1, C_2, and C_3 are constants. This simply parameterizes a vertical line segment (of length $\frac{\pi}{2}$ since v varies between 0 and 2π).

On the other hand, if v is a constant, then $\vec{r}(u,v)$ has the form $\langle \cos u, \sin u, u + C\rangle$ where C is a constant. This parameterizes a helix.

The surface which has vertical line segments as one set of grid curves and helices as the other is (I). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

(c) $\vec{r}(u,v) = \langle u\cos v, u \sin v, uv\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

Solution. If u is a constant, then $\vec{r}(u,v)$ has the form $\langle C\cos v, C\sin v, Cv\rangle = C\langle \cos v, \sin v, v\rangle$ where C is a constant. This parameterizes a helix. Notice that, unlike in the two previous parts, the helices here can be different sizes.

On the other hand, if v is a constant, then $\vec{r}(u,v)$ has the form $\langle C_1 u, C_2 u, C_3 u\rangle$ where C_1, C_2, and C_3 are constants. You should recognize this as parameterizing a line segment; since u starts at 0, this line segment always starts at the origin.

The matching surface is (III). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.