FOURIEROV RED

TEORIJA:

- INTERVAL → MOŽE BITI
  - SIMETRIČAN $[-\pi, \pi]$, $[-2, 2]$
  - NESIMETRIČAN $[-5, 0]$, $[0, \pi]$
- AKO JE SIMETRIČAN, ONDA PROVJERAVAMO PAR/NEPAR
- PARNA F-JA: SIMETRIČNA S OBZIROM NA Y-OS
- NEPARNA F-JA: SIMETRIČNA S OBZIROM NA ISHODIŠTE

primjer: sinus - NEPARNA, cosinus - PARNA

→ ako je zadan razvoj po sin/cos → odmah radimo po formulama za parne (cos) ili neparne (sin)

ZADATAK 1: Funkciju $f(x) = \frac{x}{2}$ razvij u Fourierov red na $\langle 0, 2\rangle$ po sin. f-ji.

→ koristimo formule iz tablice za neparnu f-ju:

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad ; \text{ uvrstimo u formulu:}$$

$$b_n = \frac{2}{2}\int_0^2 \frac{x}{2}\cdot\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_0^2 x\sin\left(\frac{n\pi}{2}x\right)dx \quad ; \text{ parcijalna integracija}$$

$$\left| \begin{matrix} u = x & dv = \sin\left(\frac{n\pi}{2}x\right)dx \\ du = dx & v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right) \end{matrix} \right| = \frac{1}{2}\left[\frac{-2x}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^2 + \int_0^2 \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)x\,dx\right] =$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}\cdot\cos(n\pi) + \frac{2}{n\pi}\cdot\frac{2}{n\pi}\cdot\underbrace{\sin\left(\frac{n\pi}{2}x\right)\Big|_0^2}_{0}\right] =$$

- po formulama imamo:

$\underbrace{\cos(n\pi)}_{(-1)^n} \to n=1 \to \cos\pi \to -1$	$\underbrace{\sin(n\pi)}_{0} \to n=1 \to \sin\pi = 0$
$\to n=2 \to \cos 2\pi \to 1$	$\to n=2 \to \sin 2\pi = 0$
$\to n=3 \to \cos 3\pi \to -1$	$\to n=3 \to \sin 3\pi = 0$

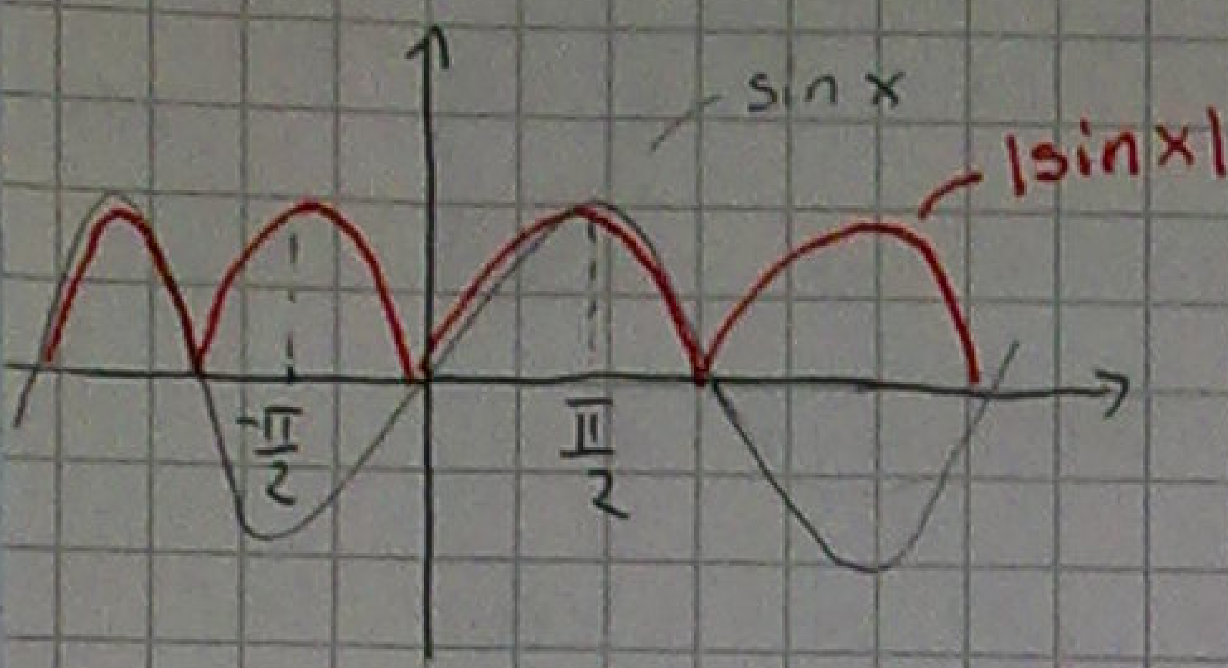
$$\to \quad \frac{1}{2}\left[\frac{-4}{n\pi}\cdot(-1)^n\right] = \frac{-2(-1)^n}{\pi\cdot n} = \frac{2\cdot(-1)^{n+1}}{n\pi}$$

→ sada ubacimo u formulu za $s(x)$

$$s(x) = \sum_{n=1}^{\infty} b_n\cdot\sin\frac{n\pi}{L}x = \boxed{\sum_{n=1}^{\infty}\frac{2\cdot(-1)^{n+1}}{n\pi}\cdot\sin\frac{n\pi}{2}x}$$

ZADATAK 2: $f(x)=|\sin x|$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ razvij u F. red. Pomoću toga izračunaj $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

- prvo skiciramo:



- iz skice vidimo da je funkcija POZITIVNA na danom intervalu i da je to parna f-ja (simetrična na y-os) $b_n=0$, računamo a_0 i a_n

$$a_0=\frac{2}{L}\int_0^L f(x)\,dx=\frac{2}{\frac{\pi}{2}}\int_0^{\pi/2}\sin x\,dx=\frac{4}{\pi}\cdot(-\cos x)\Big|_0^{\pi/2}=-\frac{4}{\pi}\left(\cos\frac{\pi}{2}-\cos 0\right)=\frac{4}{\pi}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx=\frac{4}{\pi}\int_0^{\pi/2}\sin x\cos(2nx)\,dx \Rightarrow \{\text{formula } \sin x\cdot\cos y\}$$

$$=\overset{2}{\cancel{\frac{4}{\pi}}}\int_0^{\pi/2}\frac{\sin(x+2nx)+\sin(x-2nx)}{\cancel{2}}=\frac{2}{\pi}\left[\int_0^{\pi/2}\sin(x(1+2n))\,dx+\int_0^{\pi/2}\sin(x(1-2n))\,dx\right]=$$

$$=\frac{2}{\pi}\left[\frac{-1}{1+2n}\cos(x(1+2n))\Big|_0^{\pi/2}-\frac{1}{1-2n}\cos(x(1-2n))\Big|_0^{\pi/2}\right]=$$

$$=\frac{2}{\pi}\left[\frac{-1}{1+2n}\left(\cos\left(\frac{\pi}{2}+n\pi\right)-\cos(0)\right)-\frac{1}{1-2n}\left(\cos\left(\frac{\pi}{2}-n\pi\right)-\cos(0)\right)\right]$$

→ $\cos(0)$ je 1, a ovo drugo računamo po formuli $\cos(x+y)$

dakle: $\cos\left(\frac{\pi}{2}+n\pi\right)=\overset{0}{\cancel{\cos\frac{\pi}{2}}}\cdot\cos n\pi-\sin\frac{\pi}{2}\cdot\overset{0}{\cancel{\sin n\pi}}=$

$= 0 - 0 = 0$

$$\frac{2}{\pi}\left[\frac{-1}{1+2n}(0-1)-\frac{1}{1-2n}(0-1)\right]=\frac{2}{\pi}\left[\frac{1}{1+2n}+\frac{1}{1-2n}\right]=\frac{2}{\pi}\left[\frac{2}{1-4n^2}\right]=\frac{4}{\pi(1-4n^2)}$$

→ sad imamo a_0, a_n i $b_n=0$; uvrstimo u $s(x)$

$$s(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{4}{\pi(1-4n^2)}\cos 2nx=$$

$$=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{1-4n^2}$$; moramo se riješiti cos; stavimo $x=0$ jer je $\cos(0)=1$; računamo $s(0)$

$$s(0)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-4n^2}$$; izlučimo minus; $=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

izjednačimo $s(0)=f(0)$; $f(0)=|\sin 0|=0$

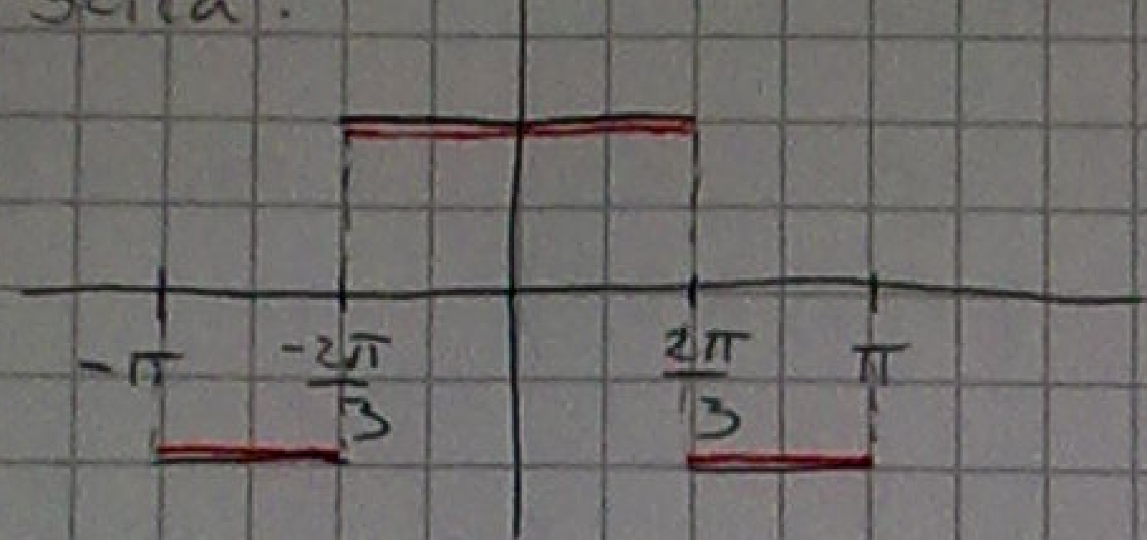
$$0=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$;

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{2}{\pi}$$; $\boxed{\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}}$

ZADATAK 3: $f(x)$ razvijte u F. red na $\langle -\pi, \pi \rangle$, skiciraj red i izračunaj sumu $\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - - -$

$$f(x) = \begin{cases} 1, & -\frac{2\pi}{3} \leq x \leq \frac{2\pi}{3} \\ -1, & -\pi < x < -\frac{2\pi}{3} \;;\; \frac{2\pi}{3} < x < \pi \end{cases}$$

Skica:



$L \rightarrow$ poluperiod $= \pi$

- iz skice $\rightarrow$ f-ja je parna (simetrična na y os)

$b_n = 0$; tražimo a_n i a_0 i $S(x)$

$$a_0 = \frac{2}{L} \int_0^L f(x)\,dx = \frac{2}{\pi} \int_0^\pi f(x)\,dx \;;$$ $\rightarrow$ dijelimo na 2 integrala jer $f(x)$ ima 2 vrijednosti (-1 i 1)

$$= \frac{2}{\pi}\left[\int_0^{2\pi/3} dx - \int_{2\pi/3}^{\pi} dx\right] = \frac{2}{\pi}\left[\frac{2\pi}{3} - \pi + \frac{2\pi}{3}\right] = \frac{2}{\pi}\left(\frac{\pi}{3}\right) = \frac{2}{3}$$

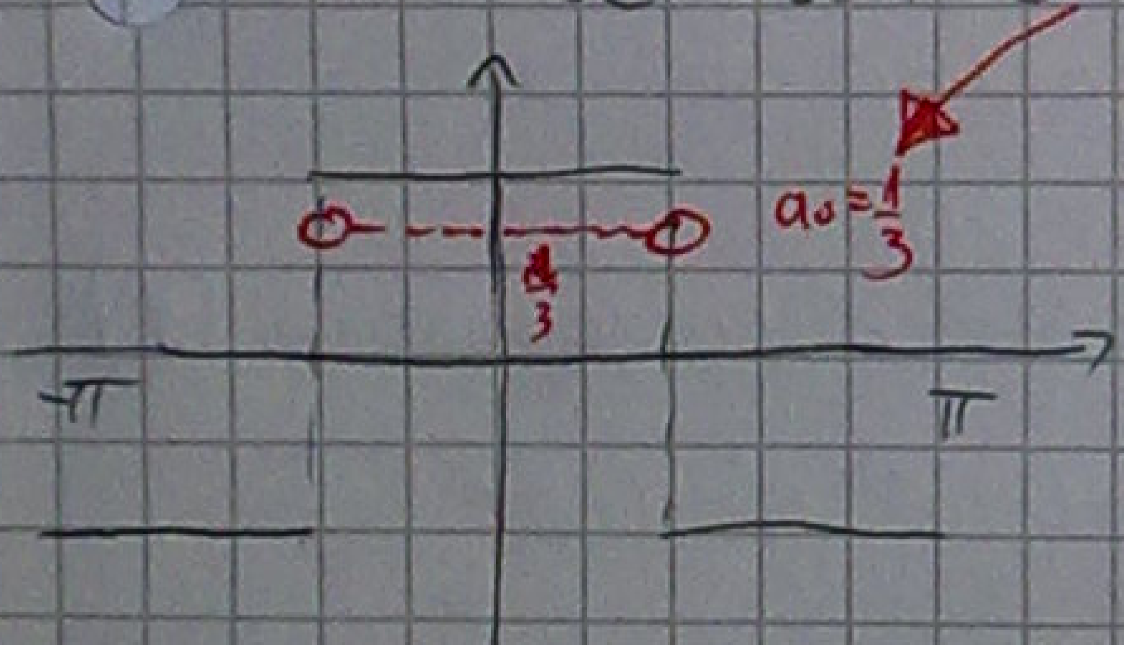
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\left[\int_0^{2\pi/3} \cos nx\,dx - \int_{2\pi/3}^{\pi} \cos nx\,dx\right] =$$

$$= \frac{2}{\pi}\left(\frac{1}{n}\sin(nx)\Big|_0^{2\pi/3} - \frac{1}{n}\sin(nx)\Big|_{2\pi/3}^{\pi}\right) = \frac{2}{\pi}\left(\frac{1}{n}\sin\left(\frac{2n\pi}{3}\right) + \frac{1}{n}\sin\left(\frac{2n\pi}{3}\right)\right) =$$

$$= \frac{4\sin\frac{2n\pi}{3}}{n\pi}$$

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4\sin\left(\frac{2n\pi}{3}\right)\cos(nx)}{n\pi}$$

$\rightarrow$ kaže da trebamo skicirati dobiveni red:



$\rightarrow$ skiciramo graf početne f-je $f(x)$ i nacrtamo kružiće di je $\frac{a_0}{2}$

$\rightarrow$ još moramo izračunati sumu zadanog reda!

$\rightarrow$ u $S(x)$ stavimo da je $n=1$; $n=2$, $n=3$, $n=4$

$$S(x) = \frac{1}{3} + \frac{4}{\pi}\left[\left(\frac{\sqrt{3}}{2}\cos x\right) + \frac{-\frac{\sqrt{3}}{2}\cos 2x}{2} + - - - + - - -\right]$$

- pratimo događanja na trig. kružnici

$\rightarrow$ trebamo se riješit (cos) ; stavimo $x = 0$ jer je $\cos(0) = 1$

$$S(0) = \frac{1}{3} + \frac{4\sqrt{3}}{2\pi}\left(\frac{1}{1} \overbrace{- \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7}}^{S'} - - -\right) \;;$$ izjednačimo $f(0) = S(0)$

$$0 = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} S' \Rightarrow S' = \frac{\pi}{3\sqrt{3}}$$ $\rightarrow$ to je ono što tražimo

ZADATAK 4: $S(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$; $f(x) = x^2$; $x \in (-1,1) \rightarrow T = 2$

a) Pomoću danog razvoja i Parsevalove jedn. izračunaj $\sum_{n=1}^{\infty} \frac{1}{n^4}$

PARSEVAL: $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 = \frac{2}{T} \int_a^b |f(x)|^2 dx$

$\Rightarrow$ iz zadanog F. reda išćitavamo komponente:

$\frac{a_0}{2} = \frac{1}{3} \Rightarrow a_0 = \frac{2}{3}$; pošto imamo cos $\Rightarrow$ zaključujemo da je f-ja parna!

$b_n = 0$; $a_n = \frac{4}{\pi^2} \cdot \frac{(-1)^n}{n^2}$ $\quad (f(x) = x^2)$

- sad uvrstimo u Parsevalovu jednakost:

$$\frac{\frac{4}{9}}{\frac{2}{1}} + \sum_{n=1}^{\infty} \frac{16 \cdot 1}{\pi^4 n^4} + 0 = \frac{2}{2} \int_{-1}^{1} x^4 dx =$$

$$= \frac{2}{9} + \frac{16}{\pi^4} \cdot \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5} ; \quad \frac{16}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8}{45} ;$$ ovo trebamo izračunati u zadatku;

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\frac{8}{45}}{\frac{16}{\pi^4}} = \frac{8 \cdot \pi^4}{16 \cdot 45} = \frac{\pi^4}{90}$$

b) nađi Fourierov red za $f(x) = x^3$, $x \in (-1,1)$

$\rightarrow$ u zadatku je zadano da je $f(x) = x^2$ i $f(x) = S(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$;

dakle, $x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$ tražimo x^3

*

$\int$ - idemo integrirat ovo gore da dobijemo x^3

D

$$\frac{x^3}{3} = \frac{1}{3} x + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \frac{1}{n\pi x} \sin(n\pi x) + C \quad / \text{množimo sa } 3$$

Δ $x^3 = x + \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{1}{n\pi x} \sin(n\pi x) + C$; fali nam x, pa deriviramo (*)

$$2x = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - n\pi \cdot \sin(n\pi x) ; \quad x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-(-1)^n}{n} \sin(n\pi x)$$

C još trebamo izračunat C a to je a_0 pa uzimamo općenitu formulu:

$$a_0 = \frac{2}{T} \int_a^b f(x) dx = \frac{2}{2} \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4}(1-1) = \emptyset$$

$\rightarrow$ sad sve ubacimo u Δ i sredimo:

$$x^3 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-(-1)^n}{n} \sin(n\pi x) + \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{1}{n\pi x} \sin(n\pi x) + 0$$

$$= \sum_{n=1}^{\infty} \frac{-2n^2\pi^2(-1)^n}{n^2 n^3} \sin(n\pi x) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{-n^2\pi^2 + 6(-1)^n \sin(n\pi x)}{n^3}$$