1. MI 2007

$\boxed{4. \text{ZAD}}$

$$\int_0^\infty \underset{e^{-st}}{e^{-\frac{x}{2}}} \underbrace{x^2 \cos x}_{f(x)}\, dx$$

$f(x) = x^2 \cos x$

$$F''(x) = \frac{s}{s^2+1} = \left(\frac{s^2+1 - 2s^2}{(s^2+1)^2}\right)' =$$

$\cos \omega x \circ\!\!- F(x)$

$f(x) - F(s)$

$f'(x) = s F(s) - f(0)$

$\omega \sin \omega x \circ\!\!- s(F - f(0))$

$-\omega^2 \cos \omega x \circ\!\!- s(sF(s) - f(\omega)) - f'(0)$

$= s^2 F(s) - s$

$= -\omega^2 F(s)$

$s^2 F(s) - s = -\omega^2 F(s)$

$F(s)[s^2 + \omega^2] = s$

$$F(s) = \frac{s}{s^2+\omega^2}$$

$F(s) \bullet\!\!-\!\!\circ f(x)$

$F'(s) \bullet\!\!-\!\!\circ -x f(x)$

$F''(s) \bullet\!\!-\!\!\circ x^2 f(x)$

$$\left[\frac{1-s^2}{(1+s^2)^2}\right]' = \frac{-2s(1+s^2)^2 - (1-s^2)\,2(1+s^2)\,2s}{(1+s^2)^4}$$

$$= \dots = \frac{2s\,(s^2-3)}{(s^2+1)^3} = F(s)$$

$$F\left(\frac{1}{2}\right) = \frac{2\cdot\frac{1}{2}\left(\frac{1}{4}-3\right)}{\left(\frac{1}{4}+1\right)^3} = \frac{-\frac{11}{4}}{\left(\frac{5}{4}\right)^3} =$$

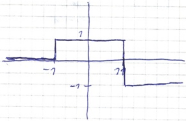
$$F\left(\frac{1}{2}\right) = -\frac{176}{125}$$

1.MI 2007

$\boxed{5.240}$ $y'' + y = f(t)$

$y(0) = 1$

$y'(0) = 0$



$f(t) = 1 \cdot g[0,1) - 1 \cdot g[1, \infty]$

$= [u(t-0) - u(t-1)] - [u(t-1) - u(t-\infty)]$

$= u(t) - u(t-1) - u(t-1)$

$= u(t) - 2u(t-1)$

$s^2 Y(s) - s\,y(0) - \underbrace{y'(0)}_{=0} + Y(s) = F(s)$

$Y(s^2+1) = F(s) + s$

$F(s) = \frac{1}{s} - 2 \cdot \frac{1}{s} e^{-s}$

$Y = \frac{1}{s(s^2+1)} - \frac{2e^{-s}}{s(s^2+1)} + \frac{s}{s^2+1}$

$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

$1 = A(s^2+1) + (Bs+C) \cdot s$

$s = 0 \qquad 1 = A$

$s = -1 \qquad 1 = A \cdot 2 + (-B + C)(-1)$

$1 = 2 + B - C$

$B = C - 1$

$B = -1$

$s = 1 \qquad 1 = 2A + B + C$

$1 = 2 + C - 1 + C$

$2C = 0$

$C = 0$

$$\boxed{\begin{array}{l} A = 1 \\ B = -1 \\ C = 0 \end{array}}$$

$$\frac{2}{s(s^2+1)} = \frac{D}{s} + \frac{Es+F}{s^2+1}$$

$$2 = D(s^2+1) + (Es+F)(s)$$

$s = 0 \qquad 2 = D$

$s = -1 \qquad 2 = D \cdot 2 + (-E+F)(-1)$

$2 = 4 + E - F$

$E = F - 2$

$s = 1 \qquad 2 = D \cdot 2 + E + F$

$2 = 4 + F - 2 + F$

$2F = 2 - 4 + 2$

$2F = 0$

$F = 0$

$$\boxed{\begin{array}{l} D = 2 \\ E = -2 \\ F = 0 \end{array}}$$

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+1} - \frac{2}{s}e^{-s} + \frac{2s}{s^2+1}e^{-s} + \frac{s}{s^2+1}$$

$$y(t) = u(t) - \cancel{\cos t\, u(t)} - 2\, u(t-1) + 2 \cos(t-1)\, u(t-1)$$
$$+ \cancel{\cos t\, u(t)}$$

9. MI 2007

$\boxed{6. ZAD}$

$$f(t) * g(t)$$

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$f(x) \le M e^{at}$$

$$|f * g| = \left| \int_0^t f(\tau)\, g(t-\tau)\, d\tau \right|$$

$$= \int_0^t \left| f(\tau)\, g(t-\tau)\, d\tau \right|$$

$$\le \int_0^t |f(\tau)|\, |g(t-\tau)|\, d\tau$$

$$|f(t)| \le A e^{\alpha t}$$

$$|g(t)| \le B e^{a t}$$

$$\le \int_0^t A e^{\alpha \tau}\, B e^{a(t-\tau)}\, d\tau$$

$$= A B e^{a t} \int_0^t e^{\alpha \tau} e^{-a\tau}\, d\tau$$

$$= A B e^{a \tau} \int_0^t e^{\tau(\alpha - a)}\, d\tau$$

$$= A B e^{a t} \frac{1}{\alpha - a} e^{\tau(\alpha - a)} \Big|_0^t$$

$$= \frac{A B e^{a t} \left(e^{\alpha t} e^{-a t} - 1\right)}{\alpha - a}$$

$$= \frac{A\,B}{a-b}\left[e^{at} - e^{bt}\right] \leq C e^{ct}$$

$$|f * g| \leq C e^{ct}$$

$$|A+B| \leq |A| + |B|$$
$$|A \cdot B| = |A| \cdot |B|$$

ako je $a = b$

c) $$y(t) = 3 \sin t + 2 \int_0^t \cos(t-\tau)\, y(\tau)\, d\tau \quad / \mathcal{L}$$

$$\cos(t) * y(t)$$

$$f * g \circ\!\!-\!\!\bullet F \cdot G$$

$$Y(s) = 3 \frac{1}{s^2+1} + 2 \frac{s}{s^2+1} \cdot Y(s)$$

$$Y(s)\left[1 - \frac{2s}{s^2+1}\right] = \frac{3}{s^2+1}$$

$$Y(s)\left[\frac{s^2+1-2s}{s^2+1}\right] = \frac{3}{s^2+1}$$

$$Y(s) = \frac{3}{s^2-2s+1} = \frac{3}{(s-1)^2}$$

$$y(t) = 3\, t\, e^t$$

$\boxed{7. \text{ZAD}}$

$$e(t) = 1 + ch\,2t \qquad L = 1 \quad C = 1$$

$$E(s) = \frac{1}{s} + \frac{s}{s^2+4}$$

$$Z(s) = \frac{s}{s^2+1}$$

$$I = \frac{E}{Z} = \frac{\frac{1}{s} + \frac{s}{s^2+4}}{\frac{s}{s^2+1}} = \frac{\frac{s^2+4+s^2}{s(s^2+4)}}{\frac{s}{s^2+1}}$$

$$= \frac{(s^2+1)(s^2+s^2+4)}{s^2(s^2+4)}$$

$$\frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$(2s^4+6s^2+4) \,/\, (s^4+4s^2)$$

$$2s^2 + ss^2$$

$$\cancel{2} + \frac{-2s^2+4}{s^2(s^2+4)} = \cancel{2} + \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$-2s^2+4 = (As+B)(s^2+4) + (Cs+D)s^2$$

$$-2s^2+4 = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2$$

$$= s^3(A+C) + s^2(B+D) + s(4A) + 4B$$

$A + C = 0$

$A = -C$ $\boxed{C = 0}$

$B + D = -2$

$4A = 0$ $D = -2 - 1$

$\boxed{A = 0}$ $\boxed{D = -3}$

$4B = 4$

$\boxed{B = 1}$

$$I = 2 + \frac{1}{s^2} - 3 \cdot \frac{1}{2} \frac{2}{s^2 + 2^2}$$

$$i(t) = 2\,\delta(t) + t\,u(t) - \frac{3}{2} \sin t\; u(t)$$

1. MI 2006

$\boxed{4.\ ZAD}$

$$e^{-\alpha t} f(t)$$

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} e^{-\alpha t} f(t)\,dt$$

$$= \int_0^\infty e^{-t(s+\alpha)} f(t)\,dt$$

$$= F(s+\alpha)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = \left| \begin{array}{ll} u = e^{-st} & du = -s e^{-st} \\ dv = f'(t)\,dt & v = f(t) \end{array} \right|$$

$$= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty (-s)\, e^{-st} f(t)\,dt$$

$$= e^{-st} f(t) \Big|_0^\infty + s \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{F(s)}$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + sF(s)$$

$$= \underbrace{\lim_{t\to\infty} \frac{f(t)}{e^{st}}}_{=0} - f(0) + sF(s)$$

e^{st} brže raste od $f(t)$ pa je ne jednako 0

$$= sF(s) - f(0)$$

1.MI 2006

$\boxed{5. \text{Z4D}}$

$$f(t) = (t-2)^3 e^{-t} u(t-2)$$

$$t \circ\!\!- \frac{1}{s}$$

$$t^n \circ\!\!- \frac{n!}{s^{n+1}}$$

$$t^3 \circ\!\!- \frac{3!}{s^4} = \frac{6}{s^4}$$

$$F(s) = \frac{6}{(s+1)^4} e^{-2(s+1)}$$

$\boxed{6. \text{ ZAD}}$ 1. MI 2006

$$\int_0^\infty e^{-t}\, t^{100}\, dt$$

$$\int_0^\infty e^{-st} f(t)\, dt$$

$$f(t) = t^{100} \qquad s = 1$$

$$t^n \circ\!\!- \frac{n!}{s^{n+1}}$$

$$t^{100} \circ\!\!- \frac{100!}{s^{101}} = F(s)$$

$$F(1) = \frac{100!}{1^{101}} = 100!$$

$\boxed{7. \text{ ZAD}}$

$$y(t) = \sin t + \int_0^t t \sin(t-\tau)\, d\tau \qquad / \cdot \mathcal{L}$$

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s^2} Y(s)$$

$$Y(s)\left[1 - \frac{1}{s^2}\right] = \frac{1}{s^2+1}$$

$$Y(s)\left[\frac{s^2-1}{s^2}\right] = \frac{1}{s^2+1}$$

$$Y(s) = \frac{s^2}{(s-1)(s+1)(s^2+1)}$$

$$\frac{s^2}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$s^2 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1)$$

$s = 1$ $\qquad 1 = A \cdot 2 \cdot 2$

$$A = \frac{1}{4}$$

$s = -1$ $\qquad 1 = B(-2) \cdot 2$

$$B = -\frac{1}{4}$$

$s = 0$ $\qquad 0 = A \cdot 1 \cdot 1 + B \cdot (-1) \cdot 1 + D(-1) \cdot 1$

$$0 = \frac{1}{4} + \frac{1}{4} - D$$

$$D = \frac{2}{4} = \frac{1}{2}$$

$s = 2$ $\qquad 4 = A \cdot 3 \cdot 5 + B \cdot 1 \cdot 5 + (2C+D) \cdot 1 \cdot 3$

$$4 = \frac{15}{4} - \frac{5}{4} + 6C + \frac{3}{2}$$

$$6C = \frac{16}{4} - \frac{15}{4} + \frac{5}{4} - \frac{6}{4}$$

$$6C =$$

$$C = 0$$

$$Y(s) = \frac{1}{4} \cdot \frac{1}{s-1} - \frac{1}{4} \frac{1}{s+1} + \frac{1}{2} \frac{1}{s^2+1}$$

1. MI 2007

1. ZAD

a) prekid 1. vrste - limes s lijeva i s desna je konačan broj

b) $f(x) = \frac{1}{1-x}$ $\qquad x \in [0,2]$

$x = 1 \qquad \frac{1}{0} = \infty \Rightarrow$ prekid nije 1. vrste

ne zadovoljava Dirichletove uvjete

c) teorem o konvergenciji Fourierovog reda

GLATKA FUNK.

NIJE GLATKA FUNK.

$s(x) = f(x)$

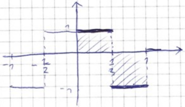
$s(x) = \frac{1}{2}\left[f(x-0) + f(x+0)\right]$

2. ZAD

$$f(x) = \begin{cases} 1 & x \in \langle 0, \frac{1}{2} \rangle \\ -1 & x \in \langle \frac{1}{2}, 1 \rangle \end{cases}$$

po cos funkcijama

PARNO PROŠIRENJE



simetričan interval

$[-L, L]$

$b_n = 0$

$a_0 = 0$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{1} \left[\int_0^{\frac{1}{2}} 1 \cdot \cos \frac{n\pi x}{1} \, dx - 1 \cdot \int_{\frac{1}{2}}^{1} \cos \frac{n\pi x}{1} \, dx \right]$$

$$= 2 \left[\int_0^{\frac{1}{2}} \cos n\pi x \, dx - \int_{\frac{1}{2}}^{1} \cos n\pi x \, dx \right]$$

$$= 2 \left[\frac{1}{n\pi} \sin n\pi x \Big|_0^{\frac{1}{2}} - \frac{1}{n\pi} \sin n\pi x \Big|_{\frac{1}{2}}^{1} \right]$$

$$= 2 \left[\frac{1}{n\pi} \sin \frac{n\pi}{2} - \frac{1}{n\pi} \left[\underbrace{\sin n\pi}_{=0} - \sin \frac{n\pi}{2} \right] \right]$$

$$= 2 \left[\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$\sin\frac{n\pi}{2}$ za $n=1$ $\sin\frac{\pi}{2}=1$

za $n=2$ $\sin\pi=0$

za $n=3$ $\sin\frac{3\pi}{2}=-1$

za $n=4$ $\sin 2\pi=0$

RED NEMA PARNE ČLANOVE

$$a_n=\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$n=2k+1$ NEPARNI

$$a_{2k+1}=\frac{4}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi}{2}\right)$$

$$=\frac{4}{(2k+1)\pi}\sin\left(\frac{2k\pi}{2}+\frac{\pi}{2}\right)$$

$$=\frac{4}{(2k+1)\pi}\sin\left(k\pi+\frac{\pi}{2}\right)$$

$$=\frac{4}{(2k+1)\pi}\cos\left(k\pi\right)$$

$$=\frac{4}{(2k+1)\pi}(-1)^k$$

$a_0=0$

$$f(x)=\sum_{k=0}^{\infty}\frac{4}{(2k+1)\pi}(-1)^k\cos\left(\frac{(2k+1)\pi x}{1}\right)$$

$n=2\cdot 0+1$

$n=1$

b) Izračunaj $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ pomoću

Parsevalove jednakosti.

$$\underbrace{\frac{1}{2} a_0^2}_{=0} + \sum_{n=1}^{\infty} (a_n)^2 + \underbrace{\sum_{n=1}^{\infty} (b_n)^2}_{=0} = \frac{2}{T} \int_a^b |f(x)|^2 \, dx$$

$T=2$

$$\sum_{k=0}^{\infty} \left(\frac{4}{(2k+1)\pi} (-1)^k \right)^2 = 1 \int_{-1}^{1} 1^2 \, dx$$

zbog proširenja

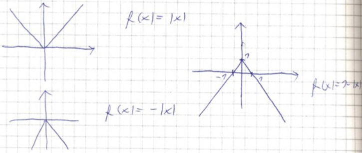
$(-1)^{2k} = 1$ $\qquad 1 \cdot (1-(-1))$

$$\frac{4^2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 2 \quad / \cdot \frac{\pi^2}{4^2}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{2\pi^2}{4^2} = \frac{\pi^2}{8}$$

3. ZAD $\quad f(x)=\begin{cases} 1-|x| & , \ x\in[-1,1] \\ 0 & , \ \text{inače} \end{cases}$

a) Prikaži pomoću Fourierovog integrala



PARNA - simetrična s obzirom na y-os

NEPARNA - simetrična s obzirom na ishodište

PARNA $\Rightarrow$ $B(\lambda)=0$

$$A(\lambda)=\frac{1}{\pi}\int_{-1}^{1} f(\xi)\cos\lambda\xi \, d\xi$$

$$=2\cdot\frac{1}{\pi}\int_0^1 (1-\xi)\cos\lambda\xi \, d\xi$$

$$=\frac{2}{\pi}\left[\int_0^1 \cos\lambda\xi \, d\xi - \int_0^1 \xi\cos\lambda\xi \, d\xi\right] - \left| \begin{matrix} u=\xi & du=d\xi \\ dv=\cos\lambda\xi \, d\xi & v=\frac{1}{\lambda}\sin\lambda\xi \end{matrix}\right.$$

$$=\frac{2}{\pi}\left[\frac{1}{\lambda}\sin\lambda - \left(\frac{\xi}{\lambda}\sin\lambda\xi\Big|_0^1 - \int_0^1 \frac{1}{\lambda}\sin\lambda\xi \, d\xi\right)\right]$$

$$=\frac{2}{\pi}\left[\frac{1}{\lambda}\sin\lambda - \left(\frac{1}{\lambda}\sin\lambda + \frac{1}{\lambda^2}\cos\lambda\xi\Big|_0^1\right)\right]$$

$= \frac{2}{\pi}\left[\cancel{\frac{1}{\lambda}\sin\lambda} - \cancel{\frac{1}{\lambda}\sin\lambda} - \left(\frac{1}{\lambda^2}\cos\lambda - \frac{1}{\lambda^2}\right)\right.$

$= \frac{2}{\pi\lambda^2}\left(1 - \cos\lambda\right)$

$f(x) = \int_0^\infty \frac{2}{\pi\lambda^2}(1-\cos\lambda)\cos\lambda x \, d\lambda$

b) izračunaj integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

za $x=0$

$f(0) = \int_0^\infty \frac{2}{\pi}\cdot\frac{1-\cos\lambda}{\lambda^2}\,d\lambda$

$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

$1 = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{\lambda}{2}}{\lambda^2}\,d\lambda \quad = |$

$1 = \frac{2}{\pi}\int_0^\infty 4\frac{\sin^2 t}{4t^2}\,dt$

$\frac{\pi}{2} = \int_0^\infty \frac{\sin^2 t}{t^2}\,dt$

1.M 2008

1. ZAD

a) $f(x) = \sin mx$

$g(x) = \sin nx$

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \frac{1}{2} \left[\cos(m-n)x + \cos(m+n)x \right] dx = 0$$

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x \, dx = 0$$

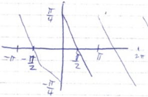
$$\frac{1}{2} \frac{1}{m-n} \sin(m-n)x \Big|_{-\pi}^{\pi} + \frac{1}{2}\frac{1}{m+n} \sin(m+n)x \Big|_{-\pi}^{\pi} = 0$$

2. ZAD Razvij u Fourierov red

a) $f(x) = \frac{\pi}{4} - \frac{x}{2}$ na $\langle 0, \pi \rangle$

po sinus funkcijama.

proširenje na $\langle -\pi, \pi \rangle$



$$b_n = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{4} - \frac{x}{2}\right) \sin\frac{n\pi x}{\pi}\, dx \quad \left| \begin{array}{ll} u = x & du = dx \\ dv = \sin nx\, dx & v = -\frac{1}{n}\cos nx \end{array}\right.$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4} \sin nx\, dx - \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} x \sin nx\, dx$$

$$= -\frac{1}{2n}\cos nx\Big|_0^{\pi} - \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos nx\, dx\right]$$

$$= -\frac{1}{2n}\left[\cos n\pi - 1\right] - \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \underbrace{\frac{1}{n}\frac{1}{n}\sin nx\Big|_0^{\pi}}_{=0}\right] \qquad \begin{array}{l}\sin\pi = 0\\ \sin 0 = 0\end{array}$$

$$= \frac{1}{2n}\left[1 - \cos n\pi\right] + \frac{1}{n}\cos n\pi$$

$(-1)^n$

$(-1)^{n+1}$

za $n=0$ $\quad 1 + (-1)^1$

$1 - 1 = 0$

$n = 1 \quad 1 + (-1)^2 = 2$

$n = 2 \quad 1 + (-1)^3 = 0$

$$= \frac{1}{2n} - \frac{1}{2n}\cos n\pi + \frac{1}{n}\cos n\pi$$

$$= \frac{1}{2n} + \cos(n\pi)\cdot\left(\frac{1}{n} - \frac{1}{2n}\right) \qquad \frac{2-1}{2n}$$

$$= \frac{1}{2n} + \frac{1}{2n}\cos n\pi$$

$$= \frac{1}{2n}\left(1 + \cos n\pi\right) = \frac{1}{2n}\left(1 + (-1)^n\right)$$

za $n=0$

za parne $\frac{1}{n}$

za neparne 0

$n = 2k$

$b_{2k} = \frac{1}{2k}$

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2k} \sin \frac{2k\pi x}{\pi}$$

b) Izračunaj $S(3\pi)$

$S(x) = f(x)$ kada je funkcija NEPREKINUTA

$S(x) = \frac{1}{2}\left[f(x-0) + f(x-0) \right]$ kada je funkcija prekinuta u točki x

funkcija je u točki 3π prekinuta (može se očitati iz slike)

$$S(3\pi) = \frac{1}{2}\left[f(3\pi) + f(3\pi) \right]$$

$$f(x) = \frac{\pi}{4} - \frac{x}{2}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{3\pi}{2} + \frac{\pi}{4} - \frac{3\pi}{2} \right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 3\pi \right] = \frac{\pi}{4} - \frac{3\pi}{2} = \frac{\pi}{4} - \frac{6\pi}{4} = \frac{-5\pi}{4}$$