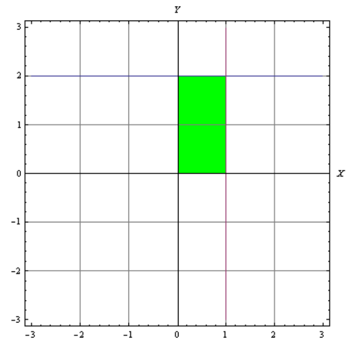
Rješenja 4. domaće zadaće MAT3E

v1.0

1. **Izračunajte dvostruki integral $\iint_P x dx dy$ pri čemu je P pravokutnik omedjen pravcima $x = 1$, $y = 2$ i koordinatnim osima.**

   Rješenje:

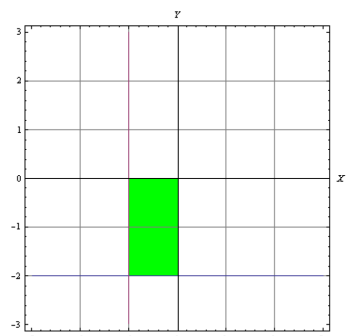
$$S^2 + x^2 \tag{1}$$



$$\int_0^1 x\ dx \int_0^2 dy = \int_0^1 x \cdot y\Big|_0^2\ dx = 2\int_0^2 x\ dx = 2 \cdot \frac{x^2}{2}\Big|_0^1 = 1$$

2. **Izračunajte dvostruki integral $\iint_P x dx dy$ pri čemu je P pravokutnik omedjen pravcima $x = -1$, $y = -2$ i koordinatnim osima.**

   Rješenje:

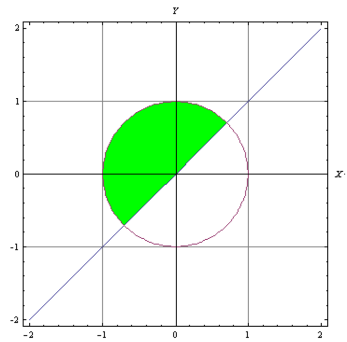


$$\int_{-1}^0 dx \int_{-2}^0 y\ dy = \int_{-1}^0 \frac{y^2}{2}\Big|_{-2}^0\ dx = \int_{-1}^0 (0-2)\ dx = -2 \cdot x\Big|_{-1}^0 = -2$$

3. **Postavite granice integracije u integralu $\iint_D f(x,y) dx dy$ ako je D područje odredjeno nejednadžbama $y \geq x$ i $x^2 \leq 1 - y^2$**

   Rješenje:

Sve eventualne pogreške molim da javite isključivo privatnom porukom.



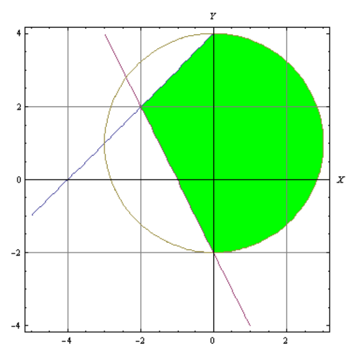
$$y = \pm\sqrt{1-x^2}$$
$$y \geq x$$

Povučete još jedan pravac u točki $x = \frac{\sqrt{2}}{2}$ i razdjelite na 2 dijela zadanu površinu radi lakše integracije.

$$\int_{-1}^{-\frac{\sqrt{2}}{2}} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\ dy + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} dx \int_{x}^{\sqrt{1-x^2}} f(x,y)\ dy$$

4. **Postavite granice integracije u integralu $\iint_D f(x,y)dxdy$ ako je D područje odredjeno nejednadžbama $y \leq x+4$, $y \geq -2x-2$ i $x^2 \leq 9-(y-1)^2$**

Rješenje:



$$y = x+4$$
$$y = -2x-2$$
$$x+4 = -2x-2$$
$$3x = -6$$
$$x = -2$$

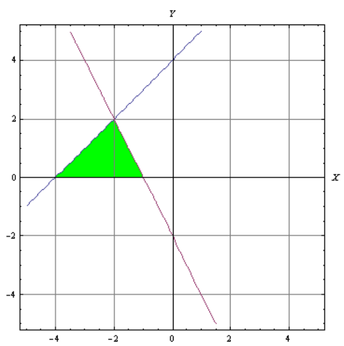
$$(y-1)^2 = 9-x^2$$
$$y-1 = \pm\sqrt{9-x^2}$$
$$y = \pm\sqrt{9-x^2}+1$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

$$\int_{-2}^{0} dx \int_{-2x-2}^{x+4} f(x,y)\ dy + \int_{0}^{3} dx \int_{-\sqrt{9-x^2}+1}^{\sqrt{9-x^2}+1} f(x,y)\ dy$$

5. **Postavite granice integracije u integralu** $\iint_D f(x,y)dxdy$ **ako je područje** D **omedjeno i ograničeno pravcima** $y = x + 4$**,** $y = -2x - 2$ **i osi** y**.**

Rješenje:



$$x + 4 = -2x - 2$$
$$3x = -6$$
$$x = -2$$

$$\int_{-2}^{0} dx \int_{-2x-2}^{x+4} f(x,y)\ dy$$

6. **U integral** $\iint_D f(x,y)dxdy$ **po području D omedjenom pravcima** $x + y = 1$**,** $x + y = 3$**,** $x - y = -1$**,** $x - y = 1$ **uvedite promjenu koordinata** $u = x + y$**,** $v = x - y$

Rješenje:

$$u = x + y,\ v = x - y\ \Rightarrow\ x = \frac{u+v}{2},\ y = \frac{u-v}{2}$$
$$1 \le u \le 3$$
$$-1 \le v \le 1$$

$$\iint_D f(x,y)dxdy = \iint f(x(u,v), y(u,v))|J|dudv$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\int_{1}^{3} du \int_{-1}^{1} f(x(u,v), y(u,v))\frac{1}{2}dv$$

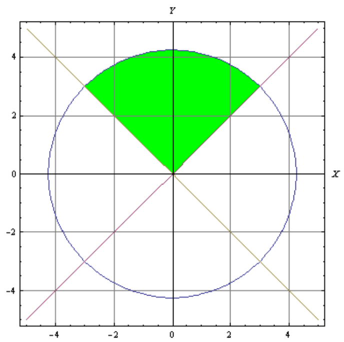
Sve eventualne pogreške molim da javite isključivo privatnom porukom.

7. **U integral $\iint_D$ po području D omedjenom krivuljama $x^2-y^2=1$, $x^2-y^2=4$, $xy=1$, $xy=3$ uvedite promjenu koordinata $u=xy$, $v=x^2-y^2$**

   Rješenje:

8. **Postavite granice integracije u polarnim koordinatama u integralu $\iint_D f(x,y)dxdy$ ako je D kružni isječak OAB sa središtem u $O(0,0)$, i krajevima u točkama $A(3,3)$ i $B(-3,3)$.**

   Rješenje:

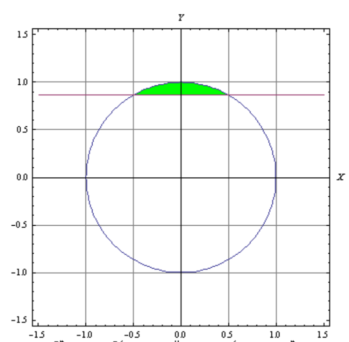


$$x^2+y^2=r^2 \Rightarrow 3^2+3^2=r^2 \Rightarrow r=3\sqrt{2}$$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_0^{3\sqrt{2}} f(r\cos\varphi, r\sin\varphi)rdr$$

9. **Postavite granice integracije u polarnim koordinatama u integralu $\iint_D f(x,y)dxdy$ ako je područje D omedjeno kružnicom $x^2+y^2=1$ i pravcem $y=\frac{\sqrt{3}}{2}$ za $y \geq \frac{\sqrt{3}}{2}$**

   Rješenje:



$$y=\frac{\sqrt{3}}{2}=r\sin\varphi \;\Rightarrow\; r=\frac{\sqrt{3}}{2\sin\varphi}$$

Ako uvrstimo vrijednost $y=\frac{\sqrt{3}}{2}$ u $x^2+y^2=1$ dobijemo x:

$$x_{1,2}=\pm\frac{1}{2}$$

$$arctg\varphi_1=\frac{y}{x_1} \;\Rightarrow\; arctg\varphi_1=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

$$\varphi_1 = \frac{\pi}{3}$$

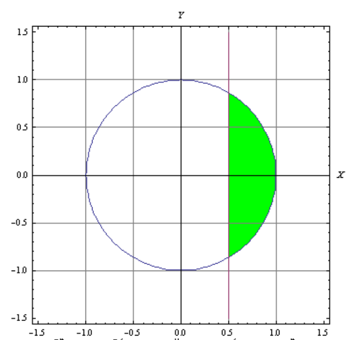
$$arctg\varphi_2 = \frac{y}{x_2} \;\Rightarrow\; arctg\varphi_2 = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

$$\varphi_1 = \frac{2\pi}{3}$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} d\varphi \int_{\frac{\sqrt{3}}{2\sin\varphi}}^{1} f(r\cos\varphi, r\sin\varphi) r \; dr$$

10. **Postavite granice integracije u polarnim koordinatama u integralu $\iint_D f(x,y)dxdy$ ako je područje D omedjeno kružnicom $x^2+y^2=1$ i pravcem $x=\frac{1}{2}$ za $x \geq \frac{1}{2}$**

Rješenje:



$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$\frac{1}{2} = r\cos\varphi \;\Rightarrow\; r = \frac{1}{2\cos\varphi}$$

Uvrštavanjem vrijednosti $x=\frac{1}{2}$ u jednadzbu $x^2+y^2=1$ dobijemo $y = \pm\frac{\sqrt{3}}{2}$ pa je točka $T(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$
Kut dobijemo pomou $\varphi_1 = arctg\frac{y}{x} = arctg\sqrt{3} = \frac{\pi}{3}$ $\varphi_2 = -\frac{\pi}{3}$
Konačno:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{1}{2\cos\varphi}}^{1} f(r\cos\varphi, r\sin\varphi) r \; dr$$

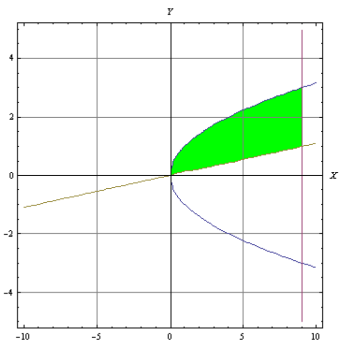
11. **Izračunajte dvostruki integral**

$$\iint_D y^2 dxdy$$

pri čemu je područje D omedjeno krivuljom $y=\sqrt{x}$ i pravcima $x=9$ i $y=\frac{x}{9}$

Rješenje:

Sve eventualne pogreške molim da javite isključivo privatnom porukom.



$$\int_0^9 dx \int_{\frac{x}{9}}^{\sqrt{x}} y^2\ dy = \frac{1}{3}\int_0^9 dx \cdot y^3\Big|_{\frac{x}{9}}^{\sqrt{x}} = \frac{1}{3}\int_0^9 \left(x^{\frac{3}{2}} - \frac{1}{729}x^3\right) dx = \frac{1}{3}\left(\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{2916}x^4\right)\Big|_0^9$$

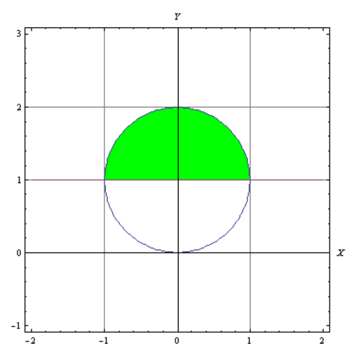
$$= \frac{1}{3}\left(\frac{2}{5}\cdot 243 - \frac{1}{2916}\cdot 6561\right) = \frac{633}{20}$$

12. **Izračunajte dvostruki integral**

$$\iint_D x^2 dxdy$$

pri čemu je područje D omedjeno kružnicom $x^2 + (y-1)^2 = 1$ i pravcem $y = 1$ za $y \geq 1$.

Rješenje:



Zamjena:

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$\int_0^\pi \cos^2\varphi\ d\varphi \int_0^1 r^3\ dr = \frac{1}{4}\int_0^\pi \cos^2\varphi\ d\varphi$$
$$\frac{1}{4}\int_0^\pi \frac{1+\cos 2\varphi}{2}\ d\varphi = \frac{1}{8}\varphi\Big|_0^\pi + \frac{1}{4}\sin 2\varphi\Big|_0^\pi = \frac{\pi}{8}$$

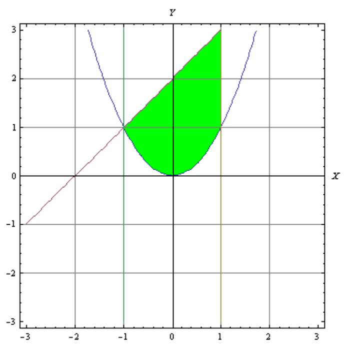
13. **Izračunajte dvostruki integral**

$$\iint_D (x^2 - y) dxdy$$

pri čemu je $D = \{(x,y) : -1 \leq x \leq 1, x^2 \leq y \leq x+2\}$.

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

Rješenje:



$$\int_{-1}^{1} dx \int_{x^2}^{x+2} (x^2-1)\ dy = \int_{-1}^{1} dx \left(yx^2 - \frac{1}{2}y^2 \right)\Big|_{x^2}^{x+2} = \int_{-1}^{1} \left[(x+2)x^2 - \frac{1}{2}(x+2)^2 - x^4 + \frac{1}{2}x^4 \right] dx$$

$$\int_{-1}^{1} \left(x^3 + 2x^2 - \frac{1}{2}x^2 - x - 2 - \frac{1}{2}x^4 \right) dx = \int_{-1}^{1} \left(-\frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 - x - 2 \right) dx$$

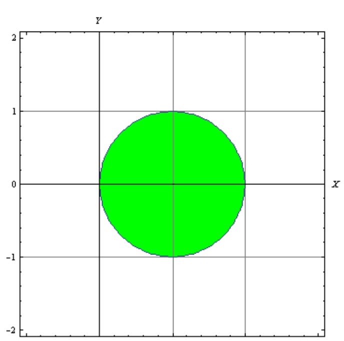
$$\left(-\frac{1}{2}\cdot\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{3}{2}\cdot\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \right)\Big|_{-1}^{1} = -\frac{1}{10} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} - 2 - \left(\frac{1}{10} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 2 \right) = -\frac{16}{5}$$

14. **Izračunajte dvostruki integral**

$$\iint_D (x^2+y^2)dxdy$$

pri čemu je područje D **omedjeno kružnicom** $(x-1)^2+y^2=1$**.**

Rješenje:



$$x - 1 = r\cos\varphi \ \Rightarrow\ x^2 = (1+r\cos\varphi)^2 \ \Rightarrow\ x^2 = 1 + 2r\cos\varphi + r^2\cos^2\varphi$$

$$y = r\sin\varphi$$

$$\iint_D (x^2+y^2)dxdy = \int_0^{2\pi} d\varphi \int_0^1 (r + 2r^2\cos\varphi + r^3\cos^2\varphi + r^3\sin^2\varphi)\ dr = \int_0^{2\pi} d\varphi \int_0^1 (r + 2r^2\cos\varphi + r^3)\ dr$$

$$= \int_0^{2\pi} d\varphi \left(\frac{1}{2}r^2 + 2\cdot\frac{1}{3}r^3\cos\varphi + \frac{1}{4}r^4 \right)\Big|_0^1 = \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\cos\varphi + \frac{1}{4} \right) d\varphi$$

$$= \left(\frac{3}{4}\varphi + \frac{2}{3}\sin\varphi \right)\Big|_0^{2\pi} = \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}$$

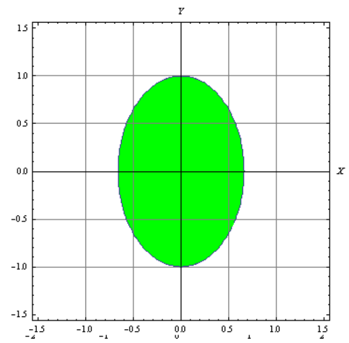
Sve eventualne pogreške molim da javite isključivo privatnom porukom.

15. **Izračunajte dvostruki integral**

$$\iint_D (9x^2 + 4y^2) dxdy$$

pri čemu je područje D **omedjeno elipsom** $9x^2 + 4y^2 = 4$**.**

Rješenje:



$$9x^2 + 4y^2 = 4$$

$$\frac{x^2}{\frac{4}{9}} + y^2 = 1$$

$$\frac{9}{4}x^2 = r^2 \cos^2\varphi \;\Rightarrow\; x = \frac{2}{3} r\cos\varphi$$

$$y^2 = r^2 \sin^2\varphi \;\Rightarrow\; y = r\sin\varphi$$

$$J = abr = \frac{2}{3}r$$

$$\int_0^{2\pi} d\varphi \int_0^1 (4r^2\cos^2\varphi + 4r^2\sin^2\varphi)\frac{2}{3}r\ dr = \frac{8}{3}\int_0^{2\pi} d\varphi \int_0^1 r^3\ dr =$$

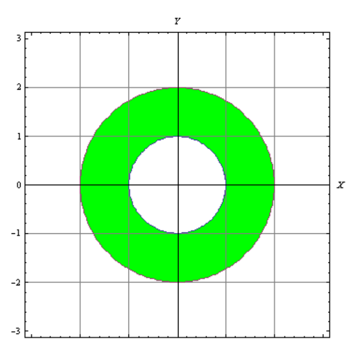
$$\frac{8}{3}\int_0^{2\pi} d\varphi \frac{1}{4}r^4\Big|_0^1 = \frac{2}{3}\int_0^{2\pi} d\varphi = \frac{2}{3}\varphi\Big|_0^{2\pi} = \frac{2}{3}2\pi = \frac{4\pi}{3}$$

16. **Izračunajte dvostruki integral**

$$\iint_D \frac{dxdy}{(x^2+y^2)^{\frac{3}{2}}}$$

pri čemu je područje D **omedjeno nejednadžbama** $1 \leq x^2 + y^2 \leq 4$.

Rješenje:



Sve eventualne pogreške molim da javite isključivo privatnom porukom.

Zamjena:

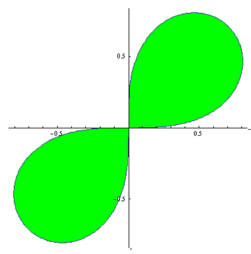
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

Preuredimo nazivnik:

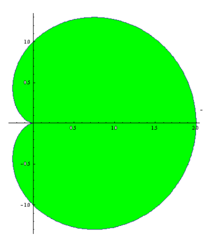
$$(x^2+y^2)^{\frac{3}{2}} = (r^2)^{\frac{3}{2}} = r^3$$

$$\int_0^{2\pi} d\varphi \int_1^2 r^{-2} dr = -\int_0^{2\pi} \frac{1}{r}\Big|_1^2 d\varphi = \frac{1}{2}\int_0^{2\pi} = \frac{1}{2}\varphi\Big|_0^{2\pi} = \pi$$

17. **Izračunajte površinu omedjenu krivuljom $r = \sqrt{\sin 2\varphi}$ u polarnim koordinatama.**
Rješenje:



18. **Izračunajte površinu omedjenu krivuljom $r = 1 + \cos\varphi$ u polarnim koordinatama.**
Rješenje:



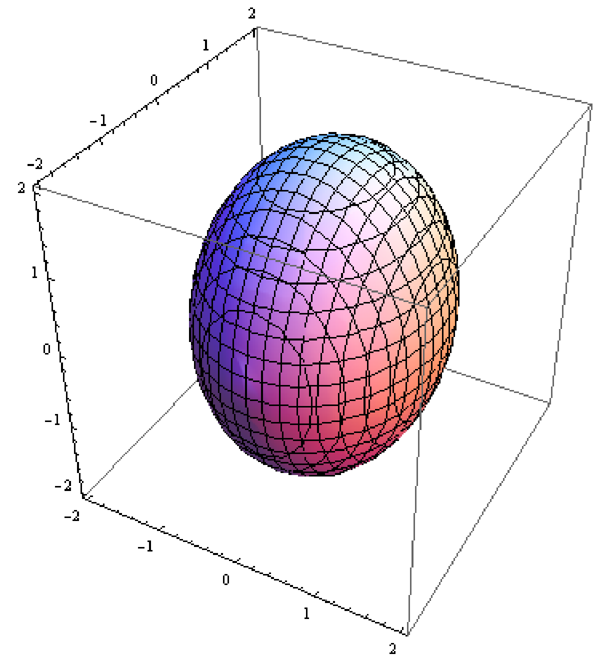
$$P = 2\cdot\int_0^{\pi} d\varphi \int_0^{1+\cos\varphi} r\ dr = 2\cdot\int_0^{\pi} d\varphi \cdot \frac{r^2}{2}\Big|_0^{1+\cos\varphi} = \int_0^{\pi} (1+\cos\varphi)^2\ d\varphi = \int_0^{\pi} (1+2\cos\varphi+\cos^2\varphi)\ d\varphi$$
$$= \int_0^{\pi} d\varphi + 2\int_0^{\pi}\cos\varphi\ d\varphi + \int_0^{\pi}\cos^2\varphi\ d\varphi = \pi + I_1$$
$$I_1 = \int_0^{\pi}\cos^2\varphi\ d\varphi = \int_0^{\pi}\frac{1+\cos 2\varphi}{2}\ d\varphi = \frac{1}{2}\int_0^{\pi} d\varphi + \int_0^{\pi}\cos 2\varphi\ d\varphi = \frac{\pi}{2}$$

19. **Izračunajte volumen tijela omedjenog plohom**

$$\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{4} = 1$$

Rješenje:

Sve eventualne pogreške molim da javite isključivo privatnom porukom.



$$\frac{z^2}{4} = 1 - \frac{x^2}{2} - \frac{y^2}{3}$$

$$z^2 = 4\left(1 - \frac{x^2}{2} - \frac{y^2}{3}\right)$$

$$z = 2\sqrt{1 - \frac{x^2}{2} - \frac{y^2}{3}} = 2\sqrt{1 - \left(\frac{x^2}{2} + \frac{y^2}{3}\right)} = 2\sqrt{1 - r^2}$$

$$V = \int_0^{2\pi} d\varphi \int_0^1 2\sqrt{6} r\sqrt{1-r^2}\ dr = 4\sqrt{6}\int_0^{2\pi} d\varphi \int_0^1 r\sqrt{1-r^2}\ dr = \begin{vmatrix} r = \sin x \\ dr = \cos\ dx \end{vmatrix} =$$

$$4\sqrt{6}\int_0^{2\pi} d\varphi \int_0^1 \sin x\sqrt{1-\sin^2 x}\cos x\ dx = 4\sqrt{6}\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin x \cdot \cos^2 x\ dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin x \cdot \cos^2 x\ dx = \int_0^{\frac{\pi}{2}} \sin x \cdot (1 - \sin^2 x)\ dx = \int_0^{\frac{\pi}{2}} \sin x - \sin^3 x\ dx =$$

$$= \int_0^{\frac{\pi}{2}} \sin x - \int_0^{\frac{\pi}{2}} \sin^3 x\ dx = -\cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin^3 x\ dx = 1 - \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\sin x\ dx =$$

$$= 1 - \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin x - \frac{1}{2}\sin x\cos x\right)\ dx = 1 - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\ dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\cos 2x\ dx =$$

$$= 1 + \frac{1}{2}\cos x\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\int_0^{\frac{\pi}{2}} (\sin x - \sin 3x)\ dx = \frac{1}{2} + \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin 3x\ dx - \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin x\ dx =$$

$$= \frac{1}{2} - \frac{1}{12}\cos 3x\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\cos x\Big|_0^{\frac{\pi}{2}} = \frac{1}{2} + \frac{1}{12} - \frac{1}{4} = \frac{1}{3}$$

$$V = \frac{2\sqrt{6}}{3}\int_0^{2\pi} d\varphi = \frac{4\sqrt{6}}{3}\cdot 2\pi = \frac{8\sqrt{6}}{3}\pi$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.

20. **Izračunajte volumen tijela omedjenog sferom $x^2+y^2+z^2=R^2$ i paralelnim ravninama $z=a$ i $z=b$, $0\le a<b<R$.**

Rješenje:

$$x^2+y^2+z^2=R^2,\ z=a,\ z=b$$
$$z=\pm\sqrt{R^2-x^2-y^2}$$

$$V=\int_0^{2\pi}d\varphi\int_0^{\sqrt{R^2-a^2}}r(\sqrt{R^2-r^2}-a)dr-\int_0^{2\pi}d\varphi\int_0^{\sqrt{R^2-b^2}}r(\sqrt{R^2-r^2}-b)dr=$$

Zamjena:

$$\begin{vmatrix} u=R^2-r^2 \\ du=-2r\ dr \\ dr=-\frac{du}{2r}\end{vmatrix}=\int-\frac{r}{2r}\sqrt{u}\ du-a\frac{r^2}{2}=-\frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}-\frac{a}{2}r^2=-\frac{1}{3}\sqrt{(R^2-r^2)^3}-\frac{a}{2}r^2$$

Nastavak:

$$=\int_0^{2\pi}\left[-\frac{1}{3}\sqrt{(R^2-r^2)^3}\Big|_0^{\sqrt{R^2-a^2}}-\frac{a}{2}r^2\Big|_0^{\sqrt{R^2-a^2}}\right]d\varphi-\left[-\frac{1}{3}\sqrt{(R^2-r^2)^3}\Big|_0^{\sqrt{R^2-b^2}}-\frac{b}{2}r^2\Big|_0^{\sqrt{R^2-b^2}}\right]d\varphi=$$
$$=\int_0^{2\pi}\left[-\frac{1}{3}\sqrt{(R^2-R^2+a^2)^3}+\frac{1}{3}\sqrt{R^6}-\frac{a}{2}(R^2-a^2)\right]d\varphi-\int_0^{2\pi}\left[-\frac{1}{3}\sqrt{(R^2-R^2+b^2)^3}+\frac{1}{3}\sqrt{R^6}-\frac{a}{2}(R^2-b^2)\right]d\varphi=$$
$$\int_0^{2\pi}\left[-\frac{1}{3}a^3+\frac{1}{3}R^3-\frac{a}{2}(R^2-a^2)\right]d\varphi-\int_0^{2\pi}\left[-\frac{1}{3}b^3+\frac{1}{3}R^3-\frac{b}{2}(R^2-b^2)\right]d\varphi=$$
$$\int_0^{2\pi}\left[-\frac{1}{3}a^3+\frac{1}{3}b^3-\frac{a}{2}(R^2-a^2)+\frac{b}{2}(R^2-b^2)\right]d\varphi=\int_0^{2\pi}\left[-\frac{a^3}{3}+\frac{b^3}{3}-\frac{a\cdot R^2}{2}+\frac{a^3}{3}+\frac{b\cdot R^2}{2}-\frac{b^3}{3}\right]d\varphi=$$
$$\int_0^{2\pi}\left[\frac{a^3}{6}-\frac{b^3}{6}+\frac{R^2}{2}(b-a)\right]d\varphi=2\pi\left[\frac{a^3}{6}-\frac{b^3}{6}+\frac{R^2}{2}(b-a)\right]=\frac{\pi}{3}(a^3-b^3)+R^2\pi(b-a)$$

Sve eventualne pogreške molim da javite isključivo privatnom porukom.