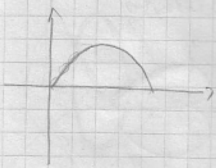
MASS - 3. cikljus (by Bürić) a.k.a. Britni

krivuljni integral - računanje duljine luka krivulje



I vrste

$$\int_\Gamma f(x,y,z)\,ds$$ – diferencijal luka

$$ds = \sqrt{(x')^2+(y')^2+(z')^2}\,dt$$

$$\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$$ - parametrizirati krivulju

polarna parametrizacija

$$ds = \sqrt{r^2+(r')^2}\,d\varphi$$ - samo u 2D se može koristit

ZZV 5) $\int_\Gamma \frac{ds}{x^2+y^2+z^2}$, $\Gamma \ldots \begin{cases} x=a\cos t \\ y=a\sin t \\ z=bt \end{cases}$, $t\in(0,2\pi)$

$$x' = -a\sin t$$

$$y' = a\cos t$$

$$z' = b$$

$$ds = \sqrt{a^2\sin^2 + a^2\cos^2 + b^2}\,dt$$

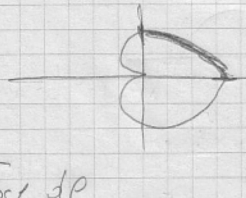
$$ds = \sqrt{a^2+b^2}\,dt$$

$$I = \int_0^{2\pi}\frac{\sqrt{a^2+b^2}\,dt}{a^2\cos^2 t + a^2\sin^2 t + b^2t^2} = \int_0^{2\pi}\frac{\sqrt{a^2+b^2}\,dt}{a^2+b^2t^2}\,dt = \sqrt{a^2+b^2}\int_0^{2\pi}\frac{dt}{a^2+b^2t^2}$$

$$= \frac{\sqrt{a^2+b^2}}{b^2}\int_0^{2\pi}\frac{dt}{(\frac{a}{b})^2+t^2} = \frac{\sqrt{a^2+b^2}}{b^2}\cdot\frac{b}{a}\,\mathrm{arctg}\,\frac{b}{a}t\Big|_0^{2\pi}$$

$= \frac{\sqrt{a^2+b^2}}{b^2} \operatorname{arctg} \frac{2\pi b}{a}$

2I 2008 - 2) $\int_\Gamma y\,ds$, $\Gamma \ldots r = 1+\cos\varphi$ u I. kvadrantu



$ds = \sqrt{r^2+(r')^2}\,d\varphi$

$r' = -\sin\varphi$

$ds = \sqrt{1+2\cos\varphi+\cos^2\varphi+\sin^2\varphi}\,d\varphi = \sqrt{2+2\cos\varphi}\,d\varphi$

$x = r\cos\varphi$

$\boxed{y = r\sin\varphi = (1+\cos\varphi)\sin\varphi}$

$I = \int_0^{\frac{\pi}{2}} (1+\cos\varphi)\sin\varphi \cdot \sqrt{2}\sqrt{1+\cos\varphi}\,d\varphi = \left| \begin{array}{l} 1+\cos\varphi = t \\ -\sin\varphi d\varphi = dt \end{array} \right|$

$= \int_2^1 t\sqrt{t}(-\sqrt{2})\,dt = \ldots = \frac{16}{5} - \frac{2\sqrt{2}}{5}$

PZI - 2009 - 3) Izračunaj duljinu luka krivulje koja je presječnica ploha $x^2 = 3y$ $2xy = 9z$ između točaka $A(0, y_A, z_A)$ i $B(3, y_B, z_B)$

parametrizacija po x

$\cancel{y = t \quad x = \sqrt{3t}}$

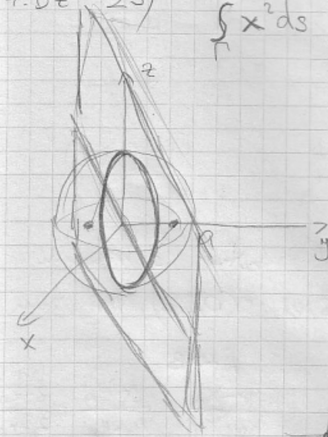
$\begin{cases} x = t & x' = 1 \\ y = \frac{1}{3}t^2 & y' = \frac{2}{3}t \\ z = \frac{2}{27}t^3 & z' = \frac{2}{9}t^2 \end{cases}$ $ds = \sqrt{1+\frac{4}{9}t^2+\frac{4}{81}t^4}$

$l = \int_\Gamma ds = \int_0^3 \frac{1}{9}\sqrt{81+36t^2+4t^4}\,dt = \frac{1}{9}\int_0^3 \sqrt{(2t^2+9)^2}\,dt$

$= \frac{1}{9}\int_0^3 (2t^2+3)\,dt = 5$

7. DZ 25)

$$\int_\Gamma x^2\,ds \quad , \quad \Gamma \dots x^2+y^2+z^2=a^2, \; x-y=0$$

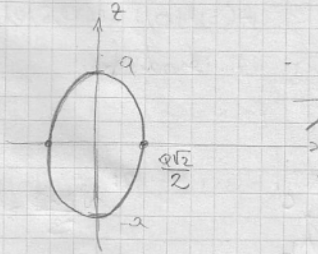


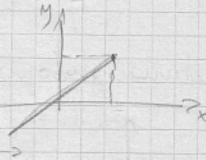
moguća parametrizacija

$x=t$

$y=t$

$z=\pm\sqrt{a^2-2t^2}$





$$2y^2+z^2=a^2 \quad /:a^2$$

$$\frac{y^2}{\frac{a^2}{2}}+\frac{z^2}{a^2}=1$$

$\frac{a\sqrt{2}}{2}$ $\quad a$

eliptičke koordinate

$$\begin{cases} y=\frac{a}{\sqrt{2}}\, r\cos\varphi = \frac{a}{\sqrt{2}}\cos\varphi \quad r=1 \\ z=a r\sin\varphi = a\sin\varphi \\ x=\frac{a}{\sqrt{2}}\cos\varphi \quad (x=y) \end{cases}$$

$$I=\int_0^{2\pi}\frac{a^2}{2}\cos^2\varphi\cdot\sqrt{\underbrace{\frac{a^2}{2}\sin^2\varphi+\frac{a^2}{2}\sin^2\varphi+a^2\cos^2\varphi}_{a^2}}\,d\varphi=\frac{a^3}{2}\int_0^{2\pi}\cos^2\varphi\,d\varphi$$

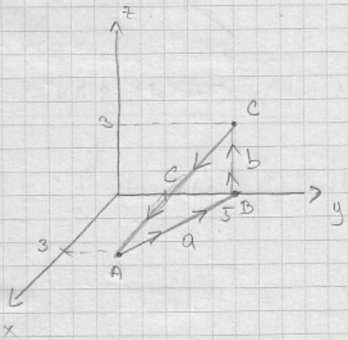
$$=\frac{a^3}{2}\int_0^{2\pi}\frac{1+\cos2\varphi}{2}\,d\varphi=\frac{a^3\pi}{2}$$

II vrste

- bitna orjentacija

$$\int_{\Gamma} \vec{a} \cdot d\vec{r} = \int_{\Gamma} a_1 dx + a_2 dy + a_3 dz$$

P21 - 2007 - 6) $\oint_{\Gamma} yz\,dx + 4x\,dz$, Γ... rub trokuta s vrhovima $A(3,2,0)$ $B(0,5,0)$ $C(0,5,3)$ orjentiran negativno gledajući iz ishodišta



$A \to B$

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}$$

a: $$\frac{x-3}{-3} = \frac{y-2}{3} = \frac{z}{0} = t$$

$$\begin{cases} x = -3t + 3 \to dx = -3dt \\ y = 3t + 2 \to dy = 3dt \\ z = 0 \to dz = 0 \end{cases}$$

a: $$\int_0^1 (3t+2) \cdot 0 \cdot (-3dt) + 4(-3t+3) \cdot 0 = 0$$

b: $B \to C$

$$\frac{x}{0} = \frac{y-5}{0} = \frac{z}{3} = t$$

$$\begin{cases} x = 0 \to dx = 0 \\ y = 5 \to dy = 0 \\ z = 3t \to dz = 3dt \end{cases}$$

b: $$\int_0^1 0 + 0 = 0$$

C: $C \to A$

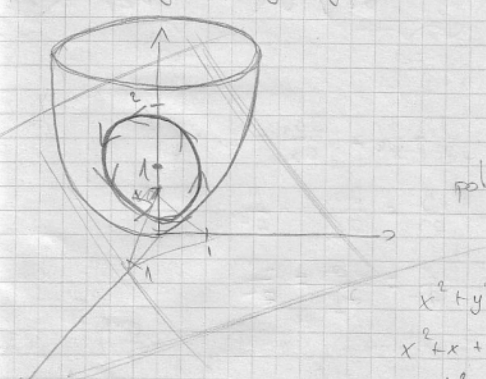
$$\frac{x}{3} = \frac{y-5}{-3} = \frac{z-3}{-3} = t$$

$$\begin{cases} x = 3t & dx = 3dt \\ y = -3t+5 & dy = -3dt \\ z = -3t+3 & dz = -3dt \end{cases}$$

C: $\int_0^1 (-3t+5)(-3t+3)\,3dt + 4\cdot 3t(-3dt) = \dots = 0$

7.DZ - 5a) $\int_\Gamma (2-z)\,dy$, $\Gamma \dots z = x^2+y^2$; $z = 1-x-y$

pozitivno orijentiran gledajući iz $(0,0,2)$. Nacrtaj sliku



$z = 1-x-y$

$x+y+z=1$

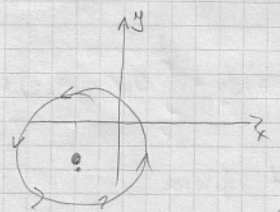
polarne koordinate

$$x^2+y^2 = 1-x-y$$

$$x^2+x+y^2+y = 1$$

$$\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{3}{2}$$

$r = \sqrt{\frac{3}{2}}$



$$x+\frac{1}{2} = r\cos\varphi = \sqrt{\frac{3}{2}}\cos\varphi$$

$$y+\frac{1}{2} = r\sin\varphi = \sqrt{\frac{3}{2}}\sin\varphi \Rightarrow dy = \sqrt{\frac{3}{2}}\cos\varphi\,d\varphi$$

$$z = 1 - \left(\sqrt{\frac{3}{2}}\cos\varphi - \frac{1}{2}\right) - \left(\sqrt{\frac{3}{2}}\sin\varphi - \frac{1}{2}\right)$$

$$z = 2 - \sqrt{\frac{3}{2}}(\cos\varphi + \sin\varphi)$$

$$I = \int_0^{2\pi} \left(\sqrt{\frac{3}{2}}(\cos\varphi+\sin\varphi)\right)\cdot\sqrt{\frac{3}{2}}\cos\varphi\,d\varphi = \frac{3}{2}\int_0^{2\pi}\left(\frac{1+\cos 2\varphi}{2} + \frac{1}{2}\sin 2\varphi\right)d\varphi =$$

$$= \frac{3\pi}{2}$$

Potencijalna polja

PZI - 2009 - 4)

a) Def. pot. polje

b) Iskazi nužan i dovoljan uvjet da polje bude potencijalno

$$\operatorname{rot} \vec{a} = \vec{0}$$

Provjeri da je polje pot. i da je izračunaj potencijal

$$\vec{a} = \left(\frac{y}{z} + \frac{y}{x^2}\right)\vec{i} + \left(1 - \frac{1}{x} + \frac{x}{z}\right)\vec{j} - \frac{xy}{z^2}\vec{k}$$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = i\underbrace{\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)}_{=0} - j\underbrace{\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right)}_{=0} + \vec{k}\underbrace{\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)}_{=0}$$

$$\frac{\partial a_3}{\partial y} = \frac{\partial a_2}{\partial z}$$

$$\frac{\partial a_3}{\partial x} = \frac{\partial a_1}{\partial z}$$

$$\frac{\partial a_1}{\partial y} = \frac{\partial a_2}{\partial x}$$

$$\frac{\partial a_1}{\partial y} = \frac{1}{z} + \frac{1}{x^2} = \frac{\partial a_2}{\partial x} = \frac{1}{x^2} + \frac{1}{z}$$

$$\frac{\partial a_1}{\partial z} = -\frac{y}{z^2} = \frac{\partial a_3}{\partial x} = -\frac{y}{z^2}$$

$$\frac{\partial a_2}{\partial z} = -\frac{x}{z^2} = \frac{\partial a_3}{\partial y} = \frac{-x}{z^2}$$

$\Rightarrow$ polje je potencijalno

Izračunaj potencijal

$$p(x,y,z) = \int_{x_0}^{x} a_1(x,y,z)\,dx + \int_{y_0}^{y} a_2(x_0,y,z)\,dy + \int_{z_0}^{z} a_3(x_0,y_0,z)\,dz$$

x_0, y_0, z_0 - po volji odabrana točka u kojoj je f definirana

$$p(x,y,z) = \int_1^x \left(\frac{y}{z} + \frac{y}{x^2}\right)dx + \int_0^y \left(1 - \frac{1}{1} + \frac{1}{z}\right)dy + \int_1^z \frac{1 \cdot 0}{z}\,dz =$$

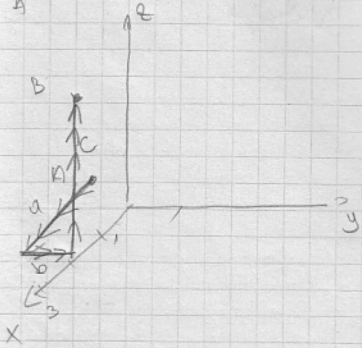
$$= \left(\frac{yx}{z} - \frac{y}{x}\right)\Big|_1^x + \frac{y}{z} = \frac{xy}{z} - \frac{y}{x} + y + C$$

Izračun $\int \vec{a}\, d\vec{r}$

$$\int_{(1,0,1)}^{(3,1,4)} \vec{a}\, d\vec{r} = ?$$

integral po polju ne ovisi o putu integracije

$$\int_A^B \left(\frac{y}{z} + \frac{y}{x^2}\right) dx + \left(1 - \frac{1}{x} + \frac{x}{z}\right) dy - \frac{xy}{z^2}\, dz$$



a: $x = t \quad dx = dt$

$y = 0 \quad dy = 0$

$z = 1 \quad dz = 0$

b: $y = t \quad dy = dt$

$x = 3 \quad dx = 0$

$z = 1 \quad dz = 0$

c: $z = t \quad dz = dt$

$x = 3 \quad dx = 0$

$y = 1 \quad dy = 0$

$$I = \int_1^3 \cancel{\left(\frac{0}{1} + \frac{0}{t^2}\right)} dt + \int_0^1 \left(1 - \frac{1}{3} + \frac{3}{1}\right) dt + \int_1^4 -\frac{3 \cdot 1}{t^2}\, dt =$$

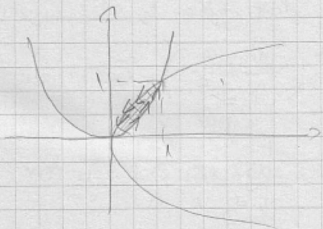
$$= \ldots = \frac{17}{12}$$

2I - 2008 - 3) a) Iskaži Green-ov teorem

$$\oint_{\Gamma} P dx + Q dy = \iint_{\Omega_{xy}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

b) $\oint_{C} x^2 y\,dx + (y + xy^2)\,dy \qquad C \ldots y = x^2,\ x = y^2$

pozitivno orijentacije

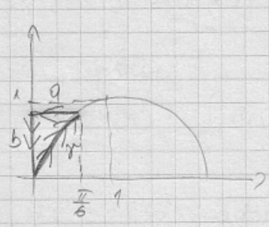


$$= \iint_D (y^2 - x^2)\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (y^2 - x^2)\,dy$$

2ZV - 22

$$\int_{\gamma} (y + \sin x \sin y)\,dx - \cos x \cos y\,dy$$

γ - luk krivulje $y = \sin x$, od $x = 0$ do $x = \frac{\pi}{6}$



$$\int_{\gamma} + \int_a + \int_b = \oint \overset{\text{Green}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_{\gamma} = \iint_D - \int_a - \int_b$$

a: $y = \frac{1}{2} \Rightarrow dy = 0$
$x = t \Rightarrow dx = dt$

$$\int_{\frac{\pi}{6}}^{0} \left(\frac{1}{2} + \sin t \sin\frac{1}{2}\right) dt = -\frac{\pi}{12} + \sin\frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$$

b: $x = 0 \Rightarrow dx = 0$
$y = t \Rightarrow dy = dt$

$$-\int_{\frac{1}{2}}^{0} \cos t\,dt = \sin\frac{1}{2}$$

$$\iint (\sin x \cos y - 1 - \sin x \cos y)\,dx\,dy = -\int_0^{\frac{\pi}{6}} dx \int_{\sin x}^{\frac{1}{2}} dy = -\frac{\pi}{12} - \frac{\sqrt{3}}{2} + 1$$

$$\int_{\gamma} = \iint_D - \int_a - \int_b = 1 - \frac{\sqrt{3}}{2}\left(1 + \sin\frac{1}{2}\right)$$