1. MI 2011/2012

1. a) Kažemo da f zadovoljava Dirichletove uvjete na intervalu $[a,b]$, ako vrijedi

1) Funkcija f je po dijelovima neprekinuta i njezini su prekidi prve vrste

2) f je monotona ili ima najviše konačan broj strogih ekstrema

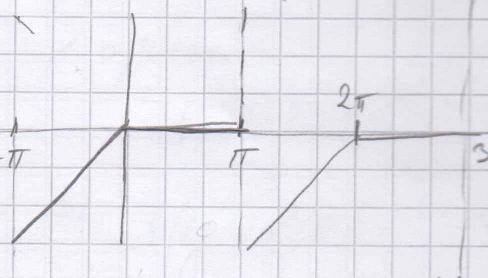
b) Neka je f po dijelovima glatka periodična f-ja s periodom 2π koja zadovoljava Dirichletove uvjete. Tada njezin Fourierov red konvergira u svakoj točki $x \in [-\pi, \pi]$ i za sumu $S(x)$ tada vrijedi:

(i) $S(x) = f(x)$, ako je f neprekinuta u točki x

(ii) $S(x) = \frac{1}{2}\left(f(x-0) + f(x+0)\right)$, ako je x točka prekida za f

c)

$$f(x) = \begin{cases} x, & x \in (-\pi, 0) \\ 0, & x \in [0, \pi) \end{cases}$$

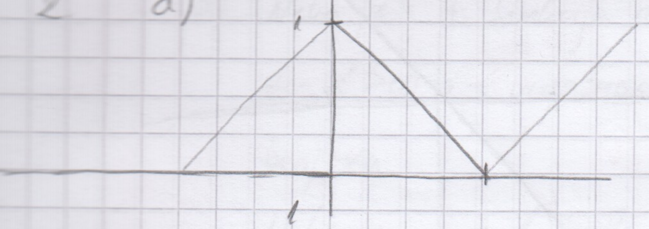


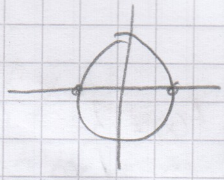
ni parna ni neparna

$2011 \cdot \pi \rightarrow$ točka prekida

$= S(\pi) = \frac{1}{2}(0 - \pi) = -\frac{\pi}{2}$

2 a)





$\langle 0,1]$

$L=1$

$$a_0 = \frac{2}{1}\int_0^1 (1-x)\,dx = 2\left(x-\frac{x^2}{2}\right)\Big|_0^1 = (2x-x^2)\Big|_0^1 = 1$$

$u=x \quad dv=\cos(n\pi x)$

$du=dx \quad v=\frac{1}{n\pi}\sin(n\pi x)$

$$a_n = 2\int_0^1 (1-x)\cos(n\pi x)\,dx = 2\frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 - 2\int_0^1 x\cos(n\pi x)\,dx =$$

$$= \overbrace{\frac{2}{n\pi}\sin(n\pi)}^{0} - 2\left[\cancel{\frac{x}{n\pi}\sin(n\pi x)\Big|_0^1} - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx\right] =$$

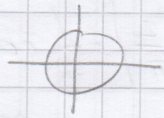
$$= \frac{-2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 = \frac{-2}{n^2\pi^2}\cos(n\pi) + \frac{2}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left(1-\cos(n\pi)\right)$$

$$= \frac{2(1-\cos(n\pi))}{n^2\pi^2}$$

parni n-ovi

$2n \Rightarrow \cos(n\pi) = 1$

$\Rightarrow (1-\cos(n\pi)) = 0$



$2n+1 \Rightarrow \cos[(2n+1)\pi] = -1$

$$= \frac{4}{(2n+1)^2\pi^2} = a_n$$

$$S(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2\pi^2}\cos((2n+1)\pi x) \quad //$$

b)

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 = \frac{2}{T}\int_a^b |f(x)|^2\,dx$$

$f(x) = 1-x$

c)

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$$

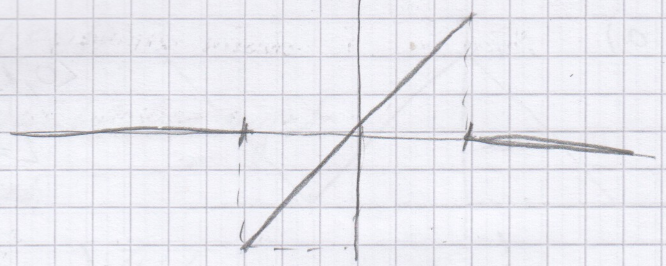
$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{16}{(2n+1)^4\pi^4} = 2\int_0^1 (1-x)^2\,dx = 2\left(\frac{x^3}{3} - x^2 + x\right)\Big|_0^1$$

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{16}{(2n+1)^4\pi^4} = \frac{2}{3}\cancel{-2+2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} = \left(\frac{2}{3} - \frac{1}{2}\right)\frac{\pi^4}{16} = \frac{4-3}{6}\cdot\frac{\pi^4}{16} = \frac{\pi^4}{96} \quad //$$

3.

$$f(x) = \begin{cases} x, & -1 \le x \le 1 \\ 0, & \text{inače} \end{cases}$$



a) Fourierov integral

neparna

$$B(\lambda) = \frac{2}{\pi}\int_0^1 x \sin\lambda x \, dx = \left| \begin{matrix} u = x & dv = \sin\lambda x \, dx \\ du = dx & v = -\frac{1}{\lambda}\cos\lambda x \end{matrix} \right| =$$

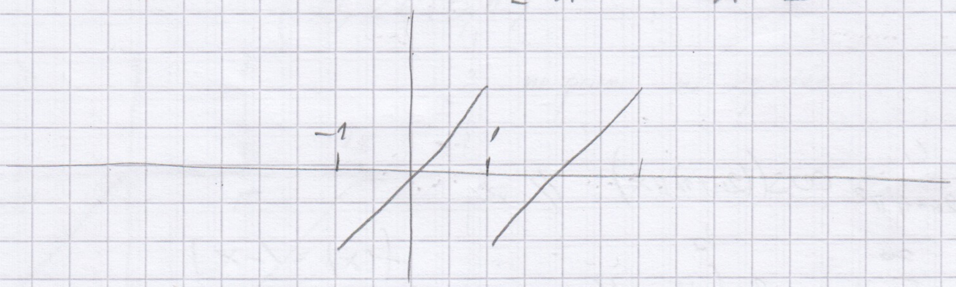
$$= \frac{2}{\pi}\left[-\frac{x}{\lambda}\cos\lambda x \Big|_0^1 + \frac{1}{\lambda}\int_0^1 \cos\lambda x \, dx \right] =$$

$$= \frac{2}{\pi}\left[-\frac{1}{\lambda}\cos\lambda + \frac{1}{\lambda^2}\sin\lambda x \Big|_0^1 \right] =$$

$$= \frac{2}{\pi}\left[-\frac{1}{\lambda}\cos\lambda + \frac{1}{\lambda^2}\sin\lambda \right]$$

$$\Rightarrow f(x) = \int_0^\infty \frac{2}{\pi}\left[\frac{\sin\lambda}{\lambda^2} - \frac{\cos\lambda}{\lambda} \right] \sin(\lambda x) \, d\lambda$$

$$am(\lambda) = |B(\lambda)| = \frac{2}{\pi}\left[\frac{\sin\lambda}{\lambda^2} - \frac{\cos\lambda}{\lambda} \right]$$



4. a) Neka je f funkcija realnog argumenta t, definirana za $t>0$ i s vrijednostima u skupu realnih ili kompleksnih brojeva

Neka je s realni ili kompleksni broj. Laplaceov transformat funkcije f je funkcija F definirana s

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt$$

za svaki s za koji ovaj nepravi integral konvergira.

b) $\mathcal{L}(e^{\alpha t}),\ \alpha \in \mathbb{R}$

$$f(t) = e^{\alpha t}$$

$$F(s) = \int_0^\infty e^{-st} e^{\alpha t}\,dt = \int_0^\infty e^{(\alpha - s)t}\,dt =$$

$$= \left.\frac{e^{(\alpha - s)t}}{\alpha - s}\right|_0^\infty = \frac{1}{s-\alpha}, \quad s > \alpha$$

c) $sh(\omega t)$

$ch(\omega t)$

$$sh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2} = \frac{1}{2} e^{\omega t} - \frac{1}{2} e^{-\omega t} \circ\!\!-\!\!\bullet$$

$$\circ\!\!-\!\!\bullet\ \frac{1}{2}\frac{1}{s-\omega} - \frac{1}{2}\frac{1}{s+\omega} = \frac{1}{\cancel{2}}\frac{\cancel{2}\omega}{s^2-\omega^2} = \frac{\omega}{s^2-\omega^2}$$

$$ch(\omega t) = \frac{e^{\omega t} + e^{-\omega t}}{2} = \frac{1}{2} e^{\omega t} + \frac{1}{2} e^{\omega t} \circ\!\!-\!\!\bullet$$

$$\circ\!\!-\!\!\bullet\ \frac{1}{2}\frac{1}{s-\omega} + \frac{1}{2}\frac{1}{s+\omega} = \frac{1}{\cancel{2}}\frac{\cancel{2}s}{s^2-\omega^2} = \frac{s}{s^2-\omega^2}$$

5. a) Konvolucija originala f_1 i f_2 definirana je integralom

$$(f_1 * f_2)(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

b) Konvoluciji u gornjem području odgovara umnožak slike u donjem području

$$(f_1 * f_2)(t) ○\!\!-\!\!\bullet F_1(s) \cdot F_2(s)$$

c) $y''(t) + 2y'(t) + 5y(t) = f(t)$, $y(0) = 1$ $y'(0) = -2$

$y ○\!\!-\!\!\bullet Y(s)$

$y' ○\!\!-\!\!\bullet sY(s) - y(0) = sY(s) - 1$

$y'' ○\!\!-\!\!\bullet s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 2$

$f(t) ○\!\!-\!\!\bullet F(s)$

$s^2Y(s) - s + \cancel{2} + 2sY(s) - \cancel{2} + 5Y(s) = F(s)$

$Y(s)(s^2 + 2s + 5) = F(s) + s$

$Y(s) = \underbrace{F(s) \cdot \frac{1}{(s+1)^2 + 4}}_{\text{konvolucija}} + \frac{s}{(s+1)^2 + 4}$

$\cancel{Y(s)} =$

$y(t) = f(t) * \frac{e^{-t}}{2} \sin t + \cos 2t e^{-t}$

$y(t) = \frac{1}{2}\left(f(t) * e^{-t} \sin t\right) + e^{-t} \cos 2t$

6. $F(s) = \dfrac{e^{-2s}}{s^3+2s^2+2s+1}$

$-1 \Rightarrow$ nultočka

$s^3+2s^2+2s+1 : s+1 = s^2+s+1$

$\underline{-s^3-s^2}$

s^2+2s+1

$\underline{-s^2-s}$

$s+1$

$$F(s) = \frac{e^{-2s}}{(s^2+s+1)(s+1)} = \frac{As+B}{s^2+s+1} + \frac{C}{s+1}$$

$(As+B)(s+1) + Cs^2+Cs+C = 1$

$As^2+As+Bs+B+Cs^2+Cs+C = 1$

$s^2(A+C)+s(A+B+C)+B+C = 1$

$A+C = 0 \quad \Rightarrow A = -1$

$A+B+C = 0 \quad \Rightarrow B = 0$

$B+C = 1 \quad \Rightarrow C = 1$ $\qquad$ ~~-1 ± A~~

pomak

$$F(s) = e^{-2s}\left[\frac{-s}{\underbrace{s^2+s+1}_{(s+\frac{1}{2})^2+\frac{3}{4}}} + \frac{1}{s+1}\right] = e^{-2s}\left[-\frac{s}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{s+1}\right] =$$

$$= e^{-2s}\left[-\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{s+1}\right] \bullet\!\!-\!\!\circ$$

$$\circ\!\!-\!\!\bullet \; -e^{-\frac{1}{2}(t-2)}\cos\left[\frac{\sqrt{3}}{2}(t-2)\right]u(t-2) + \frac{e^{-\frac{1}{2}(t-2)}}{\sqrt{3}}\sin\left[\frac{\sqrt{3}}{2}(t-2)\right]u(t-2)$$

$$+ e^{-(t-2)}u(t-2)$$

7. a) Kažemo da je skup A EKVIPOTENTAN (jednakobrojan) za skupom B ako postoji bijekcija $f: A \to B$

b) Za beskonačan skup A kažemo da je prebrojiv ako se skup njegovih elemenata može poredati u beskonačan slijed
$A = \{a_1, a_2, a_3, \dots\}$

~~d)~~ Skup svih konačnih slijedova je prebrojiv

c) $\mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3$

$p_1 = 2 \quad p_2 = 3 \quad p_3 = 5 \quad p_4 = 7 \quad p_5 = 11$

$f: A \to \mathbb{N}$

$f(n_1, \dots, n_k) = p_1^{n_1} \cdots p_k^{n_k}$

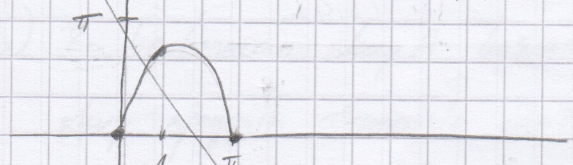
<u>Injekcija</u> $f(n_1, \dots, n_k) = f(m_1, \dots, m_j)$

$\Leftrightarrow \; p_1^{n_1} \cdots p_k^{n_k} = p_1^{m_1} \cdots p_j^{m_j}$ $\Rightarrow$ jednakost brojeva koji su rastavljeni na proste faktore, prema Osnovnom teoremu aritmetike slijedi da je $k = j$ i $n_i = m_i$ za sve $i = 1, \dots, k$

Slika preslikavanja f je očevidno beskonačan skup (već za „jednočlane" slijedove n je skup pripadnih vrijednosti oblika $f(n) = 2^n$, dakle beskonačni skup

ZI 15. 2. 2012

1. a) $f(x) = x(\pi - x)$ $[0, \pi]$



$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi-x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi (x\pi - x^2)\sin(nx)\,dx =$$

$$= 2\int_0^\pi x\sin(nx)\,dx - \frac{2}{\pi}\int_0^\pi x^2\sin(nx)\,dx = \left| \begin{array}{ll} u = x \,/\, x^2 & dv = \sin nx\,dx \\ du = dx \,/\, x\,dx & v = -\frac{1}{n}\cos nx \end{array} \right|$$

$$= 2\left[-\frac{x}{n}\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right] - \frac{2}{\pi}\left[-\frac{1}{n}x^2\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi x\cos nx\,dx\right] =$$

$$= 2\left[-\frac{\pi}{n}\cos n\pi + \cancel{\frac{1}{n^2}\sin nx\Big|_0^\pi}\right] - \frac{2}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{1}{n}\left[\cancel{-\frac{x}{n}\sin nx\Big|_0^\pi} - \frac{1}{n}\int_0^\pi \sin nx\,dx\right.\right.$$

$$= \cancel{-\frac{2\pi}{n}\cos n\pi} + \cancel{\frac{2\pi}{n}\cos n\pi} + \frac{1}{n^3}\cos(nx)\Big|_0^\pi =$$

$$= \frac{\cos(n\pi) - 1}{n^3}$$

$2n+1$

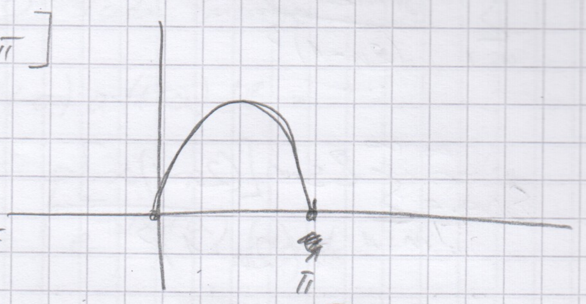
$$= \frac{2}{(2n-1)^3}$$

$x = 1$

$$\sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}\sin(2n-1) = \pi - 1$$

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)}{(2n-1)^3} = \frac{\pi-1}{2}$$

1. a) $f(x)=x(\pi-x)$ $\quad [0,\pi]$



$$b_n=\frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin(nx)\,dx=$$

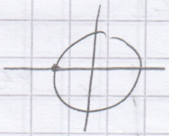
$$=\frac{2}{\pi}\int_0^{\pi}(x\pi-x^2)\sin(nx)\,dx=\frac{2}{\pi}\left[\pi\int_0^{\pi}x\sin(nx)dx-\int_0^{\pi}x^2\sin(nx)dx\right]=$$

$$\left|\begin{array}{ll} u=x/x^2 & dv=\sin(nx)dx \\ du=dx/2xdx & v=-\frac{1}{n}\cos(nx)\end{array}\right|=$$

$$=\frac{2}{\pi}\left[-\frac{x\pi}{n}\cos(nx)\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos(nx)dx-\left(-\frac{x^2}{n}\cos(nx)\Big|_0^{\pi}+\frac{2}{n}\int_0^{\pi}x\cos(nx)dx\right)\right]$$

$$=\frac{2}{\pi}\left[\cancel{-\frac{\pi^2}{n}\cos(n\pi)}+\cancel{\frac{1}{n^2}\sin(nx)\Big|_0^{\pi}}+\cancel{\frac{\pi^2}{n}\cos(n\pi)}-\frac{2}{n}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi}-\frac{1}{n}\int_0^{\pi}\sin(nx)dx\right]\right]$$

$$=\frac{2}{\pi}\left[\frac{-2}{n}\left(\cancel{\frac{\pi}{n}\sin(n\pi)}+\frac{1}{n^2}\cos(n\pi)-\frac{1}{n^2}\right)\right]=$$



$$=\frac{2}{\pi}\left(\frac{-2\cos(n\pi)+2}{n^3}\right)=\frac{4(1-\cos(n\pi))}{\pi n^3}$$

$\boxed{2n+1}$ $\quad \frac{8}{\pi(2n+1)^3}=b_{2n+1}$

$$S(x)=\sum_{n=0}^{\infty}\frac{8}{\pi(2n+1)^3}\sin[(2n+1)x]$$

b) $\sum\limits_{n=1}^{\infty} \frac{\sin(2n-1)}{(2n-1)^3}$

$$S(x) = \sum_{n=1}^{\infty} \frac{8\sin[(2n-1)x]}{\pi(2n-1)^3}$$

$$x=1 \qquad \sum_{n=1}^{\infty} \frac{8\sin(2n-1)}{\pi(2n-1)^3} = \pi - 1$$

$$f(1) = \cancel{x}(\pi - x) = 1(\pi-1)$$

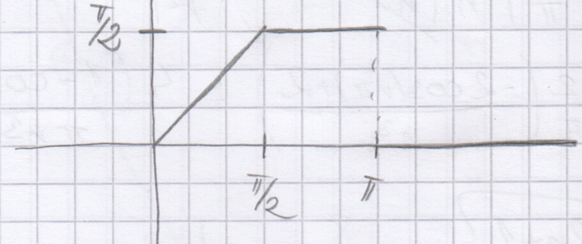
$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)}{(2n-1)^3} = \frac{(\pi-1)\pi}{8}$$

2. a) Ako je $f: \mathbb{R} \to \mathbb{R}$ po dijelovima glatka na svakom konačnom intervalu i apsolutno integrabilna, tada postoji njezin Fourierov integral i vrijedi

$$\frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi)\cos\lambda(x-\xi)\, d\xi$$

$$= \begin{cases} f(x) & \text{, ako je } f \text{ neprekinuta u } x \\ \frac{1}{2}\left[f(x-0)+f(x+0)\right] & \text{, ako je } x \text{ točka prekida za } f \end{cases}$$

b)
$$f(x) = \begin{cases} x, & x \in [0, \frac{\pi}{2}] \\ \pi/2, & x \in [\pi/2, \pi] \\ 0, & x > \pi \end{cases}$$



parno proširenje $\Rightarrow$ kosinusi

$$\frac{2}{\pi}\int_0^{\infty} f(x)\cos\lambda x\, dx \Rightarrow \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos(\lambda x)dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\cos(\lambda x)dx\right] = \left| \begin{matrix} u = x & d= \\ du = dx & v = \frac{1}{\lambda} \end{matrix} \right.$$

$$= \frac{2}{\pi}\left[\frac{x}{\lambda}\sin(\lambda x)\Big|_0^{\pi/2} - \frac{1}{\lambda}\int_0^{\pi/2}\sin(\lambda x)dx + \frac{\pi}{2\lambda}\sin(\lambda x)\Big|_{\pi/2}^{\pi}\right] =$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2\lambda}\sin\left(\lambda\frac{\pi}{2}\right) + \frac{1}{\lambda^2}\cos(\lambda x)\Big|_0^{\pi/2} + \frac{\pi}{2\lambda}\left(\sin(\lambda\pi) - \sin\left(\lambda\frac{\pi}{2}\right)\right)\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi}{2\lambda}\sin\left(\lambda\frac{\pi}{2}\right)} + \frac{1}{\lambda^2}\cos\left(\lambda\frac{\pi}{2}\right) - \frac{1}{\lambda^2} + \frac{\pi}{2\lambda}\sin(\lambda\pi) - \cancel{\frac{\pi}{2\lambda}\sin\left(\lambda\frac{\pi}{2}\right)}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2\lambda}\sin(\lambda\pi) + \frac{1}{\lambda^2}\left(\cos\left(\lambda\frac{\pi}{2}\right) - 1\right)\right] \Rightarrow \tilde{f}(x) = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\pi}{2\lambda}\sin(\lambda\pi) + \frac{1}{\lambda^2}\left(\cos\left(\lambda\frac{\pi}{2}\right)-1\right)\right)\cos$$

3. a) Neka je $f(t) \circ\!\!-\!\!\bullet F(s)$ i $a>0$. Tada imamo

$$\mathcal{L}\Big(f(t-a)\,u(t-a)\Big) = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt =$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt = \left| \begin{matrix} u = t-a \\ du = dt \end{matrix} \right| = \int_0^\infty e^{-s(u+a)} f(u)\,du =$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\,du$$

b) $g_{[a,b]}(t)$, $0 \leq a \leq b$

$$g_{[a,b]} = u(t-a) - u(t-b)$$

$$\mathcal{L}\left[u(t-a) - u(t-b)\right] = \int_0^\infty e^{-st} u(t-a)\,dt - \int_0^\infty e^{-st} u(t-b)\,dt =$$

$$= \int_a^\infty e^{-st}\,dt - \int_b^\infty e^{-st}\,dt =$$

4. $2y(t) + \underbrace{\int_0^{2t} \sin\tau \cdot e^{2t-\tau}\, d\tau}_{\text{konvolucija}} = y'(t) \qquad y(0)=0$

konvolucija $\mathcal{L}(\sin 2t * e^{2t})$

$y(t) \circ\!\!-\!\!\bullet\, Y(s)$

$y'(t) \circ\!\!-\!\!\bullet\, sY(s) - \cancel{y(0)} = sY(s)$

$\sin 2t \circ\!\!-\!\!\bullet\, \frac{2}{s^2+4}$

$e^{2t} \circ\!\!-\!\!\bullet\, \frac{1}{s-2}$

$2Y(s) + \mathcal{L}(\sin 2t * e^{2t}) = sY(s)$

$2Y(s) + \frac{2}{s^2+4} \cdot \frac{1}{s-2} = sY(s)$

$Y(s)(2-s) = -\frac{2}{(s^2+4)(s-2)}$

$Y(s) = \frac{2}{(s^2+4)(s-2)^2} = \frac{2}{s^2+4} \cdot \frac{1}{(s-2)^2} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s-2)^2}$

$(As+B)(s^2-4s+4) + (Cs+D)(s^2+4) = 2$

$As^3 - 4As^2 + 4As + Bs^2 - 4Bs + 4B + Cs^3 + 4Cs + Ds^2 + 4D = 2$

$A + C = 0 \qquad C = -\frac{1}{8}$

$-4A + B + D = 0 \qquad 4A = \frac{1}{2} \quad A = \frac{1}{8}$

$4A - 4B + 4C = 0 \quad A - B + C = 0 \Rightarrow B = 0$

$4B + 4D = 2. \qquad D = \frac{1}{2}$

$Y(s) = \frac{1}{8}\frac{s}{s^2+4} + \frac{1}{8}\frac{s}{(s-2)^2} + \frac{1}{2}\frac{1}{(s-2)^2}$

??

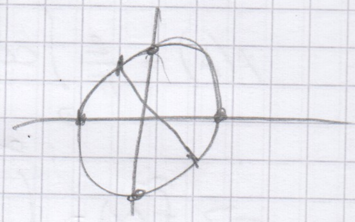
21.10.2010

I. MI

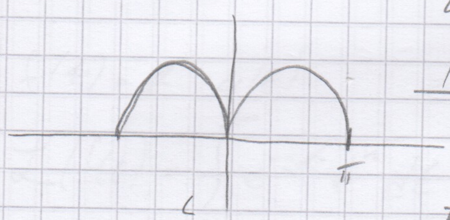
1. $f(x)=A+\sum_{n=1}^{\infty}\left(C_n\cos\frac{3n\pi x}{4}+D_n\sin\frac{3n\pi x}{4}\right.$

$$S(x)=A+\sum_{n=1}^{\infty}\left(A_n\cos\frac{2n\pi x}{T}+B_n\sin\frac{2n\pi x}{T}\right)$$

$$\frac{2n\pi x}{T}=\frac{3n\pi x}{4}\Rightarrow T=\frac{8}{3}$$



2 $f(x)=|\sin x|$ $\quad \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$



parna

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$a_0=\frac{2}{L}\int_0^L f(x)\,dx=\frac{2}{\pi}\int_0^\pi \sin x\,dx=\frac{-2}{\pi}\cos x\Big|_0^\pi=-\frac{2}{\pi}\cdot(-1-1)=\frac{4}{\pi}$$

$$a_n=\frac{2}{L}\int_0 f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{\pi}\int_0^\pi \sin x\cos(nx)\,dx=$$

$$\frac{1}{\pi}\int_0^\pi\Big(\sin(x(1+n))+\sin(x(1-n))\Big)dx=\frac{-1}{\pi}\left[\frac{1}{1+n}\cos(x+xn)\Big|_0^\pi+\frac{1}{1-n}\cos(x-xn)\Big|_0^\pi\right]$$

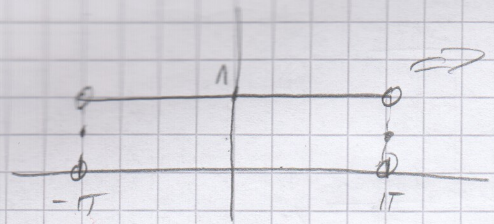
$$=-\frac{1}{\pi}\left[\frac{-\cos(n\pi)-1}{1+n}+\frac{-\cos(n\pi)-1}{1-n}\right]=-\frac{1}{\pi}\,\frac{-\cos(n\pi)(1-n)+\cancel{n}-1-\cos(n\pi)(1+n)-\cancel{n}-1}{1-n^2}$$

$$=-\frac{1}{\pi}\left[\cos(n\pi)\frac{\cancel{n}-1-\cancel{n}-1}{1-n^2}-\frac{2}{1-n^2}\right]=\frac{2}{\pi(1-n^2)}\Big(\cos(n\pi)-1\Big)$$

$$a_{2n-1}=\frac{-4}{\pi(1-}$$ $\quad (2n-1)^2=4n^2-4n+1$

$\cancel{1}-4n^2+4n-\cancel{1}$ $\quad (2n-1)(2n+1)$

3. a) $f(x) = g_{[-\pi,\pi]}(x)$



$\Rightarrow$ parna $\Rightarrow$ $B(\lambda) = 0$

$$A(\lambda) = \frac{2}{\pi}\int_0^{\pi} \cos \lambda x \, dx = \frac{2}{\pi} \cdot \frac{1}{\lambda} \sin \lambda x \Big|_0^{\pi} = \frac{2}{\lambda\pi} \sin \lambda\pi$$

$$\Rightarrow \tilde{f}(x) = \int_0^{\infty} \frac{2\sin\lambda\pi}{\lambda\pi} \cos\lambda x \, d\lambda$$

c) $\tilde{f}(0) = \int_0^{\infty} \frac{2\sin\lambda\pi}{\lambda\pi} = f(0) = 1$

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin\lambda\pi}{\lambda\pi} = 1 \quad \Rightarrow \int_0^{\infty} \frac{\sin\lambda\pi}{\lambda\pi} = \frac{\pi}{2}$$

4. $f(t) = t^n \cdot u(t)$

a) funkcija je original ako:

1) $f(t) = 0$, z $t < 0$

2) $f(t)$ je na svakom konačnom intervalu po dijelovima neprekinuta

3) f je eksponencijalnog rasta, tj. postoje konstante $M > 0$ i $a > 0$ takve da za sve $t > 0$ vrijedi

$$|f(t)| \leq Me^{at}$$

Infimum ovih konstanti a za koje vrijedi nejednakost naziva se eksponent rasta fje i označava se s a_0

b) $f(t) = t^n \cdot u(t) \quad \Rightarrow \frac{n!}{s^{n+1}}$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} t^n \, dt = \left| \begin{array}{ll} u = t^n & dv = e^{-st} dt \\ du = n t^{n-1} dt & v = -\frac{1}{s} e^{-st} \end{array} \right|$$

$$= -\frac{t^n}{s} e^{-st} \Big|_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$$ $\Rightarrow$ za $s>0$ prvi član je jednak 0

$\Rightarrow$ REKURZIVNA FORMULA

$$\mathcal{L}(t^n) = \frac{n}{s} \mathcal{L}(t^{n-1}) = \ldots = \frac{n!}{s^n} \mathcal{L}(1) = \frac{n!}{s^{n+1}}$$

b) Teorem o deriviranju slike

Deriviranju u donjem područiju odgovara množenje s $(-t)$ u gornjem područiju:

$(-t) f(t) \circ\!\!-\!\!\bullet\, F'(s)$

općenito $t^n f(t) \circ\!\!-\!\!\bullet\, (-1)^n F'(s)$

$\Rightarrow \quad u(t) \circ\!\!-\!\!\bullet\, \frac{1}{s}$

$$(-t)^n u(t) \circ\!\!-\!\!\bullet\, \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = \frac{(-1)(-2)(-3)\cdots(-n)}{s^{n+1}} = \frac{(-1)^n n!}{s^{n+1}}$$

$$\Rightarrow t^n \circ\!\!-\!\!\bullet\, \frac{n!}{s^{n+1}}$$

5.

$$\underbrace{y'(t)+\int_0^t y(\tau)d\tau}_{y(t) * u(t)} = \sin t \,, \quad y(0)=2$$

$$y'(t) = sY(s) - 2$$

$$sY(s) - 2 + Y(s)\cdot\frac{1}{s} = \frac{1}{s^2+1}$$

$$Y(s)\overbrace{\left(s+\frac{1}{s}\right)}^{\frac{s^2+1}{s}} = \frac{1}{s^2+1} + 2$$

$$Y(s) = \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} + \frac{2s}{s^2+1}$$

$$y(t) = \int_0^t \sin(\tau)\cos(t-\tau)\,d\tau + 2\cos t$$

$$y(t) = \frac{1}{2}\int_0^t \left(\sin(\tau+t-\tau) + \sin(\tau-t+\tau)\right)d\tau + 2\cos t$$

$$y(t) = \frac{1}{2}t\sin t + \frac{1}{2}\int_0^t \sin(2\tau - t)\,d\tau + 2\cos t$$

$$y(t) = \frac{1}{2}t\sin t - \frac{1}{2(2\tau-t)}\cos(2\tau-t)\Big|_0^t + 2\cos t$$

$$y(t) = \frac{1}{2}t\sin t - \cancel{\frac{1}{2t}\cos t} + \cancel{\frac{1}{2t}\cos t} + 2\cos t$$

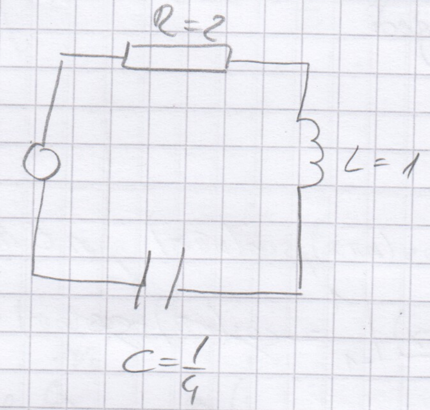
$$y(t) = \left(\frac{1}{2}t\sin t + 2\cos t\right)u(t)$$

6.

$e(t) = ?$

$i(t) = e^{-t}\left(\cos\sqrt{3}t - \frac{\sqrt{3}}{3}\sin\sqrt{3}t\right)$

$e(t) = ?$



$$I(s) = \frac{E(s)}{Z(s)} \Rightarrow E(s) = I(s)\cdot Z(s)$$

$$I(s) = \frac{s+1}{(s+1)^2+3} - \frac{1}{(s+1)^2+3} = \frac{s}{(s+1)^2+3}$$

$$Z(s) = R + Ls + \frac{1}{Cs} = 2 + s + \frac{4}{s} = \frac{2s+s^2+4}{s} = \frac{(s+1)^2+3}{s}$$

$$E(s) = \cancel{\frac{s}{(s+1)^2+3}} \cdot \cancel{\frac{(s+1)^2+3}{s}} = 1$$

$$1 \bullet\!\!-\!\!\circ\ \delta(t) \Rightarrow e(t) = \delta(t)$$

21.10.2009

I. MF

1. Za funkcije $f,g:[a,b]$ kažemo da su ortogonalne na intervalu $[a,b]$ ako vrijedi

$$\int_a^b f(x)g(x)\,dx=0$$

$$1, \cos(\pi x), \sin(\pi x), \dots, \cos(n\pi x), \sin(n\pi x), \quad n\in \mathbb{N}$$

$$\int_{-1}^{1} 1\cdot \sin(n\pi x)\,dx = \frac{-\cos(n\pi x)}{n\pi}\Big|_{-1}^{1} = -\cos(n\pi)+\cos(n\pi)=0$$

$$\int_{-1}^{1} 1\cdot\cos(n\pi x)\,dx = \frac{\sin(n\pi x)}{n\pi}\Big|_{-1}^{1} = \frac{\overbrace{\sin(n\pi)}^{0}+\overbrace{\sin(n\pi)}^{0}}{n\pi}=0$$

$$\int_{-1}^{1} \cos(n\pi x)\sin(m\pi x)\,dx = \frac{1}{2}\int_{-1}^{1}\Big[\sin(\pi x(m+n))+\sin(\pi x(m-n))\Big]\,dx$$

$$=\cancel{\frac{-1}{2\pi(m+n)}\cos(\pi(m+n))} + \cancel{\frac{1}{2\pi(m+n)}\cos(\pi(m+n))} + \cancel{\frac{-1}{2\pi(m-n)}\cos(\pi(m-n))} + \cancel{\frac{1}{2\pi(m-n)}\cos(\pi(m-n))}$$

$$=0$$

$$\int_{-1}^{1} \cos(n\pi x)\cos(m\pi x)\,dx = \frac{1}{2}\int_{-1}^{1} \cos(\pi x(n+m))+\cos(\pi x(n-m)) =$$

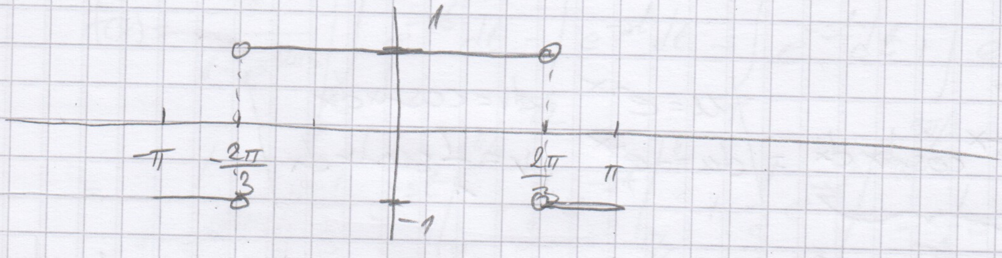
$$=\overbrace{\frac{1}{2\pi(n+m)}\sin(\pi(n+m))}^{0}+\overbrace{\frac{1}{2\pi(n+m)}\sin(\pi(n+m))}^{0} + \frac{1}{2\pi(n-m)}\sin(\pi(n-m)) + \frac{1}{2\pi(n-m)}\sin(\pi(n-m))$$

$$=0, \quad m\neq n$$

$$\int_{-1}^{1} \sin n\pi x \, \sin m\pi x\,dx = \frac{1}{2}\int$$

2 a)

$$f(x)=\begin{cases} 1, & -\frac{2\pi}{3}\le x\le\frac{2\pi}{3} \\ -1 & -\pi<x<-\frac{2\pi}{3}\ ,\ \frac{2\pi}{3}<x<\pi \end{cases}$$

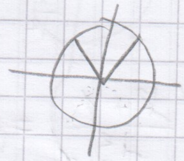


$\Rightarrow$ parna

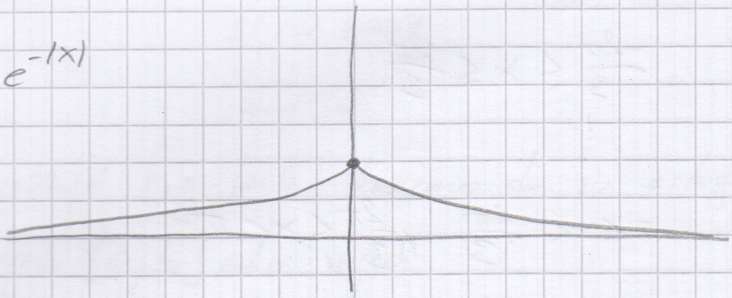
$$a_0=\frac{2}{\pi}\int_0^{2\pi/3}dx-\frac{2}{\pi}\int_{2\pi/3}^{\pi}dx=\frac{2}{\pi}\cdot\frac{2\pi}{3}-\frac{2}{\pi}\left(\pi-\frac{2\pi}{3}\right)=\frac{4}{3}-\frac{2}{3}=\frac{2}{3}$$

$$a_n=\frac{2}{\pi}\left[\int_0^{2\pi/3}\cos nx\,dx-\int_{2\pi/3}^{\pi}\cos nx\,dx\right]=\frac{2}{n\pi}\sin nx\Big|_0^{2\pi/3}-\frac{2}{n\pi}\sin nx\Big|_{2\pi/3}^{\pi}=$$

$$=\frac{2}{n\pi}\left[\sin\frac{2\pi}{3}n+\sin\frac{2\pi}{3}n\right]=\frac{4\sin\frac{2\pi}{3}n}{n\pi}$$



3. $f(x)=e^{-|x|}$



$\Rightarrow$ parna

$$A(\lambda)=\frac{2}{\pi}\underbrace{\int_0^\infty e^{-x}\cos\lambda x\,dx}_{I}=\left|\begin{matrix}u=e^{-x} & dv=\cos\lambda x\,dx\\ du=-e^{-x} & v=\frac{1}{\lambda}\sin\lambda x\,dx\end{matrix}\right|$$

$$I=\cancel{\frac{e^{-x}}{\lambda}\sin\lambda x}\Big|_0^\infty+\frac{1}{\lambda}\int_0^\infty e^{-x}\sin\lambda x\,dx=\left|\begin{matrix}u=e^{-x} & dv=\sin\lambda x\,dx\\ du=-e^{-x} & v=-\frac{1}{\lambda}\cos\lambda x\end{matrix}\right|$$

$$=\underbrace{-\frac{e^{-x}}{\lambda^2}\cos\lambda x\Big|_0^\infty}_{\frac{1}{\lambda^2}}-\frac{1}{\lambda^2}\underbrace{\int_0^\infty e^{-x}\cos\lambda x\,dx}_{I}=\frac{1}{\lambda^2}-\frac{1}{\lambda^2}I$$

$$I\left(\frac{\lambda^2+1}{\lambda^2}\right)=\frac{1}{\lambda^2}\Rightarrow I=\frac{1}{\lambda^2+1}$$

$$A(\lambda)=\frac{2}{\pi}\frac{1}{\lambda^2+1}$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\cos\lambda x}{\lambda^2+1}\,d\lambda$$

4. Za funkciju kažemo da je eksponencijalnog rasta kada postoje konstante $M>0$ i $a>0$ takve da za sve $t>0$ vrijedi

$$|f(t)|\le Me^{at}$$

5. a) Za original ćemo reći da je periodična fja kada $f(t)=0$, za $t<0$ i $f(t)=f(t+T)$ za neki $T>0$ i sve $t>0$

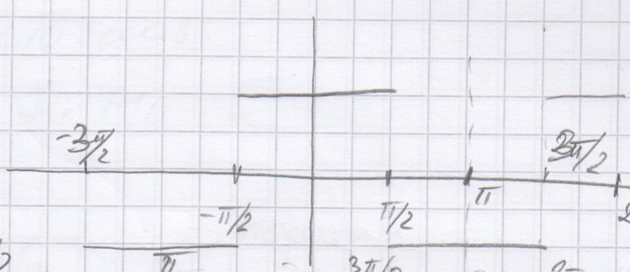
$$F(s)=\int_0^\infty e^{-st}f(t)\,dt=\sum_{n=0}^\infty\int_{nT}^{(n+1)T}e^{-st}f(t)\,dt=:\sum_{n=0}^\infty I_n$$

$$I_n=\int_{nT}^{(n+1)T}e^{-st}f(t)\,dt=\int_0^T e^{-s(\tau+nT)}f(\tau+nT)\,d\tau=e^{-nsT}\int_0^T e^{-s\tau}f(\tau+nT)\,d\tau$$

$$=e^{-nsT}\int_0^T e^{-s\tau}f(\tau)\,d\tau=e^{-nsT}I_0$$

$$F(s)=\sum_{n=0}^\infty e^{-nsT}I_0=\frac{1}{1-e^{-sT}}I_0$$

b) $f(x) = \operatorname{sgn}(\cos x)$



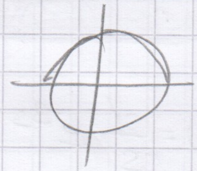
$T = 2\pi$

$$f(x) = \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi/2} e^{-st}\,dt - \int_{\pi/2}^{\pi} e^{-st}\,dt - \int_{\pi}^{3\pi/2} e^{-st}\,dt + \int_{3\pi/2}^{2\pi} e^{-st}\,dt\right] =$$

$$= \frac{1}{1-e^{-2\pi s}}\left[-\frac{1}{s}e^{-st}\Big|_0^{\pi/2} + \frac{1}{s}e^{-st}\Big|_{\pi/2}^{\pi} + \frac{1}{s}e^{-st}\Big|_{\pi}^{3\pi/2} - \frac{1}{s}e^{-st}\Big|_{3\pi/2}^{2\pi}\right] =$$

$$= \frac{1}{s(1-e^{-2\pi s})}\left[-e^{-s\frac{\pi}{2}} + 1 + \cancel{e^{-s\pi}} - e^{-s\frac{\pi}{2}} + e^{-s\frac{3\pi}{2}} - \cancel{e^{-s\pi}} - e^{-2s\pi} + e^{-s\frac{3\pi}{2}}\right]$$

$$= \frac{1}{s(1-e^{-2\pi s})}\left[1 - 2e^{-s\frac{\pi}{2}} + 2e^{-s\frac{3\pi}{2}} - e^{-2s\pi}\right]$$



6. $y''(t) + y(t) = \underbrace{2\cos t \cdot g_{[0,\pi]}(t)}$ $\quad y(0)=0 \quad y'(0)=1$

$y(t) \circ\!\!-\!\!\bullet\, Y(s) \qquad (2\cos t\, u(t) - 2\cos t\, u(t-\pi)) = 2\cos t\, u(t) + 2\cos(t-\pi)\, u(t-\pi)$

?

$y'(t) \circ\!\!-\!\!\bullet\, sY(s) - y(0) = sY(s)$

$y''(t) \circ\!\!-\!\!\bullet\, s^2Y(s) - s\,y(0) - y'(0) = s^2Y(s) - 1 \qquad \frac{2s}{s^2+1} + \frac{2s}{s^2+1}e^{-\pi s}$

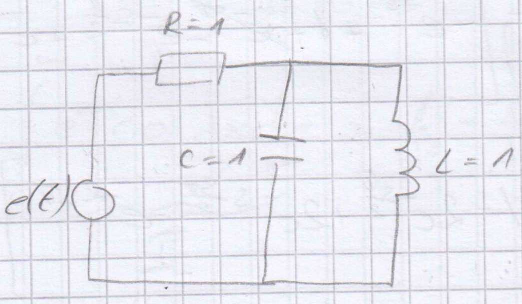
$$s^2Y(s) - 1 + Y(s) = \frac{2s}{s^2+1} + \frac{2s}{s^2+1}e^{-\pi s}$$

$$Y(s)(s^2+1) = \frac{2s}{s^2+1}\left(1 + e^{-\pi s}\right) + 1$$

$$Y(s) = \frac{2s}{s^2+1}\left(1+e^{-\pi s}\right)\cdot\frac{1}{s^2+1} + \frac{1}{s^2+1}$$

$$Y(s) = 2\frac{s}{s^2+1}\cdot\frac{1}{s^2+1} + 2\frac{se^{-\pi s}}{s^2+1}\cdot\frac{1}{s^2+1} + \frac{1}{s^2+1}$$

7. $e(t)=u(t-3)$



$$E(s)=\frac{1}{s}e^{-3s}$$

$$Z(s) = R + \frac{\frac{1}{Cs}\cdot Ls}{\frac{1}{Cs}+Ls} = 1+\frac{\frac{1}{s}\cdot s}{\frac{1}{s}+s} = 1+\frac{s}{s^2+1} = \frac{s^2+s+1}{s^2+1}$$

$$I(s) = \frac{\frac{1}{s}e^{-3s}}{\frac{s^2+s+1}{s^2+1}} = \frac{s^2+1}{s(s^2+s+1)}e^{-3s}$$

$$\frac{A}{s}+\frac{Bs+C}{s^2+s+1} = \frac{s^2+1}{\sim} \qquad \frac{1}{s}-\frac{1}{\underbrace{s^2+s+1}_{(s+\frac{1}{2})^2+\frac{3}{4}}}$$

$$As^2+As+A+Bs^2+Cs=s^2+1$$

$A+B=1 \Rightarrow B=0$

$A+C=0 \Rightarrow C=-1$

$A=1$

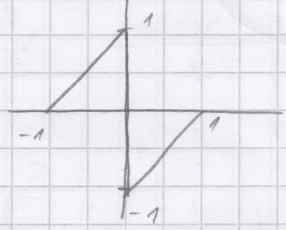
$$I(s)=\left(\frac{1}{s}-\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right)e^{-3s}$$

$$i(t)=u(t-3)-\frac{2}{\sqrt{3}}\sin\left[\frac{\sqrt{3}}{2}(t-3)\right]u(t-3)e^{-\frac{1}{2}t}$$

16.10.2008

I. MI

1 $T=2$



$\Rightarrow$ neparna

$f(x)=x-1$

$$b_n=\frac{2}{1}\int_0^1 (x-1)\sin(n\pi x)\,dx = \left| \begin{matrix} u=x & dv=\sin(n\pi x)\,dx \\ du=dx & v=-\frac{1}{n\pi}\cos n\pi x \end{matrix} \right| =$$

$$=2\left[-\frac{x}{n\pi}\cos(n\pi x)\Big|_0^1+\frac{1}{n\pi}\int_0^1\cos(n\pi x)\,dx+\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1\right]$$

$$=-\frac{2}{n\pi}\cos(n\pi)+\cancel{\frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^1}+\frac{1}{n\pi}\cos n\pi-\frac{1}{n\pi}=$$

$$=\frac{1}{n\pi}\left[-\cos(n\pi)-1\right]=\frac{-2}{n\pi}=b_n$$

$$S(x)=\sum_{n=0}^{\infty}-\frac{2}{n\pi}\sin(n\pi x)\,dx \quad //$$

2. $S(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$ $\quad f(x)=x^2,\ -1<x<1$

a) Parsevalova jednakost $\qquad \sum_{n=1}^{\infty}\frac{1}{n^4}=?$

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{2}{T}\int_a^b |f(x)|^2\,dx$$

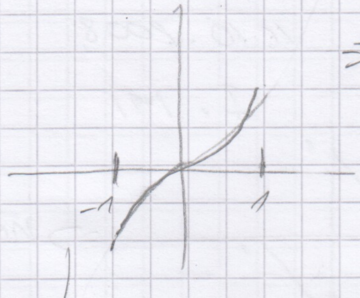
$a_0=\frac{2}{3}$; $a_n=\frac{(-1)^n 4}{n^2\pi^2}$

$$\frac{2}{9}+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}=\int_{-1}^{1}x^4\,dx=\frac{x^5}{5}\Big|_{-1}^{1}=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

$$\frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2}{5}-\frac{2}{9}=\frac{18-10}{45}=\frac{8}{45}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\overset{1}{\cancel{8}}}{45}\cdot\frac{\pi^4}{\underset{2}{\cancel{16}}}=\frac{\pi^4}{90} \quad //$$

b) $f(x)=x^3,\ -1<x<1$ $\Rightarrow$ neparna



$$b_n=\frac{2}{1}\int_0^1 x^3\sin n\pi x\,dx=$$

$$=\left|\begin{matrix} u=x^3 & dv=\sin n\pi x\,dx \\ du=3x^2dx & v=-\frac{1}{n\pi}\cos n\pi x\end{matrix}\right|=$$

$$=2\left[-\frac{x^3}{n\pi}\cos n\pi x\Big|_0^1+\frac{3}{n\pi}\int_0^1 x^2\cos n\pi x\,dx\right]=\left|\begin{matrix} u=x^2 & dv=\cos n\pi x \\ du=2x\,dx & v=\frac{1}{n\pi}\sin n\pi\end{matrix}\right|$$

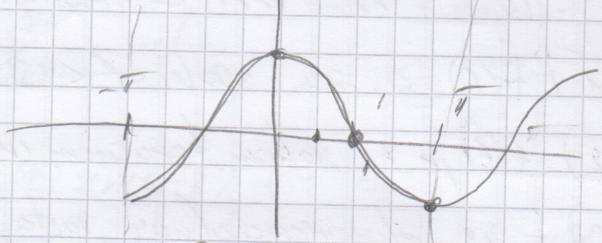
$$=2\left[-\frac{\cos(n\pi)}{n\pi}+\frac{3}{n\pi}\left(\cancel{\frac{x^2}{n\pi}\sin n\pi x\Big|_0^1}-\frac{2}{n\pi}\int_0^1 x\sin n\pi x\,dx\right)\right]=\left|\begin{matrix} u=x & dv=\sin n\pi x \\ du=dx & v=\frac{-1}{n\pi}\cos n\pi\end{matrix}\right|$$

$$=2\left[-\frac{\cos(n\pi)}{n\pi}-\frac{6}{n^2\pi^2}\left(-\frac{x}{n\pi}\cos n\pi x\Big|_0^1+\frac{1}{n\pi}\int_0^1\cos n\pi x\,dx\right)\right]=$$

$$=2\left[-\frac{\cos(n\pi)}{n\pi}+\frac{6}{n^3\pi^3}\cos(n\pi)-\cancel{\frac{6}{n^4\pi^4}\sin n\pi x\Big|_0^1}\right]=$$

$$=2\cos(n\pi)\left[\frac{6-n^2\pi^2}{n^3\pi^3}\right]=2\cos(n\pi)\left(\frac{6}{n^3\pi^3}-\frac{1}{n\pi}\right)$$

$$f(x) = \begin{cases} \cos\frac{\pi}{2}x, & |x| \le 3 \\ 0, & \text{inače} \end{cases}$$



$\Rightarrow$ parna

$$A(\lambda) = \frac{2}{\pi}\int_0^3 \cos\frac{\pi}{2}x \cos\lambda x \, dx = \frac{1}{\pi}\int_0^3 \left(\cos\left(x\left(\frac{\pi}{2}+\lambda\right)\right) + \cos\left(x\left(\frac{\pi}{2}-\lambda\right)\right)\right) dx =$$

$$= \frac{1}{\pi}\,\frac{1}{\frac{\pi}{2}+\lambda}\sin\left(x\frac{\pi}{2}+\lambda x\right)\Big|_0^3 + \frac{1}{\pi}\,\frac{1}{\frac{\pi}{2}-\lambda}\sin\left(x\frac{\pi}{2}-\lambda x\right)\Big|_0^3$$

$$= \frac{1}{\pi\left(\frac{\pi}{2}+\lambda\right)}\underbrace{\sin\left(3\frac{\pi}{2}+3\lambda\right)}_{-\cos(3\lambda)} + \frac{1}{\pi\left(\frac{\pi}{2}-\lambda\right)}\underbrace{\sin\left(3\frac{\pi}{2}-3\lambda\right)}_{-\cos(3\lambda)} =$$

$$= \frac{-\cos(3\lambda)}{\pi}\left(\frac{\frac{\pi}{2}-\lambda+\frac{\pi}{2}+\lambda}{\frac{\pi^2}{4}-\lambda^2}\right) = \frac{-\cos 3\lambda}{\frac{\pi^2}{4}-\lambda^2} = \frac{4\cos 3\lambda}{4\lambda^2-\pi^2}$$

$$\tilde{f}(x) = \int_0^\infty \frac{4\cos 3\lambda}{4\lambda^2-\pi^2}\cos\lambda x \, d\lambda$$

$$\int_0^\infty \frac{\cos 3u}{4u^2-\pi^2} \Rightarrow x=0 \Rightarrow \overset{f(0)}{\overset{\shortparallel}{1}} = \int_0^\infty \frac{4\cos 3\lambda}{4\lambda^2-\pi^2} \Rightarrow \int_0^\infty \frac{\cos 3\lambda}{4\lambda^2-\pi^2} = \frac{1}{4}$$

4. a) Za funkciju f ćemo reći da je original ako:

1) $f(t) = 0$, za svaki $t < 0$

2) $f(t)$ je na svakom konačnom intervalu po dijelovima neprekidna

3) f je eksponencijalnog rasta, tj. postoje $M > 0$ i $a > 0$ takvi da je

$$|f(t)| < Me^{at}$$

Infimum svih konstanti a za koje vrijedi nejednakost naziva se eksponent rasta i označava se s a_0

b) $$\int_0^{\infty} e^{-2t} \frac{sh t}{t} \qquad \frac{sh t}{t} \circ\!\!-\!\!\bullet \int_s^{\infty} \frac{1}{p^2-1} dp = \frac{1}{2} \ln\left|\frac{p-1}{p+1}\right| \Big|_s^{\infty} =$$

$$= \frac{1}{2} \ln\left|\frac{s+1}{s-1}\right|$$

$$F'(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} (-t) f(t) dt$$

$$I = \frac{1}{2} \ln \frac{s+1}{s-1} \qquad F(2) = \frac{1}{2} \ln 3$$

5. a) Konvolucija originala f_1 i f_2 definirana je integralom

$$(f_1 * f_2)(t) \circ\!\!-\!\!\bullet \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

Konvoluciji u gornjem području odgovara umnožak slika u donjem području $(f_1 * f_2)(t) \circ\!\!-\!\!\bullet F_1(s) \cdot F_2(s)$

b) $F(s) = \frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$

$$(\cos t * \sin t) = \int_0^t \cos(\tau)\sin(t-\tau)\,d\tau =$$

$$= \frac{1}{2}\int_0^t (\sin(t) + \sin(t-2\tau))\,d\tau = \frac{1}{2} t\sin t + \frac{1}{4}\cos(2\tau - t)\Big|_0^t =$$

$$= \frac{1}{2} t \sin t + \cancel{\frac{1}{4}\cos t} - \cancel{\frac{1}{4}\cos t} = \frac{1}{2} t \sin t \, u(t)$$

c) $F(s) = \frac{s \cdot e^{-4s}}{(s^2+1)^2}$ $\Rightarrow$ gornji original, ali s pomakom

$$\bullet\!\!-\!\!\circ \frac{1}{2}(t-4)\sin(t-4)\,u(t-4)$$

6. Teorem o derivaciji originala

$$y^n(t) = s^n Y(s) - s^{n-1} y(0) - \dots - s\, y^{(n-2)}(0) - y^{(n-1)}(0)$$

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{F(s)}$$

Ako je a_0 eksponent rasta od f tada za sve $s > a_0$ integral $\int_0^\infty e^{-st} f(t)\,dt$ konvergira i zato je nužno $\lim_{t\to\infty} e^{-st} f(t) = 0$

$$\Rightarrow f'(t) \circ\!\!-\!\!\bullet\, sF(s) - f(0)$$

b) $y'(t) - 5y(t) = e^{1-t}$ $\quad y(0) = 3$

$y(t) \circ\!\!-\!\!\bullet\, Y(s)$

$y'(t) \circ\!\!-\!\!\bullet\, sY(s) - 3$

$e^{1-t} = e \cdot e^{-t} \circ\!\!-\!\!\bullet\, e \cdot \frac{1}{s+1}$

$A(s+1) + B(s-5) = e$

$sY(s) - 3 - 5Y(s) = \frac{e}{s+1}$

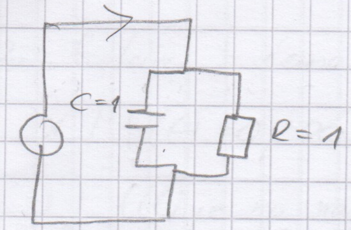
$-6B = e \Rightarrow B = -\frac{e}{6}$

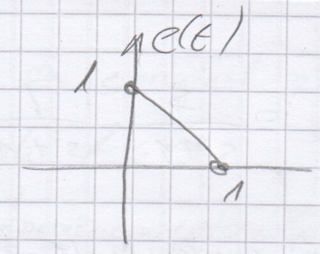
$Y(s)(s-5) = \frac{e}{s+1} + 3$

$6A = e \quad A = \frac{e}{6}$

$$Y(s) = \frac{e}{(s-5)(s+1)} + \frac{3}{s-5} = \frac{e}{6}\cdot\frac{1}{s-5} - \frac{e}{6}\frac{1}{s+1} + \frac{3}{s-5} \bullet\!\!-\!\!\circ \frac{e}{6}\left(e^{5t} - e^{-t}\right) + 3e^{5t}$$

7.





$$e(t) = (1-t)[u(t)-u(t-1)] = u(t) - u(t-1) - tu(t) + (t-1+1)u(t-1) =$$

$$= u(t) - tu(t) - \cancel{u(t-1)} + \cancel{u(t-1)} + (t-1)u(t-1)$$

$$= \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}$$

$$(1-t)[u(t)-u(t-1)] = (1-t)u(t) + (t-1)u(t-1)$$

$$Z(s) = \frac{\frac{1}{Cs} \cdot R}{\frac{1}{Cs} + R} = \frac{\frac{1}{\cancel{s}}}{\frac{1+s}{\cancel{s}}} = \frac{1}{s+1}$$

2007. I MI

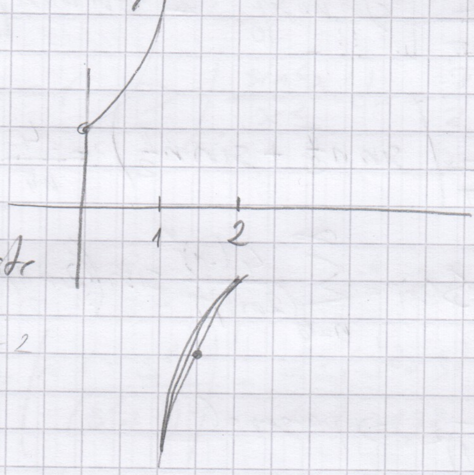
a) Kažemo da f zadovoljava Dirichletove uvjete na intervalu $[a,b]$, ako vrijedi

1) f je po dijelovima neprekinuta i njezini su prekidi prve vrste

2) f je monotona ili ima najviše konačan broj strogih ekstrema

$(t-1)=$

b) $f(x) = \frac{1}{1-x}$ $[0,2]$



$\Rightarrow$ prekid nije prve vrste

$\Rightarrow$ monotona

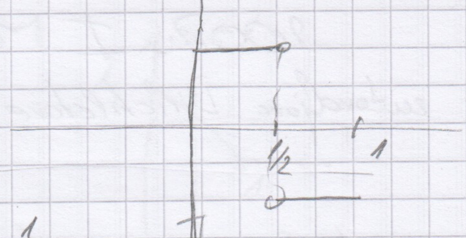
$\Rightarrow$ NG!

c) Neka je f po dijelovima glatka periodična fja s periodom 2π koja zadovoljava Dirichletove uvjete. Tada njezin Fourierov red konvergira u svakoj točki $x \in [-\pi, \pi]$ i za sum $S(x)$ reda vrijedi

1) $S(x) = f(x)$ ako je f neprekinuta u točki x

2) $S(x) = \frac{1}{2}\left[f(x-0) + f(x+0)\right]$ ako je x točka prekida za f

2. $f(x) = \begin{cases} 1, & x \in \langle 0, \frac{1}{2} \rangle \\ -1, & x \in \langle \frac{1}{2}, 1 \rangle \end{cases}$



a) $$2\left[\int_0^{1/2} \cos n\pi x \, dx - \int_{1/2}^{1} \cos n\pi x \, dx\right] =$$

$$= \frac{2}{n\pi}\left[\sin n\pi x \Big|_0^{1/2} - \sin n\pi x \Big|_{1/2}^{1}\right] =$$

$$= \frac{2}{n\pi}\left(\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right) = \frac{4}{n\pi} \sin n\frac{\pi}{2} \qquad 2n+1$$

$$\Rightarrow S(x) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} \sin\left((2n+1)\frac{\pi}{2}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \Rightarrow \text{parseral}$$

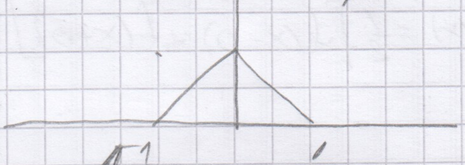
$$a_0 = 2\left(\int_0^{1/2} dx - \int_{1/2}^{1} dx\right) = 2x\Big|_0^{1/2} - 2x\Big|_{1/2}^{1} = 1 - 2 + 1 = 0$$

b) $$\sum_{n=0}^{\infty} \frac{16}{(2n+1)^2\pi^2} = \frac{2}{2}\int_{-1}^{1} 1^2 \, dx = x\Big|_{-1}^{1} = 2$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \quad //$$

3. $f(x) = \begin{cases} 1 - |x|, & x \in [-1, 1] \\ 0, & \text{inače} \end{cases}$

parna



$$A(\lambda) = \frac{2}{\pi}\int_0^1 (1-x)\cos \lambda x \, dx = \frac{2}{\pi}\int_0^1 (\cos \lambda x - x\cos \lambda x)\, dx =$$

$$= \left| \begin{matrix} u = x & dv = \cos \lambda x \, dx \\ du = dx & v = \frac{1}{\lambda}\sin \lambda x \end{matrix} \right| = \frac{2}{\pi}\left[\frac{1}{\lambda}\sin \lambda x\Big|_0^1 - \frac{x}{\lambda}\sin \lambda x\Big|_0^1 + \frac{1}{\lambda}\int_0^1 \sin \lambda x \, dx\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{1}{\lambda}\sin \lambda x} - \cancel{\frac{1}{\lambda}\sin \lambda x} - \frac{1}{\lambda^2}\cos \lambda x \Big|_0^1\right] =$$

$$= \frac{-2}{\lambda^2\pi}(\cos \lambda - 1) = \frac{2}{\lambda^2\pi}(1 - \cos \lambda)$$

$$\tilde{f}(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\cos\lambda x \, d\lambda \qquad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = ?$$

$x = 0$

$$1 \overset{\Downarrow}{=} \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^2}\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{\lambda}{2}}{\lambda^2}\,d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}\,d\left(\frac{\lambda}{2}\right) =$$

$$\Rightarrow \frac{\lambda}{2} = x \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = 1 \Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

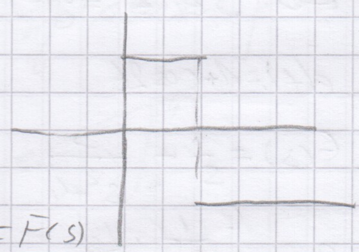
4 $\int_0^\infty e^{-\frac{x}{2}}\underbrace{x^2\cos x}\,dx \qquad \cos t \circ\!-\!\bullet \frac{s}{s^2+1} \qquad s^4+2s^2+1$

$$t^2\cos t \circ\!-\!\bullet \left(\frac{s}{s^2+1}\right)'' = \left(\frac{1-s^2}{(s^2+1)^2}\right)' = \frac{2s(s^2-3)}{(s^2+1)^3} = F(s)$$

$$f(t) \circ\!-\!\bullet F(s) \Rightarrow F(s) = \int_0^\infty e^{-st} f(t)\,dt \Rightarrow s = -\frac{1}{2}$$

$$\frac{-1\left(\frac{1}{4}-3\right)}{\left(\frac{1}{4}+1\right)^3} = \frac{\frac{\cancel{12}\,11}{4}}{\frac{125}{\cancel{64}\,16}} = \frac{16\cdot 11}{125}$$

5 $y''+y = f(t) \qquad y(0) = 1 \qquad y'(0) = 0$



$f(t) = (u(t) - u(t-1)) - u(t-1)$

$f(t) = u(t) - 2u(t-1) \circ\!-\!\bullet \frac{1}{s} - \frac{2e^{-s}}{s} = F(s)$

$\frac{A}{s} \quad \frac{Bs+C}{s^2+1}$

$As^2 + A + Bs^2 + Cs = 1$

$A+B = 0$

$C = 0 \qquad B = -1$

$A = 1$

$y'' = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$

$s^2Y(s) - s + Y(s) = \frac{1-2e^{-s}}{s}$

$Y(s)(s^2+1) = \frac{1-2e^{-s}}{s} + s \Rightarrow Y(s) = \frac{1}{s(s^2+1)}(1-2e^{-s}) + \frac{s}{s^2+1}$

$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)(1-2e^{-s}) + \frac{s}{s^2+1} = \frac{1}{s} - \cancel{\frac{s}{s^2+1}} - \frac{2e^{-s}}{s} + \frac{2e^{-s}s}{s^2+1} + \cancel{\frac{s}{s^2+1}}$

$\bullet\!-\!\circ\; u(t) - 2u(t-1) + 2\cos(t-2)\,u(t-2)$

6. a) Konvolucija originala f_1 i f_2 definirana je integralom

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$

b) f i g su originali $\Rightarrow$ f i g su eksponencijalnog rasta

$$|(f*g)(t)| \leq \int_0^t |f(\tau)||g(t-\tau)|\, d\tau \leq \begin{bmatrix} \exists M_1, M_2 \\ \exists a_1, a_2 \end{bmatrix}$$

$$\leq M_1 M_2 \int_0^t e^{a_1\tau} e^{a_2(t-\tau)}\, d\tau$$

c)

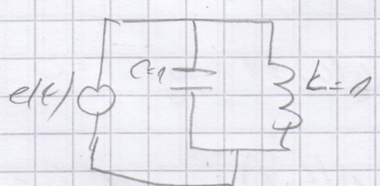
$$y(t) = 3\sin t + 2\int_0^t \cos(t-\tau)\, y(\tau)\, d\tau$$

$$Y(s) = \frac{3}{s^2+1} + 2\frac{s}{s^2+1} Y(s)$$

$$Y(s)\left(1 - \frac{2s}{s^2+1}\right) = \frac{3}{s^2+1}$$

$$Y(s)\left(\frac{s^2-2s+1}{s^2+1}\right) = \frac{3}{s^2+1} \Rightarrow Y(s) = \frac{3}{s^2-2s+1} = \frac{3}{(s-1)^2}$$

$$\circ\!\!- 3e^t t = y(t)$$



7. $e(t) = 1 + \cos 2t$

$$E(s) = \frac{1}{s} + \frac{s}{s^2+4}$$

$$Z(s) = \frac{\frac{1}{Cs} \cdot Ls}{\frac{1}{Cs} + Ls} = \frac{s}{s^2+1}$$

$$I(s) = \frac{s^2+1}{s}\left(\frac{1}{s} + \frac{s}{s^2+4}\right) = \frac{s^2+1}{s^2} + \frac{s^2+1}{s^2+4} = 1 + \frac{1}{s^2} + 1 - \frac{3}{s^2+4}$$

$$= 2 + \frac{1}{s^2} - \frac{2 \cdot 3}{2 \cdot s^2+4} \circ\!\!- 2\delta(t) + t u(t) - \frac{3}{2}\sin 2t\, u(t) = i(t)$$

I.19.1 2006.

1. a) Za funkcije $f,g:[a,b] \to \mathbb{R}$ kažemo da su ortogonalne na intervalu $[a,b]$ ako vrijedi $\int_a^b f(x)g(x)\,dx = 0$

b) $f(x) = \sin mx$
$g(x) = \sin nx$ $\quad m,n \in \mathbb{N}$, $[-\pi,\pi]$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos((m-n)x) - \cos((m+n)x)\right]dx =$$

$$\frac{1}{2}\frac{1}{m-n}\sin((m-n)x)\Big|_{-\pi}^{\pi} - \frac{1}{2}\frac{1}{m+n}\sin((m+n)x)\Big|_{-\pi}^{\pi} = 0$$

2. $f(x) = \frac{\pi}{4} - \frac{x}{2}$ $\quad \langle 0,\pi \rangle$ $\quad$ neparan razvoj

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4} - \frac{x}{2}\right)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi}\frac{\pi}{4}\sin nx\,dx - \int_0^{\pi}\frac{x}{2}\sin nx\,dx\right] =$$

$$-\frac{1}{2n}\cos nx\Big|_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}x\sin nx\,dx = \left|\begin{matrix} u = x & dv = \sin nx \\ du = dx & v = -\frac{1}{n}\cos nx\end{matrix}\right|$$

$$= -\frac{1}{2n}(\cos n\pi - 1) + \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] =$$

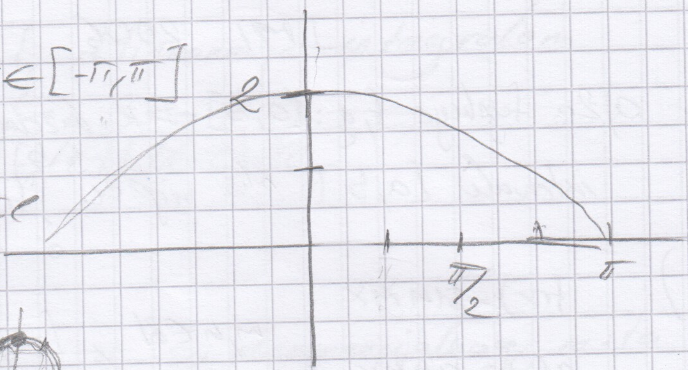
$$= \frac{1}{2n}(\cos n\pi - 1) + \frac{1}{n}\cos(n\pi) + \cancel{\frac{1}{n^2\pi}\sin nx\Big|_0^{\pi}} =$$

$$= \frac{-\cos n\pi + 1 + 2\cos n\pi}{2n} = \frac{1 + \cos n\pi}{2n} \qquad a_{2n}$$

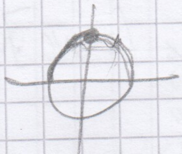
$b_n = \frac{1}{n}$ $\qquad S(x) = \sum_{n=1}^{\infty}\frac{1}{2n}\sin(2nx)$

$$S(3\pi) = \frac{1}{2}\left(f(3\pi+0) + f(3\pi-0)\right) = 0$$

3. $f(x)=\begin{cases} 2\cos\frac{x}{2}, & x\in[-\pi,\pi] \\ 0, & \text{inače}\end{cases}$



$\int_0^\infty \frac{\cos \pi t}{1-4t^2}\,dt$



$\Rightarrow$ parna

$$A(\lambda)=\frac{2}{\pi}\int_0^\pi 2\cos\frac{x}{2}\cos\lambda x = \frac{2}{\pi}\int_0^\pi\left(\cos\left(\left(\tfrac12+\lambda\right)x\right)+\cos\left(\left(\tfrac12-\lambda\right)x\right)\right)dx$$

$$A(\lambda)=\frac{2}{\pi}\left[\frac{1}{\frac12+\lambda}\underbrace{\sin\left(\left(\tfrac12+\lambda\right)x\right)}_{\cos\lambda\pi}+\frac{1}{\frac12-\lambda}\sin\left(\left(\tfrac12-\lambda\right)x\right)\right]\Bigg|_0^\pi$$

$$=\frac{2}{\pi}\left[\frac{2}{1+2\lambda}\cos\lambda\pi+\frac{2}{1-2\lambda}\cos\lambda\pi\right]=$$

$$=\frac{2\cos\lambda\pi}{\pi}\,\frac{4}{1-4\lambda^2}=\frac{8\cos\lambda\pi}{\pi(1-4\lambda^2)} \qquad \boxed{\lambda\neq\pm\frac12}$$

$$\tilde f(x)=\frac{8}{\pi}\int_0^\infty\frac{\cos\lambda\pi}{\cancel{\pi}(1-4\lambda^2)}\cos\lambda x\,d\lambda \quad, x=0 \quad f(0)=2$$

$$\frac{8}{\pi}\int_0^\infty\frac{\cos\lambda\pi}{1-4\lambda^2}=2 \Rightarrow \int_0^\infty\frac{\cos\lambda\pi}{1-4\lambda^2}=\frac{\pi}{4}$$

4. a) $e^{-\alpha t}f(t) \circ\!-\!\bullet \int_0^\infty e^{-st}e^{-\alpha t}f(t)\,dt=\int_0^\infty e^{-t(s+\alpha)}f(t)\,dt=$

$=F(s+\alpha)$

b) $f'(t) \circ\!-\!\bullet \int_0^\infty e^{-st}f'(t)\,dt = \left|\begin{matrix} u=e^{-st} & dv=f'(t)dt\\ du=-se^{-st} & v=f(t)\end{matrix}\right|=$

$$=e^{-st}f(t)\Big|_0^\infty+s\underbrace{\int_0^\infty e^{-st}f(t)\,dt}_{F(s)}=sF(s)+\underbrace{\lim_{t\to\infty}e^{-st}f(t)}_{=0}-f(0)$$

$=sF(s)-f(0)$

5. $f(t) = (t-2)^3 e^{-t} u(t-2)$

$= (t-2)^3 u(t-2) \cdot e^{-t} \circ\!\!-\!\!\bullet \frac{6}{(s+1)^4} \cdot e^{-2(s+1)}$

6. $\int_0^\infty e^{-t} t^{100} \, dt \qquad t^{100} \circ\!\!-\!\!\bullet \frac{100!}{s^{101}} = F(s)$

$\Downarrow$

$s = 1 \qquad F(1) = 100!$

7. $y(t) = \sin t + \int_0^t \tau + y(t-\tau) \, d\tau$

$Y(s) = \frac{1}{s^2+1} + \frac{1}{s^2} \cdot Y(s)$

$Y(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{1}{s^2+1} \cdot \frac{s^2}{s^2-1} = \frac{s^2}{(s^2+1)(s^2-1)}$

$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1} = \frac{s^2}{\sim}$

$As^3 - As + Bs^2 - B + Cs^3 + Cs + Ds^2 + D = s^2$

$A + C = 0 \quad \Rightarrow A = C = 0$

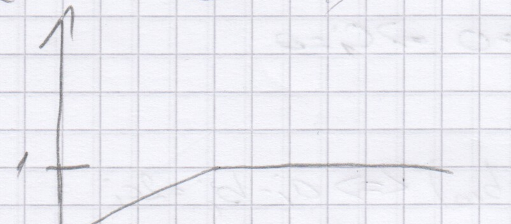
$B + D = 1$

$-A + C = 0$

$-B + D = 0 \Rightarrow B = D = \frac{1}{2} \qquad Y(s) = \frac{1}{2} \frac{1}{s^2+1} + \frac{1}{2} \frac{1}{s^2-1} \bullet\!\!-\!\!\circ$

$\bullet\!\!-\!\!\circ \frac{1}{2} \sin t + \frac{1}{2} \operatorname{sh} t$ //

8.



$e(t) = \frac{t}{2}\left(u(t) - u(t-2)\right) + u(t-2) = \frac{1}{2} t u(t) - \frac{1}{2}(t-2+2) u(t-2) +$

$+ u(t-2) = \frac{1}{2} t u(t) - \frac{1}{2}(t-2) u(t-2) \cancel{- u(t-2) + u(t-2)}$

$= \frac{1}{2} t u(t) - \frac{1}{2}(t-2) u(t-2)$ //

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