

Prvi međuspit iz Matematike 3E i 3R

31.10.2006.

1. (2 boda)

- a) (1b) Napišite definiciju ortogonalnosti funkcija $y = f(x)$ i $y = g(x)$ na intervalu $[a, b]$.
b) (1b) Dokažite da su funkcije $f(x) = \sin mx$ i $g(x) = \sin nx$ ortogonalne za svaki $m, n \in \mathbb{N}$ na intervalu $[-\pi, \pi]$.

2. (3 boda)

- a) (2b) Razvijte u Fourierov red $S(x)$ funkciju $f(x) = \frac{\pi}{4} - \frac{x}{2}$ na intervalu $\langle 0, \pi \rangle$ u red po sinus funkcijama.
b) (1b) Izračunajte $S(3\pi)$.

3. (3 boda) Pomoću prikaza funkcije

$$f(x) = \begin{cases} 2 \cos \frac{x}{2}, & x \in [-\pi, \pi], \\ 0, & \text{inače,} \end{cases} \quad (1)$$

u obliku Fourierovog integrala, izračunajte $\int_0^\infty \frac{\cos \pi t}{1-4t^2} dt$.

4. (2 boda) Neka je $F(s)$ Laplaceov transformat zadanog originala $f(t)$.

Pomoću definicije Laplaceovog transformata nađite Laplaceov transformat funkcije:

- a) (1b) $e^{-\alpha t} f(t)$,
b) (1b) $f'(t)$.

5. (2 boda) Nađite Laplaceov transformat funkcije $f(t) = (t-2)^3 e^{-t} u(t-2)$.

6. (2 boda) Izračunajte integral $\int_0^\infty e^{-t} t^{100} dt$.

7. (3 boda) Riješite integralnu jednačinu

$$y(t) = \sin t + \int_0^t \tau \cdot y(t - \tau) d\tau.$$

8. (3 boda) Pomoću Laplaceove transformacije odredite struju $i(t)$ strujnog kruga sa slike 1 uz priključeni napon $e(t)$ zadan slikom 2.

Rješenja 1. međuspita iz Matematike 3E i 3R

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1. (2 boda)

a) (1 bod) Za funkcije $f, g : [a, b] \rightarrow \mathbb{R}$ kažemo da su ortogonalne na intervalu $[a, b]$ ako vrijedi $\int_a^b f(x)g(x) dx = 0$.

b) (1 bod) $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x) dx = \frac{1}{2} \left[\frac{1}{m-n} \sin(m-n)x \Big|_{-\pi}^{\pi} - \frac{1}{m+n} \sin(m+n)x \Big|_{-\pi}^{\pi} \right] = 0$

2. (3 boda)

a) (2 boda) $f(x) = \frac{\pi}{4} - \frac{x}{2}$ na intervalu $\langle 0, \pi \rangle$ u red po sinus funkcijama \Rightarrow neparna funkcija $\Rightarrow a_n = 0$

$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{4} - \frac{x}{2} \right) \sin nx dx = -\frac{1}{2n} \cos nx \Big|_0^{\pi} + \frac{1}{n\pi} x \cos nx \Big|_0^{\pi} - \frac{1}{n\pi} \int_0^{\pi} \cos nx dx = \dots = \frac{\cos \pi n + 1}{2n}$
za n neparan je $b_n = 0$, a za n paran je $b_n = \frac{1}{n}$ te je Fourierov red
 $f(x) = \sum_{n \geq 1} \frac{1}{2n} \sin(2nx)$

b) (1 bod) $S(3\pi) = \frac{1}{2}(f(3\pi + 0) + f(3\pi - 0)) = 0$.

3. (3 boda) funkcija $f(x)$ je parna $\Rightarrow B(\lambda) = 0$

$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \cos \lambda \xi d\xi = \frac{1}{\pi} \int_{-\pi}^{\pi} 2 \cos \frac{\xi}{2} \cos \lambda \xi d\xi = \frac{2}{\pi} \int_0^{\pi} (\cos \xi (\lambda + \frac{1}{2}) + \cos \xi (\lambda - \frac{1}{2})) d\xi = \frac{2}{\pi} \left(\frac{2}{2\lambda+1} \sin \xi (\lambda + \frac{1}{2}) \Big|_0^{\pi} + \frac{2}{2\lambda-1} \sin \xi (\lambda - \frac{1}{2}) \Big|_0^{\pi} \right) = \frac{4}{\pi} \left(\frac{\sin \pi (\lambda + \frac{1}{2})}{2\lambda+1} + \frac{\sin \pi (\lambda - \frac{1}{2})}{2\lambda-1} \right) = \frac{4}{\pi} \left(\frac{\cos \pi \lambda}{2\lambda+1} - \frac{\cos \pi \lambda}{2\lambda-1} \right) = \frac{8}{\pi} \frac{\cos \pi \lambda}{1-4\lambda^2}, \lambda \neq \pm \frac{1}{2}$
Fourierov integral $f(x) = \frac{8}{\pi} \int_0^{\infty} \frac{\cos \pi \lambda}{1-4\lambda^2} \cos \lambda x d\lambda$
 $\int_0^{\infty} \frac{\cos \pi t}{1-4t^2} dt = \frac{\pi}{8} f(0) = \frac{\pi}{4}$.

4. (2 boda)

a) (1 bod) $\mathcal{L}(f(t)) = F(s)$

$\mathcal{L}(e^{-\alpha t} f(t)) = \int_0^{\infty} e^{-st} e^{-\alpha t} f(t) dt = \int_0^{\infty} e^{-(s+\alpha)t} f(t) dt = F(s + \alpha)$

b) (1 bod) $\mathcal{L}(f'(t)) = \int_0^{\infty} e^{-st} f'(t) dt = (PI) = e^{-st} f(t) \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt = sF(s) + \lim_{t \rightarrow \infty} e^{-st} f(t) - f(0) = sF(s) - f(0)$

5. (2 boda) $f(t) = (t-2)^3 e^{-t} u(t-2)$

$t^3 \circ - \bullet \frac{3!}{s^4},$
 $(t-2)^3 u(t-2) \circ - \bullet \frac{6}{s^4} e^{-2s}$
 $f(t) = (t-2)^3 e^{-t} u(t-2) \circ - \bullet \frac{6}{(s+1)^4} e^{-2(s+1)}$

6. (2 boda) $f(t) = t^{100} \circ - \bullet \frac{100!}{s^{101}} = F(s)$

$\int_0^{\infty} e^{-t} t^{100} dt = F(1) = 100!.$

$$\begin{aligned}
& \mathbf{7. (3 \text{ boda}) } y(t) = \sin t + \int_0^t \tau y(t - \tau) d\tau \\
& \int_0^t \tau y(t - \tau) d\tau = t * y(t) \circ - \bullet \frac{1}{s^2} Y(s) \\
& Y(s) = \frac{1}{1+s^2} + \frac{1}{s^2} Y(s) \Rightarrow Y(s) = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1} = \frac{1}{2} \frac{1}{s^2+1} + \frac{1}{4} \frac{1}{s-1} - \frac{1}{4} \frac{1}{s+1} \\
& y(t) = (\frac{1}{2} \sin t + \frac{1}{4} e^t - \frac{1}{4} e^{-t}) u(t)
\end{aligned}$$

$$\begin{aligned}
& \mathbf{8. (3 \text{ boda}) } e(t) = \frac{t}{2}(u(t) - u(t-2)) + u(t-2) = \frac{1}{2} t u(t) - \frac{1}{2} (t-2) u(t-2) \\
& E(s) = \frac{1}{2s^2} - \frac{1}{2s^2} e^{-2s} \\
& Z(s) = \frac{\frac{1}{s}}{1+\frac{1}{s}} = \frac{1}{s+1} \\
& I(s) = \frac{E(s)}{Z(s)} = \frac{s+1}{2s^2} (1 - e^{-2s}) = \frac{1}{2s} + \frac{1}{2s^2} - \frac{1}{2s} e^{-2s} - \frac{1}{2s^2} e^{-2s} \\
& i(t) = \frac{1}{2} (u(t) + t u(t) - u(t-2) - (t-2) u(t-2)) = \frac{1}{2} [(1+t) u(t) + (1-t) u(t-2)]
\end{aligned}$$