Uncertainties & Error Analysis Tutorial

Reporting Measurements

When we report a measured value of some parameter, X, we write it as

$$X = X_{best} \pm \Delta X \tag{1}$$

where X_{best} represents our **best estimate** of the measured parameter, and ΔX is the **uncertainty** that we associate with the measurement. The statement in Eq. 1 says that if we make a measurement of X, it is likely to fall within the range (X_{best} - ΔX) to (X_{best} + ΔX). To represent this measurement graphically, the best estimate of the value is denoted by a data point, and the uncertainty $\pm \Delta X$ is indicated by error bars drawn on both sides of the measured value.

Alternatively, with some algebraic manipulation, Eq. 1 can be written as

$$X = X_{best} \cdot \left[1 \pm \left(\frac{\Delta X}{X_{best}} \right) \right] \tag{2}$$

where the ratio of the uncertainty, ΔX, to our best estimate, X_{best}, is referred to as the ***fractional uncertainty***. Typically, the uncertainty is small compared to the measured value, so it is convenient to multiply the fractional uncertainty by 100 and report the **percent uncertainty**

$$Percent\ Uncertainty = Fractional\ Uncertainty \times 100 \tag{3}$$

If multiple trials are performed to measure X, the best estimate is the average value, $\overline{X}$, of all the trials (the bar over the variable denotes an average value). The **average value** is computed by summing up all the measured values and then dividing by the number of trials, N. Mathematically, we write this as

$$\overline{X} = \frac{\sum_{i=1}^{N} X_i}{N} = \frac{X_1 + X_2 + ... + X_N}{N} \tag{4}$$

When experimental or time limitations only permit one trial for a measurement, then the best estimate is the most careful measurement you can perform.

Even the best measurements have some degree of uncertainty associated with them. Uncertainty in measurements arises from many sources including the limited precision of the measurement tool, the manner in which the measurement is performed, and the person performing the measurement.

The inclusion of an estimate of the uncertainty when reporting an experimental measurement permits others to determine how much confidence should be placed in the measured value. Additionally, measurements should also be accompanied by a statement of how the uncertainty was determined. Without uncertainties and an explanation of how they were obtained, measured parameters have little meaning.

A simple approach for estimating the uncertainty in a measurement is to report the limiting precision of the measurement tool. For example, if a balance is calibrated to report masses to 0.1 g, then the actual mass of a sample could be up to 0.05 g greater or less than the measured mass, and the balance would still read out the same value. Thus, the uncertainty associated with mass measurements using this balance would be ± 0.05 g. This method for estimating the uncertainty of a measurement is a good choice when only a single trial is performed.

> *NOTE:* **A useful rule of thumb for reporting the uncertainty associated with a measurement tool is to determine the smallest increment the device can measure and divide that value by 2.**

If on the other hand, the best estimate of a parameter is determined by making repeated measurements and computing the average value from the multiple trials, the uncertainty associated with each measurement can be determined from the ***standard deviation,*** σ. Mathematically, the standard deviation can be expressed as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(X_i - \overline{X}\right)^2}{N-1}} = \sqrt{\frac{\left(X_1 - \overline{X}\right)^2 + \left(X_2 - \overline{X}\right)^2 + ... + \left(X_N - \overline{X}\right)^2}{N-1}} \qquad (5)$$

where N is the number of times the measurement is performed, X_i corresponds to the ith measurement of the parameter $\overline{X}$, and is the average value of X.

The standard deviation provides an estimate of the average uncertainty associated with any one of the N measurements that were performed. When the uncertainties are random and multiple trials are performed to obtain the best estimate of a parameter, the standard deviation is an appropriate choice for describing the uncertainty in the measurement. Thus, in this case, the measured value would be reported as

$$X = \overline{X} \pm \sigma$$

Uncertainties in Calculations Involving Basic Math Operations

The previous section addressed how to associate uncertainty with a measured quantity. In most experiments, the parameter of interest is not necessarily measured directly, but rather is the result of a calculation that involves a number of measured quantities.

For example, you may be interested in determining the area of a rectangle, A, based on measurements of the length, l, and the width, w, both of which have some uncertainty associated with them. In general, the area of a rectangle is found from the product of the length and width, $A = l \cdot w$. As usual, the calculated area will be reported as

$$A = A_{best} \pm \Delta A$$

where A_{best} is our best estimate of the area and ΔA is the uncertainty in the area.

The best estimate of the area is simply the product of the best estimates of the measured length and width

$$A_{best} = l_{best} \cdot w_{best}$$

where l_{best} and w_{best} might be average values from multiple measurements or a single careful measurement of each dimension.
Because the measured parameters used to calculate the area have uncertainty associated with them, the calculated area will also have some uncertainty, ΔA, associated with it. The rest of this section addresses how to determine the uncertainty associated with a calculated value when it depends on measured parameters that are added, subtracted, multiplied, or divided to obtain the parameter of interest.

__ADDITION/SUBTRACTION:__ First, consider the situation where two measured values are added together to obtain the desired parameter. For the sake of illustration, we let $q = x + y$ and measure x and y to be $x = 3.0 \pm 0.1$ and $y = 8.0 \pm 1.0$. We will write our final calculation of q in the form $q = q_{best} \pm \Delta q$. The best estimate of q will be

$$
\begin{aligned}
q_{best} &= x_{best} + y_{best} \\
&= 3.0 + 8.0 \\
&= 11.0
\end{aligned}
$$

Now we need to determine the uncertainty in q, which we will denote by Δq. **When two measured values are added, the uncertainty associated with the sum is computed by taking the sum of the squares of the uncertainty associated with each measured value, and then taking the square root of the sum.** This process is called *summing in quadrature*, which refers to the squares of numbers, or quadratics. For the example above, we write the uncertainty in q as

$$\begin{aligned} \Delta q &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(0.1)^2 + (1.0)^2} \\ &= \sqrt{0.01 + 1.0} \\ &= \sqrt{1.01} \\ &= 1.0 \end{aligned}$$

The approach for calculating the uncertainty of a difference is identical to the approach used when summing measured values. Further, if the sum or difference involves more than two measured parameters, an uncertainty term for each measured value is included in the calculation of Δq. The general rule for propagating the uncertainty in sums and differences is summarized in Appendix 1.

MULTIPLICATION/DIVISION: Handling uncertainties in calculations that require multiplication and division is similar to that for addition and subtraction, but the uncertainties are replaced with <u>fractional</u> uncertainties.

To illustrate this point, let the desired calculation be $q = x \cdot y$, where x and y still have the same measured values as in the previous example. We still want to report our calculation of q in the form $q = q_{best} \pm \Delta q$. Again, q_{best} is obtained by using our best estimates of x and y, so

$$\begin{aligned} q_{best} &= x_{best} \cdot y_{best} \\ &= 3.0 \times 8.0 \\ &= 24.0 \end{aligned}$$

Because we are multiplying two parameters with uncertainty to obtain q, it is the fractional uncertainties that are now added in quadrature, so

$$\begin{aligned} \frac{\Delta q}{q_{best}} &= \sqrt{\left(\frac{\Delta x}{x_{best}}\right)^2 + \left(\frac{\Delta y}{y_{best}}\right)^2} \\ &= \sqrt{\left(\frac{0.1}{3.0}\right)^2 + \left(\frac{1.0}{8.0}\right)^2} \\ &= \sqrt{0.001 + 0.016} \\ &= \sqrt{0.017} \\ &= 0.1 \end{aligned}$$

The equation above says that the uncertainty (Δq) associated with q is one-tenth the size of q. Said another way, the value of q computed by multiplying x and y has a 10% uncertainty (i.e., $\Delta q = 0.1 q_{best}$).

In order to report the absolute uncertainty in q, we must solve Eq. 6 for Δq, so

$$
\begin{aligned}
\Delta q &= q_{best} \cdot \sqrt{\left(\frac{\Delta x}{x_{best}}\right)^2 + \left(\frac{\Delta y}{y_{best}}\right)^2} \\
&= (24.0)(0.1) \\
&= 2.4
\end{aligned}
$$

and finally

$$ q = q_{best} \pm \Delta q = 24.0 \pm 2.4 $$

If an exact value like 2 or π is multiplied by a measured value, the exact number does not influence the fractional uncertainty of the calculation, but it does affect the absolute uncertainty. As an example, consider the uncertainty associated with calculating the circumference of a circle from a measurement of the radius. The best estimate for the circumference is

$$ q_{best} = 2\pi r_{best} $$

and the fractional uncertainty in the circumference is obtained from

$$ \frac{\Delta q}{q_{best}} = \sqrt{\left[\frac{\Delta(2\pi)}{2\pi}\right]^2 + \left[\frac{\Delta r_{best}}{r_{best}}\right]^2} $$

The exact number 2π has negligible uncertainty, so $\Delta(2\pi)=0$. Thus,

$$ \frac{\Delta q_{best}}{q_{best}} = \frac{\Delta r_{best}}{r_{best}} $$

The fractional uncertainty of the circumference is only influenced by the uncertainty in the radius measurement. The absolute uncertainty of the circumference, however, is

$$ \Delta q = q_{best}\left(\frac{\Delta r_{best}}{r_{best}}\right) = 2\pi r_{best}\left(\frac{\Delta r_{best}}{r_{best}}\right) = 2\pi\Delta r_{best} $$

and it is affected by the exact factor of 2π.

When measured values are divided to obtain a calculated result, the uncertainties are handled in an identical fashion as for multiplication. Further, if the multiplication or division involves more than two measured parameters, a fractional uncertainty term for each measured value is included in the calculation of $\Delta q/q_{best}$. The general rule for propagating the uncertainty for multiplication and division is summarized in Appendix 1.

Note: **Exact numbers like 2, π, and $g = 9.81$ m/s^2 do not contribute to the fractional uncertainty of a calculation that contains them.**

Uncertainties for Calculations Involving Functions

When calculations involve more complicated operations than addition, subtraction, multiplication, or division, we need to consider a more general rule for propagating errors. In fact, the rules given in Appendix 1 are special cases of what we are about to discuss.

If the calculated parameter q is a function of the measured value x, then q is said to be a function of x, and it is often written as $q(x)$. When this is the case, the uncertainty associated with q is obtained by

$$\Delta q = \left|\frac{dq}{dx}\right| \cdot \Delta x \tag{6}$$

where $|dq/dx|$ is the absolute value of the derivative of q with respect to x, and Δx is the uncertainty in the measurement of x. The expression in Equation 6 is valid for calculating the uncertainty in q when it is only a function of a single variable.

For example, we could measure the angle $\theta = 1.5 \pm 0.2$ radians, so we could calculate $q(\theta) = \sin(\theta)$. The uncertainty associated with q would be

$$
\begin{aligned}
\Delta q &= \left|\frac{dq}{d\theta}\right| \cdot \Delta\theta \\
&= \left|\frac{d}{d\theta}\sin\theta\right| \cdot \Delta\theta \\
&= |\cos(\theta)| \cdot \Delta\theta \\
&= |\cos(1.5)| \cdot (0.2) \\
&= 0.014
\end{aligned}
$$

Then, we would report our calculation of $\sin(\theta)$ as

$$
\begin{aligned}
\sin\theta &= q \\
&= q_{best} \pm \Delta q \\
&= \sin(\theta_{best}) \pm \Delta q \\
&= \sin(1.5) \pm \Delta q \\
&= 0.998 \pm 0.014
\end{aligned}
$$

If the calculated quantity q is a function of more than one variable, then the uncertainty, Δq, associated with q is obtained by generalizing Equation 6. When the calculated quantity q is a function of the measured values $x,...,z$ with uncertainties $\Delta x,...,\Delta z$ then the uncertainty in $q(x,...,z)$ is obtained from

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x} \cdot \Delta x\right)^2 + ... + \left(\frac{\partial q}{\partial z} \cdot \Delta z\right)^2} \tag{7}$$

In words, Equation 7 says that the uncertainty in the calculated parameter q(x,...,z) depends on the uncertainty in each of the measured values that go into the determination of q. The first term in parentheses represents the amount of uncertainty that the measured parameter x contributes to the uncertainty in q. Each subsequent term describes the uncertainty contributed by the other measured values.

When q is a function of more than one variable, the derivatives become partial derivatives. If $q = x^2 y$, then $\partial q/\partial x$ is the derivative of q with respect to x holding y constant, so $\partial q/\partial x = 2xy$. Similarly, $\partial q/\partial y$ is the derivative of q with respect to y holding x constant, so $\partial q/\partial y = x^2$.

Equation 7 is the most general rule for propagating errors through calculations, and all of the previous rules discussed in this tutorial are special cases of this general one. For example, if x is the only measured parameter in the calculation of q, the general form for error propagation given by Eq. 7 reduces to that for functions of a single variable given in Eq. 6.

The general rule given by Equation 7 must be used if the calculation is a function of more than one variable. Equation 7 must also be used if any variable is raised to a power larger than one.

For example, let $q(x,y) = 2x^3y - xy^2$, and x and y are measured with uncertainties to be $x = 4.0 \pm 0.2$ and $y = 3.0 \pm 0.1$. Because $q(x,y)$ is a function of more than one variable, we need to use Equation 7 to determine the uncertainty in $q(x,y)$,

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x} \cdot \Delta x\right)^2 + \left(\frac{\partial q}{\partial y} \cdot \Delta y\right)^2}$$

where

$$\frac{\partial q}{\partial x} = 6x^2y - y^2$$
$$\Delta x = 0.2$$
$$\frac{\partial q}{\partial y} = 2x^3 - 2xy$$
$$\Delta y = 0.1$$

For the values of x and y we take our best estimates, namely $x = 4.0$ and $y = 3.0$, so

$$q(4.0,3.0) = 2(4.0)^3 3.0 - 4.0(3.0)^2 = 348$$
$$\Delta q = \sqrt{(279 \cdot 0.2)^2 + (104 \cdot 0.1)^2} = 57$$

Appendix 2 lists some common single variable derivatives that you will encounter in physics.

Comparing the Equivalence of Quantities with Uncertainties

Once we have calculated the quantities of interest and propagated the uncertainties in an experiment, you will sometimes be asked to compare your results with that of a well-known value. For example, in lab you may have measured the acceleration of gravity to be 10.5 ± 0.75 m/s^2, and you now need to assess whether your experimental measurements with their associated uncertainties are reasonable compared to the well-known value of the acceleration of gravity, $g = 9.81$ m/s^2.

One qualitative approach to answer this question is to consider the lowest and highest likely values you could have measured based on your uncertainty. In the example above, the lowest likely value you would expect to measure is

$$(10.5 - 0.75)m/s^2 = 9.75m/s^2$$

The highest likely value you would have measured would, similarly, be

$$(10.5 + 0.75)m/s^2 = 11.25m/s^2$$

Because the well-known value of 9.81 m/s^2 falls within this range, you could conclude that your measurements were reasonably close to the commonly accepted value for the acceleration of gravity.

A second, more quantitative approach to assessing the equivalence of two quantities is to determine the percent difference between the two values. The percent difference between two values x_1 and x_2 is obtained from

$$Percent\ Difference = \left| \frac{x_1 - x_2}{x_1} \right| \cdot 100 \qquad (8)$$

If you have reason to believe that one of the two values exhibits less uncertainty than the other, take x_1 to be the measurement with the smallest uncertainty. For example, if you were comparing your experimental measurements of the acceleration of gravity to the well-known value of 9.81 m/s^2, let x_1 be the well-known value. Most introductory lab measurements are considered reasonably good if the results agree with well-known values to within 10%.

Appendix 1: Rules for Propagation of Uncertainties

- **ADDITION & SUBTRACTION**: If several quantities $x,...,w$ are measured with uncertainties $\Delta x,...,\Delta w$, and the measured values are used to compute

$$q = x + ... + z - (u + ... + w),$$

  then the uncertainty in the computed value of q is the quadratic sum of the measured uncertainties:
$$\Delta q = \sqrt{(\Delta x)^2 + ... + (\Delta z)^2 + (\Delta u)^2 + ... + (\Delta w)^2}$$

- **MULTIPLICATION & DIVISION**: If several quantities $x,...,w$ are measured with uncertainties $\Delta x,...,\Delta w$, and the measured values are used to compute

$$q = \frac{x \cdot ... \cdot z}{u \cdot ... \cdot w},$$

  then the uncertainty in the computed value of q is the quadratic sum of the fractional uncertainties in $x,...,w$:

$$\frac{\Delta q}{q} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + ... + \left(\frac{\Delta z}{z}\right)^2 + \left(\frac{\Delta u}{u}\right)^2 + ... + \left(\frac{\Delta w}{w}\right)^2}$$

- **FUNCTIONS OF ONE VARIABLE**: If the quantity x is measured with uncertainty Δx, and the measured value is used to compute $q(x)$, then the uncertainty in the value of $q(x)$ is given by

$$\Delta q = \left| \frac{dq}{dx} \right| \cdot \Delta x$$

- **FUNCTIONS OF MORE THAN ONE VARIABLE**: If several quantities $x,...,w$ are measured with uncertainties $\Delta x,...,\Delta w$, and the measured values are used to compute $q(x,...,w)$, then the uncertainty in the computed value of q is the quadratic sum

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x} \cdot \Delta x\right)^2 + ... + \left(\frac{\partial q}{\partial w} \cdot \Delta w\right)^2}$$

  where $\partial q/\partial i$ is the partial derivative of q with respect to the ith measured variable

Appendix 2: Derivatives of Common Single Variable Functions

Function	Derivative
$q(x) = ax^n$	$\dfrac{dq}{dx} = nax^{n-1}$
$q(x) = \sin(ax)$	$\dfrac{dq}{dx} = a\cos(ax)$
$q(x) = \cos(ax)$	$\dfrac{dq}{dx} = -a\sin(ax)$
$q(x) = e^{ax}$	$\dfrac{dq}{dx} = ae^{ax}$
$q(x) = \ln(ax)$	$\dfrac{dq}{dx} = \left(\dfrac{1}{ax}\right)a = \dfrac{1}{x} = x^{-1}$

Note: The variable 'n' represents any positive or negative integer. The symbol 'a' can be any constant.