2. MEĐUISPIT IZ NUM. MAT.

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Z1.

	$i=0$	$i=1$	$i=2$
x	2	2.5	4
y	0.5	0.4	0.25

$f(x) = \frac{1}{x}$ $\qquad \xi_x$

(a) $p \in \mathcal{P}_2$

$$p(x) = 0.5 \cdot \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} + 0.4 \cdot \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} + 0.25 \cdot \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)}$$

(b) $f(x) - p(x) = \dfrac{f^{(III)}(\xi_x)}{3!}\, \omega_3(x)$

- $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = (-1)(-2)x^{-3}$, $f'''(x) = -6x^{-4} = -\dfrac{6}{x^4}$
- $\|f^{(3)}\|_\infty = \max\limits_{x \in [2,4]} |f'''(x)| = |f'''(2)| = \dfrac{6}{2^4} = \dfrac{3}{8}$
- $x = 3 \qquad |\omega_3(3)| = |(3-2)(3-2.5)(3-4)| = 0.5$
- $|f(3) - p(3)| \le \dfrac{1}{6} \cdot \dfrac{3}{8} \cdot 0.5 = \dfrac{1}{32}$

(c) izbjegli rješavanje lin. sustava sa loše uvjetovanom matricom.

Z2. $x_0, x_1, \dots, x_n$

$y_0, y_1, \dots, y_n$

$x_0 \quad x_1 \quad x_2 \quad x_{n-1} \quad x_n$

- $s(x_i) = y_i \quad i = 0, \dots, n$ uvjeti interpolacije $n+1$
- $s(x_i-) = s(x_i+) \quad i = 1, \dots, n-1$ neprek. $n-1$
- $s'(x_i-) = s'(x_i+) \quad i = 1, \dots, n-1$ neprek. 1. der. $n-1$
- $s''(x_i-) = s''(x_i+) \quad i = 1, \dots, n-1$ neprek. 2. der. $n-1$

$3(n-1) + n + 1 = 4n - 2$ uvjeta

(b) $s''(x_0) = s''(x_n) = 0$

(a) $$s(x) = \begin{cases} x^3 + 2x + 1, & 0 \le x < 1 \\ x^2 + 2, & 1 \le x \le 2 \end{cases}$$

- neprekidnost : $s(1-) = s(1+)$

$-1-2+1 \neq 3$ nije spline !

Z2. $$s(x) = \begin{cases} -\frac{9}{2}x^3 + x, & 0 \le x \le \frac{1}{3} \\ \frac{9}{2}x^3 - 9x^2 + 4x - \frac{1}{3}, & \frac{1}{3} \le x \le \frac{2}{3} \\ -2x + 1, & \frac{2}{3} \le x \le 1 \end{cases}$$

$0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

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- $s\left(\frac{1}{3}-\right) = s\left(\frac{1}{3}+\right), \quad s\left(\frac{2}{3}-\right) = s\left(\frac{2}{3}+\right)$

$s'\left(\frac{1}{3}-\right) = s'\left(\frac{1}{3}+\right), \quad s'\left(\frac{2}{3}-\right) = s'\left(\frac{2}{3}+\right)$

$s''\left(\frac{1}{3}-\right) = s''\left(\frac{1}{3}+\right), \quad s''\left(\frac{2}{3}-\right) = s''\left(\frac{2}{3}+\right)$

$$s'(x) = \begin{cases} -\frac{27}{2}x^2 + 1, & 0 \le x \le 1/3 \\ \frac{27}{2}x^2 - 18x + 4, & 1/3 \le x \le 2/3 \\ -2, & 2/3 \le x \le 1 \end{cases}$$

$$s''(x) = \begin{cases} -27x, & 0 \le x \le 1/3 \\ 27x - 18, & 1/3 \le x \le 2/3 \\ 0, & 2/3 \le x \le 1 \end{cases}$$

Z3. $f(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}\,dt$

- $x_i=\frac{i}{n}$, $i=0,..,n$
- po dijelovima linearna

$x\in\left[\frac{i}{n},\frac{i+1}{n}\right]$ $\quad \exists\, \xi_x\in\left(\frac{i}{n},\frac{i+1}{n}\right)$ td $\quad f(x)-s(x)=\frac{f''(\xi_x)}{2!}\left(x-\frac{i}{n}\right)\left(x-\frac{i+1}{n}\right)$

$$\Rightarrow |f(x)-s(x)|\leq \frac{1}{2}\max_{x\in[\frac{i}{n},\frac{i+1}{n}]}|f''(x)|\cdot\Big|\underbrace{\left(x-\tfrac{i}{n}\right)\left(x-\tfrac{i+1}{n}\right)}_{g(x)}\Big|$$

- $g(x)=\left(x-\frac{i}{n}\right)\left(x-\frac{i+1}{n}\right)$

$|g(x)|\leq\frac{1}{4n^2}$, $x\in\left[\frac{i}{n},\frac{i+1}{n}\right]$

- $\max\limits_{x\in[\frac{i}{n},\frac{i+1}{n}]}|f''(x)|\leq\max\limits_{x\in[0,1]}|f''(x)|=|f''(1)|=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}}\cdot 1=\frac{1}{\sqrt{2\pi e}}$

$f'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $\quad f''(x)=-\frac{x e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$, $\quad f'''(x)=-\frac{1}{\sqrt{2\pi}}(1-x^2)e^{-\frac{x^2}{2}}$

$f'''(x)=0 \Rightarrow x=\pm1$ stac. točka za ekstreme fje. $f''(x)$

$$f''(0)=0,\quad \lim_{x\to+\infty}f''(x)=\lim_{x\to+\infty}\left(-\frac{1}{\sqrt{2\pi}}\right)x e^{-\frac{x^2}{2}}=-\frac{1}{\sqrt{2\pi}}\lim_{x\to\infty}\frac{x}{e^{x^2/2}}\overset{L'H}{=}$$

$$=-\frac{1}{\sqrt{2\pi}}\lim_{x\to\infty}\frac{1}{xe^{x^2/2}}=0$$

$\Rightarrow$ f'' u $x=\pm1$ postiže ekstreme

Dobivamo: $\quad \frac{1}{2}\cdot\frac{1}{4n^2}\cdot\frac{1}{\sqrt{2\pi e}}\leq 10^{-4} \iff 10^4\leq 8n^2\cdot\sqrt{2\pi e}$

$$n^2\geq\frac{10^4}{8\cdot\sqrt{2\pi e}}=302{,}4634$$

$$n\overset{\approx}{\geq}17{,}3915 \qquad \boxed{n_0=18}$$

Z4. $I(f) = \int_0^{\pi} e^x \cos x \, dx$

$$I(f) = M_n(f) + \sum_{i=1}^{n} \frac{h^3}{24} f''(\tau_i), \quad \tau_i \in (x_{i-1}, x_i) \qquad h = \frac{b-a}{n}$$

$$|I(f) - M_n(f)| \le \frac{h^3}{24} \|f''\|_\infty \cdot n = \frac{h^2}{24} \|f''\|_\infty (b-a)$$

$f(x) = e^x \cos x$

$f'(x) = e^x (\cos x - \sin x)$

$f''(x) = -2 \sin x \cdot e^x$, $\quad g(x) = |-2 \sin x \cdot e^x| = 2 \sin x \cdot e^x, \quad x \in [0, \pi]$

$g'(x) = 2e^x \cdot (\sin x + \cos x) = 0 \Rightarrow x = \frac{3\pi}{4}$

$g''(x) = 4e^x \cdot \cos x, \quad g''\left(\frac{3\pi}{4}\right) < 0$

$$\max_{x \in [0,\pi]} |f''(x)| = g\left(\frac{3\pi}{4}\right) = \sqrt{2} \cdot e^{\frac{3\pi}{4}}$$

$$\left(\frac{b-a}{n}\right)^2 \cdot \sqrt{2} \cdot e^{\frac{3\pi}{4}} \cdot \frac{b-a}{24} < 10^{-4}$$

$$\frac{\pi^3}{n^2} \cdot \frac{\sqrt{2}}{24} e^{\frac{3\pi}{4}} < 10^{-4} \iff n^2 > 10^4 \cdot \frac{\pi^3 \cdot \sqrt{2}}{24} e^{\frac{3\pi}{4}} \approx 1.9277 \cdot 10^5$$

$$n \gtrapprox 439,05 \qquad \boxed{n = 440}$$

Z5. (i) N

(ii) T

(iii) najbrže : MSF

najsporije : WGF

$$E = |u - u_{ref}| \text{ se ponaša kao } Ch^p$$

$$E \approx Ch^p \;/\; \ln$$

$$p \ldots \quad \ln E = \ln C + p \ln h$$

- red konv. $\rightsquigarrow$ nagib pravca $p \ldots \; Y = D + pH$
- loglog$(\vec{h}, \vec{E})$, $\vec{h} = \left[\frac{1}{N_1}, \ldots, \frac{1}{N_k}\right]$, $\vec{E}$ pripadni vektor grešaka

(iv) primjer iz skripte $z = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$z^T z = [1 \;\; i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + i^2 = 0 \quad \text{a} \quad z \neq 0$$

(v) f 2π-periodička

$N = 3$ točke $\qquad x_j = \frac{2\pi}{3} j, \; j = 0, 1, 2$

$N = 2M + 1 \Rightarrow M = 1$

$$w = e^{i \cdot \frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$\psi(x) = a_0 + a_1 \cos x + b_1 \sin x$ trig. I.P.

$c_0 = a_0, \; c_1 = \frac{a_1 - i b_1}{2}, \; c_{-1} = c_{3-1} = \overline{c_1}$

$p(x) = \sum_{k=0}^{2} c_k e^{ikx}$ formi I.P.

- $e^{-ikx_j} = e^{i(3-k)x_j} \Rightarrow \psi(x_j) = p(x_j) = Y_j \quad j = 0, 1, 2$
- $a_0 = c_0, \; a_k = c_k + c_{3-k}, \; b_k = i(c_k - c_{3-k}), \; k = 1$

TRIG. INTERP. ZADAĆA

- zadano Y_0, Y_1, Y_2 i mreža x_0, x_1, x_2
- odrediti c_0, c_1, c_2 td

$$\sum_{k=0}^{2} c_k e^{ikx_j} = Y_j \qquad j = 0, 1, 2$$

$\rightsquigarrow$ lin. sistem $F_3 c = y$

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix} \qquad w = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

- $\widetilde{F_3} = \frac{1}{\sqrt{3}} F_3$ unitarna, ker je $F_3^* F_3 = 3\mathbb{I} \Rightarrow F_3^{-1} = \frac{1}{3} F_3^* = \frac{1}{3}\overline{F_3}$

- $c = F_3^{-1} y = \frac{1}{3}\overline{F_3}\, y$

$$c_j = \frac{1}{3}\sum_{k=0}^{2} y_k w^{-jk} = \frac{1}{3}\sum_{k=0}^{2} y_k e^{-i\frac{2\pi}{3}jk}$$