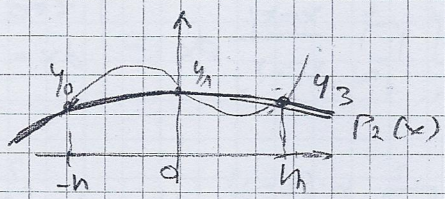
Drugi ciklus

Izvod: Simpsonovo pravilo

$$S(f) = \frac{h}{3}\cdot(y_0 + 4y_1 + y_2)$$

$$\int_{-h}^{h} p_2(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$



$$p_2(x) = ax^2 + bx + c$$

$$\int_{-h}^{h} p_2(x) = \int_{-h}^{h} (ax^2+bx+c)\,dx = \left(\frac{ax^3}{3} + \cancel{\frac{bx^2}{2}} + cx\right)\Bigg|_{-h}^{h} = \frac{2a}{3}\cdot h^3 + 2ch = \frac{h}{3}(2ah^2 + 6c) = *$$

$$p_2(0) = y_1 \quad , \quad p_2(-h) = y_0 \quad , \quad p_2(h) = y_2$$

$$y_1 = c \quad , \quad y_0 = ah^2 - bh + c \quad , \quad y_2 = ah^2 + bh + c$$

$$y_0 + y_2 = 2ah^2 + 2c$$

$$2ah^2 = y_0 + y_2 - 2y_1$$

uvrstimo se u $*$

$$S(f) = \frac{h}{3}\cdot(y_0 + y_2 - 2y_1 + 6y_1) = \frac{h}{3}\cdot(y_0 + 4y_1 + y_2)$$

$$\int_a^b f(x)\,dx \quad ; \quad a = x_0 \quad ; \quad x_1 = \frac{a+b}{2} \quad ; \quad x_2 = b \quad ; \quad h = \frac{b-a}{2}$$

$$S(f) = \frac{b-a}{6}\cdot\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Sami odrediti da je $S(f)$ st. egzaktnosti 3

$$I(f) = \int_a^b f(x)\,dx \doteq \int_a^b p_n(x)\,dx$$

NEWTON-COTESOVE FORMULE

Želimo izvesti Simpsonovu formulu na način da izvod možemo lako generalizirati za proizvoljni n.

$$p_2(x) = \sum_{i=0}^{2} f(x_i)\,\varphi_i(x) \quad , \quad \varphi_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x - x_j}{x_i - x_j}$$

- interpolacijske točke:

$x_0 = a \quad , \quad x_1 = a + h \quad , \quad x_2 = b \qquad h = \frac{b-a}{2}$

$$I(f) = \int_a^b f(x)\,dx \approx \int_a^b p_2(x)\,dx = \int_a^b \sum_{i=0}^{2} f(x_i)\,\varphi_i(x)\,dx = \sum_{i=0}^{2} f(x_i) \underbrace{\int_a^b \varphi_i(x)\,dx}$$

$$\int_a^b \varphi_i(x)\,dx = \int_a^b \prod_{j=0}^{2} \frac{x-x_j}{x_i-x_j}\,dx = \left| \begin{array}{l} x = a+th \\ dx = h\cdot dt \end{array} \right| = \int_0^{\frac{b-a}{h}} \prod_{j=0}^{2} \frac{a+th-(a+jh)}{a+ih-(a+jh)}\,h\,dt$$

$$= \int_0^2 \prod_{j=0}^{2} \frac{h\cdot(t-j)}{i-j}\cdot dt = h\cdot \underbrace{\int_0^2 \prod \frac{t-j}{i-j}\,dt}_{\lambda_{2,i}}$$

$$\lambda_{2,0} = \int_0^2 \prod_{j=0}^{2} \frac{t-j}{0-j}\,dt = \int_0^2 \prod_{j=1}^{2} \frac{t-j}{-j}\,dt = \int_0^2 \frac{(t-1)\cdot(t-2)}{-1\cdot(-2)}\,dt = \ldots = \frac{1}{3}$$

$i=1,\ \lambda_{2,1} = \ldots = \frac{4}{3}$ $\qquad$ $i=2,\ \lambda_{2,2} = \ldots = \frac{1}{3}$

konačno:

$$\int_a^b f(x)\,dx \doteq \int_a^b p_2(x)\,dx = \sum_{i=0}^{2} f(x_i)\cdot h\cdot\lambda_{2,i} = \ldots = \underbrace{\frac{b-a}{6}\cdot\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}_{\text{SIMPSONOVA FORMULA}}$$

Generalizacija : $n \in \mathbb{N}$

$$\int_a^b f(x)\,dx \approx h\cdot \sum_{i=0}^{n} f(x_i)\cdot\lambda_{n,i}$$

$$\lambda_{n,i} = \int_0^n \prod_{j=0}^{n} \frac{t-j}{i-j}\,dt$$

PR.

SIMPSONOVO FORMULA $\frac{3}{8}$

$$\frac{3h}{8}\cdot\left(f_0 + 3f_1 + 3f_2 + f_3\right) \longrightarrow \text{ne treba pamtiti}$$

MATLAB : quad $\rightarrow$ koristi simpsonovu formulu

OCJENE GREŠKE

- $M(f)$, $T(f)$, $S(f)$
- $I(f) = \int_a^b f(x)\,dx$

$$\left.\begin{array}{l} |I(f) - M(f)| \\ |I(f) - T(f)| \\ |I(f) - S(f)| \end{array}\right\} \; O(h^s)$$

s → red točnosti kvadraturne formule

$p_n(x) \in P_n$

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \cdot \omega_{n+1}(x)$$

$\xi_x \in (a,b)$

$$\omega_{n+1}(x) = (x-x_0)\cdot(x-x_1)\cdot\ldots\cdot(x-x_n)$$

$$\underbrace{\int_a^b f(x)\,dx}_{I(f)} - \underbrace{\int_a^b p_n(x)\,dx}_{Q_n(f)} = \frac{1}{(n+1)!}\cdot\int_a^b f^{(n+1)}(\xi_x)\,\omega_{n+1}(x)\,dx$$

$Q_n(f)$ → kvadraturna formula

$$|I(f) - Q_n(f)| = \frac{1}{(n+1)!}\cdot\left|\int \ldots\right| \leq \frac{1}{(n+1)!}\int_a^b \left|f^{(n+1)}(\xi_x)\right|\cdot|\omega_{n+1}(x)|\,dx$$

$$\leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot\int_a^b |\omega_{n+1}(x)|\,dx$$

— gruba ocjena $(b-a)^{n+2}$
imamo $n+1$ član i još integriramo pa bude $n+2$

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

$$|I(f) - Q_n(f)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot(b-a)^{n+2}$$

fina ocjena

za $n=0$, $n=1$, $n=2$

PROP 1.

$f \in C^2(a,b)$. Tada $\exists c \in (a,b)$ t.d. $I(f) - M(f) = \frac{h^3}{24} f''(\tau)$, $h = b-a$

blaže C^2 znači da f, a f ima 2 derivacije

Dokaz

$$M(f) = (b-a)\cdot f\left(\frac{a+b}{2}\right)$$

T. Taylorov razvoj $f(x)$ oko $c = \frac{a+b}{2}$

$$f(x) = f(c) + f'(c)\cdot(x-c) + \frac{f''(\xi_x)}{2!}\cdot(x-c)^2 \quad , \xi_x \text{ između } x \text{ i } c \quad \Big/ \int_a^b dx$$

$$I(f) = \underbrace{f(c)(b-a)}_{M(f)} + f'(c)\cdot\int_a^b (x-c)\,dx + \int_a^b \frac{f''(\xi_x)}{2!}\cdot(x-c)^2\,dx$$

$$\int_a^b (x-c)\,dx = 0$$

f g iz ☆

$$I(f) - M(f) = \int_a^b \frac{f''(\xi_x)}{2!}\cdot(x-c)^2\,dx = *$$

iskoristit ćemo Lagrangeov teorem srednje vrijednosti u malo ~~proompenjenoj~~ formi.

f neprekidna na $[a,b]$ tada je $\exists\, \xi \in (a,b)$ t.d. $\int_a^b f(x)\,dx = f(\xi)(b-a)$

Verzija koja nama treba:

$g(x) \geq 0 \quad , \forall x \in [a,b]$ + f nepreb. na $[a,b]$

tada $\exists\, \xi \in (a,b)$ t.d. $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$

Skica:
$$\underset{m}{\min} \leq f(x) \leq \underset{M}{\max} \quad \Big/ \cdot g(x)$$

$$m\cdot g(x) \leq f(x)\cdot g(x) \leq M\cdot g(x) \quad \Big/ \int$$

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

$$\exists\,\xi \text{ t.d. } f(\xi) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \quad ☆$$

$$* = \frac{1}{2!}f''(\tau)\int_a^b (x-c)^2\,dx = \frac{1}{2}f''(\tau)\,\frac{(x-c)^3}{3}\Big|_a^b = \frac{f''(\tau)}{6}\left[\left(b-\frac{a+b}{2}\right)^3 - \left(a-\frac{a+b}{2}\right)^3\right]$$

$$= \frac{f''(\tau)}{24}\cdot(b-a)^3 = \frac{f''(\tau)}{24}\cdot h^3 \rightarrow \text{formula je reda točnosti 3 zbog } h^3$$

PROP 2 $f \in C^2(a,b)$. Tada $\exists\, \tau \in (a,b)$ t.d. $I(f) - T(f) = -\frac{h^3}{12} \cdot f''(\tau)$, $h = b-a$

PROP 3 $f \in C^4(a,b)$. Tada $\exists\, \tau \in (a,b)$ t.d. $I(f) - S(f) = -\frac{h^5}{90} \cdot f^{(IV)}(\tau)$, $h = \frac{b-a}{2}$

$$|I(f) - M(f)| \le \frac{h^3}{24} \cdot \|f''\|_\infty \quad , h = b-a$$

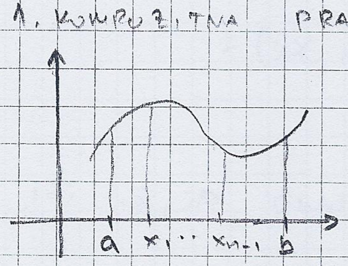
$$|I(f) - T(f)| \le \frac{h^3}{12} \cdot \|f''\|_\infty \quad , h = b-a$$

$$|I(f) - S(f)| \le \frac{h^5}{90} \cdot \|f^{IV}\|_\infty \quad , h = \frac{b-a}{2}$$

KOMPOZITNE FORMULE ZA NUMERIČKU INTEGRACIJU

IDEJA: segment $[a,b]$ podijeli na podintervale te na svakom od njih primjeni osnovnu formulu.

1. KOMPOZITNA PRAVOKUTNA FORMULA



$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = (x_i - x_{i-1}) \cdot f\left(\frac{x_i + x_{i-1}}{2}\right) + \overset{\substack{\text{error} \\ \uparrow}}{E_i^M(f)}$$

$$E_i^M(f) = \frac{h^3}{24} f''(\tau_i) \quad , \tau_i \in (x_{i-1}, x_i) \quad , h = x_i - x_{i-1}$$

$$I(f) = (x_1 - x_0) f\left(\frac{x_0 + x_1}{2}\right) + E_1^M(f) + (x_2 - x_1) \cdot f\left(\frac{x_1 + x_2}{2}\right) + E_2^M(f) + \dots + (x_n - x_{n-1}) f\left(\frac{x_{n-1} + x_n}{2}\right) + E_n^M$$

$$= \underbrace{h \cdot \left[f\left(\frac{x_0 + x_1}{2}\right) + \dots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right]}_{M_n(f)} + \sum_{i=1}^{n} \frac{h^3}{24} f''(\tau_i)$$

svaki od njih je $\le \|f''\|_\infty$

$$|I(f) - M_n(f)| = \left| \sum_{i=1}^{n} \frac{h^3}{24} f''(\tau_i) \right| \underset{\substack{\uparrow \\ \text{nejednakost trokuta}}}{\le} \frac{h^3}{24} \cdot \sum_{i=1}^{n} |f''(\tau_i)| \le \frac{h^3}{24} \cdot \|f''\|_\infty \cdot \underset{\substack{\downarrow \\ n = \frac{b-a}{h}}}{n} = \frac{h^2}{24} \cdot \|f''\|_\infty \cdot (b-a)$$

→ ovim smo izgubili jedan red točnosti jer je došlo do akumulacije pogreške.

na koliko intervala treba podijeliti da je pogreška $< \varepsilon$

$$\frac{(b-a)^3}{24 n^2} \cdot \|f''\|_\infty \le \varepsilon \sim 10^{-5} \text{ npr.}$$

$\dots$ $n = \sqrt{\dots}$, a treba izračunat $\|f''\|_\infty$

→ eventualno na ispitu treba znat izvest kompozitnu trapeznu, Simpson.

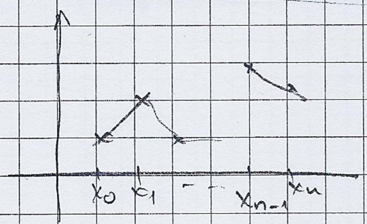
na grafu: koju fju ima najveći prirast na $[a,b]$, ima najveću točnost zbog $O(h^5)$

Gaussove formule → nisu bitne za nas

adaptivna integracija → zanimljivo

Interpolacija splajnovima

LINEARNI SPLAJN



tražimo $s \in S_{\Delta,1}^{\ell}$ takvu da $s(x_i) = y_i,\ i=0,\dots,n$

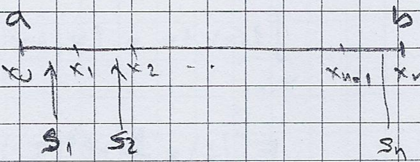
na svakom podintervalu (x_{i-1}, x_i):

$s(x) = a_i \cdot x + b_i\ ,\ i=1,\dots,n$

$2n$ stupnjeva slobode (nepoznati $a_i, b_i\ \ i=1,\dots,n$)

UVJETI:

① Neprekidnost u čvorovima



$s(x_i-) = s(x_i+)$

$i = 1,\dots,n-1$

② Uvjeti interpolacije $s(x_i) = y_i\ ,\ i=0,\dots,n$

ukupno uvjeta: $(n-1) + (n+1) = 2n$

PR 1.

x_i	1	2	3	4	5	6
y_i	16	18	21	17	15	12

$f(4,5) = ?$

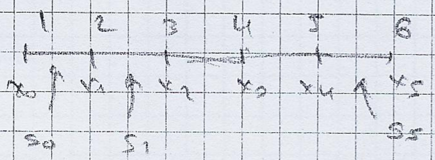
$s \in S_{\Delta,1}$ td $s(x_i) = y_i,\ i=0,\dots,5$

na (x_{i-1}, x_i) $\quad s(x) = a_i x + b\ ,\ i=1,5$ $\qquad s(x_i-) = s(x_i+)\ \ i=1,\dots,4$

(1). $a_i x_i + b_i = y_i \qquad i=0,\dots,5$

$$\left.\begin{array}{ll} i=0 & a_0 \cdot 1 + b_0 = 16 \\ \vdots & \\ i=5 & a_5 \cdot 6 + b = 12 \end{array}\right\} *$$

$f(4.5) =$



$s(x_i-) = s(x_i+)$, $i = 1, \dots, 4$

$$\left.\begin{aligned} a_0 \cdot x_1 + b_0 &= a_1 \cdot x_1 + b_1 \\ &\vdots \\ a_3 \cdot x_4 + b_3 &= a_4 \cdot x_4 + b_4 \end{aligned}\right\} \ (**)$$

$(*) + (**)$ lin sustav

10 jedn s 10 nepoznanica

<u>Bolji način</u>:

$s_i(x) = a_i \cdot (x - x_{i-1}) + b_i$, $i = 1, \dots, n$

- uvjet interpolacije + uvjet neprekidnosti:

$s_i(x_{i-1}) = y_{i-1}$, $i = 1, \dots, n$

$\underline{s_i(x_i) = y_i}$, $i = 1, \dots, n$

npr. $i=1$ $s_1(x_0) = y_0$ $\quad$ $i=2$ $s_2(x_1) = y_1$

$s_1(x_1) = y_1$ $\quad$ $s_2(x_2) = y_2$

uvjet neprekidnosti

$a_i (x_{i-1} - x_{i-1}) + b_i = y_{i-1} \Rightarrow b_i = y_{i-1}$

$a_i (x_i - x_{i-1}) + b_i = y_i \Rightarrow a_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$

- ocjena greške za linearni spline <u>preskačemo</u>

KUBNI SPLAJN

$l=3$

$s \in C^2(a,b)$ t.d. $s\big|_{(x_i,x_{i+1})} \in P_3$

uvjeti interpolacije

uvjeti neprekidnosti na s, s' i s'' u čvorovima mreže

x_i	x_0	x_1	x_2	---	x_n
y_i	y_0	y_1	y_2	---	y_n

$$\underset{x_0}{|} \;s_1\; \underset{x_1}{|} \;s_2\; \underset{x_2}{|} \;-\; \cdots \;\underset{x_{n-1}}{|}\; s_n \;\underset{x_n}{|}$$

Sustav mora biti opisan regularnom matricom koja se rješava metodom Choleskog

Parametri: $a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 \rightarrow 4$ koeficijenta

n podintervala $\rightarrow 4n$ parametra

uvjeti: neprekidnost $s(x_i-) = s(x_i+)$ $i=1,\dots,n-1$ $\quad n-1$

interpolacije $s(x_i) = y_i$, $i=0,\dots,n$ $\quad n+1$

neprekidnost s': $s'(x_i-) = s'(x_i+)$ $i=1,\dots,n-1$ $\quad n-1$

neprekidnost s'': $s''(x_i-) = s''(x_i+)$ $i=1,\dots,n-1$ $\quad n-1$

$3(n-1) + n+1) = 4n-2$ uvjeta

uvjeta: $4n-2 \Rightarrow 2$ stupnja slobode !

Dodatni rubni uvjeti:

(i) $s'(x_0) = f'(x_0)$, $s'(x_n) = f'(x_n)$ — Potpuni splajn

(ii) $s'(x_0) = s'(x_n)$, $s''(x_0) = s''(x_n)$ — Periodički splajn

(iii) $s''(x_0) = s''(x_n) = 0$ — prirodni splajn

iv) matlabova naredba spline koristi "NOT-A-KNOT" uvjet

$s'''(x_1-) = s'''(x_1+)$

$s'''(x_{n-1}-) = s'''(x_{n-1}+)$

Konstrukcija (prirodnog) kubnog splajna ⇝ vidi slajd

⇝ lin. sustav za tridijagonalnom matricom

Struktura matrice sustava:

1) Tridijagonalna, simetrična

2) Strogo dijagonalno dominantna (po retcima)

DEF. $A \in \mathbb{R}^{n\times n}$ je strogo dijag. dom. po retcima ako vrijedi:

$$\sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}| < |a_{kk}| \quad , \forall k=1,\dots,n$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |a_{11}| > a_{12} + a_{13}$$

$$|a_{22}| > a_{21} + a_{23}$$

PROPOZICIJA : $A \in \mathbb{R}^{n\times n}$ strogo dijag. dom. Tada je A regularna.

dokaz: pretp. da je A singularna matrica ⇒ ~~nema jedinstveno rješenje~~

Tada postoji $\exists x \in \mathbb{R}^{n\times n}$, $x\neq 0$ td $Ax=0$

Odaberemo indeks $k \in \{1,\dots,n\}$ td $|x_k| = \max\limits_{j=1,\dots,n} |x_j|$

Imamo:

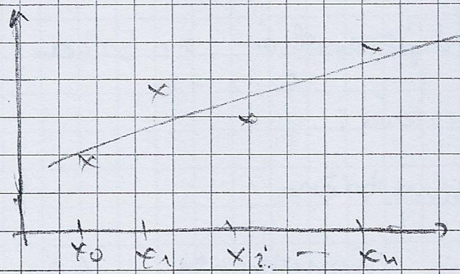
$$0 = (Ax)_k = \sum_{j=1}^{n} a_{kj}\cdot x_j = a_{kk}\cdot x_k + \sum_{\substack{j=1\\ j\neq k}}^{n} a_{kj}\cdot x_j$$

$$|a_{kk} x_k| = \left|\sum_{\substack{j=1\\ j\neq k}}^{n} a_{kj}\cdot x_j\right| \leq \sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}||x_j| \quad /: |x_k|$$

$$|a_{kk}| \leq \sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}|\cdot \underbrace{\frac{|x_j|}{|x_k|}}_{\leq 1} \rightarrow \text{kontradikcija.} \quad > \sum |a_{kj}| < |a_{kk}|$$

Metoda najmanjih kvadrata

x	x_0	x_1	...	x_n
y	y_0	y_1	...	y_n



$x_i \neq x_j$ za $i \neq j$

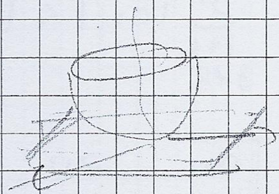
Želje: naći f-ju $\tilde{f}(x)$ t.d. je ukupna greška interpolacije što manja.

Problem aproksimacije

$$\sum_{i=0}^{n} |y_i - \tilde{f}(x_i)|^2 \to min$$

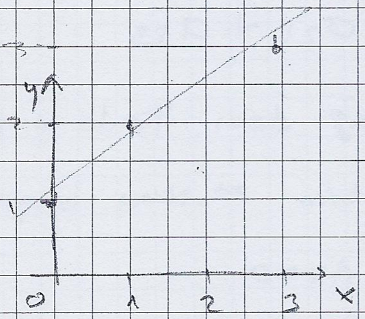
$$\sum |y_i - (c_0 + c_1 \cdot x_i)|$$

$$E(c_0, c_1) \to min$$



Pr. 1

x	0	1	3
y	1	2	3



Postoji li pravac koji prolazi zadanim točkama?

$k \cdot 0 + l = 1$

$k \cdot 1 + l = 2$

$k \cdot 3 + l = 3$

em. sustav 3 jedn. s 2 nep.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A\vec{x} = \vec{b}$$

MAT 1:

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 1 & 3 \end{array}\right] \sim \dots \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$\left.\begin{array}{l} r(A) = 2 \\ r(A_{pros}) = 3 \end{array}\right\} \neq$$ sustav nema egzaktno rješenje

MATLAB: $x = A \backslash b \rightsquigarrow \begin{bmatrix} 0.64286 \\ 1.14286 \end{bmatrix}$

$\|Ax - b\|_2 \to min$ $\qquad \|Ax - b\|^2 \to min$ LMNK

$r = Ax - b$
rezidual.

Važno znati razliku između linearne (L) i nelinearne MNK

$$\|Ax-b\|_2^2 = \sum_{i=1}^{2} (k\cdot x_i + \ell - y_i)^2 \to \min$$

LINEARNA REGRESIJA

$$\tilde{f}(x) = c_0 + c_1 \cdot x$$

$$\sum_{i=0}^{n} (y_i - \tilde{f}(x_i))^2 \to \min$$

greška

$$E(c_0, c_1) = \sum_{i=0}^{n} [y_i - (c_0 + c_1 \cdot x)]^2$$

$$\left.\begin{array}{l} E(c_0, c_1) \to \min \\ c_0, c_1 \in \mathbb{R} \end{array}\right\}$$

• nužan uvjet ekstrema: $\nabla E(c_0, c_1) = 0$

$$0 = \frac{\partial E}{\partial c_0} = 2 \cdot \sum_{i=0}^{n} (y_i - c_0 - c_1 \cdot x_i) \cdot (-1)$$

$$0 = \frac{\partial E}{\partial c_1} = 2 \sum_{i=0}^{n} (y_i - c_0 - c_1 \cdot x_i) \cdot (-x_i)$$

$$\left.\begin{array}{l} \sum_{i=0}^{n} y_i - c_0 \sum_{i=0}^{n} 1 - c_1 \cdot \sum_{i=0}^{n} x_i = 0 \\ \sum_{i=0}^{n} x_i \cdot y_i - c_0 \cdot \sum_{i=0}^{n} x_i - c_1 \cdot \sum_{i=0}^{n} x_i^2 = 0 \end{array}\right\} \text{sve znamo osim } c_0 \text{ i } c_1$$

$$\left.\begin{array}{l} c_0 (n+1) + c_1 \cdot \sum_{i=0}^{n} x_i = \sum_{i=0}^{n} y_i \\ c_0 \cdot \sum_{i=0}^{n} x_i + c_1 \cdot \sum_{i=0}^{n} x_i^2 = \sum_{i=0}^{n} x_i \cdot y_i \end{array}\right\} (1)$$

---

lin. sustav 2×2. Je li matrica sustava $\begin{bmatrix} n+1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$ regularna?

$$\underbrace{\begin{bmatrix} s_0 & s_1 \\ s_1 & s_2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \ell_0 \\ \ell_1 \end{bmatrix}$$

$\det A \neq 0$?

skalarni produkt

$$\det A = s_0 \cdot s_2 - s_1^2 = \underbrace{(n+1)}_{\sum_{i=0}^{n} 1^2} \cdot \sum_{n=0}^{n} x_i^2 - \left(\sum_{i=0}^{n} x_i\right)^2 = \|e\|_2^2 \cdot \|x\|_2^2 - (x, e)^2$$

$$x=\begin{bmatrix}x_0\\ \vdots\\ x_n\end{bmatrix},\quad e=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

Nejednakost SCB

(Schwarz - Cauchy - Bunjakovski)

$\forall x,y\in\mathbb{R}^n$

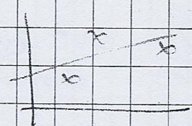
$|(x,y)|\le \|x\|_2\cdot\|y\|_2$

Predavanje 3.6.2016

Razlika linearne i ne linearne regresije

$$c_0\cdot\overset{s_0}{(n+1)}+c_1\cdot\overset{s_1}{\sum_{i=0}^{n}x_i}=\sum_{i=0}^{n}y_i$$

$$c_0\cdot\overset{s_1}{\sum_{i=0}^{n}x_i}+c_1\cdot\overset{s_2}{\sum_{i=0}^{n}x_i^2}=\sum_{i=0}^{n}x_i\cdot y_i$$



$f(x)=c_0+c_1\cdot x$

x	x_0 ... x_n
y	y_0 ... y_n

$Ax=b$

$\det A\neq 0 \quad A=\begin{bmatrix}s_0 & s_1\\ s_1 & s_2\end{bmatrix}$

$$\det A=s_0\cdot s_2-s_1^2=(n+1)\cdot\sum_{i=0}^{n}x_i^2-\left(\sum_{i=0}^{n}x_i\right)^2$$

$$=\left(\sum_{i=0}^{n}1^2\right)\cdot\left(\sum x_i^2\right)-\left(\sum x_i\cdot 1\right)^2$$

$$=\|e\|_2^2\cdot\|x\|_2^2-(x,e)^2$$

↑ skalarni produkt

$$x=\begin{bmatrix}x_0\\ \vdots\\ x_n\end{bmatrix},\quad e=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

SCB: $\forall x,y\in\mathbb{R}^n$ vrijedi:

$|(x,y)|\le\|x\|_2\cdot\|y\|_2$

- Jednakost vrijedi akko su x i y kolinearni vektori.

Dokaz SCB: $x, y \in \mathbb{R}^n$

- pretp. $\exists \lambda \in \mathbb{R}$ td $x = \lambda y$

$$\boxed{|(x,y)| = |(\lambda y, y)| = |\lambda| \cdot \|y\|_2^2 = \|x\|_2 \cdot \|y\|_2 \quad \text{QED}}$$

- pretp. da x i y nisu kolinearni

$f(\lambda) = (\underbrace{x - \lambda y}_{\neq 0 \text{ jer nisu kolinearni}}, x - \lambda y) \qquad f: \mathbb{R} \to \mathbb{R}^{+}$; jer je norma uvijek poz.

$f(\lambda) = \|x - \lambda y\|_2^2 > 0$ jer je $x \neq \lambda y$

$0 < f(\lambda) = (x - \lambda y, x - \lambda y) = (x,x) - \lambda(x,y) - \lambda(y,x) + \lambda^2 \cdot (y,y)$
skalamo

$= \|x\|_2^2 - 2\lambda(x,y) + \lambda^2 \cdot \|y\|_2^2 \quad , \forall \lambda \in \mathbb{R}$

diskriminanta $\quad D = 4\lambda^2 (x,y)^2 - 4\lambda^2 \cdot \|x\|_2^2 \cdot \|y\|_2^2 < 0$

$|(x,y)|^2 < \|x\|_2^2 \cdot \|y\|_2^2$

$$\boxed{|(x,y)| < \|x\|_2 \cdot \|y\|_2 \quad \forall x, y \in \mathbb{R}^n}$$

uzmem $\|e\|_2^2 \cdot \|x\|_2^2 - (x,e)^2 > 0 \quad \rightarrow$ ne možem uzet ≥ 0 jer je tu samo ako su x i e kolinearni a oni nisu jer u vektoru x su različiti elementi.

<u>Kvadratna regresija</u> : kratka napomena, na zadaćama

$f(x) = c_0 + c_1 \cdot x + c_2 \cdot x^2$

$E(c_0, c_1, c_2) = \|y_i - (c_0 + c_1 \cdot x + c_2 \cdot x^2)\|_2^2 \rightarrow \min$

↳ ennearizacija opet dobijemo linearni sustav 3×3

MATLAB : polyfit

NELINEARNI PROBLEM NAJMANJIH KVADRATA

$f(x) = c_0 \cdot e^{c_1 \cdot x} \rightarrow$ exponential fitting

$(c_0, c_1) = \sum_{i=0}^{n} \left[y_i - (c_0 \cdot e^{c_1 \cdot x_i}) \right]^2 \rightarrow \min$

$\frac{\partial E}{\partial c_0} = 2 \sum [y_i - c_0 e^{c_1 x_i}] \cdot (-e^{c_1 x_i}) = 0$

$\frac{\partial E}{\partial c_1} = 2 \sum (y_i - c_0 e^{c_1 x_i}) \cdot (-c_0 c_1 \cdot e^{c_1 x_i}) = 0$

} uočavamo da je ovo nelinearni sustav za c_0 i c_1

Kako efektivno riješiti problem najmanjih kvadrata?

(n+1 jednadžbi, k+1 nepoznanica) k - stupanj aproks.

broj nepoznanica $\rightsquigarrow$ $c_0, c_1, \ldots, c_k$ stupanj aprok. fje

broj jednadžbi $\rightsquigarrow$ broj mjerenja $\begin{array}{c|c} x & x_0 \ldots x_n \\ \hline y & y_0 \ldots y_n \end{array}$ $\Big\}$ $n > k$

① NORMALNE JEDNADŽBE

② QR, SVD FAKTORIZACIJE

$\|Ax - b\|_2^2 \to \min.$

Pr.

x	0	1	3
y	1	2	3

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\|Ax - b\|_2^2 \to \min \ldots (1)$

svelo se na rješavanje lin. sustava: $\begin{bmatrix} (n+1) & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i \cdot y_i \end{bmatrix} \ldots (2)$

Konveksnost fje nam pomaže pri određivanju min. što nam govori da je lokalni min i globalni.

$f(x) = \|Ax - b\|_2^2 \to \min, \quad A \in \mathbb{R}^{n \times k}, \quad k < n$

$$\left.\begin{array}{l} f(x) \to \min \\ x \in \mathbb{R}^k \end{array}\right\}$$

- $f(x+h) \approx f(x) + \underbrace{\nabla f(x)}_{=0 \text{ za min}} \cdot h + \frac{1}{2} (\nabla^2 f(x) \cdot h, h)$

$\uparrow$ kvadratna aproksimacija

$(x, y) = y^T x$

$(Ax, y) = (x, A^T y)$

$y^T A x = y^T \cdot \underbrace{A^{T^T}}_{A} \cdot y$

$\nabla f = ? \qquad \nabla^2 f(x) = ?$

$$f(x) = \|Ax - b\|_2^2 = (Ax - b) \cdot (Ax - b) = (Ax, Ax) - (Ax, b) - (b, Ax) + (b, b)$$

$$= x^T \cdot A^T A x - b^T A x - x^T A^T \cdot b + \|b\|_2^2$$

$$= x^T \cdot A^T \cdot A \cdot x - 2 \cdot b^T \cdot A \cdot x + \|b\|_2^2$$

Uvedimo oznake:

$Q = A^T \cdot A \in \mathbb{R}^{k \times k}$ $\qquad \gamma = \|b\|_2^2$

$d^T = -2 \underset{1/n}{b^T} \underset{n \times k}{A} \in \mathbb{R}^{1 \times k}$

$f(x) = x^T \cdot Q x + d^T \cdot x + \gamma$ $\to$ ovo je skalar

$f : \mathbb{R}^k \to \mathbb{R}$ kvadratna forma

$$x^T Q x = (Qx, x) = \sum_{i,j=1}^{k} q_{ij} \cdot x_j \cdot x_i$$ $\to$ elementi matrice Q

$\nabla(x^T Q x) = ?$

$$\frac{\partial}{\partial x_\ell}\left(\sum_{i,j=1}^{k} q_{ij} \cdot x_j \cdot x_i\right) = \frac{\partial}{\partial x_\ell}\left(\sum_{\substack{i=1 \\ i \neq \ell}}^{k} q_{i\ell} \cdot x_\ell \cdot x_i + \sum_{\substack{j=1 \\ j \neq \ell}}^{k} q_{\ell j} \cdot x_\ell \cdot x_j + q_{\ell\ell} \cdot x_\ell^2\right)$$

$$= \sum_{\substack{i=1 \\ i \neq \ell}}^{k} q_{i\ell} \cdot x_i + \sum_{\substack{j=1 \\ j \neq \ell}}^{k} q_{\ell j} \cdot x_j + 2 q_{\ell\ell} \cdot x_\ell$$

$$= \sum_{i=1}^{k} q_{i\ell} \cdot x_i + \sum_{j=1}^{k} q_{\ell j} \cdot x_j = \sum_{i=1}^{k} (q_{i\ell} + q_{\ell i}) \cdot x_i$$

$$= [(Q^T + Q)x]_\ell$$

Dobili smo grad. forme prvog člana

$$\nabla(x^T Q x) = (Q^T + Q)x$$

Uoči: u primjeru Q je simetrična $\qquad \nabla(x^T Q x) = 2Qx$

Sami izračunajte $\nabla(d^T x) = d$

$\boxed{\text{Dobili smo: } \nabla f(x) = 2Qx + d}$

$\nabla^2 f(x) = ?$

$$[\nabla(Qx)]_{ij} = \frac{\partial}{\partial x_j}(Qx)_i = \frac{\partial}{\partial x_j}\left(\sum_{\ell=1}^{k} q_{i\ell} \cdot x_\ell\right) = \ldots \underset{\text{identično kao i prije}}{=} \frac{\partial}{\partial x_j}\left(\sum_{\substack{\ell=1 \\ \ell \neq j}}^{k} q_{i\ell} x_\ell + q_{ij} x_j\right) = q_{ij}$$

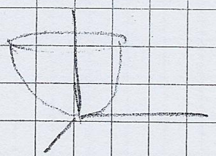
$\underbrace{\nabla(Qx)}_{\text{vektor}} = \underset{\text{matrica}}{Q}$

$$\min_{x \in \mathbb{R}^k} f(x) = x^T Q x + d^T x + \gamma$$

Što je fundamentalno kod traženja min. kvadratne forme?

To je pojam $\boxed{\text{KONVEKSNOSTI}}$

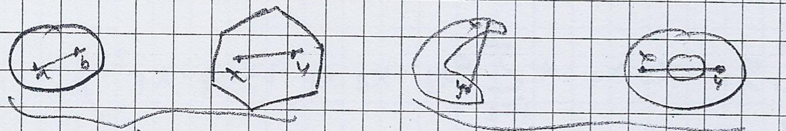
$f(x,y)=x^2+y^2$ $\qquad\qquad$ $f(x,y)=x^2-y^2$



sedlo → nije konveksna

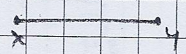
Kada je f konveksna, tada je lokalni min. ujedno i globalni.

def. $S\subseteq\mathbb{R}^d$ je konveksan ako



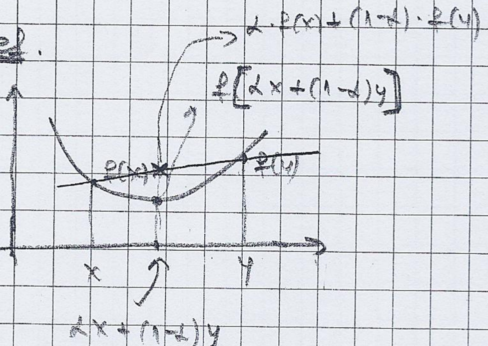
konv. $\qquad\qquad$ nisu konv.

$\alpha x+(1-\alpha)y$ za $\alpha=[0,1]$



Skup S će biti konv. ako $\forall x,y\in S$ i $\forall\alpha\in[0,1]$

def.



DEF. neka je $S\subseteq\mathbb{R}^d$ konv. skup

$f:S\to\mathbb{R}$ je konveksna na S ako $\forall x,y\in S$ vrijedi

$f(\alpha x+(1-\alpha)y)\le\alpha f(x)+(1-\alpha)\cdot f(y)$, $\forall\alpha\in[0,1]$

Za naš problem: $f(x)=x^T\cdot Q\cdot x+d^T\cdot x+r$ KVADRATNA FORMA

f je konv. $\Longleftrightarrow$ $\nabla^2 f(x)$ poz. semidef $(Ax,x)\ge 0$

f je strogo konv. akko $\nabla^2 f(x)$ poz. def $(Ax,x)>0$ ☆

U ☆ je važno da je f kvadratna f-ja.

$f(x)=x^4$, $f''(0)=0$ $\quad$ f je strogo konv.

Za općenite f $\quad\nabla^2 f(x)$ poz. def. $\Rightarrow$ f strogo konv.

Vratimo se na MNK!

$f(x) = \|Ax - b\|_2^2 \rightarrow \min$

$A \in \mathbb{R}^{n \times k}$, $k < n$

za $Ax = b$, $n < k$ $\rightarrow$ nema jedinstveno rješenje

$\nabla f(x) = 0$

$2Qx + d = 0$, $Q = A^T \cdot A$; $d^T = -2b^T \cdot A$

$Qx = \frac{-d}{2}$ $\boxed{A^T A x = A^T b}$ $\rightarrow$ sustav normalnih jedn. $(*)$

- Egzistencija i jedinstvenost rješenja normalnih jednadžbi?

TEOREM

Neka je $r(A) = k$

Stupci matrice A su lin. nezavisni

Tada je rješenje problema NMK $f(x) \rightarrow \min$ $x \in \mathbb{R}^k$ ujedno i jedinstveno rješenje sustava normalnih jednadžbi $(*)$

Dokaz uzmemo $y \in \mathbb{R}^k$

$h = y - x$

$f(y) = y^T \cdot Q \cdot y + d^T \cdot y + r = (x+h)^T \cdot Q \cdot (x+h) + d^T \cdot (x+h) + r$

$= x^T Q x + x^T \cdot Q \cdot h + h^T \cdot Q \cdot x + h^T \cdot Q \cdot h + d^T \cdot x + d^T \cdot h + r = \heartsuit$

$Qx = \frac{-d}{2}$ $h^T \cdot Q \cdot x = \frac{-1}{2} h^T \cdot d$ $Q^T = (A^T \cdot A)^T = A^T \cdot A = Q$

$x^T Q h = (x^T Q h)^T = h^T \cdot Q^T \cdot x = h^T \cdot Q \cdot x = \frac{-1}{2} h^T \cdot d$

$\heartsuit = x^T Q x + \left(\frac{-1}{2} h^T d\right) + \left(\frac{-1}{2} h^T d\right) + h^T Q h + d^T x + \cancel{d^T h} + r$

$f(y) = f(x) + \underbrace{h^T Q h}_{\geq 0 \text{ za min}}$

Izraz $h^T Q h \geq 0$ jer je: $h^T Q h = h^T \cdot A^T A \cdot h = (Ah)^T \cdot Ah = \|Ah\|_2^2$

dakle, $\forall y \in \mathbb{R}^k$ vrijedi da je $f(y) = f(x) + \underbrace{\|Ah\|_2^2}_{\geq 0} \geq f(x)$

Jedinstvenost:

predp. $f(x) = f(y)$ za $x \neq y$

nulvektori

To znači da je $h^T Q h = 0$, odnosno $\|Ah\|_2 = 0$ tj $Ah = 0$ za $h \neq 0$

Što ne može jer dolazi do kontradikcije jer su stupci + lin. nez.

$Ah = 0$, $A \in \mathbb{R}^{n \times k}$, $n > k$

$h \in \mathbb{R}^k$

$$h_1\left[\vec{a}_1\right] + h_2\left[\vec{a}_2\right] + \dots + h_k\left[\vec{a}_k\right] = \vec{0}$$

↳ to bi značilo da su $\vec{a}$-ovi zavisni, a iz predp. oni su ne zavisni

Predavanje

$$\min_{x \in \mathbb{R}^k} \|Ax-b\|$$

- $Ax \approx b$, $A \in \mathbb{R}^{n \times k}$, $b \in \mathbb{R}^n$

- $n > k$, $r(A) = k$

$\boxed{A^T \cdot A \cdot x = A^T b}$ SUSTAV NORMALNIH JEDNADŽBI

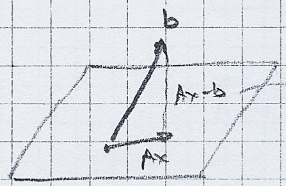
$\longrightarrow$ slika od A

$b \notin R(A)$

jer su stupci od A lin. nezavisni

$$Ax = x_1 \cdot [\vec{a}_1] + x_2 \cdot [\vec{a}_2] + \dots + x_k \cdot [\vec{a}_k] = 0$$

$Ax \in R(A)$ - podprostor od R^n koji je razapet stupcima $\vec{a}_1, \dots, \vec{a}_k$



$R(A) = \mathbb{R}^k$ $\rightarrow$ trebam da b se ne može prikazat kao komb. stupaca Ax $\rightarrow$ to znači da $b \notin R(A)$

$\|Ax-b\| \Leftrightarrow \min$ kada su ovi vektori ortogonalni

$$\|Ax-b\| \longrightarrow \min \Leftrightarrow (Ax-b, Ay) = 0, \quad \forall y \in \mathbb{R}^k$$

slika od A, neki proizvoljni vektor iz

$$\Leftrightarrow (A^T(Ax-b), y) = 0, \quad \forall y \in \mathbb{R}^k$$

$$\Leftrightarrow A^T(Ax-b) = 0 \qquad (x, Ay) = (A^T x, y)$$

$$\boxed{A^T \cdot Ax = A^T \cdot b}$$

- $(Ax-b) \cdot Ax = \dots = 0 \rightarrow$ DZ

Pr.

x	0	1	3
y	1	2	3

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix} \rightarrow c_0, c_1$$

$$A^T A = \ldots = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$A^T A x = A^T b$

$A \in \mathbb{R}^{n \times k}$, $n > k$, $r(A) = k$

ima jedinstveno rješenje $\Longleftrightarrow$ $A^T A$ regularna matrica

- $A^T A$ simetrična
- $r(A) = k$, $A^T A$ poz. definitna to znači $(Ax, x) > 0$ $\forall x \in \mathbb{R}^n$, $x \neq 0$

$$(A^T A x, x) = (Ax, Ax) = \|Ax\|^2 > 0 \quad \forall x \neq 0$$

$Ax = 0$ ne može imat netrivijalno rješenje zbog $r(A) = k$

MNK $\rightsquigarrow$ matrična formulacija $\rightsquigarrow$ sustav normalnih jedn.

Kako numerički riješiti sustav normalnih jednadžbi?

- zbog sim, poz def matrice $\rightarrow$ FAKTORIZACIJA CHOLESKOG $\rightarrow$ problem s uvjetovanošću $A^T A$
- konačna aritmetika računala

$$A = \begin{bmatrix} 1 & 1 \\ \delta & 0 \\ 0 & \delta \end{bmatrix}, \quad \delta < \sqrt{\varepsilon} \doteq 10^{-8}, \quad \delta > 0$$

$$A^T A = \begin{bmatrix} 1 & \delta & 0 \\ 1 & 0 & \delta \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ \delta & 0 \\ 0 & \delta \end{bmatrix} = \begin{bmatrix} 1+\delta^2 & 1 \\ 1 & 1+\delta^2 \end{bmatrix} \rightarrow \text{dok u računalu je to } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ a to je singularna matrica}$$

U praksi se koristi: QR faktorizacija

SVD faktorizacija

QR faktorizacija

$A \in \mathbb{R}^{n \times k}$

$A = QR$, $Q \in \mathbb{R}^{n \times n}$ ortogonalna

$R \in \mathbb{R}^{n \times n}$ gornjetrokutasta

MAT 1: Q je ortogonalna ako je $Q \cdot Q^T = Q^T \cdot Q = I$

Binet - Chauchy $\Rightarrow \det |Q| = \pm 1$

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}}_{Q^T} = \begin{bmatrix} 1 & & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

ortogonalne matrice su matrice sastavljene od ortogonalnih vektora

$$\alpha_i \alpha_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$ regularne!

PR 1.

$$Q = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$ → matrica rotacije u ravnini, ovo je linearni operator

matrica rotacije u $\mathbb{R}^2$ za kut φ.

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$$

ZAD. $x = [x_1 \ \ x_2]^T$

$\|x\| = \sqrt{x_1^2 + x_2^2}$ $\qquad A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

$\|Ax\| = \|x\|$ → ortogonalne matrice čuvaju euklidsku normu.

PR.

$\|x\|^2 = (x, x) = x^T x$, $x \in \mathbb{R}^n$

$Q \in \mathbb{R}^{n \times n}$ ortogonalna $\rightsquigarrow \|Qx\| = \|x\|$

$\|Qx\|^2 = (Qx, Qx) = (Q^T Q x, x) = (x, x) = \|x\|^2 \quad \forall x \in \mathbb{R}^n$

$R \in \mathbb{R}^{n \times k}$ gornje trokutasta

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$$

npr. za 4×3

Kako se računa QR faktorizacija?

- "na ruke": Gram-Schmidtov postupak ortogonalizacije
- Householderove transf. } u računalu se tako
  Givensove rotacije } implementira

MATLAB: $[Q, R] = qr(A)$

Primjena QR faktorizacije na rješavanje LMNK

$\|Ax - b\| \rightarrow \min$

$A \in \mathbb{R}^{n \times k}$, $n > k$, $b \in \mathbb{R}^n$

$r(A) = k$

$$A = QR = [Q' \ Q''] \cdot \begin{bmatrix} R' \\ 0 \end{bmatrix} = Q' \cdot R'$$

$A = Q' \cdot R'$ gdje je $Q' \in \mathbb{R}^{n \times k}$ ortogonalna prvih k stupaca matrice Q

$R' \in \mathbb{R}^{k \times k}$ gornje trokutasta, prvih k redaka matrice R

r - rezidual $\qquad Q \cdot Q^T = I = 1$

$$\|r\| = \|Ax - b\| = \|Q \cdot R \cdot x - b\| = \|Q \cdot (Rx - Q^T \cdot b)\| = \|Rx - Q^T \cdot b\|$$

$$Q^T \cdot b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \begin{array}{l} b_1 \in \mathbb{R}^k \\ b_1 = (Q')^T \cdot b \end{array}$$

$$\|Ax - b\|^2 = \|Rx - Q^T b\|^2 = \left\| \begin{bmatrix} R'x \\ 0 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\|^2 = \|R'x - b_1\|^2 + \underbrace{\|b_2\|^2}_{\text{ovog ne možemo kontrolirati}}$$

tražimo x da bude $\rightarrow \min$

Izaberemo x takav da je $R'x - b_1 = 0$ tj. $R'x = b_1$

$$R'x = (Q')^T \cdot b$$

Rješavanje LMNK svodi se na rješavanje jednog gornjetrokutastog linearnog sustava $R'x = (Q')^T \cdot b$

Regularno je rješenje jer smo pretp. $r(A) = k$, a onda R' je $k \times k$

ALGORITAM:

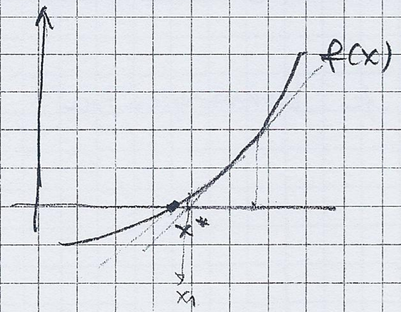
ULAZ: $A \in \mathbb{R}^{n \times k}$, $r(A)=k$, $n > k$, $b \in \mathbb{R}^n$

IZLAZ: $x \in \mathbb{R}^k$ koji minimizira $\|Ax-b\|$

1. Izračunaj $A = Q' \cdot R'$
2. Izračunat $(Q')^T \cdot b \in R^k$
3. riješi: $R'x = (Q')^T \cdot b$

↳ pomoću povratnih supstitucija

Newtonova metoda



$f(x^*) = 0$

neki niz konvergira ako je monoton i ograničen.

- iterativna metoda
  - x_0 zadan → treba odrediti $x_1, x_2, \dots, x_n$

x_n, $n \in N$

$f : I \subseteq \mathbb{R} \to \mathbb{R}$

f je derivabilna

$f(x^*) = 0$

$f'(x) \neq 0$ na nekoj okolini od x^*

$f(x) = 0 \Rightarrow x = ?$

x_0 zadan

$y - f(x_0) = f'(x_0) \cdot (x - x_0)$

$y = 0$: $-f(x_0) = f'(x_0) \cdot (x - x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x)} = \cancel{x}_1$

$f'(x) \to f'(x_0) \neq 0$

⋮

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $n \in N_0$, ali x_n mora biti zadan

MAT 1: $(x_n)_{n \in N}$ niz realnih brojeva

- konvergira li niz prema x^*
- ako da, koliko brzo

PR 1. Newtonovom metodom izračunajmo $\sqrt{a}$, $a>0$

Rj.

$$f(x) = x^2 - a = 0$$

$$f'(x) = 2x \quad , \quad x^* = \sqrt{a} \rightsquigarrow \text{tražimo}$$

Newtonova metoda: x_0 zadan

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}\left(x_n - \frac{a}{x_n}\right) = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad , n \in \mathbb{N}_0 \quad \ldots \quad (1)$$

MAT1: Ograničen i monoton niz realnih brojeva je konvergentan

(1) ograničenost niza $(x_n)_{n\in\mathbb{N}}$

(2) monotonost niza $(x_n)_{n\in\mathbb{N}}$

$\rightsquigarrow$ rješenje na Moodleu

Zaključak: niz x_n dan formulom (1) je konvergentan pa postoji limes

$$\exists \lim_{n\to\infty} x_n$$

Označimo: $x^* = \lim\limits_{n\to\infty} x_n$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad \lim_{n\to\infty}$$

$$x^* = \frac{1}{2}\left(x^* + \frac{a}{x^*}\right)$$

$$\hookrightarrow x^* = \sqrt{a}$$

Brzina konvergencije $(x_n)_{n\in\mathbb{N}}$? KVADRATIČNO

Općenito o brzini konvergencije niza.

(x_n), $x_n \to x^*$, $n\to\infty$

$e_n = |x_n - x^*| \to 0 \quad , n \to \infty$

DEF 1. Kažemo da niz (x_n) konvergira prema x^* <u>linearno</u> ako je

$\exists c$, $0<c<1$ i $\exists n_0 \in \mathbb{N}$

zašto manja od 1 → nema konvergencije

td $\forall n \geq n_0$ vrijedi:

$$|x_{n+1}-x^*| \leq c\cdot|x_n-x^*|$$

<u>Pr 1</u>

$e_0 = |x_0-x^*| = 1$, $c=10^{-1}$ (i)

$e_1 = |x_1-x^*| \leq c\cdot \overbrace{|x_0-x^*|}^{1} \leq 10^{-1}$

$e_2 = |x_2-x^*| \leq 10^{-1}\cdot 10^{-1} = 10^{-2}$

⋮

$e_7 = |x_7-x^*| \leq 10^{-7}$

(ii) $e_0=1$, $c=0.99$ → koliko će nam biti iteracija? (više)

DEF 2. Kažemo da niz (x_n) konvergira prema x^* <u>kvadratično</u> ako $\exists c>0$ i $\exists n_0\in\mathbb{N}$

$$|x_{n+1}-x^*| \leq c\cdot|x_n-x^*|^2$$

za pr (1) sada trebaju 3 koraka da $e<10^{-7}$

<u>Predavanje 10.6.2016.</u>

- Pokažimo da Newtonova metoda konvergira kvadratično (ako konvergira)

$f: \subseteq\mathbb{R}\to\mathbb{R}$, $f(x)=0$, x^* nultočka $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n\in\mathbb{N}_0$

- f razvijemo u Taylorov red oko x_n, te uzmemo $x=x^*$

$f(x) = f(x_n)+f'(x_n)(x-x_n)+\frac{1}{2}\cdot f''(\xi_n)\cdot(x-x_n)^2$, ξ_n između x i x_n

$0 = f(x^*) = f(x_n)+f'(x_n)\cdot(x^*-x_n)+\frac{1}{2} f''(\xi_n)(x^*-x_n)^2 \quad / : f'(x_n)$

$0 = \frac{f(x_n)}{f'(x_n)} + (x^*-x_n) + \frac{1}{2}(x^*-x_n)^2\cdot\frac{f''(\xi_n)}{f'(x_n)}$

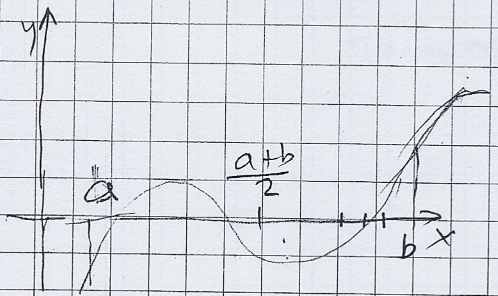
x_0 zadan

$|x_{n+1}-x^*| \leq C\cdot|x_n-x^*|^2$

$x_{n+1}-x^* = \frac{1}{2}\cdot(x^*-x_n)^2\cdot\frac{f''(\xi_n)}{f'(x_n)}$

$|x_{n+1}-x^*| \leq C\cdot|x^*-x_n|^2$

Newtonova metoda je osjetljiva na odabir početne iteracije. Kako odrediti zadovoljavajući x_0?

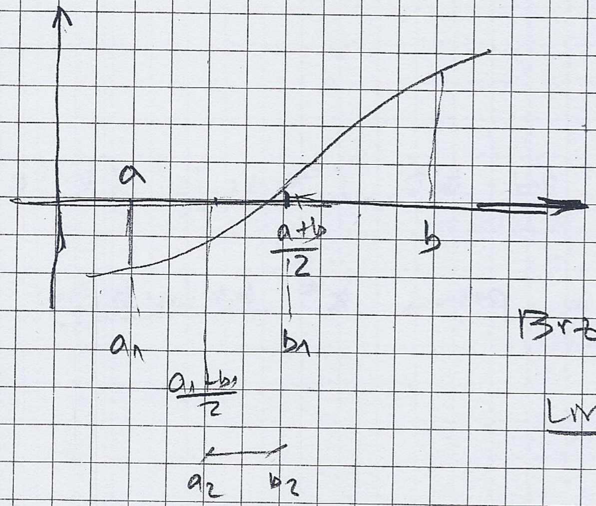


→ metoda bisekcije → sporije konvergira od Newtonove

Metoda Bisekcije:

MAT 1: $f: [a,b] \to \mathbb{R}$, neprekidna i neka je $f(a) \cdot f(b) \leq 0$

tada postoji barem jedna nultočka od f u $[a,b]$.



Brzina konvergencije?

Linearna jer je $e_{n+1} \leq \frac{1}{2} e_n$

SLAJD: usporedba metode bisekcije i newtona

34 5 iteracija

10.6.2016.

Pokažimo da Newtonova metoda konvergira kvadratično (ako konvergira)

$$f : I \subseteq \mathbb{R} \to \mathbb{R} \qquad f(x) = 0 \qquad x^* \text{ nultočka}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}_0$$

x_0 zadan

$$|x_{n+1} - x^*| \leq C\,|x_n - x^*|^2$$

ξ_n između x i x_n

f razvijamo u Taylorov red do x_n te uzmemo $x = x^*$

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f_n''(\xi_n)(x - x_n)^2$$

$$0 = f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{1}{2} f_n''(\xi_n)(x^* - x_n)^2 \; / f'(x_n)$$

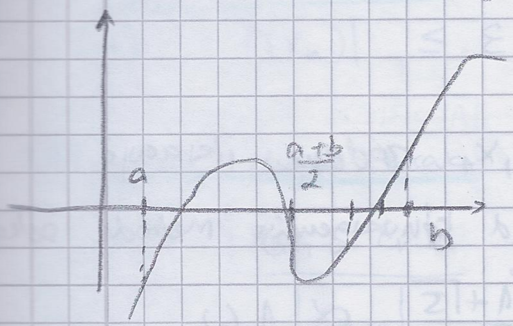
$$0 = \frac{f(x_n)}{f'(x_n)} + (x^* - x_n) + \frac{1}{2} \frac{f_n''(\xi_n)}{f_n'(x_n)} (x^* - x_n)^2$$

$$x_{n+1} - x^* = \frac{1}{2} (x^* - x_n)^2 \cdot \frac{f_n''(\xi_n)}{f_n'(x_n)}$$

$$|x_{n+1} - x^*| \leq C\,|x^* - x_n|^2$$

Newtonova metoda je osjetljiva na odabir početne iteracije

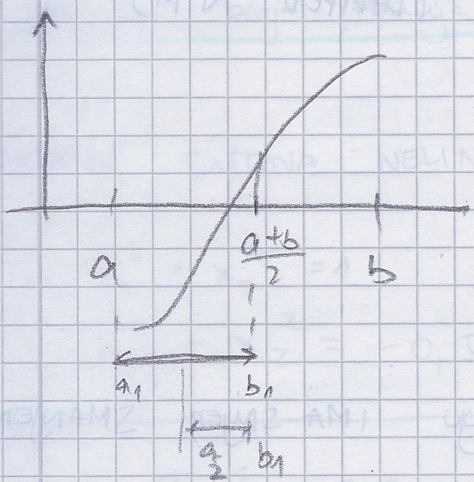
Kako odrediti zadovoljavajući x_0?



METODA BISEKCIJE

MAT1 : $f : [a,b] \to \mathbb{R}$
neprekinuta i neka je $f(a) \cdot f(b) \leq 0$

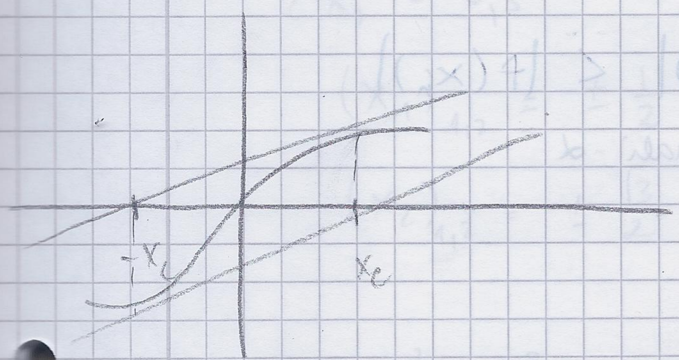
Tada postoji barem jedna nultočka od f
u $[a,b]$



brzina konvergencije?

linearna

SLAJD: usporedba metode bisekcije i Newtonove
34 5



$$y - f(x_c) = f'(x_c)(x - x_c)$$

$$-f(x_c) = f'(x_c)(x - x_c)$$

$$x - x_c = -\frac{f(x_c)}{f'(x_c)} \Rightarrow -2x_c = \frac{-\operatorname{arctg} x_c}{\frac{1}{1+x_c^2}}$$

$$\Leftrightarrow \quad 2x_c = (1+x_c^2) \operatorname{arctg} x_c$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n + \alpha s_n \qquad 0 < \alpha < 1$$

prednosti i nedostaci Newtonove metode

⊕ kvadratična konvergencija

⊖ odabir početne iteracije
potreba računanja f'

METODA SEKANTE

x_0 zadano

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}_0$$

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

x_0, x_1 zadane iteracije

Red konvergencije metode sekante

$$\frac{1+\sqrt{5}}{2} \approx 1{,}62$$

SKRAĆIVANJE KORAKA U NEWTONOVOJ METODI (DAMPED N.M)

$$x_{n+1} = x_n \underbrace{- \frac{f(x_n)}{f'(x_n)}}_{s_n \text{ korak}}, \quad n \in \mathbb{N}_0$$

x_0 zadano

$$\boxed{x_{n+1} = x_n + s_n}$$

NEWTONOV KORAK KOJI VODI PREMA RJEŠENJU IMA SMJER SMANJENJA VRIJEDNOSTI $|f(x)|$ LOKALNO OKO $x = x_n$

$$|f(x_n + \alpha s_n)| < |f(x_n)|$$

za dovoljno mali α

SKRAĆIVANJE KORAKA!

$$\boxed{x_{n+1} = x_n + \alpha s_n}$$

$0 < \alpha < 1$

pr. $f(x_0) > 0$ $\quad f'(x_0) < 0$

$$x_1 = x_0 - \frac{f(x_0)^{>0}}{f'(x_0)_{<0}} > 0 \Rightarrow x_n > x_0$$

$x_0 > 0$

$$|f(x_1)| > |f(x_0)|$$

↳ ako da, skrati korak

KRITERIJI ZAUSTAVLJANJA NEWTONOVE METODE

(1) ocjena reziduala : $|f(x_n)| \leq \varepsilon$

1.1. $|f(x_n)| \leq \varepsilon\, |f(x_0)|$

(2) ocjena duljine koraka $|x_{n+1} - x_n| < \varepsilon$

2.1. $\dfrac{|x_{n+1} - x_n|}{|x_{n+1}|} < \varepsilon$

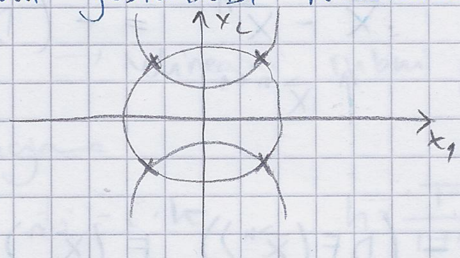
(3) kontrola broja iteracija ITER < MAX_ITER

NA ISPITU

RJEŠAVANJE SUSTAVA NELINEARNIH JEDNADŽBI NEWTONOVOM METODOM

pr.

$$\left.\begin{aligned} x_1^2 + x_2^2 &= 1 \\ x_1^2 - x_2^2 &= -0,5 \end{aligned}\right\} +$$



$$2x_1^2 = 0,5$$

$$(x_1)_{1,2} = \pm \frac{1}{2}$$

$$(x_2)_{1,2} = \pm \frac{\sqrt{3}}{2}$$

NEWTONOVA METODA u $\mathbb{R}^2$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

$$F(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$$

$$f_2(x_1, x_2) = x_1^2 - x_2^2 + 0,5$$

$F(\vec{x}) = 0$ $\quad F: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad (F: \mathbb{R}^d \to \mathbb{R}^d)$ za d jednadž

↑ NELINEARNI SUSTAV

u 1d: $f_{LIN}(x) = f(x_n) + f'(x_n)(x - x_n)$

$$f_{LIN}(x) = 0$$

$$f(x_n) + f'(x_n)(x - x_n) = 0 \quad / : f'(x_n)$$

$$x - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

u 2d:

$$F_{LIN}(\vec{x}) = F(\vec{x}^n) + DF(\vec{x}^n)(\vec{x} - \vec{x}^n) \qquad \text{(MAT2)}$$

$$[DF(\vec{x}^n)]_{ij} = \frac{\partial F_i}{\partial x_j}(\vec{x}^n) \qquad \text{JAKOBIJEVA MATRICA}$$

$$F_{LIN}(\vec{x}) = 0$$

$$F(\vec{x}^n) + DF(\vec{x}^n)(\vec{x} - \vec{x}^n) = 0$$

$$(DF(\vec{x}^n))^{-1} / \; DF(\vec{x}^n)(\vec{x} - \vec{x}^n) = -F(\vec{x}^n)$$

$$\underbrace{\vec{x}}_{\vec{x}^{n+1}} - \vec{x}^n = -(DF(\vec{x}^n))^{-1} F(\vec{x}^n)$$

$$\vec{x}^{n+1} = \vec{x}^n - (DF(\vec{x}^n))^{-1} F(\vec{x}^n) \qquad \vec{x}^0 \text{ zadano}$$

$$\vec{s}^n = \vec{x}^{n+1} - \vec{x}^n \qquad \text{Newtonov korak}$$

$$DF(\vec{x}^n)\,\vec{s}^n = -F(\vec{x}^n) \qquad \text{linearni sustav} \qquad \text{ISPIT}$$

$$\vec{x}^{n+1} = \vec{x}^n + \vec{s}^n$$

Izračunajmo "na prste" naš primjer

$$F(\vec{x}) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ x_1^2 - x_2^2 + 0{,}5 \end{bmatrix}, \quad \vec{x}^0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$DF(\vec{x}) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}$$

1. KORAK:

$$DF(\vec{x}^0)\,\vec{s}^0 = -F(\vec{x}^0)$$

$$\begin{bmatrix} 2 & 6 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} s_1^0 \\ s_2^0 \end{bmatrix} = -\begin{bmatrix} 9 \\ -7{,}5 \end{bmatrix}$$

$$\rightsquigarrow \quad \vec{s}^0 = \begin{bmatrix} -0{,}375 \\ -1{,}375 \end{bmatrix}$$

$$\vec{x}^1 = \vec{x}^0 + \vec{s}^0 = \begin{bmatrix} 0,625 \\ 1,625 \end{bmatrix}$$

2. KORAK:

$$DF(\vec{x}^1)\,\vec{s}^1 = -F(\vec{x}^1)$$

$$\vec{x}^2 = \vec{x}^1 + \vec{s}^1$$

$$\vdots$$

Završni primjer:

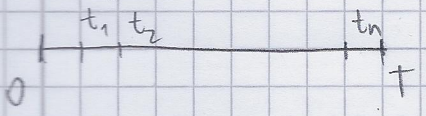
$$\varphi'' + \sin\varphi = 0 \quad \text{na } [0, T]$$

$$\varphi(0) = \alpha, \quad \varphi(T) = \beta$$

---

nepoznato: $\varphi(t)$ → nelinearni rubni problem za ODJ

Diskretizacija centralnim diferencijama



$$t_i = i \cdot h \qquad h = \frac{T}{n+1}$$

$$i = 0, \dots n+1$$

$$\frac{1}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right) + \sin\varphi_i = 0 \qquad i = 1, \dots n$$

$$\varphi_0 = \alpha \qquad \varphi_{n+1} = \beta$$

---

dobije se linearni sustav sa trodijagonalnom matricom

nelinearna jednadžba $\xrightarrow{\text{diskretizacija}}$ nelinearni sustav $\xrightarrow{\text{Newtonova metoda}}$ linearni sustav $\xrightarrow{\text{direktne metode (LU, PLU, Chole..., Thomas)}}$ rješenje