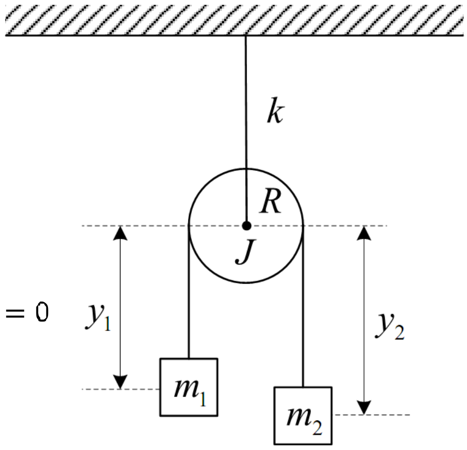
Zad 1.) Mase m1 i m2 vezane su nerastezljivom niti duljine l preko koloture momenta tromosti J i polumjera R (slika 1). Uz pretpostavku da nema klizanja između niti i koloture odrediti jednadžbu gibanja mase m1 primjenom lagrangeove jednadžbe.

$y_1 = c - y_2 \qquad \omega = \left(\frac{v}{r}\right)^2 = \frac{\dot{y}_1}{R^2}$



Slika 1

$\dot{y}_1 = -\dot{y}_2$

$$E_k = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_1^2 + \frac{1}{2}J\frac{\dot{y}_1^2}{R^2}$$

$$E_p = -m_1 g y_1 - m_2 g y_2 = -m_1 g y_1 - m_2 g(c - y_1) \qquad E_p(y_1 = 0) = 0$$

$$L = E_k - E_p = \frac{1}{2}\dot{y}_1^2\left(m_1 + m_2 + \frac{J}{R^2}\right) + y_1 g(m_1 - m_2) + m_2 g c$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} = (m_1 + m_2 + \frac{J}{R^2})\ddot{y}_1$$

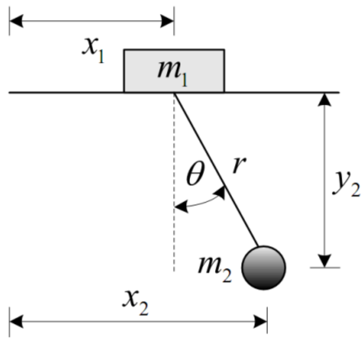
$$\frac{\partial L}{\partial y_1} = g(m_1 - m_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = \left(m_1 + m_2 + \frac{J}{R^2}\right)\ddot{y}_1 - g(m_1 - m_2) = 0$$

Zad 2.) Na slici 2 prikazan je mehanički sustav koji se sastoji od mase m1 koja može slobodno kliziti po podlozi (bez trenja) i na nju vezana m2. Potrebno je korištenjem Lagrangeove jednadžbe odrediti jednadže gibanja pojedinih masa. Pritom kao poopćene koordinate potrebno je uzeti pomak x1 i kut θ.

$y_2 = r\cos\varphi \qquad \dot{y}_2^2 = r^2\dot{\varphi}^2\sin^2\varphi$

$x_2 = x_1 + r\sin\varphi \qquad \dot{x}_2 = \dot{x}_1 + r\dot{\varphi}\cos\varphi$



Slika 2

$$\dot{x}_2^2 = \dot{x}_1^2 + 2\dot{x}_1\dot{\varphi}r\cos\varphi + r^2\dot{\varphi}^2\cos^2\varphi$$

$$E_k = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{y}_2^2 + \dot{x}_2^2\right) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\varphi}r\cos\varphi + r^2\dot{\varphi}^2\right)$$

$$E_p = -m_2 g h = -m_2 g r\cos\varphi \qquad E_p(y_2 = 0) = 0$$

$$L = E_k - E_p = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + 2\dot{x}_1\dot{\varphi}r\cos\varphi + r^2\dot{\varphi}^2\right) + m_2 g r\cos\varphi$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2}m_2 2\dot{x}_1 r\cos\varphi + \frac{1}{2}m_2 r^2 2\dot{\varphi} = m_2\dot{x}_1 r\cos\varphi + m_2 r^2\dot{\varphi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = m_2 r(\ddot{x}_1\cos\varphi - \dot{x}_1\dot{\varphi}\sin\varphi) + m_2 r^2\ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = \frac{1}{2}m_2 2\dot{x}_1\dot{\varphi}r(-\sin\varphi) + m_2 g r(-\sin\varphi) = -m_2\dot{x}_1\dot{\varphi}r\sin\varphi - m_2 g r\sin\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = m_2 r\ddot{x}_1\cos\varphi + m_2 r^2\ddot{\varphi} + m_2 g r\sin\varphi = 0$$

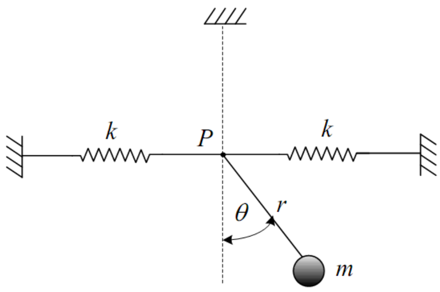
$$\frac{\partial L}{\partial \dot{x}_1}=\frac{1}{2}m_1 2\dot{x}_1+\frac{1}{2}m_2 2\dot{x}_1+\frac{1}{2}m_2 2\dot{\varphi} r\cos\varphi=m_1\dot{x}_1+m_2\dot{x}_1+m_2\dot{\varphi} r\cos\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}=m_1\ddot{x}_1+m_2\ddot{x}_1+m_2 r(\ddot{\varphi}\cos\varphi-\dot{\varphi}\dot{\varphi}\sin\varphi)=\ddot{x}_1(m_1+m_2)+m_2 r(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi)$$

$$\frac{\partial L}{\partial x_1}=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}-\frac{\partial L}{\partial x_1}=(m_1+m_2)\ddot{x}_1+m_2 r\ddot{\varphi}\cos\varphi-m_2 r\dot{\varphi}^2\sin\varphi=0$$

Zad 3.) Primjenom Lagrangeove jednadžbe odrediti jednadžbe gibanja mase m i vrha njihala (točka P) x prikazanog na slici 3. Pritom je gibanje vrha njihala moguće samo u smjeru x osi. Pretpostaviti idealne karakteristike opruge. Za poopćene koordinate uzeti kut θ i pomak vrha njihala (točke P) x.



Slika 3

$$x'=r\sin\varphi+x \qquad \dot{x}'=r\dot{\varphi}\cos\varphi+\dot{x}$$
$$(\dot{x}')^2=r^2\dot{\varphi}^2\cos^2\varphi+\dot{x}^2+2r\dot{x}\dot{\varphi}\cos\varphi$$

$$y'=r\cos\varphi \qquad \dot{y}'=-r\dot{\varphi}\sin\varphi \qquad \left(\dot{y}'\right)^2=r^2\dot{\varphi}^2\sin^2\varphi$$

$$E_k=\frac{1}{2}m\left(\dot{y}'^2+\dot{x}'^2\right)=$$

$$=\frac{1}{2}m(r^2\dot{\varphi}^2\cos^2\varphi+\dot{x}^2+2r\dot{x}\dot{\varphi}\cos\varphi+r^2\dot{\varphi}^2\sin^2\varphi)$$

$$E_p=-mgh+2\frac{1}{2}kx^2=-mgr\cos\varphi+kx^2$$

$$L=E_k-E_p=\frac{1}{2}m(r^2\dot{\varphi}^2+2r\dot{x}\dot{\varphi}\cos\varphi+\dot{x}^2)+mgr\cos\varphi-kx^2$$

$$\frac{\partial L}{\partial \dot{x}}=\frac{1}{2}m2r\dot{\varphi}\cos\varphi+\frac{1}{2}m2\dot{x}=mr\dot{\varphi}\cos\varphi+m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=mr(\ddot{\varphi}\cos\varphi-\dot{\varphi}\dot{\varphi}\sin\varphi)+m\ddot{x}$$

$$\frac{\partial L}{\partial x}=-2kx$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=m\ddot{x}+\ mr\ddot{\varphi}\cos\varphi-mr\dot{\varphi}^2\sin\varphi+2kx=0$$

$$\frac{\partial L}{\partial \dot{\varphi}}=\frac{1}{2}m2r^2\dot{\varphi}+\frac{1}{2}m2r\dot{x}\cos\varphi=mr^2\dot{\varphi}+mr\dot{x}\cos\varphi$$

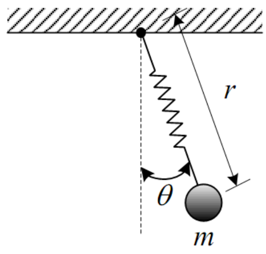
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}}=\ mr^2\ddot{\varphi}+mr(\ddot{x}\cos\varphi-\dot{x}\dot{\varphi}\sin\varphi)$$

$$\frac{\partial L}{\partial \varphi}=-mr\dot{x}\dot{\varphi}\sin\varphi-mgr\sin\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = mr^2\ddot{\varphi} + mr\ddot{x}\cos\varphi + mgr\sin\varphi = 0$$

$$r\ddot{\varphi} + \ddot{x}\cos\varphi + g\sin\varphi = 0$$

Zad 4.) Odrediti jednadžbe gibanja mehaničkog sustava prikazanog na slici 4 uz poopćene koordinate θ i r.



Slika4

$x = r\sin\varphi \qquad \dot{x} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$
$\dot{x}^2 = \dot{r}^2\sin^2\varphi + 2\dot{r}r\dot{\varphi}\sin\varphi\cos\varphi + r^2\dot{\varphi}^2\cos^2\varphi$

$y = r\cos\varphi \qquad \dot{y} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$
$\dot{y}^2 = \dot{r}^2\cos^2\varphi - 2\dot{r}r\dot{\varphi}\sin\varphi\cos\varphi + r^2\dot{\varphi}^2\sin^2\varphi$

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$E_p = -mgh + \frac{1}{2}k(r - l_0)^2 = -mgr\cos\varphi + \frac{1}{2}k(r - l_0)^2$$

$$L = E_k - E_p = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + mgr\cos\varphi - \frac{1}{2}k(r - l_0)^2$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m2\dot{r} = m\dot{r} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = m\dot{\varphi}^2 r + mg\cos\varphi - k(r - l_0)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\varphi}^2 - mg\cos\varphi + k(r - l_0) = 0$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2}m2r^2\dot{\varphi} = mr^2\dot{\varphi} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = mr^2\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} \qquad \frac{\partial L}{\partial \varphi} = -mgr\sin\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = mr^2\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} + mgr\sin\varphi = 0$$