Službeni podsjetnik za završni ispit (Teorija estimacije)

Matrični diferencijalni račun

$$\frac{\partial (x^{\mathrm{T}} y)}{\partial x} = y^{\mathrm{T}}, \qquad \frac{\partial (x^{\mathrm{T}} y)}{\partial y} = x^{\mathrm{T}}$$
$$\frac{\partial (x^{\mathrm{T}} A x)}{\partial x} = x^{\mathrm{T}} A^{\mathrm{T}} + x^{\mathrm{T}} A$$
$$\frac{\partial (A x)}{\partial x} = A, \qquad \frac{\partial (x^{\mathrm{T}} A)}{\partial x} = A$$
$$\frac{\partial \operatorname{Tr}(A B A^{\mathrm{T}})}{\partial A} = A B^{\mathrm{T}} + A B,$$
$$\frac{\partial \operatorname{Tr}(A B)}{\partial A} = B^{\mathrm{T}}, \qquad \frac{\partial \operatorname{Tr}(A^{\mathrm{T}} B)}{\partial A} = B, \qquad \frac{\partial \operatorname{Tr}(B A^{\mathrm{T}})}{\partial A} = B$$

Diskretizacija linearnih sustava

$$\Phi = e^{AT} = I + AT + \frac{A^2 T^2}{2!} + \frac{A^3 T^3}{3!} + \ldots = \sum_{v=0}^{\infty} A^v \frac{T^v}{v!}$$
$$\Gamma = e^{AT} \left(\int_0^T e^{-A\eta} d\eta \right) B = (e^{AT} - I) A^{-1} B$$
$$\Phi \approx I + AT, \quad T \ll$$
$$\Gamma \approx BT, \quad T \ll$$

Nerekurzivni LS estimator

$$\hat{x} = (H^{\mathrm{T}} H)^{-1} H^{\mathrm{T}} y$$
$$\hat{x} = (H^{\mathrm{T}} R^{-1} H)^{-1} H^{\mathrm{T}} R^{-1} y$$

Rekurzivni LS estimator

$$K_k = P_{k-1} H_k^{\mathrm{T}} (R_k + H_k P_{k-1} H_k^{\mathrm{T}})^{-1}$$
$$K_k = P_k H_k^{\mathrm{T}} R_k^{-1}$$
$$\hat{x}_k = \hat{x}_{k-1} + K_k (y_k - H_k \hat{x}_{k-1})$$
$$P_k = (I - K_k H_k) P_{k-1} (I - K_k H_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}}$$
$$P_k = (P_{k-1}^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k)^{-1}$$
$$P_k = (I - K_k H_k) P_{k-1}$$

Diskretni Kalmanov filtar

$$x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1} + w_{k-1}$$
$$y_k = H_k x_k + v_k$$
$$w \sim (0, Q_k)$$
$$v \sim (0, R_k)$$
$$P_k^- = \Phi_{k-1} P_{k-1}^+ \Phi_{k-1}^{\mathrm{T}} + Q_{k-1}$$
$$K_k = P_k^- H_k^{\mathrm{T}} (H_k P_k^- H_k^{\mathrm{T}} + R_k)^{-1}$$
$$K_k = P_k^+ H_k^{\mathrm{T}} R_k^{-1}$$
$$\hat{x}_k^- = \Phi_{k-1} \hat{x}_{k-1}^+ + \Gamma_{k-1} u_{k-1}$$
$$\hat{x}_k^+ = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-)$$
$$P_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}}$$
$$P_k^+ = \left[(P_k^-)^{-1} + H_k^{\mathrm{T}} R_k^{-1} H_k \right]^{-1}$$
$$P_k^+ = (I - K_k H_k) P_k^-$$

Kontinuirani Kalmanov filtar

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + v$$
$$w \sim (0, Q_c)$$
$$v \sim (0, R_c)$$
$$K = PC^{\mathrm{T}} R_c^{-1}$$
$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$
$$\dot{P} = -PC^{\mathrm{T}} R_c^{-1} CP + AP + PA^{\mathrm{T}} + Q_c$$

Kontinuirani prošireni Kalmanov filtar

$$\dot{x} = f(x, u, w, t)$$
$$y = h(x, v, t)$$
$$w \sim (0, Q_c)$$
$$v \sim (0, R_c)$$
$$A = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}}; \quad L = \left.\frac{\partial f}{\partial w}\right|_{\hat{x}}; \quad C = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}}; \quad M = \left.\frac{\partial h}{\partial v}\right|_{\hat{x}}$$
$$\tilde{Q}_c = L Q_c L^{\mathrm{T}}, \quad \tilde{R}_c = M R_c M^{\mathrm{T}}$$
$$\dot{\hat{x}} = f(\hat{x}, u, w_0, t) + K \left[y - h(\hat{x}, v_0, t) \right]$$
$$K = PC^{\mathrm{T}} \tilde{R}^{-1}$$
$$\dot{P} = -PC^{\mathrm{T}} \tilde{R}_c^{-1} CP + AP + PA^{\mathrm{T}} + \tilde{Q}_c$$

Diskretni prošireni Kalmanov filtar

$$
\begin{gathered}
x_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\
y_k = h_k(x_k, v_k) \\
w \sim (0, Q_k) \\
v \sim (0, R_k) \\
\Phi_{k-1} = \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^{+}} ; \quad L_{k-1} = \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^{+}} ; \quad H_k = \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}_k^{-}} ; \quad M_k = \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}_k^{-}} ; \\
\hat{x}_k^{-} = f_{k-1}(\hat{x}_{k-1}^{+}, u_{k-1}, 0) \\
P_k^{-} = \Phi_{k-1} P_{k-1}^{+} \Phi_{k-1}^{\mathrm{T}} + L_{k-1} Q_{k-1} L_{k-1}^{\mathrm{T}} \\
K_k = P_k^{-} H_k^{\mathrm{T}} (H_k P_k^{-} H_k^{\mathrm{T}} + M_k R_k M_k^{\mathrm{T}})^{-1} \\
\hat{x}_k^{+} = \hat{x}_k^{-} + K_k(y_k - h_k(\hat{x}_k^{-}, 0)) \\
P_k^{+} = (I - K_k H_k) P_k^{-}
\end{gathered}
$$

Iterativni prošireni Kalmanov filtar

$$
\begin{gathered}
x_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\
y_k = h_k(x_k, v_k) \\
w \sim (0, Q_k) \\
v \sim (0, R_k) \\
\Phi_{k-1} = \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^{+}} ; \quad L_{k-1} = \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^{+}} \\
\hat{x}_k^{-} = f_{k-1}(\hat{x}_{k-1}^{+}, u_{k-1}, 0) \\
P_k^{-} = \Phi_{k-1} P_{k-1}^{+} \Phi_{k-1}^{\mathrm{T}} + L_{k-1} Q_{k-1} L_{k-1}^{\mathrm{T}} \\
H_{k,i} = \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}_{k,i}^{+}} ; \quad M_{k,i} = \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}_{k,i}^{+}} ; \\
K_{k,i} = P_k^{-} H_{k,i}^{\mathrm{T}} (H_{k,i} P_k^{-} H_{k,i}^{\mathrm{T}} + M_{k,i} R_k M_{k,i}^{\mathrm{T}})^{-1} \\
\hat{x}_{k,i+1}^{+} = \hat{x}_k^{-} + K_{k,i}(y_k - h_k(\hat{x}_{k,i}^{+}, 0) - H_{k,i}(\hat{x}_k^{-} - \hat{x}_{k,i}^{+})) \\
P_{k,i+1}^{+} = (I - K_{k,i} H_{k,i}) P_k^{-}
\end{gathered}
$$

Informacijski filtar

$$
\begin{gathered}
x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1} + w_{k-1} \\
y_k = H_k x_k + v_k \\
w \sim (0, Q_k) \\
v \sim (0, R_k) \\
\mathcal{I}_k^{-} = Q_{k-1}^{-1} - Q_{k-1}^{-1} \Phi_{k-1} (\mathcal{I}_{k-1}^{+} + \Phi_{k-1}^{\mathrm{T}} Q_{k-1}^{-1} \Phi_{k-1})^{-1} \Phi_{k-1}^{\mathrm{T}} Q_{k-1}^{-1} \\
\mathcal{I}_k^{+} = \mathcal{I}_k^{-} + H_k^{\mathrm{T}} R_k^{-1} H_k \\
K_k = (\mathcal{I}_k^{+})^{-1} H_k^{\mathrm{T}} R_k^{-1} \\
\hat{x}_k^{-} = \Phi_{k-1} \hat{x}_{k-1}^{+} + \Gamma_{k-1} u_{k-1} \\
\hat{x}_k^{+} = \hat{x}_k^{-} + K_k(y_k - H_k \hat{x}_k^{-})
\end{gathered}
$$

Ustaljeni Kalmanov filtar

$$
\begin{gathered}
x_k = \Phi x_{k-1} + \Gamma u_{k-1} + w_{k-1} \\
y_k = H x_k + v_k \\
w \sim (0, Q_k) \\
v \sim (0, R_k) \\
\hat{x}_k^+ = (I - K_\infty H)\Phi \hat{x}_{k-1}^+ + K_\infty y_k \\
P_\infty = \Phi P_\infty \Phi^{\mathrm{T}} - \Phi P_\infty H^{\mathrm{T}}(H P_\infty H^{\mathrm{T}} + R)^{-1} H P_\infty \Phi^{\mathrm{T}} + Q \\
K_\infty = P_\infty H^{\mathrm{T}}(H P_\infty H^{\mathrm{T}} + R)^{-1}
\end{gathered}
$$

Alfa-beta filtar

$$
\begin{gathered}
x_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} w_{k-1} \\
y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k \\
w \sim (0, \sigma_w^2) \\
v \sim (0, R) \\
K_1 = -\frac{1}{8}(\lambda^2 + 8\lambda - (\lambda + 4)\sqrt{\lambda^2 + 8\lambda}) = \alpha \\
K_2 = \frac{1}{4T}(\lambda^2 + 4\lambda - \lambda\sqrt{\lambda^2 + 8\lambda}) = \frac{\beta}{T} \\
\lambda = \frac{\sigma_w^2 T^2}{R} \\
P_{11}^+ = K_1 R, \qquad P_{12}^+ = K_2 R, \qquad P_{22}^+ = \left(\frac{K_1}{T} - \frac{K_2}{2}\right)\frac{K_2 \sigma_w^2}{1 - K_1}
\end{gathered}
$$

Diskretni Kalmanov filtar za estimaciju sustava s međukoreliranim procesnim i mjernim šumom

$$
\begin{gathered}
x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1} + w_{k-1} \\
y_k = H_k x_k + v_k \\
w \sim (0, Q_k) \\
v \sim (0, R_k) \\
E[w_{k-1} v_j^{\mathrm{T}}] = M_k \delta_{k-j} \\
\hat{x}_k^- = \Phi_{k-1}\hat{x}_{k-1}^+ + \Gamma_{k-1} u_{k-1} \\
P_k^- = \Phi_{k-1} P_{k-1}^+ \Phi_{k-1}^{\mathrm{T}} + Q_{k-1} \\
K_k = (P_k^- H_k^{\mathrm{T}} + M_k)(H_k P_k^- H_k^{\mathrm{T}} + H_k M_k + M_k^{\mathrm{T}} H_k^{\mathrm{T}} + R_k)^{-1} \\
\hat{x}_k^+ = \hat{x}_k^- + K_k(y_k - H_k \hat{x}_k^-) \\
P_k^+ = P_k^- - K_k(H_k P_k^- + M_k^{\mathrm{T}})
\end{gathered}
$$

Diskretni Kalmanov filtar s obojenim mjernim šumom

$$
\begin{gathered}
x_k = \Phi_{k-1} x_{k-1} + w_{k-1} \\
y_k = H_k x_k + v_k \\
v_k = \Psi_{k-1} v_{k-1} + \zeta_{k-1} \\
w \sim (0, Q_k) \\
\zeta \sim (0, Q_{\zeta,k})
\end{gathered}
$$

$$
\begin{gathered}
{y'}_{k-1} = y_k - \Psi_{k-1} y_{k-1} \\
{H'}_{k-1} = H_k \Phi_{k-1} - \Psi_{k-1} H_{k-1} \\
{v'}_{k-1} = H_k w_{k-1} + \zeta_{k-1} \\
{y'}_k = {H'}_k x_k + {v'}_k
\end{gathered}
$$

$$
\begin{gathered}
\hat{x}_k^+ = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-) \\
\hat{x}_{k+1}^- = \Phi_k \hat{x}_k^+ + C_k ({y'}_k - {H'}_k \hat{x}_k^-) \\
K_k = P_k^- H_k'^{\mathrm{T}} ({H'}_k P_k^- H_k'^{\mathrm{T}} + R_k)^{-1} \\
M_k = Q_k H_{k+1}^{\mathrm{T}} \\
C_k = M_k ({H'}_k P_k^- H_k'^{\mathrm{T}} + R_k)^{-1} \\
P_k^+ = (I - K_k {H'}_k) P_k^- (I - K_k {H'}_k)^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}} \\
P_{k+1}^- = \Phi_k P_k^+ \Phi_k + Q_k - C_k M_k^{\mathrm{T}} - \Phi_k K_k M_k^{\mathrm{T}} - M_k K_k^{\mathrm{T}} \Phi_k^{\mathrm{T}}
\end{gathered}
$$