1. a) $S_0 = S_{uu}(0) = \int_{-\infty}^{\infty} R_{uu}(\tau) e^{-j\cdot 0\tau} d\tau = \int_{-\infty}^{\infty} R_{uu}(\tau) d\tau$

S_0 je površina ispod autokorelacijske funkcije

Budući je u bijeli šum, njegova je spektralna snaga konstant.

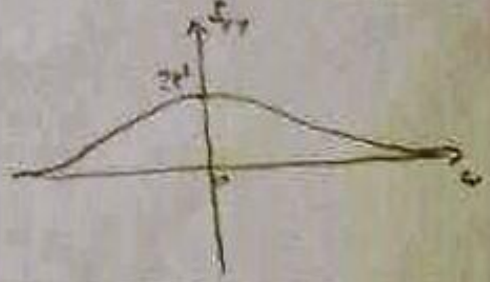
$$S_0 = 2$$

b) $S_{yy}(\omega) = |G(j\omega)|^2 \cdot S_{uu}(\omega)$

$$= \left| \frac{K^2}{(Tj\omega+1)^2} \right| \cdot 2 = \frac{2K^2}{|Tj\omega+1|^2} = \frac{2K^2}{1+(T\omega)^2}$$

$S_{yy}(0) = 2K^2$

$S_{yy}(\pm\infty) = 0$



c) Srednja snaga signala je

$$P_y = R_{yy}(0) = \frac{1}{\pi}\int_0^{\infty} S_{yy}(\omega) d\omega = \frac{1}{\pi}\int_0^{\infty} \frac{2K^2}{1+T^2\omega^2} d\omega = \frac{2K^2}{\pi}\int_0^{\infty}\frac{d(T\omega)}{1+(T\omega)^2}\cdot\frac{1}{T}$$

$$= \frac{2K^2}{T\pi}\cdot \operatorname{arctg}(T\omega)\Big|_0^{\infty} = \frac{2K^2}{T\pi}\left(\frac{\pi}{2}-0\right) = \frac{K^2}{T}$$

2. $u(t) = u^*(t) + w(t)$

$v(t) = v^*(t) + w(t)$

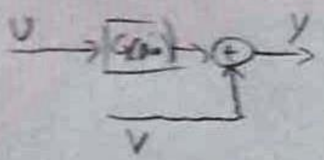
$R_{u^*v^*}(\tau) = 0$

u^* i v^* su bijeli šumovi

$$R_{uv}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t) v(t+\tau) dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [u^*(t) + w(t)][v^*(t+\tau) + w(t+\tau)] dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \Big[\int_{-T}^{T} u^*(t) v^*(t+\tau) dt + \int_{-T}^{T} u^*(t) w(t+\tau) dt + \int_{-T}^{T} w(t) v^*(t+\tau) dt + \int_{-T}^{T} w(t) w(t+\tau) dt \Big]$$



$$= R_{u^*v^*}(\tau) + R_{u^*w}(\tau) + R_{wv^*}(\tau) + R_{ww}(\tau)$$

$$= 0 + 0 + 0 + R_{ww}(\tau)$$

$$R_{uv}(\tau) = R_{ww}(\tau)$$

$$R_{yu}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t) u(t+\tau) dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big[\int_0^{\infty} g(\sigma) u(t-\sigma) d\sigma + v(t) \Big] \cdot u(t+\tau) dt$$

$$R_{yu}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^{\infty} g(\sigma) u(t-\sigma) u(t+\tau) \, d\sigma dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t) u(t+\tau) dt$$

$$R_{yu}(\tau) = \int_0^{\infty} g(\sigma) \Big[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t-\sigma) u(t+\tau) dt \Big] d\sigma + R_{vu}(\tau)$$

$$R_{yu}(\tau) = \int_0^{\infty} g(\sigma) R_w(\tau+\sigma) d\sigma + R_w(\tau)$$

$R_w(\tau) = R_w(-\tau)$

$$R_{uy}(\tau) = R_{yu}(-\tau) = \int_0^{\infty} g(\sigma) R_w(\sigma-\tau) d\sigma + R_w(-\tau)$$

$$R_{uy}(\tau) = \int_0^{\infty} g(\sigma) R_w(\tau-\sigma) + R_w(\tau) \quad / \mathcal{F}\{\}$$

$$S_{uy}(j\omega) = G(j\omega) \cdot S_{w}(\omega) + S_{w}(j\omega)$$

$$S_{uv}(j\omega) = \mathcal{F}\{R_{uv}(\tau)\} = \mathcal{F}\{R_{uw}(\tau)\} = S_{uw}(\omega)$$

Uz zanemarenje korelacije signala u i v:

$$R_{uv}(\tau) = 0 \Rightarrow S_{uv}(\omega) = 0$$

$$S_{uy}(j\omega) = \hat{G}(j\omega) \cdot S_{uu}(\omega)$$

$$\hat{G}(j\omega) = \frac{S_{uy}(j\omega)}{S_{uu}(\omega)}$$

Bez zanemarenja

$$G(j\omega) S_{uu}(\omega) = S_{uy}(j\omega) - S_{uv}(j\omega) = S_{uy}(j\omega) - S_{uw}(\omega)$$

$$G(j\omega) = \frac{S_{uy}(j\omega) - S_{uw}(\omega)}{S_{uu}(\omega)}$$

Pogreška estimacije:

$$\hat{G}(j\omega) - G(j\omega) = \frac{S_{uy}(j\omega)}{S_{uu}(\omega)} - \frac{S_{uy}(j\omega) - S_{uw}(\omega)}{S_{uu}(\omega)} = \frac{S_{uw}(\omega)}{S_{uu}(\omega)}$$

$$S_{uu}(\omega) = \mathcal{F}\{R_{uu}(\tau)\} = \mathcal{F}\left\{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)u(t+\tau)\,dt\right\}$$

$$= \mathcal{F}\left\{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [u^*(t)+w(t)][u^*(t+\tau)+w(t+\tau)]\,dt\right\}$$

$$= \mathcal{F}\{R_{u^*u^*}(\tau) + \cancel{R_{u^*w}(\tau)}^{0} + \cancel{R_{wu^*}(\tau)}^{0} + R_{ww}(\tau)\}$$

$$= \mathcal{F}\{c \cdot \delta(t) + R_{ww}(\tau)\} = c + S_{ww}(\omega)$$

$$\hat{G}(j\omega) - G(j\omega) = \frac{S_{uw}(\omega)}{c + S_{ww}(\omega)}$$

3. $y(k) = \alpha + \beta u(k) + \varepsilon(k)$

a) $y(k) - \alpha - \beta \cdot u(k) = \varepsilon(k)$

$$y(1) - \alpha - \beta \cdot u(1) = \varepsilon(1)$$
$$y(2) - \alpha - \beta \cdot u(2) = \varepsilon(2)$$
$$\vdots$$
$$y(N) - \alpha - \beta \cdot u(N) = \varepsilon(N)$$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} - \begin{bmatrix} -1 & u(1) \\ -1 & u(2) \\ \vdots & \vdots \\ -1 & u(N) \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \varepsilon(1) \\ \varepsilon(2) \\ \vdots \\ \varepsilon(N) \end{bmatrix}$$

$$\underline{Y}(N) - \underline{\Phi}(N) \cdot \underline{\Theta} = \underline{\varepsilon}(N)$$

b) $f(\varepsilon(k), \alpha, \beta) = \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} e^{-\frac{(\varepsilon(k)-0)^2}{2\sigma_\varepsilon^2}} = \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} e^{-\frac{\varepsilon(k)^2}{2\sigma_\varepsilon^2}}$

$$L = f(\varepsilon(1), \alpha, \beta) \cdot \ldots \cdot f(\varepsilon(N), \alpha, \beta)$$

$$L = \left(\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\right)^N e^{-\frac{1}{2\sigma_\varepsilon^2}\sum_{k=1}^{N} \varepsilon^2(k)}$$

$$\ln L = -\frac{N}{2}\ln(2\pi\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\sum_{k=1}^{N}\varepsilon^2(k)$$

$$\frac{\partial}{\partial\alpha}\ln L = -\frac{1}{2\sigma_\varepsilon^2}\sum_{k=1}^{N}\frac{\partial}{\partial\alpha}\varepsilon^2(k) = 0$$

$$\frac{\partial}{\partial\beta}\ln L = -\frac{1}{2\sigma_\varepsilon^2}\sum_{k=1}\frac{\partial}{\partial\beta}\varepsilon^2(k) = 0$$

$$\frac{\partial}{\partial\sigma_\varepsilon}\ln L = -\frac{N}{2}\frac{4\pi\sigma_\varepsilon}{2\pi\sigma_\varepsilon^2} + \frac{1}{\sigma_\varepsilon^3}\sum_{k=1}^{N}\varepsilon^2(k) = 0$$

$$\frac{1}{\sigma_\varepsilon^2}\sum_{k=1}^{N}\varepsilon^2(k) = N \cdot \frac{1}{\sigma_\varepsilon}$$

$$\sigma_\varepsilon = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\varepsilon^2(k)}$$

$$\varepsilon(k) = y(k) - \alpha - \beta u(k)$$

$$\sum_{k=1}^{N}\frac{\partial}{\partial\alpha}\varepsilon^2(k) = \sum_{k=1}^{N}\frac{\partial}{\partial\alpha}\left(y(k)-\alpha-\beta u(k)\right)^2 = 0$$

$$\sum_{k=1}^{N} -2\left(y(k)-\alpha-\beta u(k)\right) = 0$$

I $$\sum_{k=1}^{N} y(k) - \hat{\beta}\sum_{k=1}^{N} u(k) = N\hat{\alpha}$$

$$\sum_{k=1}^{N}\frac{\partial}{\partial\beta}\varepsilon^2(k) = \sum_{k=1}^{N} -2u(k)\left(y(k)-\alpha-\beta u(k)\right) = 0$$

II $$\sum_{k=1}^{N} u(k)y(k) - \hat{\alpha}\sum_{k=1}^{N} u(k) = \hat{\beta}\sum_{k=1}^{N} u^2(k)$$

$$\hat{\beta} = \frac{\sum u(k)y(k)}{\sum u^2(k)} - \hat{\alpha}\frac{\sum u(k)}{\sum u^2(k)}$$

$$\sum y(k) - \left[\frac{\sum u(k)y(k)}{\sum u^2(k)} - \hat{\alpha}\frac{\sum u(k)}{\sum u^2(k)}\right]\cdot\sum u(k) = N\hat{\alpha}$$

$$\sum y(k) - \frac{\sum u(k)\sum u(k)y(k)}{\sum u^2(k)} = \left[N - \frac{\left(\sum u(k)\right)^2}{\sum u^2(k)}\right]\hat{\alpha}$$

$$\hat{\alpha} = \frac{\sum u^2(k)\sum y(k)}{N\sum u^2(k) - \left(\sum u(k)\right)^2} - \frac{\sum u(k)\sum u(k)y(k)}{N\sum u^2(k) - \left(\sum u(k)\right)^2}$$

c) $\underline{y} - \underline{\Phi} \cdot \underline{\theta} = \underline{\varepsilon} \Rightarrow \Phi\theta = y - \underline{\varepsilon}$

$$\Phi^T\Phi\theta = \Phi^T(y-\underline{\varepsilon})$$

$$\theta = (\Phi^T\Phi)^{-1}\Phi^T(y-\underline{\varepsilon})$$

$E([\hat{\theta}-\theta][\hat{\theta}-\theta]^T) = ?$

- Za metodu LS vrijedi:

$$\hat{\theta} = [\Phi^T\Phi]^{-1} \cdot \Phi^T y$$

$$E([\hat{\theta}-\theta][\hat{\theta}-\theta]^T) = E\left\{\left[(\Phi^T\Phi)^{-1}\Phi^T y - (\Phi^T\Phi)^{-1}\Phi^T(y-\underline{\varepsilon})\right]\cdot[\dots]^T\right\}$$

$$= E\left\{\left[(\Phi^T\Phi)^{-1}\Phi^T\underline{\varepsilon}\right]\cdot\left[(\Phi^T\Phi)^{-1}\Phi^T\underline{\varepsilon}\right]^T\right\}$$

$$= E\left\{\left[(\Phi^T\Phi)^{-1}\Phi^T\underline{\varepsilon}\right]\cdot\left[\underline{\varepsilon}^T\Phi(\Phi^T\Phi)^{-1}\right]\right\}$$

$$= E\left[(\Phi^T\Phi)^{-1}\Phi^T\underline{\varepsilon}\cdot\underline{\varepsilon}^T\Phi(\Phi^T\Phi)^{-1}\right]$$

$$= E\left[\Phi^{*-1}\Phi^T\underline{\varepsilon}\cdot\underline{\varepsilon}^T\Phi\Phi^{*-1}\right]$$

$$= \Phi^{*-1}\Phi^T E[\underline{\varepsilon}\cdot\underline{\varepsilon}^T]\cdot\Phi\cdot\Phi^{*-1}$$

$$= \Phi^{*-1}\Phi^T\cdot\sigma_\varepsilon^2 I\cdot\Phi\cdot\Phi^{*-1}$$

$$= \sigma_\varepsilon^2\cdot\Phi^{*-1}\cdot\Phi^T\cdot\Phi\cdot\Phi^{*-1}$$

$$= \sigma_\varepsilon^2\cdot\Phi^{*-1} = \sigma_\varepsilon^2\cdot(\Phi^T\Phi)^{-1}$$

4. 6)

$A(z^{-1}) = 1 - 4z^{-1} + 4z^{-2}$

$B(z^{-1}) = z^{-1} - 2z^{-2}$

$A(z^{-1}) = (1 - 2z^{-1})^2$

$B(z^{-1}) = z^{-1}(1 - 2z^{-1})$

Nule:	Polovi
$z_{n1} = 0$	$z_{p1} = 2$
$~~z_{n2} = 2~~$	$~~z_{p2} = 2~~$

KRATE SE

Red modela je 1

6. a)

$$\Delta x_{p,k+1} = x_{p,k+1} - x_{p,k} = \frac{1}{2} x_{g,k} - \frac{1}{2} x_{p,k} + w_{p,k}$$

$$\Delta x_{g,k+1} = x_{g,k+1} - x_{g,k} = -\frac{1}{2} x_{p,k} + u_k + w_{g,k}$$

$$x_{p,k+1} = \frac{1}{2} x_{p,k} + \frac{1}{2} x_{g,k} + w_{p,k}$$

$$x_{g,k+1} = -\frac{1}{2} x_{p,k} + x_{g,k} + u_k + w_{g,k}$$

$$\begin{bmatrix} x_{p,k+1} \\ x_{g,k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{p,k} \\ x_{g,k} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_{p,k} \\ w_{g,k} \end{bmatrix}$$

$$\begin{bmatrix} y_{p,k} \\ y_{g,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{p,k} \\ x_{g,k} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v_k$$

$$\Phi_k = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \Gamma_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

---

b) $P_0^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$P_1^- = \Phi_0 P_0^+ \Phi_0^T + Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_1 = P_1^- \cdot H_1^T \cdot \left(H_1 \cdot P_1^- \cdot H_1^T + R_1\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$P_1^+ = (1 - K_1 H_1) P_1^- (1 - K_1 H_1)^T + K_1 R_1 K_1^T$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot (\ldots)^T + \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$P_1^+ = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

c)

$$x_{p,k+1} = x_{p,k} = x_p$$

$$x_{g,k+1} = x_{g,k} = x_g$$

$$w_k = 0$$

$$x_p = \frac{1}{2}x_p + \frac{1}{2}x_g$$

$$x_g = -\frac{1}{2}x_p + x_g + u$$

$$x_p = 2 \cdot u$$

$$2u - u = \frac{1}{2}x_g$$

$$x_g = 2u$$

$$\frac{x_p}{x_g} = 1$$

7. a)

$$x_k = \frac{1}{2}x_{k-1} + 25\cdot\frac{x_{k-1}}{1+x_{k-1}^2} + 10\cos(1.2k-1.2) + w_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1})$$

$$w_k \sim \mathcal{N}(0,1)$$

$$y_k = \frac{x_k^2}{20} + v_k = h_k(x_k, v_k)$$

$$v_k \sim \mathcal{N}(0,1)$$

$$\phi_{k-1} = \frac{\partial f_{k-1}}{\partial x_{k-1}}\bigg|_{\hat{x}_{k-1}^+} = \left(\frac{1}{2} + 25\cdot\frac{1+x_{k-1}^2 - 2x_{k-1}^2}{(1+x_{k-1}^2)^2}\right)\bigg|_{\hat{x}_{k-1}^+} = \frac{1}{2} + 25\cdot\frac{1-\hat{x}_{k-1}^{+2}}{(1+\hat{x}_{k-1}^{+2})^2}$$

$$L_{k-1} = \frac{\partial f_{k-1}}{\partial w_{k-1}}\bigg|_{\hat{x}_{k-1}^-} = 1$$

$$H_k = \frac{\partial h_k}{\partial x_k}\bigg|_{\hat{x}_k^-} = \frac{\hat{x}_k^-}{10}$$

$$M_k = \frac{\partial h_k}{\partial v_k}\bigg|_{\hat{x}_k^-} = 1$$

b) $\hat{x}_0^+ = 10 \quad P_0^+ = 1 \quad \hat{x}_1^- = ? \quad P_1^- = ?$

$$\hat{x}_1^- = f_0(\hat{x}_0^+, 0, 0) = \frac{1}{2}\cdot 10 + 25\cdot\frac{10}{1+100} + 10\cdot\cos(1.2-1.2) + 0$$

$$= 5 + \frac{250}{101} + 10 = \frac{505+1010+250}{101} = \frac{1765}{101} = 17.475$$

$$P_1^- = \phi_0 P_0^+ \phi_0^T + L_0 Q_0 L_0 = \left(\frac{1}{2} + 25\cdot\frac{1-100}{(1+100)^2}\right)^2\cdot 1 + 1^2\cdot 1$$

$$= \left(\frac{1}{2} - 25\cdot\frac{99}{101^2}\right)^2 + 1 = \left(\frac{10201 - 50\cdot 99}{2\cdot 10201}\right)^2 + 1 = \left(\frac{10201-4950}{2\cdot 10201}\right)^2 + 1$$

$$= 1.066$$

) $\hat{x}_1^+ = ?$ $P_1^+ = ?$ $\quad x_1 = 18$, $v_1 = 1$

$$y_1 = \frac{18^2}{20} + 0.1 = 16{,}3$$

$H_1 = 1.7475$
$M_1 = 1$, $R_1 = 1$

$$K_1 = P_1^- \cdot H_1^T \cdot \left(H_1 P_1^- H_1^T + M_1 R_1 M_1^T \right)^{-1}$$

$$= 1.066 \cdot 1.7475 \cdot \left(1.7475^2 \cdot 1.066 + 1 \right)^{-1} = 0.4378$$

$$\hat{x}_1^+ = \hat{x}_1^- + K_1 \left(y_1 - h_1(\hat{x}_1^-, 0) \right)$$

$$= 17.475 + 0.4378 \left(16.3 - \frac{17.475^2}{20} \right)$$

$$\hat{x}_1^+ = 17.926$$

$$P_1^+ = (1 - K_1 H_1) \cdot P_1^- = (1 - 0.4378 \cdot 1.7475) \cdot 1.066$$

$$P_1^+ = 0.2505$$