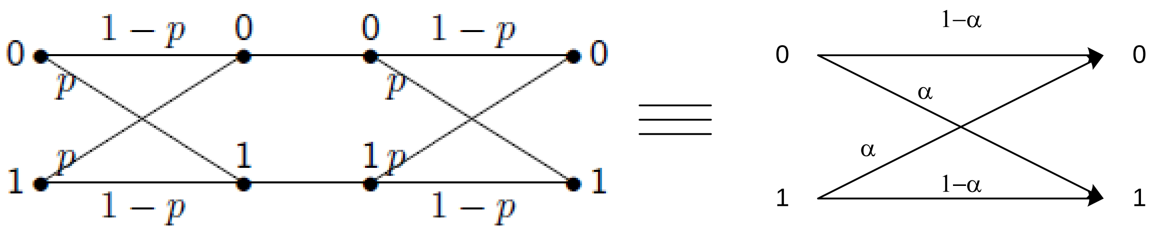
**Usmeni ispit - ponovljeni zimski rok (2012./2013.)**

**Zadatak 1.**

Dva binarna simetrična kanala povezana su kao na slici. Neka je *p* vjerojatnost pogrešnog prijenosa u svakom od kanala.




Odredite:

i) vjerojatnost pogrešnog prijenosa, α, za jedan ekvivalntni binarni simetrični kanal.

ii) kapacitet ekvivalentnog binarnog simetričnog kanala.

*Rješenje*:

i)

$\alpha = ?$

$$\left[p\left(y_j|x_i\right)\right] = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} = \left[p\left(z_k|y_j\right)\right]$$

$$\left[p\left(z_k|y_j\right)\right] = \left[p\left(y_j|x_i\right)\right] * \left[p\left(z_k|y_j\right)\right] = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} * \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} =$$

$$= \begin{bmatrix} (1-p)^2 + p^2 & p*(1.p)*2 \\ p*(1.p)*2 & (1-p)^2 + p^2 \end{bmatrix} = \begin{bmatrix} 2p^2 - 2p + 1 & 2p*(1-p) \\ 2p*(1-p) & 2p^2 - 2p + 1 \end{bmatrix}$$

Iz matrice uvjetnih vjerojatnosti slijedi:

$\alpha = 2p*(1-p)$

ii)

$C = ?$

$$C = \max_{\{p(x_i)\}} I(x;z) \quad \left[\frac{\text{bit}}{\text{simbol}}\right]$$

$$H(x) = -\sum_{i=1}^{2} p(x_i) * log_2 p(x_i)$$
$$= -\big(p(x_1) * log_2 p(x_1) + p(x_2) * log_2 p(x_2)\big)\left[\frac{\text{bit}}{\text{simbol}}\right]$$

$$p(z_k) = \sum_{k=1}^{2} p(z_k|x_i) * p(x_i)$$
$$p(z_1) = p(z_1|x_1) * p(x_1) + p(z_1|x_2) * p(x_2)$$
$$= (2p^2 - 2p + 1) * p(x_1) + 2p * (1-p) * p(x_2)$$

$$p(z_2) = p(z_2|x_1) * p(x_1) + p(z_2|x_2) * p(x_2)$$
$$= 2p * (1-p) * p(x_2) + (2p^2 - 2p + 1) * p(x_1)$$

$$p(z_1) = p(z_2)$$

$$p(x_i|z_k) = \frac{p(z_k|x_i)p(x_i)}{p(z_k)} = \frac{p(x_i, z_k)}{p(z_k)}$$

$$p(x_1|z_1) = \frac{p(x_1, z_1)}{p(z_1)} = \frac{p(x_1) * (2p^2 - 2p + 1)}{(2p^2 - 2p + 1) * p(x_1) + 2p * (1-p) * p(x_2)}$$
$$= A$$

$$p(x_1|z_2) = \frac{p(x_1, z_2)}{p(z_2)} = \frac{p(x_1) * 2p * (1-p)}{(2p^2 - 2p + 1) * p(x_1) + 2p * (1-p) * p(x_2)}$$
$$= B$$

$$p(x_2|z_1) = \frac{p(x_2, z_1)}{p(z_1)} = \frac{p(x_2) * 2p * (1-p)}{(2p^2 - 2p + 1) * p(x_1) + 2p * (1-p) * p(x_2)}$$
$$= C$$

$$p(x_2|z_2) = \frac{p(x_2, z_2)}{p(z_2)} = \frac{p(x_2) * (2p^2 - 2p + 1)}{(2p^2 - 2p + 1) * p(x_1) + 2p * (1-p) * p(x_2)}$$
$$= D$$

$$p(x_1, z_1) = p(x_1) * p(z_1|x_1)$$
$$= p(x_1) * (2p^2 - 2p + 1)$$

$$p(x_1, z_2) = p(x_1)p(z_2|x_1)$$
$$= p(x_1) * 2p * (1-p)$$

$$p(x_2, z_1) = p(x_2) * p(z_1|x_2)$$
$$= p(x_2) * 2p * (1-p)$$

$$p(x_2, z_2) = p(x_2) * p(z_2|x_2)$$
$$= p(x_2) * (2p^2 - 2p + 1)$$

$I(x;z) = H(x) - H(x|z) \left[\frac{\text{bit}}{\text{simbol}}\right]$

$C = \max_{\{p(x_i)\}} \left(H(X) - H(X|Z)\right) \left[\frac{\text{bit}}{\text{simbol}}\right]$

Maksimum kapaciteta je za :

$p(x_1) = p(x_2) = 0.5$ jer je tada $H(X) = 1 \left[\frac{\text{bit}}{\text{simbol}}\right]$

$$H(x|z) = -p(x_1)(1-\alpha)log_2(A) - \alpha * p(x_1)log_2(B) - \alpha * p(x_2)log_2(C) - p(x_2)(1-\alpha)log_2(D)$$
$$= -p(x_1)log_2(A) - p(x_2)log_2(D) + \alpha * p(x_1)log_2\left(\frac{A}{B}\right) + \alpha * p(x_2)log_2\left(\frac{D}{C}\right) \left[\frac{\text{bit}}{\text{simbol}}\right]$$

$$C = \underset{\{p(x_i)\}}{max}\{I(x;z)\} = \underset{\{p(x_i)\}}{max}\{H(x) - H(x|z)\} \quad \left[\frac{\text{bit}}{\text{simbol}}\right]$$

$$I(x;z) = H(x) - H(x|z)$$
$$= -p(x_1)log_2p(x_1) - p(x_2)log_2p(x_2) + p(x_1)log_2(A) + p(x_2)log_2(D) - \alpha * p(x_1)log_2\left(\frac{A}{B}\right) - \alpha * p(x_2)log_2\left(\frac{D}{C}\right)$$
$$= p(x_1)log_2\left(\frac{A}{p(x_1)}\right) + p(x_2)log_2\left(\frac{D}{p(x_2)}\right) - \alpha * p(x_1)log_2\left(\frac{A}{B}\right) - \alpha * p(x_2)log_2\left(\frac{D}{C}\right) \left[\frac{\text{bit}}{\text{simbol}}\right]$$

*Zamjenom izraza A,B,C i D sa vjerojatnostima $p(x_1)=p(x_2)=1/2$ u zadnjem koraka i pojednostavljivanjem dobivamo kapacitet koji iznosi:*

$$C = log_2\left(2 * (1-\alpha)\right) - \alpha * log_2\left(\frac{1-\alpha}{\alpha}\right)$$
$$= 1 + \alpha * log_2(\alpha) + (1-\alpha) * log_2(1-\alpha) \quad \left[\frac{\text{bit}}{\text{simbol}}\right]$$

**Zadatak 2.**
Dokažite da je I(X; yj)>=0 uz uvijet ln(x) <= x-1 za svaki x>0.
Imate diskretnu slučajnu varijablu X={$x_1$, $x_2$, … $x_n$}, a na izazu je slučajna varijabla Y.

$$I(X;y_j) = \sum_{i=1}^{n} p(x_i|y_j) * \log_2 \frac{p(x_i)}{p(x_i|y_j)}$$
$$= \sum_{i=1}^{n} p(x_i|y_j) * \frac{\ln \frac{p(x_i)}{p(x_i|y_j)}}{\ln 2}$$
$$= \frac{1}{\ln 2} \sum_{i=1}^{n} p(x_i|y_j) * \ln \frac{p(x_i)}{p(x_i|y_j)}$$

Ovdje iskoristimo onu nejednakost koja je zadana: ln(x) ≤ x-1

$$= \frac{1}{\ln 2} \sum_{i=1}^{n} p(x_i|y_j) * \left(1 - \frac{p(x_i)}{p(x_i|y_j)}\right)$$
$$= \frac{1}{\ln 2} \sum_{i=1}^{n} \left(p(x_i|y_j) - p(x_i)\right)$$
$$= \frac{1}{\ln 2} \sum_{i=1}^{n} p(x_i|y_j) - \frac{1}{\ln 2} \sum_{i=1}^{n} p(x_i)$$
$$= 0 \ \frac{bit}{simbol}$$

Napomena:
S obzirom da je $I(X;y_j)$, vrijedi da je $\sum_{i=1}^{n} p(x_i|y_j) = 1$.

**Zadatak 3.**
Izvedite formulu za određivanje kapaciteta kanala uz prisutstvo Gausssovog šuma.
(ovo nismo sigurni je li točno jer taj zadatak je riješavao samo jedan kolega i Ilić to nije izvodio na ploči)

$$C = \max I(X;Y) = H(Y) - H(Y|X)$$

$$H(X) = \sum_{k=1}^{n} \log\left(\sigma_{x_k}\sqrt{2\pi e}\right)$$

$$H(Y) = \sum_{k=1}^{n} \log\left(\sigma_{y_k}\sqrt{2\pi e}\right)$$

$$H(Y|X) = H(Z) = \sum_{k=1}^{n} \log\left(\sigma_{z_k}\sqrt{2\pi e}\right)$$

$$C = \max I(X;Y) = \sum_{k=1}^{n} \log\left(\sigma_{y_k}\sqrt{2\pi e}\right) - \sum_{k=1}^{n} \log\left(\sigma_{z_k}\sqrt{2\pi e}\right)$$

$$= \sum_{k=1}^{n} \log \frac{\sigma_{y_k}\sqrt{2\pi e}}{\sigma_{z_k}\sqrt{2\pi e}} = \sum_{k=1}^{n} \log \frac{\sigma_{y_k}}{\sigma_{z_k}} = \frac{1}{2}\sum_{k=1}^{n} \log\left(\frac{\sigma_{y_k}}{\sigma_{z_k}}\right)^2 = \frac{1}{2}\sum_{k=1}^{n} \log\left(1 - \frac{{\sigma_{x_k}}^2}{{\sigma_{z_k}}^2}\right)$$

Sada zamijenimo: $S = {\sigma_{x_k}}^2$ , $N = {\sigma_{z_k}}^2$ $i$ $n = 2B$

$$C = \frac{n}{2}\log\left(1 - \frac{S}{N}\right) = B\log\left(1 - \frac{S}{N}\right) \quad \left[\frac{bit}{s}\right]$$