3. Domaća zadaća iz Teorije informacije

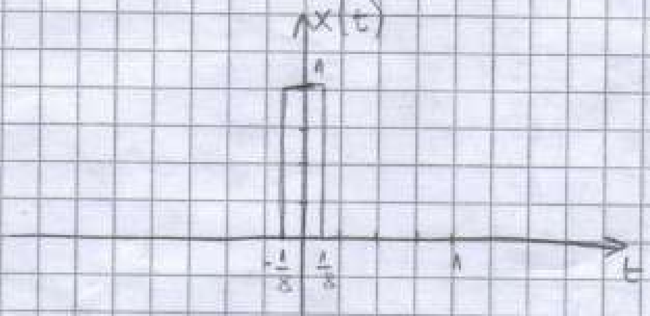
1)

$$x(t) = \begin{cases} A, & t \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right] \\ 0, & \text{inače} \end{cases}$$

$A = 1$

$\tau = \frac{1}{4}, \frac{1}{2}, 1.$

Ako je $\tau = \frac{1}{4}$: $x(t) = \begin{cases} 1, & t \in \left[-\frac{1}{8}, \frac{1}{8}\right] \\ 0, & \text{inače} \end{cases}$



Spektar pravokutnog impulsa:

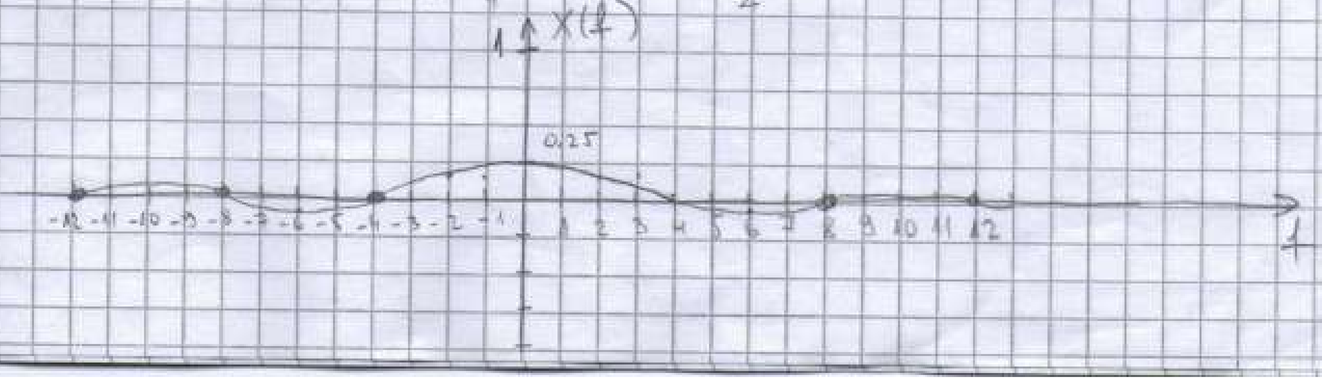
$$X(f) = A\tau \frac{\sin(2\pi f \tau/2)}{2\pi f \tau/2} = \frac{1}{4} \cdot \frac{\sin\left(2\pi f \cdot \frac{1}{8}\right)}{2\pi f \cdot \frac{1}{8}} = \frac{1}{4} \cdot \frac{\sin\left(f \cdot \frac{\pi}{4}\right)}{f \cdot \frac{\pi}{4}}$$

$X(0) = 1 \cdot \frac{1}{4} = \frac{1}{4}$

$X(f)$ će biti 0 za $f = \pm 4, \pm 8, \pm 12 \ldots$

$$X(2) = X(-2) = \frac{1}{4} \cdot \frac{\sin\left(2 \cdot \frac{\pi}{4}\right)}{2 \cdot \frac{\pi}{4}} = \frac{1}{4} \cdot \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{1}{\cancel{4}_2} \cdot \frac{\cancel{2}}{\pi} = \frac{1}{2\pi} \approx 0.16$$

$$X(6) = X(-6) = \frac{1}{4} \cdot \frac{\sin\left(6 \cdot \frac{\pi}{4}\right)}{6 \cdot \frac{\pi}{4}} = \frac{1}{4} \cdot \frac{\sin\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}} = -\frac{1}{\cancel{4}_2} \cdot \frac{\cancel{2}}{3\pi} = -\frac{1}{6\pi} \approx 0.053$$



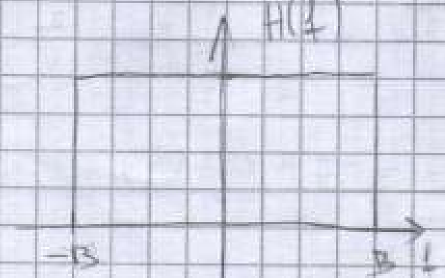
2.) Teorem o uzorkovanju:

Vremenski kontinuirani signal $x(t)$, $\forall t \in \mathbb{R}$, s frekvencijama ne većim od B (pojasno ograničen signal), može biti egzaktno rekonstruiran iz svojih očitaka $x(n) \triangleq x(nT)$, $\forall n \in \mathbb{Z}$ ako je očitanje provedeno frekvencijom $f_s \geq 2B$

Aliasing: pojava koja dolazi do preklapanja spektra signala zbog premale frekvencije uzorkovanja.

a) Ako signal uzorkujemo sukladno teoremu o uzorkovanju njegovu rekonstrukciju provodimo njegovim propuštanjem kroz niskopropusni filtar koji ima $h(t)$ koji je funkcija $\mathrm{sinc}(x)$, a

$$H(f) = \begin{cases} 1 & , |f| < B \\ 0 \end{cases}$$

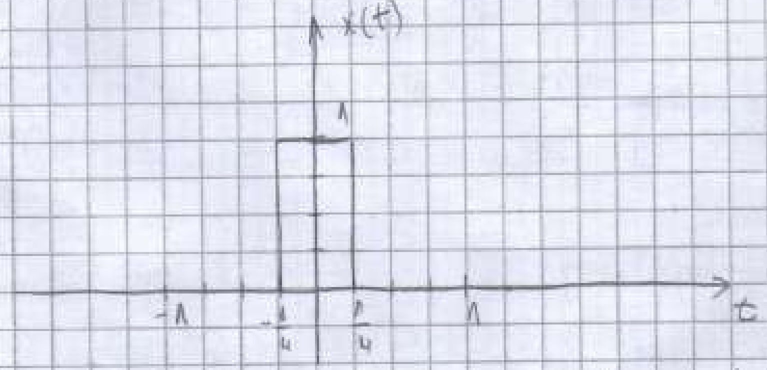


Vidljivo je da ćemo time dobiti signal $x(t)$, koji je pojasno ograničen frekvencijom B, a $A=1$ omogućuje da se frekvencijski spektar signala u tom području ne mijenja te tako dobijemo originalni signal $x(t)$

Compressed sensing, odnosno sažeto uzorkovanje je metoda u otipkavanju signala sa smanjenim brojem uzoraka s obzirom na teorem o uzorkovanju. Ova metoda se može primjenjivati samo na određenim klasama signala, u određenim uvjetima. Dakle, [illegible] ne vrijedi uvijek.

$= \frac{1}{2}$

$x(t) = \begin{cases} 1, & t \in [-\frac{1}{4}, \frac{1}{4}] \\ 0, & \text{inače} \end{cases}$



$X(f) = A\tau \cdot \frac{\sin(2\pi f \tau/2)}{2\pi f \tau/2} = 1 \cdot \frac{1}{2} \cdot \frac{\sin(2\pi f \cdot \frac{1}{4})}{2\pi f \cdot \frac{1}{4}} = \frac{1}{2} \frac{\sin(f \cdot \frac{\pi}{2})}{f \cdot \frac{\pi}{2}}$

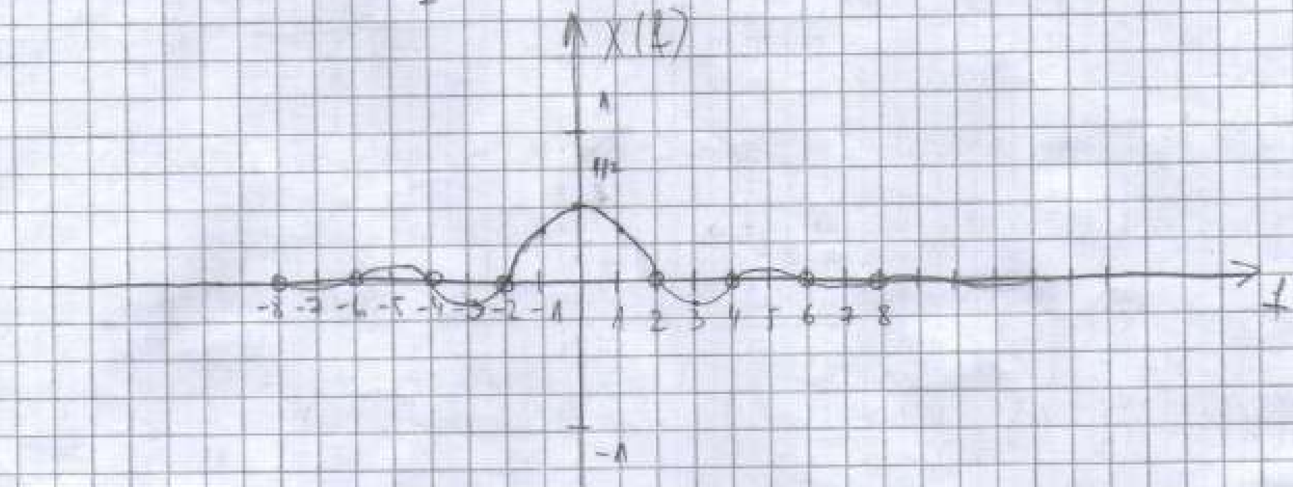
$X(0) = \frac{1}{2}$

$X(f)$ je 0 za $f = \pm 2, \pm 4, \pm 6, \pm 8 \ldots$

$X(1) = X(-1) = \frac{1}{2} \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{1}{2} \cdot \frac{2}{\pi} = \frac{1}{\pi} \approx 0.32$

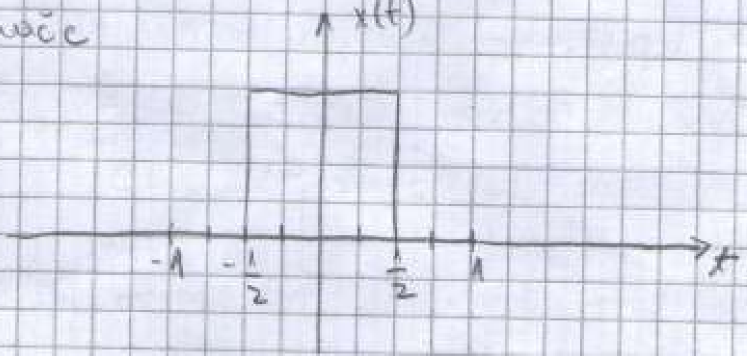
$X(3) = X(-3) = \frac{1}{2} \frac{\sin(\frac{3\pi}{2})}{\frac{3\pi}{2}} = -\frac{1}{2} \frac{2}{3\pi} = -\frac{1}{3\pi} \approx -0.11$

$X(5) = X(-5) = \frac{1}{2} \frac{\sin(\frac{5\pi}{2})}{\frac{5\pi}{2}} = \frac{1}{2} \frac{2}{5\pi} = \frac{1}{5\pi} \approx 0.06$



$\tau = 1$

$$x(t) = \begin{cases} 1, & t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{inače} \end{cases}$$



$$X(f) = A\,\tau \cdot \frac{\sin(2\pi f \tau/2)}{2\pi f \tau/2} = \frac{\sin\left(2\pi f \cdot \frac{1}{2}\right)}{2\pi f \cdot \frac{1}{2}} = \frac{\sin(f \cdot \pi)}{f\,\pi}$$

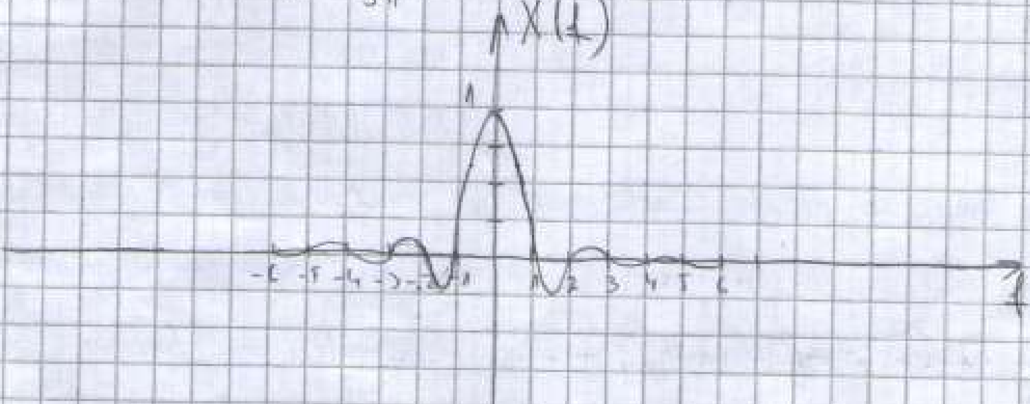
$X(0) = 1$

$X(f)$ je 0 za $f = \pm 1, \pm 2, \pm 3, \pm 4 \ldots$

$$X\left(\frac{1}{2}\right) = X\left(-\frac{1}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \approx 0.64$$

$$X\left(\frac{3}{2}\right) = X\left(-\frac{3}{2}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}} = -\frac{2}{3\pi} \approx -0.21$$

$$X\left(\frac{5}{2}\right) = X\left(-\frac{5}{2}\right) = \frac{2}{5\pi} \approx 0.13$$



Zaključak: Ako produljimo impuls duljine trajanja (veći τ), spektar signala se sužava, tj. nultočke su sve gušće raspoređene.

Heisenbergovo načelo neodređenosti govori da se ne može točno odrediti signal i vremenski i frekvencijski, tj. da se ne može reći koje frekvencijske komponente postoje u kojem vremenskom intervalu. Može se znati vremenske intervale u kojima postoje određeni frekvencijski pojasevi.

4) Kapacitet kanala C je definiran izrazom

$$C = \frac{2B}{2} I_{max}(X,Y) = B \log\left(1+\frac{S}{N}\right) \ [Bit/s]$$

maksimalna transinformacija

gornja granična frekvencija

Teorem o prijenosu informacije kanalom sa smetnjama

Pretpostavimo kanal kapaciteta C i izvor karakteriziran entropijom H. Ako je $H \leq C$, tada postoji takav sustav kodiranja da se poruke s izvora mogu prenositi kanalom uz proizvoljno malu učestalost pogreške. Ako je $H > C$ tada je moguće kodirati tako da ekvivokacija bude manja od $H - C + \varepsilon$ [bit/simbol] gdje je ε proizvoljno mala veličina. Ne postoji metoda kodiranja kojom bi se postigla ekvivokacija manja od $H - C$.

a)

$B = 1000 Hz$

$H = 10^6 b$

$S_N(f) = 10^{-18} W/Hz$

$t = 1000 s$

$$C = \frac{H}{t} = 1000 bit/s$$

$$C = B \log_2\left(1+\frac{S}{N}\right)$$

$S_N(f) = \frac{N_0}{2}$

$N_0 = 2 \cdot 10^{-18} W/Hz$

$N = N_0 \cdot B = 2 \cdot 10^{-15} W$

$$C = B\log_2\left(1+\frac{S}{N}\right)$$

$$1000 = 1000 \cdot \log_2\left(1+\frac{S}{2\cdot 10^{-15}}\right)$$

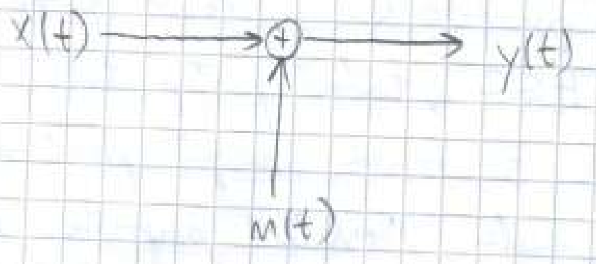
$$1 = \log_2\left(1+\frac{S}{2\cdot 10^{-15}}\right)$$

$$2 = 1 + \frac{S}{2\cdot 10^{-15}}$$

$$1 = \frac{S}{2\cdot 10^{-15}}$$

$$S = 2\cdot 10^{-15}\ W$$

b)



$$I(X;Y) = H(Y) - H(Y|X)$$

$$f_X(y|x) = f_X(x+z|x) = \Phi(z)$$

$$H(Y|X) = \int_{-\infty}^{+\infty} \Phi(z)\log\frac{1}{\Phi(z)}\,dz = H(z) = \sum_{k=1}^{n} \log \sigma_z\sqrt{2\pi e}$$

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = H(Y) - \sum_{k=1}^{n}\log(\sigma_{zk}\sqrt{2\pi e})$$

$$Y_k = X_k + Z_k$$

$$\sigma_{yk}^2 = \sigma_{zk}^2 + \sigma_{xk}^2 \qquad k = 1,2,\dots,n$$

$$I_{MAX}(X;Y) = \sum_{k=1}^{n}\log(\sigma_{yk}\sqrt{2\pi e}) - \sum_{k=1}^{n}\log(\sigma_{zk}\sqrt{2\pi e})$$

$$= \sum_{k=1}^{n}\log\left(\frac{\sigma_{yk}}{\sigma_{zk}}\right) = \sum_{k=1}^{n}\frac{1}{2}\log\left(1+\frac{\sigma_{xk}^2}{\sigma_{zk}^2}\right)$$

$$= \frac{n}{2}\log\left(1+\frac{\sigma_x^2}{\sigma_z^2}\right) = \frac{n}{2}\log\left(1+\frac{S}{N}\right) = \frac{2B}{2}\log\left(1+\frac{S}{N}\right)$$

$$= B\log\left(1+\frac{S}{N}\right)$$

2.)

b) $x_1(t) = \cos(2\pi t)$

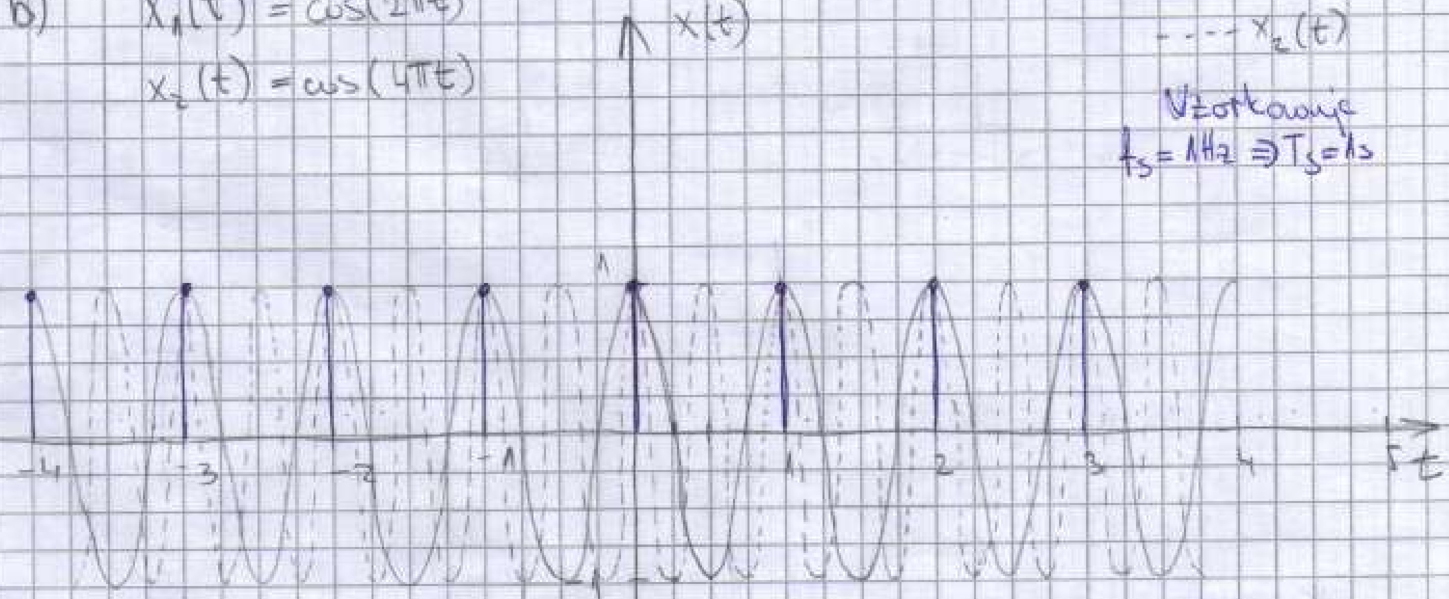
$x_2(t) = \cos(4\pi t)$

—— $x_1(t)$

- - - - $x_2(t)$

Vzorkovanje

$f_s = 1\,Hz \Rightarrow T_s = 1\,s$



$x_1(t) = \cos(2\pi t)$

$\omega_1 = 2\pi$

$2\pi f_1 = 2\pi$

$f_1 = 1\,Hz$

$x_2(t) = \cos(4\pi t)$

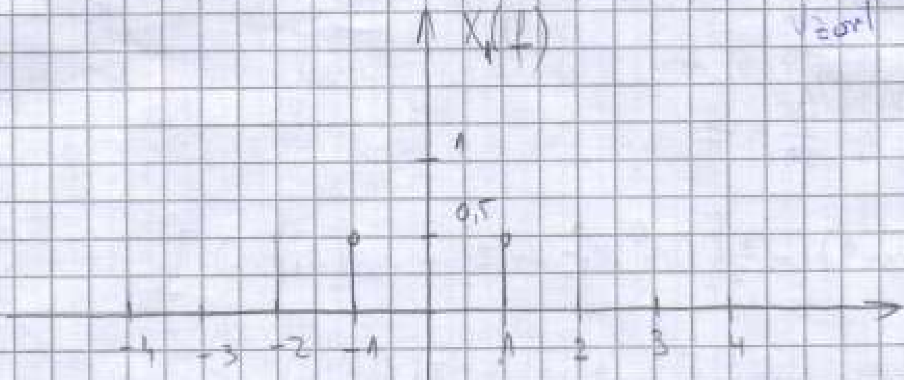
$\omega_2 = 4\pi$

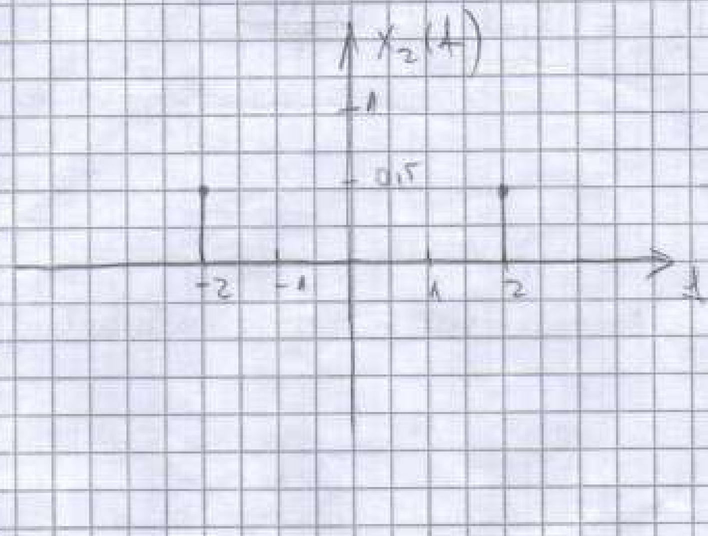
$2\cancel{\pi} f_2 = 4\cancel{\pi}$

$f_2 = 2\,Hz$

Vzorci

Spektri:





3.) $x(t) = A\cos(2\pi f t)$

$A = 3$

$f = 1\,kHz$

$f_s = 4\,kHz$

$L = 32$

$L = 2^r$ $\qquad\qquad m_{max} = 3$

$32 = 2^r \quad /\log$

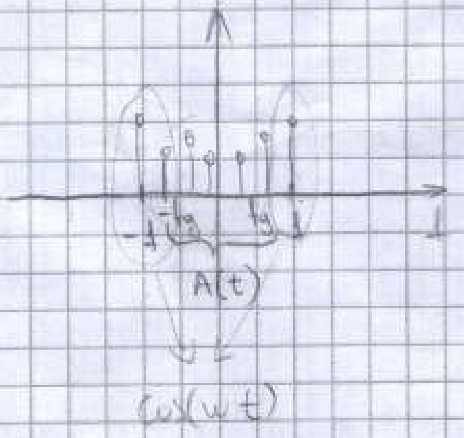
$\log 32 = r \log 2$

$r = 5$

$S = \frac{A^2}{2} = \frac{9}{2} = 4.5\,W$

$N = \frac{1}{3} \cdot m_{max}^2 \cdot 2^{-2r} = \frac{1}{3} \cdot 9 \cdot 2^{-10} = 3 \cdot 2^{-10} = 2.9296875 \cdot 10^{-3}\,W$

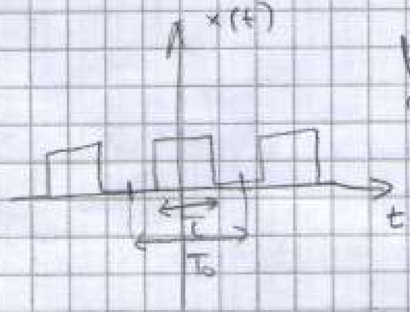
$\left(\frac{S}{N}\right) = \frac{4.5}{2.9296875 \cdot 10^{-3}} = 1536$

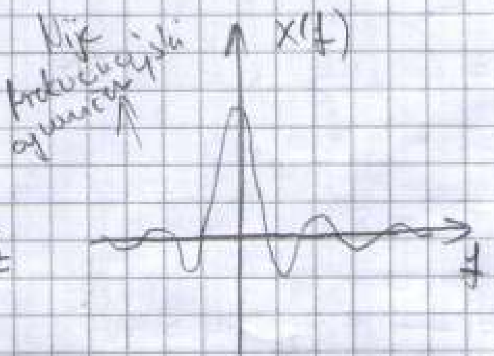


$f = \frac{w}{2\pi}$

Konvolucijom ta 2 signala dobijemo signal frekvencije $f_{MAX} = \frac{w + w_g}{2\pi}$

$f_s = 2 \cdot f_{MAX} = \frac{w + w_g}{\pi}$





(4) $T = 10$

$\tau = 1$

$A = 1$

Periodični signal pravokutnih impulsa ima neograničen frekvencijski spektar. Zbog toga nije zadovoljen uvjet teorema uzorkovanja (signal mora biti pojasno ograničen) pa je nemoguće rekonstruirati taj signal ako uzorkujemo s bilo kojim f_s

d) $f_s = 2$

Ω je slučajna varijabla $[0, 2]$

$$x(t) = \cos(\Omega t) + \cos(\Omega^2 t)$$

Najveća vrijednost koju omega može poprimiti je 2.

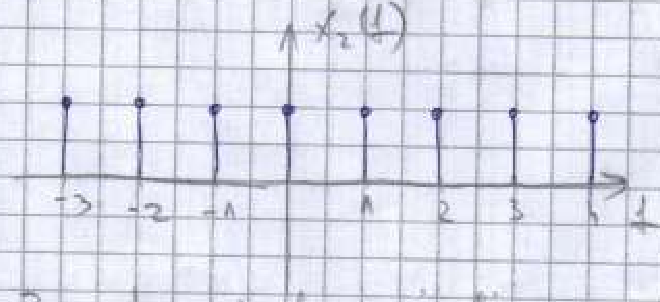
U tom slučaju:

$$x(t) = \cos(2t) + \cos(4t)$$

$$f_{MAX} = \frac{4}{2\pi} = \frac{2}{\pi}$$

$f_s > 2 f_{MAX}$, pa je vjerojatnost da neće doći do aliasinga 1.

Spektri nakon uzorkovanja



fs je premala za dva signala – javlja se aliasing.

c)

(1) $x(t) = \cos(\omega t)$

$\omega = 2\pi f$

$f = \frac{\omega}{2\pi}$

$f_s = 2f = 2 \cdot \frac{\omega}{2\pi} = \frac{\omega}{\pi}$

(2) $\cos(\omega_1 t)\cos(\omega_2 t) = \frac{1}{2}\left[\cos[(\omega_1 - \omega_2)t] + \cos[(\omega_1 + \omega_2)t]\right]$

Najveća frekvencija $\omega_{max} = \omega_1 + \omega_2 = 2\pi(f_1 + f_2)$

$f_s = 2(f_1 + f_2) = \frac{\omega_1 + \omega_2}{\pi}$

3) $A(t)\cos(\omega t)$

$\omega_A < \omega$

Umnožak u vremenskoj domeni se preslikava u konvoluciju u frekvencijskoj.

$X_A(f)$