1. DOMAĆA ZADAĆA

1. Diskretna slučajna varijabla je varijabla odabrana iz skupa od n mogućih vrijednosti, gdje je za svaku vrijednost poznata vjerojatnost odabira. Entropija diskretne slučajne varijable definirana je kao: $H(X) = -\sum_{i=1}^{n} p(x_i) \cdot \log p(x_i)$ [bit/simbol].

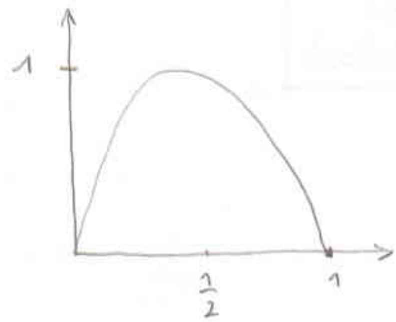
Uzajamni sadržaj informacije (transinformacija) između slučajnih varijabli X i Y je definiran kao relativna entropija između razdiobe njihovih združenih vrijednosti i razdiobe umnoška njihovih pojedinačnih vrijednosti:

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \cdot \log \frac{p(x_i, y_j)}{p(x_i)\,p(y_j)} =$$

$$= H(X) - H(X|Y) = H(Y) - H(Y|X)$$

a) $p(X=x) = \begin{cases} p & , x=1 \\ 1-p & , x=0 \end{cases}$

$$H(x) = -(p \log_2 p + (1-p)\log_2(1-p)) \text{ bit/simbolu}$$



$$\log_2 p + \frac{\cancel{p}}{\cancel{p}\ln 2} - \log_2(1-p) + \frac{\cancel{1-p}}{\cancel{\ln 2(1-p)}} = 0$$

$$\log_2\left(\frac{p}{1-p}\right) = 0$$

$$\frac{p}{1-p} = 1 \Rightarrow p = 1-p$$

$$2p = 1 \Rightarrow \boxed{p = \frac{1}{2}}$$

$E(H(x)) = ?$

$$E(H(x)) = \sum_{i=1}^{3} H(x, p=p_i) \cdot \frac{1}{3}$$

$H(X, p=0) = 0$

$H(X, p=\frac{1}{2}) = 1$

$H(X, p=1) = 0$

$$\boxed{E(H(X)) = \frac{1}{3} \text{ bit/simbol}}$$

b) $p(X=k) = (1-p)^{k-1} \cdot p$

$$E(x) = \sum_{k=1}^{\infty} -(1-p)^{k-1} p \log_2((1-p)^{k-1} p) =$$

$$= -p\log_2(p) \cdot \sum_{k=1}^{\infty} (1-p)^{k-1} - p\log_2(1-p) \sum_{k=1}^{\infty} (k-1)(1-p)^{k-1} =$$

$$\boxed{= -\log_2(p) - \frac{(1-p)\log_2(1-p)}{p}}$$

c) $X \sim \{-1, 0, 1\}$ $\quad X^2 \sim \begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix}$

$p(x_i) = \frac{1}{3}$

$$\boxed{H(x) = 1{,}584 \ \frac{\text{bit}}{\text{simbol}}} \quad \boxed{H(X^2) = 0{,}918 \ \frac{\text{bit}}{\text{simbol}}}$$

$$2^X \sim \begin{pmatrix} 1/2 & 1 & 2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\boxed{H(2^X) = 1{,}584 \text{ bit/simbol}}$$

Može se pretpostaviti $H(U) \geq H(f(U))$

vrijedi: (1.) $H(U, f(U)) = H(U) + H(f(U)|U) =$

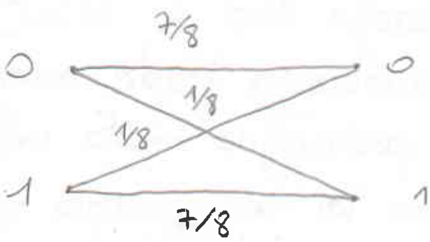
$$= H(f(U)) + H(U|f(U))$$

(2.) $H(f(U), U) = 0$ jer je $f(U)$ neodređen

(3.) $H(U|f(U)) \geq 0$

$$H(U) \geq H(f(U))$$

d) $X^2 \sim \begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix}$



$I(X;Y) = ?$

$$[p(y_j|x_i)] = \begin{bmatrix} 7/8 & 1/8 \\ 1/8 & 7/8 \end{bmatrix}$$

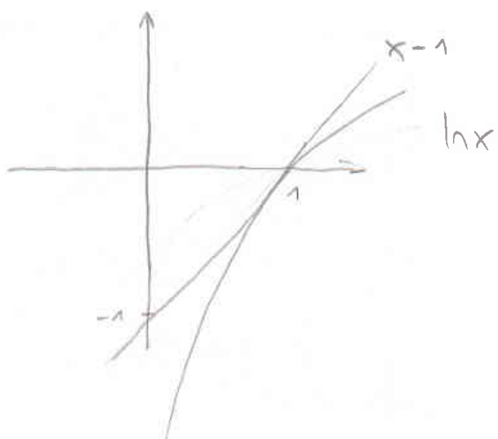
$$p(x_i, y_j) = \begin{bmatrix} 7/24 & 1/24 \\ 2/24 & 14/24 \end{bmatrix} \quad [p(y_j)] = [\,9/24 \quad 15/24\,]$$

$H(Y) = 0,9544$

$H(Y|X) = 0,5435$

$I(X;Y) = H(Y) - H(Y|X) = \boxed{0,4108 \text{ bit/simbol}}$

e)



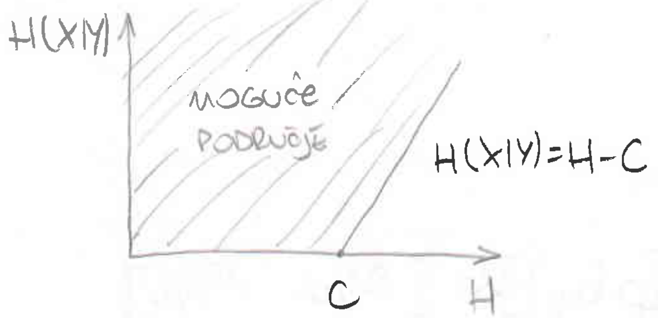
$\ln x \leq x - 1$

2. Kapacitet kanala je definiran kao maksimum transinformacije, gdje se maksimizacija provodi po svim mogućim razdiobama vjerojatnosti pojave simbola na ulazu:

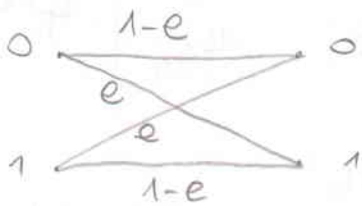
$$C = \max_{p(x_i)} I(X;Y) \text{ bit/simbol}$$

OSNOVNI TEOREM KANALA SA SMETNJAMA

Pretpostavimo kanal kapaciteta C [bit/simbol] i izvor karakteriziran entropijom H [bit/simbolu]. Ako je $H \leq C$, tada postoji takav sustav kodiranja da se poruke s izvora mogu prenositi kanalom uz proizvoljno malu učestalost pogreške, tj. proizvoljno malu ekvivokaciju. Ako je $H > C$ tada je moguće kodirati tako da ekvivokacija bude manja od $H - C + \varepsilon$ [bit/simbol], gdje je ε proizvoljno mala veličina. Ne postoji metoda kodiranja kojom bi se postigla ekvivokacija manja od $H-C$.



a) $[p(x_i)] = [1-p \quad p]$



$$[p(y_j|x_i)] = \begin{bmatrix} 1-e & e \\ e & 1-e \end{bmatrix}$$

$$p(x_i, y_j) = \begin{bmatrix} (1-p)(1-e) & (1-p)e \\ ep & (1-e)p \end{bmatrix}$$

$$[p(y_j)] = [(1-p)(1-e)+ep \quad (1-p)e+(1-e)p]$$

$$H(Y) = -\Big[\big((1-p)(1-e)+ep\big)\log_2\big((1-p)(1-e)+ep\big) + \big((1-p)e+(1-e)p\big)\log_2\big((1-p)e+(1-e)p\big)\Big]$$

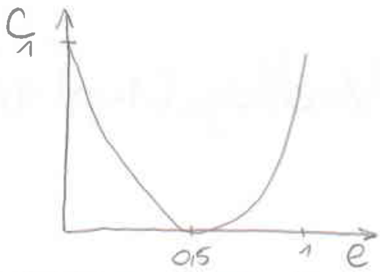
$$H(Y|X) = -\Big[(1-p)(1-e)\log_2(1-e) + (1-p)e\log_2(e) + ep\log_2(e) + (1-e)p\log_2(1-e)\Big] =$$

$$= -\Big[(1-e-\cancel{p}+\cancel{pe}+\cancel{p}-\cancel{pe})\log_2(1-e) + (e-\cancel{ep}+\cancel{ep})\log_2(e)\Big] =$$

$$= -\big[(1-e)\log_2(1-e) + e\log_2(e)\big]$$

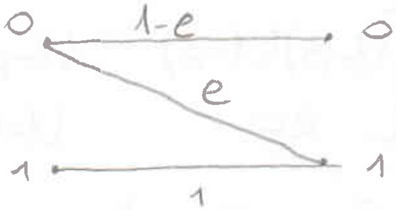
- za $p = \frac{1}{2}$

$$H(Y) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}\right) = 1$$



$$\boxed{C = \max(I(X;Y)) = \max\big(1 + e\log_2(e) + (1-e)\log_2(1-e)\big)}$$

b) $[p(x_i)] = [1-p \quad p]$



$$[p(y_j|x_i)] = \begin{bmatrix} 1-e & e \\ 0 & 1 \end{bmatrix}$$

$$[p(x_i, y_j)] = \begin{bmatrix} (1-p)(1-e) & (1-p)e \\ 0 & p \end{bmatrix} \qquad [p(y_j)] = [(1-p)(1-e) \quad (1-p)e+p]$$

$$H(Y) = -[(1-p)(1-e)\log_2((1-p)(1-e)) + ((1-p)e+p)\log_2((1-p)e+p)]$$

$$H(Y|X) = -[(1-p)(1-e)\log_2(1-e) + (1-p)e\log_2(e)]$$

$$I(X;Y) = H(Y) - H(Y|X)$$

za max: $H(Y) = 1$

$H(Y|X) = 0 \rightarrow \boxed{e = 0}$

$$(1-p)\log_2(1-p) + p\log_2(p) = 1 \rightarrow 1-p = p$$

$2p = 1$

$\boxed{p = \frac{1}{2}}$

c) za min: $H(Y) = H(Y|X) = 0 \rightarrow \boxed{e = 1}$

$$(1-\cancel{p}+\cancel{p})\log_2(1-\cancel{p}+\cancel{p}) = 0 \ \forall \ \boxed{p \in [0,1]}$$

d) ① pritisak bilo koje tipke generira pripadajući broj

$$H(Y) = -\left(\cancel{10}\cdot\frac{1}{\cancel{10}}\log_2 10\right) = 3,321 \text{ bit/simbol} \qquad H(Y|X)=0$$

$$\boxed{C = H(Y) = 3,321 \text{ bit/simbol}}$$

② pripadajući ili njemu sljedeći

$$p(y_j|x_i) = \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & & & & & & & & & & \\ 3 & & & \cdot & & & & & & & \\ 4 & & & & \cdot & & & & & & \\ 5 & & & & & \cdot & & & & & \\ 6 & & & & & & & & & & \\ 7 & & & & & & & & & & \\ 8 & & & & & & & & & & \\ 9 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{array}$$

$$p(x_i, y_j) = \begin{bmatrix} 1/20 & 1/20 & \cdots & 0 \\ 0 & & & \\ & & \ddots & \\ 1/20 & \cdots & & 1/20 \end{bmatrix} \qquad [p(y_j)] = \left[\frac{1}{10} \cdots \frac{1}{10}\right]$$

$$\left.\begin{array}{l} H(Y) = 3,321 \text{ bit/simbol} \\ H(Y|X) = 1 \end{array}\right\} \boxed{C = H(Y) - H(X|Y) = 2,321 \text{ bit/simbol}}$$

3. $H(S) = -\lambda\ln\lambda - (1-\lambda)\ln(1-\lambda) + \lambda H_A + (1-\lambda)H_B \,/'$

$$-\ln\lambda - \cancel{1} + \ln(1-\lambda) + \cancel{1} + H_A - H_B = 0$$

$$\ln\left(\frac{1-\lambda}{\lambda}\right) + H_A - H_B = 0$$

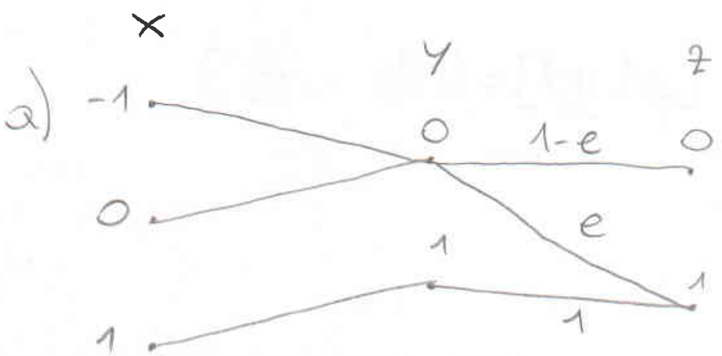
$$\frac{1-\lambda}{\lambda} = e^{-H_A + H_B}$$

$$1-\lambda = \lambda e^{-H_A + H_B}$$

$$\lambda = \frac{1}{1+e^{-H_A+H_B}} = \boxed{\frac{e^{H_A}}{e^{H_A}+e^{H_B}}}$$

4. $X \sim \{-1, 0, 1\}$ $\quad Y = u(x) = \begin{cases} 0, & x \leq 0 \\ 1, & \text{inače} \end{cases}$

a)



b) Entropija od Y će biti najveća za sve razdiobe od X oblika $X \sim \begin{pmatrix} -1 & 0 & 1 \\ p & q & r \end{pmatrix}$, gdje vrijedi $\boxed{p+q=r}$

$$[p(y_j|x_i)] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[p(x_i, y_j)] = \begin{bmatrix} p & 0 \\ q & 0 \\ 0 & r \end{bmatrix}$$

$$[p(y_j)] = [p+q \quad r]$$

c) Analogno zadatku 2c, kapacitet je minimalan za $e=1$.

d) $e = \frac{1}{2}$, $p(x_i) = \frac{1}{3}$

$$[p(y_j|x_i)] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p(x_i, y_j) = \begin{bmatrix} 1/3 & 0 \\ 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \quad [p(y_j)] = [2/3 \quad 1/3]$$

$$p(z_k|y_j) = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$p(y_j, z_k) = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{bmatrix} \quad [p(z_k)] = [1/3 \quad 2/3]$$

$$[p(z_k|x_i)] = [p(Y|X)] \cdot [p(Z|Y)] =$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$p(x_i, z_k) = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/6 \\ 0 & 1/3 \end{bmatrix}$$

$$\left.\begin{matrix} H(Z) = 0{,}918 \\ H(Z|X) = 0{,}667 \end{matrix}\right\} I(Z;X) = H(Z) - H(Z|X) = \boxed{0{,}251 \text{ bit/simbol}}$$

5. Prosječna duljina kodne riječi, L, za neki kod izračunava se tako da se zbroje duljine kodnih riječi za svaki simbol $l(x_i)$, pomnožene s vjerojatnošću pojavljivanja tog simbola, $p(x_i)$: $L = \sum_{i=1}^{n} p(x_i) \cdot l(x_i)$ bit/simbol

Za svaki prefiksni kod s abecedom od d simbola (baza koda) i duljinama kodnih riječi l_i vrijedi:

$$\sum_{i=1}^{n} d^{-l_i} \leq 1$$ KRAFTOVA NEJEDNAKOST

Optimalan kod je prefiksni kod s najmanjom mogućom prosječnom duljinom kodne riječi.

a)

					l_i
a	1/2	0			1
b	1/8	1	0	0	3
c	1/8	1	0	1	3
d	1/8	1	1	0	3
e	1/8	1	1	1	3

$L(X) = 2$
$H(X) = 2$ } $L(X) = H(X) \Rightarrow$ kod je optimalan

3/9	0	0		2
2/9	0	1		2
2/9	1	0		2
1/9	1	1	0	3
1/9	1	1	1	3

$H(X) = 2,1971$
$L_1(X) = 2,22$ } $L(X) < H(X)$, postoji bolji kod

b) A $[0, 0{,}2)$

B $[0{,}2, 1)$

ABA

B: $D = 0 + (0{,}2) \cdot 0{,}2 = 0{,}04$

$G = 0 + 0{,}2 \cdot 1 = 0{,}2$

A: $D = 0{,}04 + (0{,}2 - 0{,}04) \cdot 0 = 0{,}04$

$G = 0{,}04 + (0{,}2 - 0{,}04) \cdot 0{,}2 = 0{,}072$

$\boxed{[0{,}04, 0{,}072)}$

- primili smo $0 \rightarrow \boxed{AAAAA}$

1. simbol = 0 $D=0$ $G = 0{,}2$

2. A: $D = 0 + 0{,}2 \cdot 0 = 0$

$G = 0 + 0{,}2 \cdot 0{,}2 = 0{,}04$

3. A: $D = 0 + 0{,}04 \cdot 0 = 0$

$G = 0 + 0{,}04 \cdot 0{,}2 = 0{,}008$

4: A: $D = 0$

$G = 0{,}0016$

5: A: $D = 0$

$G = 0{,}00032$

- primili smo $0{,}21 \rightarrow \boxed{BAABA}$

$D = 0{,}2 \quad G = 1$

2.: A: $D = 0{,}2 + 0{,}8 \cdot 0 = 0{,}2$

$G = 0{,}2 + 0{,}8 \cdot 0{,}2 = 0{,}36$

3: A: $D = 0{,}2 + (0{,}36 - 0{,}2) \cdot 0 = 0{,}2$

$G = 0{,}232$

4: ~~A~~: $D = 0{,}2$

$G = 0{,}2064$

B: $D = 0{,}2064$

$G = 0{,}232$

5: A: $D = 0{,}2064$

$G = 0{,}2115$

c) AACAACABCABAA* (0,0,A)

$\boxed{A}$ACAA (1,1,C)

$\boxed{AAC}$AACA (3,3,A)

AAC$\boxed{AACA}$BCAB (0,0,B)

AACA$\boxed{ACAB}$CABA (3,3,A)

AACAACAB$\boxed{CABA}$A* (1,1,*)

d) yxyxxxxx   D[0] = x D[1] = y

yx D[2] = yx → 1
xy D[3] = xy → 0
yxx D[4] = yxx → 2
xx D[5] = xx → 0
xxx D[6] = xxx → 5
x → 0

šalje se $\boxed{102050}$

LZ77, LZ78 i LZW daju bolje rezultate pri kompresiji realnih izvora informacije u usporedbi s Huffmanovim zato što je efikasnije dinamički graditi rječnik te ga nadograđivati nadolazećim simbolima, čime se rječnik automatski prilagođava karakteristikama izvora. Pritom vrijednosti pojavljivanja ne moraju biti poznate, što je velika prednost u odnosu na ostale tipove kodiranja. Zbog ovoga svojstva ovi koderi nazivaju se i univerzalnima.