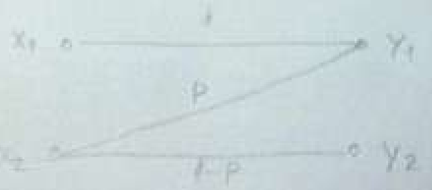
**Zadatak 2.** Binarni izvor odašilje simbole s vjerojatnostima $p(x_1)=1-\alpha$ i $p(x_2)=\alpha$ kroz tzv. Z-kanal opisan sljedećom matricom prijelaza:

$$P=\begin{bmatrix}1 & 0\\ p & 1-p\end{bmatrix}$$




a) Izračunaj $H(Y)$.

$$[p(x_i,y_j)]=[p(x_i)\cdot p(y_j|x_i)]=\begin{bmatrix}1-\alpha & 0\\ \alpha p & \alpha(1-p)\end{bmatrix}$$

$$[p(y_j)]=\begin{bmatrix}1-\alpha+\alpha p & \alpha(1-p)\end{bmatrix}=\begin{bmatrix}1-\alpha(1-p) & \alpha(1-p)\end{bmatrix}$$

$$H(Y)=-\sum_{j=1}^{m}p(y_j)\log_2 p(y_j)$$

$$H(Y)=-\left[(1-\alpha(1-p))\log_2(1-\alpha(1-p))+\alpha(1-p)\log_2\alpha(1-p)\right]\ \frac{bit}{simbol}$$

$$\boxed{H(Y)=-(1-\alpha(1-p))\log_2(1-\alpha(1-p))-\alpha(1-p)\log_2\alpha-\alpha(1-p)\log_2(1-p)\ \frac{bit}{simbol}}$$

b) količina prenesene informacije

$$I(X;Y)=H(Y)-H(Y|X)$$

$$H(Y|X)=-\sum_{i=1}^{n}\sum_{j=1}^{m}p(x_i,y_j)\log_2 p(y_j|x_i)$$

$$=-\left[\underbrace{(1-\alpha)\log_2 1}_{\emptyset}+\alpha p\log_2 p+\alpha(1-p)\log_2(1-p)\right]$$

$$=-\alpha p\log_2 p-\alpha(1-p)\log_2(1-p)$$

$$I(X;Y)=-(1-\alpha(1-p))\log_2(1-\alpha(1-p))-\alpha(1-p)\log_2\alpha-\cancel{\alpha(1-p)\log_2(1-p)}+$$
$$+\alpha p\log_2 p+\cancel{\alpha(1-p)\log_2(1-p)}$$

$$\boxed{I(X;Y)=-(1-\alpha+\alpha p)\log_2(1-\alpha+\alpha p)-\alpha(1-p)\log_2\alpha+\alpha p\log_2 p}$$

c) kapacitet kanala

$$C = \max_{\{p(x_i)\}} I(X;Y) = \max_{\{p(x_i)\}} \left( H(Y) - H(Y|X=x_2)\cdot p(X=x_2) \right)$$

$$I(X;Y) = -(1-\alpha(1-p))\log_2(1-\alpha(1-p)) - \alpha(1-p)\log_2\alpha - \cancel{\alpha(1-p)\log_2(1-p)}$$
$$+ \alpha p\log_2 p + \cancel{\alpha(1-p)\log_2(1-p)}$$

$$\frac{\partial I(X;Y)}{\partial\alpha} = (1-p)\log_2(1-\alpha(1-p)) - \frac{\cancel{(1-\alpha(1-p))}\cdot(1-p)}{\cancel{(1-\alpha(1-p))}\ln 2} - (1-p)\log_2\alpha$$
$$- \cancel{\alpha(1-p)\frac{1}{\alpha\ln 2}} + p\log_2 p = 0$$

$$\Rightarrow (1-p)\left(\log_2\alpha - \log_2(1-\alpha(1-p))\right) = p\log_2 p$$

$$\log_2\frac{\alpha}{1-\alpha(1-p)} = \frac{p}{1-p}\log_2 p$$

$$\frac{1-\alpha(1-p)}{\alpha} = 2^{-\frac{p\log_2 p}{1-p}}$$

$$1-\alpha(1-p) = \alpha\cdot 2^{-\frac{p\log_2 p}{1-p}}$$

$$\alpha\left(2^{-\frac{p\log_2 p}{1-p}} + (1-p)\right) = 1$$

$$\alpha = \frac{1}{2^{-\frac{p\log_2 p}{1-p}} + (1-p)}$$

$$\alpha = \frac{1}{2^{-\frac{p\log_2 p}{1-p}}\cdot 2^{\log_2(1-p)}\cdot\frac{1}{1-p} + (1-p)} = \frac{1}{2^{-\frac{p\log_2 p}{1-p}}\cdot 2^{-\frac{(1-p)\log_2(1-p)}{1-p}}\cdot\frac{1}{(1-p)^{-1}} + (1-p)}$$

$$= \frac{1}{2^{\frac{H(p)}{1-p}}\cdot(1-p) + (1-p)} = \boxed{\frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)} = \alpha}$$

$$C = -\left(1 - \frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)} + p\cdot\frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)}\right)\log_2\left(1 - \frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)} + p\cdot\frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)}\right)$$
$$-(1-p)\cdot\frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)}\log_2\frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)} + p\cdot\frac{1}{(1-p)\left(1+2^{\frac{H(p)}{1-p}}\right)}\log_2 p$$

Nakon sređivanja: $\boxed{C = \log_2\left(1+(1-p)\cdot p^{\frac{p}{1-p}}\right)\ \frac{\text{bit}}{\text{simbol}}}$