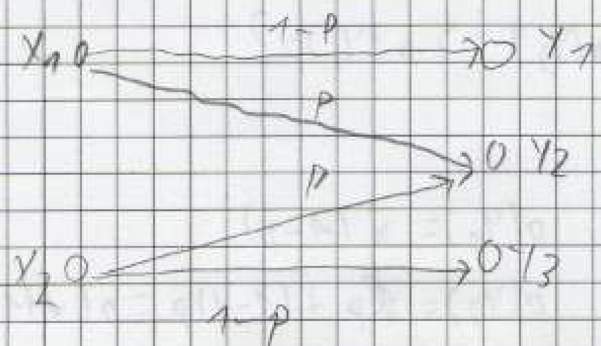
(1.) $p(x_1) = \alpha$
$p(x_2) = 1-\alpha$

$$P = p(y_j|x_i) = \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$



a) $H(Y|X) = ?$

$$H(Y|X) = -\sum_{i=1}^{2}\sum_{j=1}^{3} p(x_i, y_j) \log_2 p(y_j|x_i) \Rightarrow p(x_i, y_j) = ?$$

$$p(x_i, y_j) = p(y_j|x_i) \cdot p(x_i)$$

$$p(x_i, y_j) = \begin{bmatrix} \alpha(1-p) & \alpha p & 0 \\ 0 & (1-\alpha)p & (1-\alpha)(1-p) \end{bmatrix}$$

$$H(Y|X) = -\alpha(1-p)\log(1-p) - \alpha p \log p - (1-\alpha)p\log p - (1-\alpha)(1-p)\log(1-p)$$

$$H(Y|X) = -\log(1-p)\left[\alpha(1-p) + (1-\alpha)(1-p)\right] - \log p\left[\alpha p + (1-\alpha)p\right] =$$

$$= -\log(1-p)\left[(1-p)(\cancel{\alpha} + 1 - \cancel{\alpha})\right] - \log p\left[p(\cancel{\alpha} + 1 - \cancel{\alpha})\right] =$$

$$\boxed{H(Y|X) = -(1-p)\log(1-p) - p\log p}$$

b) $I(X;Y) = ?$

$$p(x_i,y_j) = \begin{bmatrix} \alpha(1-p) & \alpha p & 0 \\ 0 & (1-\alpha)p & (1-\alpha)(1-p) \end{bmatrix}$$

$$I(X;Y) = H(Y) - H(Y/X) \Rightarrow H(Y) = ?$$

$$H(Y) = -\sum_{j=1}^{3} p(y_j) \log p(y_j) \Rightarrow p(y_j) = ?$$

$$p(y_j) = \sum_{i=1}^{2} p(x_i, y_j) \Rightarrow \quad p(y_1) = \alpha(1-p)$$

$$p(y_2) = \alpha p + (1-\alpha)p = p(\cancel{\alpha} + 1 - \cancel{\alpha}) = p$$

$$p(y_3) = (1-\alpha)(1-p)$$

$$H(Y) = -\alpha(1-p)\log[\alpha(1-p)] - p\log p - (1-\alpha)(1-p)\log[(1-\alpha)(1-p)] =$$

$$= -\alpha(1-p)\log\alpha - \alpha(1-p)\log(1-p) - p\log p - (1-\alpha)(1-p)\log(1-\alpha) - (1-\alpha)(1-p)\log(1-p) =$$

$$= -\alpha(1-p)\log\alpha - \log(1-p)[\alpha(1-p) + (1-\alpha)(1-p)] - p\log p - (1-\alpha)(1-p)\log(1-\alpha) =$$

$$= -\alpha(1-p)\log\alpha - \log(1-p)[(1-p)(\cancel{\alpha} + 1 - \cancel{\alpha})] - p\log p - (1-\alpha)(1-p)\log(1-\alpha) =$$

$$\underline{H(Y) = -\alpha(1-p)\log\alpha - (1-p)\log(1-p) - p\log p - (1-\alpha)(1-p)\log(1-\alpha)}$$

$$H(Y/X) = -(1-p)\log(1-p) - p\log p$$

$$I(X;Y) = -\alpha(1-p)\log\alpha - \cancel{(1-p)\log(1-p)} - \cancel{p\log p} - (1-\alpha)(1-p)\log(1-\alpha) + \cancel{(1-p)\log(1-p)} + \cancel{p\log p}$$

$$\boxed{I(X;Y) = -\alpha(1-p)\log\alpha - (1-\alpha)(1-p)\log(1-\alpha)}$$

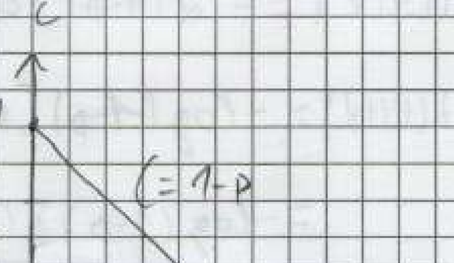
c)

$$C = \max_{\{p(x_i)\}} I(X;Y) \Rightarrow \quad p(x_1) = p(x_2)$$

$$\alpha = 1 - \alpha$$

$$2\alpha = 1$$

$$\alpha = \frac{1}{2}$$



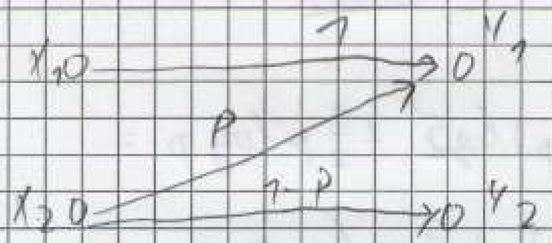
$$C = -\frac{1}{2}(1-p)\log 2^{-1} - \frac{1}{2}(1-p)\log 2^{-1} =$$

$$= -(1-p)\log 2^{-1} = -(1-p)\cdot(-1) = 1-p$$

$$\boxed{C = 1-p}$$

(2) $p(x_1) = 1-\alpha$ $\qquad P = p(y_j|x_i) = \begin{bmatrix} 1 & 0 \\ p & 1-p \end{bmatrix}$

$p(x_2) = \alpha$



a) $H(Y) = ?$

$$H(Y) = -\sum_{j=1}^{2} p(y_j) \log p(y_j) \Rightarrow p(y_j) = ?$$

$$p(y_j) = \sum_{i=1}^{2} p(x_i, y_j) \Rightarrow p(x_i, y_j) = ?$$

$$p(x_i, y_j) = p(y_j|x_i) \cdot p(x_i) \Rightarrow p(x_i, y_j) = \begin{bmatrix} 1-\alpha & 0 \\ \alpha p & \alpha(1-p) \end{bmatrix}$$

$\underline{p(y_1) = 1-\alpha+\alpha p \quad , \quad p(y_2) = \alpha(1-p)}$

$$H(Y) = -(1-\alpha+\alpha p) \log(1-\alpha+\alpha p) - \alpha(1-p) \log \alpha(1-p)$$

$$\boxed{H(Y) = -(1-\alpha+\alpha p) \log(1-\alpha+\alpha p) - \alpha(1-p) \log \alpha - \alpha(1-p) \log(1-p)}$$

b) $I(X;Y) = ?$

$$I(X;Y) = H(Y) - H(Y|X) \Rightarrow H(Y|X) = ?$$

$$H(Y|X) = -\sum_{i=1}^{2} \sum_{j=1}^{2} p(x_i, y_j) \log p(y_j|x_i)$$

$\underline{H(Y|X) = -\alpha p \log p - \alpha(1-p) \log(1-p)}$

$$I(X;Y) = -(1-\alpha+\alpha p) \log(1-\alpha+\alpha p) - \alpha(1-p) \log \alpha - \cancel{\alpha(1-p) \log(1-p)} + \alpha p \log p + \cancel{\alpha(1-p) \log(1-p)}$$

$$\boxed{I(X;Y) = -(1-\alpha+\alpha p) \log(1-\alpha+\alpha p) - \alpha(1-p) \log \alpha + \alpha p \log p}$$

c) $C = ?$

$$C = \max_{\{p(x_i)\}} I(X;Y) \Rightarrow p(x_1) = p(x_2) \Rightarrow \underline{\alpha = \frac{1}{2}}$$

$$C = -\left(1-\frac{1}{2}+\frac{1}{2}p\right)\log\left(1-\frac{1}{2}+\frac{1}{2}p\right) - \frac{1}{2}(1-p)\log 2^{-1} + \frac{1}{2}p\log p =$$

$$= -\frac{1}{2}(1+p)\log\left[\frac{1}{2}(1+p)\right] + \frac{1}{2}(1-p) + \frac{1}{2}p\log p =$$

$$= -\frac{1}{2}(1+p)\log 2^{-1} - \frac{1}{2}(1+p)\log(1+p) + \frac{1}{2}(1-p) + \frac{1}{2}p\log p =$$

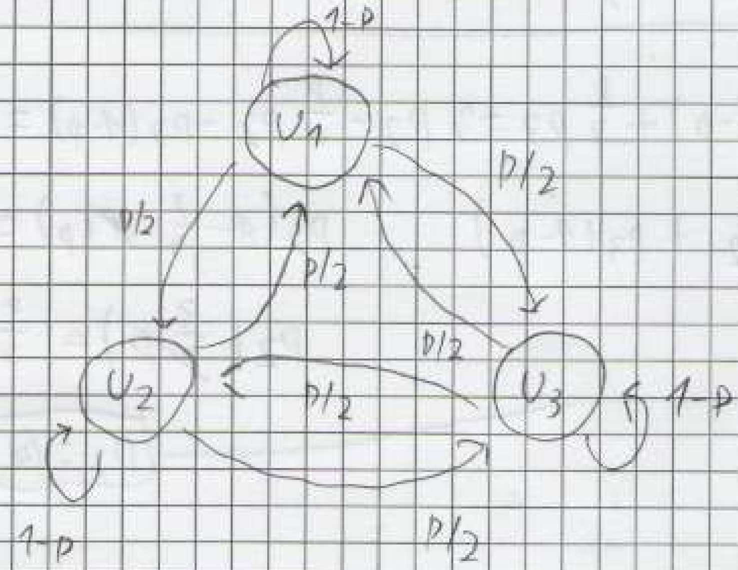
$$= \frac{1}{2}(1+p) - \frac{1}{2}(1+p)\log(1+p) + \frac{1}{2}(1-p) + \frac{1}{2}p\log p =$$

$$\boxed{C = 1 - \frac{1}{2}(1+p)\log(1+p) + \frac{1}{2}p\log p} =$$

(3) $U = \{U_1, U_2, U_3\}$

$p(U_j | U_i) = p/2 \quad , \; i \neq j$

a)



b) $p(U_1) = ? = p_1$

$p(U_2) = ? = p_2$

$p(U_3) = ? = p_3$

$$\underline{p(U_j | U_i)} = \begin{bmatrix} 1-p & p/2 & p/2 \\ p/2 & 1-p & p/2 \\ p/2 & p/2 & 1-p \end{bmatrix}$$

$$p(U_i) = \sum_{j=1}^{3} p(U_i, U_j) \Rightarrow p(U_i, U_j) = ?$$

$$p(U_i, U_j) = p(U_j | U_i) p(U_i) \Rightarrow p(U_j | U_i) = ?$$

$$p(U_i, U_j) = \begin{bmatrix} p_1(1-p) & \frac{p}{2} p_2 & \frac{p}{2} p_3 \\ \frac{p}{2} p_1 & p_2(1-p) & \frac{p}{2} p_3 \\ \frac{p}{2} p_1 & \frac{p}{2} p_2 & (1-p) p_3 \end{bmatrix}$$

$\underline{p_1 + p_2 + p_3 = 1}$

$$p_1 = p_1(1-p) + \frac{p}{2} p_2 + \frac{p}{2} p_3 \Rightarrow p_1 \cancel{p} = \cancel{p} \frac{(p_2 + p_3)}{2}$$

$$p_2 = \frac{p}{2} p_1 + p_2(1-p) + \frac{p}{2} p_3$$

$$p_3 = \frac{p}{2} p_1 + \frac{p}{2} p_2 + p_3(1-p)$$

$$p_2 = \frac{p(p_2+p_3)}{4} + p_2(1-p) + \frac{p}{2} p_3 \Rightarrow p_2 - \frac{p}{4} p_2 - p_2(1-p) = \frac{p}{4} p_3 + \frac{p}{2} p_3$$

$$p_3 = \frac{p(p_2+p_3)}{4} + \frac{p}{2} p_2 + p_3(1-p)$$

$$p_2\left(\cancel{1} - \frac{p}{4} - \cancel{1} + p\right) = \frac{3p}{4} p_3$$

$$p_2\left(\frac{3}{4} \cancel{p}\right) = \frac{3}{4} \cancel{p}\, p_3$$

$$p_1 = \frac{p_2 + p_3}{2}$$

$$p_2 = p_3$$

$$\boxed{p_1 = \frac{2p_2}{2} = p_2}$$

$$p_1 + p_2 + p_3 = 1$$

$$p_2 + p_2 + p_2 = 1$$

$$3p_2 = 1 \Rightarrow \boxed{p_2 = p_1 = p_3 = \frac{1}{3}} \Rightarrow p(U_i, U_j) = \begin{bmatrix} \frac{1-p}{3} & \frac{p}{6} & \frac{p}{6} \\ \frac{p}{6} & \frac{1-p}{3} & \frac{p}{6} \\ \frac{p}{6} & \frac{p}{6} & \frac{1-p}{3} \end{bmatrix}$$

c) $H(U_j / U_i) = ?$

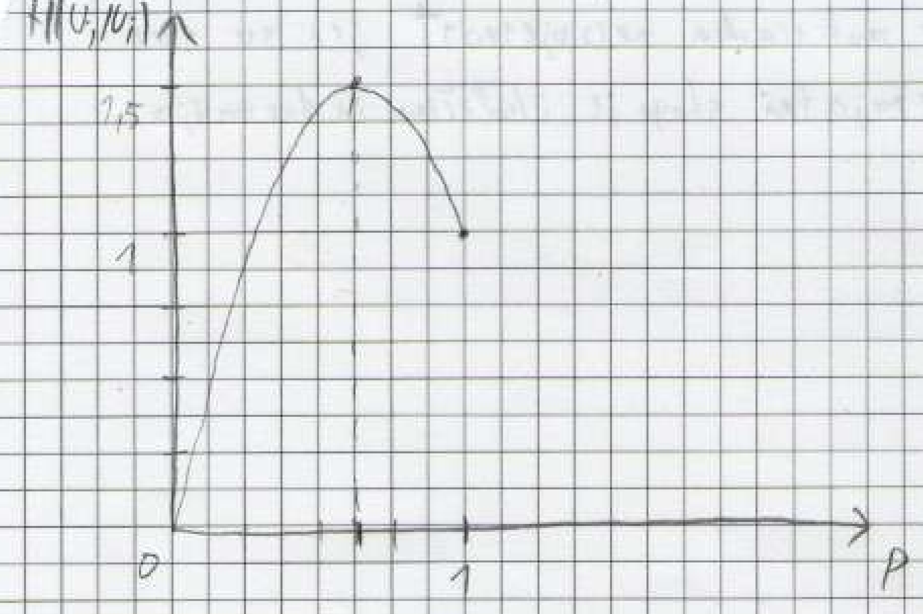
$$H(U_j / U_i) = -\sum_{i=1}^{3} \sum_{j=1}^{3} p(U_i, U_j) \log p(U_j / U_i) =$$

$$= -\left(\left(\frac{1-p}{3}\right) \log(1-p) - \frac{p}{6} \log \frac{p}{2} - \frac{p}{6} \log \frac{p}{2}\right) \cdot 3 =$$

$$= -\left(\frac{1-p}{3} \log(1-p) - \frac{p}{3} \log \frac{p}{2}\right) \cdot 3 =$$

$$= -(1-p)\log(1-p) - p \log \frac{p}{2} = -(1-p)\log(1-p) - p\log p + p \log 2 =$$

$$\boxed{H(U_j / U_i) = p - p\log p - (1-p)\log(1-p)}$$



d) $p = ?$ $(\max H(U_j|U_i))$

$$H(U_j|U_i) = p - p\log p - (1-p)\log(1-p)$$

$$(H(U_j|U_i))' = 1 - \log p - \cancel{p}\frac{1}{\cancel{p}\ln 2} + \log(1-p) + \cancel{(1-p)}\frac{1}{\cancel{(1-p)}\ln 2} =$$

$$= 1 - \log p + \log(1-p)$$

$$(H(U_j|U_i))' = 0 \Rightarrow 1 - \log p + \log(1-p) = 0$$

$$\log 2 - \log p + \log(1-p) = \log 1$$

$$\log\left[\frac{2}{p}(1-p)\right] = \log 1$$

$$\frac{2}{p}(1-p) = 1$$

$$2 - 2p = p$$

$$3p = 2$$

$$\boxed{p = \frac{2}{3}}$$

e) Za $p=0$ svaki simbol može jedino prijeći sam u sebe stoga ne postoji nikakva neizvjesnost pa je količina informacije 0 bit/simbol.

Za $p=1$ svaki simol može jedino prijeći u jedan od preostala dva simbola sa jednakom vjerojatnošću, u tom slučaju je količina informacije 1 bit/simol.

»»

Za $p = \frac{2}{3}$ ostvaruje se maksimalna neizvjesnost jer su svi prijelazi jednako vjerojatni stoga je količina informacija maksimalna.

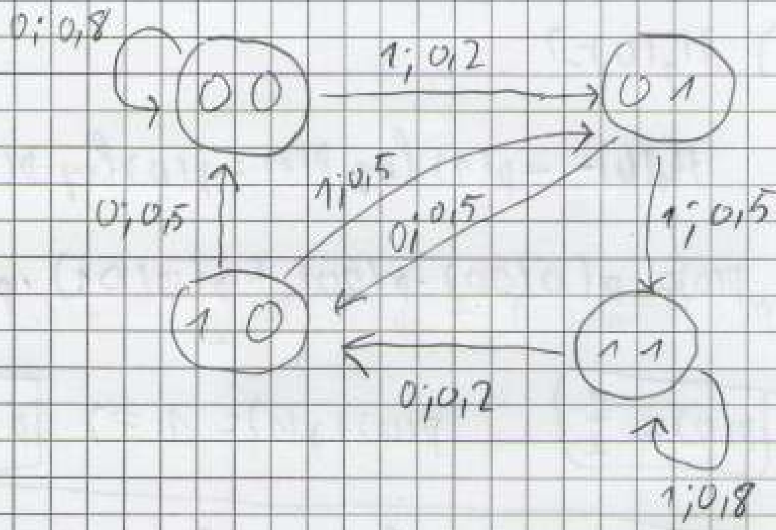
$U = \{0, 1\}$

(1.) $p(0|00) = 0{,}8$

$p(0|01) = 0{,}5$

$p(0|10) = 0{,}5$

$p(0|11) = 0{,}2$



a) $H_3(tri) = ?$

$H_3(U) = ?$

$p(00) = p(11) = \frac{5}{14}$

$p(01) = p(10) = \frac{1}{7}$

$p(000) = p(00) \cdot p(0|00) = \frac{2}{7}$

$p(001) = p(00) \cdot p(1|00) = \frac{1}{14}$

$p(010) = p(01) \cdot p(0|01) = \frac{1}{14}$

$p(011) = p(01) \cdot p(1|01) = \frac{1}{14}$

$p(100) = p(10) \cdot p(0|10) = \frac{1}{14}$

$p(101) = p(10) \cdot p(1|10) = \frac{1}{14}$

$p(110) = p(11) \cdot p(0|11) = \frac{1}{14}$

$p(111) = p(11) \cdot p(1|11) = \frac{2}{7}$

$H_3(tri) = -2 \cdot \frac{2}{7} \log \frac{2}{7} - 6 \cdot \frac{1}{14} \log \frac{1}{14} = \boxed{2{,}664 \text{ bit/trigram}}$

$\boxed{H_3(U) = \frac{H_3(tri)}{3} = 0{,}888 \text{ bit/simbol}}$

b) $H_2(bi) = ?$

$H_2(U) = ?$

$H_2(bi) = -2 \cdot \frac{5}{14} \log \frac{5}{14} - 2 \cdot \frac{1}{7} \log \frac{1}{7}$

$\boxed{H_2(bi) = 1{,}863 \text{ bit/bigram}}$

$\boxed{H_2(U) = \frac{H_2(bi)}{2} = 0{,}932 \text{ bit/simbol}}$

c) $H_1(U) = ?$

$H_1(U) = -p(1)\log p(1) - p(0)\log p(0) \Rightarrow p(0) = ?,\ p(1) = ?$

$p(0) = p(0/00)\cdot p(00) + p(0/01)\cdot p(01) + p(0/10)\cdot p(10) + p(0/11)\cdot p(11)$

$\boxed{p(0) = \frac{1}{2}}$ $\quad p(0) + p(1) = 1 \Rightarrow \boxed{p(1) = \frac{1}{2}}$

$$\boxed{H_1(U) = -2\cdot\frac{1}{2}\log 2^{-1} = 1 \text{ bit/simbol}}$$

d) $F_1(U), F_2(U), F_3(U) = ?$

$F_N(U) = N H_N(U) - (N-1) H_{N-1}(U)$

$$\boxed{\begin{aligned} F_1(U) &= H_1(U) = 1 \text{ bit/simbol} \\ F_2(U) &= 2H_2(U) - H_1(U) = 0{,}863 \text{ bit/simbol} \\ F_3(U) &= 3H_3(U) - 2H_2(U) = 0{,}801 \text{ bit/simbol} \end{aligned}}$$

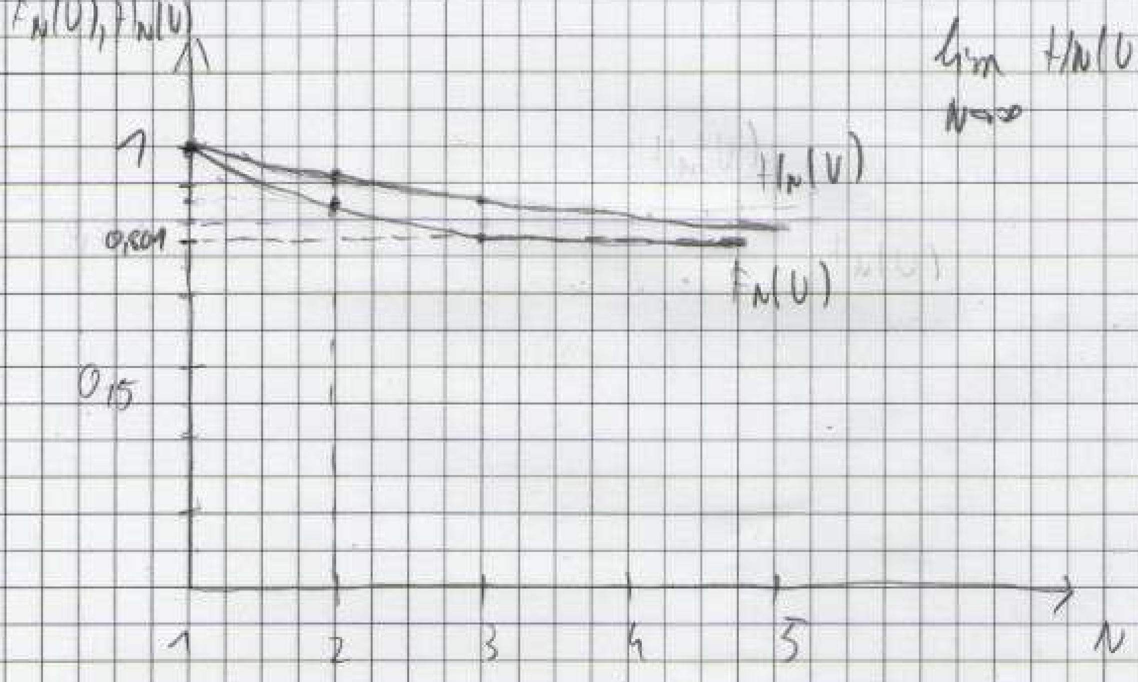
e) $F_1(U) < F_2(U) < F_3(U)$

Za rastući N, $F_N(U)$ će biti određen vjerojatnostima prijelaza čija vrijednost ovisi o broju prethodnih simbola. Povećanjem broja prethodnih simbola neizvjesnost se smanjuje stoga će se i $F_N(U)$ također smanjiti.

f) $F_4(U)$ će biti jednak $F_3(U)$ jer je markov lanac drugog reda, stoga samo dva prethodna simbola utječu na vjerojatnosti prijelaza. Zato za $N \geq 3$ se vrijednost $F_N(U)$ više ne mijenja.

$H_4(U)$ će biti manji od $H_3(U)$, jer prva dva simbola i dalje zadržavaju utjecaj u količini informacije poruke.

g)



$\lim_{N\to\infty} H_N(U) = F_3(U)$

9. LZW

poruka = a bbbbbb ababab aaa b

D[1] = a   D[2] = b

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a	b	b	b	b	b	b	a	b	a	b	a	b	a	a	a	b

korak	mjesto	sadržaj rječnika	izlaz iz kodera
1.	1	(1) a	(1)
2.	2	(2) b	(2)
3.	3	(3) ab	(4)
4.	5	(4) bb	(5)
5.	8	(5) bbb	(3)
6.	10	(6) bbba	(7)
7.	13	(7) aba	(2)
8.	14	(8) abab	(1)
9.	15	(9) ba	(10)
10.	17	(10) aa	(2)
		(11) aab	