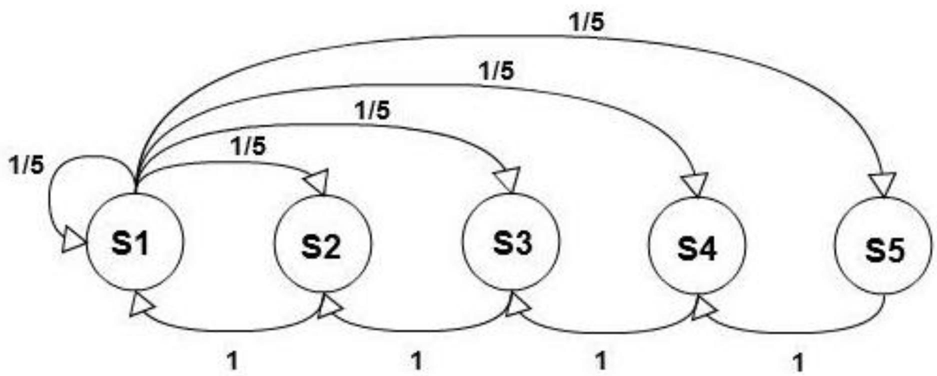
1. ZADATAK 1 Informacijski izvor u pravilnim vremenskim intervalima generira jedan od 5 simbola: s1, s2, s3, s4, s5. Ako je u generiran simbol s1, u sljedećem koraku bit će generiran bilo koji od 5 simbola s jednakom vjerojatnošću (uključujući s1). Ako je generiran simbol si, i > 1, sljedeći simbol koji će se generirati je si1. Pretpostavite da se ovaj izvor može modelirati Markovljevim lancem s diskretnim parametrom.

a) Dijagram stanja izvora



b) Matrica prijelaznih vjerojatnosti izvora

$$P = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

c) Postavljena matrična jednadžbu oblika $A \cdot \hat{p} = \hat{b}$ čijim rješavanjem dolazimo do vektor-stupa stacionarnih vjerojatnosti.

$$P^\top \cdot \hat{p} = \hat{p} \quad \Rightarrow \quad (P^\top - I) \cdot \hat{p} = 0 \qquad \sum \hat{p}_i = 1$$

$$A \cdot \hat{p} = \hat{b}$$

$$(P^\top - I) = \begin{bmatrix} -\frac{4}{5} & 1 & 0 & 0 & 0 \\ \frac{1}{5} & -1 & 1 & 0 & 0 \\ \frac{1}{5} & 0 & -1 & 1 & 0 \\ \frac{1}{5} & 0 & 0 & -1 & 1 \\ \frac{1}{5} & 0 & 0 & 0 & -1 \end{bmatrix}$$
$$1$$

$$A = \begin{bmatrix} -\frac{4}{5} & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \frac{1}{5} & 0 & -1 & 1 & 0 \\ \frac{1}{5} & 0 & 0 & -1 & 1 \\ \frac{1}{5} & 0 & 0 & 0 & -1 \end{bmatrix} \quad \hat{p} = \begin{bmatrix} \hat{p_1} \\ \hat{p_2} \\ \hat{p_3} \\ \hat{p_4} \\ \hat{p_5} \end{bmatrix} \quad \hat{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{4}{5} & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ \frac{1}{5} & 0 & -1 & 1 & 0 \\ \frac{1}{5} & 0 & 0 & -1 & 1 \\ \frac{1}{5} & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \hat{p_1} \\ \hat{p_2} \\ \hat{p_3} \\ \hat{p_4} \\ \hat{p_5} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

d) Postavljen nehomogen sustav jednadžbi čijim rješavanjem bismo dobili stacionarne vjerojatnosti stanja lanca.

$$P^\top \cdot \hat{p} = \hat{p}$$

$$\begin{bmatrix} \hat{p_1} \\ \hat{p_2} \\ \hat{p_3} \\ \hat{p_4} \\ \hat{p_5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 1 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 1 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 1 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & 1 \\ \frac{1}{5} & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{p_1} \\ \hat{p_2} \\ \hat{p_3} \\ \hat{p_4} \\ \hat{p_5} \end{bmatrix}$$

$\hat{p_1} = \frac{1}{5}\hat{p_1} + \hat{p_2}$

$\hat{p_2} = \frac{1}{5}\hat{p_1} + \hat{p_3}$

$\hat{p_3} = \frac{1}{5}\hat{p_1} + \hat{p_4}$

$\hat{p_4} = \frac{1}{5}\hat{p_1} + \hat{p_5}$

$\hat{p_5} = \frac{1}{5}\hat{p_1}$

$\hat{p_1} + \hat{p_2} + \hat{p_3} + \hat{p_4} + \hat{p_5} = 1$

$\hat{p_2} = \frac{4}{5}\hat{p_1}$

$\hat{p_3} = \frac{3}{5}\hat{p_1}$

$\hat{p_4} = \frac{2}{5}\hat{p_1}$

$\hat{p_5} = \frac{1}{5}\hat{p_1}$

e) Stacionarne vjerojatnosti stanja lanca

$$(1 + \frac{4}{5} + \frac{3}{5} + \frac{2}{5} + \frac{1}{5}) = \frac{1}{\hat{p}}$$

$$\hat{p_1} = \frac{1}{3} \quad \hat{p_2} = \frac{4}{15} \quad \hat{p_3} = \frac{1}{5}$$

$$\hat{p_4} = \frac{2}{15} \quad \hat{p_5} = \frac{1}{15}$$

f) Funkcija vjerojatnosti varijable koja mjeri broj intervala koje lanac provodi u istom stanju bez prekida za svako od stanja.

$$P\left(A_i\right) = q^{i-1} \cdot p$$

$$q = \frac{1}{5} \quad p = \frac{4}{5}$$

$$P\left(A_i\right) = \frac{4}{5} \quad za \quad i = 1$$

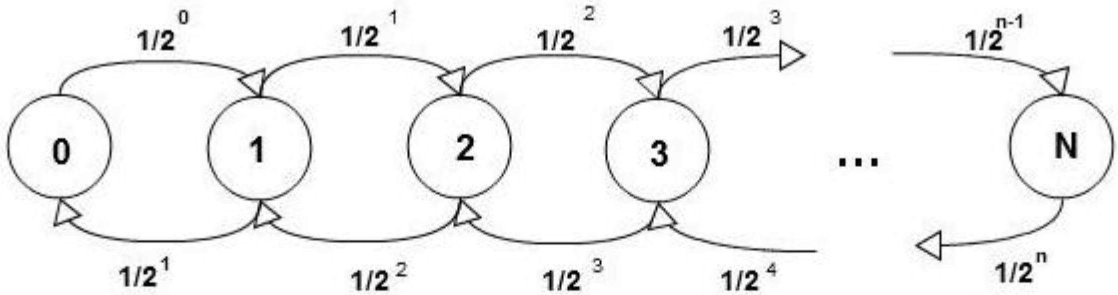
$$P\left(A_i\right) = 0 \quad za \quad i > 1$$

g) Prosječan broj koraka koje lanac bez prekida boravi u istom stanju

$$E\left[N\right] = \frac{p}{(1-q)^2} = \frac{1}{p} = \frac{5}{4} \quad za \quad i = 1$$

$$E\left[N\right] = 1 \quad za \quad i > 1$$

2. ZADATAK 2 Promatrajte Markovljev lanac diskretan po parametru sa stanjima 0, 1, 2,…, N. Vjerojatnost prijelaza iz stanja i u stanje i + 1 kada se lanac nalazi u stanju i je $(1/2)^i$, i < N. Vjerojatnost prijelaza iz stanja i u stanje i+1 je također $(1/2)^i$, i > 0. Ostali prijelazi nisu mogući. Izračunajte stacionarne vjerojatnosti ovog lanca.



$$p_0 = \frac{1}{2}p_1 \qquad p_1 = 2p_0$$

$$(\frac{1}{2}+\frac{1}{2})p_1 = \frac{1}{4}p_2 + p_0 \qquad p_2 = 4p_0$$

$$(\frac{1}{4}+\frac{1}{4})p_2 = \frac{1}{2}p_1 + \frac{1}{8}p_3 \qquad p_3 = 8p_0$$

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$$(\frac{1}{2})^n p_n = (\frac{1}{2})^{n-1}p_{n-1} \qquad p_n = 2p_{n-1} = 2^n p_0$$

$$p_0 + p_1 + p_2 + p_3 + ... + p_n = 1$$

$$p_0 + 2p_0 + 4p_0 + 8p_0 + ... + 2^n p_0 = 1$$

$$p_0 \cdot \sum_0^n 2^n = 1$$

$$p_0 = \frac{1}{\sum_0^n 2^n} = \frac{1}{2^{n+1}-1}$$

$$p_i = \frac{2^i}{2^{n+1}-1}$$