MODULNI OPTIMUM

Optimiranje vremenski diskretnih sustava

Prijenosna funkcija linearnog diskretnog sustava:

$$G(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + \cdots + b_1 z + b_0}{a_n z^n + \cdots + a_1 z + a_0}, \quad m < n$$

Potrebno je odrediti parametre sustava tako da vrijede sljedeće relacije:

$$\lim_{\omega \to 0} \frac{d^i \left| G\left(e^{j\omega T}\right) \right|}{d\omega^i}, \quad i = 1, \ldots, l < n$$

Pritom je jednostavnije koristiti relaciju:

$$\lim_{\omega^2 \to 0} \frac{d^i H\left(\omega^2\right)}{d\left(\omega^2\right)^i} = 0,$$

gdje je:

$$H\left(\omega^2\right) = \left|G\left(e^{j\omega T}\right)\right|^2 = G\left(e^{j\omega T}\right)G\left(e^{-j\omega T}\right)$$

$$H\left(\omega^2\right)=G\left(e^{j\omega T}\right)G\left(e^{-j\omega T}\right)=\frac{B\left(e^{j\omega T}\right)B\left(e^{-j\omega T}\right)}{A\left(e^{j\omega T}\right)A\left(e^{-j\omega T}\right)}$$

Za početak promotrimo samo nazivnik $A\left(e^{j\omega T}\right)A\left(e^{-j\omega T}\right)$:

$$\left(a_0+a_1e^{j\omega T}+a_2e^{j2\omega T}+\cdots+a_ne^{jn\omega T}\right)\left(a_0+a_1e^{-j\omega T}+a_2e^{-j2\omega T}+\cdots+a_ne^{-jn\omega T}\right)=$$

$$\sum_{k=0}^{n}a_k^2+e^{j\omega T}\left(a_1a_0+a_2a_1+\cdots a_na_{n-1}\right)+e^{j2\omega T}\left(a_2a_0+a_3a_1+\cdots a_na_{n-2}\right)+\cdots$$

$$H\left(\omega^2\right)=G\left(e^{j\omega T}\right)G\left(e^{-j\omega T}\right)=\frac{B\left(e^{j\omega T}\right)B\left(e^{-j\omega T}\right)}{A\left(e^{j\omega T}\right)A\left(e^{-j\omega T}\right)}$$

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$$+e^{-j\omega T}\left(a_1a_0+a_2a_1+\cdots a_na_{n-1}\right)+e^{-j2\omega T}\left(a_2a_0+a_3a_1+\cdots a_na_{n-2}\right)+\cdots$$

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$$\sum_{k=1}^{n}a_ka_{k-1}=a_1^*$$

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$$\sum_{k=1}^{n}a_ka_{k-1}=a_1^* \qquad \sum_{k=2}^{n}a_ka_{k-2}=a_2^*$$

$$+e^{-j\omega T}\left(a_1a_0+a_2a_1+\cdots a_na_{n-1}\right)+e^{-j2\omega T}\left(a_2a_0+a_3a_1+\cdots a_na_{n-2}\right)+\cdots$$

Korištenjem relacije:

$$e^{jk\omega T} + e^{-jk\omega T} = 2\cos k\omega T,$$

dobivamo:

$$A\left(e^{j\omega T}\right)A\left(e^{-j\omega T}\right) = \sum_{k=0}^{n} a_k^2 + 2a_1^* \cos\omega T + 2a_2^* \cos 2\omega T + \cdots + 2a_n^* \cos n\omega T$$

Analogno tome:

$$B\left(e^{j\omega T}\right)B\left(e^{-j\omega T}\right) = \sum_{k=0}^{n} b_k^2 + 2b_1^* \cos\omega T + 2b_2^* \cos 2\omega T + \cdots + 2b_m^* \cos m\omega T$$

$$H\left(\omega^2\right)=\frac{B_H}{A_H}=\frac{\sum_{k=0}^{n} b_k^2+2b_1^* \cos\omega+\cdots+2b_m^* \cos m\omega}{\sum_{k=0}^{n} a_k^2+2a_1^* \cos\omega+\cdots+2a_n^* \cos n\omega}$$

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Uvedimo oznake:

$$x=\cos\omega T$$

$$T_k\left(x\right)=\cos k\omega T$$

Nađimo prvu derivaciju:

$$\frac{dH}{d\left(\omega^{2}\right)}=\frac{1}{2\omega}\cdot\frac{dH}{d\omega}=\frac{1}{2\omega}\cdot\frac{dx}{d\omega}\cdot\frac{dH}{dx}$$

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$$\frac{dx}{d\omega} = \frac{d\cos\omega T}{d\omega} = -T\sin\omega T$$

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$$\frac{dx}{d\omega} = \frac{d\cos\omega T}{d\omega} = -T\sin\omega T$$

$$\lim_{\omega^2\to 0}\frac{dH\left(\omega^2\right)}{d\left(\omega^2\right)} = \lim_{\substack{\omega\to 0\\ x\to 1}}\frac{-T\sin\omega T}{2\omega}\cdot\frac{dH}{dx} = 0$$

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$$\neq 0$$

Iz prethodne relacije je očito da mora vrijediti:

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Uvedimo oznake:

$$H' = \frac{dH}{dx}$$

$$\vdots$$

$$H^{(i)} = \frac{d^i H}{dx^i}$$

Promotrimo sada i drugu derivaciju:

$$\frac{d^2 H}{d\left(\omega^2\right)^2} = \frac{1}{2\omega} \cdot \frac{d}{d\omega}\left(-\frac{T \sin \omega T}{2\omega} H'\right)$$

$$= \frac{-1}{2\omega} \cdot \frac{d}{d\omega}\left(\frac{T \sin \omega T}{2\omega}\right) H' + \left(\frac{T \sin \omega T}{2\omega}\right)^2 H''$$

$$= T \frac{\sin \omega T - \omega T \cos \omega T}{4\omega^3} H' + \left(\frac{T \sin \omega T}{2\omega}\right)^2 H''$$

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$$=\frac{-1}{2\omega}\cdot\frac{d}{d\omega}\left(\frac{T\sin\omega T}{2\omega}\right)H'+\left(\frac{T\sin\omega T}{2\omega}\right)^2H''$$

$$=\underbrace{T\frac{\sin\omega T-\omega T\cos\omega T}{4\omega^3}H'}_{\substack{\omega\to 0\\ x\to 1\\ \downarrow\\ 0}}+\left(\frac{T\sin\omega T}{2\omega}\right)^2H''$$

Iz prethodnog izraza se vidi da je

$$\lim_{\omega^2 \to 0} \frac{d^2 H}{d\left(\omega^2\right)^2} = 0$$

samo ako vrijedi:

$$\lim_{x \to 1} H'' = \lim_{x \to 1} \frac{d^2 H}{dx^2} = 0$$

Općenito možemo reći da će relacija

$$\lim_{\omega^2 \to 0} \frac{d^i H\left(\omega^2\right)}{d\left(\omega^2\right)^i} = 0$$

biti zadovoljena ako vrijedi:

$$\lim_{x \to 1} H^{(i)} = \lim_{x \to 1} \frac{d^i H}{dx^i} = 0.$$

Podsjetimo se:

$$H = \frac{B_H}{A_H} = \frac{\sum_{k=0}^{n} b_k^2 + 2b_1^* T_1(x) + \cdots + 2b_m^* T_m(x)}{\sum_{k=0}^{n} a_k^2 + 2a_1^* T_1(x) + \cdots + 2a_n^* T_n(x)},$$

gdje je:

$$x = \cos \omega T$$

$$T_k(x) = \cos k\omega T$$

Iz uvjeta da prva derivacija bude jednaka nuli dobivamo:

$$\frac{dH}{dx} = H' = \frac{B_H' - HA_H'}{A_H} \xrightarrow{x \to 1} 0$$

$$B_H' \Big|_{x=1} = \left(HA_H' \right) \Big|_{x=1}$$

Iz uvjeta da druga derivacija bude jednaka nuli dobivamo:

$$\frac{d^2H}{dx^2} = H'' = \frac{B_H'' - HA_H'' - 2H'A_H'}{A_H} \xrightarrow{x \to 1} 0$$

Iz uvjeta da druga derivacija bude jednaka nuli dobivamo:

$$\frac{d^2H}{dx^2} = H'' = \frac{B_H'' - HA_H'' - \overbrace{2H'A_H'}^{0}}{A_H} \xrightarrow{x \to 1} 0$$

$$B_H''\Big|_{x=1} = \left(HA_H''\right)\Big|_{x=1}$$

Općenito možemo pisati da je

$$\lim_{x\to 1}\frac{d^{i}H}{dx^{i}}=0$$

samo ako vrijedi:

$$B_H^{(i)}\Big|_{x=1}=\left(HA_H^{(i)}\right)\Big|_{x=1}$$

Promotrimo sada što nam dobiveni uvjeti predstavljaju:

$$B_H^{(i)}\Big|_{x=1} = \left(HA_H^{(i)}\right)\Big|_{x=1}\quad , \; i = 1,..,l < n$$

Promotrimo sada što nam dobiveni uvjeti predstavljaju:

$$\left. B_H^{(i)} \right|_{x=1} = \left. \left(H A_H^{(i)} \right) \right|_{x=1} \quad , \; i = 1,..,l < n$$

$$\left. H \right|_{x=1} = \left| G\left(e^{j\omega T}\right) \right|^2_{\omega=0} = \left| G(1) \right|^2 = \left(\frac{\sum_{k=0}^{n} b_k}{\sum_{k=0}^{n} a_k} \right)^2 = K^2$$

Derivacije polinoma A_H i B_H možemo zapisati na sljedeći način:

$$A_H^{(i)} = \frac{d^i A_H}{dx^i} = 2\sum_{k=i}^{n} a_k^* T_k^i$$

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gdje je:

$$T_k^i = \frac{d^i T_k}{dx^i}$$

Za Čebiševljeve polinome $T_k\left(x\right)$ vrijedi sljedeća diferencijalna jednadžba:

$$\frac{d^2 T_k}{d\omega^2} + k^2 T_k = 0$$

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Ili drukčije zapisano:

$$\left(1-x^2\right)T_k^{''}-xT_k^{'}+k^2T_k=0$$

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Ili drukčije zapisano:

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Uvrstimo li x=1, dobijemo:

$$T_k^{'}\left(1\right) = k^2T_k\left(1\right)$$

Pošto za Čebiševljeve polinome također vrijedi:

$$T_k\left(1\right)=1,$$

za prvu derivaciju Čebiševljevih polinoma za x=1 možemo pisati:

$$T_k^{'}\left(1\right)=k^2$$

Pošto za Čebiševljeve polinome također vrijedi:

$$T_k(1) = 1,$$

za prvu derivaciju Čebiševljevih polinoma za x=1 možemo pisati:

$$T_k'(1) = k^2$$

Sada uvjet $B_H'\big|_{x=1} = \left(HA_H'\right)\big|_{x=1}$ možemo pisati na sljedeći način:

$$\boxed{b_1^* + 4b_2^* + \cdots + m^2 b_m^* = K^2\left(a_1^* + 4a_2^* + \cdots + n^2 a_n^*\right)}$$

Da bismo odredili ostale derivacije Čebiševljevih polinoma, vratimo se opet na diferencijalnu jednadžbu:

$$\left(1-x^2\right)T_k'' - xT_k' + k^2 T_k = 0$$

Derivirajmo cijeli izraz (i-1) puta:

$$\frac{d^{i-1}}{dx^{i-1}}\left[\left(1-x^2\right)T_k'' - xT_k' + k^2 T_k\right] = 0$$

$$\left(1-x^2\right)T_k^{i+1} - \left[2x\left(i-1\right)+x\right]T_k^{i} + \left[k^2 - \left(i-1\right)^2\right]T_k^{i-1} = 0$$

Uvrstimo li u prethodni izraz x=1, dobijemo:

$$\left(2i-1\right)T_k^i = k^2 T_k^{i-1} - \left(i-1\right)^2 T_k^{i-1}$$

Pošto nam je otprije poznato $T_k^{'}\left(1\right) = k^2$, ovim smo dokazali da je $T_k^i\left(1\right)$ polinom i-tog stupnja po k^2, tj:

$$T_k^i\left(1\right) \sim k^{2i}$$

Iz prethodne relacije proizlazi slijedeće:

Za neki proizvoljan i=1,..,n-1, linearnom kombinacijom prvih *i* derivacija Čebiševljevog polinoma u točki x=1, moguće je dobiti:

$$\sum_{j=1}^{i} c_j T_k^j = k^{2i} \quad , c_j \in \mathbb{R}$$

Iz toga proizlazi sljedeći zapis uvjeta na promatrani sustav:

$$b_1^* + 2^2 b_2^* + \cdots m^2 b_m^* = K^2 \left(a_1^* + 2^2 a_2^* + \cdots n^2 a_n^* \right)$$

$$\vdots$$

$$b_1^* + 2^{2i} b_2^* + \cdots m^{2i} b_m^* = K^2 \left(a_1^* + 2^{2i} a_2^* + \cdots n^{2i} a_n^* \right)$$

$$\vdots$$

$$b_1^* + 2^{2(n-1)} b_2^* + \cdots m^{2(n-1)} b_m^* = K^2 \left(a_1^* + 2^{2(n-1)} a_2^* + \cdots n^{2(n-1)} a_n^* \right)$$

$$G(z) = \frac{\sum_{k=0}^{m} b_k z^k}{\sum_{k=0}^{n} a_k z^k}$$

$$a_j^* = \sum_{k=j}^{n} a_k a_{k-j}$$

$$b_j^* = \sum_{k=j}^{n} b_k b_{k-j}$$

$$K = G(1) = \frac{\sum_{k=0}^{n} b_k}{\sum_{k=0}^{n} a_k}$$