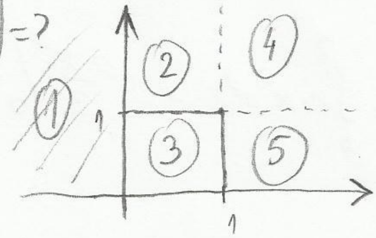
2.) $F(x,y) = \begin{cases} 0, & \text{za } x<0 \text{ ili } y<0 \\ 0, & x+y<1 \\ 1, & \text{ostalo} \end{cases}$

$F(X,Y) = ?$



① $F(x,y) = \emptyset \quad x \leq 0,\ y \leq 0$

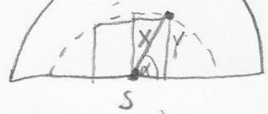
② $F(x,y) = \int_0^x dx \cdot \int_0^1 dy = x \quad , 0 \leq x \leq 1,\ y > 1$

③ $F(x,y) = \int_0^x dx \int_0^y dy = xy \quad , 0 \leq x \leq 1,\ 0 \leq y \leq 1$

④ $F(x,y) = \int_0^1 dx \cdot \int_0^1 dy = 1 \quad , x > 1,\ y > 1$

⑤ $F(x,y) = \int_0^1 dx \cdot \int_0^y dy = y \quad , x > 1,\ 0 \leq y < 1$

3.)

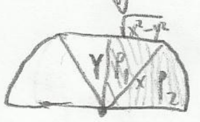


$x \in [0, R]$

$y \in [0, R]$

$x \geq y$

$f(x,y) = ?$



$P_1 = \frac{1}{2} \cdot y\sqrt{x^2-y^2}$

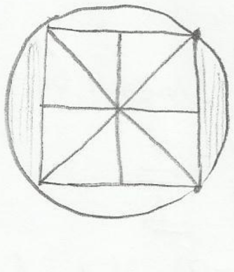
$P_2 = R^2\pi \frac{\alpha}{2\pi}$

$$F(x,y) = P(X<x, Y<y) = \frac{2 \cdot \left[\frac{1}{2} \cdot y\sqrt{x^2-y^2} + x^2 \cdot \frac{1}{2} \arcsin\frac{y}{x}\right]}{\frac{1}{2}R^2 \cdot \pi}$$

$$f(x,y) = \frac{\partial F(x,y)}{\partial x \partial y} = \frac{4x}{R^2 \cdot \pi \cdot \sqrt{x^2-y^2}} \qquad 0 \leq y \leq x \in R$$

4.) jedinična kružnica $(x^2+y^2=1)$

$\varphi = \operatorname{arctg}\left(\frac{y}{x}\right) \qquad y^2 = 1-x^2 \Rightarrow y = \sqrt{1-x^2}$



duljina luka: $L = \frac{2r \cdot \pi \cdot \varphi}{2\pi} = r \cdot \varphi = 1 \cdot \operatorname{arctg}\left(\frac{x}{y}\right)$

$$F(x,y) = \frac{2\pi - 2 \cdot (\operatorname{arctg}(\frac{x}{y}))}{2\pi} = 1 - \frac{\operatorname{arctg}\frac{x}{\sqrt{1-x^2}}}{\pi}$$

$$f(x,y) = \frac{\partial F(x,y)}{\partial x \partial y} = -\frac{1}{\pi} \cdot \frac{1}{\frac{1-x^2}{x^2}} \cdot \frac{1}{x^2 \cdot \sqrt{1-x^2}} = -\frac{1}{\pi} \cdot \frac{\cancel{x^2}}{1-x^2} \cdot \frac{1}{\cancel{x^2}\sqrt{1-x^2}}$$

$$f(x,y) = \frac{1}{\pi \cdot \sqrt{1-x^2}}$$

5.) $f(x) = ax^2, \quad 0 \le x \le 2$

$Y \sim U[-1,1] \to f(y) = \frac{1}{1-(-1)} = \frac{1}{2}, \quad -1 \le y \le 1$

$P(X+Y) \ge 1 = ?$

$\iint \frac{1}{2} a x^2 = 1$

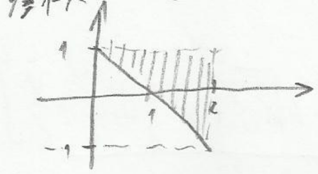
$\frac{1}{2} \int_0^2 ax^2 dx \cdot \int_{-1}^{1} dy = \frac{1}{2} \int_0^2 ax^2 dx \cdot x \Big|_{-1}^{1} = \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot \int_0^2 ax^2 dx = \frac{ax^3}{3} \Big|_0^2 = \frac{a}{3} \cdot 8$

$\frac{a}{3} \cdot 8 = 1 \Rightarrow \boxed{a = \frac{3}{8}}$

$f(x,y) = \frac{1}{2} a x^2 = \frac{3}{16} x^2$

$P(X+Y \ge 1) = \iint_G f(x,y)\,dx\,dy = \frac{3}{16} \int_0^2 x^2 dx \cdot \int_{1-x}^{1} dy = \frac{3}{16} \cdot \int_0^2 x^3 dx = \frac{3}{16} \cdot \frac{x^4}{4} \Big|_0^2$

$y \ge 1-x$

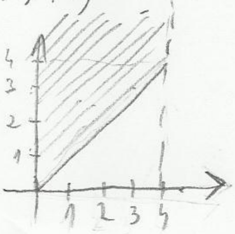


$= \frac{3}{64} \cdot 16 = \frac{3}{4} = 0{,}75$

6.) $X \sim U[0,4] \quad f(x) = \frac{1}{4}$

$Y \sim g(y) = e^{-y}, \; y > \emptyset$

$P(Y \ge X) = ?$



$x \in [0,4]$
$y \in [0,4]$
$P(Y \ge X)$

$P(Y \ge X) = \int_0^4 \frac{1}{4} dx \cdot \int_x^{\infty} e^{-y} dy = \frac{1}{4} \int_0^4 \frac{e^{-y}}{-1} \Big|_x^{\infty} =$

$= \frac{1}{4} \int_0^4 (-e^{-\infty} + e^{-x}) = \frac{1}{4} \int_0^4 e^{-x} = \frac{1}{4} \frac{e^{-x}}{-1} \Big|_0^4$

$= -\frac{1}{4} (e^{-4} - 1) = 0{,}245$

7.) $X \sim N(0,2) \to \sigma^2 = 2 \Rightarrow \sigma = \sqrt{2}$

$Y \sim g(y) = ay, \; y \in [0,2]$

$P(X \le 1, Y \le 1) = ?$

$\int_0^2 ay\,dy = 1$

$a \frac{y^2}{2} \Big|_0^2 \Rightarrow 2a = 1 \Rightarrow \boxed{a = \frac{1}{2}}$

$f(y) = \frac{1}{2} y$

$P(X \le 1, Y \le 1) = P(X \le 1) \cdot P(Y \le 1)$! NEZAVISNE

$P(Y \le 1) = \int_0^1 \frac{1}{2} y\,dy = \frac{1}{2} \int_0^1 y\,dy = \frac{1}{2} \cdot \frac{y^2}{2} \Big|_0^1 = \frac{1}{4}$

$P(X \le 1) = (\text{normalna razdioba}) = P\left(\frac{x-a}{\sigma} \le \frac{1-0}{\sqrt{2}}\right) = P\left(X^* \le \frac{1}{\sqrt{2}}\right)$

$= P\left(X^* \le \frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2} \Phi^*\left(\frac{\sqrt{2}}{2}\right) = 0{,}76$

$P(X \le 1, Y \le 1) = \frac{1}{4} \cdot 0{,}76 = 0{,}19$

8.) $X \sim N\left(0, \frac{1}{2}\right) \Rightarrow \sigma^2 = \frac{1}{2},\ \sigma = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$Y \sim g(y) = 2y,\ 0 \leq y \leq 1$

$f(x,y) = ?,\ P(X>1, Y<0.5) = ?,\ E(XY) = ?$

$$P(X \geq 1) = \frac{1}{2} - \frac{1}{2}\phi^*\left(\frac{1-0}{\frac{\sqrt{2}}{2}}\right) = \frac{1}{2} - \frac{1}{2}\phi^*(\sqrt{2}) = 0{,}07927$$

$$P(Y \leq 0.5) = \int_0^{0.5} 2y\,dy = y^2\Big|_0^{0.5} = \frac{1}{4}$$

$$P(X>1, Y\leq 0.5) = P(X \geq 1) \cdot P(Y \leq 0.5) = 0.07927 \cdot \frac{1}{4} = 0{,}0198$$

$$E(XY) = E(X) \cdot E(Y) = 0 \cdot E(Y) = 0$$

$$f(x,y) = f(x) \cdot f(y)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} = \frac{2}{\sqrt{2}\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2 \cdot \frac{2}{4}}} = \frac{1}{\sqrt{\pi}} \cdot e^{-x^2}$$

$$f(x,y) = \frac{2y}{\sqrt{\pi}} \cdot e^{-x^2} \quad y \in [0,1]$$

9.) III. kvadrant, $x<0,\ y<0$

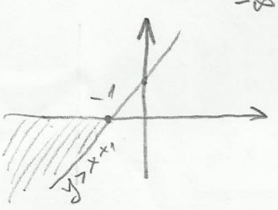
$f(x,y) = k \cdot e^{2x+3y}$

$P(A = Y > X+1) = ?$

$$\int_{-\infty}^{0}\int_{-\infty}^{0} k \cdot e^{2x+3y}\,dx\,dy = 1 \Rightarrow k \cdot \int_{-\infty}^{0} e^{2x}dx \int_{-\infty}^{0} e^{3y}dy = k \cdot \frac{e^{2x}}{2}\Big|_{-\infty}^{0} \frac{e^{3y}}{3}\Big|_{-\infty}^{0} = \frac{k}{6}(1-0)(1-0)$$

$$\frac{k}{6} = 1 \Rightarrow \boxed{k=6}$$

$$P(Y>X+1) = 6\int_{-\infty}^{-1} e^{2x}dx \cdot \int_{x+1}^{0} e^{3y}dy = 6 \cdot \int_{-\infty}^{-1} e^{2x} \cdot \frac{1}{3}(1 - e^{3x}e^{3}) = 2\int_{-\infty}^{-1}(e^{2x} - e^{5x}e^{3})dx =$$



$$= 2\left(\frac{1}{2}(e^{-2} - 0) - e^{3} \cdot \frac{1}{5}(e^{-5} - 0)\right) = e^{-2} - \frac{2}{5}e^{-2} = \frac{3}{5}e^{-2}$$

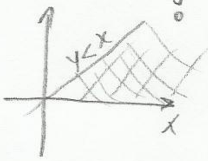
10.) $f(x,y) = C e^{-(2x+y)}$, $x>0$, $y>0$

$P(X>Y)=?$, $P(X=Y)=?$

$$\int_0^\infty\int_0^\infty Ce^{-(2x+y)}\,dx\,dy = 1 \Rightarrow C\int_0^\infty e^{-2x}dx\cdot\int_0^\infty e^{-y}dy = C\cdot\frac{e^{-2x}}{-2}\Big|_0^\infty\cdot\frac{e^{-y}}{-1}\Big|_0^\infty = 1$$

$$C\left(0+\frac{1}{2}\right)(0+1) = 1 \Rightarrow \frac{1}{2}C = 1 \Rightarrow \boxed{C=2}$$

$$P(X>Y) = 2\int_0^\infty e^{-2x}dx\cdot\int_0^x e^{-y}dy = 2\cdot\int_0^\infty e^{-2x}dx\cdot\frac{e^{-y}}{-1}\Big|_0^x = 2\int_0^\infty e^{-2x}dx\cdot(e^{-x}+1) = 2\int_0^\infty(e^{-3x}+e^{-2x})dx$$



$$= 2\cdot\left(\frac{1}{3}(0-1) - \frac{1}{2}(0-1)\right) = \underline{\underline{\frac{1}{3}}}$$

$\underline{P(X=Y) = \emptyset}$, NEZAVISNOST: $f(x,y) = f(x)\cdot f(y)$

$$f(x) = \int_0^\infty f(x,y)\,dy = 2\cdot e^{-2x}\int_0^\infty e^{-y}dy = 2e^{-2x}\cdot 1 = \underline{\underline{2e^{-2x}}}$$

$$f(y) = \int_0^\infty f(x,y)\,dx = 2e^{-y}\int_0^\infty e^{-2x}dx = -e^{-y}(0-1) = \underline{\underline{e^{-y}}}$$

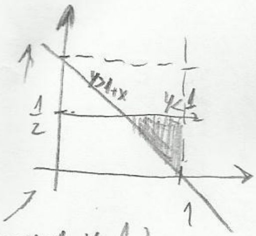
$$f(x)\cdot f(y) = \underline{\underline{2e^{-2x}\cdot e^{-y}}} \rightarrow \text{NEZAVISNE!}$$

12.) $f(x,y) = Cx(1-x)y$, $0<x<1$, $0<y<1$

$P(X+Y>1 \mid Y<\frac{1}{2}) = ?$

$$\int_0^1\int_0^1 Cx(1-x)y\,dx\,dy = 1 \Rightarrow C\int_0^1 x(1-x)\cdot\int_0^1 y\,dy = C\int_0^1 x(1-x)\cdot\frac{y^2}{2}\Big|_0^1 = \frac{1}{2}C\int_0^1(x-x^2)dx$$

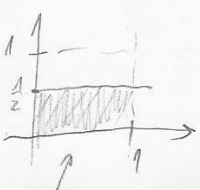
$$= \frac{1}{2}C\left[\int_0^1 x - \int_0^1 x^2\right] = \frac{1}{2}C\cdot\left[\frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1\right] = \frac{1}{2}C\left(\frac{1}{2}-\frac{1}{3}\right) \Rightarrow \frac{1}{12}C = 1 \Rightarrow \boxed{C=12}$$



$P(X+Y>1, Y<\frac{1}{2})$

$$P(X+Y>1 \mid Y<\tfrac{1}{2}) = \frac{P(X+Y>1, Y<\frac{1}{2})}{P(Y<\frac{1}{2})} = \frac{\cancel{12}\cdot\int_{0.5}^1(x-x^2)\cdot\int_{1+x}^{\frac{1}{2}} y\,dy}{\cancel{12}\cdot\int_0^1(x-x^2)\cdot\int_0^{0.5} y\,dy}$$

$$= \frac{\int_{0.5}^1(x^2-x)\cdot\frac{1}{2}\left(\frac{1}{4}-1+2x-x^2\right)dx}{\left(\frac{1}{3}-\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\frac{1}{4}} = \frac{\frac{1}{2}\int_{0.5}^1\left(-\frac{3}{4}x^2+2x^3-x^4+\frac{3}{4}x-\cancel{x}\right)}{-\frac{1}{6}\cdot\frac{1}{8}}$$



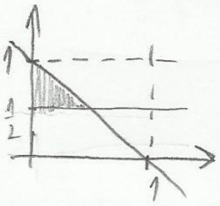
$P(Y<\frac{1}{2})$

$$= \frac{\frac{1}{2}\int_{0.5}^1\left(3x^3-x^4-\frac{11}{4}x^2+\frac{3}{4}x\right)dx}{-\frac{1}{48}} = -24\left(\frac{3}{4}\left(1-\frac{1}{16}\right)-\frac{1}{5}\left(1-\frac{1}{32}\right)-\frac{11}{12}\left(1-\frac{1}{8}\right)+\frac{3}{8}\left(1-\frac{1}{4}\right)\right)$$

$$= \underline{\underline{0{,}275}}$$

13.) $f(x,y) = 12x(1-y)(1-x)$, $x \in [0,1]$, $y \in [0,1]$

$P(X+Y<1 \mid Y>\frac{1}{2}) = ?$



$$P(X+Y<1 \mid Y>\tfrac{1}{2}) = \frac{P(X+Y<1,\ Y>\frac{1}{2})}{P(Y>\frac{1}{2})} = \frac{\int_0^{1/2}(x-x^2)dx \cdot \int_{1/2}^{1-x}(1-y)\,dy}{12\int_0^1(x-x^2)dx \cdot \int_{1/2}^1(1-y)\,dy} =$$

$$= \frac{\int(x-x^2)\left(\frac{1}{2}-x-\frac{1}{2}y^2\right)}{\left(\frac{1}{2}-\frac{1}{3}\right)\cdot\left(\frac{1}{2}-\frac{1}{2}\cdot\frac{3}{4}\right)} = \frac{\int(x-x^2)\left(\frac{1}{2}-x-\frac{1}{2}\left(1-2x+x^2-\frac{1}{4}\right)\right)dx}{\frac{1}{6}\cdot\left(\frac{1}{2}-\frac{3}{8}\right)}$$

$$= \frac{\int(x-x^2)\left(\cancel{-x+x}-\frac{1}{2}x^2+\frac{1}{8}\right)}{\frac{1}{48}} = \frac{\int\left(-\frac{1}{2}x^3+\frac{1}{8}x+\frac{1}{2}x^4-\frac{1}{8}x^2\right)dx}{\frac{1}{48}}$$

$$= 48\cdot\left(-\frac{1}{8}\cdot\frac{1}{16}+\frac{1}{16}\cdot\frac{1}{4}+\frac{1}{10}\cdot\frac{1}{32}-\frac{1}{24}\cdot\frac{1}{8}\right) = \frac{11}{40} = \underline{0{,}275}$$

14.) $f(x,y) = Cxy$

$0 \le y \le x \le 1$

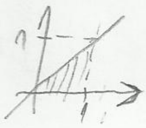
$f(x), f(y) = ?$, $P(X>\frac{1}{2} \mid X+Y>\frac{1}{2}) = ?$

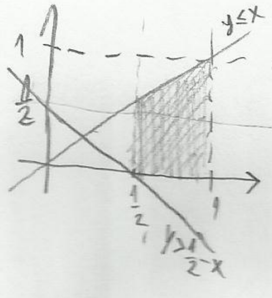
$$C\cdot\int_0^1 x\,dx\cdot\int_0^x y\,dy = C\cdot\int_0^1 x\cdot\frac{1}{2}x^2dx = \frac{C}{2}\cdot\frac{x^4}{4}\Big|_0^1 \Rightarrow \frac{C}{8} = 1 \Rightarrow \boxed{C=8} \Rightarrow \underline{f(x,y) = 8xy}$$

$f_X(x) = \int_0^x 8xy\,dy = 8x\cdot\frac{x^2}{2} = \underline{4x^3}$ $x \in [0,1]$

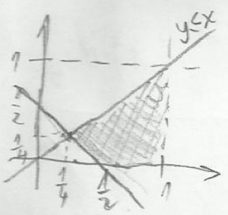
$f_Y(y) = \int_y^1 8xy\,dx = 8y\cdot\frac{x^2}{2}\Big|_y^1 = \underline{4y-4y^3}$ $y \in [0,1]$

$\Big\}$ ZAVISNE!





$$\frac{P(X>\frac{1}{2},\ X+Y>\frac{1}{2})}{P(X+Y>\frac{1}{2})} = \frac{8\int_{1/2}^1 x\,dx\cdot\int_0^x y\,dy}{\int_0^{1/4}dy\cdot\int_{\frac{1}{2}-y}^1 8xy\,dx + \int_{1/4}^1 dy\cdot\int_y^1 8xy\,dx} = \frac{\int_{1/2}^1 4x^3dx}{8\int_0^{1/4} y\frac{1}{2}\left(1-\frac{1}{4}+y-y^2\right)dy + 8\int_{1/4}^1 y\frac{1}{2}(1-y^2)\,dy}$$

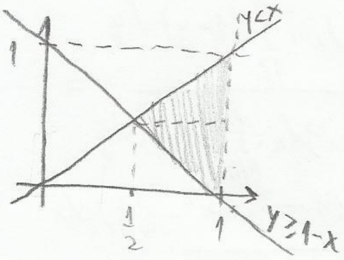


$$= \frac{\frac{15}{16}}{4\int_0^{1/4}\left(\frac{3}{4}y+y^2-y^3\right)dy + 4\int_{1/4}^1(y-y^3)\,dy} = \frac{\frac{15}{16}}{4\cdot\frac{3}{8}\cdot\frac{1}{16}+4\cdot\frac{1}{3}\cdot\frac{1}{64}-\cancel{\frac{1}{16\cdot16}}+2\cdot\frac{15}{16}-1+\cancel{\frac{1}{16\cdot16}}}$$

$$= \frac{18}{19} = 0{,}9473$$

15.) $f(x,y)=8xy(1-y^2)$, $x\in[0,1], y\in[0,1]$

$P(X+Y\geq 1 \mid Y<X)=?$



$$\frac{P(X+Y\geq1,\ Y<X)}{P(Y<X)} = \frac{\int_0^{1/2}(y-y^3)\,dy\cdot\int_{1-y}^1 x\,dx + \int_{1/2}^1 (y-y^3)\,dy \cdot \int_y^1 x\,dx}{\int_0^1 x\,dx \cdot \int_0^x (y-y^3)\,dy}$$

$$= \frac{\int_0^{1/2}(y-y^3)\cdot\frac{x^2}{2}\Big|_{1-y}^1 + \int_{1/2}^1 (y-y^3)\,dy\cdot \frac{x^2}{2}\Big|_y^1}{\int_0^1 x\,dx\cdot\left(\frac{y^2}{2}\Big|_0^x - \frac{y^4}{4}\Big|_0^x\right)} = \frac{\int_0^{1/2}(y-y^3)\frac{1}{2}(1-1+2y-y^2)+\int_{1/2}^1(y-y^3)\cdot\left(\frac12-\frac{y^2}{2}\right)}{\int_0^1 x\,dx\cdot\left(x^2-\frac{x^4}{4}\right)}$$

$$= \frac{\frac12\int_0^{1/2}(2y^2-y^3-2y^4+y^5)\,dy+\int_{1/2}^1\left(\frac12 y-\frac12 y^3-\frac12 y^3+\frac12 y^5\right)dy}{\int_0^1\left(x^3-\frac{x^5}{4}\right)dx} = \frac{\frac12\left(\frac23\cdot\frac18-\frac14\cdot\frac1{16}-\frac25\cdot\frac1{32}+\frac16\cdot\frac1{64}\right)+\frac14\left(1-\frac14\right)-\frac14\left(1-\frac1{16}\right)+\frac1{12}\left(1-\frac1{64}\right)}{\frac{3}{40}}$$

$=\underline{0,854}$ ✱ (u rješenjima ispada 0,768)

16.) $f(x,y)=8xy(1-x^2)$

$0\leq x,y\leq 1$

NEZAVISNOST=?

$f_X(x)=\int_0^1 8xy(1-x^2)\,dy = 8x\cdot(1-x^2)\cdot\int_0^1 y\,dy = 8x(1-x^2)\cdot\frac{y^2}{2}\Big|_0^1 = \underline{4x(1-x^2)}$

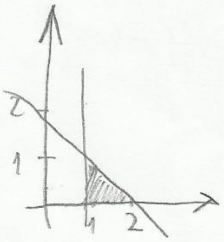
$f_Y(y)=\int_0^1 y(8x-8x^3)\,dx = y\int_0^1(8x-8x^3)\,dx = y(4-2)=\underline{2y}$

$f(x,y)=f(x)\cdot f(y) = \underline{8xy(1-x^2)}$ → NEZAVISNE!

17.) $X\sim E(\lambda)$ $f_X(x)=\lambda e^{-\lambda x}$

$Y\sim E(\lambda)$ $f_Y(y)=\lambda e^{-\lambda y}$ $\Big\}$ $f(x,y)=\lambda^2 e^{-\lambda(x+y)}$

$P(X+Y<2\mid X>1)=?$



$$\frac{P(X+Y<2,\ X>1)}{P(X>1)} = \frac{\lambda^2\int_1^2 e^{-\lambda x}dx\cdot\int_0^{2-x}e^{-\lambda y}dy}{\lambda^2\int_1^\infty e^{-\lambda x}dx\cdot\int_0^\infty e^{-\lambda y}dy} = \frac{\lambda^2\left[\int_1^2 e^{-\lambda x}\cdot\frac{-1}{\lambda}\left(e^{-2\lambda}e^{\lambda x}-1\right)dx\right]}{\cancel{\lambda^2}\frac{-1}{\cancel{\lambda}}(e^{-\infty}-e^{-\lambda})\frac{-1}{\cancel{\lambda}}(e^{-\lambda\infty}-e^{0})}$$

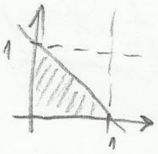
$$=\frac{\lambda^2\cdot\frac{-1}{\lambda}\left[\int_1^2(e^{-2\lambda}-e^{-\lambda x})\,dx\right]}{e^{-\lambda}} = \frac{-\lambda\left[e^{-2\lambda}+\frac1\lambda(e^{-2\lambda}-e^{-\lambda})\right]}{e^{-\lambda}} =$$

$$=-\lambda\left[e^{-\lambda}+\frac1\lambda(e^{-\lambda}-e^{0})\right] = -\lambda e^{-\lambda}-e^{-\lambda}+1 = \underline{1-e^{-\lambda}(1+\lambda)}$$

18.) $f(x,y) = Cxy$ $\quad y \le 1-x$

$0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1$

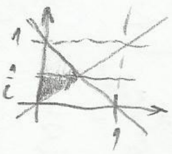
$C=?,\ f_X(x)=?,\ P(X>Y \mid Y<\frac{1}{2})=?$



$$C\int_0^1 x\,dx \int_0^{1-x} y\,dy = \frac{C}{2}\int_0^1 (x-2x^2+x^3)\,dx = \frac{C}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) \Rightarrow \boxed{C=24}$$

$$f(x,y)=24xy$$

$$f_X(x) = \int_0^{1-x} f(x,y)\,dy = 24x\int_0^{1-x} y\,dy = 24\frac{1}{2}x(1-2x+x^2) = 12x(1-x)^2,\ x\in[0,1]$$



$$\frac{P(X>Y,\ Y<\frac{1}{2})}{P(Y<\frac{1}{2})} = \frac{\int_0^{1/2} y\,dy \cdot \int_y^{1-y} x\,dx}{\int_0^{1/2} y\,dy \cdot \int_0^{1-y} x\,dx} = \frac{\int_0^{1/2} y\cdot\frac{1}{2}(1-y^2+\cancel{y^2}-\cancel{y^2})\,dy}{\int_0^{1/2} y\frac{1}{2}(1-2y+y^2)\,dy} = \frac{\frac{1}{2}\cdot\frac{1}{4}-\frac{2}{3}\cdot\frac{1}{8}}{\frac{1}{2}\cdot\frac{1}{4}-2\cdot\frac{1}{3}\cdot\frac{1}{8}+\frac{1}{4}\cdot\frac{1}{16}}$$

$$P = \frac{8}{11} = 0,7\dot{2}\dot{7}$$

19.) $f(x,y) = 2(x+y-3xy^2),\quad 0<x,y<1$

NEZAVISNOST=?

$$f_X(x) = 2\int_0^1 (x+y-3xy^2)\,dy = 2\left(x+\frac{1}{2}-x\right) = 1 \quad x\in[0,1]$$

$$f_Y(y) = 2\int_0^1 (x+y-3xy^2)\,dx = 2\cdot\left(\frac{1}{2}x^2+xy-\frac{3}{2}x^2y^2\right)\Big|_0^1 = 2\left(\frac{1}{2}+y-\frac{3}{2}y^2\right) = 1+2y-3y^2,\ y\in[0,1]$$

$$f(x)\cdot f(y) \neq f(x,y) \rightarrow \text{ZAVISNE!}$$

21.) $f(x,y) = xe^{-(1+y)x},\ x>0,\ y>0$

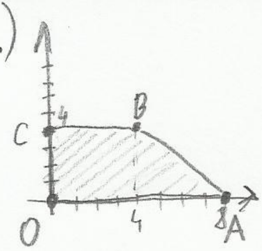
$$f_X(x) = \int_0^\infty xe^{-(1+y)x}\,dy = x\int_0^\infty e^{-x}\cdot e^{-xy}\,dy = xe^{-x}\cdot -\frac{1}{x}e^{-xy}\Big|_0^\infty = -e^{-x}(0-1) = e^{-x},\ x\in[0,\infty)$$

$$f_Y(y) = \int_0^\infty xe^{-(1+y)x}\,dx = \left|\begin{matrix} u=x & du=dx \\ dv=e^{-(1+y)x}dx & v=-\frac{1}{1+y}\cdot e^{-(1+y)x}\end{matrix}\right| = -\frac{x}{1+y}e^{-(1+y)x}\Big|_0^\infty + \frac{1}{1+y}\int_0^\infty e^{-x(1+y)}\,dx =$$

$$= \frac{1}{1+y}\cdot\underbrace{\left(-\frac{x}{e^{x(1+y)}}\Big|_0^\infty\right)}_{\emptyset} + \frac{1}{1+y}\cdot\frac{-1}{1+y}\cdot e^{-x(1+y)}\Big|_0^\infty = -\frac{1}{(1+y)^2}(0-1) = \frac{1}{(1+y)^2}$$

$$f(x)\cdot f(y) \neq f(X,Y) \rightarrow \text{ZAVISNE!}$$

22.)



O(0,0)
A(8,0)
B(4,4)
C(0,4)

$f(x), f(y) = ?$

$$f(x) = \frac{1}{P_{\triangle}} = \frac{1}{\frac{8+4}{2}\cdot 4} = \underline{\underline{\frac{1}{24}}}$$

$$f_X(x) = \frac{1}{24}\int dy \rightarrow (1)\ x \in [0,4] \Rightarrow \frac{1}{24}\int_0^4 dy = \underline{\underline{\frac{1}{6}}}$$

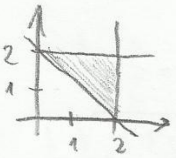
$$(2)\ x \in [4,8] \rightarrow \frac{1}{24}\int_4^{8-x} dy = \underline{\underline{\frac{1}{24}(8-x)}}$$

$$f_Y(y) = \frac{1}{24}\int_0^{8-y} dx = \underline{\underline{\frac{1}{24}(8-y)}}\ ,\ y \in [0,4]$$

25.) $x=2,\ y=2,\ x+y=2$, jednolika razdioba

$f_X(x) = ?,\ E(x) = ?$

$y = 2-x$



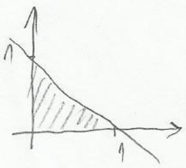
$$f(x,y) = \frac{1}{P_{\triangle}} = \frac{1}{\frac{2\cdot 2}{2}} = \underline{\underline{\frac{1}{2}}}$$

$$f_X(x) = \frac{1}{2}\int_{2-x}^{2} dy = \frac{1}{2}(2-2+x) = \underline{\underline{\frac{x}{2}}}\ ,\ x\in[0,2]$$

$$E(x) = \int_0^2 x\cdot f_X(x)\,dx = \int_0^2 \frac{1}{2}x^2\,dx = \frac{1}{6}x^3\Big|_0^2 = \underline{\underline{\frac{4}{3}}}$$

26.) $x>0,\ y>0,\ x+y<1$, jednolika razdioba

$f(x), f(y)$, NEZAVISNOST $= ?$



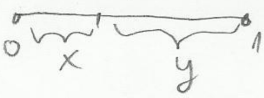
$$f(x,y) = \frac{1}{P_{\triangle}} = \frac{1}{\frac{1\cdot 1}{2}} = \underline{\underline{2}}$$

$$f_X(x) = 2\cdot\int_0^{1-x} dy = 2(1-x)\ ,\ x\in[0,1]$$

$$f_Y(y) = 2\cdot\int_0^{1-y} dx = 2(1-y)\ ,\ y\in[0,1]$$

$$f(x)\cdot f(y) \neq f(x,y) \rightarrow \text{ZAVISNE!}$$

27.) $[0,1] \longrightarrow f(x,y) = 1$



$y > x$

$$x \in \left[0, \tfrac{1}{2}\right] \rightarrow f_X(x) = \frac{1}{\frac{1}{2}-0} = \underline{\underline{2}}$$

$$y \in \left[\tfrac{1}{2}, 1\right] \rightarrow f_Y(y) = \frac{1}{1-\frac{1}{2}} = \underline{\underline{2}}$$

$$f_X(x)\cdot f_Y(y) = 4 \neq f(x,y)$$

$$E(X) = \int_0^{1/2} x f(x)\,dx = x^2\Big|_0^{1/2} = \underline{\underline{\frac{1}{4}}}$$

$$E(Y) = \int_{1/2}^{1} y f(y)\,dy = 1 - \frac{1}{4} = \underline{\underline{\frac{3}{4}}}$$

$$\boxed{E(X^2) = \int_0^{1/2} x^2 f(x)\,dx = \frac{2x^3}{3}\Big|_0^{1/2} = \underline{\underline{\frac{1}{12}}}}$$

$E(XY) = ?$ → pošto su X i Y zavisne jedna varijabla nam je X, a druga $Y = 1-X$, tj. $1-X$:

$$E(XY) = E[X\cdot(1-x)] = E(X - X^2) = E(X) - E(X^2) = \frac{1}{4} - \frac{1}{12} = \frac{2}{12} = \underline{\underline{\frac{1}{6}}}$$

28.) $f(x,y) = A\sin(x+y)\ ,\ 0<x<\frac{\pi}{2},\ 0<y<\frac{\pi}{2}$

$A=?\ ,\ f_X(x), f_Y(y)=?\ ,\ P(X<\frac{\pi}{4}, Y<\frac{\pi}{4})$

$$A\cdot\int_0^{\pi/2} dx \int_0^{\pi/2}\sin(x+y)\,dy = A\int_0^{\pi/2} dx\left(-\cos(x+y)\Big|_0^{\pi/2}\right) = -A\int_0^{\pi/2}\left(\cos(x+\tfrac{\pi}{2})-\cos x\right)dx$$

$$= -A\left(\sin(x+\tfrac{\pi}{2})-\sin x\right)\Big|_0^{\pi/2} = -A(-1-1) \Rightarrow 2A=1 \Rightarrow \boxed{A=\tfrac{1}{2}}$$

$$f_X(x) = \int_0^{\pi/2}\tfrac{1}{2}\sin(x+y)\,dy = -\tfrac{1}{2}\cos(x+y)\Big|_0^{\frac{\pi}{2}} = -\tfrac{1}{2}\left(\cos(x+\tfrac{\pi}{2})-\cos x\right) = \tfrac{1}{2}(\sin x+\cos x)$$

$$F_X(x) = \int_0^x f_X(x)\,dx = \tfrac{1}{2}(-\cos x+\sin x)\Big|_0^x = \tfrac{1}{2}(-\cos x+\sin x+1)$$

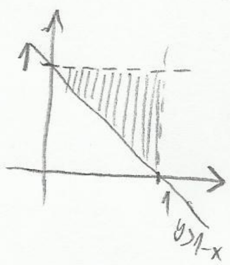
$$f_Y(y) = \int_0^{\pi/2}\tfrac{1}{2}\sin(x+y)\,dx = -\tfrac{1}{2}\cos(x+y)\Big|_0^{\pi/2} = -\tfrac{1}{2}(\cos y+\sin y)$$

$$F_Y(y) = \tfrac{1}{2}(1+\sin y-\cos y)$$

$$P(X<\tfrac{\pi}{4}, Y<\tfrac{\pi}{4}) = \tfrac{1}{2}\int_0^{\pi/4}dx\int_0^{\pi/4}\sin(x+y)\,dy = -\tfrac{1}{2}\int_0^{\pi/4}\left(\cos(x+\tfrac{\pi}{4})-\cos x\right)dx =$$

$$= -\tfrac{1}{2}\left[\sin\tfrac{\pi}{2}-\sin\tfrac{\pi}{4}-\sin\tfrac{\pi}{4}+\sin 0\right] = -\tfrac{1}{2}\left[1-\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}+0\right] = 0{,}2071$$

29.) $f(x,y) = C(x+y)\ ,\ x<1,\ y<1,\ x+y>1$



$f_X(x), E(X)=?$

$$C\cdot\int_0^1 dx\int_{1-x}^1 (x+y)\,dy = C\int_0^1\left(xy+\tfrac{y^2}{2}\right)\Big|_{1-x}^1 dx = C\int_0^1\left(\cancel{x}+\tfrac{1}{2}-\cancel{x}+x^2-\tfrac{1}{2}(1-2x+x^2)\right)dx$$

$$C\int_0^1\left(x^2+x-\tfrac{1}{2}x^2\right)dx = C\cdot\left(\tfrac{x^3}{3}+\tfrac{x^2}{2}-\tfrac{1}{2}\cdot\tfrac{x^3}{3}\right)\Big|_0^1 \Rightarrow C\cdot\left(\tfrac{1}{6}+\tfrac{1}{2}\right)=1$$

$$\Rightarrow C\cdot\tfrac{4}{6}=1 \Rightarrow \boxed{C=\tfrac{6}{4}=\tfrac{2}{3}}$$

$$f_X(x) = \int_{1-x}^1 f(x,y)\,dy = \int_{1-x}^1\tfrac{3}{2}(x+y)\,dy = \tfrac{3}{2}\left(xy+\tfrac{1}{2}y^2\right)\Big|_{1-x}^1 = \tfrac{3}{2}\left(x+\tfrac{1}{2}-x(1-x)-\tfrac{1}{2}(1-2x+x^2)\right)$$

$$= \tfrac{3}{2}\left(\cancel{\tfrac{1}{2}}+x^2-\cancel{\tfrac{1}{2}}+x-\tfrac{1}{2}x^2\right) = \tfrac{3}{4}x^2+\tfrac{3}{2}x\ ,\ x\in[0,1]$$

$$E(X) = \int_0^1 x\cdot f_X(x)\,dx = \int_0^1\left(\tfrac{3}{4}x^3+\tfrac{3}{2}x^2\right)dx = \tfrac{3}{4}\cdot\tfrac{x^4}{4}+\tfrac{3}{2}\cdot\tfrac{x^3}{3}\Big|_0^1 = \tfrac{3}{16}+\tfrac{1}{2} = \tfrac{11}{16} = 0.6875$$

SLUŽBENA RJEŠENJA:

§ 7. Slučajni vektori

1. Ne.

2. F(x,y)=

$$\begin{cases} 0 & , x \leqslant 0 \text{ ili } y \leqslant 0, \\ xy & , 0 < x \leqslant 1, 0 < y \leqslant 1, \\ x & , 0 \leqslant x < 1, 1 < y, \\ y & , 1 < x, 0 \leqslant y < 1, \\ 1 & , 1 < x, 1 < y. \end{cases}$$

3. $f(x,y) = \dfrac{4xy}{R^2\pi\sqrt{x^2-y^2}}$, $0 \leqslant y \leqslant x \leqslant R$.

4. a) $\dfrac{1}{\pi\sqrt{1-x^2}}$; b) $\dfrac{1}{2}$, $|x| \leqslant 1$.

5. $\dfrac{3}{4}$.

6. 0.245.

7. 0.190.

8. $f(x,y) = \dfrac{2y}{\sqrt{\pi}} e^{-x^2}$, $0 \leqslant y \leqslant 1$;
$p = 0.0198$; $E(XY) = 0$.

9. $k = 6$; $\dfrac{3}{5e^2}$.

10. X i Y su nezavisne, $P(A) = \frac{1}{3}$, $P(B) = 0$.

11. $F(x,y) = 1 - e^{-x}$, ako je $x^2 < y$;
$1 - e^{-\sqrt{y}}$, ako je $y < x^2$.

12. 0.275.

13. $\dfrac{11}{40}$

14. $f_X(x) = 4x^3$, $x \in [0,1]$;
$f_Y(y) = 4y - 4y^3$, $y \in [0,1]$;
X i Y su zavisne; $\dfrac{18}{19}$

15. $P(A \mid B) = \dfrac{123}{160}$.

17. $1 - (\lambda + 1)e^{-\lambda}$.

18. $C = 24$; $f_X(x) = 12x(1-x)^2$, $x \in [0,1]$;
$p = \dfrac{8}{11}$.

19. $f_X(x) = 1$, $0 < x < 1$,
$f_Y(y) = 1 + 2y - 3y^2$, $0 < y < 1$.

20. $f_X(x) = f_Y(x) = e^{-x}$, $x > 0$.

21. $f_X(x) = e^{-x}$, $x > 0$;
$f_Y(y) = \dfrac{1}{(1+y)^2}$, $y > 0$.

22. $f_X(x) = \begin{cases} \dfrac{1}{6}, & x \in [0,4], \\ \dfrac{8-x}{24}, & x \in [4,8], \end{cases}$
$f_Y(y) = \dfrac{8-y}{24}$, $y \in [0,4]$.

23. $f_X(x) = f_Y(x) = \dfrac{2}{\pi R^2}\sqrt{R^2 - x^2}$,
$|x| < R$; $f_Z(z) = \dfrac{1}{2H}$, $-H < z < H$.
Zavisne su.

25. $F_X(x) = \dfrac{x^2}{4}$, $0 < x < 2$; $E(X) = \dfrac{4}{3}$.

26. $f_X(x) = \frac{1}{2}(1-x) = f_Y(x)$.

27. $f_X(x) = 2$, $0 \leqslant x \leqslant \frac{1}{2}$; $E(X) = \frac{1}{4}$;
$f_Y(y) = 2$, $\frac{1}{2} \leqslant y \leqslant 1$; $E(Y) = \dfrac{3}{4}$;
X i Y su zavisne, $E(XY) = \dfrac{1}{6}$.

28. $A = 0.5$; $F_X(x) = \frac{1}{2}(1 + \sin x - \cos x)$;
$F_Y(y) = \frac{1}{2}(1 + \sin y - \cos y)$; 0.2071.

29. $f_X(x) = \frac{3}{4}x^2 + \frac{3}{2}x$, $x \in [0,1]$; $E(X) = \dfrac{11}{16}$.

30. $F(x,y) = F_X(x) \cdot F_Y(y)$, $F_X(x) = 1 - e^{-\alpha x}$,
$F_Y(y) = 1 - e^{-\beta y}$; $E(X) = \dfrac{1}{\alpha}$, $E(Y) = \dfrac{1}{\beta}$,
$D(X) = \dfrac{1}{\alpha^2}$, $D(Y) = \dfrac{1}{\beta^2}$. Nezavisne su.

31. $-\dfrac{1}{2}$.

32. $\operatorname{cov}(Y_1, Y_2) = 0$. Zavisne su.

33. $E(Y) = 3$, $D(Y) = 4.246$.

34. 0, $\dfrac{\sqrt{21}}{5}$.

35. $D(X) = \dfrac{1}{2}$, $D(Y) = \dfrac{1}{4n-1}$,
$r_{xy} = \dfrac{\sqrt{3(4n-1)}}{2n+1}$.

37. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

38. $\cos x \cos y$, $\begin{pmatrix} \pi-3 & 0 \\ 0 & \pi-3 \end{pmatrix}$.

39. $\dfrac{3\sqrt{5}}{7}$.

40. 0.9597.

41. $\dfrac{407}{9}$.

62. $E(X) = \dfrac{1}{3}$.

63. $\mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

64. $\mathcal{N}(0, 2\sigma^2(1+r))$.

65. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

66. $\dfrac{1}{2\pi\sqrt{6\pi}} \exp(-\frac{1}{6}(5x^2 + 2y^2 + 2z^2 - 2xy + 4xz -$
$2yz - 12x + 6y - 6z + 5))$.

67. $\mathcal{N}(\frac{1}{2}, 1)$.

68. 0.44 , 0.642 .

70. $\dfrac{2}{\pi\sqrt{3}[\frac{1}{3}(2z-1)^2 + 1]}$.

71. $\sqrt{\dfrac{2}{5\pi}} \exp(-\frac{2}{5}x^2)$, $\sqrt{\dfrac{2}{\pi}} \exp(-2y^2)$,
$\dfrac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)$,
$\dfrac{1}{2\pi} \exp(-\frac{1}{2}(u^2 - 2uv + 2v^2))$,

72. $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

74. $f_Z(z) = \dfrac{1}{\pi\sqrt{1-r^2}} \displaystyle\int_{-\infty}^{z} \exp(-\frac{u^2 - 2ruz + z^2}{2(1-r^2)})du$,
$E(Z) = \sqrt{(1-r^2)/\pi}$

LITERATURA:

[1] Neven Elezović: Slučajne varijable, *Element 2010.godine*