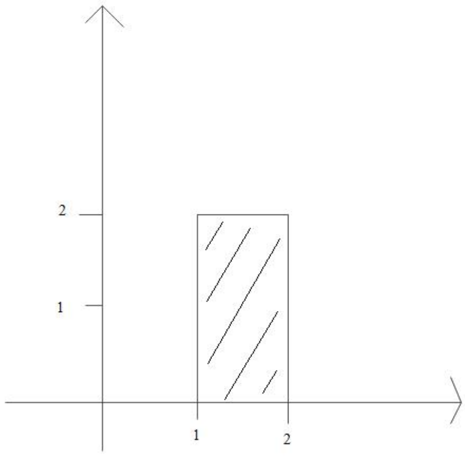
SEDMI PUT ☺

UVOD U DVOSTRUKE INTEGRALE $\iint_{\Omega} f(x,y)dxdy$ **; Ω –**područje integracije

Primjer 1)

Ω: { 1 ≤ x ≤ 2

0 ≤ y ≤ 2 } $\iint_{\Omega} (xy+x^2)dxdy = ?$



1.korak: SKICIRANJE Ω

2.korak: razdjeljivanje integrala

3.korak: izračun

RAZDJELJIVANJE INTEGRALA:

$$\int_1^2 dx \int_0^2 (xy+x^2)dy = \int_0^2 dy \int_1^2 (xy+x^2)dx$$

<u>Granice kod prvog integrala su konstante, a kod drugog funkcije od onoga po čemu ne integriramo u tom integralu</u> ☺ (u ovom slučaju su to ponovno konstante)

IZRAČUN:

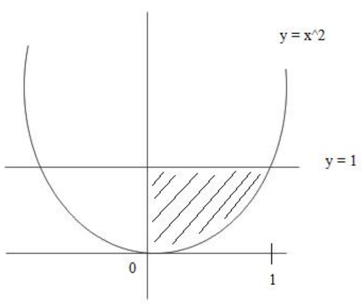
$$\int_1^2 dx \int_0^2 (xy+x^2)dy = \int_1^2 \left(\frac{xy^2}{2} + x^2y\right)\Big|_0^2 dx = \int_1^2 (2x+2x^2)dx = x^2 + \frac{2x^3}{3}\Big|_1^2 = \frac{23}{3}$$

$$(granice\ 2\ i\ 0\ uvrštavamo\ u\ \boldsymbol{IPSILON}!)$$

$$\int_0^2 dy \int_1^2 (xy+x^2)dx = \int_0^2 \left(\frac{x^2y}{2} + \frac{x^3}{3}\right)\Bigg|_1^2 dy = \int_0^2 \left(2y + \frac{8}{3} - \frac{y}{2} - \frac{1}{3}\right)dy = \int_0^2 \left(\frac{3}{2}y + \frac{7}{3}\right)dy =$$

$$= \left(\frac{3}{4}y^2 + \frac{7}{3}y\right)\Bigg|_0^2 = \frac{23}{3}$$

Primjer 2)



Površina područja Ω:

$\mathbf{P} = \iint_{\Omega} \mathbf{dxdy}$ - bez funkcije!!

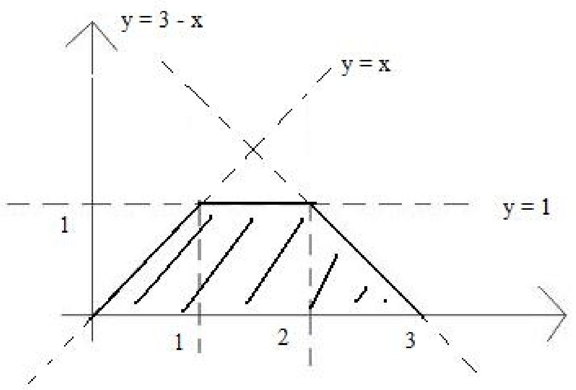
$$P(\Omega) = \iint_{\Omega} dxdy = \int_0^1 dx \int_{x^2}^1 dy = \int_0^1 (1 - x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3}$$

(kako bismo otkrili granice ovog integrala, sliku gledamo odozdola prema gore i opet su kod prvog granice konstante, a kod drugog funkcije)

$$P(\Omega) = \iint_{\Omega} dxdy = \int_0^1 dy \int_0^{\sqrt{y}} dx = \int_0^1 \sqrt{y} dy = \frac{2}{3}$$

(gledamo slijeva na desno ili okrenemo sliku ☺)

Primjer 3)



Ako gledamo preko ipsilona, puno nam je brze za izracunati ☺

$$\iint_{\Omega} f(x,y)dxdy = \int_0^1 dy \int_y^{3-y} dx$$

f(x,y) nam je zadano inace.....

x ide od 0 do 3, ali onda y ima 3 intervala:

$$\iint_{\Omega} f(x,y)dxdy = = \int_0^1 dx \int_0^x f(x,y)dy + \int_1^2 dx \int_0^1 f(x,y)dy + \int_2^3 dx \int_0^{3-x} f(x,y)dy$$

SLUČAJNI VEKTORI (X, Y)

- funkcija razdiobe: $F(x,y) = P(X < x, Y < y)$
- funkcija gustoće: $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$ → parcijalno deriviranje ☺

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1}$$

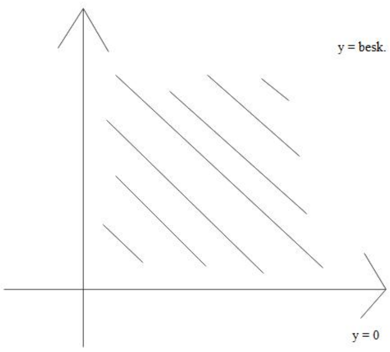
Ovo više nije površina ispod krivulje, osim ako je f(x,y) konstanta ☺

Marginalne gustoće: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$ $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$

Uvjet za nezavisnost: $f(x,y) = f_X(x) \cdot f_Y(y)$

Vjerojatnost: $P\big((X,Y) \in G\big) = \iint_G f(x,y)dxdy \quad \neq površina$

Zadatak 5. (DZ5) – tipičan zadatak za MI (4b)



$f(x,y) = C \cdot e^{-2x-y}, x > 0, y > 0$

Funkcije intervala:

Gornja je y = ∞ , a donja je y = 0.

a) C = ?

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$

$$\int_0^{\infty} dx \int_0^{\infty} C \cdot e^{-2x-y}dy = C\int_0^{\infty} e^{-2x} \int_0^{\infty} e^{-y}dy = C\int_0^{\infty} e^{-2x} \cdot 1dy = C \cdot \frac{e^{-2x}}{-2}\bigg|_0^{\infty} = \frac{C}{2} \rightarrow$$

$\frac{C}{2} = 1 \ \rightarrow \quad C = 2$

b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} 2e^{-2x-y}dy = 2e^{-2x}\int_0^{\infty} e^{-y}dy = 2e^{-2x} \cdot \frac{e^{-y}}{-1}\Big|_0^{\infty} = 2e^{-2x},$

$x > 0$ → nikako zaboraviti interval ☺

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} 2e^{-2x-y}dx = 2e^{-y}\int_0^{\infty} e^{-2x}dx = 2e^{-y} \cdot \frac{e^{-2x}}{-2}\bigg|_0^{\infty} = e^{-y},$$

$y > 0$ → ne zaboraviti!!!

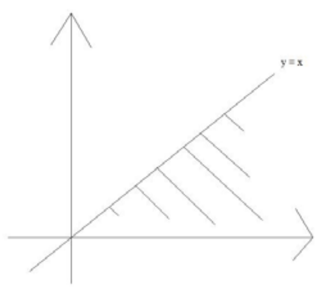
($granice\ ovih\ integrala\ su\ FUNKCIJE$!!)

$f(x,y) = f_X(x) \cdot f_Y(y) \rightarrow \quad 2e^{-2x-y} = 2e^{-2x} \cdot e^{-y} \quad \rightarrow varijable\ su\ nezavisne$!!

c) Vjerojatnost! $P(\{X > Y\}) =?$

$$P(\{X > Y\}) = \iint_G 2e^{-2x-y}dxdy$$

Najprije crtamo da vidimo koji nam je G:



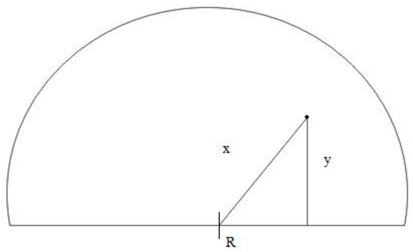
$$P(\{X > Y\}) = \int_0^{\infty} dx \int_0^{x} 2e^{-2x-y}dy =$$

$$= 2\int_0^{\infty} \left(e^{-2x} \cdot \frac{e^{-y}}{-1}\right)\bigg|_0^{x} dy =$$

$$= 2\int_0^{\infty} -e^{-2x} \cdot (e^{-x} - 1)dx =$$

$$= -2\int_0^{\infty} e^{-3x}dx + 2\int_0^{\infty} e^{-2x}dx = -2 \cdot \frac{e^{-3x}}{-3}\bigg|_0^{\infty} + 2 \cdot \frac{e^{-2x}}{-2}\bigg|_0^{\infty} = \frac{1}{3}$$

Zadatak 2. (5DZ)



$X = udaljenost\ točke\ od\ središta$

$Y = udaljenost\ točke\ od\ promjera$

$(X, Y) = ?$

Najprije gledamo vrijednosti koje x i y mogu poprimiti:

$x \in (0, R), \quad y \in (0, R)$ $\qquad F(x,y) = P(X < x, Y < y)$

Imamo pet slučajeva: 1.slučaj x<y, y<0: F(x,y)=0

2.slučaj x>R, y>R: F(x,y)=1

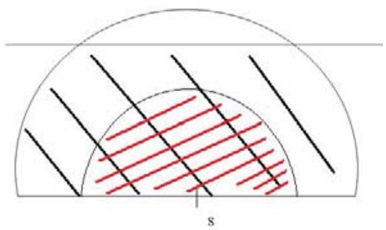
3.slučaj x<y

4.slučaj x>y

5.slučaj x=y - to je granični slučaj i ne zanima nas

3.slučaj: x<y

Recimo da je radijus R = 1, a y = 0.8, očito x mora biti manji od 0.8.



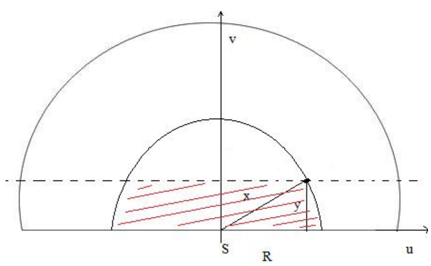
$F(x,y) = P(X < x, Y < y) =$ geom. vjerojatnost =

$$= \frac{\frac{1}{2}x^2\pi}{\frac{1}{2}R^2\pi} = \frac{x^2}{R^2} \qquad \frac{\partial^2 F(x,y)}{\partial x \partial y} = 0$$

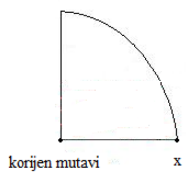
4.slučaj: x>y

$$F(x,y) = P(X < x, Y < y) =$$

$$= \frac{1}{\frac{1}{2}R^2\pi}\left[2y\sqrt{x^2 - y^2} + 2\int_{\sqrt{x^2-y^2}}^{x} \sqrt{x^2 - u^2}\,du\right]$$



Prvi dio jednakosti je površina pravokutnika a drugi površine ispod krivulje oko pravokutnika ☺



Gornja fija je kružnica: $u^2 + v^2 = x^2$

$\rightarrow v = \sqrt{x^2 - u^2}$

Donja je nula.

Ovaj integral rješavamo susptitucijom $u = x \cdot \sin t$ ☺☺☺☺☺☺☺☺