PZI - 2008. - 10)

$Z = 1 - \min\{x_1, \dots, x_n\}$ procjena duljine intervala $[\alpha, 1]$ $\rightarrow$ stvarna vrijednost duljine intervala: $1-\alpha$

- uvjet za nepristranost: $E(Z) = 1 - \alpha$

- neka je $X_m = \min\{x_1, \dots, x_n\}$, $x_1, \dots, x_n \in [\alpha, 1]$

$\Rightarrow$ X_i ima jednoliku razdiobu $\Rightarrow$ $F_{X_i} = \dfrac{x-\alpha}{1-\alpha}$, $x \in (\alpha, 1)$

$$F_{X_m}(x) = P(X_m < x) = P(\min\{X_1, \dots, X_n\} < x) =$$
$$= 1 - P(\min\{X_1, \dots, X_n\} > x) = 1 - P(X_1 > x, \dots, X_n > x)$$
$$\overset{nez.}{=} 1 - P(X_i > x)^n = 1 - (1 - F_{X_i}(x))^n$$

$$F_{X_m}(x) = 1 - \left(\frac{1-x}{1-\alpha}\right)^n \Rightarrow f_{X_m}(x) = n\left(\frac{1-x}{1-\alpha}\right)^{n-1} \cdot \frac{1}{1-\alpha}$$

$$E(Z) = 1 - E(\min\{x_1, \dots, x_n\}) = 1 - E(X_m)$$
$$= 1 - \int_{-\infty}^{+\infty} x \cdot f_{X_m}(x)\,dx = 1 - \frac{n}{(1-\alpha)^n} \cdot \int_{\alpha}^{1} x(1-x)^{n-1}\,dx$$
$$= \left| \begin{array}{l} 1-x = t \\ -dx = dt \end{array} \right| = 1 + \frac{n}{(1-\alpha)^n} \cdot \int_{1-\alpha}^{0} (1-t)t^{n-1}\,dt$$
$$= 1 + \frac{n}{(1-\alpha)^n} \cdot \int_{1-\alpha}^{0} (t^{n-1} - t^n)\,dt = 1 + \frac{n}{(1-\alpha)^n} \cdot \left[\frac{t^n}{n} - \frac{t^{n+1}}{n+1}\right]\Bigg|_{1-\alpha}^{0}$$
$$= 1 + \frac{\cancel{n}}{\cancel{(1-\alpha)^n}} \cdot \frac{\cancel{(1-\alpha)^n}}{\cancel{n}(n+1)} \left(n(1-\alpha) - (n+1)\right)$$

$\Rightarrow$ $E(Z) = \dfrac{n}{n+1}(1-\alpha)$ $\Rightarrow$ procjena nije nepristrana!

$\Rightarrow$ Da bi bila nepristrana moramo pomnožiti s $\underline{\underline{\dfrac{n+1}{n}}}$

6. DZ - 10)

$$X \sim P(\lambda) \quad , \quad P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$P(X=5) = \frac{\lambda^5}{5!} e^{-\lambda} \quad , \quad P(X=7) = \frac{\lambda^7}{7!} e^{-\lambda} \quad , \quad P(X=3) = \frac{\lambda^3}{3!} e^{-\lambda}$$

$$L(\lambda, x_1, x_2, x_3) = P(X=5) \cdot P(X=7) \cdot P(X=3) = \frac{\lambda^{15}}{3!\,5!\,7!} e^{-3\lambda}$$

$$\ln L = \ln \lambda^{15} + \ln e^{-3\lambda} - \ln 3!\,5!\,7! = 15 \ln \lambda - 3\lambda - \ln 3!\,5!\,7!$$

$$\frac{d}{d\lambda} \ln L = \frac{15}{\lambda} - 3 = 0 \quad \Rightarrow \quad \underline{\underline{\lambda = 5}}$$

PZ1 - 2007. - 3)

$$f(x) = \frac{x}{a^2} e^{-\frac{x}{a}} \quad , \; x > 0$$

$$L(a, x_1, x_2, \dots, x_n) = f(x_1) \cdot \ldots \cdot f(x_n) = \frac{x_1}{a^2} e^{-\frac{x_1}{a}} \cdot \ldots \cdot \frac{x_n}{a^2} e^{-\frac{x_n}{a}}$$

$$= \frac{x_1 \cdot \ldots \cdot x_n}{a^{2n}} e^{-\frac{1}{a}(x_1 + \ldots + x_n)}$$

$$\ln L = \ln (x_1 \cdot \ldots \cdot x_n) - \frac{1}{a}(x_1 + \ldots + x_n) - 2n \ln a \quad \Big/ \frac{\partial}{\partial a}$$

$$\frac{x_1 + \ldots + x_n}{a^2} - \frac{2n}{a} = 0 \quad \Big/ \cdot a$$

$$\Rightarrow \quad a = \frac{x_1 + \ldots + x_n}{2n} = \frac{\overline{x}}{2} \quad \Rightarrow \quad \underline{\underline{a = \frac{\overline{X}}{2}}}$$

- provjera nepristranosti :

$$E\left(\frac{\overline{X}}{2}\right) = \frac{1}{2} E\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) = \frac{1}{2} \cdot \frac{1}{n} \cdot n \cdot E(x_i) = \frac{1}{2} E(x_i)$$

$$= \frac{1}{2} \cdot \int_{-\infty}^{+\infty} x \cdot f(x)\, dx = \frac{1}{2a^2} \int_0^{\infty} x^2 e^{-\frac{x}{a}} dx = \left| \begin{array}{l} \text{dvije parcijalne} \\ \text{integracije :)} \end{array} \right|$$

$$= \frac{1}{2} \cdot 2a = a \quad \Rightarrow$$ procjena je nepristrana!

6.DZ - 17)

x_j	110	115	120	125	130	135
n_j	2	3	6	5	2	2

$\Rightarrow \quad n = \sum n_j = 20$

- točkaste procjene za sredinu i disperziju (uz nepoznato očekivanje)

$$\bar{x} = \frac{1}{n}\sum x_j \cdot n_j = \underline{\underline{122}} \qquad \hat{s}^2 = \frac{1}{n-1}\sum n_j (x_j - \bar{x})^2 = \underline{\underline{51.05}}$$

- 90% interval za očekivanje (uz nepoznatu disperziju) - 34. str.

$p = 0.9 \Rightarrow \alpha = 1 - p = 0.1$

$t_{1-\frac{\alpha}{2}} = 1.729$ uz 19 stupnjeva slobode (tablica studentove razdiobe)

$t_{1-\frac{\alpha}{2}} \cdot \frac{\hat{s}}{\sqrt{n}} = 2.762$

$$\Rightarrow P\left(\bar{x} - t_{1-\frac{\alpha}{2}}\frac{\hat{s}}{\sqrt{n}} < a < \bar{x} + t_{1-\frac{\alpha}{2}} \cdot \frac{\hat{s}}{\sqrt{n}}\right) = 0.9$$

$$\underline{\underline{P(119.238 < a < 124.762) = 0.9}}$$

- 90% interval za disperziju (uz nepoznato očekivanje) - 35. str.

$p = 0.9 \Rightarrow \alpha = 1 - p = 0.1$

- uz 19 stupnjeva slobode (tablica hi-kvadrat razdiobe):

$$c_1 = \chi^2_{\alpha/2} = \chi^2_{0.05} = 10.117 \,, \quad c_2 = \chi^2_{1-\frac{\alpha}{2}} = \chi^2_{0.95} = 30.144$$

$$\beta_1 = \frac{(n-1)\hat{s}^2}{c_2} = 32.177 \,, \quad \beta_2 = \frac{(n-1)\hat{s}^2}{c_1} = 95.873$$

$$\Rightarrow \qquad P(\beta_1 < \sigma^2 < \beta_2) = 0.9$$

$$\underline{\underline{P(32.177 < \sigma^2 < 95.873) = 0.9}}$$