VJEROJATNOST

slučajni pokus – svaki pokus kojem unaprijed ne znamo ishod

elementarni događaj – ishod pokusa $\omega_1, \omega_2, \omega_3$

Ω skup svih elementarnih događaja $\Omega = \{\omega_1, \omega_2 \omega_3\}$

$\emptyset$ prazan skup

događaj – podskup od Ω / A,B,C

UNIJA $A \cup B$ $A+B$ **PRESJEK** $A \cap B$ AB **RAZLIKA** $\setminus$

De Morganov $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (1) *dokaz $\omega \in \overline{A \cup B} \Leftrightarrow \omega \notin A \cup B \Rightarrow \omega \notin A, \omega \notin B$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ (2) $\overline{\overline{A} \cup \overline{B}} = \overline{\overline{A}} \cap \overline{\overline{B}} = A \cap B$ $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$

ALGEBRA DOGAĐAJA je svaka familija $\mathcal{F}$ podskupova od Ω na kojoj su definirani binarni operatori zbrajanja $+$ $F \times F \to F$ i unarna operacija komplementiranja sa svojstvima

1) $\Omega \in F$, $\emptyset \in F$
2) $A \in F \to \overline{A} \in F$
3) $A, B \in F \to A+B \in F$

elementi algebre F zovemo događaji

$A \cdot B = \overline{\overline{A} + \overline{B}} \in F$ $A \setminus B = A \cdot \overline{B} \in F$

VJEROJATNOST je preslikavanje $P: F \to [0,1]$ definirano na algebri događaja F, koje ima svojstva

1) $P(\Omega) = 1$ $P(\emptyset) = 0$ **normiranost**
2) ako je $A \subset B$, vrijedi $P(A) \leq P(B)$ **monotonost**
3) ako su A, B disjunktivni događaji $P(A \cup B) = P(A) + P(B)$ **aditivnost**

Broj **P(A)** nazivamo vjerojatnost događaja A

$P(\overline{A}) = 1 - P(A)$ $P(A \cup B) = P(A) + P(B) - P(AB)$ $\to P(\Omega) = P(A \cup \overline{A}) = P(A) + P(\overline{A}) = 1$

* $P(A \cup B) = P(A) + P(B\overline{A})$ $P(A\overline{B}) = P(A) - P(AB)$ * $A \cup B = A \cup B\overline{A}$

* $P(B) = P(AB) + P(B\overline{A})$ $P(\overline{A}B) = P(B) - P(AB)$ $B = AB \cup B\overline{A}$

KONAČNI VJEROJATNOSNI PROSTOR vjerojatnosni prostor Ω koji posjeduje konačno mnogo elementarnih događaja

$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ $P(\omega_1) = p_1$ $P(\omega_2) = p_2$ $P(\omega_n) = p_n$ $= \boxed{\sum_{i=1}^{n} p_i = 1}$

Svi elementarni događaji su jednako vjerojatni $p_1 = p_2 = p_i = p_n = p$

$\sum p_i = np = 1$ $p = \frac{1}{n}$ događaj $A\{\omega_1, \dots \omega_m\} = P(A) = mp = m \cdot \frac{1}{n} = \frac{\text{br. povoljnih}}{\text{ukupni ishodi}}$

BESKONAČNI VJEROJATNOSNI PROSTOR

Ω - prostor = beskonačan : prebrojiv i neprebrojiv

Algebra događaja $\mathcal{F}$ je σ "sigma" algebri ako vrijedi:

$A_1, A_2, A_3 \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Tada vjerojatnost P mora zadovoljiti uvjete: σ - aditivnosti : $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$ za disjunktivne A_1, A_2

Svojstvo neprekinutosti : $A = \bigcup_{n=1}^{\infty} A_n \quad \lim_{n \to \infty} P(A_n) = P(A)$

GEOMETRIJSKA VJEROJATNOST

definiramo kao $P(A) = \frac{m(A)}{m(\Omega)}$ mjera događaja i skupa svih

UVJETNA VJEROJATNOST

Neka je B događaj takav da je njegova vjerojatnost strogo veća od nule, $P(B) > 0$. Uvjetna vjerojatnost događaja A uz uvjet događaja B je $\mathcal{F} \to [0,1]$ definirana $P(A|B) = \frac{P(AB)}{P(B)}$

$P(AB) = P(B)\,P(A|B) = P(A)\,P(B|A)$

NEZAVISNOST DOGAĐAJA

za A i B kažemo da su nezavisni ako je $P(A) = P(A|B)$ ili $P(B) = P(B$

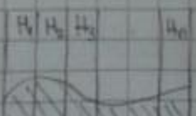
$P(A|B) = \frac{P(AB)}{P(B)} = P(A) \qquad P(AB) = P(A)P(B)$ nužan i dovoljan uvjet

događaj $A_1, \ldots, A_n$ su nezavisni ako za svaki izbor $A_{i1}, \ldots, A_{ik}$ vrijedi

$P(A_{i1} \ldots A_{ik}) = P(A_{i1}) \ldots P(A_{ik})$

POTPUNA VJEROJATNOST i BAYESOVA FORMULA

$P(H_1) = P(A|H_1) + P(A|H_2) \cdot P(H_2) \qquad P(H_1|A) = P(H_1) \cdot P$



$\Omega = H_1 \cup H_2 \cup H_3 \ldots \cup H_n$ } particije skupa

$H_1 \cap H_2 = \emptyset$

$A = AH_1 \cup AH_2 \cup \ldots AH_n$

$P(A) = P(AH_1) + \ldots + P(H_nA)$

$P(A) = P(H_1)P(A|H_1) + \ldots + P(H_n)P(A|H_n)$

DISKRETNE SLUČAJNE VARIJABLE

- neka je $S=\{x_1, x_2, \ldots\}$ konačan ili prebrojiv skup te neka variabla x svakom elementarnom događaju pridruženje neku vjerojatnosti iz S
- DISKRETNA SLUČAJNA VARIJABLA je preslikavanje $x: \Omega \to S$ ako za $\forall x_k \in S$ je podskup $A_k = \{\omega \in \Omega : x(\omega) = x_k\}$ događaj. $p_k = P(x=x_k)$, $p_k > 0$, $\sum p_k = 1$
- ZAKON RAZDIOBE SLUČAJNE VARIJABLE $x \sim \begin{pmatrix} x_1 & x_2 & \ldots & x_k \\ p_1 & p_2 & \ldots & p_k \end{pmatrix}$
- slučajne varijable $x, y: \Omega \to S$ su **nezavisne** ako za sve $x_j, y_j \in S$ vrijedi $P(x=x_k, y=y_j) = P(x=x_k) \cdot P(y=y_j)$
- slučajne varijable $x_1, \ldots x_n$ definirane na istom Ω su nezavisne ako $\forall A_1, \ldots A_n \in S \subset \Omega$ vrijedi $P(x_1 \in A_1, \ldots x_n \in A_n) = P(x_1 \in A_1) \cdot \ldots \cdot P(x_n \in A_n)$

KARAKTERISTIKE SLUČAJNIH VARIJABLI

- OČEKIVANJE DISKRETNIH S.V je $E(x) = \sum_k x_k p_k$ (prosjek srednje vrijednost) ako suma konvergira. Oznake: μ, μ_x, m_x
- SVOJSTVO OČEKIVANJA

  - za $\forall s, t \in R$: $E(sX+tY) = sE(X) + tE(Y)$

    $E(sX) = \sum (sx_k) p_k = s \sum x_k p_k = sE(X)$

    $E(x+y) = \sum_{kj} (x_j + y_k) p_{kj} = \sum_{kj} x_j p_{kj} + \sum_{kj} y_k p_{kj} = \sum_j x_j \sum_k p_{kj} + \sum_k y_k \sum_j p_{kj} = \sum_j x_j p_j + \sum_k y_k q_k$

  - za X i Y nezavisne $E(xy) = E(x)E(y)$

    $E(xy) = \sum_{j,k} x_j y_k p_{jk} = \sum_{j,k} x_j y_k p_j q_k = \sum_j x_j p_j \sum_k y_k q_k = E(x)E(y)$

$E(x^2) \neq E(x)^2$ $\quad E(\varphi(x)) = \sum \varphi(x_k) \cdot p_k$

$E(x^2) = \sum_k x_k^2 \cdot p_k$

- ISHODIŠNI MOMENT REDA n od X: $E(x^n) = \sum_k x_k^n p_k$
- CENTRALNI MOMENT REDA n: $E[(x - \underbrace{E(x)}_{m})^n] = \sum_k (x_k - m_x)^n p_k$
- DISPERZIJA ili VARIJACIJA $D(X) = E[(x-m_x)^2]$ $\quad D_x \geq 0$
- STANDARDNA DEVIJACIJA ILI ODSTUPANJE $\sigma = \sqrt{D(x)}$
- $D(x) = E(x^2) - E(x)^2 = \sum_k x_k^2 p_k - m_x^2$

$D(x) = E[x - E(x)]^2 = E(x^2 - 2xE(x) + E(x)^2] = E(x^2) - 2E(xE(x)) + E(x)^2 =$

$= E(x^2) - 2E(x)E(x) + E^2(x) = E(x^2) - E(x)^2$

BINOMNA RAZDIOBA

ponavljamo n puta, slučajna varijabla x bilježi koliko se put se dogodio neki događaj)

$x \sim B(n,p)$ $\quad P(x=k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad k=0,1,\dots,n$ $\quad E(x) = np$

karakteristična funkcija

$$\varphi_x(t) = E(e^{itx}) = E(e^{itx}) = \sum_{k=0}^{n} e^{itk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (e^{it}p)^k (1-p)^{n-k}$$

$$= (pe^{it} + 1-p)^n = [1 + p(e^{it}-1)]^n$$

$E(x) = -i\varphi'(0) = -i\,np\,i\,e^{i\cdot 0} = np$

$D(x) = -\varphi''(0) + \varphi'(0)^2 = \dots = np(1-p)$

SVOJSTVO STABILNOSTI nezavisne $\begin{cases} x_1 \sim B(n_1,p) \\ x_2 \sim B(n_2,p) \end{cases}$ $\quad x_1 + x_2 \sim B(n_1+n_2, p)$

$\varphi_{x_1+x_2}(t) = \varphi_{x_1}(t)\cdot\varphi_{x_2}(t) = (pe^{it}+1-p)^{n_1}(pe^{it}+1-p)^{n_2} = (pe^{it}+1-p)^{n_1+n_2}$

BERNOULLIJEVA SLUČAJNA VARIJABLA (INDIKATORSKI)

$x_i \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$ $\quad E(x_i) = p$ $\quad D(x_i) = p(1-p)$

nezavisne

$x \sim B(n,p)$ $\quad x = \sum_{i=1}^{n} x_i$

$E(x) = E(\Sigma x_i) = \sum_{i=1}^{n} E(x_i) = np$

$D(x) = D(\Sigma x_i) = \sum_{i=1}^{n} D(x_i) = np(1-p)$

POISSONOVA RAZDIOBA

Neka je n veliki, a p malen. Ako je $\lambda = np$ tada:

$$\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}$$

DOKAZ: $p = \frac{\lambda}{n}$

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right) = \frac{\lambda^k}{k!}\,\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!}\,\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots}_{1}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda} \quad n \to \infty$$

$P(x=k) = \frac{\lambda^k}{k!} e^{-\lambda}$

karakteristična funkcija $\varphi(t) = \sum_{k=0}^{\infty} e^{itk}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{it})^k}{k!} = e^{-\lambda}e^{\lambda e^{it}}$

$\varphi(t) = e^{\lambda(e^{it}-1)}$

$E(x) = -i\varphi'(0) = -i\,e^{-\lambda}(-i)e^{\lambda}e^{i(t-1)} = \lambda$

SVOJSTVO STABILNOSTI

nezavisne $\begin{cases} X_1 \sim P(\lambda_1) \\ X_2 \sim P(\lambda_2) \end{cases} \rightarrow X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$

$\vartheta_{X_1+X_2}(t) = \vartheta_{X_1}(t) \cdot \vartheta_{X_2}(t) = e^{\lambda_1(e^{it}-1)} \cdot e^{\lambda_2(e^{it}-1)} = e^{(\lambda_1+\lambda_2)(e^{-it}-1)}$

NEPREKINUTE SLUČAJNE VARIJABLE

- Preslikavanje $X: \Omega \to \mathbb{R}$ je slučajna varijabla ako za $\forall x \in \mathbb{R}$ je:

$A_x = \{\omega \in \Omega : X(\omega) < x\} \in \mathcal{F}$

Funkcija razdiobe varijable X je $F: \mathbb{R} \to [0,1]$ def s $F(x) = P(X < x)$

- SVOJSTVA: $\lim_{x \to -\infty} F(x) = 0 \quad \lim_{x \to \infty} F(x) = 1$

$F(x)$ je rastuća (prva der > 0)

$F(x)$ je neprekinuta s lijeva $F(x-0) = F(x)$

$P(x_1 \leq X < x_2) = F(x_2) - F(x_1)$

- $F(x)$ diskretne slučajne variable je stepenasta funkcija sa skokovima u x_1, x_2, x_3 a veličina skoka je p_1, p_2, p_3.

- za slučajnu varijablu X kažemo da je neprekinuta ako postoji nenegativna funkcija $f: \mathbb{R} \to \mathbb{R}^+$ tako da je

$F(x) = \int_{-\infty}^{x} f(t)\,dt$, tj $f(x) = F'(x)$, $f(x)$ - funkcija gustoće vjergatnosti

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx \qquad D(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - E(x)^2 \qquad \sigma = \sqrt{D(x)}$

- svojstva

$E(ax + by) = aE(x) + bE(y) \qquad D(aX) = a^2 D(X)$

$\vartheta_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$

jednolika (uniformna) razdioba

$F(x) = \frac{x-a}{b-a} \qquad f(x) = \frac{1}{b-a} \qquad E(x) = \frac{a+b}{2} \qquad D(x) = \frac{(b-a)^2}{12}$

FUNKCIJE SLUČAJNIH VARIJABLI

$\psi : \mathbb{R} \to \mathbb{R}$, X slučajna varijabla $Y = \psi(X)$

$$G(y) = P(Y < y) = P(\psi(X) < y) = P(X \in A_1) + P(X \in A_2) + P(X \in A_3)$$

ψ - monotono rastuća

$$G(y) = P(\psi(X) < y) = P(X < \psi^{-1}(y)) = \int_{-\infty}^{\psi^{-1}(y)} f(x)\,dx = F(\psi^{-1}(y))$$

$$g(y) = G'(y) = f(\psi^{-1}(y)) \cdot \frac{d\,\psi^{-1}(y)}{dy}$$

$$g(y) = f(x)\frac{dx}{dy}, \quad x = \psi^{-1}(y)$$

EKSPONENCIJALNA RAZDIOBA

neka je $Z \sim P(\lambda)$, tada je $Z_x \sim P(\lambda x)$ broj pojavljivanja u int. $[0, x]$

X - vrijeme do prve pojave promatranog događaja

$$F(x) = P(X < x) = 1 - P(Z_x = 0) = 1 - \frac{(\lambda x)^0}{0!} e^{-\lambda x} = 1 - e^{-\lambda x}, \quad x \in \langle 0, +\infty \rangle$$

$X \sim E(\lambda)$ $\quad f(x) = \lambda e^{-\lambda x}$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \qquad D(X) = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} = \frac{1}{\lambda^2}$$

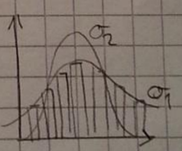
karakt. fj^a:

$$\varphi_X(t) = \int_0^\infty e^{itx} \cdot \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - it}$$

ODSUDSTVO PAMĆENJA $\quad P(X < x+t \mid X > t) = P(X < x)$

$$P(X < x+t \mid X > t) = \frac{P(t < X < x+t)}{P(X > t)} = \frac{F(x+t) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda x - \lambda t} - 1 - e^{-\lambda t}}{1 - (1 - e^{-\lambda t})}$$

$$= 1 - e^{-\lambda x} = F(x) = P(X < x)$$



NORMALNA (GAUSSOVA) RAZDIOBA

$X \sim N(\mu, \sigma^2)$

μ - očekivanje razdiobe

σ^2 - varijacija, disperzija

σ - standardna devijacija $\quad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$

širina razdiobe

$$E(X) = \int_{-\infty}^{\infty} \frac{x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{2\sqrt{2\pi}} dx = \dots \mu$$

$$D(X) = \sigma^2$$

jedinična normalna razdioba

$X^* \sim N(0,1)$

$$\Phi(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt = \frac{1}{2\pi}\int_{-\infty}^{0} e^{-\frac{1}{2}t^2}dt + \frac{1}{2\pi}\int_{0}^{x} e^{-\frac{1}{2}t^2}dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\int_{-x}^{x} e^{-\frac{1}{2}t^2}dt}{2}$$

$$= \frac{1}{2}(1+\Phi^*(x))$$

$$\Phi(x) = \frac{1}{2}(1+\Phi^*(x))$$

za $N(0,1)$: $P(-a < X^* < a) = \Phi^*(a)$

$P(X^* < a) = \frac{1}{2} + \frac{1}{2}\Phi^*(a)$

$P(X^* > a) = \frac{1}{2} - \frac{1}{2}\Phi^*(a)$

$P(a < X^* < b) = \frac{1}{2}(\Phi^*(b) - \Phi^*(a))$

$\Phi^*(-a) = -\Phi(a)$

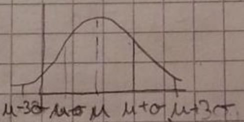
– neka je $X \sim N(\mu, \sigma^2)$ → svest na $N(0,1)$

$X^* = \frac{X-\mu}{\sigma}$ $\quad E(X^*) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{\mu-\mu}{\sigma} = 0$

$D(X^*) = E(X^2) = E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{\sigma^2}{\sigma^2} = 1$

PRAVLO 3σ

Neka je $X \sim N(\mu, \sigma^2)$



$P(|X-a| < k\sigma) = P(-k\sigma < X-a < k\sigma) = P(-k < \tilde{X} < k) = \Phi(k)$

karakteristična funkcija $\vartheta(x) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$

za jedičnu $N(0,1)$: $\vartheta(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \quad / \cdot \frac{d}{dt}$

$$\vartheta'(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} ix\underbrace{e^{-\frac{1}{2}x^2}}_{dv}\,\underbrace{e^{itx}}_{u} \overset{\text{par}}{=} -\frac{t}{2\pi}\int e^{itx}\cdot e^{-\frac{1}{2}x}dx = -t\cdot\vartheta(t)$$

$\vartheta'(t) = -t\vartheta(t) \quad \vartheta_{X^*}(t) = e^{-\frac{1}{2}t^2}$

tada $X \sim N(\mu, \sigma^2)$

$X^* = \frac{X-\mu}{\sigma} \quad X = \mu + \sigma X^* \quad \vartheta_{\mu+\sigma X^*}(t) = e^{it\mu}\vartheta_{X^*}(\sigma t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$

$E(X) = -i\vartheta'(0) = -i(i\mu - \sigma^2 t)e^{it\mu - \frac{1}{2}\sigma^2 t^2}\Big|_0 = \mu$

$D(X) = -\vartheta''(0) + \vartheta'(0)^2 = \sigma^2$

□ SVOJSTVO STABILNOSTI : Neka su $X_1 \sim N(\mu_1, \sigma_1^2)$ i $X_2 \sim N(\mu_2, \sigma_2^2)$ nezavisne. Tada $\alpha X_1 + \beta X_2 \sim N(\alpha\mu_1 + \beta\mu_2, \alpha^2\sigma_1^2 + \beta^2\sigma_2^2)$

DOKAZ $\varphi_{\alpha X_1 + \beta X_2}(t) = \varphi_{\alpha X_1}(t) \cdot \varphi_{\beta X_2}(t) = e^{it\alpha\mu_1 - \frac{1}{2}\alpha^2\sigma^2 t^2} \cdot e^{t\beta\mu_2 - \frac{1}{2}\beta^2\sigma^2 t^2}$

$= e^{it(\alpha\mu_1 + \beta\mu_2) - \frac{1}{2}t^2(\alpha^2\sigma^2 + \beta^2\sigma^2)}$

APROKSIMACIJA BINOMNE RAZDIOBE

□ (Moivre-Laplace) PRVI CENTRALNI GRANIČNI TEOREM : $B(n,p) \approx N(np, np$

za dovoljno velik n