2. MI 2006/2007

1) $\int_{-\infty}^{\infty} f(x)dx = 1$

$\int_0^2 Cx^2dx = C*\frac{1}{3}*x^3 \Big|_0^2 = C*\frac{8}{3} = 1 \rightarrow C = \frac{3}{8}$

Za x<0 → F(x) = 0

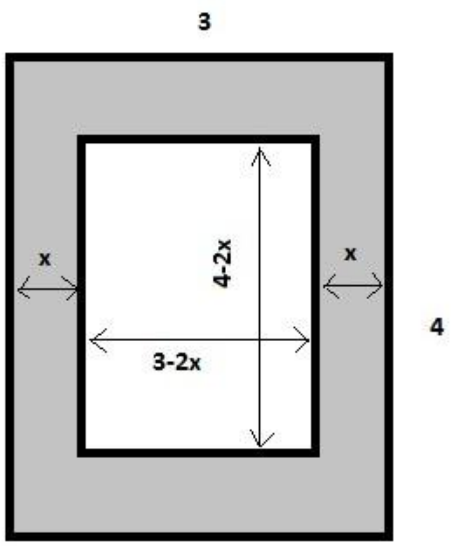
Za x>2 → F(x) = 1

Za 0<x<2 →$F(x) = \int_{-\infty}^{x} f(x)dx = \int_0^x \frac{3}{8}\, x^2dx = \frac{3}{8}\int_0^x x^2\ dx = \frac{3}{8}\frac{1}{3}x^3 \Big|_0^x = \frac{1}{8}x^3$

$$F(X) = \begin{cases} 0, x < 0 \\ \frac{1}{8}\, x^3, 0 < x < 2 \\ 1, x > 2 \end{cases}$$

$P(\{0 < x < 1\}) = F(1) - F(0) = \frac{1}{8} - 0 = \frac{1}{8}$

2)

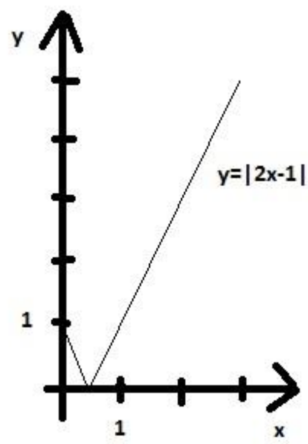


$F(x) = \frac{3*4-(3-2x)(4-2x)}{3*4} = \frac{-1}{3}x^2 + \frac{7}{6}x$

$f(x) = \frac{dF(x)}{d(x)} = \frac{-2}{3}x + \frac{7}{6}, x \in (0, \frac{3}{2})$

$E(X) = \int_0^{\frac{3}{2}} xf(x)dx = \int_0^{\frac{3}{2}} x(\frac{-2}{3}x + \frac{7}{6})dx = \frac{9}{16}$

3) funkcija nije injekcija (povučemo pravac paralelan sa x-osi i ako taj pravac siječe funkciju u točno jednoj točki onda je ta funkcija injekcija)



Znači da funkciju moramo rastaviti na dva intervala na kojima je ona injekcija:

I) $x \in (0, \frac{1}{2})$

$Y = -2x+1 \rightarrow x = \frac{1-y}{2}$

$\left|\frac{dx}{dy}\right| = \frac{1}{2}$

$g_I(\mathrm{y}) = f(x) * \left|\frac{dx}{dy}\right| = e^{-2\frac{1-y}{2}} = e^{y-1}$

II) x € ($\frac{1}{2}$, ∞)

$Y = 2x - 1 \rightarrow x = \frac{(y+1)}{2}$

$\left|\frac{dx}{dy}\right| = \frac{1}{2}$

$g_I(\mathrm{y}) = \mathrm{f(x)} * \left|\frac{dx}{dy}\right| = e^{-2\frac{y+1}{2}} = e^{-y-1}$

$g(y) = \begin{cases} g_I(y) + g_{II}(y), 0 < y < 1 \\ g_{II}(y), y > 1 \end{cases} = \begin{cases} e^{y-1} + e^{-y-1}, 0 < y < 1 \\ e^{-y-1}, y > 1 \end{cases}$

4) a) Slučajna varijabla X je neprekinuta ako postoji funkcija f:R→R takva da vrijedi:

$F(x) = \int_{-\infty}^{x} f(t)dt$

b) Slučajna varijabla X ima eksponencijalnu razdiobu s parametrom λ>0 ako joj je gustoća

$f(x) = \lambda e^{-\lambda x}, x > 0$

c) $E(X) = \int_0^\infty x\lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty x e^{-\lambda x}\, dx =$ parcijalno deriviramo $= \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda}$

5) a) Gustoća opće normalne razdiobe: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$

Gustoća jedinične normalne razdiobe: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

b) $a = 2, \sigma^2 = 8 \rightarrow X \sim N(2{,}8)$

$Y = aX + b = \frac{X-a}{\sigma} = \frac{X-2}{\sqrt{8}} = \frac{X}{2\sqrt{2}} - \frac{1}{\sqrt{2}}$

$a = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}, b = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

c) stranica 47

d) $n = 15000$

$p = \frac{1}{2}, q = \frac{1}{2}$

$B(n,p) \sim N(np, npq) = B(15000, \frac{1}{2}) \sim N(7500, 3750)$

$P(X > 7600) = 1 - P(X <= 7600) = 1 - \frac{1}{2}(1+\phi(\frac{7600-7500}{\sqrt{3750}})) = 1 - \frac{1}{2}(1+\phi(1.633)) =$

$= 1 - \frac{1}{2}(1+0.89690) = 0.0515$

6) a) $\int_0^1 dx \int_0^1 kxydy = k\int_0^1 xdx \int_0^1 ydy = \cdots = \frac{1}{4}k = 1 \rightarrow k = 4$

$$f(x,y) = 4xy, \qquad 0 < x,y < 1$$

b) $f_Y(y) = \int_0^1 4xydx = 4y\int_0^1 xdx = \cdots = 2y$

c) $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{4xy}{2y} = 2x\ ,\ \ 0 < x, y, < 1$

7) Y € [0,2], X € [Y,2]

$$f_Y(y) = \frac{1}{2-0} = \frac{1}{2}$$

$$f_{X|Y=y}(x) = \frac{1}{2-y}$$

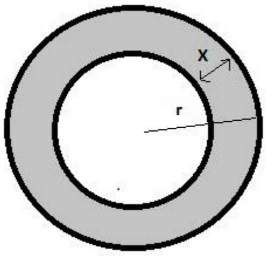
$$f_X(x) = \int_0^x f_{X|Y=y}(x) f_Y(y)dy = \frac{1}{2}\int_0^x \frac{1}{2-y}dy = \frac{1}{2}\ln|2-y|\Big|_0^x = \frac{1}{2}(\ln|2-x| - \ln 2)$$

$$E(X|Y=y) = \int_y^2 x\frac{1}{2-y}dx = \frac{1}{2-y}\int_y^2 xdx = \cdots = \frac{1}{2}(y+2)$$

$$E(X) = \int_0^2 E(X|Y=y)f_Y(y)dy = \int_0^2 \frac{1}{2}(y+2)\frac{1}{2}dy = \frac{1}{4}\int_0^2 (y+2)dy = \cdots = \frac{3}{2}$$

2.MI 2007/2008

1)



$$r = 1$$

$$F(x) = P = (X < x) = \frac{r^2\pi - (r-x)^2\pi}{r^2\pi} = 2x - x^2$$

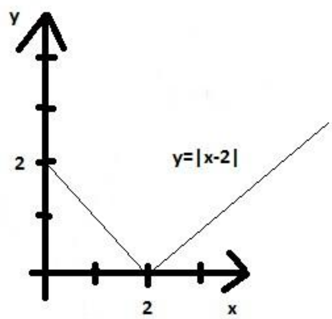
$$f(x) = 2 - 2x$$

$$E(X) = \int_0^1 xf(x)dx =$$

$$= \int_0^1 x(2-2x)dx = 2\int_0^1 (x - x^2)dx = \cdots = \frac{1}{3}$$

2) $Y = |X - 2|$

$f(x) = e^{-x}, x > 0$



Nije injekcija pa oopet moramo podijeliti na dva intervala

I) x € (0, 2)

y € (0, 2)

$y = -x + 2 \rightarrow x = 2 - y$

$\left|\frac{dx}{dy}\right| = 1$

$g_I(y) = e^{-(2-y)} * 1 = e^{y-2}$

II) x € (2, ∞)

y € (0, ∞)

$y = x - 2 \rightarrow x = y + 2$

$\left|\frac{dx}{dy}\right| = 1$

$g_{II}(y) = e^{-y-2}$

$$g(y) = \begin{cases} e^{y-2} + e^{-y-2}, 0 < y < 2 \\ e^{-y-2},\ y \geq 2 \end{cases}$$

3) a) $P(X < E(X)) = P\left(X < \frac{1}{\lambda}\right) = F\left(\frac{1}{\lambda}\right) = 1 - e^{-\lambda\frac{1}{\lambda}} = 1 - e^{-1}$

b) $E(X) = 3 = \frac{1}{\lambda} \rightarrow \lambda = \frac{1}{3}$

$$P(2 < X < 3|X > 2) = \frac{P(2 < X < 3, X > 2)}{P(X > 2)} = \frac{P(2 < X < 3)}{P(X > 2)} = \frac{F(3) - F(2)}{1 - P(X < 2)} =$$

$$= \frac{F(3) - F(2)}{1 - F(2)} = \frac{1 - e^{-\frac{1}{3}3} - 1 + e^{-\frac{1}{3}2}}{1 - 1 + e^{-\frac{1}{3}2}} = \frac{-e^{-1} + e^{-\frac{2}{3}}}{e^{-\frac{2}{3}}} = 1 - e^{-\frac{1}{3}}$$

4) a) stranica 44

b) $a = 370$

$P(10 < X < 730) = 0.9973$

$0.9973 = \frac{1}{2}\left[\phi\left(\frac{730-370}{\sigma}\right) - \phi\left(\frac{10-370}{\sigma}\right)\right] = \frac{1}{2}2\phi\left(\frac{360}{\sigma}\right) = \phi\left(\frac{360}{\sigma}\right)$

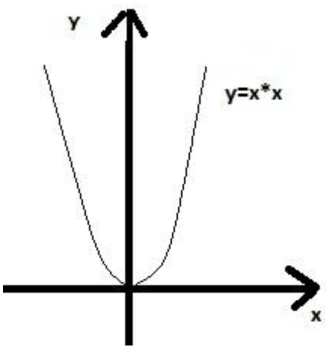
$$\frac{360}{\sigma} = 3 \rightarrow \sigma = 120$$

$$P(X > 450) = \frac{1}{2}\left[1 - \phi\left(\frac{450 - 370}{120}\right)\right] = \frac{1}{2}[1 - \phi(0.667)] = 0.252385$$

c) $X \sim N(0,1), Y = X^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$x = \pm\sqrt{y}$$



Nije injekcija, pa opet dijelimo na dva intervala

I) x € (-∞, 0)
y € (0, ∞)
$x = -\sqrt{y}$
$\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$

$$g_I(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}}$$

Ista stvar je i sa drugim intervalom pa dobijemo da je

$$g_I(y) = g_{II}(y)$$

I konačno na kraju imamo

$$g(y) = g_I(y) + g_{II}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{\sqrt{y}}, \qquad y \in (0, \infty)$$

5) $f(x, y) = C$

$x^2 + y^2 \leq 4,\ y > 0$

Moramo prijeći na polarne koordinate inače će ilko dobit slom živaca ☺

$$\int_0^{\pi} d\varphi \int_0^2 Crdr = 2\pi C = 1 \rightarrow C = \frac{1}{2\pi}$$

$$f(x, y) = \frac{1}{2\pi}$$

$$f_Y(y) = \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)dx = \cdots = \frac{1}{\pi}\sqrt{4 - y^2}, \qquad 0 \leq y \leq 2$$

$$E(Y) = \int_0^2 y \frac{1}{\pi}\sqrt{4 - y^2}dy = \begin{vmatrix} t = 4 - y^2 \\ dt = 2ydy \end{vmatrix} = \frac{1}{\pi}\int_4^0 -\frac{1}{2}\sqrt{t}dt = \cdots = \frac{8}{3\pi}$$