1. a) Funkcija razd. biti > 1 NE

b) gustoća > 1 DA ! $f(x) > 1$

c) F-ju razd. < 0 NE $\int f(x) = 1$

d) gustoća < 0 NE

e) f-ja razd. prekidna (prebrojivo puta) NE

f) gustoća razdioba prekidna 24

2. a) $\frac{3}{4} + \frac{1}{2\pi} \operatorname{arctg} x$ $-\frac{\pi}{2}, +\frac{\pi}{2}$ – ne pomaknuta prema gore

b) $\frac{1}{2} + \frac{1}{\pi} \operatorname{arctg} x$ ✓

$\frac{1}{2} - \frac{1}{2} = 0$ $\frac{1}{2} + \frac{1}{2} = 1$

c) $\frac{x}{1+x}$ $x > 0$ $\lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = 1$ ✓

d) $2^{-e^{-x}}$ ✓

$2^{-e^{-\infty}} = 2^{0} = 1$ $2^{-e^{\infty}} = 2^{-\infty} = 0$

e) $1 - e^{-x}$, $x > 1$ ✓

$x < 1 \Rightarrow 0$ ✓ $1 - e^{-\infty} = 1 - 0 = 1$ ✓

3. Dokazati da su f-je gustoće

a) $1-|1-x|$, $0<x<2$

- monotona

$1-|1-0|=0$ (pozitivno) $\quad 1-|1-2|=1-1=0$ (pozitivno)

$$\int_0^2 1-|1-x|dx = \int_0^1 1-(1-x)\,dx + \int_1^2 1+(1-x)\,dx =$$

$$= \int_0^1 x\,dx + \int_1^2 (2-x)\,dx = \frac{x^2}{2}\Big|_0^1 + \left(2x-\frac{x^2}{2}\right)\Big|_1^2 =$$

$$= \frac{1}{2} + \left(2\cdot 2 - \frac{2^2}{2}\right) - \left(2-\frac{1}{2}\right) = \frac{1}{2} + \cancel{2} - \cancel{2} + \frac{1}{2} = 1 \checkmark$$

b) $|x|$, $-1<x<1$

↳ uvijek ≥ 0 ✓

$$\int_{-1}^1 |x|\,dx = -\int_{-1}^0 x\,dx + \int_0^1 x\,dx = -\frac{x^2}{2}\Big|_{-1}^0 + \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}+\frac{1}{2} = 1 \checkmark$$

c) $\frac{e^x}{(1+e^x)^2}$ - nenegativna je

$$\int_{-\infty}^{\infty} \frac{e^x}{(1+e^x)^2}dx = \left| \begin{array}{l} 1+e^x = u \\ e^x dx = du \end{array} \right| = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{1+e^x}\Big|_{-\infty}^{\infty}$$

$$= -\underbrace{\frac{1}{1+\infty}}_{0} + \underbrace{\frac{1}{1+e^{-\infty}}}_{1} = 1 \checkmark$$

✓ d) $\frac{2}{\pi}\cdot\frac{e^x}{1+e^{2x}}$, nenegativna

$$\frac{2}{\pi}\int\frac{e^x}{1+e^{2x}} = \frac{2}{\pi}\int\frac{du}{1+u^2} = \frac{2}{\pi}\,\mathrm{arctg}(e^x)\Big|_{-\infty}^{+\infty} = \frac{2}{\pi}\left(\mathrm{arctg}(\infty) - \mathrm{arctg}(0)\right)$$

$e^x = u$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2} = \underset{\sim}{1} \checkmark$$

4. a)

$$f(x) = C \quad x \in [a, b]$$

$$C\int_a^b dx = Cx\Big|_a^b = C(b-a) = 1 \Rightarrow C = \frac{1}{b-a}$$

b) $$f(x) = C|x-a| \quad x \in [c, d]$$

$$\int_c^a C(x-a)\,dx - \int_a^d C(x-a)\,dx =$$

$$= \int_c^a Cx - Ca\,dx - \int_a^d Cx - Ca\,dx =$$

$$= C\int_c^a x\,dx - Ca\int_c^a dx - C\int_a^d x\,dx + Ca\int_a^d dx =$$

$$= C\left(\frac{x^2}{2}\Big|_c^a - ax\Big|_c^a - \frac{x^2}{2}\Big|_a^d + ax\Big|_a^d\right) =$$

$$= C\left(\frac{a^2}{2} - \frac{c^2}{2} - a^2 + ac - \frac{d^2}{2} + \frac{a}{2} + ad - ad\right) =$$

- - -

6. $f(x) = \frac{C}{x^2} \quad x > 1$

$$C\int_1^\infty \frac{1}{x^2} = C \left. \frac{-1}{x} \right|_1^\infty = C(0+1) = 1 \Rightarrow \underline{C = 1}$$

$$P(1 < X < 2) = F(2) - F(1) = 1 - \frac{1}{2} - \left(1 - \frac{1}{1}\right) = \frac{1}{2}$$

$$F_x = \int_1^x \frac{1}{x^2} = \left. \frac{-1}{x} \right|_1^x = \frac{-1}{x} - \frac{-1}{1} = -\frac{1}{x} + 1 = 1 - \frac{1}{x}$$

8. a) $f(x) = \sin x \quad , 0 < x < \frac{\pi}{2}$

$$F(x) = \int_0^x \sin x = -\cos x \Big|_0^x = -\cos x + 1 = 1 - \cos x \checkmark$$

b) $f(x) = x - \frac{1}{2}, \quad 1 < x < 2$

$$F(x) = \int_1^x \left(x - \frac{1}{2}\right) dx = \int_1^x x\,dx - \frac{1}{2}\int_1^x dx = \left. \frac{x^2}{2} \right|_1^x - \left. \frac{1}{2} x \right|_1^x =$$

$$= \frac{x^2}{2} - \frac{1}{2} - \frac{1}{2}(x - 1) = \frac{x^2}{2} - \cancel{\frac{1}{2}} - \frac{x}{2} + \cancel{\frac{1}{2}} = \frac{1}{2}(x^2 - x)$$

c) $3\sin 3x \quad , \frac{\pi}{6} < x < \frac{\pi}{3}$

$$F_x = 3\int_{\frac{\pi}{6}}^x \sin 3x = -3 \cdot \frac{1}{3} \cos 3x \Big|_{\frac{\pi}{6}}^x = +\cos\frac{\pi}{2} - \cos 3x = -\cos 3x$$

9.

$$X \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0.1 & 0.2 & 0.2 & 0.3 & 0.2 \end{pmatrix}$$

$p = 0.7$

$$F(x) \begin{cases} 0 & x \leq -2 \\ 0.1 & -1 < x \leq -1 \\ 0.3 & -1 < x \leq 0 \\ 0.5 & 0 < x \leq 1 \\ 0.8 & 1 < x \leq 2 \\ 1 & 2 < x \end{cases}$$

???

$$P(|X| \leq 1) = P(-1 < X < 1) = F(1) - F(-1) = 0.8 - 0.1 = \underline{0.7}$$

10.

$$f(x) = \frac{2}{\pi} \cos^2 x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Od tri realizacije dvije padnu unutar $\left(0, \frac{\pi}{4}\right)$

$$F(x) = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{x} \cos^2 x = \ldots$$

$$\cos^2 = \frac{1}{2}(1 + \cos 2x)$$

$$\int \cos^2 x = \frac{1}{2}\int 1 + \cos 2x \, dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$F(x) = \frac{2}{\pi}\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right)\Bigg|_{-\frac{\pi}{2}}^{x} = \frac{2}{\pi}\left(\frac{x}{2} + \frac{1}{4}\sin^2 x - \left(\frac{-\frac{\pi}{2}}{2} + \underbrace{\frac{1}{4}\sin 2\frac{\pi}{2}}_{0}\right)\right)$$

$$= \frac{2}{\pi}\left(\frac{x}{2} + \frac{1}{4}\sin 2x + \frac{\pi}{4}\right) = \frac{x}{\pi} + \frac{1}{2\pi}\sin 2x + \frac{1}{2}$$

$$P\left(0 < X < \frac{\pi}{4}\right) = F\left(\frac{\pi}{4}\right) - F(0) = \frac{\frac{\pi}{4}}{\pi} + \frac{1}{2\pi}\sin 2\cdot\frac{\pi}{4} + \cancel{\frac{1}{2}} - \frac{0}{\pi} - 0 - \cancel{\frac{1}{2}}$$

$$= \frac{1}{4} + \frac{1}{2\pi} = \frac{\pi + 2}{4\pi}$$

$$P(2 \text{ unutra}, 1 \text{ van}) = 3 \cdot \left(\frac{\pi + 2}{4\pi}\right)^2 \left(1 - \frac{\pi + 2}{4\pi}\right)$$

$$= 3 \cdot \left(\frac{\pi + 2}{4\pi}\right) \frac{3\pi - 2}{4\pi}$$

11. $P(X<1) = 0.9$

$P(Y<2) = 0.5$

$\{X<1,\ Y<2\} \subset \{X+Y<3\}$

$P(X+Y<3) \geq P(X<1,\ Y<2) = P(X<1) + P(Y<2) - 1 = 0.4$

12. $P(0<X<1) = 0.3$ $\qquad AB \subset C$

$P(-1<Y<0) = 0.9$ $\qquad P(C) \geq P(A) + P(B) - P(A+B) \geq P(A) + P(B) - 1$

$P(-1<X+Y<1) \geq 0.2$

$P(-1<X+Y<1) \geq P(-1<Y<0) + P(0<X<1) - \underbrace{P(0<X<1) \cup P(-1<Y<0)}_{\leq 1} =$

$= 0.3 + 0.9 - 1 \geq 0.2$

13. a) $f(x) = Cx \quad 0<x<1$

$C\int_0^1 x = 1 \quad C\frac{x^2}{2}\Big|_0^1 = 1 \quad C\left(\frac{1}{2} - 0\right) = 1 \quad C = 2$

$E(X) = 2\int_0^1 x \cdot x\,dx = 2\int_0^1 x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$

$D(X) = 2\int_0^1 x^2 \cdot x\,dx - \left(\frac{2}{3}\right)^2 = 2\int_0^1 x^3\,dx - \left(\frac{2}{3}\right)^2 = 2\,\frac{x^4}{4}\Big|_0^1 - \left(\frac{2}{3}\right)^2 =$

$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

b) $f(x) = Cx, \quad 0<x<2$

$C\int_0^2 x = 1 \quad C\frac{x^2}{2}\Big|_0^2 = 1 \quad C\left(\frac{4}{2} - 0\right) = 1 \quad 2C = 1 \quad C = \frac{1}{2}$

$E(X) = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^2 = \frac{1}{6}(8-0) = \frac{4}{3}$

$D(X) = \frac{1}{2}\int_0^2 x^3\,dx = \frac{1}{2}\,\frac{x^4}{4}\Big|_0^2 = \frac{1}{2}\,\frac{2^4}{4} - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$

14. a) $F(x) = \frac{1}{4}x \qquad 0<x<4$

$f(x) = \frac{1}{4}$

$E(x) = \frac{1}{4}\int_0^4 x\,dx = \frac{1}{4}\frac{x^2}{2}\Big|_0^4 = \frac{1}{4}\frac{4^2}{2} = 2$

b) $F(x) = 1 - e^{-\lambda x} \qquad x>0$

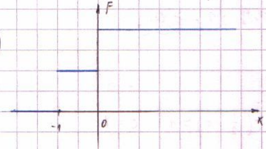
$f(x) = 0 + \lambda e^{-\lambda x}$

$E(x) = \lambda\int_0^\infty x\, e^{-\lambda x} = \lambda\left(\dots \right) = \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda}$

$u = x \quad dv = e^{-\lambda x}$

$du = dx \quad v = -\frac{e^{-\lambda x}}{\lambda}$

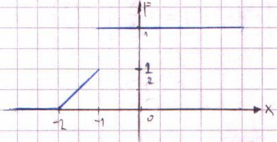
15. a)



$E(x) = \int_{-\infty}^{\infty} x\,dF(x) = -1\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = -\frac{1}{2}$

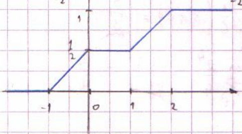
(samo dva skoka, integrali su 0)

b)



$E(x) = \frac{1}{2}\int_{-2}^{-1} x + (-1)\cdot\frac{1}{2} = \frac{1}{2}\frac{x^2}{2}\Big|_{-2}^{-1} - \frac{1}{2} = \frac{1}{2}\left(\frac{1}{2} - \frac{4}{2}\right) - \frac{1}{2} = -\frac{5}{4}$

c)



$E(x) = \frac{1}{2}\int_{-1}^{0} x\,dx + \frac{1}{2}\int_{1}^{2} x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_{-1}^{0} + \frac{1}{2}\frac{x^2}{2}\Big|_{1}^{2} = \frac{x^2}{4}\Big|_{-1}^{0} + \frac{x^2}{4}\Big|_{1}^{2} = 0 - \frac{1}{4} + \frac{4}{4} - \frac{1}{4} = \frac{1}{2}$

16. JEDNOLIKA RAZDOBA $\qquad f(x) = \frac{1}{b-a}$

$E(x) = 4 \qquad D(x) = 3$

$\frac{a+b}{2} = 4 \qquad \frac{(b-a)^2}{12} = 3$

$a+b = 8 \qquad (b-a)^2 = 36$

$a = 8-b \qquad b^2 - 2ab + a^2 = 36$

$b^2 - 2(8-b)b + (8-b)^2 = 36$

$b^2 - 2(8b - b^2) + 64 - 16b + b^2 = 36$

$b^2 - 16b + 2b^2 + 64 - 16b + b^2 = 36$

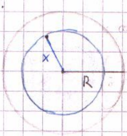
$4b^2 - 32b + 28 = 0$

$b_1 = 7 \qquad b_2 = 1$

$a_1 = 1 \qquad a_2 = 7$

$f(x) = \frac{1}{7-1} = \frac{1}{6} \qquad 1 < x < 7$

18.



$\Omega = R^2 \pi$

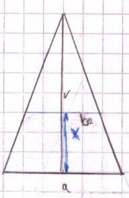
$G = x^2 \pi \qquad F(x) = \left(\frac{x}{R}\right)^2$

$f(x) = \frac{2}{R} x$

$E(x) = \frac{2}{R^2} \int_0^R x \cdot x \, dx = \frac{2}{R^2} \frac{x^3}{3} \Big|_0^R = \frac{2R^{\cancel{3}}}{\cancel{R^2} 3} = \frac{2}{3} R$

$D(x) = \frac{2}{R^2} \int_0^R x^3 dx - \frac{2}{R^2} \frac{R^4}{4} - \left(\frac{2}{3} R\right)^2 = \frac{R^2}{2} - \frac{4}{9} R^2 = \frac{1}{18} R^2$

19.



$\Omega = \frac{a \cdot v}{2} \qquad P = \frac{av}{2} - \frac{a(v-x)}{v}$

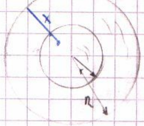
$\frac{v}{a} = \frac{v-x}{b} \qquad F_x = 1 - \frac{\frac{a(v-x)}{v}}{\frac{av}{2}} = 1 - \frac{\cancel{2}a(v-x)}{\cancel{2}va}$

$bv = a(v-x)$

$b = \frac{a(v-x)}{v} \qquad f(x) = \frac{x}{v^2}$

$E(x) = \int_0^v x \cdot \frac{x}{v^2} dx = \int_0^v \frac{x^2}{v^2} dx = \frac{x^3}{3v^2} \Big|_0^v = \frac{v^{\cancel{3}}}{3\cancel{v^2}} = \frac{v}{3}$

20.



bīramo točku unutar male kugle

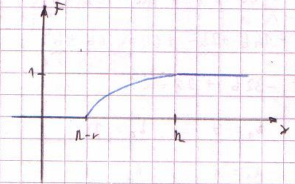
$X \equiv$ udaljenost do veće kugle

$$V = \frac{4}{3}\pi r^3$$

$$\Omega = \frac{4}{3}\pi r^3$$

$$G = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi (R-x)^3$$

$$F_x = 1 - \frac{(R-x)^3}{r^3} \qquad R-r \le x \le R$$



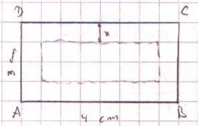
$$f_x = -\frac{3}{r^3}(R-x)^2(-1) = \frac{3}{r^3}(R-x)^2$$

$$E(x) = \int_{R-r}^{R} \frac{3}{r^3}(R-x^2)\cdot x = \frac{3}{r^3}\int_{R-r}^{R} Rx - x^3\,dx =$$

$$= \frac{3}{r^3}\left(R\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{R-r}^{R} = \frac{3}{r^3}R\left(\frac{R^2}{2} - \frac{(R-r)^2}{2}\right) - \frac{3}{r^3}R\left(\frac{R^4}{4} - \frac{(R-r)^5}{4}\right)$$

$$= \dots = R - \frac{3}{4}r$$

21.



X – udaljenost do najbliže stranice

$x = [0,\ 1{,}5]$

$\Omega = 3 \cdot 4 = 12$

$$F_X = \frac{12 - (4-2x)(3-2x)}{12} = \frac{12 - (12 - 8x - 6x + 4x^2)}{12} =$$

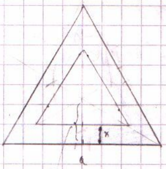
$$= \frac{\cancel{12} - 12 + 8x + 6x - 4x^2}{12} = \frac{7}{6}x - \frac{1}{3}x^2$$

$$f_X = \frac{7}{6} - \frac{2}{3}x$$

$$EX = \int_0^{1.5} x\left(\frac{7}{6} - \frac{2}{3}x\right)dx = \int_0^{1.5} \frac{7}{6}x - \frac{2}{3}x^2 = \frac{7}{6}\cdot\frac{x^2}{2}\Big|_0^{1.5} - \frac{2}{3}\cdot\frac{x^3}{3}\Big|_0^{1.5} =$$

$$= \frac{7}{6}\cdot\frac{(1{,}5)^2}{2} - \frac{2}{3}\cdot\frac{(1.5)^3}{3} = \frac{9}{16}$$

22.



X = udaljenost do najbliže stranice

$$\Omega = \frac{a \cdot v}{2} = \frac{a \cdot \frac{\sqrt{3}}{2}a}{2} = \frac{\sqrt{3}}{4}a^2$$

$$v = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2}a$$

$\frac{1}{3}\cdot\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{6}a$

(izmedu je linearno)

$$\left.\begin{array}{ll} x = 0 \ \ldots\ldots & a' = a \\ x = \frac{1}{3}\cdot\frac{\sqrt{3}}{2}a \ \ldots & a' = 0 \end{array}\right\} \ a' = a - \frac{6}{\sqrt{3}}x$$

$$\frac{a}{\frac{1}{3}\cdot\frac{\sqrt{3}}{2}a} = \frac{a'}{\frac{1}{3}\cdot\frac{\sqrt{3}}{2}a - x}$$

$$a\left(\frac{\sqrt{3}}{6}a - x\right) = a'\ \frac{\sqrt{3}}{6}a$$

$$a' = \frac{\frac{\sqrt{3}}{6}a \ldots}{\frac{\sqrt{3}}{6}} = a - \frac{6}{\sqrt{3}}x$$

$$P = \frac{\sqrt{3}}{4}\left(a - \frac{6}{\sqrt{3}}x\right)^2$$

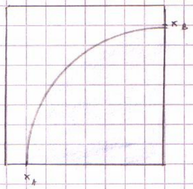
$$F_X = 1 - \frac{\left(a - \frac{6}{\sqrt{3}}x\right)^2}{a^2}$$

$$f_X = \frac{4(\sqrt{3}a - 6x)}{a^2}$$

$$EX = \int_0^{\frac{\sqrt{3}}{6}a} x\left(\frac{4(\sqrt{3}a - 6x)}{a^2}\right)dx = \ldots$$

$$= \frac{a}{6\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{a\sqrt{3}}{6\cdot 3} = \frac{a\sqrt{3}}{18}$$

42.



Za $x < 1$

$$\sqrt{x_A^2 + x_B^2} \leq y$$

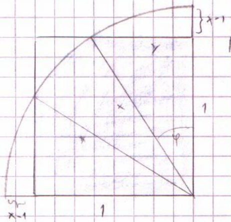
$$\Omega = 1 \cdot 1 = 1$$

$$F_x = \frac{\frac{1}{4} x^2 \pi}{1} = \frac{\pi}{4} x^2$$

$$f(x) = F_x' = 2 \cdot \frac{\pi}{4} x = \frac{\pi}{2} x$$

Za $1 < x \leq \sqrt{2}$

$$\boxed{P_{\text{isječak}} = \frac{1}{2} r^2 \Theta}$$



$$F_x = \frac{\pi}{4} x^2 - 2 \cdot \frac{1}{2} x^2 \arccos \frac{1}{x} + 2 \cdot \frac{1}{2} \cdot 1 \cdot x \sin \varphi$$

$$\sin \varphi = \frac{y}{x} \Rightarrow y = x \sin \varphi$$

$$\cos \varphi = \frac{1}{x} \Rightarrow \varphi = \arccos \frac{1}{x}$$

$$x \sin \varphi = x \sqrt{1 - \frac{1}{x^2}}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\sin^2 x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(\frac{1}{x}\right)^2} = \sqrt{1 - \frac{1}{x^2}}$$

$$F_x = \frac{\pi}{4} x^2 - x^2 \arccos \frac{1}{x} + x \sqrt{1 - \frac{1}{x^2}}$$

$$f(x) = \frac{\pi}{2} x - \left(2x \arccos \frac{1}{x} + x^2 \frac{1}{\sqrt{1 - \frac{1}{x^2}} x^2}\right) + \frac{1}{\sqrt{1 - \frac{1}{x^2}}}$$

$$= \frac{\pi}{2} x - 2x \arccos \frac{1}{x} - \cancel{\frac{1}{\sqrt{1 - \frac{1}{x^2}}}} + \cancel{\frac{1}{\sqrt{1 - \frac{1}{x^2}}}}$$

$$f(x) = \begin{cases} \frac{\pi}{2} x, & 0 < x < 1 \\ \frac{\pi}{2} - 2x \arccos \frac{1}{x}, & 1 < x \leq \sqrt{2} \end{cases}$$