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2. PREDMEĐUISPIT

by Burić

1.

$$f(x) = \begin{cases} a, & -2a \le x \le 0 \\ a - \frac{1}{4}x, & 0 \le x \le 4a \end{cases}$$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-2a}^{0} a\,dx + \int_{0}^{4a} \left(a - \frac{1}{4}x\right) dx = ax\Big|_{-2a}^{0} + \left(ax - \frac{x^2}{8}\right)\Big|_{0}^{4a} = 4a^2$

$4a^2 = 1$

$a^2 = \frac{1}{4}$ $\quad a = \pm\frac{1}{2}$ → fja gustoće je pozitivna fja $\quad a = \frac{1}{2}$

$$f(x) = \begin{cases} \frac{1}{2}, & -1 \le x \le 0 \\ \frac{1}{2} - \frac{1}{4}x, & 0 \le x \le 2 \end{cases}$$

b) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{0} \frac{1}{2}x\,dx + \int_{0}^{2} \left(\frac{1}{2} - \frac{1}{4}x\right)x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_{-1}^{0} + \left(\frac{1}{2}\frac{x^2}{2} - \frac{1}{4}\frac{x^3}{3}\right)\Big|_{0}^{2} = \frac{1}{12}$

c) $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{1}{2}\,dt = \frac{1}{2}x + \frac{1}{2}$

$F(x) = 0, \; x \le -1$

$F(x) = 1, \; x \ge 2$

$x \in (0,2)$: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \cancel{\int_{0}^{x}\left(\frac{1}{2} - \frac{1}{4}t\right)dt} =$ → potrebno je pobrati sve intervale

$= \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{0} \frac{1}{2}\,dt + \int_{0}^{x} \left(\frac{1}{2} - \frac{1}{4}t\right)dt = \frac{1}{2} + \frac{1}{2}x - \frac{1}{8}x^2$

Rješenje

$$F(x) = \begin{cases} 0, & x \le 1 \\ \frac{1}{2}x + \frac{1}{2}, & -1 \le x \le 0 \\ \frac{1}{2} + \frac{1}{2}x - \frac{1}{8}x^2, & 0 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

ovisno o intervalu u kojem se nalazi

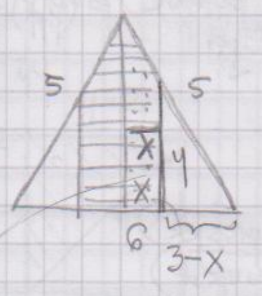
d) $P\left(-\frac{1}{2} \le X \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{15}{32}$

$= \int_{-\frac{1}{2}}^{1/2} f(x)\,dx = \int_{-\frac{1}{2}}^{0} \frac{1}{2}\,dx + \int_{0}^{1/2} \left(\frac{1}{2} - \frac{1}{4}x\right)dx$

→ pazi na granice zadane u zadatku

Geometrijski zadatak

(2)



X ~ udaljenost do visine na osnovici trokuta

$x \in (0,3)$ $\quad F(x) = 0 \,, x<0$

$F(x) = 1 \,, x > 3$

$$F(x) = P(X<x) = \frac{12 - \cancel{2} \cdot \frac{1}{\cancel{2}}(3-x)\cdot \frac{4(3-x)}{3}}{\frac{1}{\cancel{2}}\cdot \cancel{4}_2 \cdot 6} = \frac{12 - \frac{4}{3}(3-x)^2}{12} =$$

$$\frac{y}{3-x} = \frac{4}{3} \Rightarrow y = \frac{4(3-x)}{3} \qquad = -\frac{x^2}{9} + \frac{2}{3}x \,, x \in (0,3) \;//$$

$$f(x) = F'(x) = \frac{-2x}{9} + \frac{2}{3}$$

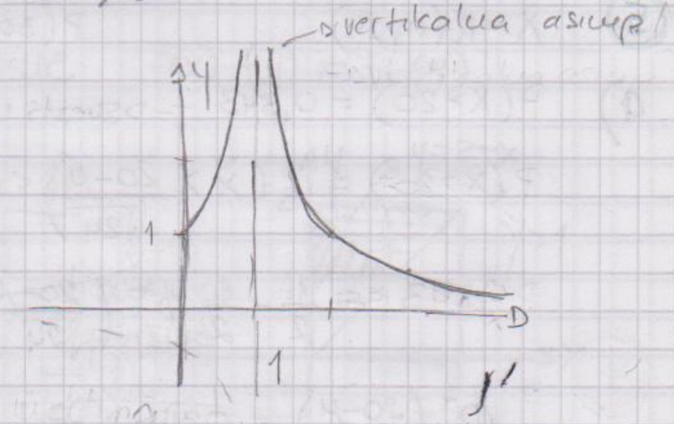
$$E(X) = \int_0^3 x\left(-\frac{2x}{9} + \frac{2}{3}\right)dx = \left(\frac{-2x^3}{27} + \frac{2}{3}x\right)\Big|_0^3 = 1 \;//$$

(3) $X \sim \mathcal{E}(2)$

$$f(x) = \lambda \cdot e^{-\lambda x} = 2e^{-2x} \,, x > 0$$

$$Y = \frac{1}{|X-1|} \qquad g(y) = ?$$

$$g(y) = f(x)\left|\frac{dx}{dy}\right| \rightarrow \text{ako je } \psi$$



1. sl.) $x \in (0,1)$
$y \in (1, +\infty)$

$$y = \frac{1}{1-x}$$

$$x = 1 - \frac{1}{y} \qquad \left|\frac{dx}{dy}\right| = \frac{1}{y^2}$$

$$g_1(y) = 2e^{-2\left(1-\frac{1}{y}\right)} \cdot \frac{1}{y^2} \,, y \in (1, +\infty)$$

2. sl.) $x \in (1, +\infty)$
$y \in (0, +\infty)$

$$y = \frac{1}{x-1}$$

$$x = 1 + \frac{1}{y} \qquad \left|\frac{dx}{dy}\right| = \frac{1}{y^2}$$

$$g_2(y) = 2e^{-2\left(1+\frac{1}{y}\right)} \cdot \frac{1}{y^2} \qquad y \in (0, +\infty)$$

na intervalu gdje se poklapa → zbrojiti!

$$g(y) = \begin{cases} g_1 + g_2 \,, & y \in (1, +\infty) \\ g_2 \,, & y \in (0,1) \end{cases} = \begin{cases} 2e \\ 2e^{-2\left(1+\frac{1}{y}\right)}\frac{1}{y^2} \,, & y \in (1, +\infty) \end{cases}$$

$\cancel{0, +\infty}$

(4) $X \sim \mathcal{E}(\lambda)$ exponencijalna razdioba - vrijeme do pojave prvog događaja

$E(X) = 200$ ⇒ Paziti!!

$X \sim \mathcal{E}\left(\frac{1}{200}\right)$ $F(x) = 1 - e^{-\lambda x}$

a) $P(X<100) = F(100) = 1 - e^{-\frac{100}{200}} = 1 - e^{-\frac{1}{2}}$ → $\frac{P(\text{presjek})}{P(\text{uvjet})}$

b) $P(X<200 \mid X>100) = \frac{P(X<200, X>100)}{P(X>100)} = \frac{P(100<X<200)}{P(X>100)} =$

1. način $= \frac{F(200)-F(100)}{1-F(100)} = \frac{1-e^{-1}-(1-e^{-\frac{1}{2}})}{1-(1-e^{-1/2})} = 1-e^{-1/2}$

2. način → odsutstvo pamćenja! → nije bitno što se dogodilo prije

$P(X<200 \mid X>100) = P(X<100)$

→ 6. POGLAVLJE! → TEORIJA

(5) $X \sim N(a, 24)$ $P(30<X<40) = ?$ ϕ^* iz tablice

a) $P(X>20) = 0{,}343$ → svesti na

$P(X>20) = P\left(\tilde{X} > \frac{20-\mu}{\sqrt{24}}\right)$

$0{,}343 = \frac{1}{2} - \frac{1}{2}\phi^*\left(\frac{20-\mu}{\sqrt{24}}\right)$

$\phi^*\left(\frac{20-\mu}{\sqrt{24}}\right) = 0{,}314$

$\frac{20-\mu}{\sqrt{24}} = 0{,}405$ ⇒ $\mu = 18$

$P(a<\tilde{X}<b) = \frac{1}{2}\left[\phi^*(b) - \phi^*(a)\right]$

$P(\tilde{X}>a) = \frac{1}{2} - \frac{1}{2}\phi^*(a)$

$P(\tilde{X}<a) = \frac{1}{2} + \frac{1}{2}\phi^*(a)$

$P(30<X<40) = \frac{1}{2}\left[\phi^*\underbrace{\left(\frac{40-18}{\sqrt{24}}\right)}_{4{,}49} - \phi^*\underbrace{\left(\frac{30-18}{\sqrt{24}}\right)}_{2{,}45}\right] = \frac{1}{2}(1-0{,}98571) = 0{,}007$

→ vrijednost koje nema u tablici jednaka je 1

b) $p = 23\% = 0{,}23$, $n = 700$ $q = 1-p = 0{,}77$ $B(700, 0{,}23)$ za velike n-ove → aproksimirati normalnom razd.

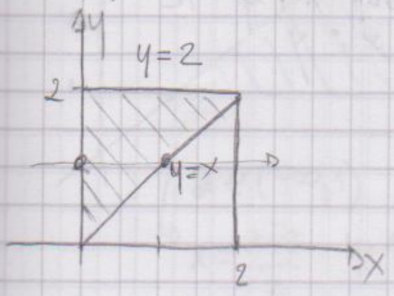
$X \sim N(np, npq) \sim N(161, 123{,}97)$

$P(X>150) = \frac{1}{2} - \frac{1}{2}\phi^*\underbrace{\left(\frac{150-161}{\sqrt{123{,}97}}\right)}_{-0{,}988} = \frac{1}{2} + \frac{1}{2}\phi^*(0{,}988) =$

$= \frac{1}{2} + \frac{1}{2}\cdot 0{,}67685 = 83{,}3\%$

6) $f(x,y) = C(y+x^2)$ na $0 \le x \le y \le 2$ $\qquad f(x,y) = \frac{1}{4}(y+x^2)$

$x \le y$



a) $\iint f(x,y)\,dx\,dy = \int_0^2 dx \int_x^2 C(y+x^2)\,dy =$

$= C\int_0^2 \left(\frac{y^2}{2} + x^2 y\right)\Big|_x^2 dx = 4C = 1$

$C = \frac{1}{4}$

b) marg. gustoće, zavisnost

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^2 \frac{1}{4}(y+x^2)\,dy = \frac{1}{2} + \frac{3}{8}x^2 - \frac{1}{4}x^3, \quad x \in (0,2)$

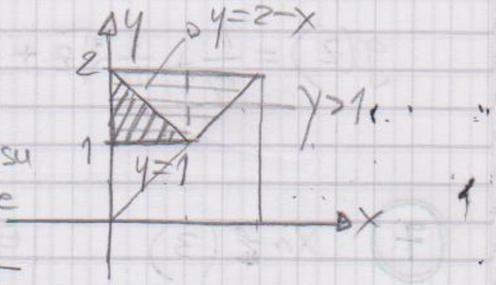
→ INTERVAL! obavezno

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \frac{1}{4}(y+x^2)\,dx = \frac{1}{4}y^2 + \frac{1}{12}y^3, \quad y \in (0,2)$

$\boxed{f_X(x) \cdot f_Y(y) = f(x,y)}$ → uvjet za nezavisnost

X, Y → su zavisne $\quad f(x) \cdot f(y) \neq f(x,y)$

c) $P(X+Y<2 \mid Y>1) = \frac{P(X+Y<2,\, Y>1)}{P(Y>1)}$



↓ ovo nisu 0 površine

$= \frac{\iint_G f(x,y)\,dx\,dy}{\iint_{G_1} f(x,y)\,dx\,dy} = \frac{\int_0^1 dx \int_1^{2-x} \frac{1}{4}(y+x^2)\,dy}{\int_0^1 dx \int_1^2 \frac{1}{4}(y+x^2) + \int_1^2 dx \int_1^2 \frac{1}{4}(y+x^2)}$

ili promijeniti redoslijed

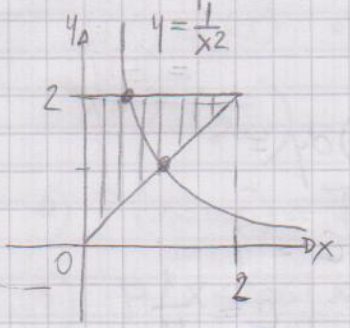
$\int_1^2 dy \int_0^y \frac{1}{4}(y+x^2)\,dx$

3. način

$\int_1^2 f_Y(y)\,dy$

$P(X+Y<2 \mid Y>1) = \frac{\frac{3}{16}}{\frac{43}{48}} = \frac{9}{43}$

d) $Z = Y \cdot X^2$



1.) $z \in (0, 8)$ → uvrstiti najmanji x i y te najveći x i y

2.) $Z = y \cdot x^2 \Rightarrow y = \frac{z}{x^2}$

$$\left|\frac{\partial y}{\partial z}\right| = \frac{1}{x^2}$$

$$g(z) = \int_{-\infty}^{\infty} f(x,y)\left|\frac{\partial y}{\partial z}\right| dx$$

$$g(z) = \int_{\frac{\sqrt{z}}{2}}^{\sqrt[3]{z}} \frac{1}{4}\left(\frac{z}{x^2} + x^2\right) \cdot \frac{1}{x^2}\, dx =$$

↳ uvrstiti $y = \frac{z}{x^2}$

$$= -\frac{1}{4x}\left(\frac{z}{x} + \frac{x^3}{3}\right)\Big|_{\frac{\sqrt{z}}{2}}^{\sqrt[3]{z}} =$$

$$g(z) = \frac{1}{12} + \frac{1}{4}\sqrt[3]{z} + \frac{1}{\sqrt[3]{2z}} - \frac{\sqrt{z}}{4\sqrt{2}}, \quad z \in (0,8)$$

3) „Burićev princip" ⓒ granice:

- nacrtati $y = \frac{z}{x^2}$
- u koju točku ulazi, a u koju izlazi x

↳ uvrštavanje $y = \frac{z}{x^2}$ u naša područja $y = 2$ i u $y = x$.

$\frac{z}{x^2} = 2$ | $x = \frac{z}{x^2}$
$x^2 = \frac{z}{2}$ | $x^3 = z$
$x = \frac{\sqrt{z}}{2}$ | $x = \sqrt[3]{z}$

7. $X \sim \mathcal{E}(3)$ $\quad E(X) = \frac{1}{3}$ $\quad \lambda = 3$

$Y \sim U[1,4]$

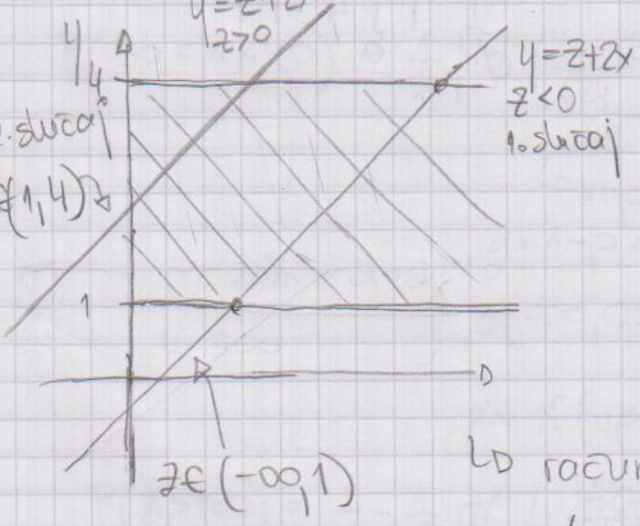
$Z = Y - 2X$

$$\left.\begin{aligned} f(x) &= 3e^{-3x} \\ f(y) &= \frac{1}{3} \end{aligned}\right\}$$

⇒ nezavisne su

$f(x,y) = f(x) f(y) = \frac{1}{3} \cdot 3e^{-3x}$

$f(x,y) = e^{-3x}$ $\quad \underline{x>0,\ y>0}$



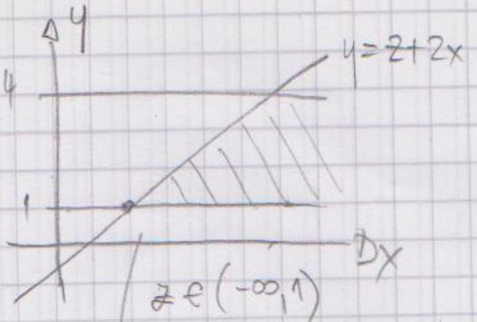
1. $z \in (-\infty, 4)$

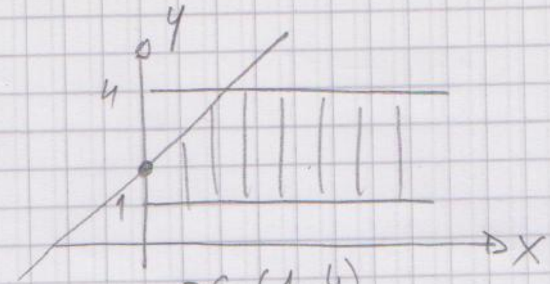
2. $z = y - 2x \Rightarrow y = z + 2x$

$$\left|\frac{\partial y}{\partial z}\right| = 1$$

↳ računanje treba podijeliti na 2 slučaja (bitno je crtanje)

ulazimo i izlazimo na različitim mjestima





$z \in (-\infty, 1)$

$z \in (1, 4)$

$\rightarrow x = \frac{1-z}{2} \quad y = 1$

$$g(z) = \int_{\frac{1-z}{2}}^{\frac{4-z}{2}} e^{-3x} dx =$$

$$= \frac{1}{3} - \frac{1}{3}\left(e^{-z+\frac{3z}{2}} - e^{-\frac{3}{2}+\frac{3}{2}z}\right)$$

$z \in ($

$$g(z) = \int_{0}^{\frac{4-z}{2}} e^{-3x} dx = \frac{1}{3} - \frac{1}{3} e^{-3\left(2-\frac{z}{2}\right)}$$

$z \in (1, 4)$

$\rightarrow$ to je rješenje zadatka $\rightarrow$ dok su gustoće