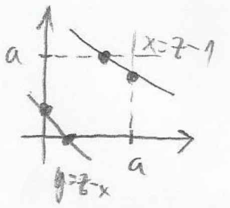
1.) jednolika razdioba $[0,a] \Rightarrow f(x)=\frac{1}{a}$, $f(y)=\frac{1}{a}$, $f(x,y)=\frac{1}{a^2}$

a) $X+Y=Z \rightarrow Y=Z-X$



$$g_z(z)=\int f(x,y)\left|\frac{\partial y}{\partial z}\right|dx=\int\frac{1}{a^2}dx\ ,\ z\in[0,2a]$$

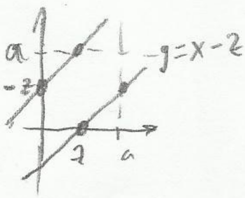
1° $z\in[0,a]$

$$g(z)=\int_0^z\frac{1}{a^2}dx=\frac{1}{a^2}x\Big|_0^z=\underline{\underline{\frac{z}{a^2}}}$$

2° $x\in[a,2a]$

$$g(z)=\int_{z-1}^{a}\frac{1}{a^2}dx=\frac{1}{a^2}(a-z+a)=\underline{\underline{\frac{2a-z}{a^2}}}$$

b) $X-Y=Z$, $z\in[-a,a]$



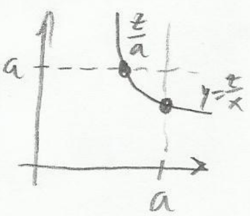
1° $z\in[-a,0]$

$$g_z(z)=\int_0^{a+z}\frac{1}{a^2}\cdot 1\cdot dx=\frac{1}{a^2}(a+z)$$

2° $x\in[0,a]$

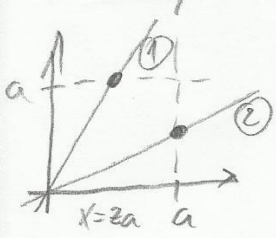
$$g_z(z)=\int_z^a\frac{1}{a^2}dx=\underline{\underline{\frac{a-z}{a^2}}}$$

c) $z=XY \rightarrow Y=\frac{z}{X}$, $z\in[0,a^2]$



$$g_z(z)=\int_{\frac{z}{a}}^{a}\frac{1}{a^2}\cdot\frac{1}{x}dx=\frac{1}{a^2}\left(\ln a-\ln\frac{z}{a}\right)=\underline{\underline{\frac{1}{a^2}\ln\frac{a^2}{z}}},\ z\in[0,a^2]$$

d) $Z=\frac{X}{Y} \rightarrow Y=\frac{X}{Z}=\frac{1}{Z}\cdot X$



1° $z\in[0,1]$

$$g_z(z)=\int_0^{za}\frac{1}{a^2}\cdot\frac{x}{z^2}dx=\frac{1}{a^2z^2}\cdot\frac{1}{2}a^2z^2=\underline{\underline{\frac{1}{2}}}$$

2° $z\in[1,\infty\rangle$

$$g_z(z)=\frac{1}{a^2z^2}\int_0^a x\,dx=\frac{1}{a^2z^2}\cdot\frac{a^2}{2}=\underline{\underline{\frac{1}{2z^2}}}$$

2.) X i $Y \sim U[0,1]$

$f(x) = f(y) = 1$

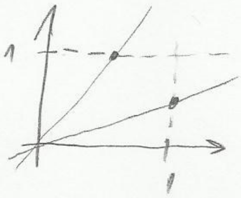
$f(x,y) = 1$

$Z = \frac{X}{X+Y} = ?$

$(x+y)z = x$

$yz = x - xz$

$y = \frac{1}{z}(x - xz) = \frac{x}{z}(1-z)\,,\ z \in [0,1]$



$1^\circ\ z \in [0, \frac{1}{2}]$

$$G(z) = \int_0^{\frac{z}{1-z}} dx \cdot \int_{\frac{1-z}{z}x}^{1} dy = \int \left(1 - \frac{1-z}{z}x\right)dx = x - \frac{1-z}{2z}x^2 \Big|_0^{\frac{z}{1-z}}$$

$$= \frac{z}{1-z} - \frac{1-z}{2z} \cdot \frac{z^2}{(1-z)^2}$$

$$= \frac{z}{1-z} - \frac{z}{2(1-z)} = \frac{z}{2(1-z)}$$

$$g(z) = G(z)' = \frac{(1-z)+z}{2(1-z)^2} = \frac{1}{2(1-z)^2}$$

$2^\circ\ z \in [\frac{1}{2}, 1]$

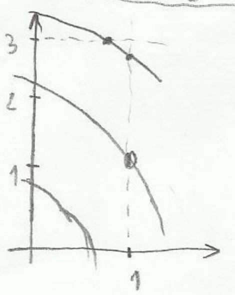
$$G(z) = \int_0^1 dx \cdot \int_{\frac{1-z}{z}x}^{1} dy = \int_0^1 \left(1 - \frac{1-z}{z}x\right)dx = \int_0^1 \frac{z-1+z}{z}x = \frac{2z-1}{z} \cdot \frac{1}{2}$$

$$g(z) = G(z)' = \frac{1}{2} \cdot \frac{2z - (2z-1)}{z^2} = \frac{1}{2} \cdot \frac{1}{z^2} = \frac{1}{2z^2}$$

3.) $f(x) = 1\ ,\ 0 \le x \le 1$

$g(y) = \frac{2y}{9}\ ,\ 0 \le y \le 3$

$\Big\}\ f(x,y) = \frac{2y}{9} = \frac{2r\sin\varphi}{9}$



$z = \sqrt{x^2+y^2}\ ,\ g(z) = ?$

$y = \sqrt{z^2 - x^2} \longrightarrow$ $x = r\cdot\cos\varphi$, $y = r\cdot\sin\varphi$

$1^\circ\ z \in [0,1]$

$$F(z) = \int_0^{\pi/2} d\varphi \cdot \int_0^z \frac{2r^2\sin\varphi}{9} = \int_0^{\pi/2} d\varphi\, \frac{2}{9} \cdot \frac{z^3}{3}\sin\varphi = \frac{2}{27}z^3\left(-\cos\frac{\pi}{2} + \cos 0\right) = \frac{2}{27}z^3$$

$$g(z) = \frac{2}{9}z^2$$

$2^\circ\ z \in [1,3]$

$$F(z) = \int_0^1 dx \cdot \int_0^{\sqrt{z^2-x^2}} \frac{2y}{9}dy = \int_0^1 \frac{2}{9 \cdot 2}(z^2-x^2)dx = \frac{1}{9}(z^2 - x^2) \Rightarrow g(z) = \frac{2}{9}z$$

$3^\circ\ z \in [3, \sqrt{10}]$

$$F(z) = \int_0^{\sqrt{z^2-9}} \frac{2y}{9}dx \cdot \int_0^3 dy + \int_{\sqrt{z^2-9}}^{1} \frac{2y}{9}dx \int_0^{\sqrt{z^2-x^2}} dy = {}^2$$

$y = 3 \rightarrow 9 = z^2 - x^2$

$x^2 = z^2 - 9$

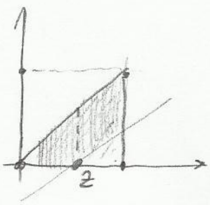
$x = \sqrt{z^2 - 9}$

4.) ZADATAK ISTI KAO 1. c) $\Rightarrow z = xy$

5.) X, Y jednolika razdioba

$O(0,0),\ A(1,0),\ B(1,1)$

$z = X - Y,\ G(z) = ?$



$$f(x,y) = \frac{1}{P_\Delta} = \frac{1}{\frac{1\cdot 1}{2}} = 2$$

$$G(z) = P(X - Y < z) = P(Y > X - z) = 2\int_0^z dx \cdot \int_0^x dy + 2\int_z^1 dx \cdot \int_{x-z}^x dy =$$

$$= 2\int_0^z x\,dx + 2\int_z^1 z\,dx = z^2 + 2z(1-z) = z^2 + 2z - 2z^2$$

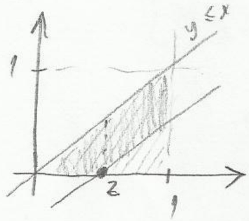
$$= \underline{2z - z^2}\ ,\ z \in [0,1]$$

6.) X, Y, jednolika razdioba

$0 \le y \le x \le 1$

$z = x - y \Rightarrow y = x - z$

$P(z < \frac{1}{2})$



$$G(z) = P(X - Y < z) = P(Y > X - z) = 2\int_0^z dx \cdot \int_0^x dy + 2\int_z^1 dx \cdot \int_{x-z}^x dy$$

$$= 2\int_0^z x\,dx + 2\int_z^1 z\,dx = z^2 + 2z(1-z) = z^2 + 2z - 2z^2$$

$$= \underline{2z - z^2}\ ,\ z \in [0,1]$$

$$G\left(\frac{1}{2}\right) = 2\cdot\frac{1}{2} - \frac{1}{4} = 1 - \frac{1}{4} = \underline{\frac{3}{4} = 0{,}75}$$

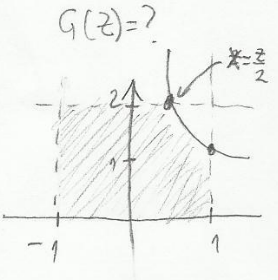
7.) $|x| \le 1,\ 0 \le y \le 2$

$z = XY \Rightarrow y = \frac{z}{x}$ (hiperbola)

$G(z) = ?$

$z \in [-2, 2]$

$$\left.\begin{aligned} f(x) &= \frac{1}{1-(-1)} = \frac{1}{2} \\ f(y) &= \frac{1}{2-0} = \frac{1}{2} \end{aligned}\right\}\ f(x,y) = \frac{1}{2}\cdot\frac{1}{2} = \underline{\frac{1}{4}}$$



$$G(z) = P(Z < z) = P(XY < z) = P\left(y < \frac{z}{x}\right)$$

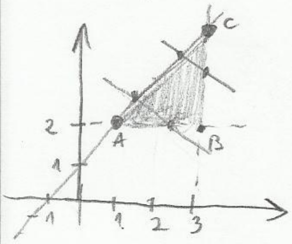
$$g_z(z) = \int_{\frac{z}{2}}^{1} \frac{1}{4}\cdot\frac{1}{x}\,dx = \frac{1}{4}\left(\ln(1) - \ln\left(\frac{z}{2}\right)\right) = \underline{\frac{1}{4}\ln\frac{2}{|z|}\ ,\ z \in [-2, 2]}$$

10.) $X, Y \sim$ jednoliko distribuiran

$y>2,\ x<3,\ y-x<1 \Rightarrow y<1+x$

$Z = X+2Y \rightarrow 2y = z-x \Rightarrow y = -\frac{1}{2}x + \frac{1}{2}z$, $z \in [5,11]$

$g(z) = ?$



$A(1,2)$, $B(3,2)$, $C(3,4)$ $\Big\}$ $f(x,y) = \frac{1}{P_\Delta} = \frac{1}{\frac{2\cdot 2}{2}} = \underline{\underline{\frac{1}{2}}}$

min: $z = x+2y$, $z = 1+2\cdot 2$, $\underline{\underline{z=5}}$ (pogledati sliku)

max: $z = x+2y$, $z = 3+2\cdot 4$, $\underline{\underline{z=11}}$

granica $(B(3,2))$: $z = 3+2\cdot 2 = \underline{\underline{7}}$

$-\frac{1}{2}x + \frac{1}{2}z = x+1$, $y = -\frac{1}{2}x+\frac{1}{2}z$

$\underline{\underline{x = \frac{z}{3} - \frac{2}{3}}}$, $\frac{x}{2} = \frac{z}{2} - 2$, $\underline{\underline{x = z-4}}$

1° $z \in [5,7]$

$$g_Z(z) = \frac{1}{2}\int_{\frac{z-2}{3}}^{z-4} \frac{1}{2}\,dx = \frac{1}{4}\int_{\frac{z-2}{3}}^{z-4} dx$$

$$= \frac{1}{4}\left(z-4-\frac{z}{3}+\frac{2}{3}\right) = \frac{1}{4}\left(\frac{2z}{3}+\frac{2}{3}-\frac{12}{3}\right)$$

$$= \frac{1}{4}\left(\frac{2z}{3}-\frac{10}{3}\right) = \underline{\underline{\frac{1}{6}(z-5)}}$$

2° $z \in [7,11]$

$$g_Z(z) = \frac{1}{4}\int_{\frac{z-2}{3}}^{3} dx = \frac{1}{4}\left(3-\frac{z}{3}+\frac{2}{3}\right) = \frac{1}{4}\left(\frac{11}{3}-\frac{z}{3}\right)$$

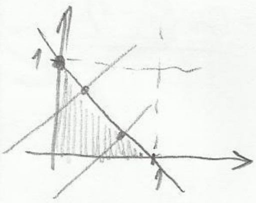
$$= \underline{\underline{\frac{1}{12}(11-z)}}$$

11.) $x>0,\ y>0,\ 1-x-y>0 \Rightarrow y<1-x$

$Z = \frac{Y}{1+X} \Rightarrow \underline{\underline{y = zx+z}}$, $z \in [0,1]$

$F(z), P\left(\frac{1}{4}<Z<\frac{3}{4}\right) = ?$

$f(x,y) = \frac{1}{P_\Delta} = \underline{\underline{2}}$



$$F_Z(z) = P\left(\frac{Y}{1+X}<z\right) = P(Y<z(1+X)) = 1-2\int_0^{\frac{1-z}{1+z}} dx \cdot \int_{zx+z}^{1-x} dy$$

$$= 1-2\int_0^{\frac{1-z}{1+z}} dx\,(1-x-zx-z) = 1-2\int_0^{\frac{1-z}{1+z}} [(1-z)-x(1+z)]\,dx =$$

$$= 1-2\cdot\left[(1-z)\frac{1-z}{1+z} - (1+z)\cdot\frac{1}{2}\cdot\frac{(1-z)^2}{(1+z)^2}\right] = 1-2\cdot\frac{(1-z)^2}{1+z} + \frac{(1-z)^2}{1+z} =$$

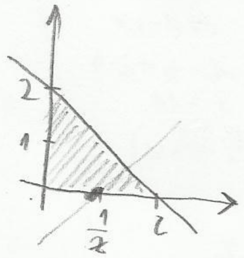
$$= 1-\frac{(1-z)^2}{1+z} = \frac{1+z-1+2z-z^2}{1+z} = \underline{\frac{z(3-z)}{1+z}}, \quad 0 \le z \le 1$$

$$P\left(\frac{1}{4}<Z<\frac{3}{4}\right) = F\left(\frac{3}{4}\right) - F\left(\frac{1}{4}\right) = \frac{\frac{3}{4}\cdot\left(3-\frac{3}{4}\right)}{1+\frac{3}{4}} - \frac{\frac{1}{4}\left(3-\frac{1}{4}\right)}{1+\frac{1}{4}} = \underline{\underline{0{,}4143}}$$

12.) $x \geq 0,\ y \geq 0,\ x+y \leq 2 \Rightarrow y \leq 2-x$

$z = \frac{y+1}{x} \Rightarrow zx = y+1 \Rightarrow y = zx-1 \quad , z \in [\frac{1}{2}, \infty>$

$g(z) = ?$



$(0,2)$
$0 = 2z-1$
$2z = 1$
$z = \frac{1}{2}$

$zx-1 = 2-x$
$x(z+1) = 3$
$x = \frac{3}{z+1}$

$f(x,y) = \frac{1}{P_\Delta} = \frac{1}{\frac{2\cdot 2}{2}} = \frac{1}{2}$

$$g(z) = \frac{1}{2}\int_{\frac{1}{z}}^{\frac{3}{z+1}} x\,dx = \frac{1}{2}\cdot\frac{1}{2}\left(\frac{9}{(z+1)^2} - \frac{1}{z^2}\right)$$

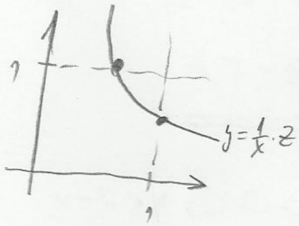
$$= \frac{1}{4}\cdot\frac{9z^2 - z^2 - 2z - 1}{z^2(z+1)^2} = \frac{8z^2-2z-1}{4z^2(z+1)^2}$$

14.) $f_X(x) = 12x^2(1-x) \quad , 0<x<1$

$f_Y(y) = 2y \quad , 0<y<1$

$z = xy \Rightarrow y = \frac{1}{x}\cdot z \quad , z \in [0,1]$

$G(z) = ?$



$f(x,y) = f(x)\cdot f(y) = 24yx^2(1-x)$

$$g(z) = 24\int_z^1 \frac{z}{x}\cdot x^2(1-x)\cdot\frac{1}{x}\,dx = 24z\int_z^1 (1-x)\,dx = 24z\left(1-z-\frac{1}{2}+\frac{1}{2}z^2\right) =$$

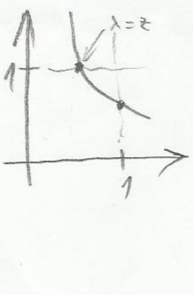
$$= 24z - 24z^2 - 12z + 12z^3 = 12z^3 - 24z^2 + 12z = 12z(z^2-2z+1) = 12z(z-1)^2$$

$$G(z) = \int_0^z g(z)\,dz = 12z^2 - 8z^3 - 6z^2 + 3z^4 = 3z^4 - 8z^3 + 6z^2$$

15.) $f(x,y) = 8xy(1-x^2) \quad , 0<x<1,\ 0<y<1$

$z = xy \Rightarrow y = \frac{1}{x}z$

$g(z) = ?$



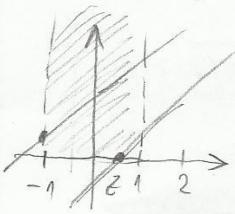
$$g_z(z) = \int_z^1 \cancel{8x}\cdot\frac{z}{\cancel{x}}(1-x^2)\cdot\frac{1}{x}\,dx = 8z\int_z^1\left(\frac{1}{x}-x\right)dx$$

$$= 8z\left(\ln 1 - \ln z - \frac{1}{2} + \frac{1}{2}z^2\right) = 4z(z^2 - 2\ln z - 1) \quad , z \in [0,1]$$

16.) $f_X(x)=\frac{1}{2}\ ,\ -1<x<1$
$f_Y(y)=e^{-y}\ ,\ y>0$ $\Big\}\ f(x,y)=\frac{1}{2}e^{-y}$

PODSJETNIK:
$\boxed{sh(x)=\frac{e^{x}-e^{-x}}{2}}$

$Z=X-Y \Rightarrow y=x-z\ ,\ z\in\langle-\infty,\infty\rangle$
$G(z)=?$



$1^\circ\ z\in\langle-\infty,-1]$

$$g_Z(z)=\frac{1}{2}\int_{-1}^{1}e^{-y}dx=\frac{1}{2}\int_{-1}^{1}e^{-x}e^{z}dx=\frac{1}{2}e^{z}\cdot\frac{e^{-x}}{-1}\Big|_{-1}^{1}=-\frac{1}{2}e^{z}(e^{-1}-e^{1})$$

$$=\frac{1}{2}e^{z}(e^{1}-e^{-1})=e^{z}\cdot sh(1)\Rightarrow G(z)=sh(1)\int_{-\infty}^{z}e^{z}=sh(1)\cdot e^{z}$$

$2^\circ\ z\in[-1,1]$

$$g_Z(z)=-\frac{1}{2}e^{z}e^{-x}\Big|_{z}^{1}=-\frac{1}{2}e^{z}(e^{-1}-e^{-z})=-\frac{1}{2e}e^{z}+\frac{1}{2}$$

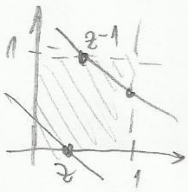
$$G(z)=\int_{-1}^{z}g(z)\,dz=-\frac{1}{2e}(e^{z}-e^{-1})+\frac{1}{2}(z+1)=-\frac{1}{2e}e^{z}+\frac{1}{2e}+\frac{1}{2}z+1$$

$3^\circ\ z\in[1,\infty\rangle$

? $G(z)=1$

17.) $f(x,y)=ax+y\ ,\ 0<x<1,\ 0<y<1$
$Z=X+Y\Rightarrow y=z-x\ ,\ z\in[0,2]$
$a=?\ ,\ g(z)=?$

$f(x,y)=1\cdot x+y=x+y$



$$\int_0^1\int_0^1(ax+y)dxdy=\int_0^1 dx\left(ax+\frac{1}{2}\right)\Rightarrow\frac{a}{2}+\frac{1}{2}=1\Rightarrow\boxed{a=1}$$

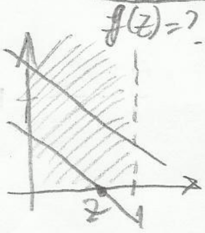
$1^\circ\ z\in[0,1]$

$$g_Z(z)=\int_0^z(x+y)\,dx=\int_0^z z\,dx=z^2$$

$2^\circ\ z\in[1,2]$

$$g_Z(z)=\int_{z-1}^{1}z\,dx=z(1-z+1)=z(2-z)=2z-z^2$$

18.) $f(x)=1\ ,\ 0\leq x\leq 1$
$g(y)=e^{-y}\ ,\ y\geq 0$ $\Big\}\ f(x,y)=e^{-y}$
$Z=X+Y\Rightarrow y=z-x\ ,\ z\in[0,\infty\rangle$
$g(z)=?$



$1^\circ\ z\in[0,1]$

$$g(z)=\int_0^1 e^{-y}dz=\int_0^1 e^{-z}e^{x}dz$$

$$=e^{-z}(e^{z}-1)=1-e^{-z}$$

$2^\circ\ z\in[1,\infty\rangle$

$$g(z)=\int_0^1 e^{x}e^{-z}dx=e^{-z}e^{x}\Big|_0^1=e^{-z}(e-1)$$

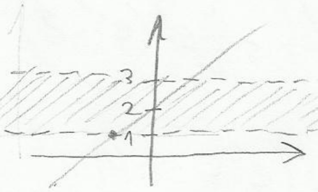
19.) $f(x)=\frac{1}{\pi(1+x^2)}$, $x\in\mathbb{R}$

$g(y)=C$, $1\le x\le 3$

$\Big\}\ f(x,y)=C\cdot\frac{1}{\pi(1+x^2)}$

$z=\frac{x}{y}\Rightarrow y=\frac{1}{z}\cdot x$

$g(z)=?$



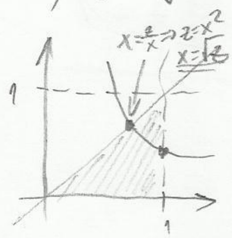
$C\cdot\int_{-\infty}^{\infty}\frac{1}{\pi(1+x^2)}dx\cdot\int_1^3 dy=C\cdot\int_{-\infty}^{\infty}\frac{2}{\pi}\cdot\frac{1}{1+x^2}=\frac{2C}{\pi}\operatorname{arctg}x\Big|_{-\infty}^{\infty}$

$\Rightarrow\frac{2C}{\pi}\cdot\pi=1\Rightarrow\boxed{C=\frac{1}{2}}\Rightarrow f(x,y)=\frac{1}{2\pi(1+x^2)}$

$g(z)=\frac{1}{2\pi}\int_z^{3z}\frac{1}{1+x^2}\cdot\frac{-x}{z^2}dx=\frac{1}{2\pi z^2}\int_z^{3z}\frac{-x}{x^2+1}dx=\left|\begin{matrix}x^2=t\\dt=2xdx\end{matrix}\right|$

$=\frac{1}{2\pi z^2}\int-\frac{1}{2}\cdot\frac{dt}{t+1}=-\frac{1}{2\pi z^2}\cdot\frac{1}{2}\ln|x^2+1|\Big|_z^{3z}=-\frac{1}{4\pi z^2}\big(\ln|3z+1|-\ln(z+1)\big)$?

20.) $f(x,y)=C(x-y)$, $0\le y\le x\le 1$



$z=xy\Rightarrow y=\frac{z}{x}$

$C\cdot\int_0^1dx\cdot\int_0^x(x-y)\,dy=1\Rightarrow C\cdot\int_0^1dx\left(xy-\frac{1}{2}y^2\right)\Big|_0^x=C\int_0^1\left(x^2-\frac{1}{2}x^2\right)dx=C\cdot\frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^1$

$\Rightarrow C\cdot\frac{1}{6}=1\Rightarrow\boxed{C=6}\Rightarrow f(x,y)=6(x-y)$

$f_X(x)=6\cdot\int_0^x(x-y)\,dy=6\left(x^2-\frac{x^2}{2}\right)=3x^2$, $x\in[0,1]$

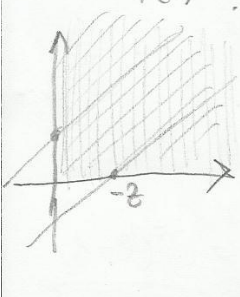
$g_Z(z)=6\cdot\int_{\sqrt{z}}^1(x-y)\cdot\frac{1}{x}\,dx=6\cdot\int_{\sqrt{z}}^1\left(1-\frac{z}{x^2}\right)dx=6\cdot\left(x+\frac{z}{x}\right)\Big|_{\sqrt{z}}^1=6\left(1+z-\sqrt{z}-\sqrt{z}\right)$

$=6\left(1+z-2\sqrt{z}\right)=6+6z-12\sqrt{z}$, $z\in[0,1]$

21.) $f(x,y)=9e^{-3x-3y}$, $x>0$, $y>0$

$z=y-x\Rightarrow y=z+x$

$G(z)=?$



$1^\circ\ z>0$

$g_Z(z)=9\int_0^{\infty}e^{-6x}e^{-3z}dx=$

$=9e^{-3z}\cdot\frac{-1}{6}\left(e^{-\infty}-1\right)$

$=\frac{3}{2}e^{-3z}$

$G(z)=\frac{3}{2}\cdot\frac{-1}{3}\left(e^{-3z}-1\right)$

$=\frac{1}{2}-\frac{1}{2}e^{-3z}$, $z>0$

$2^\circ\ z<0$

$g_Z(z)=9\cdot\int_{-z}^{\infty}e^{-6x}e^{-3z}dx=$

$=9\cdot e^{-3z}\cdot\frac{-1}{6}\cdot\left(e^{-\infty}-e^{6z}\right)=\frac{3}{2}e^{3z}$

$G(z)=\frac{3}{2}\cdot\frac{1}{3}e^{3z}=\frac{1}{2}e^{3z}$, $z<0$

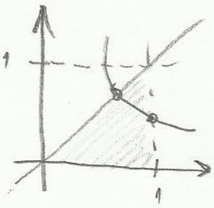
22.) $f(x,y)=C(x+y)$, $0\le y\le x\le 1$

$Z=XY \Rightarrow y=\frac{z}{x}$

$g(z)=?\,,\ f(x)=?$

$$C\cdot\int_0^1 dx\cdot\int_0^x (x+y)\,dy = C\cdot\int_0^1 dx\cdot\left(xy+\frac{1}{2}y^2\right)\Big|_0^x = C\cdot\int_0^1\left(x^2+\frac{1}{2}x^2\right)dx$$

$$= C\cdot\int_0^1 \frac{3}{2}x^2\,dx = C\cdot\frac{\cancel{3}}{2}\frac{x^3}{\cancel{3}}\Big|_0^1 \Rightarrow \frac{C}{2}=1 \Rightarrow \boxed{C=2} \quad f(x,y)=2(x+y)$$



$$f_X(x)=2\cdot\int_0^x (x+y)\,dy = 2\cdot\frac{3}{\cancel{2}}x^2 = 3x^2$$

$$g_Z(z)=\int_{\sqrt{z}}^1 2(x+y)\cdot\frac{1}{x}\,dx = 2\int_{\sqrt{z}}^1\left(x+\frac{z}{x}\right)\cdot\frac{1}{x}\,dx = 2\cdot\int_{\sqrt{z}}^1\left(1+\frac{z}{x^2}\right)dx$$

$$= 2\cdot\left(x-\frac{z}{x}\right)\Big|_{\sqrt{z}}^1 = 2\cdot\left(1-z+\cancel{\sqrt{z}}-\cancel{\sqrt{z}}\right) = 2-2z$$

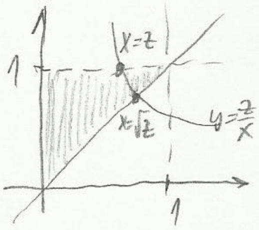
$$E(z)=\int_0^1 z\cdot g(z)\,dz = \int_0^1 (2z-2z^2)\,dz = z^2-\frac{2}{3}z^3\Big|_0^1 = 1-\frac{2}{3} = \frac{1}{3}$$

23.) $f(x,y)=C(x+y)$, $0\le x\le y\le 1$

$Z=XY \Rightarrow y=\frac{z}{x}$

$f(x), g(z), E(z)=?$

$$C\cdot\int_0^1 dx\cdot\int_x^1 (x+y)\,dy = C\cdot\int_0^1\left(x-x^2+\frac{1}{2}-\frac{x^2}{2}\right)dx = C\cdot\int_0^1\left(x+\frac{1}{2}-\frac{3}{2}x^2\right)dx$$



$$= C\left(\frac{x^2}{2}+\frac{1}{2}x-\frac{\cancel{3}}{2}\cdot\frac{x^3}{\cancel{3}}\right)\Big|_0^1 = C\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{2}\right) \Rightarrow \frac{1}{2}C=1 \Rightarrow \boxed{C=2}$$

$$f_X(x)=2\cdot\int_x^1 (x+y)\,dy = 2\left(xy+\frac{1}{2}y^2\right)\Big|_x^1 = 2\left(x-x^2+\frac{1}{2}-\frac{1}{2}x^2\right) = 1+2x-3x^2$$

$$g_Z(z)=2\cdot\int_z^{\sqrt{z}} (x+y)\,\frac{1}{x}\,dx = 2\int\left(1+\frac{z}{x^2}\right)dx = 2\left(x-\frac{z}{x}\right)\Big|_z^{\sqrt{z}} =$$

$$= 2\left(\cancel{\sqrt{z}}-z-\cancel{\sqrt{z}}+1\right) = 2(1-z)\ ,\ z\in[0,1]$$

$$E(z)=\int_0^1 z\cdot g(z)\,dz = \int_0^1 (2z-2z^2)\,dz = z^2-\frac{2}{3}z^3\Big|_0^1 = 1-\frac{2}{3} = \frac{1}{3}$$

SLUŽBENA RJEŠENJA:

§ 8. Funkcije slučajnih vektora

1. a) $\frac{1}{a}\left(1-\left|\frac{x}{a}-1\right|\right)$, $0 \leqslant x \leqslant 2a$;

b) $\frac{1}{a}\left(1-\left|\frac{x}{a}\right|\right)$, $|x| \leqslant a$;

c) $\frac{1}{a^2}\ln\frac{a^2}{x}$, $0 < x < a^2$;

d) $\frac{1}{2}$, $0 \leqslant x \leqslant 1$. $\frac{1}{2x^2}$, $x \geqslant 1$.

2. $\frac{1}{2}\min\left\{\frac{1}{x^2}, \frac{1}{(1-x)^2}\right\}$, $0 \leqslant x \leqslant 1$.

3. $f_Z(z) = \begin{cases} \frac{2}{9}z^2, & z \in [0,1], \\ \frac{2}{9}z, & z \in [1,3], \\ \frac{2}{9}z(1-\sqrt{z^2-9}), & z \in [3, \sqrt{10}]. \end{cases}$

4. $f(x) = \frac{1}{a^2}\ln\frac{a^2}{x}$, $0 < x < a^2$.

5. $2z - z^2$, $0 < z < 1$.

6. $F_Z(z) = 2z - z^2$, $z \in [0,1]$; $\frac{3}{4}$

7. $f_Z(z) = \frac{1}{4}\ln\frac{2}{|z|}$, $z \in [-2,2]$.

8. $f_Z(z) = \frac{1}{\pi(1+z^2)}$, $z \in \mathbf{R}$.

9. $f_Z(z) = \begin{cases} \frac{b}{2a}, & 0 < z < \frac{a}{b}, \\ \frac{a}{2bz^2}, & \frac{a}{b} < z < \infty. \end{cases}$

10. $g_Z(z) = \begin{cases} \frac{1}{6}(z-5), & 5 \leqslant z \leqslant 7, \\ \frac{1}{12}(11-z), & 7 \leqslant z \leqslant 11. \end{cases}$

11. $F_Z(z) = \frac{z(3-z)}{1+z}$, $0 < z < 1$; 0.4143.

12. $f_Z(z) = \frac{8z^2-2z-1}{4z^2(z+1)^2}$, $z \in [\frac{1}{2}, \infty)$.

13. $f_Z(z) = \begin{cases} \frac{1}{2}z - \frac{1}{4}, & z \in [1,2), \\ \frac{1}{2}, & z \in (2,3]. \end{cases}$

14. $12z(1-z)^2$, $0 < z < 1$.

15. $4z(z^2 - 2\ln z - 1)$, $0 < z < 1$.

16. $\begin{cases} \mathrm{sh}(1)e^z, & z \leqslant -1, \\ 1 + \frac{1}{2}z - \frac{1}{2e}e^z, & -1 < z \leqslant 1, \\ 1, & z > 1. \end{cases}$

17. $a = 1$; $f_Z(z) = \begin{cases} z^2, & 0 \leqslant z \leqslant 1, \\ 2z - z^2, & 1 \leqslant z \leqslant 2. \end{cases}$

18. $F_Z(z) = \begin{cases} \mathrm{sh}\,1 \cdot e^z, & z \leqslant -1, \\ 1 + \frac{1}{2}z + \frac{1}{2}e^{z-1}, & -1 \leqslant z \leqslant 1. \end{cases}$

20. $f_Z(z) = 6 + 6z - 12\sqrt{z}$, $0 \leqslant z \leqslant 1$.

21. $F_Z(z) = \begin{cases} \frac{1}{2}e^{3z}, & z < 0, \\ 1 - \frac{1}{2}e^{-3z}, & z > 0. \end{cases}$

22. $f_X(x) = 3x^2$, $x \in [0,1]$;
$f_Z(z) = 2 - 2z$, $z \in [0,1]$; $\boldsymbol{E}(Z) = \frac{1}{3}$.

23. $f_X(x) = -3x^2 + 2x + 1$, $0 < x < 1$;
$f_Z(z) = 2 - 2z$, $z \in [0,1]$, $\boldsymbol{E}(Z) = \frac{1}{3}$.

24. Vrijedi $F_Z(x) = F_X(x)^2$. Zato

$$\begin{aligned} \boldsymbol{E}(Z) &= 2\int_{-\infty}^{\infty} xF_X(x)dF_X(x) \\ &= \frac{1}{\pi\sigma^2}\int_{-\infty}^{\infty} x\left(\int_{-\infty}^{x} e^{-\frac{(z-a)^2}{2\sigma^2}}dz\right)dx \\ &= a + \frac{\sigma}{\sqrt{\pi}}. \end{aligned}$$

28. Normalna razdioba $\mathcal{N}(0,1)$.

30. a) $\lambda^2 xe^{-\lambda x}$, $x > 0$;

b) $\frac{\lambda}{2}e^{-\lambda|x|}$, $-\infty < x < \infty$;

c) $\lambda e^{-\lambda x}$, $x > 0$;

d) $\frac{1}{(1+x)^2}$, $x \geqslant 0$.

31. $F(z) = 1 - \frac{1}{z}$, $z > 1$.

32. $F_Z(t) = F_W(t) = (1 - e^{-\lambda t})^2$, $t > 0$.

33. $F_Y(y) = \frac{\lambda_1 x}{(\lambda_1 - \lambda_2)x + \lambda_2}$, $0 < x \leqslant 1$.

34. $(n-1)(1-x)^{n-2}$, $0 < x < 1$.

35. $\frac{1}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$.

36. $F_Y(x) = 1 - [1 - F(x)]^n$; $F_Z(x) = F(x)^n$.

37. Jednolika na $[0,2]$.

39. Dokaži najprije slučaj $n = 1$. Nakon toga primjeni indukciju, koristeći relaciju

$$\sum_{k=1}^{n+1}\frac{X_k}{2^k} + \frac{Y}{2^{n+1}} = \frac{X_1}{2} + \frac{1}{2}\left(\sum_{k=2}^{n+1}\frac{X_k}{2^{k-1}} + \frac{Y}{2^n}\right).$$

40. $f_Z(z) = \frac{1}{2\pi}\left(1 + \frac{1}{2}\sum_{n=1}^{\infty} a_n b_n \cos n(x - \alpha_n - \beta_n)\right)$.

41. a) $F_{(1)}(x) = 1 - (1 - F(x))^n$;

b) $F_{(n)}(x) = F(x)^n$;

c) $F_{(1,n)}(x,y) = F(y)^n - (F(y) - F(x))^n$, $x \leqslant y$;

d) $\frac{n!}{(k-1)!(n-k)!}F(x)^{k-1}[1-F(x)]^{n-k}f(x)$;

e) $\frac{n!}{(k-1)!(m-k-1)!(n-m)!}F(x)^{k-1} \times$
$\times [F(y) - F(x)]^{m-k-1}[1 - F(y)]^{n-m}f(x)f(y)$;

f) $n!f(x_1)f(x_2)\cdots f(x_n)$, $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$.

42. $g(r,\vartheta,\varphi) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 e^{-\frac{r^2}{2\sigma^2}} r^2 \sin\vartheta$.

43. $g(r,\vartheta,\varphi) = h(r)r^2 \sin\vartheta$, $0 < r < \infty$, $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$. Zato su R, Θ, Φ nezavisne, s gustoćama

$$g_R(r) = 4\pi h(r)r^2,$$
$$g_\Theta(\vartheta) = \frac{1}{2}\sin\vartheta,$$
$$g_\Phi(\varphi) = \frac{1}{2\pi}.$$

44. $f_X(x) = f_Y(x) = \dfrac{1}{\pi(1+x^2)}$,
$g(r,\varphi) = \dfrac{r}{2\pi\sqrt{(1+r^2)^3}}$.

45. b) $f_X(x) = f_Y(x) = f_Z(x) = \dfrac{1}{\pi(1+x^2)}$.
c) $g(r,\theta,\varphi) = 4\pi f_1(r)r^2 \cdot \frac{1}{2}\sin\vartheta \cdot \frac{1}{2\pi}$,
gdje je $f_1(r) = \dfrac{1}{\pi^2(1+r^2)^2}$.

46. a) $g(x,y) = \dfrac{1}{2}f\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}\right)$;
b) $g(x,y) = f(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$.

47. $g(u,v) = \dfrac{1}{\pi(1+v^2)} \cdot \dfrac{e^{-\frac{u}{2\sigma^2}}}{2\sigma^2} = g_U(u)g_V(v)$.
Nezavisne su.

48. $F(y_1,y_2) = 2y_1y_2 - y_1^2$, $y_1 \leqslant y_2$,
$f(y_1,y_2) = 2$, $y_1 \leqslant y_2$,
$f_1(y_1) = 2 - 2y_1$, $0 \leqslant y_1 \leqslant 1$,
$f_2(y_2) = 2y_2$, $0 \leqslant y_2 \leqslant 1$.

49. $\dfrac{\Gamma(k+\alpha)}{\lambda^k\Gamma(\alpha)}$.

50. $n(n+2)\cdots(n+2k-2)$.

51. $2^{k/2}\Gamma\left(\dfrac{k+n}{2}\right)/\Gamma\left(\dfrac{n}{2}\right)$.

54. $\dfrac{\alpha}{\alpha+\beta}$, $\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

56. Stavimo

$$Y_{11} = \frac{1}{2}(X_{11}+X_{12}), Y_{22} = \frac{1}{2}(X_{11}-X_{22}),$$
$$Y_{12} = \frac{1}{2}(X_{12}+X_{21}, \; Y_{21} = \frac{1}{2}(X_{12}-X_{21}).$$

Ove slučajne varijable imaju normalnu razdiobu $\mathcal{N}(0,\frac{1}{2})$. Jer su nekorelirane, zaključujemo da su i nezavisne. Zato $Y_{11}^2+Y_{21}^2$ i $Y_{22}^2+Y_{12}^2$ imaju eksponencijalnu razdiobu $E(1)$ (χ^2-razdiobu s dva stupnja slobode). Vrijedi pritom

$$\Delta = (Y_{11}^2+Y_{21}^2) - (Y_{22}^2 - Y_{12}^2).$$

Odavde se dobiva razdioba od Δ: $g_\Delta(x) = \frac{1}{2}e^{-|x|}$.

57. $F_Y(y) = \dfrac{\lambda_1 x}{(\lambda_1-\lambda_2)x+\lambda_2}$, $0 < x \leqslant 1$.

LITERATURA:

[1] Neven Elezović: Slučajne varijable, *Element 2010.godine*