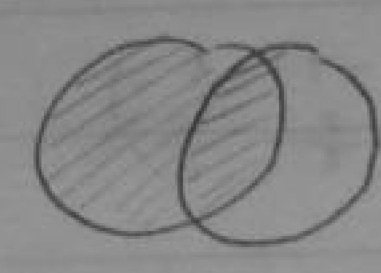
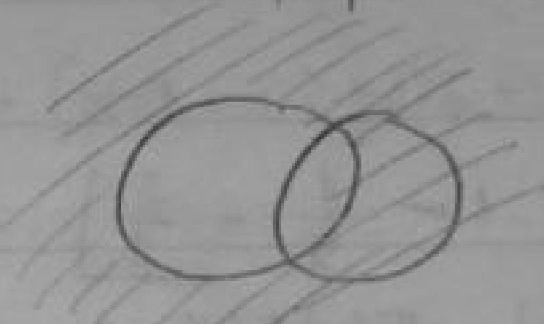
PMIA 07

1) a) $P(A \cap B) = 0{,}72$ $\qquad$  $\qquad$ $\boxed{P(A) = 0{,}9}$

$P(A \cap \bar{B}) = 0{,}18$

b) $P(B|\bar{A}) = P(B|A)$, onda su događaji A i B nezavisni , prove it ?

$P(B|\bar{A}) = \dfrac{P(\bar{A}B)}{P(\bar{A})}$ $\qquad$ $P(B|A) = \dfrac{P(AB)}{P(A)}$



$P(\bar{A})\,P(AB) = P(A)\,P(\bar{A}B)$

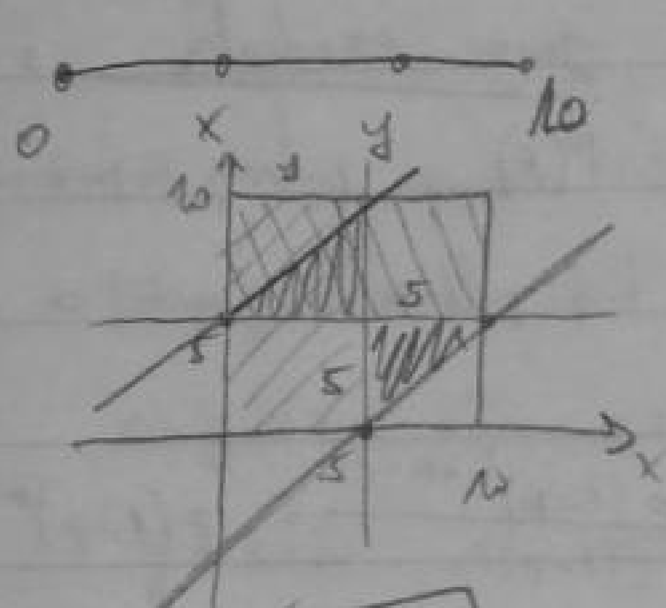
$[1 - P(A)]\,P(AB) = P(A)\,[P(B) - P(AB)]$

$P(AB) - \cancel{P(A)}P(AB) = P(A)P(B) - \cancel{P(A)}P(AB)$ $\boxtimes$

2) 5B, 10C $\quad$ 5 vučemo $\quad$ P(bar 2B)

$P(X \geq 2) = 1 - \dfrac{\binom{10}{5}}{\binom{15}{5}} - \dfrac{\binom{10}{4}\binom{5}{1}}{\binom{15}{5}} = \underline{0{,}566}$?

3)



$\boxed{P = \frac{1}{4}}$

$x < L - x$ $\qquad$ $\boxed{x < \frac{10}{2}}$

$y - x < x + (L - y) \Rightarrow 2y < 2x + L$ $\quad$ $\boxed{y < x + \frac{10}{2}}$

$L - y < y$ $\qquad$ $\boxed{y > \frac{10}{2}}$

$y < L - y$ $\qquad$ $y < 5$

$x - y < y + L - x$ $\qquad$ $y > x - 5$

$L - x < x$ $\qquad$ $x > 5$

4) 1. kutija $\quad$ 2B, 3C $\quad \xrightarrow{3}$ 2. kutija $\xrightarrow{1}$ C , P(3 crne ?)

H_1 - 2B, 1C $\qquad P(H_1) = \dfrac{\binom{3}{1}}{\binom{5}{3}} = \boxed{\dfrac{3}{10}}$

H_2 - 1B, 2C $\qquad P(H_2) = \dfrac{\binom{2}{1}\binom{3}{2}}{\binom{5}{3}} = \boxed{\dfrac{6}{10}}$

H_3 - 3C $\qquad P(H_3) = \dfrac{1}{\binom{5}{3}} = \boxed{\dfrac{1}{10}}$

A - izvučena crna kuglica

$P(A|H_1) = \dfrac{1}{\binom{3}{1}} = \boxed{\dfrac{1}{3}}$

$P(A|H_2) = \dfrac{2}{\binom{3}{1}} = \boxed{\dfrac{2}{3}}$ $\qquad$ $P(A) = \dfrac{3}{5}$

$P(A|H_3) = \dfrac{3}{3} = \boxed{1}$

$\boxed{P(H_3|A) = \dfrac{P(H_3)P(A|H_3)}{P(A)} = \dfrac{\frac{1}{10} \cdot 1}{\frac{3}{5}} = \dfrac{1}{6}}$

5) kocka bacamo do broja djeljivog s 3 $\quad$ [{3,6}]

$P(X=1) = \dfrac{2}{6}$

$P(X=2) = \dfrac{4}{6}\,\dfrac{2}{6}$

$P(X=3) = \left(\dfrac{4}{6}\right)^2 \dfrac{2}{6}$

$\vdots$

$P(X=n) = \left(\dfrac{4}{6}\right)^{n-1} \dfrac{2}{6}$

$E(X) = \sum_{k=1}^{\infty} p_k x_k = \sum_{k=1}^{\infty} k \cdot \dfrac{1}{3} \cdot \left(\dfrac{2}{3}\right)^{k-1} = \dfrac{1}{3}\,\dfrac{1}{\left(1 - \frac{2}{3}\right)^2} = \underline{3}$?

⑥ kocka, X – kvadrat dobivenog, Y – -1, broj ≤ 2, inače 1

$X \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 & 36 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$ $\quad Y \sim \begin{pmatrix} -1 & 1 \\ \frac{2}{6} & \frac{4}{6} \end{pmatrix}$

X|Y	-1	1	
1	$\frac{1}{6}$	0	$\frac{1}{6}$
4	$\frac{1}{6}$	0	$\frac{1}{6}$
9	0	$\frac{1}{6}$	$\frac{1}{6}$
16	0	$\frac{1}{6}$	$\frac{1}{6}$
25	0	$\frac{1}{6}$	$\frac{1}{6}$
36	0	$\frac{1}{6}$	$\frac{1}{6}$
	$\frac{2}{6}$	$\frac{4}{6}$	

$\mathbb{E}(X) = \frac{91}{6}$ $\qquad \mathbb{E}(Y) = \frac{1}{3}$

$\boxed{\mathbb{D}(X)} = \frac{1}{6}\cdot(1+16+81+256+625+1296) - \left(\frac{91}{6}\right)^2 = \underline{149,14}$

$\boxed{\mathbb{D}(Y)} = \frac{2}{6}\cdot 1 + \frac{4}{6}\cdot 1 - \frac{1}{9} = \boxed{\frac{8}{9}}$

$cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

$\boxed{\mathbb{E}(XY) = 13,5}$

$\boxed{cov(X,Y)} = 13,5 - \frac{91}{6}\cdot\frac{1}{3} = \boxed{8,444}$

$\boxed{r(X,Y)} = \frac{cov(X,Y)}{\sqrt{\mathbb{D}(X)}\sqrt{\mathbb{D}(Y)}} = \frac{8,444}{\text{-- " -- " --}} = 0,733$

⑦ a) Sluč. var. X koja poprima vrijednosti u skupu {1, 2, 3...} ima geom. razdiobu s parametrom p — $G(p)$. • pri izvođenju pokusa p je $P(A)$, pokus ponavljamo u NEPROMIJENJENIM UVJETIMA DO PRVE REALIZACIJE dog. A. X – broj pokusa u kojem se real. A

$p_k = P(X=k) = p(1-p)^{k-1}$

b) $\boxed{P(X=k+m \mid X>k)} = \frac{P(X=k+m,\ X>k)}{P(X>k)} \overset{k,m \geq 1}{=} \frac{P(X=k+m)}{P(X>k)} = \frac{p(1-p)^{k+m-1}}{(1-p)^k} = p(1-p)^{m-1} = \underline{P(X=m)}$

⑧ a) $\vartheta_X(t) = \sum_{k=0}^{\infty} p_k e^{itk} = \sum_{k=0}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{itk} = \sum_{k=0}^{\infty} \binom{n}{k} (pe^{it})^k q^{n-k} = \underline{(pe^{it}+q)^n}$

b) $\mathbb{E}(X) = -i\vartheta_X'(0)$

$\vartheta_X'(t) = n(pe^{it}+q)^{n-1}\cdot ipe^{it}$

$\vartheta_X'(0) = n(p+q)^{n-1}\cdot ip$ $\qquad \boxed{\mathbb{E}(X) = n\cdot p}$

MI1 07

① u žari 2Z, 3C, 4P kuglice

a) izvlačimo na sreću 2

skup elem. događaja – sve moguće kombinacije od 2 boje

$\Omega = \{ZZ, ZC, ZP, CC, CP, PP\}$, nisu jednako vjerojatni

$$\frac{\binom{2}{2}}{\binom{9}{2}} \quad \frac{\binom{2}{1}\binom{3}{1}}{\binom{9}{2}} \quad \frac{\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} \quad \frac{\binom{3}{2}}{\binom{9}{2}} \quad \frac{\binom{3}{1}\binom{4}{1}}{\binom{9}{2}} \quad \frac{\binom{4}{2}}{\binom{9}{2}}$$

raznobojne $\frac{6+8+12}{9\cdot 4} = \boxed{\frac{13}{18}}$

b) ponavljamo $10x$, $p(\text{bar } 2x \text{ istobojne}) = ?$ → binomna

A – {izvukli smo dvije istobojne} $\Rightarrow p = \frac{5}{18}$

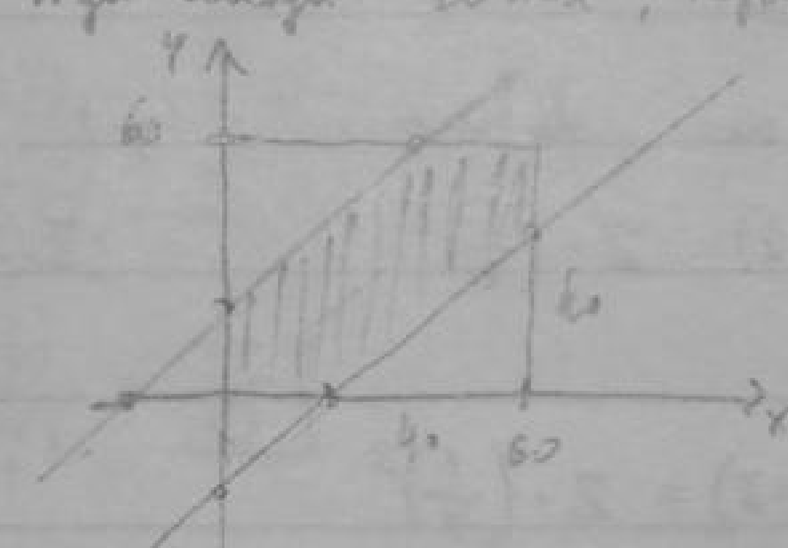
$p = 1 - P(X=0) - P(X=1) = 1 - \binom{10}{0} p^0 \left(\frac{13}{18}\right)^{10} - \binom{10}{1}\left(\frac{5}{18}\right)^1\left(\frac{13}{18}\right)^9 = \underline{0{,}8129}$

② A i M između 20 i 21h. Na trgu čekaju 20 min, najkasnije do 21

$|X - Y| < 20$ $\quad [0 - 60]$

$X - Y < 20 \;\cap\; X - Y > -20$

$Y > X - 20 \qquad Y < X + 20$



$p = \frac{60\cdot 60 - 40\cdot 40}{60\cdot 60} = \boxed{\frac{5}{9}}$

③ a) Dva događaja A i B su nezavisna ako vrijedi $P(AB) = P(A)P(B)$

(Ako vrijedi bilo koja od jednakosti $P(A|B) = P(A)$ ili $P(B|A) = P(B)$)

b) $P(AB) = P(A)P(B)$

$P(\overline{A}\,\overline{B}) = P(\overline{A+B}) = 1 - P(A+B) = 1 - P(A) - P(B) + P(AB) = 1 - P(A) - P(B) + P(A)P(B) =$

$= (1 - P(A))(1 - P(B)) = P(\overline{A})P(\overline{B})$ ■

④ $p(1) = 0{,}75$, $p(0) = 0{,}25$, 5% znakova se pogrešno interpretira $p(1 \text{ primljen, poslan})$?

A – primljen znak 1 $\qquad$ B – primljena 0

H_1 – poslan znak 1 $\qquad$ H_0 – poslana 0 $\qquad$ $p(H_1) = 0{,}75$, $p(H_0) = 0{,}25$

B – poslana 1, ako primljena 1

$p(A) = P(H_1)P(A|H_1) + P(H_0)P(A|H_0) = 0{,}75\cdot 0{,}95 + 0{,}25\cdot 0{,}05 = \boxed{\frac{29}{40}}$ $\qquad$ $p(B) = \boxed{\frac{11}{40}}$

$P(H_1|A) = \frac{P(H_1)P(A|H_1)}{P(A)} = \frac{0{,}75\cdot 0{,}95}{\frac{29}{40}} = \underline{0{,}983}$

(5)

$X \backslash Y$	0	1	
-1	1/12	1/12	$\frac{1}{6}$
0	1/12	1/4	$\frac{1}{3}$
1	1/6	1/3	$\frac{1}{2}$
	$\frac{1}{3}$	$\frac{2}{3}$	1

a) X, Y – nisu nezavisne

$X \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$ $\quad Y \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

b) $U = X^2 \sim \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ $\quad \boxed{\mathbb{E}(U) = \frac{2}{3}}$

c) $\boxed{V = X^2 + Y^2}$

$X \backslash Y$	0	1
-1	1	2
0	0	1
1	1	2

$$\boxed{V \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{12} & \frac{1}{2} & \frac{5}{12} \end{pmatrix}}$$

$W = X \cdot Y$

$X \backslash Y$	0	1
-1	0	-1
0	0	0
1	0	1

$$\boxed{W = \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{12} & \frac{7}{12} & \frac{1}{3} \end{pmatrix}}$$

d) $Z = (V, W)$

$V \backslash W$	-1	0	1	
0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
2	$\frac{1}{12}$	0	$\frac{1}{3}$	$\frac{5}{12}$
	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{1}{3}$	1

$$\sum_0 x^n = \frac{1}{1-x} \Big|'$$

$$\sum_1 n x^{n-1} = \frac{1}{(1-x)^2}$$

(6) Flip a coin dok ne dobijamo 2put zaredom isti ishod

$\omega_{2P} = PP$ $\qquad p(X=2) = 2 \cdot \left(\frac{1}{2}\right)^2$

$\omega_{2G} = GG$

$\omega_{3P} = GPP$ $\qquad p(X=3) = 2 \cdot \left(\frac{1}{2}\right)^3$

$\omega_{3G} = PGG$

$\omega_{4P} = PGPP$

$\omega_{4G} = \underbrace{GP}_{n}GG$

$\vdots$

$\omega_{NP} = PG \dots GPP$

$\omega_{NG} = \underbrace{GP \dots P}_{n-2}GG$ $\qquad p(X=n) = 2 \cdot \left(\frac{1}{2}\right)^n$

$$\mathbb{E}(X) = \sum_{k=2}^{\infty} k \cdot 2 \cdot \left(\frac{1}{2}\right)^k = \sum_{k=2}^{\infty} k \cdot \left(\frac{1}{2}\right)^{k-1} =$$

$$= \sum_1 k \left(\frac{1}{2}\right)^{k-1} - 2 \cdot \left(\frac{1}{2}\right)^{2-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} - 1 = \underline{3}$$

(7) a) Disperzija s.v. X $\qquad \mathbb{D}(X) = \mathbb{E}\left[(X - m_X)^2\right] = \mathbb{E}(X^2) - \mathbb{E}^2(X)$

b) $\mathbb{D}(X+Y) = \mathbb{E}\left[(X+Y)^2\right] - \mathbb{E}^2[X+Y] = \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - \left[\mathbb{E}(X) + \mathbb{E}(Y)\right]\left[\mathbb{E}(X) + \mathbb{E}(Y)\right] =$

$= \underline{\mathbb{E}(X^2)} + 2\mathbb{E}(XY) + \underline{\mathbb{E}(Y^2)} - \underline{\mathbb{E}^2(X)} - \underline{\mathbb{E}^2(Y)} - 2\mathbb{E}(X)\mathbb{E}(Y) =$

$= \underline{\mathbb{D}(X) + \mathbb{D}(Y) + 2\operatorname{cov}(X, Y)}$

(8) a) $\vartheta_X(t) = \sum p_k e^{itk} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{-\lambda} e^{\lambda e^{it}} = \underline{e^{\lambda(e^{it}-1)}}$

b) $\lambda_1 = 3$, $\lambda_2 = 4$ $\qquad P(X+Y) = 10$

$\lambda = \lambda_1 + \lambda_2 = 7$ $\qquad P(X+Y=10) = \frac{7^{10}}{10!} e^{-7} = \underline{0{,}071}$

$\boxed{MI\,1\ \ 08}$

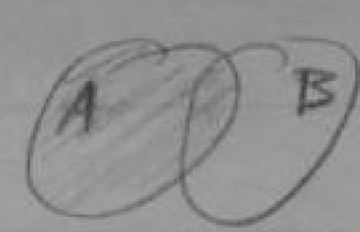
① a) Preslikavanje $\mathbb{P}: \mathcal{F} \to [0,1]$ definirano na $\mathcal{F}$, sa svojstvima

1) $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$

2) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

3) A i B disjunktni $\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

b) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$



$A \cup B = A \cup B\bar{A}$ } disjunktni

$B = AB \cup B\bar{A}$

$\mathbb{P}(B) = \mathbb{P}(AB) + \mathbb{P}(B\bar{A})$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B\bar{A})$

$\underline{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)}$

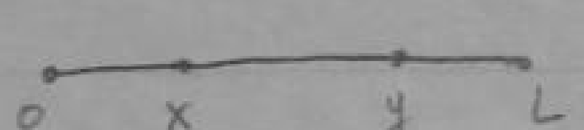
② 12 ping-pong balls, 4 defektna. Na sreću vadimo 7 komada

a) točno 1 defektna $\quad p_1 = \dfrac{\binom{8}{6}\cdot\binom{4}{1}}{\binom{12}{7}} = \dfrac{14}{99}$

b) max 1 defektna $\quad p_2 = p_1 + \dfrac{\binom{8}{7}}{\binom{12}{7}} = \dfrac{5}{33}$

c) barem 1 defektna $\quad p = 1 - \dfrac{\binom{8}{7}}{\binom{12}{7}} = \dfrac{98}{99}$

③ štap L prelomljen na 3 dijela, p(svaki dulji od $\frac{1}{4}L$)

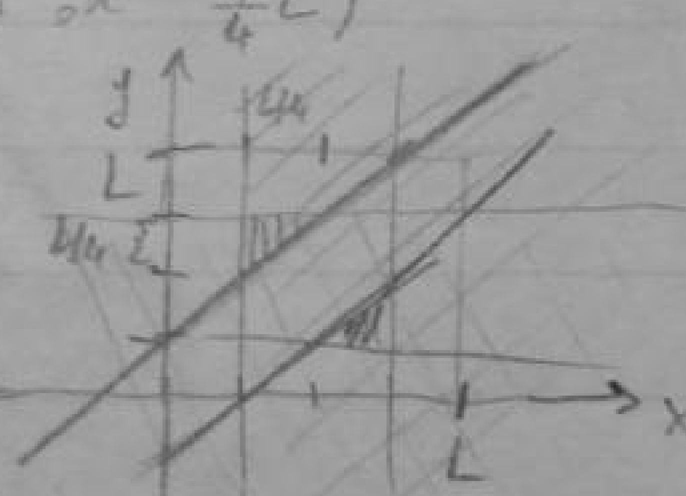


$x > \frac{1}{4}L$

$y - x > \frac{1}{4}L \quad y > x + \frac{1}{4}L$

$L - y > \frac{1}{4}L$

$y < \frac{3}{4}L$



$y > \frac{1}{4}L$

$x - y > \frac{1}{4}L$

$L - x > \frac{1}{4}L$

$p = \dfrac{m(A)}{m(\Omega)} = \dfrac{\left(\frac{1}{4}\right)^2 \cdot \frac{1}{2} \cdot 2}{L^2} = \dfrac{2}{32} = \dfrac{1}{16}$

④ Kocka na sreću jednom, potom onoliko puta koji broj padne. Pale točno 2 petice, p(u prvom)

A – pale 2 petice

$X_1 = 1 \quad \frac{1}{6} \quad p(A) = 0$

$X_2 = 2 \quad \frac{1}{6} \quad p(A) = \left(\frac{1}{6}\right)^2$

$X_3 = 3 \quad \frac{1}{6} \quad p(A) = \left(\frac{1}{6}\right)^2 \frac{5}{6} \cdot \binom{3}{2}$

$X_4 = 4 \quad \frac{1}{6} \quad p(A) = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 \cdot \binom{4}{2}$

$X_5 = 5 \quad \frac{1}{6} \quad p(A) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4 \binom{5}{1}$

$X_6 = 6 \quad \frac{1}{6} \quad p(A) = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 \binom{6}{2}$

$\boxed{P(A) = 0{,}13596}$

$\boxed{P(X_5 \mid A) = \dfrac{P(X_5)\,P(A \mid X_5)}{P(A)} = 0{,}4926}$

$X \backslash Y$	0	1	
0	$\frac{1}{4}$	$\frac{5}{12}$	
1	$\frac{5}{24}$	$\frac{1}{8}$	

$$X+Y \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{15}{24} & \frac{1}{8} \end{pmatrix} \qquad \boxed{\mathbb{E}(X+Y) = \frac{7}{8}}$$

$$\mathbb{D}(X) = 1\cdot\frac{15}{24} + 4\cdot\frac{1}{8} - \left(\frac{7}{8}\right)^2 = \boxed{\frac{23}{64}}$$

$X_1, X_2 \sim \mathcal{G}(p)$

a) $$\sum_{i=1}^{\infty} \mathbb{P}(X_1 = i, X_2 = n-i) = \sum_{i=1}^{n-1} \mathbb{P}(X_1=i)\,\mathbb{P}(X_2=n-i) = \sum_{i=1}^{n-1} pq^{i-1}\cdot pq^{n-i-1} = \sum_{i=1}^{n-1} pq^{n-2} = \boxed{(n-1)\,p^2 q^{n-2}} \; \mathrm{P}$$

$\mathbb{P}(X_1+X_2=n)$

$(q = 1-p)$

$n = 2, 3, \dots$

b) $\mathbb{P}(Y>3,\ p=\frac{1}{3})$ $$\mathbb{P}(Y>3) = 1 - \mathbb{P}(Y=2) - \mathbb{P}(Y=3) = 1 - 1\cdot\frac{1}{9}\left(\frac{2}{3}\right)^0 - 2\cdot\frac{1}{9}\left(\frac{2}{3}\right)^1 = \boxed{\frac{20}{27}}$$

a) $$\vartheta_X(t) = \sum_{k=0}^{\infty} p_k e^{itk} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{-\lambda} e^{\lambda e^{it}} = \boxed{e^{\lambda(e^{it}-1)}} \; \mathrm{P}$$

b) $$\vartheta_{X_1+X_2}(t) = \vartheta_{X_1}(t)\cdot\vartheta_{X_2}(t) = e^{\lambda_1(e^{it}-1)} e^{\lambda_2(e^{it}-1)} = e^{(\lambda_1+\lambda_2)(e^{it}-1)} \Rightarrow X_1+X_2 \sim \mathcal{P}(\lambda_1+\lambda_2)$$

c) $\lambda = 3$

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2) =$$

$$= 1 - \frac{3^0}{0!}e^{-3} - \frac{3^1}{1!}e^{-3} - \frac{3^2}{2!}e^{-3} = \boxed{1 - \frac{17}{2}e^{-3}} \; \mathrm{P}$$