8. FUNKCIJE SLUČAJNIH VEKTORA

motivacija: - ako znamo razdiobu od X i Y, kolika je razdioba od $Z=\psi(X,Y)$?

Zad.) Neka su X i Y nezavisne s eksponencijalnom razdiobom s parametrom 2. Nađi razdiobu od $Z=X+Y$.

$$f_X(x)=\lambda e^{-\lambda x}=2e^{-2x}, \quad x>0$$

$$f_Y(y)=2e^{-2y}, \quad y>0$$

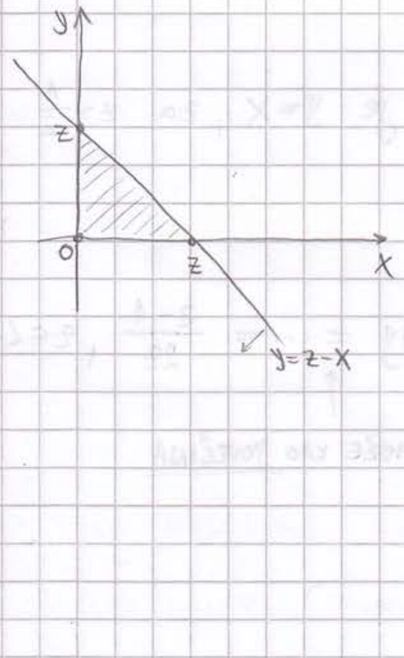
$$f(x,y)=f_X(x)\underset{\substack{\uparrow\\ \text{nezavisne}}}{\cdot} f_Y(y)=4e^{-2x-2y}$$

$z\in\langle 0,\infty\rangle$

$G(z)=?$

$$G(z)=P(Z<z)=P(\underbrace{X+Y<z}_{G})=\iint_G f(x,y)\,dx\,dy$$

$\to y<z-x$ (z ↳ konst.)



$$G(z)=\int_0^z dx\int_0^{z-x}4e^{-2x-2y}\,dy=\dots$$

$$G(z)=1-(1+2z)e^{-2z}, \quad z>0$$

$$g(z)=G'(z)=4ze^{-2z}, \quad z\in\langle 0,\infty\rangle$$

5.DZ – 15.) Slučajni vektor (X,Y) ima jednoliku razdiobu na $[0,1]\times[0,1]$.

Odredi gustoću od varijable $Z = \frac{X}{X+Y}$.

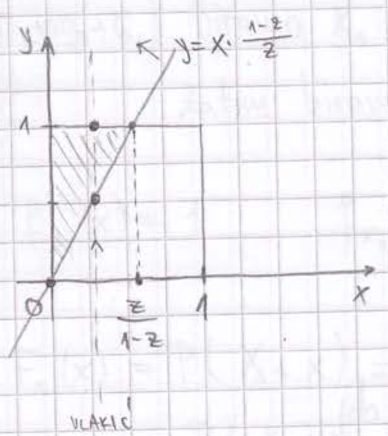
$$f(x,y) = \frac{1}{P} = \frac{1}{1} = 1 \,//$$

$$z \in [0,1]$$

$$G(z) = P(Z < z) = P\left(\frac{X}{X+Y} < z\right) =$$

$$= P\left(Y > X\cdot\frac{1-z}{z}\right)$$

1. SLUČAJ: $z \in \langle 0, \frac{1}{2} \rangle$

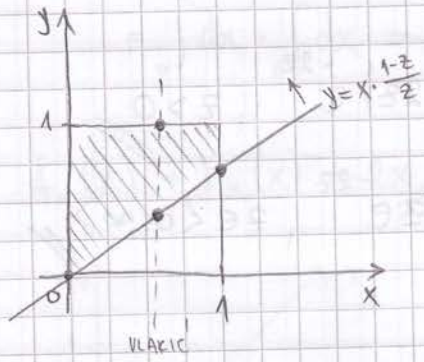


$$y = 1 = x\cdot\frac{1-z}{z} \Rightarrow x = \frac{z}{1-z} \,//$$

$$\Rightarrow G(z) = \int_0^{\frac{z}{1-z}} dx \int_{x\cdot\frac{1-z}{z}}^{1} 1\, dy = \dots = \frac{z}{2(1-z)}, \quad z \in \langle 0, \tfrac{1}{2} \rangle$$

2. SLUČAJ: $z \in \langle \frac{1}{2}, 1 \rangle$

⇝ ako je $z = \frac{1}{2}$ onda je $y = x$, za $z > \frac{1}{2}$ površina drugačija



$$\Rightarrow G(z) = \int_0^1 dx \int_{x\cdot\frac{1-z}{z}}^{1} 1\, dy = \dots = \frac{z-1}{2z}, \quad z \in \langle \tfrac{1}{2}, \dots$$

MOŽE KAO POVRŠINA

1].

$$g(z) = G'(z)$$

$$g(z) = \begin{cases} \frac{1}{2} \cdot \frac{1}{(1-z)^2}, & z \in \langle 0, \frac{1}{2} \rangle \\ \frac{1}{2} \cdot \frac{1}{z^2}, & z \in \langle \frac{1}{2}, 1 \rangle \end{cases}$$

*PODSJETNIK:

$$\text{(1D)}: \quad y=\Psi(x) \Rightarrow g(y)=f(x)\cdot\left|\frac{dx}{dy}\right|, \quad x=\Psi^{-1}(y)$$

(n-D)

⇝ znamo funkciju gustoće $f(x_1,\dots,x_n)$, kolika je $g(y_1,\dots,y_n)=?$

Vrijedi: $P\left((x_1,\dots,x_n)\in G\right)=P\left((y_1,\dots,y_n)\in G'\right)$

$$\underbrace{\int\cdots\int}_{\text{višestruki integral}} f(x_1,\dots,x_n)\,dx_1\dots dx_n=\left|\begin{array}{l} x_1=\Psi_1(y_1,\dots,y_n)\\ \vdots\\ x_n=\Psi_n(y_1,\dots,y_n)\end{array}\right|=$$

$$=\int\cdots\int_{G'} f(x_1,\dots,x_n)\cdot\underbrace{|J|}_{\text{JAKOBIJAN}}\,dy_1\dots dy_n=$$

$$=\int\cdots\int_{G'} g(y_1,\dots,y_n)\,dy_1\dots dy_n=$$

STARE VAR.

$$J=\frac{\partial(x_1,\dots,x_n)}{\partial(y_1,\dots,y_n)}=\begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n}\\ \vdots & & \vdots\\ \frac{\partial x_n}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n}\end{vmatrix}$$

NOVE VAR.

$$\Rightarrow \boxed{g(y_1,\dots,y_n)=f(x_1,\dots,x_n)\left|\frac{\partial(x_1,\dots,x_n)}{\partial(y_1,\dots,y_n)}\right|}$$

⇝ mi, **NAŽALOST**, vadimo samo slučaj $n=2$ i to ovaj specifični slučaj specifičnog slučaja

$$\boxed{\begin{array}{l} x=X\\ y=\Psi(x,z)\end{array}}$$

„BILO JEDNE GODINE IZVOD OVOG JAKOBIJANA ZA $n=2$"

$$\left|\frac{\partial(x,y)}{\partial(x,z)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = \left|\frac{\partial y}{\partial z}\right| =$$

UVRSTITI

$$\Rightarrow \boxed{g(x,z) = f(x,y) \cdot \left|\frac{\partial y}{\partial z}\right|, \quad y = \Psi(x,z)}$$

!

⇝ marginalna gustoća od z:

$$\boxed{g(z) = \int_{-\infty}^{+\infty} f(x,y) \cdot \left|\frac{\partial y}{\partial z}\right| dx}$$

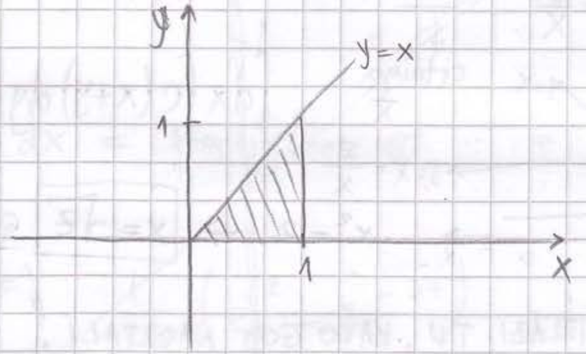
NAJVIŠE ĆEMO KORISTITI !

⇝ ZA GRANICE ⟶ BURIĆEV PRINCIP

2.MI-11-7.) Neka je zadan slučajan vektor $f(x,y) = Cx$, na intervalu $0 \leq y \leq x \leq 1$. Zadano je $z = x - y$, odredite gustoću $g(z)$.

$f(x,y) = Cx$

$0 \leq y < x \leq 1$



$$\int_0^1 dx \int_0^1 Cx\, dy = \frac{1}{3} C = 1$$

$$\Rightarrow \boxed{C = 3} \Rightarrow f(x,y) = 3x$$

1. KORAK: Koje vrijednosti z može poprimiti?

$z \in (0,1)$ (uvjet $y \leq x$!)

2. KORAK: Što je y?

$y = x - z$

3. KORAK: Derivacija.

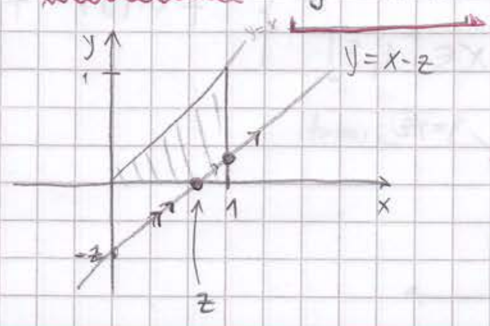
$$\left| \frac{\partial y}{\partial z} \right| = |-1| = 1$$

$$g(z) = \int_z^1 3x \cdot 1 \cdot dx$$

DA BI ODREDILI GRANICE TREBA NACRTATI $y = x - z$!!

$$g(z) = 3 \frac{x^2}{2} \Big|_z^1 = \frac{3}{2} - \frac{3}{2} z^2$$

$z \in (0,1)$



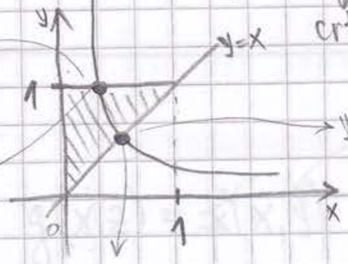
NE TREBA DIJELITI NA INTE.

ULAZI UVIJEK U z, IZLAZI U 1 !

5.DZ-19.) Neka je zadan sluč. vektor s gustoćom $f(x,y) = C(x+y)$ pri čemu je $0 \leq x < y \leq 1$. Odredi gustoću od $z = x \cdot y$.

$\rightarrow y = \frac{z}{x}$, $z \rightarrow$ const.

crtamo $\frac{1}{x}$



UVIJEK ULAZI TU

$y = 1$

$\frac{z}{x} = 1$

$\boxed{x = z}$ DONJA GRANICA !!

$$\int_0^1 dx \int_0^1 c(x+y)\,dy = \dots = \frac{1}{2}C = 1 \Rightarrow \boxed{C = 2}$$

$$y = x = \frac{z}{x}$$

$$x^2 = z \Rightarrow \boxed{x = \sqrt{z}} \text{ GORNJA GRANICA !!}$$

UVIJEK IZLAZI TU, KAKO GOD NACRTALI!

PAZI !! UVRSTI !!!

1.] $z \in (0,1)$

2.] $y = \frac{z}{x}$

3.] $\left|\frac{\partial y}{\partial z}\right| = \frac{1}{x}$

$$\Rightarrow g(z) = \int_z^{\sqrt{z}} 2 \cdot \left(x + \frac{z}{x}\right) \cdot \frac{1}{x}\,dx =$$

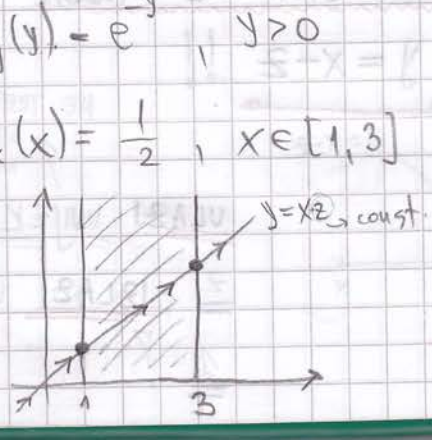
$$g(z) = \dots = 2 - 2z \qquad z \in (0,1)$$

$$\Rightarrow G(z) = \int g(z)\,dz = 2z - z^2$$

2.MI-08-7.) X i Y su nezavisne sluč. varijable. $Y \sim E(1)$, $X \sim U[1,3]$. Odredite razdiobu od $z = \frac{y}{x}$ i vjerojatnost $P\left(\frac{y}{x} < 1\right) =$?

$G(z) = ?$

$$\left.\begin{aligned} f_y(y) &= e^{-y}, \quad y > 0 \\ f_x(x) &= \frac{1}{2}, \quad x \in [1,3] \end{aligned}\right\} f(x,y) = f_x \cdot f_y = \frac{1}{2}e^{-y}$$



1.] $z \in (0, \infty)$

2.) $y = x \cdot z$

3.) $\left|\frac{\partial y}{\partial z}\right| = x$

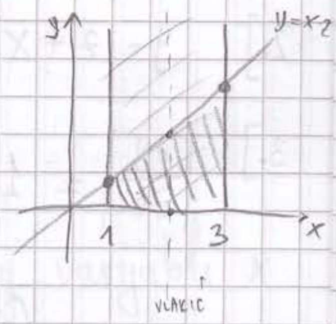
$$\Rightarrow \quad g(z) = \int_1^3 \frac{1}{2} e^{-(xz)} \cdot x \cdot dx = \text{parcijalna int. } \ldots =$$

$$= \frac{1}{2z^2}\left(e^{-z} - e^{-3z}\right) + \frac{1}{2z}\left(e^{-z} - 3e^{-3z}\right), \quad z \in (0, \infty)$$

$G(z) = ?$ $\quad \int \frac{e^{-x}}{x^2} dx$??

NE!

BOLJE PO DEFINICIJI !!



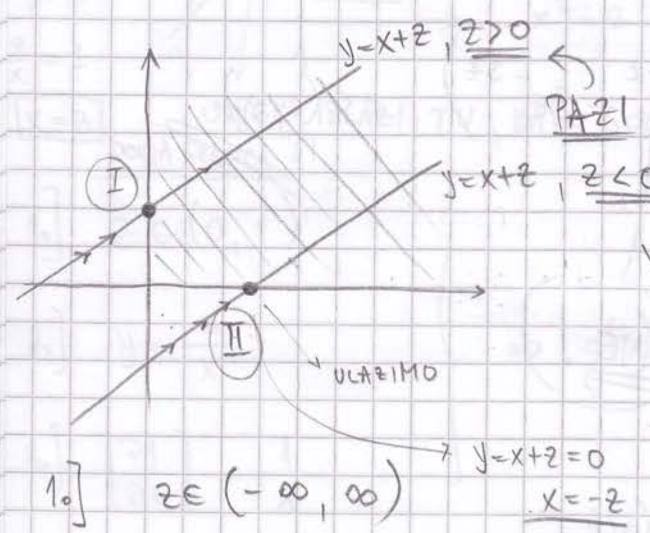
$$G(z) = P(Z < z) = P\left(\frac{y}{x} < z\right) = P(y < z \cdot x)$$

$$= \int_1^3 dx \int_0^{xz} \frac{1}{2} e^{-y} dy = 1 + \frac{1}{2z}\left(e^{-3z} - e^{-z}\right)$$

$$P\left(\frac{y}{x} < 1\right) = P(Z < 1) = G(1)$$

2.MI-10-6.) Neka su X i Y nez. sluč. var. s eksp. razdiobom s očekivanjem 2. Odredite gustoću $Z = Y - X$.

$$\left.\begin{array}{l} X \sim E\left(\frac{1}{2}\right) \\ Y \sim E\left(\frac{1}{2}\right) \end{array}\right\} f(x,y) = f_X \cdot f_Y = \frac{1}{4} e^{-\frac{1}{2}x - \frac{1}{2}y}$$



PAZI!! eksp. razd. def. samo za $x, y > 0$

$\Downarrow$

2 SLUČAJA !!

1.] $z \in (-\infty, \infty)$

2.] $y = z + x$

3.] $\left|\frac{\partial y}{\partial z}\right| = 1$

$$\Rightarrow g(z) = \int \frac{1}{4} e^{-\frac{1}{2}x - \frac{1}{2}(z+x)} \cdot 1 \cdot dx$$

(I) $g(z) = \int_0^\infty \frac{1}{4} e^{-\frac{1}{2}x - \frac{1}{2}(z+x)} dx = \dots = \frac{1}{4} e^{-\frac{1}{2}z}$, $z > 0$

(II) $g(z) = \int_{-z}^\infty \frac{1}{4} e^{-\frac{1}{2}x - \frac{1}{2}(z+x)} dx = \dots = \frac{1}{4} e^{\frac{1}{2}z}$, $z < 0$

ZADACI: 7. 8. : OD 1 - 23.

7., 8., 9. → SK