Dokazi iz Vjerojatnosti i statistike

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Sadržaj

Poglavlje 1

Vjerojatnost

1.1 Vjerojatnost komplementa

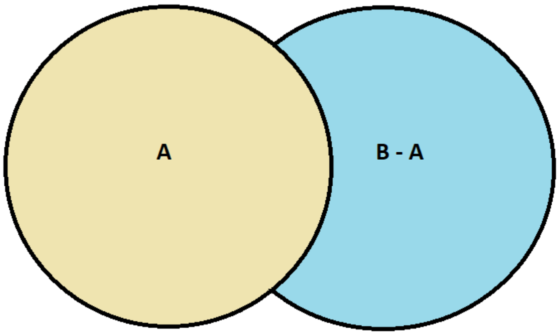
$$P(\overline{A}) = 1 - P(A)$$

Dokaz:
A je neki odabrani događaj iz skupa elementarnih događaja $P(\Omega) = 1$, (normiranost)
$A \cup \overline{A} = \Omega$, A i $\overline{A}$ disjunktni
$P(\Omega) = P(A \cup \overline{A}) = \text{(aditivnost)} = P(A) + P(\overline{A}) = 1$
$\Rightarrow P(\overline{A}) = 1 - P(A)$

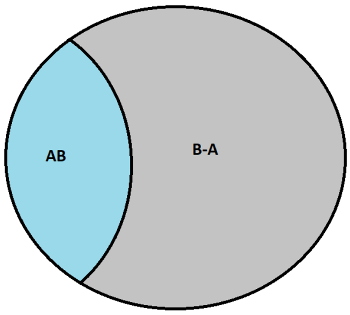
1.2 Vjerojatnost unije

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Dokaz:
A i B su bilo koja dva događaja iz skupa elementarnih događaja Podijelimo $A \cup B$ na disjunktne skupove na sljedeći način:
1)A
2)$B - A = B \cap \overline{A} = B - (A \cap B)$



$$P(A \cup B) = P(A \cup (B - A)) = (aditivnost) = P(A) + P(B - A)$$



$$P(B) = P(B - A) + P(A \cap B)$$
$$-P(B - A) = -P(B) + P(A \cap B) / \cdot (-1)$$
$$P(B - A) = P(B) - P(A \cap B)$$

sada se umjesto $P(B - A)$ uvrsti $P(B) - P(A \cap B)$ u prvu jednakost

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Poglavlje 2

Uvjetna vjerojatnost

2.1 Svojstva vjerojatnosti na uvjetnoj vjerojatnosti

1) normiranost, $P(\Omega) = 1$, $P(\emptyset) = 0$

Dokaz:

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$$P(\emptyset|B) = \frac{P(\emptyset \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0$$

2) monotonost, ako je $C \subset A$ onda vrijedi $P(C|B) \leq P(A|B)$

Dokaz:
vrijedi: $P(C \cap B) \leq P(A \cap B)$, jer je C $\subset$ A

$$P(C \cap B) \leq P(A \cap B) / \cdot \frac{1}{P(B)}, \mathbf{P(B) \neq 0}$$

$$\frac{P(C \cap B)}{P(B)} \leq \frac{P(A \cap B)}{P(B)} \Rightarrow P(C|B) \leq P(A|B)$$

3) aditivnost, ako su A i C disjunktni vrijedi $P(A \cup C|B) = P(A|B) + P(C|B)$

Dokaz:

$$P(A \cup C|B) = \frac{P((A \cup C) \cap B)}{P(B)} = \frac{P((A \cap B) \cup (C \cap B))}{P(B)} =$$

$A \cap B$, $C \cap B$ su disjunktni, vrijedi $P((A \cap B) \cup (C \cap B)) = P(A \cap B) + P(C \cap B)$

$$\frac{P(A \cap B) + P(C \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} + \frac{P(C \cap B)}{P(B)} = P(A|B) + P(C|B)$$

2.2 Nuždan i dovoljan uvjet za nezavisnost događaja

$$P(AB) = P(A) \cdot P(B)$$

Dokaz:
Događaji A i B su nezavisni ako vrijedi $P(A|B) = P(A)$ ili $P(B|A) = P(B)$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(B) \cdot P(A|B)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B)$$

2.3 Komplement dvaju nezavisnih događaja je nezavisan

$$P(\overline{A}) \cdot P(\overline{B}) = P(\overline{A} \cap \overline{B})$$

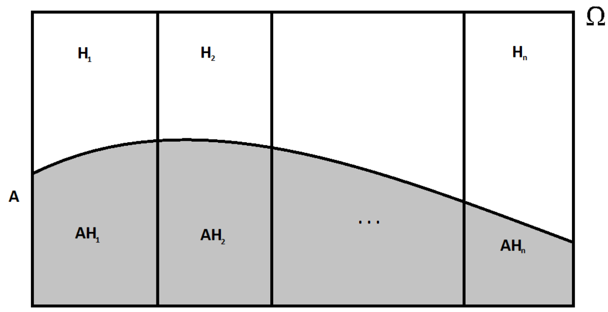
Dokaz:
A i B su nezavisni događaji iz skupa Ω

$$\begin{aligned}
P(\overline{A}) \cdot P(\overline{B}) &= (1 - P(A))(1 - P(B)) \\
&= 1 - P(A) - P(B) + P(AB) \\
&= 1 - (P(A) + P(B) - P(AB)) \\
&= 1 - (P(A) + P(B) - P(A)P(B)) \\
&= 1 - P(A \cup B) \\
&= P(\overline{A \cup B}) \\
&= P(\overline{A} \cap \overline{B})
\end{aligned}$$

2.4 Formula potpune vjerojatnosti

$$P(A) = \sum_{i=1}^{\infty} P(H_i) \cdot P(A|H_i)$$

Dokaz:
$P(H_i) > 0$, za $i \in \{1, 2, ..., n\}$
$\Omega = H_1 \cup H_2 \cup ... \cup H_n$
$H_i \cap H_j = \emptyset$, za $i \neq j$



$A = AH_1 \cup AH_2 \cup ... \cup AH_n$

vjerojatnost umnoška: $P(AH_i) = P(H_i) \cdot P(A|H_i)$

$$\begin{aligned} P(A) &= P(AH_1) + P(AH_2) + ... + P(AH_n) \\ &= P(H_1) \cdot P(A|H_1) + P(H_2) \cdot P(A|H_2) + ... + P(H_n) \cdot P(A|H_n) \\ &= \sum_{i=1}^{n} P(H_i) \cdot P(A|H_i) \end{aligned}$$

2.5 Bayesova formula

$$P(H_i|A) = \frac{P(H_i) \cdot P(A|H_i)}{\sum\limits_{j=1}^{n} P(H_j) \cdot P(A|H_j)}$$

Dokaz:

$P(H_i) > 0$, za i = 1, 2, ..., n
$\Omega = H_1 \cup H_2 \cup ... \cup H_n$
$H_i \cap H_j = \emptyset$, za $i \neq j$

$$P(A \cap H_i) = P(H_i) \cdot P(A|H_i) = P(H_i) \cdot P(A|H_i)$$

$$P(H_i|A) = \frac{P(A \cap H_i)}{P(A)} = \frac{P(H_i) \cdot P(A|H_i)}{P(A)}$$

$\mathbf{P(A) \neq 0}$, računa se pomoću formule potpune vjerojatnosti:

$$P(A) = \sum_{j=1}^{n} P(H_j) \cdot P(A|H_j)$$

uvrstimo to u izraz $P(H_i|A)$ i dobivamo Bayesovu formulu:

$$\Rightarrow P(H_i|A) \frac{P(H_i) \cdot P(A|H_i)}{\sum_{j=1}^{n} P(H_j) \cdot P(A|H_j)}$$

Poglavlje 3

Diskretne slučajne varijable i vektori

3.1 Nezavisne slučajne varijable

Slučajne varijable $X, Y : \Omega \to S$ su nezavisne ako za svaki x_k i y_j vrijedi:

$$P(X = x_k, Y = y_j) = P(X = x_k)P(Y = y_j)$$

općenito, za sve A, B $\subset$ S:

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

Dokaz:
Slučajne varijable X i Y su nezavisne i označimo A i B:

$$A = \{x_1, x_2, ..., x_n\}, B = \{y_1, y_2, ..., y_m\}$$

$$
\begin{aligned}
P(X \in A, Y \in B) &= P(X \in \{x_1, x_2, ..., x_n\}, Y \in \{y_1, y_2, ..., y_m\}) \\
&= P\Big(\bigcup_{\substack{1\leq k\leq n \\ 1\leq j\leq m}} \{X = x_k, Y = y_j\}\Big) \\
&= \sum_{\substack{1\leq k\leq n \\ 1\leq j\leq m}} P(X = x_k, Y = y_j) \\
&= \sum_{\substack{1\leq k\leq n \\ 1\leq j\leq m}} P(X = x_k)P(Y = y_j) \\
&= \sum_{1\leq k\leq n} P(X = x_k) \cdot \Big(\sum_{1\leq k\leq m} P(X = y_j)\Big) \\
&= P\Big(\bigcup_{1\leq k\leq n} \{X = x_k\}\Big) \cdot P\Big(\bigcup_{1\leq j\leq m} \{Y = y_j\}\Big) \\
&= P(X \in \{x_1, x_2, ..., x_n\})P(Y \in \{y_1, y_2, ..., y_m\}) \\
&= P(X \in A)P(Y \in B)
\end{aligned}
$$

3.2 Svojstva očekivanja

Za dvije slučajne varijable X i Y definirane na istom vjerojatnosnom prostoru vrijedi:

$$E(sX + tY) = sE(X) + tE(Y)$$

Ukoliko su X i Y nezavisne vrijedi dodatno:

$$E(XY) = E(X)E(Y)$$

Dokaz: 1) svojstvo $E(sX) = sE(X)$

$$\begin{aligned}
E(sX) &= \sum_{k=1}^{n} s x_k p_k \\
&= s x_1 p_1 + s x_2 p_2 + ... + s x_n p_n \\
&= s(x_1 p_1 + x_2 p_2 + ... + x_n p_n) \\
&= s \sum_{k=1}^{n} x_k p_k \\
&= sE(X)
\end{aligned}$$

2) svojstvo $E(X+Y) = E(X) + (EY)$

$$\begin{aligned}
E(X+Y) &= \sum_{k,j} (x_k + y_j) p_{jk} \\
&= \sum_{k,j} (x_k p_{jk} + y_j p_{jk}) \\
&= \Big(\sum_{k,j} (x_k p_{jk}) \Big) + \Big(\sum_{j,k} (y_j p_{jk}) \Big) \\
&= \sum_{k} x_k \cdot \sum_{j} p_{jk} + \sum_{j} y_j \cdot \sum_{k} p_{jk} \\
&= \sum_{k} x_k p_k + \sum_{j} y_j p_j \\
&= E(X) + E(Y)
\end{aligned}$$

konačno:

$$E(sX + tY) = E(sX) + E(tY) = sE(X) + tE(Y)$$

$$
\begin{aligned}
E(XY) &= \sum_{k,j} x_k y_j p_{jk} \\
&= (nezavisnost) \\
&= \sum_{k,j} x_k y_j p_k p_j \\
&= \Big(\sum_k x_k p_k\Big) \cdot \Big(\sum_j y_j p_j\Big) \\
&= E(X)E(Y)
\end{aligned}
$$

3.3 Disperzija slučajne varijable

Disperzija slučajne varijable X se može računati formulom:

$$D(X) = E(X^2) - E(X)^2$$

Dokaz:

$$
\begin{aligned}
D(X) &= E\Big[X - E(X)\Big]^2 \\
&= E\Big[X^2 - 2XE(X) + E(X)^2\Big] \\
&= E(X^2) - 2E(X)^2 + E(X)^2 \\
&= E(X^2) - E(X)^2
\end{aligned}
$$

3.4 Svojstva disperzije

Za slučajnu varijablu X i realni broj s vrijedi:

$$D(sX) = s^2 D(X)$$

Ako su X i Y nezavisne slučajne varijable, onda dodatno vrijedi:

$$D(X+Y) = D(X) + D(Y)$$

Dokaz:

1) svojstvo $D(sX) = s^2 D(X)$

$$\begin{aligned} D(sX) &= E\big[(sX)^2\big] - \big[E(sX)\big]^2 \\ &= E(s^2X^2) - \big[sE(X)\big]^2 \\ &= s^2E(X^2) - s^2E(X)^2 \\ &= s^2\big[E(X^2) - E(X)^2\big] \\ &= s^2D(X) \end{aligned}$$

2) svojstvo $D(X+Y) = D(X) + D(Y)$

$$\begin{aligned} D(X+Y) &= E\big[(X+Y)^2\big] - \big[E(X+Y)\big]^2 \\ &= E\big[X^2 + 2XY + Y^2\big] - \big[E(X) + E(Y)\big]^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - \big[E(X)^2 + 2E(X)E(Y) + E(Y^2)\big] \\ &= \big[E(X^2) - E(X)^2\big] + \big[E(Y^2) - E(Y)^2\big] + 2E(XY) - 2E(X)E(Y) \\ &= D(X) + D(Y) + 2E(X)E(Y) - 2E(X)E(Y) \\ &= D(X) + D(Y) \end{aligned}$$

3.5 Centrirane i normirane slučajne varijable

Za bilo koju slučajnu varijablu X vrijedi:

$$E(X-a) = E(X) - a$$

$$D(X-a) = D(X)$$

$$cov(X-a, Y-b) = cov(X, Y)$$

Za normiranu slučajnu varijablu $X^* = \frac{X-m_x}{\sigma_x} = \frac{X-E(X)}{\sqrt{D(X)}}$ vrijedi:

$$E(X^*) = 0$$

$$D(X^*) = 1$$

Dokaz:
1) svojstvo $E(X-a) = E(X) - a$

$$\begin{aligned} E(X-a) &= \sum_k (x_k - a)p_k \\ &= \sum_k x_k p_k - ap_k \\ &= \Big(\sum_k x_k p_k\Big) - \Big(\sum_k ap_k\Big) \\ &= E(X) - a\sum_k p_k \\ &= E(X) - a \end{aligned}$$

2) svojstvo $D(X-a) = D(X)$

$$\begin{aligned} D(X-a) &= E\big[(X-a)^2\big] - \big[E(X-a)\big]^2 \\ &= E\big[X^2 - 2aX + a^2\big] - \big[E(X) - a\big]^2 \\ &= E(X^2) - 2aE(X) + a^2 - \big[E(X^2) - 2aE(X) + a^2\big] \\ &= \big[E(X^2) - E(X)^2\big] + 2aE(X) - 2aE(X) + a^2 - a^2 \\ &= D(X) \end{aligned}$$

3) svojstvo $cov(X-a, Y-b) = cov(X,Y)$

$$\begin{aligned}
&cov(X-a, Y-b) = E\big[(X-a)(Y-a)\big] - \big[E(X-a)\big]\big[E(Y-a)\big] \\
&= E\big[XY - aX - aY + a^2)\big] - \big[E(X) - a\big]\big[E(Y) - a\big] \\
&= E(XY) - aE(X) - aE(Y) + a^2 - \big[E(X)E(Y) - aE(X) - aE(Y) + a^2\big] \\
&= \big[E(XY) - E(X)E(Y)\big] - aE(X) - aE(Y) + a^2 + aE(X) + aE(Y) - a^2 \\
&= cov(X,Y)
\end{aligned}$$

4) svojstvo $E(X^*) = 0$

$$E(X^*) = E\left(\frac{X - m_x}{\sigma_x}\right) = \frac{E(X) - m_x}{\sigma} = \frac{0}{\sigma} = 0$$

5) svojstvo $D(X^*) = 1$

$$D(X^*) = D\left(\frac{X - E(X)}{\sqrt{D(X)}}\right) = \frac{D\big[X - E(X)\big]}{D\big[\sqrt{D(X)}\big]} = \frac{D(X)}{D(X)} = 1$$

3.6 Svojstva normiranog koeficijenta korelacije

Za dvije normirane slučajne varijable $X^* = \frac{X-m_x}{\sigma_x}$ i $Y^* = \frac{Y-m_y}{\sigma_y}$ vrijedi:

$$r(X^*, Y^*) = r(X, Y)$$

Dokaz:

$$\begin{aligned}
r(X^*, Y^*) &= E(X^*Y^*) \\
&= E\left(\frac{X-m_x}{\sigma_x} \cdot \frac{Y-m_y}{\sigma_y}\right) \\
&= \frac{E\big[XY - m_xY - m_yX + m_xm_y\big]}{E\big[\sigma_x\sigma_y\big]} \\
&= \frac{E(XY) - m_xE(Y) - m_yE(X) + m_xm_y}{\sigma_x\sigma_y} \\
&= \frac{E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)}{\sigma_x\sigma_y} \\
&= \frac{E(XY) - E(X)E(Y)}{\sigma_x\sigma_y} \\
&= \frac{cov(X, Y)}{\sigma_x\sigma_y} \\
&= r(X, Y)
\end{aligned}$$

3.7 Disperzija zbroja slučajnih varijabli

Disperzija zbroja $S = X_1 + X_2 + ... + X_n$ slučajnih varijabli računa se formulom

$$D(S) = \sum_{i=1}^{n} D(X_i) + 2\sum_{i<j}^{n} cov(X_i, X_j)$$

Dokaz:

$m_s = m_{X1} + m_{X2} + ... + m_{Xn}$

$$\begin{aligned}
D(S) &= E\big[(S - m_s)^2\big] \\
&= E\Big(\sum_{i=1}^{n}(X_i - m_{Xi})\Big)^2 \\
&= \sum_{i=1}^{n} E(X_i - m_{Xi})^2 + \sum_{i\neq j} E\big[(X_i - m_{Xi})\big]\big[(X_j - m_{Xj})\big] \\
&= \sum_{i=1}^{n} D(X_i)^2 + 2\sum_{i<j} cov(X_i, X_j)
\end{aligned}$$

3.8 Svojstva koeficijenta korelacije

Za koeficijent korelacije vrijedi:

$$|r(X,Y)| \leq 1$$

Ako je $Y = \pm aX + b$, a, $b \in R$

$$r(X,Y) = \pm 1$$

Dokaz:

$X^* = \frac{X-m_x}{\sigma_x}$ i $Y^* = \frac{Y-m_y}{\sigma_y}$ su normirane slučajne varijable

1) svojstvo $|r(X,Y)| \leq 1$

$$\begin{aligned} D(X^* \pm Y^*) &= D(X^*) + D(Y^*) \pm 2cov(X^*, Y^*) \\ &= 1 + 1 \pm 2r(X^*, Y^*) \\ &= 2\big[1 \pm r(X,Y)\big] \end{aligned}$$

2 slučaja:

1) $D(X^* + Y^*) = 2\big[1 + r(X,Y)\big]$
$D(X^* + Y^*) \geq 0$, za vrijednosti $r(X,Y) \leq 0$ vrijedi: $r(X,Y) \in \big[-1, 0\big]$

2) $D(X^* - Y^*) = 2\big[1 - r(X,Y)\big]$
$D(X^* - Y^*) \geq 0$, za vrijednosti $r(X,Y) \geq 0$ vrijedi: $r(X,Y) \in \big[0, 1\big]$

$$\Rightarrow |r(X,Y)| \leq 1$$

2) svojstvo $r(X,Y)=\pm1$, ako je $Y=\pm aX+b$, a, $b\in R$

$Y=aX+b$
$m_Y=m_{aX+b}=E(aX+b)=aE(X)+b=am_X+b$
$\sigma_Y=\sigma_{aX+b}=\sqrt{D(aX+b)}=\sqrt{a^2D(X)}=a\sqrt{D(X)}=a\sigma_X$

$$\begin{aligned}
r(X,Y) &= r(X,aX+B)\\
&= \frac{cov(X,aX+b)}{\sigma_X\sigma_{aX+b}}\\
&= \frac{E\big[X(aX+b)\big]-m_Xm_{aX+b}}{\sigma_X\sigma_{aX+b}}=\\
&= \frac{E\big[aX^2+bX\big]-m_x(am_X+b)}{a\sigma_X^2}\\
&= \frac{aE(X^2)+bE(X)-am_X^2-bm_X}{a\sigma_X^2}\\
&= \frac{aE(X^2)+bE(X)-aE(X)^2-bE(X)}{aD(X)}\\
&= \frac{a\big[E(X^2)-E(X)^2\big]}{aD(X)}\\
&= \frac{aD(X)}{aD(X)}\\
&= 1
\end{aligned}$$

$$Y = -aX + b$$
$$m_Y = m_{-aX+b} = E(-aX+b) = -aE(X) + b = -am_X + b$$
$$\sigma_Y = \sigma_{-aX+b} = \sqrt{D(-aX+b)} = \sqrt{(-a)^2 D(X)} = a\sqrt{D(X)} = a\sigma_X$$

$$\begin{aligned}
r(X,Y) &= r(X, -aX+B) \\
&= \frac{cov(X,-aX+b)}{\sigma_X \sigma_{-aX+b}} \\
&= \frac{E\big[X(-aX+b)\big] - m_X m_{-aX+b}}{\sigma_X \sigma_{-aX+b}} = \\
&= \frac{E\big[-aX^2 + bX\big] - m_x(-am_X+b)}{a\sigma_X^2} \\
&= \frac{-aE(X^2) + bE(X) + am_X^2 - bm_X}{a\sigma_X^2} \\
&= \frac{-aE(X^2) + bE(X) + aE(X)^2 - bE(X)}{aD(X)} \\
&= \frac{-a\big[E(X^2) - E(X)^2\big]}{aD(X)} \\
&= \frac{-aD(X)}{aD(X)} \\
&= -1
\end{aligned}$$

Poglavlje 4

Primjeri diskretnih razdioba

4.1 Izvod karakteristične funkcije geometrijske razdiobe

$X \sim \mathscr{G}(p)$

$$\vartheta(t) = \frac{pe^{it}}{1 - qe^{it}}$$

Dokaz:
vrijedi $q = 1 - p$

$$\begin{aligned}
\vartheta(t) &= E(e^{itX}) \\
&= \sum_k e^{itx_k} p_k \\
&= \sum_{k=1}^{\infty} e^{itk} \cdot pq^{k-1} \\
&= pe^{it} \sum_{k=1}^{\infty} e^{it(k-1)} q^{k-1} \\
&= pe^{it} \sum_{k=0}^{\infty} e^{itk} q^k \\
&= pe^{it} \sum_{k=0}^{\infty} \left(qe^{it}\right)^k \\
&= \frac{pe^{it}}{1 - qe^{it}}
\end{aligned}$$

4.2 Očekivanje geometrijske razdiobe

$X \sim \mathscr{G}(p)$

$$E(X) = \frac{1}{p}$$

Dokaz:
vrijedi $q = 1 - p$, $p = 1 - q$

$$\begin{aligned}
E(X) &= \frac{\vartheta^{(1)}(0)}{i^1} \cdot \frac{i}{i} \\
&= -i \cdot \vartheta^{(1)}(0) \\
&= -i \cdot \left[\frac{(pe^{it})'(1 - qe^{it}) - pe^{it}(1 - qe^{it})'}{(1 - qe^{it})^2} \right] \\
&= -i \cdot \left[\frac{(ipe^{it} - ipqe^{2it}) - pe^{it}(-iqe^{it})}{(1 - qe^{it})^2} \right] \\
&= -i \cdot \left[\frac{ipe^{it} - ipqe^{2it} + ipqe^{2it}}{(1 - qe^{it})^2} \right] \\
&= \frac{pe^{it}}{(1 - qe^{it})^2} \\
&= \left[t = 0 \right] \\
&= \frac{p}{1 - q^2} \\
&= \frac{p}{p^2} \\
&= \frac{1}{p}
\end{aligned}$$

4.3 Disperzija geometrijske razdiobe

$X \sim \mathscr{G}(p)$

$$D(X) = \frac{1-p}{p^2}$$

Dokaz:
vrijedi $q = 1 - p$, $p = 1 - q$

$$\begin{aligned}
E(X^2) &= \frac{\vartheta^{(2)}(0)}{i^2} \\
&= -\Big[\vartheta^{(1)}\Big]' \\
&= -\Big[\frac{ipe^{it}}{(1-qe^{it})}\Big]' \\
&= -\Big[\frac{ip \cdot ie^{it}(1-qe^{it})^2 - ipe^{it}(-2qie^{it} + 2q^2ie^{2it})}{(1-qe^{it})^4}\Big] \\
&= \Big[t = 0\Big] \\
&= -\Big[\frac{-p(1-q)^2 - ip(-2qi + 2q^2i)}{(1-q)^4}\Big] \\
&= -\Big[\frac{-p \cdot p^2 - 2pq + 2pq^2}{p^4}\Big] \\
&= -\Big[\frac{-p^3 - 2p(1-p) + 2p(1-p)^2}{p^4}\Big] \\
&= -\Big[\frac{-p^3 - 2p(1-p) + 2p(1-2p+p^2)}{p^4}\Big] \\
&= -\Big[\frac{-p^3 - 2p + 2p^2 + 2p - 4p^2 + 2p^3}{p^4}\Big] \\
&= -\frac{p^3 - 2p^2}{p^4} \\
&= -\frac{p^2(p-2)}{p^2 \cdot p^2} \\
&= -\frac{p-2}{p^2} \\
&= \frac{2-p}{p^2}
\end{aligned}$$

$$D(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

4.4 Odsutstvo pamćenja geometrijske razdiobe

Slučajna varijabla X koja poprima vrijednosti u skupu $\{1, 2, 3, ...\}$ ima geometrijsku razdiobu onda i samo onda ako vrijedi za sve $k, m \geq 1$

$$P(X = k + m | X > k) = P(X = m)$$

Dokaz:
vrijedi $q = 1 - p$, $p = 1 - q$

$$P(X = k + m | X > k) = \frac{P(X = k + m, X > k)}{P(X > k)} = \frac{P(X = k + m)}{P(X > k)}$$

$$= \frac{p \cdot q^{k+m-1}}{q^k} = \frac{q^k}{q^k} \cdot p \cdot q^{m-1}$$

$$= P(X = m)$$

4.5 Odsutstvo pamćenja geometrijske razdiobe iz starog ispita

$X \sim \mathscr{G}(p)$

$$P(X \leq k+m|X > k) = P(X \leq m)$$

Dokaz:
vrijedi $q = 1-p$, $p = 1-q$

$$P(X \leq k+m|X > k) = \frac{P(X \leq k+m, X > k)}{P(X > k)} = \frac{P(X \leq k+m) - P(X \leq k)}{P(X > k)}$$

$$= \frac{1 - P(X > k+m) - \big[1 - P(X > k)\big]}{P(X > k)} =$$

$$= \frac{1 - q^{k+m} - 1 + q^k}{q^k} = \frac{q^k(1-q^m)}{q^k}$$

$$= 1 - q^m = 1 - P(X > m) = P(X \leq m)$$

4.6 Razdioba minimuma kod geometrijske razdiobe

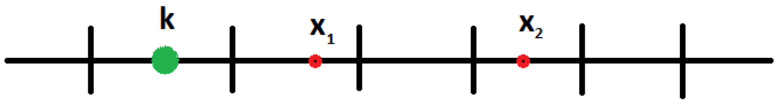
Ponavlja se pokus do realizacije bilo kojeg od 2 nezavisna događaja: A_1 ili A_2.
Ako su X_1 i X_2 međusobno nezavisne i distribuirane po geometrijskom zakonu s parametrima $p1$ i $p2$

$$X_1 \sim \mathscr{G}(p_1), X_2 \sim \mathscr{G}(p_2)$$

onda slučajna varijabla X ima geometrijsku razdiobu s parametrom $1-(1-p_1)(1-p_2)$:

$$X \sim \mathscr{G}\big(1-(1-p_1)(1-p_2)\big)$$

Dokaz:



$$P(X>k) = P(min\{X_1, X_2\} > k) = P(X_1 > k, X_2 > k)$$

$$= \big[nezavisnost\big] = P(X_1 > k)P(X_2 > k)$$

$$= q_1^{k-1} \cdot q_2^{k-1} = (q_1 q_2)^{k-1}$$

$$= \Bigg[q = q_1 q_2\Bigg] = q^{k-1}$$

$$P(X=k) = P(X>k-1) - P(X>k) =$$

$$= q^{k-1} - q^k = q^{k-1}(1-q) =$$

$$= (q_1 q_2)^{k-1}(1 - q1q2) =$$

$$= \Big[(1-p_1)(1-p_2)\Big]^{k-1} \cdot \Big(1-(1-p_1)(1-p_2)\Big)$$

$$\Rightarrow min\{X_1, X_2\} \sim \mathscr{G}\Big(1-(1-p_1)(1-p_2)\Big)$$

4.7 Izvod karakteristične funkcije binomne razdiobe

$X \sim \mathscr{B}(n,p)$

$$\vartheta(t) = (pe^{it}+q)^n$$

Dokaz:
vrijedi $q = 1-p$, $p = 1-q$

$$\begin{aligned}
\vartheta(t) &= E(e^{itX}) \\
&= \sum_k e^{itx_k} p_k \\
&= \sum_{k=0}^{n} e^{itk} \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k q^{n-k} \\
&= (pe^{it}+q)^n
\end{aligned}$$

4.8 Očekivanje binomne razdiobe

$X \sim \mathscr{B}(n, p)$

$$E(X) = np$$

Dokaz:
vrijedi $q = 1 - p$, $p = 1 - q$

$$\begin{aligned}
E(X) &= \frac{\vartheta^{(1)}(0)}{i^1} \cdot \frac{i}{i} \\
&= -i \cdot \vartheta^{(1)}(0) \\
&= -i \cdot \left[(pe^{it} + q)^n \right]' \\
&= -i \cdot \left[n \cdot (pe^{it} + q)^{n-1} \cdot ipe^{it} \right] \\
&= n(pe^{it} + q)^{n-1} \cdot pe^{it} \\
&= \left[t = 0 \right] \\
&= n(p + q)^{n-1} \cdot p \\
&= n \cdot (p + 1 - p)^{n-1} \cdot p \\
&= np
\end{aligned}$$

4.9 Disperzija binomne razdiobe

$X \sim \mathscr{B}(n,p)$

$$D(X) = npq$$

Dokaz:
vrijedi $q = 1-p$, $p = 1-q$

$$\begin{aligned}
E(X^2) &= \frac{\vartheta^{(2)}(0)}{i^2} \\
&= -\Big[\vartheta^{(1)}\Big]' \\
&= -\Big[inpe^{it}(pe^{it}+q)^{n-1}\Big]' \\
&= -\Big[npi^2e^{it}\cdot(pe^{it}+q)^{n-1} + npe^{it}\cdot(n-1)(pe^{it}+q)^{n-2}\cdot i^2pe^{it}\Big] \\
&= \Big[t=0\Big] \\
&= -\Big[npi^2(p+q)^{n-1} + np\cdot(n-1)\cdot(p+q)^{n-2}\cdot i^2p\Big] \\
&= -\Big[npi^2 + np^2i^2(n-1)\Big] \\
&= -npi^2 - np^2i^2(n-1) \\
&= np + np^2(n-1)
\end{aligned}$$

$$D(X) = E(X^2) - E(X)^2 = np + n^2p^2 - np^2 - n^2p^2 = np - np^2 = np(1-p) = npq$$

4.10 Stabilnost binomne razdiobe

$X_1 \sim \mathscr{B}(n_1, p), X_2 \sim \mathscr{B}(n_2, p)$, međusobno nezavisne

$$X_1 + X_2 \sim \mathscr{B}(n_1 + n_2, p)$$

Dokaz:
$\vartheta_{X1}(t) = (pe^{it} + q)^{n_1}$

$\vartheta_{X2}(t) = (pe^{it} + q)^{n_2}$

$$\begin{aligned} \vartheta_{X_1+X_2}(t) &= \Big[nezavisnost \Big] \\ &= \vartheta_{X_1}(t) \cdot \vartheta_{X_2}(t) \\ &= (pe^{it} + q)^{n_1} \cdot (pe^{it} + q)^{n_2} \\ &= (pe^{it} + q)^{n_1+n_2} \\ &\Rightarrow X_1 + X_2 \sim \mathscr{B}(n_1 + n_2, p) \end{aligned}$$

4.11 Karakteristike binomne preko Bernoullijevih sluč. var.

Bernoullijeva slučajna varijabla:

$$X_i \sim \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

Označimo $X = X_1 + X_2 + ... + X_n$, ako su sve X_i međusobno nezavisne, vrijedi:

$$X \sim \mathscr{B}(n, p)$$

$$E(X) = np$$

$$D(X) = npq$$

Dokaz:

$$\vartheta_{X_1} = \vartheta_{X_2} = ... = \vartheta_{X_n} = \sum_{k=0}^{1} e^{itk} p^k q^{1-k} = q \cdot e^{it\cdot 0} + p \cdot e^{it\cdot 1} = pe^{it} + q$$

$$\begin{aligned} \vartheta_X &= \vartheta_{X_1+X_2+...+X_n} \\ &= \Big[nezavisnost\Big] \\ &= \vartheta_{X_1} \cdot \vartheta_{X_2} \cdot ... \cdot \vartheta_{X_n} \\ &= (pe^{it}+q) \cdot (pe^{it}+q) \cdot ... \cdot (pe^{it}+q) \\ &= (pe^{it}+q)^n \\ &\Rightarrow X \sim \mathscr{B}(n,p) \end{aligned}$$

$$\begin{aligned} E(X) &= E(X_1 + X_2 + ... + X_n) \\ &= E(X_1) + E(X_2) + ... + E(X_n) \\ &= n \cdot E(X_i) \\ &= n \cdot \sum_{k=0}^{1} k \cdot p^k q^{1-k} \\ &= n \cdot \Big[0 \cdot p^0 q^1 + 1 \cdot p^1 q^0\Big] \\ &= np \end{aligned}$$

$$D(X_i) = E(X_i^2) - E(X_i)^2 = 0^2 \cdot p^0 q^1 + 1^2 \cdot p^1 q^0 - p^2 = p - p^2 = p(1-p) = pq$$

$$\begin{aligned} D(X) &= D(X_1 + X_2 + ... + X_n) \\ &= \Big[nezavisnost\Big] \\ &= D(X_1) + D(X_2) + ... + D(X_n) \\ &= n \cdot D(X_i) \\ &= npq \end{aligned}$$

4.12 Najvjerojatnija realizacija binomne

$$(n+1)p - 1 \leq k \leq (n+1)p$$

Dokaz:
$p_0 \leq p_1 \leq ... \leq p_{k-1} \leq p_k$
$p_n \leq p_{n-1} \leq ... \leq p_{k-1} \leq p_k$

$$1)\frac{p_k}{p_{k-1}} \geq 1$$

$$\frac{p_k}{p_{k-1}} = \frac{\frac{n!}{k\cdot(k-1)!(n-k)!}}{\frac{n!}{(k-1)!\cdot(n-k+1)(n-k)!}} \cdot \frac{p^k q^{n-k}}{p^{k-1}q^{n-k+1}} = \frac{n-k+1}{k} \cdot \frac{p}{q} \geq 1 / \cdot kq$$

$$\begin{aligned} np - kp + p &\geq kq \\ -kp - kq &\geq -np - p / \cdot (-1) \\ kp + kq &\leq np + p \\ k &\leq \frac{np+p}{p+q} \\ \Rightarrow k &\leq (n+1)p \end{aligned}$$

$$2)\frac{p_k}{p_{k+1}} \geq 1$$

$$\frac{p_k}{p_{k+1}} = \frac{\frac{n!}{k!\cdot(n-k)(n-k-1)!}}{\frac{n!}{(k+1)k!\cdot(n-k-1)!}} \cdot \frac{p^k q^{n-k}}{p^{k+1}q^{n-k-1}} = \frac{k+1}{n-k} \cdot \frac{q}{p} \geq 1 / \cdot (n-k)p$$

$$\begin{aligned} kq + q &\geq np - kp \\ kq + kp &\geq np - q \\ k(q+p) &\geq np - q \\ k &\geq np - q \\ k &\geq np - 1 - p \\ \Rightarrow k &\geq (n+1)p - 1 \end{aligned}$$

$$(n+1)p - 1 \leq k \leq (n+1)p$$

4.13 Aproksimacija binomne razdiobe Poissonovom

Neka je n velik, a p malen. $\lambda =$ np, vrijedi aproksimacija:

$$\binom{n}{k} p^k q^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}$$

Dokaz:
$\lambda = np$

$$\begin{aligned}
\binom{n}{k} p^k q^{n-k} &= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \lambda^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{1}{k!} \cdot 1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \lambda^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&\to \frac{\lambda^k}{k!} e^{-\lambda}
\end{aligned}$$

4.14 Izvod karakteristične funkcije Poissonove razdiobe

$X \sim \mathscr{P}(\lambda)$

$$\vartheta(t) = e^{\lambda(e^{it}-1)}$$

Dokaz:

$$
\begin{aligned}
\vartheta(t) &= E(e^{itX}) \\
&= \sum_k e^{itx_k} p_k \\
&= \sum_{k=0}^{\infty} e^{itk} \cdot \frac{\lambda^k}{k!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} \\
&= e^{-\lambda} \cdot e^{\lambda e^{it}} \\
&= e^{\lambda e^{it} - \lambda} \\
&= e^{\lambda(e^{it} - 1)}
\end{aligned}
$$

4.15 Očekivanje Poissonove razdiobe

$X \sim \mathscr{P}(\lambda)$

$$E(X) = \lambda$$

Dokaz:

$$
\begin{aligned}
E(X) &= \frac{\vartheta^{(1)}(0)}{i^1} \cdot \frac{i}{i} \\
&= -i \cdot \vartheta^{(1)}(0) \\
&= -i \cdot \left[e^{-\lambda} \cdot e^{\lambda e^{it}} \right]' \\
&= -i \cdot \left[e^{-\lambda} \cdot e^{it} \lambda i e^{\lambda e^{it}} \right] \\
&= e^{-\lambda} \cdot e^{it} \cdot e^{\lambda e^{it}} \cdot \lambda \\
&= \left[t = 0 \right] \\
&= e^{-\lambda} \cdot e^{\lambda} \cdot \lambda \\
&= e^{\lambda - \lambda} \cdot \lambda \\
&= \lambda
\end{aligned}
$$

4.16 Disperzija Poissonove razdiobe

$X \sim \mathscr{P}(\lambda)$

$$D(X) = \lambda$$

Dokaz:

$$\begin{aligned}
E(X^2) &= \frac{\vartheta^{(2)}(0)}{i^2} \\
&= -\Big[\vartheta^{(1)}\Big]' \\
&= -\Big[e^{-\lambda} \cdot e^{it} \lambda i e^{\lambda e^{it}}\Big]' \\
&= -e^{-\lambda} \cdot \lambda i \Big[e^{it} \cdot e^{\lambda e^{it}}\Big]' \\
&= -e^{-\lambda} \cdot \lambda i \Big[i e^{it} \cdot e^{\lambda e^{it}} + i\lambda e^{it} \cdot e^{it} \lambda e^{\lambda e^{it}}\Big] \\
&= -e^{-\lambda} \cdot \lambda i \cdot i e^{it + \lambda e^{it}} - e^{-\lambda} \cdot \lambda i \cdot i \lambda e^{2it + \lambda e^{it}} \\
&= \Big[t = 0\Big] \\
&= e^{-\lambda} \cdot \lambda e^{\lambda} + e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda} = \\
&= \lambda + \lambda^2
\end{aligned}$$

$$D(X) = E(X^2) - E(X)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

4.17 Stabilnost Poissonove razdiobe

Ako su $X_1 \sim \mathscr{P}(\lambda_1), X_2 \sim \mathscr{P}(\lambda_2)$ nezavisne slučajne varijable onda vrijedi:

$$X_1 + X_2 \sim \mathscr{P}(\lambda_1 + \lambda_2)$$

Dokaz:
$\vartheta_{X_1}(t) = e^{\lambda_1(e^{it}-1)}$
$\vartheta_{X_2}(t) = e^{\lambda_2(e^{it}-1)}$

$$\begin{aligned}
\vartheta_{X_1+X_2}(t) &= \Big[nezavisnost\Big] \\
&= \vartheta_{X_1}(t) \cdot \vartheta_{X_2}(t) \\
&= e^{\lambda_1(e^{it}-1)} \cdot e^{\lambda_2(e^{it}-1)} \\
&= e^{\lambda_1(e^{it}-1)+\lambda_2(e^{it}-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^{it}-1)} \\
&\Rightarrow X_1 + X_2 \sim \mathscr{P}(\lambda_1 + \lambda_2)
\end{aligned}$$

Poglavlje 5

Neprekinute slučajne varijable

5.1 Temeljno svojstvo funkcije razdiobe

Za sve realne brojeve a, b, $a < b$ vrijedi:

$$P(\{a \leq X < b\}) = F(b) - F(a)$$

Dokaz:

$$\begin{aligned} F(b) &= P(\{X < b\}) = P(\{X < a\} \cup \{a \leq X < b\}) \\ &= P(\{X < a\}) + P(\{a \leq X < b\}) \\ &= F(a) + P(\{a \leq X < b\}) \\ &\Rightarrow P(\{a \leq X < b\}) = F(b) - F(a) \end{aligned}$$

5.2 Izvod jednolike razdiobe

$$f(x) = \frac{1}{b-a}, \quad F(x) = \frac{x-a}{b-a}$$

Dokaz:
X uzima vrijednosti iz intervala $[a, b]$

$$1) \quad 1 = P(a \leq X < b) = F(b) - F(a) = K(b-a)$$
$$2) \quad P(a \leq X < x) = F(x) - F(a) = K(x-a)$$
$$3) \quad P(X < a) = F(a) = 0$$
$$1) \quad 1 = K(b-a) \Rightarrow K = f(x) = \frac{1}{b-a}$$
$$2) \quad P(a \leq X < x) = F(x) = K(x-a) = \frac{x-a}{b-a}$$

5.3 Karakteristična funkcija jednolike razdiobe

$X \sim \mathscr{U}(a, b)$

$$\vartheta(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}$$

Dokaz:

$$\vartheta(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx = \int_{a}^{b} e^{itx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_{a}^{b} e^{itx} dx = \left[\quad k = itx, \qquad \frac{dk}{it} = dx \quad \right]$$

$$= \frac{1}{b-a} \int e^{k} \frac{dk}{it} = \frac{1}{it(b-a)} \int e^{k} dk$$
$$= \frac{1}{it(b-a)} e^{itx} \Big|_{a}^{b} = \frac{e^{itb} - e^{ita}}{it(b-a)}$$

5.4 Očekivanje jednolike razdiobe

$X \sim \mathscr{U}(a,b)$

$$E(X) = \frac{b+a}{2}$$

Dokaz:

$$E(X) = \int_a^b xf(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{1}{b-a}\int_a^b xdx = \frac{1}{b-a}\cdot\frac{x^2}{2}\Big|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$$

5.5 Disperzija jednolike razdiobe

$X \sim \mathscr{U}(a,b)$

$$D(X) = \frac{(b-a)^2}{12}$$

Dokaz:

$$E(X^2) = \int_a^b x^2f(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{b-a}\int_a^b x^2dx = \frac{1}{b-a}\cdot\frac{x^3}{3}\Big|_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$D(X) = E(X^2) - E(X)^2 = \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

5.6 Transformacija funkcije gustoće

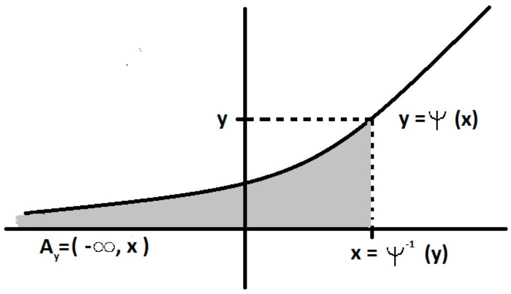
Neka je $Y = \psi(X)$. Ako je funkcija ψ monotono rastuća ili padajuća funkcija, onda vrijedi formula:

$$g(y) = f(x)\left|\frac{dx}{dy}\right|, \quad y = \psi(x)$$

tj.

$$g(y) = f(\psi^{-1}(y))\left|\frac{d\psi^{-1}(y)}{dy}\right|$$

Dokaz:

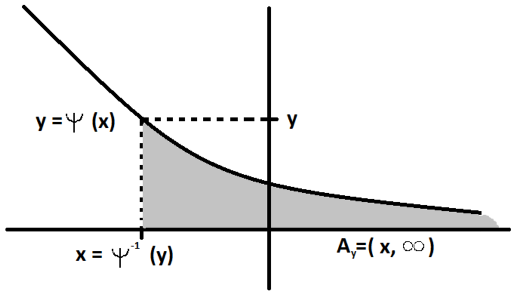


1) ψ je monotono rastuća funkcija:

$$A_y = \ \psi^{-1}\{< -\infty, y >\} \ = < -\infty, \psi^{-1}(y) > = < -\infty, x >$$

$$G(y) = P(X \in A_y) = P(X \in < -\infty, x >) = P(X < x) = F(x)$$

$$g(y) = \frac{d}{dy}G(y) = \frac{d}{dx}F(x) \cdot \frac{dx}{dy} = f(x)\frac{dx}{dy}$$



1) ψ je monotono padajuća funkcija:

$$A_y = \ \psi^{-1}\{< -\infty, y >\} \ = < \psi^{-1}(y), \infty > = < x, \infty >$$

$$G(y) = P(X \in A_y) = P(X \in < x, \infty >) = P(X > x) = -F(x)$$

$$g(y) = \frac{d}{dy}G(y) = \frac{d}{dx} - F(x) \cdot \frac{dx}{dy} = -f(x)\frac{dx}{dy}$$

Oba slučaja se mogu napisati formulom

$$g(y) = f(x)\left|\frac{dx}{dy}\right|$$

Poglavlje 6

Primjeri neprekinutih razdioba

6.1 Karakteristična funkcija eksponencijalne razdiobe

$X \sim \mathscr{E}(\lambda)$

$$\vartheta(t) = \frac{\lambda}{\lambda - it}$$

Dokaz:

$$\vartheta(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

$$= \int_{0}^{\infty} e^{itx} \cdot \lambda e^{-\lambda x} dx = \lambda \int_{0}^{\infty} e^{x(it-\lambda)} dx$$

$$= \left[\quad k = x(it - \lambda), \qquad \frac{dk}{it - \lambda} = dx \quad \right]$$

$$= \frac{\lambda}{it - \lambda} \int e^{k} dk = \frac{\lambda}{it - \lambda} \cdot e^{x(it-\lambda)} \Big|_{0}^{\infty}$$

$$= \frac{\lambda \left[e^{\infty \cdot (it-\lambda)} - e^{0 \cdot (it-\lambda)} \right]}{it - \lambda} = \frac{\lambda}{-(it - \lambda)} = \frac{\lambda}{\lambda - it}$$

6.2 Očekivanje eksponencijalne razdiobe

$X \sim \mathscr{E}(\lambda)$

$$E(X) = \frac{1}{\lambda}$$

Dokaz:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx = \lambda \int_{0}^{\infty} x e^{-\lambda x} dx \\
&= \left[\quad u = x, \quad du = dx, \qquad v' = e^{-\lambda x}, \quad v = -\frac{1}{\lambda} e^{-\lambda x} \quad \right] \\
&= \lambda \left[\frac{-x e^{-\lambda x}}{\lambda} \Bigg|_{0}^{\infty} + \frac{1}{\lambda} \int_{0}^{\infty} e^{-\lambda x} dx \right] \\
&= \ \lambda \cdot \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda x} \right] \ = \ -\frac{1}{\lambda} e^{-\infty \cdot \lambda} + \frac{1}{\lambda} e^{-0 \cdot \lambda} \ = \ \frac{1}{\lambda}
\end{aligned}$$

6.3 Disperzija eksponencijalne razdiobe

$X \sim \mathscr{E}(\lambda)$

$$D(X) = \frac{1}{\lambda^2}$$

Dokaz:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_{0}^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = \lambda \int_{0}^{\infty} x^2 e^{-\lambda x} dx$$

$$= \left[\quad u = x^2, \quad du = 2xdx, \qquad v' = e^{-\lambda x}, \quad v = -\frac{1}{\lambda} e^{-\lambda x} \quad \right]$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} \Big|_0^{\infty} - \int_0^{\infty} 2x \Big(-\frac{1}{\lambda} e^{-\lambda x} \Big) dx \right] \;=\; \lambda \left[\frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx \right]$$

$$= \left[\quad u = x, \quad du = dx, \qquad v' = e^{-\lambda x}, \quad v = -\frac{1}{\lambda} e^{-\lambda x} \quad \right]$$

$$= \;2 \left[\frac{-x e^{-\lambda x}}{\lambda} \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right] \;=\; \frac{2}{\lambda} \int_0^{\infty} e^{-\lambda x} dx$$

$$= \;\left[\quad -\lambda x = t, \qquad dx = \frac{dt}{-\lambda} \quad \right] \;=\; \frac{2}{\lambda} \left[-\frac{e^{-\lambda x}}{\lambda} \right] \Big|_0^{\infty}$$

$$= \;\frac{2}{\lambda} \left[-\frac{e^{-\infty \cdot \lambda}}{\lambda} + \frac{e^{-0 \cdot \lambda}}{\lambda} \right] \;=\; \frac{2}{\lambda} \cdot \frac{1}{\lambda} \;=\; \frac{2}{\lambda^2}$$

$$D(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

6.4 Odsutstvo pamćenja eksponencijalne razdiobe

$X \sim \mathscr{E}(\lambda)$
$x, t > 0$

$$P(X < x + t | X > t) = P(X < x)$$

Dokaz:

$$\begin{aligned}
P(X < x + t | X > t) &= \frac{P(X < x + t, X > t)}{P(X > t)} = \frac{P(t < X < x + t)}{1 - P(X < t)} \\
&= \frac{F(x + t) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda(x+t)} - \big(1 - e^{-\lambda t}\big)}{1 - \big(1 - e^{-\lambda t}\big)} \\
&= \frac{e^{-\lambda t} - e^{-\lambda(x+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda t} - e^{-\lambda x} \cdot e^{-\lambda t}}{e^{-\lambda t}} \\
&= \frac{e^{-\lambda t}\big(1 - e^{-\lambda x}\big)}{e^{-\lambda t}} = 1 - e^{-\lambda x} = \\
&= P(X < x)
\end{aligned}$$

6.5 Funkcija razdiobe normalne slučajne varijable

Funkcija razdiobe normalne slučajne varijable $X \sim \mathscr{N}(0, 1)$ je:

$$\Phi(u) = \frac{1}{2}\Big[1 + \Phi^*(u)\Big]$$

Dokaz:
Funkcija gustoće slučajne varijable $X \sim \mathscr{N}(0, 1)$:

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$$

Funkcija razdiobe:

$$\Phi(u) = \int_{-\infty}^{u} \phi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}t^2} dt$$

Svojstva funkcije gustoće:

$$\int_{-\infty}^{0} \phi(t)dt = \int_{0}^{\infty} \phi(t)dt = \frac{1}{2} \int_{-\infty}^{\infty} \phi(t)dt = \frac{1}{2}$$

$$\int_{-u}^{0} \phi(t)dt = \int_{0}^{u} \phi(t)dt = \frac{1}{2} \int_{-u}^{u} \phi(t)dt$$

Definiramo funkciju Φ^*:

$$\Phi^* = \int_{-u}^{u} \phi(t)dt$$

Računajmo funkciju razdiobe:

$$\Phi(u) = \int_{-\infty}^{u} \phi(t)dt = \int_{-\infty}^{0} \phi(t)dt + \int_{0}^{u} \phi(t)dt$$

$$= \frac{1}{2} + \frac{1}{2} \int_{-u}^{u} \phi(t)dt \;\; = \;\; \frac{1}{2}\Big[1 + \Phi^*(u)\Big]$$

6.6 Veza između jedinične i općenite normalne razdiobe

Ako vrijedi:
$X \sim \mathcal{N}(0,1), \quad a, \sigma \in R^+$
$Y = a + \sigma X$
Onda je:

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-a)^2}{2\sigma^2}}, \quad Y \sim \mathcal{N}(a, \sigma^2)$$

Ako vrijedi:
$X \sim \mathcal{N}(a, \sigma^2), \quad a, \sigma \in R^+$

$Y = \frac{X-a}{\sigma}$
Onda je:

$$g(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \quad Y \sim \mathcal{N}(0,1)$$

Dokaz:

$$\begin{aligned} a + \sigma X &= Y \\ \sigma X &= Y - a \\ X &= \frac{Y-a}{\sigma} \end{aligned}$$

$$\left|\frac{dx}{dy}\right| = \left|\frac{d(\frac{y-a}{\sigma})}{dy}\right| = \frac{1}{\sigma}$$

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = \left[X = \frac{Y-a}{\sigma}\right]$$

$$= f\left(\frac{y-a}{\sigma}\right) \cdot \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{y-a}{\sigma})^2}{2}} \cdot \frac{1}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-a)^2}{2\sigma^2}} \Rightarrow Y \sim \mathcal{N}(a, \sigma^2)$$

$$\begin{aligned} \frac{X-a}{\sigma} &= Y \\ X - a &= \sigma Y \\ X &= a + \sigma Y \end{aligned}$$

$$\left|\frac{dx}{dy}\right| = \left|\frac{d(a+\sigma y)}{dy}\right| = \sigma$$

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = \left[X = a + \sigma Y\right]$$

$$= f\big(a + \sigma y\big) \cdot \sigma = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{((a+\sigma y)-a)^2}{2\sigma^2}} \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2 y^2}{2\sigma^2}} =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \Rightarrow Y \sim \mathscr{N}(0,1)$$

6.7 Pravilo 3σ

$X \sim \mathscr{N}(a, \sigma^2)$

$$P(|X - a| < 3\sigma) = 0.9973$$

Dokaz:

$$P(|X - a| < k\sigma) = P(X^* < k) = \left[k = 3\right]$$

$$= \Phi^*(3) = 0.9973$$

6.8 Stabilnost normalne razdiobe

$X_1 \sim \mathcal{N}(a_1, \sigma_1^2), \quad X_2 \sim \mathcal{N}(a_2, \sigma_2^2), \quad s_1, s_2, c \in R$
X_1 i X_2 međusobno nezavisne

$$s_1X_1 + s_2X_2 + c \sim \mathcal{N}(s_1a_1 + s_2a_2 + c,\ s_1\sigma_1^2 + s_2\sigma_2^2)$$

Dokaz:
$\vartheta_{X_k}(t) = e^{ita_k - \frac{1}{2}\sigma^2t^2}, \quad k = 1, 2$

$\vartheta_{s_1X_1}(t) = \vartheta_{X_1}(s_1t) = e^{its_1a_1 - \frac{1}{2}s_1^2\sigma_1^2t^2}$

$\vartheta_{s_2X_2+c}(t) = e^{itc} \cdot \vartheta_{X_2}(s_2t) = e^{itc} \cdot e^{its_2a_2 - \frac{1}{2}s_2^2\sigma_2^2t^2} = e^{it(s_2a_2+c) - \frac{1}{2}s_2^2\sigma_2^2t^2}$

$$\begin{aligned}
\vartheta_{s_1X_1+s_2X_2+c}(t) &= \left[nezavisnost \right] \\
&= \vartheta_{s_1X_1}(t) \cdot \vartheta_{s_2X_2+c}(t) \\
&= e^{it(s_1a_1+s_2a_2+c) - \frac{1}{2}(s_1^2\sigma_1^2 + s_2^2\sigma_2^2)t^2}
\end{aligned}$$

$$\Rightarrow s_1X_1 + s_2X_2 + c \sim \mathcal{N}(s_1a_1 + s_2a_2 + c,\ s_1\sigma_1^2 + s_2\sigma_2^2)$$

Očekivanje:

$$\begin{aligned}
E(s_1X_1 + s_2X_2 + c) &= E(s_1X_1) + E(s_2X_2) + c \\
&= s_1E(X_1) + s_2E(X_2) + c \\
&= s_1a_1 + s_2a_2 + c
\end{aligned}$$

Disperzija:

$$\begin{aligned}
D(s_1X_1 + s_2X_2 + c) &= D(s_1X_1) + D(s_2X_2) \\
&= s_1^2E(X_1) + s_2^2E(X_2) \\
&= s_1^2\sigma_1^2 + s_2^2\sigma_2^2
\end{aligned}$$

Poglavlje 7

Slučajni vektori

7.1 Kriterij nezavisnosti za neprekinute slučajne vektore

Komponente $X_1, X_2, ..., X_n$ neprekinutog slučajnog vektora $(X_1, X_2, ..., X_n)$ su nezavisne onda i samo onda ako vrijedi:

$$f(x_1, ..., x_n) = f_1(x_1) \cdot ... \cdot f_n(x_n), \qquad \forall (x_1, ..., x_n) \in R^n$$

Dokaz:
Prvi smjer:

$$F(x_1, ..., x_n) = P(X_1 < x_1, ..., X_n < x_n)$$

$$= \left[nezavisnost\right]$$

$$= P(X_1 < x_1) \cdot ... \cdot P(X_n < x_n)$$

$$= F_1(x_1) \cdot ... \cdot F_n(x_n)$$

$$F(x_1, ..., x_n) = F_1(x_1) \cdot ... \cdot F_n(x_n) \Big/ \cdot \frac{\partial^n}{\partial x_1 \partial x_2 ... \partial x_n}$$

$$\frac{\partial^n F(x_1, ..., x_n)}{\partial x_1 \partial x_2 ... \partial x_n} = \frac{\partial F_1(x_1)}{\partial x_1} \cdot \frac{\partial F_2(x_2)}{\partial x_2} \cdot ... \cdot \frac{\partial F_n(x_n)}{\partial x_n}$$

$$\Rightarrow f(x_1, ..., x_n) = f_1(x_1) \cdot ... \cdot f_n(x_n)$$

Drugi smjer:
$G = A_1 \times A_2 \times ... \times A_n$,
A_i intervali za funkcije razdiobe slučajnih varijabli X_i

$$
\begin{aligned}
P(X_1 \in A_1, ..., X_n \in A_n) &= P((X_1, ..., X_n) \in G) \\
&= \int \underset{G}{\cdots} \int f(x_1, ..., x_n) dx_1 \cdot ... \cdot dx_n \\
&= \Bigg[nezavisnost \Bigg] \\
&= \int \underset{G}{\cdots} \int f_1(x_1) \cdot ... \cdot f_n(x_n) dx_1 \cdot ... \cdot dx_n \\
&= \int_{A_1} f_1(x_1) dx_1 \cdot ... \cdot \int_{A_n} f_n(x_n) dx_n \\
&= P(X \in A_1) \cdot ... \cdot P(X \in A_n)
\end{aligned}
$$

7.2 Svojstva očekivanja slučajnih vektora

Za svake dvije slučajne varijable $X, Y : \Omega \to R$ vrijedi:

$$E(X + Y) = E(X) + E(Y)$$

Ako su X i Y nezavisne, onda vrijedi:

$$E(XY) = E(X) \cdot E(Y)$$

Dokaz:

$$
\begin{aligned}
E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)+yf(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dydx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x\int_{-\infty}^{\infty} f(x,y)dydx + \int_{-\infty}^{\infty} y\int_{-\infty}^{\infty} f(x,y)dxdy \\
&= \left[\quad \int_{-\infty}^{\infty} f(x,y)dy = f_x(x), \quad \int_{-\infty}^{\infty} f(x,y)dx = f_y(y) \quad \right] \\
&= \int_{-\infty}^{\infty} xf_x(x)dx + \int_{-\infty}^{\infty} yf_y(y)dy \\
&= E(X)+E(Y)
\end{aligned}
$$

7.3 Disperzija zboja za nezavisne slučajne vektore

Ako su $X_1, X_2, ..., X_n$ nekorelirane slučajne varijable, tada vrijedi:

$$D(X_1+X_2+...+X_n) = D(X_1)+D(X_2)+...+D(X_n)$$

Dokaz:
vrijedi: $E(X_k)=0$

$$
\begin{aligned}
D\left(\sum_{k=1}^{n} X_k\right) &= E\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] - 0^2 \\
&= E\left(\sum_{j=1}^{n}\sum_{k=1}^{n} X_j X_k\right) \\
&= \sum_{k=1}^{n} E(X_k^2) + \sum_{j \neq k} E(X_j X_k) \\
&= \sum_{k=1}^{n} E(X_k^2) + \sum_{j \neq k} E(X_j)E(X_k) \\
&= \left[\quad E(X_k) = 0, \quad E(X_j) = 0 \quad\right] \\
&= \sum_{k=1}^{n} E(X_k^2) \\
&= \sum_{k=1}^{n} D(X_k)
\end{aligned}
$$

Poglavlje 8

Funkcije slučajnih vektora

8.1 Jakobijan transformacije kartezijevih u polarne koordinate

Jakobijan transformacije kartezijevih u polarne koordinate glasi:

$$J = r$$

Dokaz:
$x = r\ cos\ \varphi$
$y = r\ sin\ \varphi$

$$J = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} cos\ \varphi & -r\ sin\ \varphi \\ sin\ \varphi & r\ cos\ \varphi \end{vmatrix}$$

$$= cos\ \varphi \cdot r\ cos\ \varphi - sin\ \varphi \cdot (-r\ sin\ \varphi)$$

$$= r\ (cos^2\varphi + sin^2\varphi) = r$$

8.2 Izvod formule za gustoću funkcije slučajnog vektora

Gustoća slučajne varijable $Z = \psi(X, Y)$ dobiva se formulom:

$$g_z(z) = \int_{-\infty}^{\infty} f(x, y) \left| \frac{\partial y}{\partial z} \right| dx$$

Dokaz:
Sustav ovog preslikavanja je:

$$x = x$$

$$z = \psi(x, y)$$

Njegovo inverzno preslikavanje je:

$$x = x$$

$$y = \chi(x, z)$$

Jakobijan inverznog preslikavanja je:

$$J = \frac{\partial(x, y)}{\partial(x, z)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{vmatrix} = 1 \cdot \frac{\partial y}{\partial z} - 0 \cdot \frac{\partial y}{\partial x} = \frac{\partial y}{\partial z}$$

Formula gustoće vektora (X, Z) je:

$$g(x, z) = f(x, y) \left| \frac{\partial y}{\partial z} \right|$$

Konačno, integriramo funkciju gustoće $g(x, z)$ po varijabli x:

$$g_z(z) = \int_{-\infty}^{\infty} g(x, z) dx$$

$$= \int_{-\infty}^{\infty} f(x, y) \left| \frac{\partial y}{\partial z} \right| dx$$

8.3 Izvod funkcije razdiobe za min$\{X_1, X_2, ..., X_n\}$

Funkcija razdiobe slučajne varijable $min\{X_1, X_2, ..., X_n\}$ glasi:

$$F_{min\{X_1,X_2,...,X_n\}} = 1 - \left[1 - F_{X_i}(x)\right]^n$$

Dokaz:
$X_1, ..., X_n$ su međusobno nezavisne slučajne varijable s istom distribucijom

$$\begin{aligned} P(min\{X_1, X_2, ..., X_n\} < x) &= 1 - P(min\{X_1, X_2, ..., X_n\} > x) \\ &= 1 - P(X_1 > x, X_2 > x, ..., X_n > x) \\ &= \left[nezavisnost\right] \\ &= 1 - P(X_1 > x)P(X_2 > x)...P(X_n > x) \\ &= 1 - \left[1 - F_{X_1}(x)\right]\left[1 - F_{X_2}(x)\right]...\left[1 - F_{X_n}(x)\right] \\ &= 1 - \left[1 - F_{X_i}(x)\right]^n \end{aligned}$$

8.4 Izvod funkcije razdiobe za max$\{X_1, X_2, ..., X_n\}$

Funkcija razdiobe slučajne varijable $max\{X_1, X_2, ..., X_n\}$ glasi:

$$F_{max\{X_1,X_2,...,X_n\}} = F_{X_i}(x)^n$$

Dokaz:

$X_1, ..., X_n$ su međusobno nezavisne slučajne varijable s istom distribucijom

$$
\begin{aligned}
P(max\{X_1, X_2, ..., X_n\} < x) &= P(X_1 < x, X_2 < x, ..., X_n < x) \\
&= \Big[nezavisnost\Big] \\
&= P(X_1 < x)P(X_2 < x)...P(X_n < x) \\
&= \big[F_{X_1}(x)\big]\big[F_{X_2}(x)\big]...\big[F_{X_n}(x)\big] \\
&= F_{X_i}(x)^n
\end{aligned}
$$

Poglavlje 9

Zakon velikih brojeva i centralni granični teorem

9.1 Markovljeva nejednakost

Ako X poprima nenegativne vrijednosti, onda za svaki $a > 0$ vrijedi:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Dokaz:
Pretpostavimo da se radi o neprekidnoj slučajnoj varijabli:

$$E(X) = \int_0^\infty xf(x)dx \geq \int_a^\infty xf(x)dx \geq \int_0^\infty af(x)dx$$

$$= \ a \int_0^\infty f(x)dx \ = \ a\,P(X \geq a)$$

$$E(X) \geq a\,P(X \geq a) / \cdot \frac{1}{a}$$

$$\frac{E(X)}{a} \geq P(X \geq a)$$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$$

9.2 L_p nejednakost

Za slučajnu varijablu X s očekivanjem m_x i svaki $p > 0$ vrijedi:

$$P(|X - m_x| \geq a) \leq \frac{E(|X - m_x|^p)}{a^p}$$

Dokaz:
Primijeni se Markovljeva nejednakost:

$$P(|X - m_x| \geq a) = P(|X - m_x|^p \geq a^p) \leq \frac{E(|X - m_x|^p)}{a^p}$$

9.3 Čebiševljeva nejednakost

$$P(|X - m_x| \geq a) \leq \frac{D(X)}{a^2}$$

Dokaz:
Primijeni se Čebiševljeva nejednakost, $p = 2$:

$$P(|X - m_x| \geq a) \leq \frac{E(|X - m_x|^2)}{a^2} = \frac{D(X)}{a^2}$$

9.4 Dovoljni uvjeti za slabi zakon velikih brojeva

Ako varijable $X_1, X_2, \ldots$ zadovoljavaju uvjet:

$$\lim_{n\to\infty} \frac{1}{n^2} D\left(\sum_{k=1}^{n} X_k\right) = 0$$

tada on zadovoljava zakon velikih brojeva.
Taj će uvjet biti ispunjen ako su na primjer:
1) $X_1, X_2, ...$ nekorelirane, s ograničenim varijancama
2) $X_1, X_2, ...$ nezavisne s istom varijancom σ
3) $X_1, X_2, ...$ nezavisne s istom distribucijom i konačnom varijancom

Dokaz:

1)
$\varepsilon > 0$
Čebiševljeva nejednakost:

$$P\Big\{\Big|\frac{1}{n}\sum_{k=1}^{n}(X_k - E(X_k))\Big| > \varepsilon\Big\} \leq \frac{D\Big(\frac{1}{n}\sum\limits_{k=1}^{n} X_k\Big)}{\varepsilon^2}$$

$$= \frac{1}{\varepsilon^2}\frac{1}{n^2}D\Big(\sum_{k=1}^{n} X_k\Big)$$

$$\to 0, \quad kad\ n \to \infty$$

2) i 3)
Neka je $\sum\limits_{k=1}^{n} D(X_k) \leq M$

$$\frac{1}{n^2}\, D\left(\sum_{k=1}^{n} X_k\right) \;=\; \frac{1}{n^2}\,\sum_{k=1}^{n} D(X_k) \;\leq\; \frac{1}{n^2}\cdot n\cdot M \;=\; \frac{M}{n} \to 0, \quad kad\ n \to \infty$$

9.5 Centralni granični teorem

Neka je (X_n) niz identički distribuiranih nezavisnih slučajnih varijabli s očekivanjem m i varijancom σ^2. Onda za normirani zbrojii vrijedi:

$$\frac{\left(\sum_{k=1}^{n} X_k\right) - nm}{\sigma\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}(0,1)$$

Dokaz:
Pretpostavimo da vrijedi: $m = 0$
$Z_n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} X_k$

$$\begin{aligned}
\vartheta_{Z_n}(t) &= \vartheta_{\frac{1}{\sigma\sqrt{n}}X_1 + \ldots + \frac{1}{\sigma\sqrt{n}}X_n}(t) \\
&= \left[nezavisnost \right] \\
&= \vartheta_{\frac{1}{\sigma\sqrt{n}}X_1}(t) \cdot \ldots \cdot \vartheta_{\frac{1}{\sigma\sqrt{n}}X_n}(t) \\
&= \vartheta_{\frac{1}{\sigma\sqrt{n}}X_k}(t) \cdot \ldots \cdot \vartheta_{\frac{1}{\sigma\sqrt{n}}X_k}(t) \\
&= \vartheta_{\frac{1}{\sigma\sqrt{n}}X_k}(t)^n \\
&= \vartheta_{X_k}\left(\frac{t}{\sigma\sqrt{n}}\right)^n
\end{aligned}$$

Taylorov red:

$$\vartheta(t) = \vartheta(0) + \frac{\vartheta'(0)}{1!} + \frac{\vartheta''(0)}{2!} + R$$

0) $\quad \vartheta(0) = 1$

$\frac{\vartheta'(0)}{i} = E(X_i)$

1) $\vartheta'(0) = iE(X_i) = im = i \cdot 0 = 0$

$$\frac{\vartheta''(0)}{-1} = E(X_i^2)$$

2) $\vartheta''(0) = -E(X_i^2) = \left[\; E(X_i) = 0 \; \right] = -\Big[E(X_i^2) - E(X_i)^2\Big] = -D(X_i) =$
$-\sigma^2$

$$\begin{aligned}
\vartheta_{Z_n}(t) &= \left[\vartheta(0) + \frac{\vartheta'(0)}{1!} + \frac{\vartheta''(0)}{2!} + R\right]^n \\
&= \left[1 + 0 \cdot \frac{t}{1! \cdot \sigma\sqrt{n}} - \sigma^2 \cdot \frac{t^2}{2! \cdot \sigma^2 \cdot n} + R\right]^n \\
&= \left[1 - \frac{t^2}{2n} + R\right]^n \\
&\to e^{-\frac{t^2}{2}}
\end{aligned}$$

Prema Levyjevom teoremu, niz (Z_n) konvergira po distribuciji k jediničnoj normalnoj razdiobi:

$$Z_n = \frac{\left(\sum\limits_{k=1}^{n} X_k\right) - nm}{\sigma\sqrt{n}} \xrightarrow{\mathscr{D}} \mathscr{N}(0,1)$$

Poglavlje 10

Matematička statistika

10.1 Nepristrani procjenitelj za očekivanje

Statistika koja je nepristrani procjenitelj za očekivanje je:

$$\overline{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

Dokaz:
Nepristranost vrijedi ako je: $E(\overline{X}) = a$

$$\begin{aligned}
E(\overline{X}) &= E\left(\frac{X_1 + X_2 + ... + X_n}{n}\right) \\
&= \frac{E(X_1) + E(X_2) + ... + E(X_n)}{n} \\
&= \frac{n \cdot E(X_i)}{n} \\
&= E(X_i) \\
&= a
\end{aligned}$$

10.2 Valjane statistike

Statistika $\Theta_n = \Theta(X_1, X_2, ..., X_n)$ nazivamo valjanom procjenom parametra ϑ ako za svaki $\varepsilon > 0$ slučajna varijabla Θ_n konvergira prema ϑ po vjerojatnosti:

$$\lim_{n\to\infty} P(|\Theta_n - \vartheta| < \varepsilon) = 1$$

Dokaz:
Primijenimo Čebiševljevu nejednakost:

$$P(|\Theta_n - \vartheta| < \varepsilon) \geq 1 - \frac{E\big[|\Theta_n - \vartheta|\big]}{\varepsilon^2} = 1 - \frac{D(\Theta_n)}{\varepsilon^2} \to 1$$

Dovoljan uvjet za valjanost:

$$\lim_{n\to\infty} D(\Theta_n) \to 0$$

10.3 Nepristrani procjenitelj za disperziju uz nepoznato očekivanje

Statistika koja je nepristrani procjenitelj za disperziju je:

$$\Theta^* = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Dokaz:
Definiramo statistiku:

$$\Theta = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

i računamo njeno očekivanje:

$$E(\Theta) = E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\Big[(X_i - \overline{X})^2\Big]$$

$$= \Bigg[\quad X_i \ i \ \overline{X} \ zavisne, \qquad E(X_i - \overline{X}) = E(X_i) - E(\overline{X}) = 0 \quad \Bigg]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\Big[(X_i - \overline{X})^2\Big] - \Big[E(X_i - \overline{X})\Big]^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} D(X_i - \overline{X})$$

$$= \frac{1}{n}\sum_{i=1}^{n} D\left[X_i - \frac{X_1 + ... + X_{i-1} + X_i + X_{i+1} + ... + X_n}{n}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} D\left[\frac{n \cdot X_i - (X_1 + ... + X_{i-1} + X_i + X_{i+1} + ... + X_n)}{n}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} D\left[\frac{(n-1) \cdot X_i}{n} - \frac{X_1 + ... + X_{i-1} + X_{i+1} + ... + X_n}{n}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^2 \sigma^2 + D\left[\sum_{i \neq j}^{n} \frac{X_j}{n}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^2 \sigma^2 + \frac{1}{n^2} D\left[\sum_{i \neq j}^{n} X_j\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^2 \sigma^2 + \frac{n-1}{n^2}\sigma^2$$

$$= \frac{(n-1)^2}{n^2}\sigma^2 + \frac{n-1}{n^2}\sigma^2$$

$$= \frac{(n^2 - 2n + 1) + n - 1}{n^2}\sigma^2 = \frac{n^2 - n}{n^2}\sigma^2 = \frac{n(n-1)}{n^2}\sigma^2 = \frac{n-1}{n}\sigma^2$$

Ova statistika nije nepristrana pa nam je potreban koeficijent kojim pomnožiti ovu statistiku da bi bila nepristrana:

$$cE(\Theta) = \sigma^2$$

$$c = \sigma^2 \cdot \frac{1}{E(\Theta)}$$

$$c = \sigma^2 \cdot \frac{n}{\sigma^2(n-1)}$$

$$c = \frac{n}{n-1}$$

Ovo je faktor kojim moramo pomnožiti statistiku Θ da bi ona postala nepristrana:

$$\Theta^* = c \cdot \Theta$$

$$= \frac{n}{n-1} \cdot \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$