MI 2015

① Dokaži

$P: \mathcal{F} \to [0,1]$ ima svojstva:

(i) $P(\Omega) = 1$

(ii) A, B disjunktni $\longrightarrow P(A \cup B) = P(A) + P(B)$

(iii) $P(\emptyset) = 0$

(iv) $A \subset B \longrightarrow P(A) \leq P(B)$

$P(\emptyset) = 0 \Rightarrow A = \Omega, \; B = \emptyset$

A, B disj. $\longrightarrow P(A \cup B) = P(\Omega) + P(\emptyset)$

$= 1 + 0 = 1$

$A \subset B \Rightarrow P(A \cup B) = P(A) + P(B)$

$P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$

$P(B \setminus A) \geq 0 \longrightarrow P(A) \leq P(B)$

② 52 karte → vučemo 5 (poker)

13 jačina (2, 3, 4, 5, 6, 7, 8, 9, J, Q, K, A)

4 boje (♡, ♢, ♧, ♤)

a) $A = \{$Full house$\}$ (3 iste jačine + 2 neke druge iste jačine)

$$P(A) = \frac{13\binom{4}{3}12\binom{4}{2}}{\binom{52}{5}} = 1.44 \cdot 10^{-3} = 0.14\%$$

b) $B = \{$Two pair$\}$ (2 jedne jačine + 2 druge jačine + 1 različita)

$$P(B) = \frac{\binom{4}{2}\binom{4}{2} \cdot \binom{13}{2} \cdot 11 \cdot 4}{\binom{52}{5}} = 0.0475 = 4.75\%$$

c) $C = \{$One pair$\}$ (2 iste jačine, ostalo različite)

$$P(C) = \frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3}{\binom{52}{5}} = 0.4226 = 42.26\%$$

d) $D = \{$Straight$\}$ (skala, ali ne iste boje)

5 karata u 4 boje

sve karte iste boje

$$P(D) = \frac{10(4^5 - 4)}{\binom{52}{5}} = 3.924 \cdot 10^{-3} = 0.39\%$$

10 mogućih skala

③ 3 kutije

↳ #1 ⇒ 3B = C

↳ #2 i #3 ⇒ B = 3C

⇒ vrćemo 5 kuglica s vraćanjem iz slučajne kutije

⇒ izvučeno 3B i 2C → p da su iz #1 ?

$P_{(H_1|A)} = ?$

Bayesova formula

$$P(H_1|A) = \frac{P(H_1) \cdot P(A|H_1)}{P(H_1)P(A|H_1) + P(\overline{H_1})P(A|\overline{H_1})}$$

$$= \frac{\frac{1}{3}\binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2}{\frac{1}{3}\binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + \frac{2}{3}\binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2} = 0.6 = 60\%$$

uzeli kutiju #1 — 3 od 5 kuglica bijela — $P_{(bijela)}$ — $P_{(crna)}$ — obrnuto od toga

④ GEOMETRIJSKA RAZDIOBA

$$P(X=k) = p(1-p)^{k-1}$$

a) $$E(X) = \sum_{k=1}^{\infty} p_k x_k$$

$$= \sum_{m=1}^{\infty} m(1-p)^{m-1}$$

$$= p\sum_{1}^{\infty} m(1-p)^{m-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

b) Odsutstvo pamćenja

$$P(X=k+m \mid X>k) = P(X=m)$$

↳ Dokaz:

$$P(X=k+m \mid X>k) = \frac{P(X=m+k \;\&\; X>k)}{P(X>k)}$$

$$= \frac{P(X=k+m)}{P(X>k)} = \frac{p(1-p)^{k+m+1}}{(1-p)^k}$$

$$= p(1-p)^{m-1} = P(X=m)$$

c) $p = 0.4$

$X \sim$ broj bacanja

$P(X > E(X)) = ?$

$P(X=k) = p(1-p)^{k-1}$

$= 0.4\,(1-0.4)^{k-1}$

$= 0.4 \cdot 0.6^{k-1}$

$$E(X) = \sum_{k=1}^{\infty} x_k p_k$$

$$= \sum_{1}^{\infty} n \cdot 0.4 \cdot 0.6^{n-1}$$

$$= 0.4 \sum_{1}^{\infty} n \cdot 0.6^{n-1}$$

$$= 0.4 \cdot \frac{1}{(1-0.6)^2}$$

$$= 0.4 \cdot \frac{1}{0.4^2} = 2.5$$

$$\Rightarrow \sum_{1}^{\infty} x^n = \frac{1}{1-x} \quad / \frac{d}{dx}$$

$$\sum_{1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$P(X > E(X)) = P(X > 2.5)$

$= P(X \geq 3) = \sum_{3}^{\infty} 0.4\,(1-0.4)^{n-1}$

$= \sum_{3}^{n} 0.4 \cdot 0.6^{n-1}$

$= 0.4 \sum_{3}^{n} 0.6^{n-1}$

$= 0.4 \cdot \frac{0.6^2}{1-0.6} = 0.36$

5) $P(\text{pogotka}) = 0.005$

$P(\text{ruševina}) = 0.05$

$\Rightarrow$ 1000 puta gađa brod

$P(\text{uništenja}) = P(X \geq 1) = ?$

$p = 0.005 \cdot 0.05$

$= 2.5 \cdot 10^{-4}$

$pn = 2.5 \cdot 10^{-4} \cdot 1000 = 0.25$

$\Pi = 0.25$

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - \frac{0.25^0}{0!} e^{-0.25} = 1 - e^{-0.25}$

$= 0.22 = 22\%$

⑥ KOCKA

$X \to 1$ paran, -1 neparan

$Y \to$ broj na kocki

d6	X	Y
1	-1	1
2	1	2
3	-1	3
4	1	4
5	-1	5
6	1	6

$(X,Y) \sim ?$

$cov(X,Y) = ?$

$E(Z) = ?$, $D(Z) = ?$ $\Big\}$ $Z = X+Y$

$X \downarrow \backslash Y \rightarrow$	1	2	3	4	5	6	
-1	$\frac{1}{6}$	/	$\frac{1}{6}$	/	$\frac{1}{6}$	/	$\frac{1}{2}$
1	/	$\frac{1}{6}$	/	$\frac{1}{6}$	/	$\frac{1}{6}$	$\frac{1}{2}$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	

$E(X) = 0$

$E(Y) = 3.5$

$$cov(X,Y) = E[(X-E(X))\cdot(Y-E(Y))]$$
$$= E[X(Y-E(Y))] = E[X(Y-3.5)] = 0.5$$

$$Z = X+Y \sim \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 7 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$E(Z) = 3.5$

$D(Z) = E(Z^2) - (E(Z))^2$

$= 4.92$

⑦ a) Fja. razdiobe ne može biti >1

$\Rightarrow F(x) = P(X<x) \leq 1$

b) Disperzija ne može biti negativna

$\Rightarrow D(X) = E[\underbrace{(X-E(X))^2}_{>0}]$

$\hookrightarrow E(>0)$ ne može biti <0

$\Downarrow$

$D(X) \geq 0$

c) $\square ABCD$

$a = 1$

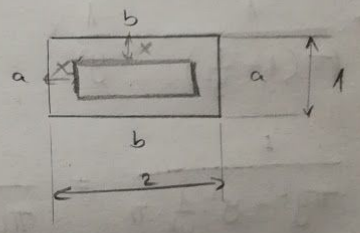
$b = 2$

X - {oddaljenost T do najbližje stranice}

$T(x,y)$ ——— točka

$f(x) = ?$

Skica:



$$F(x) = \frac{2-(1-2x)(2-2x)}{2}$$

$$= \frac{2-2+6x-4x^2}{2}$$

$$= -2x^2+3x$$

$$f(x) = F'(x) = -4x+3$$

$$E(x) = \int_0^{1/2} x\, f(x)\, dx$$

$$= \int_0^{1/2} x(-4x+3)\, dx$$

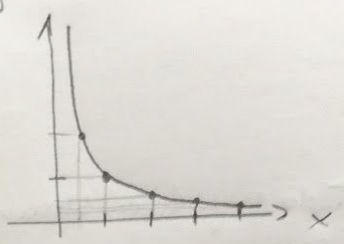
$$= \int_0^{1/2} (3x-4x^2)\, dx$$

$$= \frac{5}{24}$$

(8) $y = \frac{1}{x}$

$f(x) = \frac{1}{\pi(1+x^2)}$

$f(y) = ?$



$y = \frac{1}{x}$

$x = \frac{1}{y}$

$g(y) = f(x) \left|\frac{dx}{dy}\right|$

$\left|\frac{dx}{dy}\right| = \left|\frac{d\left(\frac{1}{y}\right)}{dy}\right| = \left|-\frac{1}{y^2}\right| = \frac{1}{y^2}$

$= \frac{1}{\pi\left(1+\frac{1}{y^2}\right)} \cdot \frac{1}{y^2}$

$= \frac{1}{\pi y^2 + y^2 \cdot \frac{1}{y^2} \cdot \pi} = \frac{1}{\pi y^2 + \pi}$

$= \frac{1}{\pi(y^2+1)}$

$f(y) = \frac{1}{\pi(1+y^2)}$ //