The Spring-Mass Walker

Assignment 6 Submission Date: 2.06.2024
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Problem statement / Introduction

In this assignment, the task is to model a bipedal walking gait of a simple system that consists of a mass mounted on top of two springs of equal stiffness (representing the legs) and analyze the behavior of the system for a range of initial conditions and system parameters.

As a walking model, it is assumed that one leg is always in contact with the ground. Therefore, the motion of the system alternates between two phases: single support, in which there is only one leg in stance mode, and double support, in which both legs are in stance mode. If a leg is in the stance phase, it is in contact with the ground, and the spring exerts an elastic force upon the mass; whereas during the swing phase, the leg is off the ground and the elastic force is null.

It is assumed for this analysis that the system starts with the left leg in stance mode and the right leg in swing mode. Therefore, each cycle of the periodic gait is composed of a single support phase for the left leg, a double support phase, and a single support phase for the right leg. The single support phases for each leg (1/2) are always followed by a double support phase in which the corresponding leg is behind leg 2/1, which is then followed by the single support of leg 2/1, and so on.

An illustration of a gait cycle is shown in Figure 1 (Geyer, Seyfarth, and Blickhan 2006), and the free body diagrams for each of its phases are given in Figures 2-4. In the following derivations, the left leg is described by spring 1, and the right leg is described by spring 2.

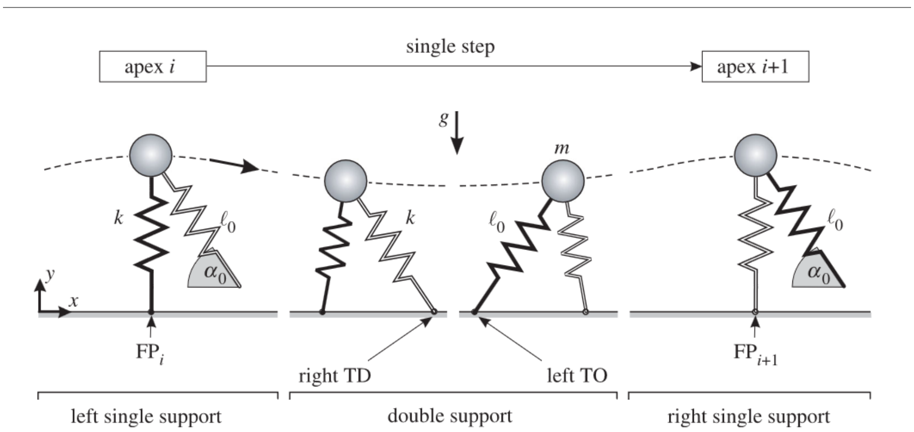

Figure 1: A single step is shown that starts at the highest COM position in left leg single support (apex i), includes the double support ranging from right leg touchdown (right TD) to left leg take-off (left TO), and ends at the next apex in right leg single support (apex i+1). FP, foot point position in single support.

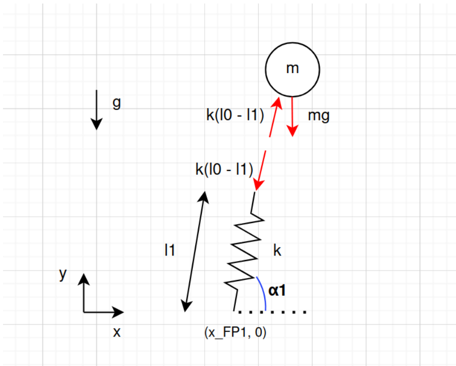

Figure 2: Free body diagram describing the left single support phase of the walking gait.

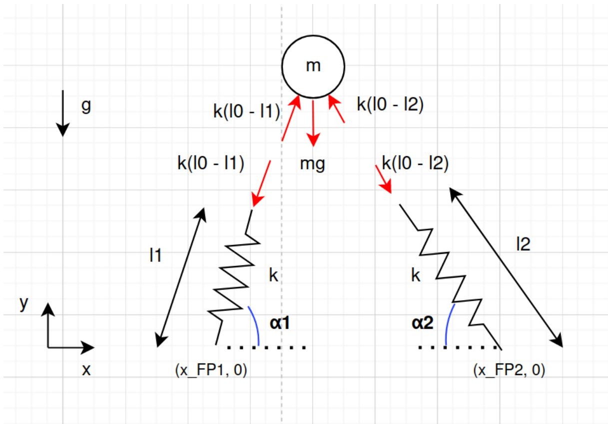

Figure 3: Free body diagram describing the double support phase of the walking gait.

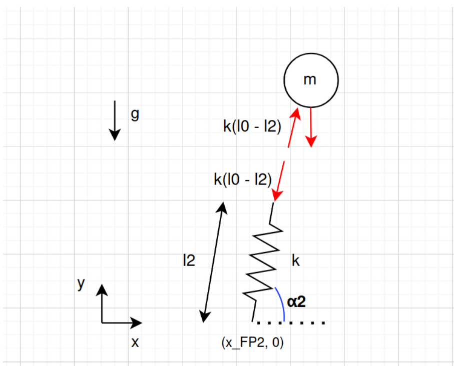

Figure 4: Free body diagram describing the right single support phase of the walking gait.

The equations of motion for each phase of the gait cycle will be derived and the results of the numerical implementation in Matlab will then be visualized in plots.

Definitions

Table 1 gives an overview to all used symbols, constants, variables and initial conditions.

Table 1: Declaration of used symbols, constants, variables and initial conditions.

Symbol	Property	Value	Unit
g	Gravitational acceleration	9.81	$[\frac{m}{s^2}]$
m	Mass of the body	80	$[kg]$
k	Stiffness of the springs	10	$[\frac{kN}{m}]$
l_0	Undeformed length of the springs	1	$[m]$
l_1	Length of the left leg	-	$[m]$
l_2	Length of the right leg	-	$[m]$
α_{TD}	Angle of attack	68	$[degrees]$
α_1	(Smallest) angle between the ground and the left leg	-	$[degrees]$
α_2	(Smallest) angle between the ground and the right leg	-	$[degrees]$
x_{FP1}	Horizontal foot point position of the left leg	-	$[m]$
x_{FP0}	Initial horizontal foot point position of the left leg	0	$[m]$
x_{FP2}	Horizontal foot point position of the right leg	-	$[m]$
x	Horizontal position of the mass	-	$[m]$
x_0	Initial horizontal position of the mass	0	$[m]$
$\dot{x}_0$	Initial horizontal velocity of the mass	1.05	$[\frac{m}{s}]$
y	Vertical position of the mass	-	$[m]$
y_0	Initial vertical position of the mass	0.97	$[m]$
$\dot{y}_0$	Initial vertical velocity of the mass	0	$[\frac{m}{s}]$
$F_{el,1}$	Elastic force applied on the mass by spring 1	-	$[N]$
$F_{el,2}$	Elastic force applied on the mass by spring 2	-	$[N]$
K	Kinetic energy of the system	-	$[J]$
V_g	Gravitational potential energy of the system	-	$[J]$
V_{el}	Elastic potential energy of the system	-	$[J]$
E_{mec}	Total energy of the system	-	$[J]$

The following assumptions are made for the upcoming derivations:

- Drag (resistance of air) and ground friction forces are neglected
- The coordinate system is located at the ground with the x-axis pointing right in the horizontal direction and the y-axis pointing upwards in the vertical direction
- The foot point position of each leg remains constant throughout the corresponding stance mode

Approach and Implementation

Equations of Motion

As explained, the forces applied to the mass are not the same throughout the gait cycle, thus the equations of motion must be derived separately for each gait phase. The first gait cycle is considered in this section, but the analysis for the subsequent ones is analogous (see implementation section).

In this section Newton's law is applied to the free body diagrams of figures 2-4 in each direction, considering the following auxiliary equations for i = 1,2 (note that the trigonometric functions are only defined in stance):

$$cos(\alpha_i) = \begin{cases} \frac{x - x_{FPi}}{l_i(x,y)}, & \text{if } x > x_{FPi} \\ \frac{x_{FPi} - x}{l_i(x,y)}, & \text{otherwise} \end{cases} \tag{1}$$

This ensures that the cosine is always positive so that the direction of the elastic force applied to the mass in the EoMs is coherent.

$$sin(\alpha_i) = \frac{y}{l_i(x,y)} \tag{2}$$

The leg length is given by

$$l_i(x,y) = \begin{cases} \sqrt{(x - x_{FPi})^2 + y^2}, & \text{if leg i is in stance } i \\ l_0, & \text{otherwise} \end{cases} \tag{3}$$

And the spring forces acting upon the mass by:

$$F_{el,i}(x,y) = \begin{cases} k(l_0 - l_i(x,y)), & \text{if leg i is in stance} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Left single support

From figure 2, it is clear that the angle between the elastic force $F_{el,1}$ and the horizontal direction is α_1. Decomposing the spring force in the horizontal and vertical directions yields:

$$m\ddot{x} = F_{el,1} cos(\alpha_1) \tag{5}$$

$$m\ddot{y} = -mg + F_{el,1} sin(\alpha_1) \tag{6}$$

Double support

In the double support phase there is the additional elastic force of the right leg acting in the opposite horizontal direction and in the same vertical direction (see figure 3). Thus:

$$m\ddot{x} = F_{el,1} cos(\alpha_1) - F_{el,1} cos(\alpha_2) \tag{7}$$

$$m\ddot{y} = -mg + F_{el,1} sin(\alpha_1) + F_{el,2} sin(\alpha_2) \tag{8}$$

Right single support

Following figure 4 yields:

$$m\ddot{x} = F_{el,2} cos(\alpha_2) \tag{9}$$

$$m\ddot{y} = -mg + F_{el,2} sin(\alpha_2) \tag{10}$$

Energy Analysis

The total mechanical energy is conserved over time due to the made assumptions and can be expressed as

$$E_{mec}(t) = V_g(t) + V_{el}(t) + K(t) = constant \tag{11}$$

The kinetic energy contains the horizontal and vertical velocity components:

$$K(t) = \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}m\dot{y}^2(t) \tag{12}$$

The gravitational potential energy is then

$$V_g(t) = mgy(t) \tag{13}$$

The elastic potential energy can be expressed as $V_{el}(t) = V_{el,1}(t) + V_{el,2}(t)$, with

$$V_{el,i} = \begin{cases} \frac{1}{2}k(l_i(x,y) - l_0)^2, & \text{if leg i is in stance} \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

Implementation considerations

The implementation was done in Matlab because I encountered some challenges in Simulink that I was unable to overcome. Using Matlab allowed me to have more control and a better understanding of the gait phases and transitions.

Regarding the equations of motion (EoMs), instead of using the cosine function as defined in equation (1) for the horizontal EoMs, the same negative sign is applied to the forces, and the cosine is written explicitly as:

$$\ddot{x} = \begin{cases} -\frac{F_{el,1}}{m}\frac{x_{FP1}-x}{l_1} & \text{spring 1 in stance} \\ -\frac{F_{el,1}}{m}\frac{x_{FP1}-x}{l_1} - \frac{F_{el,1}}{m}\frac{x_{FP2}-x}{l_2} & \text{both springs in stance} \\ -\frac{F_{el,2}}{m}\frac{x_{FP2}-x}{l_2} & \text{spring 2 in stance} \end{cases} \tag{15}$$

This ensures that the elastic forces of the legs point in the correct direction for any gait cycle.

Touchdown condition

In order to be able to correctly apply the equations 1-8, the x coordinate of the foot point needs to be updated at every touchdown of a leg. The touchdown condition for a spring is expressed by $y = l_0 sin(\alpha_{TD})$. However, this is problematic in two ways: it does not distinguish if it corresponds to a left or a right leg touchdown, and it is sensitive to numerical approximation errors (due to the finite floating point number representation precision it might be the case that y does not equal, but slightly lower than this value). The approach to resolve this was to initialize two boolean variables $stance_1$ and $stance_2$ with corresponding initial modes of each spring - true and false - and then performing the following update for i = 1, 2:
$x_{FPi} = x + l_0 cos(\alpha_{TD})$ and $stance_i$ is set true, when $y \le l_0 sin(\alpha_{TD})$ and $stance_i$ is false

That is, the mode is switched from stance to swing once it is detected that the height of the mass is equal or immediately inferior to the y projection of l_0.

Mode transitions

Each leg transitions to swing mode if $l_i(x,y) >= l_0$, it is in stance mode otherwise. This is checked at every time instant and the booleans are updated correspondingly.

Failure mode identification

Depending on the initial conditions and parameters employed, it may occur that both legs are off the ground at the same time, which would correspond to a running gait. The above model does not account for such effects, so they must be detected in the implementation. This is done at every time instant by asserting that $stance_1 + stance_2 > 0$, that is, both legs should not be in swing mode simultaneoulsy. If this happens, the assert interrupts the simulation with an error message.

Results

Given the equations of motion and transition conditions, a 4th order runge kutta solver (a standard ode45 could not be used since the EoMs vary with time) with a timestep of $1e-3$ was implemented in Matlab to simulate the model. Simulating for a time period of 10 seconds yields the following plots:

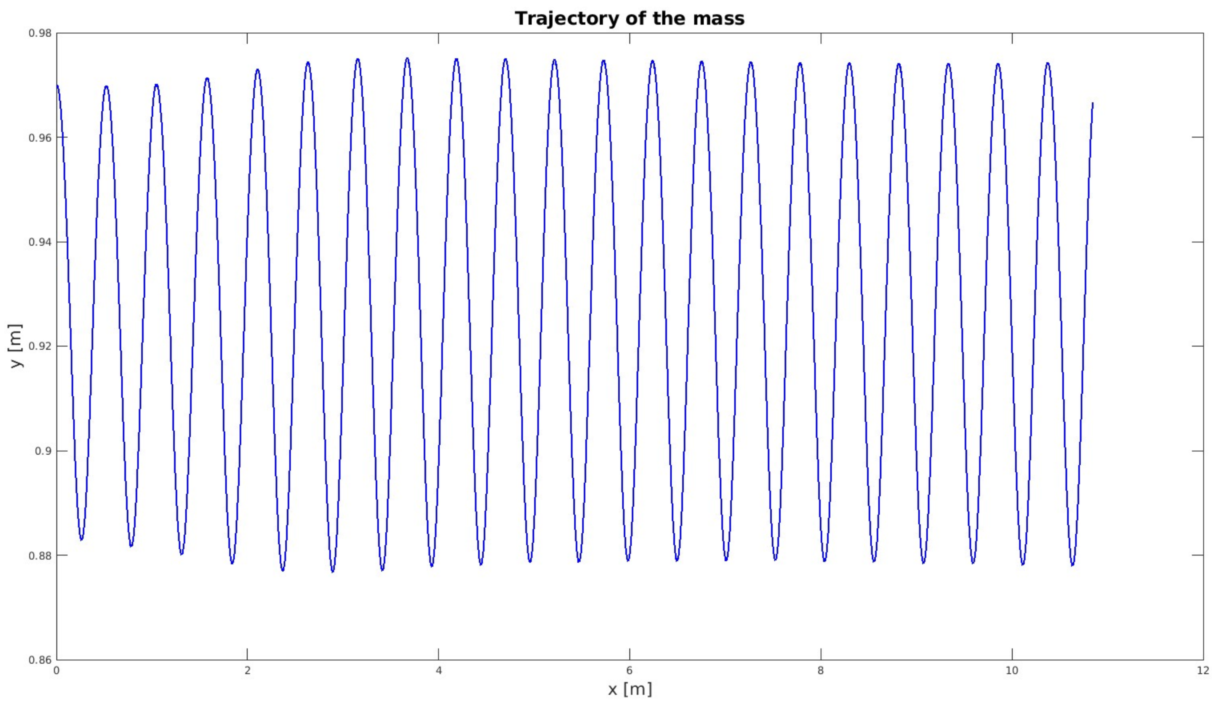

Figure 5: Trajectory of the mass in the considered time period

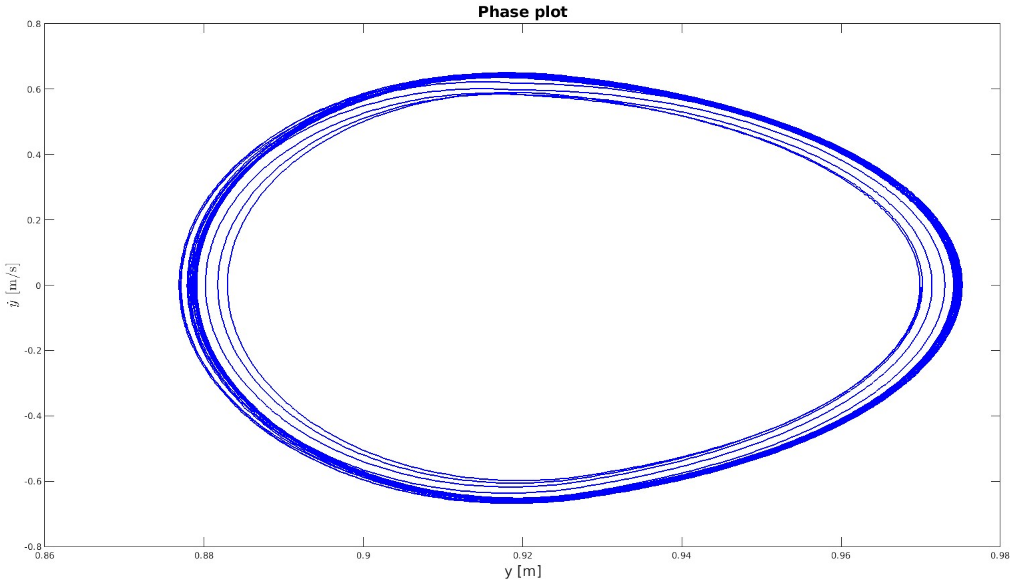

Figure 6: Phase plot for the walking model

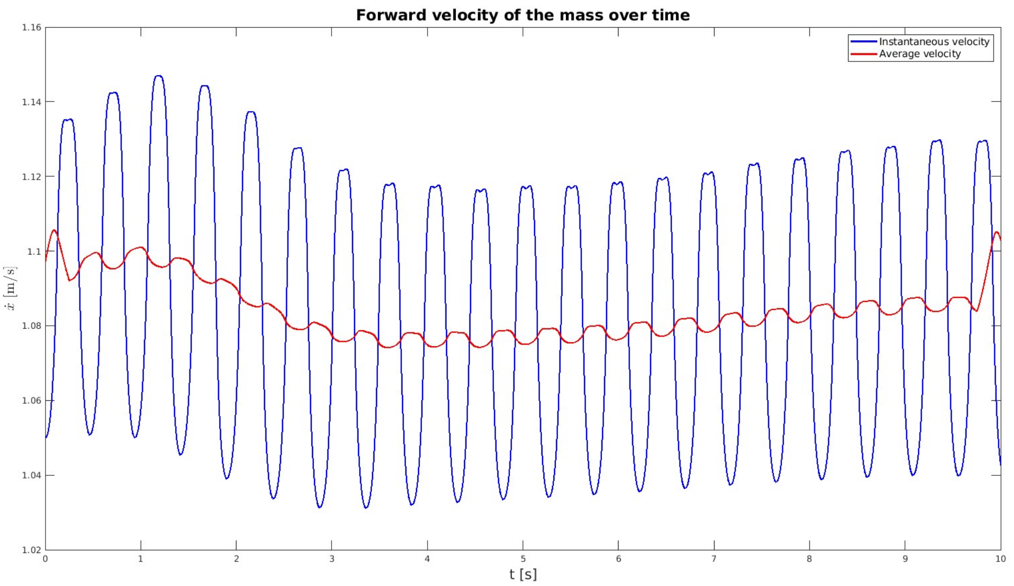

Figure 7: Horizontal velocity of the mass over time

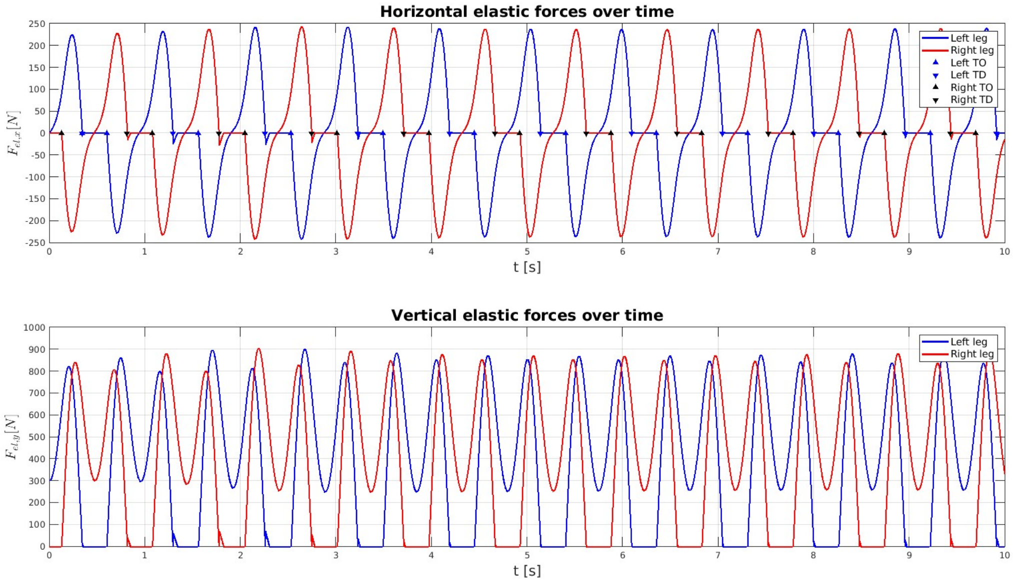

Figure 8: Horizontal and vertical elastic forces applied on the mass over time

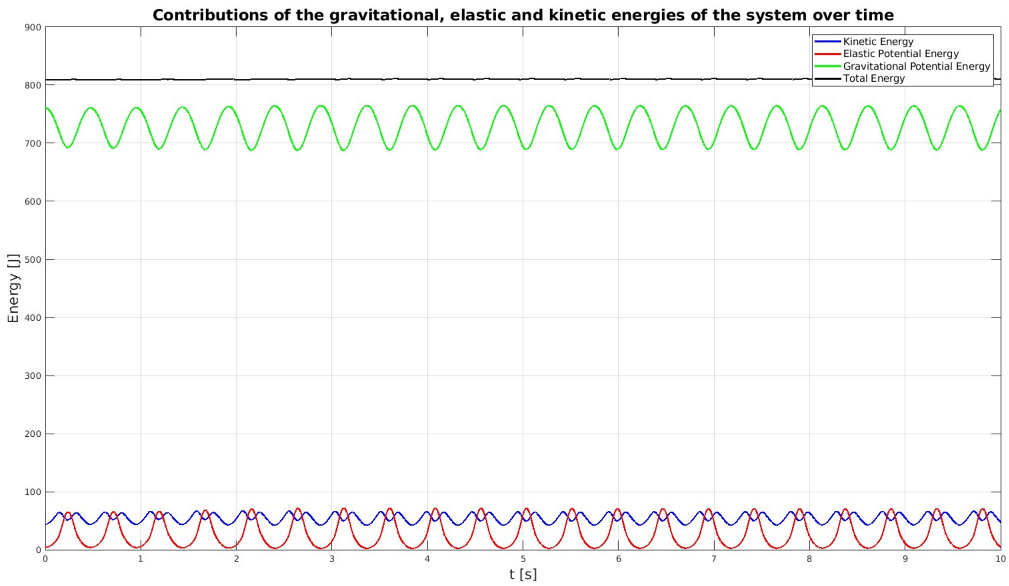

Figure 9: Different kinds of mechanical energy and their contribution to the system's total energy over time

Discussion

Trajectory

Figure 5 demonstrates the periodic movement of the mass, showing a consistent oscillation after an initial settling period. This behavior aligns with expectations for a scenario without dissipation. However, there is a slight increase in the initial amplitude, which is unexpected. This anomaly can be attributed to the numerical errors characteristic of the fixed-step Runge-Kutta solver.

Phase plot

Figure 6 contains the phase plot for the model. It demonstrates the behavior of the vertical velocity relative to the vertical position throughout the body's trajectory. At the apex of the trajectory, the vertical velocity is zero. As the body descends, the vertical velocity increases in the negative direction, reaching a maximum negative value at an intermediate height. The vertical velocity then decreases in magnitude as the body continues to descend, reaching zero at the lowest point of the trajectory. Subsequently, as the body ascends, the vertical velocity increases in the positive direction, reaching a maximum positive value at the intermediate height, before decreasing again to zero at the apex. This cycle repeats, indicating a periodic motion.

Forward velocity

Figure 7 shows the instantaneous and average (computed by applying a moving average over the instantaneous velocity vector) horizontal velocities of the mass over time. The maximum instantaneous forward speed in the simulation is $\dot{x}_m ax = 1.147$ m/s and the maximum average speed $\dot{x}_{avg,max} = 1.106$.

Elastic forces

Figure 8 shows the characteristic pattern of the horizontal and vertical elastic forces applied to the mass by each leg. The horizontal force of each spring is periodic and consists of five phases.

For the left leg: it is initially positive during the left single support phase and increases in magnitude as the mass's height diminishes, compressing the spring. The force then reaches a maximum and starts decreasing shortly after the double support phase begins, as the left spring stretches. It continues to decrease until it finally reaches zero when the leg detaches from the ground and the right single support phase starts. Following this, the double support phase begins with the left leg in front of the right leg, causing the negative force to increase in magnitude until it reaches a minimum. It then starts increasing (magnitude decreasing) shortly before the left leg is behind the right leg and continues to increase until it reaches zero and switches direction. The period then restarts with the left leg behind the right leg.

For the right leg: forces are initially zero during the left single support, become negative, and decrease to a minimum during the first double support phase. The force then starts increasing (decreasing in magnitude) shortly before the start of the right single support and continues increasing. It switches direction and continues increasing while positive until it reaches a maximum shortly after the second double support phase starts. The force then decreases and becomes null again at the second right single support.

The vertical elastic forces applied to the mass by each leg are also periodic. They are positive during the corresponding single support phase for the corresponding leg and the double support phase. Furthermore, they increase as the body's height diminishes and decrease as it increases.

Energies of the system

Finally, Figure 9 shows that the total mechanical energy of the system is constant (up to 1 J, which is due to the accumulated errors of the Runge-Kutta solver), which is consistent with the no friction assumption. It should be noted that although the elastic energy reaches its lowest values at points where the single support horizontal spring force is null, it is never completely zero. The oscillatory nature of the gait is also reflected in the kinetic and gravitational potential energies

Finding continuous gait patterns

It must be noted that the results were specific to the initial conditions and parameters given in the table. Nonetheless, it could be desired to find a range of parameters $\alpha_{TD}, y_0, \dot{y}_0, FP_0, \dot{x}_0$ such that a walking gait is achieved. A possibility would be to use a grid search approach: define plausible ranges for the parameters and scan the 5-dimensional parameter space. For each set of parameters, run the simulation, checking first if the simulation is aborted under the assertion that ensures both legs are not in swing mode simultaneously. If the simulation is not aborted, check if it converges to a stable gait pattern. This convergence can be observed by examining whether the key variables (such as horizontal and vertical velocities, forces, and energy) exhibit periodic behavior over time or if the phase plot of figure 6 converges to a limit cycle.

References

Geyer, H., Seyfarth, A., and Blickhan, R. (2006). "Compliant leg behaviour explains basic dynamics of walking and running". In: *Proceedings of the Royal Society B*. DOI: 10.1098/rspb.2006.3637.

AI usage declaration

tools used

1. ChatGPT

used prompts

1. Correct this text

comments
