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The Bouncing Mass Problem - Analytical Solution

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Problem statement / Introduction

In this assignment, the task is to describe the trajectory of a falling mass mounted on top of a vertical spring when the system is dropped from a height of 1.2m above the ground. The mass has a value of 1kg, and the spring is massless, with a resting length of 1m and a stiffness of 100N/m.

The system's behavior is dictated by two modes: flight, in which the spring is not in contact with the ground and the elastic force is null; and stance, in which the spring is in contact with the ground, thus experiencing compression and applying an elastic force to the mass.

The analytical solution to the equations of motion for each mode - stance and flight - will be derived and solved analytically, yielding the mass displacements and velocities over time. These results will then be visualized in plots using an implementation in MATLAB.

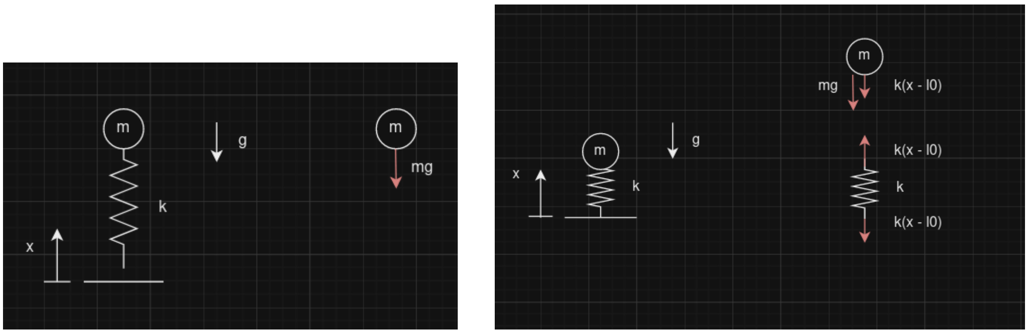

Figure 1: Illustration of the problem. The left image depicts the system (on the left) and its free body diagram (on the right) for the flight mode; the stance mode is illustrated in the right image in the same manner.

Definitions

Table 1 gives an overview to all used symbols, constants, variables and initial conditions.

Table 1: Declaration of used symbols, constants, variables and initial conditions.

Symbol	Property	Value	Unit
g	Gravitational acceleration	9.81	$[\frac{m}{s^2}]$
m	Mass of the body	1	$[kg]$
k	Stiffness of the spring	100	$[\frac{N}{m}]$
l_0	Undeformed length of the spring	1	$[m]$
x	Position of the mass	-	$[m]$
x_0	Initial position of the mass	1.2	$[m]$
$\dot{x}_0$	Initial velocity of the mass	0	$[\frac{m}{s}]$
K	Kinetic energy of the system	-	$[J]$
V_g	Gravitational potential energy of the system	-	$[J]$
V_{el}	Elastic potential energy of the system	-	$[J]$
E_{mec}	Total energy of the system	-	$[J]$

The following assumptions are made for the upcoming derivations:

- Drag (resistance of air) is neglected
- The coordinate system is located at the ground with the x-axis pointing upwards in the vertical direction
- Mass-movement is only considered in the x-direction (disturbances in horizontal direction e.g. due to wind are neglected)

Approach and Implementation

The proposed problem combines a free fall scenario (flight mode) with a spring-mass oscillator (stance mode). Since the forces acting on the mass in each mode are distinct, the equations of motion will also differ and must be derived separately. Both sets of equations can be derived using Newton's second law.

Flight

In the flight mode the mass is only subject to the gravitational field, thus:

$$m\ddot{x} = -mg \qquad \longrightarrow \qquad \ddot{x} = -g \tag{1}$$

The integration of this equation with respect to t gives the velocity of the ball (eq.(2)), a second integration the position (eq.(3)) during the flight mode:

$$\dot{x} = -gt + C_1 \tag{2}$$

$$x = -\frac{1}{2}gt^2 + C_1 t + C_2 \tag{3}$$

Substituting the initial conditions yields, for the first flight period (and for the latter ones if the time is reset to 0 in the beginning of each transition):

$$\dot{x} = -gt + \dot{x}_0 = -gt \tag{4}$$

$$\dot{x} = -\frac{1}{2}gt^2 + \dot{x}_0 t + x_0 = -\frac{1}{2}gt^2 + x_0 \tag{5}$$

Stance

In the stance mode the elastic force also acts on the spring, therefore:

$$m\ddot{x} = -mg - k(x - l_0) \qquad \longrightarrow \qquad m\ddot{x} + kx = kl_0 - mg \tag{6}$$

Eq. 6 represents a forced second-order linear differential equation. The solution to this equation is the sum of the solution to the homogeneous equation, denoted as $x_h(t)$, and the particular solution, denoted as $x_p(t)$. The particular solution $x_p(t)$ is the function that, when substituted into Eq. 6, satisfies the right-hand side of the equation: $x(t) = x_h(t) + x_p(t)$ (*Forced Second Order Differential Equations* n.d.).

Homogeneous equation

For the homogeneous equation

$$m\ddot{x} + kx = 0 \tag{7}$$

the solution is assumed to be of the form $x(t) = e^{\lambda t}$. Substituting in eq. 7 yields:

$$m\lambda^2 e^{\lambda t} + ke^{\lambda t} = 0 \quad \longrightarrow \quad \lambda^2 = -\frac{k}{m} \quad \longrightarrow \quad \lambda_1 = j\sqrt{\frac{k}{m}}, \lambda_2 = -j\sqrt{\frac{k}{m}} \tag{8}$$

The solution to the homogeneous equation can then be expressed as a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$:

$$x_h(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{9}$$

Utilizing Euler's identity ($e^{j\theta} = cos(\theta) + jsin(\theta)$) gives:

$$x_h(t) = A(\cos(\sqrt{\frac{k}{m}}t) + j\sin(\sqrt{\frac{k}{m}}t)) + B(\cos(-\sqrt{\frac{k}{m}}t) + j\sin(-\sqrt{\frac{k}{m}}t)) \tag{10}$$

Recalling that $cos(-\theta) = cos(\theta)$ and $sin(-\theta) = -sin(\theta)$ and grouping the sine and cosine terms yields (with $C_3 = A + B$, $C_4 = j(A - B)$):

$$x_h(t) = C_3 \cos\sqrt{\frac{k}{m}}t + C_4 \sin\sqrt{\frac{k}{m}}t \tag{11}$$

Particular solution

For the particular solution, it may be assumed to be constant, since there is a constant forcing term on the right-hand side of Eq. 6.

$$kx_p = kl_0 - mg \quad \longrightarrow \quad x_p = l_0 - \frac{mg}{k} \tag{12}$$

Finally, the position of the mass during stance mode is given by:

$$x(t) = C_3 \cos\sqrt{\frac{k}{m}}t + C_4 \sin\sqrt{\frac{k}{m}}t + l_0 - \frac{mg}{k} \tag{13}$$

The velocity is obtained simply by differentiating eq.(13) with respect to time:

$$\dot{x}(t) = -C_3\sqrt{\frac{k}{m}} \sin\sqrt{\frac{k}{m}}t + C_4\sqrt{\frac{k}{m}} \cos\sqrt{\frac{k}{m}}t \tag{14}$$

Now the boundary conditions for the transition from the flight to stance mode need to be determined. For the position this is simply $x(t_{FS}) = l_0$. For the velocity, energy conservation needs to be considered (see section below), which leads to $K_{SF/FS} + V_{g,SF/FS} = V_g(x = x_0)$, where the subscripts SF and FS indicate a transition from stance to flight and flight to stance, respectively. This yields

$$\frac{1}{2}m\dot{x}^2(t_{SF/FS}) = mg(x_0 - l_0) \tag{15}$$

thus $\dot{x}(t_{FS}) = -\sqrt{2g(x_0 - l_0)}$ (since in this case the mass is descending). Correspondingly in the transition from stance to flight $x(t_{SF}) = l_0$ and $\dot{x}(t_{SF}) = \sqrt{2g(x_0 - l_0)}$ hold, since the mass is then ascending.

Again, if t_{FS} is set to zero every time a transition into stance occurs, substitution of the boundary conditions in eqs. 13 and 14 yields $C_3 = \frac{mg}{k}$ and $C_4 = -\sqrt{\frac{2mg(x_0-l_0)}{k}}$. Substituting the constants in these equations again results in:

$$x(t) = \frac{mg}{k}\cos\sqrt{\frac{k}{m}}t - \sqrt{\frac{2mg(x_0-l_0)}{k}}\sin\sqrt{\frac{k}{m}}t + l_0 - \frac{mg}{k} \tag{16}$$

$$\dot{x}(t) = -\frac{mg}{k}\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t - \sqrt{2g(x_0-l_0)}\cos\sqrt{\frac{k}{m}}t \tag{17}$$

Energy Analysis

Since drag is neglected, the mechanical energy of the system is conserved throughout the movement of the mass:

$$E_{mec}(t) = V_g(t) + V_{el}(t) + K(t) = constant \tag{18}$$

In terms of the defined coordinate and reference system, the kinetic energy can be expressed as

$$K(t) = \frac{1}{2}m\dot{x}^2(t) \tag{19}$$

The gravitational potential energy is then

$$V_g(t) = mgx(t) \tag{20}$$

Lastly, the elastic potential energy is given by

$$V_{el}(t) = \begin{cases} \frac{1}{2}k(x(t)-l_0)^2, & \text{if } x \le l_0 \\ 0, & \text{if } x > l_0 \end{cases} \tag{21}$$

Implementation considerations

Regarding the implementation, the 0-10s interval was discretized in time increments of 5×10^{-3}s and the following strategies were adopted:

1. The transitions from flight to stance and stance to flight were detected by checking if $x_{t_i} <= l_0$ and $x_{t_{i-1}} > l_0$ or if $x_{t_i} > l_0$ and $x_{t_{i-1}} <= l_0$, respectively
2. The time was not reset to 0 at each transition to avoid confusion. Instead, the integration constants C_3 and C_4 were updated by substituting t_{FS} and the boundary conditions $x(t_{FS})$ and $\dot{x}(t_{FS})$ into eqs 13 and 14, respectively, and solving the linear system for C_3 and C_4. Likewise, C_1 and C_2 were updated by substituting t_{SF} and the boundary conditions $x(t_{SF})$ and $\dot{x}(t_{FS})$ into eqs. 2 and 3 and solving the system.

Results

Figure 2 illustrates the position and velocity of the mass within a 10-second time frame. Figure 3 shows the contributions of the mechanical, gravitational potential and elastic potential energy to the total mechanical energy over time.

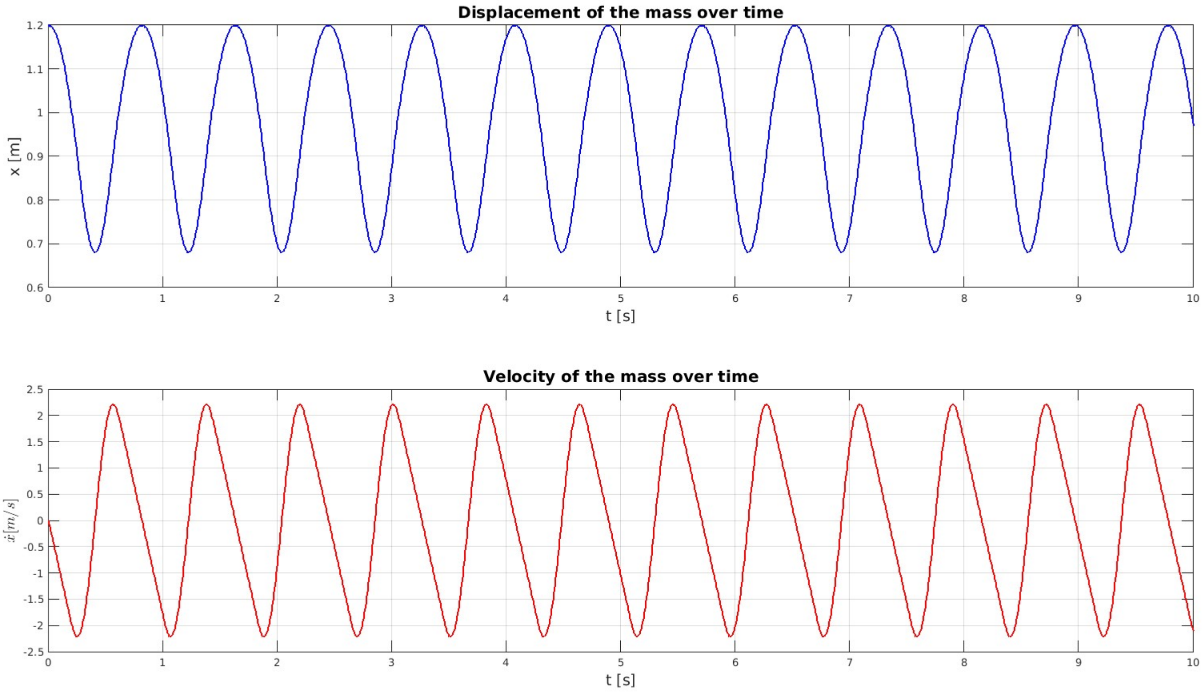

Figure 2: Position and velocity of the mass over time.

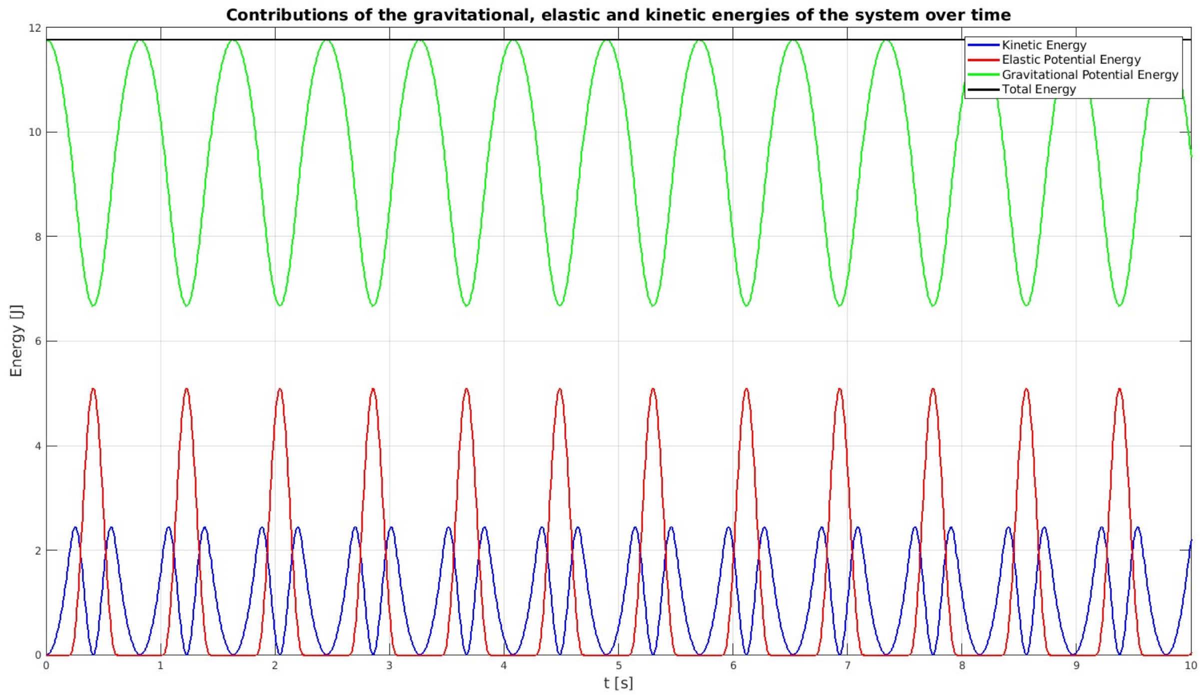

Figure 3: Different kinds of mechanical energy and their contribution to the system's total energy over time

Discussion

Figure 2 demonstrates the periodic movement of the mass, as expected due to the presence of the spring. The amplitude of the oscillations remains constant since energy dissipation is neglected.

This observation is further supported by Figure 3, where we can observe that the total mechanical energy remains constant over time. At the beginning of each period, it is solely composed of gravitational potential energy, which is then converted into kinetic energy (during flight and stance modes) and elastic potential energy (during stance).

It's important to acknowledge that neglecting drag is often unrealistic for real-life scenarios. In reality, this assumption would only be justifiable for small heights and specific shapes/materials of the mass. Mechanical energy would diminish over time due to energy dissipation.

References

Forced Second Order Differential Equations (n.d.). https://www.ncl.ac.uk/webtemplate/ask-assets/external/maths-resources/core-mathematics/calculus/forced-second-order-differential-equations.html. Accessed: 24/04/2024.

AI usage declaration

tools used

1. ChatGPT

used prompts

1. How can I add a spring in drawio?
2. How can I added text to my diagram in drawio?
3. Correct this text if there are mistakes

comments
