1) X_1 e X_2 independentes $\Leftrightarrow$ X_1 e X_2 são não correlacionados

· Provando ($\Rightarrow$)

Se X e Y são duas v.a.'s independentes, então $E(XY) = E(X)E(Y)$. Logo,

$$cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2) = E(X_1)E(X_2) - E(X_1)E(X_2) = 0$$

Como $cov(X_1, X_2) = 0$, X_1 e X_2 são não correlacionadas.

· Provando ($\Leftarrow$)

Temos que mostrar que $f_X(x) = f_{X_1}(x_1) f_{X_2}(x_2)$, onde $f_{X_1}(x_1) = (\sigma_1 \sqrt{2\pi})^{-1} \exp\left(-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right)$ e $f_{X_2}(x_2) = (\sigma_2 \sqrt{2\pi})^{-1} \exp\left(-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)$.

$$\rho = 0 \Rightarrow f_X(x) = (\sigma_1 \sigma_2)^{-1} (2\pi)^{-1} \exp\left\{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \right.\right.$$

$$\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\Big]\Big\} = (\sigma_1\sqrt{2\pi})^{-1}(\sigma_2\sqrt{2\pi})^{-1}\times$$

$$\exp\left\{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\} =$$

$$(\sigma_1\sqrt{2\pi})^{-1}\exp\left\{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right\}(\sigma_2\sqrt{\pi 2})^{-1}\exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}$$

$$= f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$\blacksquare$

2) $$f_{X_1|X_2}(x_1|x_2) = \frac{f_X(x)}{f_{X_2}(x_2)} = \frac{(\sigma_1\sigma_2\sqrt{1-\rho^2})^{-1}(2\pi)^{-1}}{\sigma_2^{-1}(2\pi)^{-1/2}}\times$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\times\right.\right.$$

$$\left.\left.\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}\times\exp\left\{+\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\} =$$

$$(\sigma_1\sqrt{1-\rho^2})^{-1}(2\pi)^{-1/2}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 +\right.\right.$$

$$\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) - (1-\rho^2)\times$$

$$\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\Big]\Big\}$$

· Manipulando a expressão no expoente

$$-\frac{1}{2(1-\rho^2)\sigma_1^2}\Big[(x_1-\mu_1)^2 + \left(\frac{\sigma_1}{\sigma_2}\right)^2(x_2-\mu_2)^2 - 2\rho\frac{\sigma_1}{\sigma_2}\times$$

$$(x_1-\mu_1)(x_2-\mu_2) - (1-\rho^2)\left(\frac{\sigma_1^2}{\sigma_2^2}\right)(x_2-\mu_2)^2\Big]$$

$$= \frac{-1}{2V^2}\Big[(x_1-\mu_1)^2 + \rho^2\left(\frac{\sigma_1}{\sigma_2}\right)^2(x_2-\mu_2)^2 - 2\rho\frac{\sigma_1}{\sigma_2}\times$$

$$(x_1-\mu_1)(x_2-\mu_2)\Big] = \frac{-1}{2V^2}\Big[x_1^2 - 2x_1\mu_1 + \mu_1^2 +$$

$$\rho^2\left(\frac{\sigma_1}{\sigma_2}\right)^2(x_2-\mu_2)^2 - 2\rho\frac{\sigma_1}{\sigma_2}x_1(x_2-\mu_2) + 2\rho\frac{\sigma_1}{\sigma_2}\mu_1\times$$

$$(x_2-\mu_2)\Big] = \frac{-1}{2V^2}\Big[x_1^2 - 2x_1\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2)\right) +$$

$$\mu_1^2 + 2\mu_1 \rho \frac{\sigma_1}{\sigma_2}(x_2-\mu_2) + \rho^2\left(\frac{\sigma_1}{\sigma_2}\right)^2 (x_2-\mu_2)^2]$$

$$= -\frac{1}{2V^2}[x_1^2 - 2x_1 M + (\mu_1 + \rho(\frac{\sigma_1}{\sigma_2})(x_2-\mu_2)]$$

$$= -\frac{1}{2V^2}(x_1^2 - 2x_1 M + M^2) = -\frac{1}{2V^2}(x_1^2 - M)^2,$$

onde $V = \sigma_1\sqrt{1-\rho^2}$ e $M = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2)$

Logo, $f_{X_1|X_2}(x_1|x_2) = V^{-1}(2\pi)^{-1/2} \exp\left\{-\frac{1}{2V^2}(x_1-M)^2\right\}$

$\Rightarrow X_1 | X_2 = x_2 \sim N(M, V^2)$ ■

3) A fgm (função geradora de momentos) identifica unicamente uma distribuição, então temos que mostrar que a fgm de X tem a forma:

$$M_X(t) = \exp\{\mu' t + \frac{1}{2}t'\Sigma t\},$$

onde $M_X(t)$ é a fgm do vetor normal multivariado.

Seja $t = (t_1, \ldots, t_n)'$, então

$M_X(t) = E[\exp\{t'X\}] = E[\exp\{\sum_{i=1}^n t_i X_i\}] =$
$E[\prod_{i=1}^n \exp\{t_i X_i\}] \overset{ind}{=} \prod_{i=1}^n E[\exp\{t_i X_i\}] =$
$\prod_{i=1}^n M_{X_i}(t_i)$

· A fgm de uma v.a. normal é
$\exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$. Logo,

$M_X(t) = \prod_{i=1}^n \exp\{\mu_i t_i + \sigma_i^2 t_i^2 / 2\} = \exp\{\sum_{i=1}^n t_i \mu_i$
$+ \frac{1}{2}\sum_{i=1}^n t_i \sigma_i^2 t_i\} = \exp\{t'\mu + \frac{1}{2} t' \Sigma t\}$, onde
$\mu = (\mu_1, \ldots, \mu_n)'$ e $\Sigma = diag(\sigma_1^2, \ldots, \sigma_n^2)$.

Portanto, $X \sim N_n(\mu, \Sigma)$ ■

4) a) Vamos usar os resultados do exercício 4.10 para resolver esse item

Fazendo $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, pode ser verificado que

$$\begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0' & I \end{bmatrix}\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} =$$

$$\begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0' & \Sigma_{22} \end{bmatrix}$$

Aplicando o determinante em ambos os lados da igualdade, temos

$$\det(I)\det(I)\det(\Sigma)\det(I)\det(I) =$$
$$\det(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\det(\Sigma_{22}) \Leftrightarrow$$

$$\det(\Sigma) = \det(\Sigma_{22})\det(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) \quad \blacksquare$$

b) Como Σ é simétrica, do exercício 4.12,

$$\Sigma^{-1} = \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}\begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0' & \Sigma_{22}^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0' & I \end{bmatrix}$$

Fazendo $B = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, temos que

$$\Sigma^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}B^{-1} & \Sigma_{22}^{-1}\Sigma_{21}B^{-1}\Sigma_{12}\Sigma_{22}^{-1} + \Sigma_{22}^{-1} \end{bmatrix}$$

então, $(x-\mu)'\Sigma^{-1}(x-\mu) = [(x_1-\mu_1)',$
$(x_2-\mu_2)']\begin{bmatrix} B^{-1} & -B^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}B^{-1} & \Sigma_{22}^{-1}\Sigma_{21}B^{-1}\Sigma_{12}\Sigma_{22}^{-1} + \Sigma_{22}^{-1} \end{bmatrix} \times$

$$\begin{bmatrix} (x_1-\mu_1) \\ (x_2-\mu_2) \end{bmatrix} = [(x_1-\mu_1)', (x_2-\mu_2)'] \times$$

$$\begin{bmatrix} B^{-1}(x_1-\mu_1) - B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) \\ -\Sigma_{22}^{-1}\Sigma_{21}B^{-1}(x_1-\mu_1) + (\Sigma_{22}^{-1}\Sigma_{21}B^{-1}\Sigma_{12}\Sigma_{22}^{-1} + \Sigma_{22}^{-1})(x_2-\mu_2) \end{bmatrix} =$$

$$(x_1-\mu_1)'B^{-1}(x_1-\mu_1) - (x_1-\mu_1)'B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)$$

$$-(x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}B^{-1}(x_1-\mu_1) +$$

$$(x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) +$$

$$(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2) =$$

$$(x_1-\mu_1)'B^{-1}(x_1-\mu_1) - (x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}B^{-1}(x_1-\mu_1)$$

$$-(x_1-\mu_1)'B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) +$$

$$(x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) +$$

$$(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2) =$$

$$[(x_1-\mu_1)' - (x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}]B^{-1}(x_1-\mu_1) -$$

$$[(x_1-\mu_1)' - (x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21}]B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)$$

$$+(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2) \ast$$

Sabendo que $(\Sigma_{22}^{-1})' = \Sigma_{22}^{-1}$ e $\Sigma_{21}' = \Sigma_{12}$, então

$$(x_2-\mu_2)'\Sigma_{22}^{-1}\Sigma_{21} = [\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)]'.$$

Além disso, definindo $a = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)$, temos que

$$\begin{aligned}
\Sigma^{-1} = {} & [(x_1-\mu_1) - \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)]' B^{-1}(x_1-\mu_1) - \\
& [(x_1-\mu_1) - \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)]' B^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) + \\
& (x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2) = [x_1-a]' B^{-1}[(x_1-\mu_1) - \\
& \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)] + (x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2) = \\
& [x_1-a]' B^{-1}[x_1-a] + (x_2-\mu_2)\Sigma_{22}^{-1}(x_2-\mu_2) \quad \blacksquare
\end{aligned}$$

c) Usando os resultados dos itens a) e b), temos

$$\begin{aligned}
f_X(x) = {} & |\Sigma|^{-1/2}(2\pi)^{-p/2}\exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} \\
= {} & (|\Sigma_{22}||\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|)^{-1/2}(2\pi)^{(n+m)/2} \times \\
& \exp\left\{-\frac{1}{2}[x_1-a]'B^{-1}[x_1-a] - \frac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\right\} \\
= {} & |\Sigma_{22}|^{-1/2}(2\pi)^{n/2}\exp\left\{-\frac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\right\} \times
\end{aligned}$$

$$|B|^{-1/2}(2\pi)^{m/2} \exp\{-\frac{1}{2}(x_1-a)' B^{-1}(x_1-a)\}, \; (\star)$$

onde n e m são as dimensões dos vetores x_1 e x_2, respectivamente. Logo,

$$f_{X_2}(x_2) = \int_{R^m} f_X(x)\, dx_1 =$$

$$|\Sigma_{22}|^{-\frac{1}{2}}(2\pi)^{\frac{n}{2}} \exp\{-\frac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\} \times$$

$$\int_{R^m} |\underline{B}|^{-1/2}(2\pi)^{m/2} \exp\{-\frac{1}{2}(x_1-\underline{a})'\underline{B}^{-1}(x_1-\underline{a})\}\, dx_1$$

O valor da integral é 1, pois é a integral sobre uma distribuição normal multivariada, com vetor de médias $\underline{a}$ e matriz de variância $\underline{B}$. Então

$$f_{X_2}(x_2) = |\Sigma_{22}|^{-1/2}(2\pi)^{n/2} \exp\{-\frac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\}$$

$$\Rightarrow X_2 \sim N_n(\mu_1, \Sigma_{22}).$$

· Podemos fatorar $f_X(x)$ como $f_{X_2}(x_2) \cdot f_{X_1|X_2}(x_1|x_2)$.

Como sabemos a forma de $f_{X_2}(x_2)$, de $(\star)$ temos que

$$f_X(x) = f_{X_2}(x_2)\,|B|^{-1/2}(2\pi)^{m/2} \exp\left\{-\frac{1}{2}(x_1-a)'B^{-1} \times (x_1-a)\right\}$$

$$\Rightarrow f_{X_1|X_2}(x_1|x_2) = |B|^{-1/2}(2\pi)^{m/2} \exp\left\{-\frac{1}{2}(x_1-a)'B^{-1} \times (x_1-a)\right\} \Rightarrow X_1|X_2 = x_2 \sim N_m(a, B) \quad \blacksquare$$

5) $M_Y(t) = E[e^{tY}] = E[e^{tX'\Sigma^{-1}X}] =$

$$\int_{R^p} |\Sigma|^{-1/2}(2\pi)^{p/2} \exp\left\{t x'\Sigma^{-1}x - \frac{1}{2}x'\Sigma^{-1}x\right\} dx =$$

$$\int_{R^p} |\Sigma|^{-1/2}(2\pi)^{p/2} \exp\left\{-\frac{1}{2}(1-2t)x'\Sigma^{-1}x\right\} dx =$$

$$\int_{R^p} |\Sigma|^{-1/2}(2\pi)^{p/2} \exp\left\{-\frac{1}{2}x'\left(\frac{1}{1-2t}\cdot\Sigma\right)^{-1}x\right\} dx =$$

$$\int_{R^p} \left[\frac{1}{1-2t}\right]^{\frac{p}{2}} \left[\frac{1}{1-2t}\right]^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}}(2\pi)^{p/2} \times$$

$$\exp\left\{-\frac{1}{2}x'\left(\frac{1}{1-2t}\cdot\Sigma\right)^{-1}x\right\} dx =$$

$$\int_{R^p} \left[\frac{1}{1-2t}\right]^{p/2} \left|\frac{1}{1-2t}\Sigma\right|^{-\frac{1}{2}}(2\pi)^{p/2} \times$$

$$\exp\left\{-\frac{1}{2}x'\left[\frac{1}{1-2t}\cdot\Sigma\right]^{-1}x\right\} dx =$$

$$\frac{1}{(1-2t)^{p/2}} \int_{R^p} \left|\frac{1}{1-2t}\cdot\Sigma\right|^{-1/2}(2\pi)^{p/2} \times$$

$$\exp\left\{-\frac{1}{2}x'\left[\frac{1}{1-2t}\Sigma\right]^{-1}x\right\} dx = \frac{1}{(1-2t)^{p/2}}$$

A fgm de uma v.a. χ^2_K é dada por $(1-2t)^{-K/2}$. Logo, $M_Y(t) = (1-2t)^{-p/2}$

$\Rightarrow \quad Y \sim \chi^2_p$ ■

6) Seja $t = (t_1', \ldots, t_n')'$, onde $t_i = (t_{i1}, \ldots, t_{ip})'$, então

$$M_X(t) = E[\exp\{t'X\}] = E[\exp\{\textstyle\sum_{i=1}^n t_i'X_i\}] =$$

$$E[\textstyle\prod_{i=1}^n \exp\{t_i'X_i\}] \overset{ind}{=} \prod_{i=1}^n E[\exp\{t_i'X_i\}] =$$

$$\prod_{i=1}^n M_{X_i}(t_i) = \prod_{i=1}^n \exp\{t_i'\mu_i + \frac{1}{2} t_i'\Sigma_i t_i\} =$$

$$\exp\{\textstyle\sum_{i=1}^n t_i'\mu_i + \frac{1}{2} \sum_{i=1}^n t_i'\Sigma_i t_i\} =$$

$$\exp\{t'\mu + \frac{1}{2} t'\Sigma t\},$$

onde $\mu = (\mu_1', \ldots, \mu_n')'$ e $\Sigma = \begin{bmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_n \end{bmatrix}$.

Portanto, $X \sim N_{np}(\mu, \Sigma)$ ■