Met. Numéricos Computacionais
Aula 2-1 Recapitulando o Cálculo

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Slides are based on the textbook “Differential Equations and Linear Algebra" by Gilbert Strang

Sugestões de leitura:

- Strang - Cap. 1.1, 1.2, 1.3;
- Boyce - Cap. 1.3

Topics

- Introductory Remarks
- Types of ODEs
- Essential Calculus
- Exponentials

- **Introductory Remarks**
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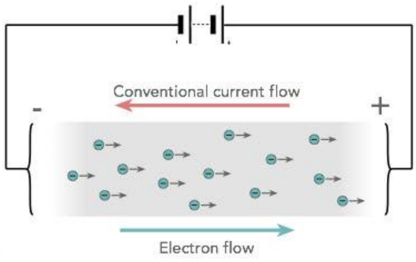
- Exponentials



Differential Equations …

- … help to understand movement: An economy grows, currents flow, the moon rises and messages travel
- … are used in models for (frequently exponential) growth, for oscillation and rotation, for equilibrium
- … are remarkably useful in a linear form
- … define/describe important functions:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y \;\Rightarrow\; y(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots = \exp(t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 \;\Rightarrow\; y(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}$$

Link to Linear Matrix Equations

All null solutions … :

$$Ay_n = 0 \qquad\qquad \frac{\mathrm{d}y_n}{\mathrm{d}t} = 0 \;\Rightarrow\; y_n(t) = C \text{ (any constant)}$$

… and one particular solution … :

$$Ay_p = b \qquad\qquad \frac{\mathrm{d}y_p}{\mathrm{d}t} = f(t) \;\Rightarrow\; y_p(t) = \int_0^t f(x)\,\mathrm{d}x$$

… give the general solution:

$$y = (\text{One } y_p) + (\text{All } y_n) \qquad\qquad y(t) = y_p(t) + C$$

- Some input functions **f(t)** are more important than others, like sines and cosines, exponentials, switches and impulses
- If **f(t)** is continuous in time, use ODEs (Ordinary Differential Equations), if the input comes in discrete time steps, use Linear Algebra

We learn to solve

- Exponential growth/decay: $\mathrm{d}y/\mathrm{d}t = ay$

- Growth/Decay with added input: $\mathrm{d}y/\mathrm{d}t = ay + q(t)$
  - Input q(t) may oscillate (complex exponential)
  - Growth factor may vary with time: a = a(t)

- The nonlinear logistic equation (limiting growth by competition):

$$\mathrm{d}y/\mathrm{d}t = ay - by^2$$

- Special types of first order nonlinear differential equations:

$$\mathrm{d}y/\mathrm{d}t = g(t)/f(y)$$

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Three Examples

- Linear ODE with constant coefficient:

$$y(t) = \exp(at) \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = a\,y$$

- Linear ODE with variable coefficient:

$$y(t) = \exp(t^2) \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = 2t\,y$$

- Nonlinear ODE:

$$y(t) = \frac{1}{1-t} \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = y^2$$

- The first two examples represent linear ODEs, i.e., a solution multiplied with a constant is still a solution.
- The given $y(t)$ is the special solution for $y(0) = 1$

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Derivatives of Key Functions

- From first principles:

$$y(t) = t^n \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = nt^{n-1}$$

- From geometrical considerations:

$$y(t) = \sin(t) \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = \cos(t)$$

$$y(t) = \cos(t) \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = -\sin(t)$$

- From definition: $\frac{\mathrm{d}y}{\mathrm{d}t} = y \;\Rightarrow\; y(t) = \exp(t)y(0)$

- From definition (inverse to exp): $y(t) = \ln(t) \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = 1/t$

Rules of Derivatives

- Sum rule (Linearity): $\dfrac{\mathrm{d}(ay_1 + by_2)}{\mathrm{d}t} = a\dfrac{\mathrm{d}y_1}{\mathrm{d}t} + b\dfrac{\mathrm{d}y_2}{\mathrm{d}t}$
- Product rule: $\dfrac{\mathrm{d}(y_2 y_1)}{\mathrm{d}t} = y_2\dfrac{\mathrm{d}y_1}{\mathrm{d}t} + y_1\dfrac{\mathrm{d}y_2}{\mathrm{d}t}$
- Reciprocal rule: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{y}\right) = -\dfrac{1}{y^2}\dfrac{\mathrm{d}y}{\mathrm{d}t}$
- Chain rule: $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}t}$

Other rules, like the quotient rule, follow. Try it!

The Fundamental Theorem of Calculus

The two operations, Differentiating and Integrating, invert each other:

$$\int_a^b \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = y(b) - y(a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t y(\tau)\,\mathrm{d}\tau = y(t)$$

Basically, the Fundamental Theorem follows from the finite difference definition of the derivative and integral:

$$\mathrm{slope}\,\Delta t = \Delta y$$

$$\mathrm{area}/\Delta t = y$$

Finite Differences Are Useful

- Linear approximation:

$$y(t + \Delta t) \approx y(t) + \text{slope}\, \Delta t$$

- Newton's Method to solve $y(t) = 0$:

$$t_{\text{new}} = t_{\text{old}} + \frac{0 - y(t_{\text{old}})}{\text{slope}}$$

- Better numerical algorithms:

$$\text{slope} = \frac{\Delta y}{\Delta t} = \frac{y\left(t + \Delta t/2\right) - y\left(t - \Delta t/2\right)}{\Delta t}$$

Gives the correct slope for straight lines AND parabolas when higher order terms in Δt beyond the first can be ignored!

Taylor Series

- Improving the linear approximation:

$$y(t+\Delta t) = \sum_{n=0}^{\infty} \frac{(\Delta t)^n}{n!}\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = y(t) + \Delta t \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{2}(\Delta t)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \cdots$$

- Two important examples (**t=0, Δt = t**):

$$y(t) = \exp(t) = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$$

$$y(t) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

Educated Guessing is Desirable

Let's guess the solution of:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y + q(t)$$

Strategy:

- At each time **s** the system receives a new input q(s)
- The corresponding contribution to y growths exponentially during the remaining time **t-s** (Input leads to output)
- Thus, at time **t** the input at **s** has contributed: $\exp(t-s)q(s)$
- Since the equation is linear, the total output can be obtained by superposition, i.e., integration:

$$y(t) = \int_0^t \exp(t-s)q(s)\,\mathrm{d}s$$

- Compute the derivative to verify

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Important Properties of Exponentials

We defined **exp(t)** as the solution of **dy/dt=y** with **y(0)=1**. All other properties follow!

- Constructing **exp(t)**:

$$y \approx 1 + t + \frac{1}{2}t^2 \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = 1 + t$$

$$y \approx 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = 1 + t + \frac{1}{2}t^2$$

$$y = \exp(t) = \sum_{i=0}^{\infty} \frac{t^n}{n!} \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = y$$

The deduced construction correspond to the *Taylor series* of **exp(t)**.

More Properties

- *Multiplying powers* by adding exponents:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (a+b)y \;\Rightarrow\; y(t) = \exp[(a+b)t]$$
$$\Rightarrow\; y(t) = \exp(at)\exp(bt)$$

Since both functions are a solution to the same ODE with the same initial condition $y(0)=1$, they must be equal!

- *Difference Equation* for compound interest:
  - Calculating the balance at time t in N finite steps $\Delta t = t/N$

$$y_{n+1} = (1 + a\Delta t)y_n \;\Rightarrow\; \exp(at) = \lim_{N\to\infty}\left(1 + \frac{at}{N}\right)^N$$

$$y_n = y_{n-1}/(1 - a\Delta t) \;\Rightarrow\; \exp(at) = \lim_{N\to\infty}\left(\frac{1}{1 - at/N}\right)^N$$

Complex Exponents

- *Euler's Formula*:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = iy \;\Rightarrow\; y(t) = \exp(it)$$
$$\Rightarrow\; y(t) = \cos(t) + i\sin(t)$$

Since both functions are a solution to the same ODE with the same initial condition **y(0)=1**, they must be equal!

One might also use the constructing rule (Taylor series) for **exp(it)** and identify the sum of the even powers of **t** as **cos(t)** and the sum of the odd powers of **t** as **sin(t)**.

Matrix Exponents

- Linear systems and the *Matrix exponential*:

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = A\vec{y} \;\Rightarrow\; \vec{y}(t) = \exp(At)\,\vec{y}(0)$$

The solution matrix can be constructed as infinite series of powers of the matrix **A**:

$$\exp(At) = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

Suppose the matrix **A** can be diagonalized, $\Lambda = V^{-1}AV$

then:

$$\exp(At) = V\exp(\Lambda t)\,V^{-1}$$

Like the eigenvalues λ_i are on the diagonal of $\boldsymbol{\Lambda}$, the numbers **exp(λ_it)** are on the diagonal of **exp(Λt)**

Example

Consider the rotation matrix $$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since: $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = -i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

we know the eigenvalues and -vectors of A.

Thus:

$$\exp(At) = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

The corresponding linear system with y(0) = [1 0] is then solved by:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

Review of the Key Ideas

- In the series for **exp(t)**, each term $t^n/n!$ is the derivative of the next term
- Then the derivative of **exp(t)** is **exp(t)**, and the exponent rule holds: $\exp(t)\exp(T) = \exp(t+T)$
- Another approach to $dy/dt = y$ is by finite differences $(Y_{n+1} - Y_n)/\Delta t = Y_n$. Note that $Y_{n+1} = Y_n + \Delta t Y_n$ is the same as compound interest. Then Y_n is close to $\exp(n\,\Delta t)\,Y_0$.
- $y=\exp(at)$ solves $y' = ay$, and $a=i$ leads to $\exp(it) = \cos(t) + i\sin(t)$ (Euler's Formula).
- $\cos(t) = 1 - t^2/2 + \ldots$ and $\sin(t) = t - t^3/6 + \ldots$ are the even and odd parts of **exp(it)**.

Problem Set

All problems can be found in our textbook
Nomenclature: ProblemSet_Problem

- Types of ODE: 1.1_7 e 8, 1.1_10, 1.1_11, 1.1_12
- Exponentials: 1.3_1, 1.3_9, 1.3_10, 1.3_11, 1.3_15, 1.3_22

All problems are summarized in the file L1.pdf

»Wissen schafft Brücken.«