

Chapter 5 Trigonometric Functions: Right Triangle Approach

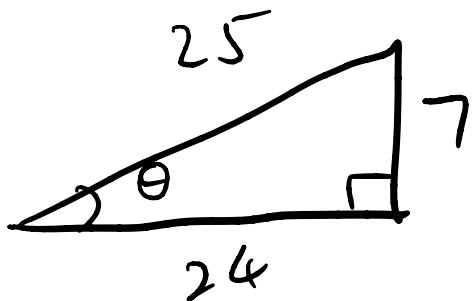
5.2 Trigonometry of Right Triangles

- ① Trigonometric Ratios
- ② Using a Calculator
- ③ Finding an Unknown Side
- ④ Evaluating an Expression
- ⑤ Solving a Right Triangle
- ⑥ Applications

① Trigonometric Ratios

3. $\sin \theta = \frac{4}{5}$, $\cos \theta = \frac{3}{5}$, $\tan \theta = \frac{4}{3}$
 $\csc \theta = \frac{5}{4}$, $\sec \theta = \frac{5}{3}$, $\cot \theta = \frac{3}{4}$

4.



$$\sin \theta = \frac{7}{25}$$

$$\csc \theta = \frac{25}{7}$$

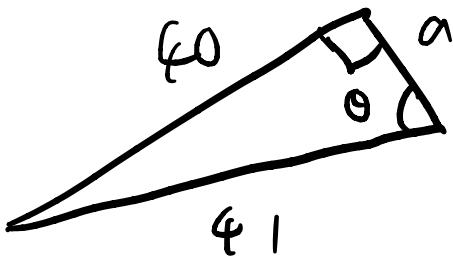
$$\cos \theta = \frac{24}{25}$$

$$\sec \theta = \frac{25}{24}$$

$$\tan \theta = \frac{7}{24}$$

$$\cot \theta = \frac{24}{7}$$

5.



$$\sin \theta = \frac{40}{41}$$

$$\csc \theta = \frac{41}{40}$$

$$40^2 + a^2 = 41^2$$

$$\sec \theta = \frac{41}{9}$$

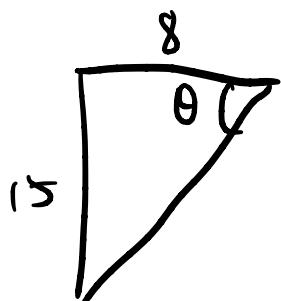
$$\begin{aligned} a &= \sqrt{41^2 - 40^2} \\ &= 9 \end{aligned}$$

$$\cot \theta = \frac{9}{40}$$

$$\cos \theta = \frac{9}{41}$$

$$\tan \theta = \frac{40}{9}$$

6.



$$8^2 + 15^2 = h^2$$

$$h = 17$$

$$\sin \theta = \frac{15}{17} \quad \csc \theta = \frac{17}{15}$$

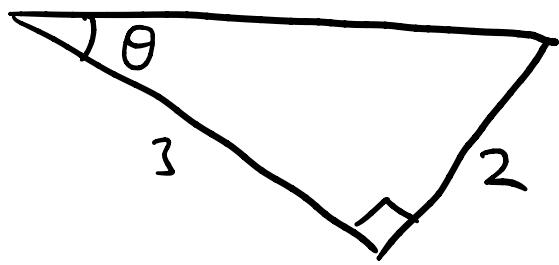
$$\cos \theta = \frac{8}{17} \quad \sec \theta = \frac{17}{8}$$

$$\tan \theta = \frac{15}{8} \quad \cot \theta = \frac{8}{15}$$

7.

$$h = \sqrt{3^2 + 2^2}$$

$$= \sqrt{13}$$

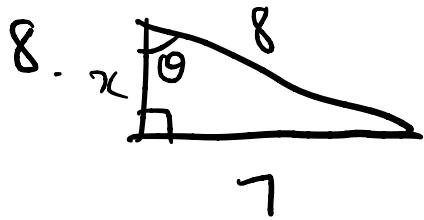


$$\sin \theta = \frac{2}{\sqrt{13}}$$

$$\csc \theta = \frac{\sqrt{13}}{2}$$

$$\cos \theta = \frac{3}{\sqrt{13}} \quad \sec \theta = \frac{\sqrt{13}}{3}$$

$$\tan \theta = \frac{2}{3} \quad \cot \theta = \frac{3}{2}$$



$$x = \sqrt{8^2 - 7^2}$$

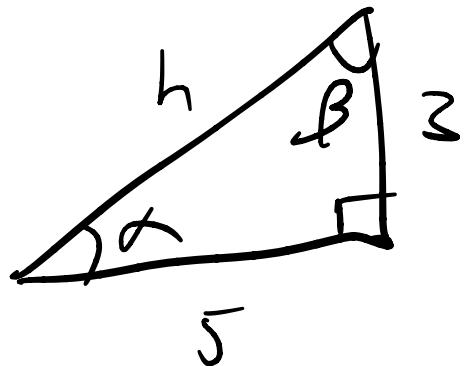
$$\csc \theta = \frac{8}{7}$$

$$\cos \theta = \frac{\sqrt{15}}{8} \quad \sec \theta = \frac{8}{\sqrt{15}}$$

$$\tan \theta = \frac{7}{\sqrt{15}} \quad \cot \theta = \frac{\sqrt{15}}{7}$$

9/4/2024

q.



$$h = \sqrt{3^2 + 5^2}$$
$$= \sqrt{34}$$

(a) $\sin \alpha = \frac{3}{\sqrt{34}}$

$$\cos \beta = \frac{3}{\sqrt{34}}$$

(b) $\tan \alpha = \frac{3}{5}$

$$\cot \beta = \frac{3}{5}$$

(c) $\sec \alpha = \frac{1}{\cos \alpha}$

$$= \frac{\sqrt{34}}{5}$$

$$\csc \beta = \frac{\sqrt{34}}{5}$$

② Using a Calculator

11. (a) $\sin 22^\circ$, (b) $\cot 23^\circ$

$$\begin{aligned}\sin 22^\circ \\ = 0.37461\end{aligned}$$

$$\begin{aligned}\cot 23^\circ \\ = 2.35585\end{aligned}$$

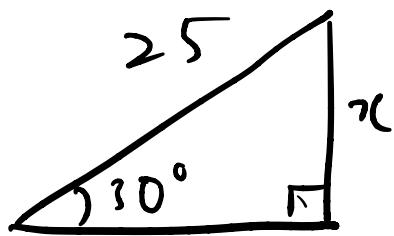
12. (a) $\cos 37^\circ$ (b) $\csc 48^\circ$
= 0.79864 = 1.34563

13. (a) $\sec 13^\circ$ (b) $\tan 51^\circ$
= 1.02630 = 1.23490

14. (a) $\csc 10^\circ$ (b) $\sin 46^\circ$
= 5.75877 = 0.71934

③ Finding an Unknown Side

15.



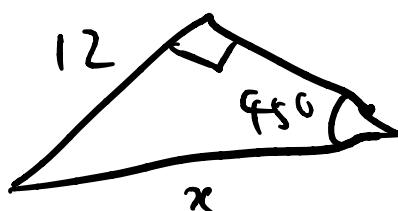
$$\sin 30^\circ = \frac{x}{25} = \frac{1}{2}$$

$$x = \frac{25}{2}$$

[2]

$$\begin{aligned} x &= 25 \sin 30^\circ \\ &= 25 \left(\frac{1}{2}\right) \\ &= \frac{25}{2} \end{aligned}$$

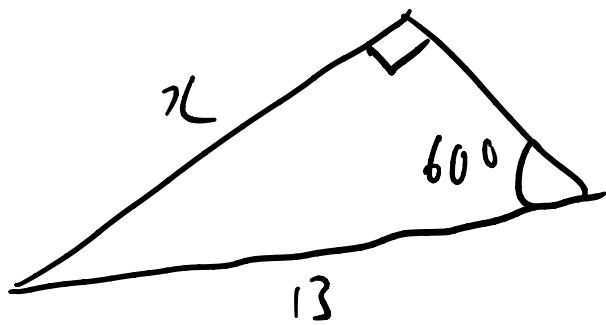
16.



$$\sin 45^\circ = \frac{12}{x} = \frac{1}{\sqrt{2}}$$

$$x = 12\sqrt{2}$$

17.

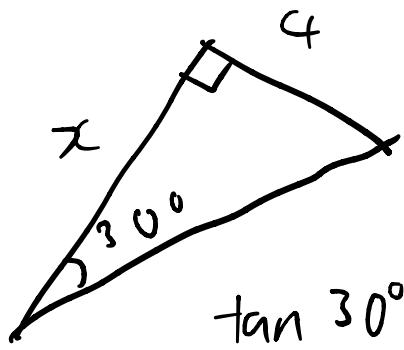


$$\sin 60^\circ = \frac{x}{13} = \frac{\sqrt{3}}{2}$$

$$x = \frac{13\sqrt{3}}{2}$$

$$= 11.25833$$

18.



$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

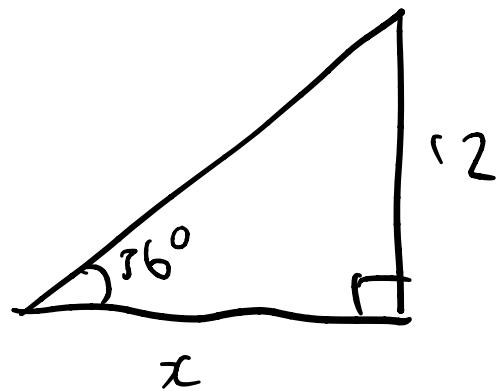
$$\tan 30^\circ = \frac{4}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{4}{x}$$

$$x = 4\sqrt{3}$$

$$= 6.92820$$

19.

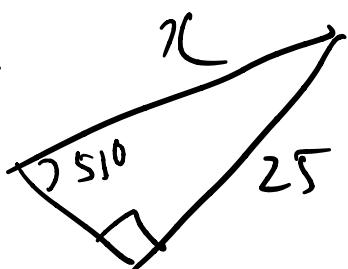


$$\tan 36^\circ = \frac{12}{x}$$

$$x = \frac{12}{\tan 36^\circ}$$

$$= 16.51658$$

20.

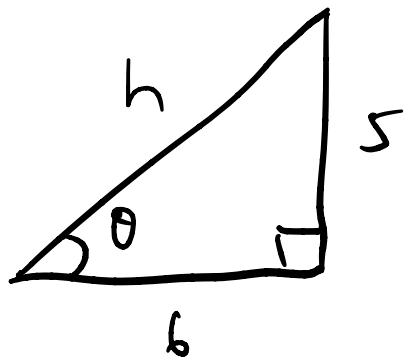


$$\sin 53^\circ = \frac{25}{x}$$

$$x = \frac{25}{\sin 53^\circ}$$

$$= 31.3$$

$$23. \tan \theta = \frac{5}{6}$$



$$\begin{aligned}5^2 + b^2 &= h^2 \\h &= \sqrt{25+36} \\&= \sqrt{61}\end{aligned}$$

$$\sin \theta = \frac{5\sqrt{61}}{61}$$

$$\cos \theta = \frac{6\sqrt{61}}{61}$$

$$\csc \theta = \frac{\sqrt{61}}{5}$$

$$\sec \theta = \frac{\sqrt{61}}{6}$$

$$\cot \theta = \frac{6}{5}$$

29. $\sin \frac{\pi}{6} + \cos \frac{\pi}{6}$ ④ Evaluating an Expression

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{1+\sqrt{3}}{2}$$

30. $\sin 30^\circ \csc 30^\circ$

$$= \frac{1}{2} (2)$$

$$= 1$$

31. $\sin 30^\circ \cos 60^\circ + \sin 60^\circ \cos 30^\circ$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4} + \frac{3}{4}$$

$$= 1$$

32. $(\sin 60^\circ)^2 + (\cos 60^\circ)^2 = 1$

$$\begin{aligned} (\sin 60^\circ)^2 + (\cos 60^\circ)^2 &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1 \end{aligned}$$

$$\begin{aligned}
 33. \quad & (\cos 30^\circ)^2 - (\sin 30^\circ)^2 \\
 &= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \\
 &= \frac{3}{4} - \frac{1}{4} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad & \left(\sin \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{3} \right)^2 \\
 &= \left(\left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{2}\right) \right)^2 \\
 &= \left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right)^2 \\
 &= \frac{(\sqrt{3}-1)^2}{(2\sqrt{2})^2} \\
 &= \frac{3-2\sqrt{3}+1}{8} \\
 &= \frac{4-2\sqrt{3}}{8} \\
 &= \frac{2-\sqrt{3}}{4}
 \end{aligned}$$

$$35. \left(\cos \frac{\pi}{4} + \sin \frac{\pi}{6} \right)^2$$

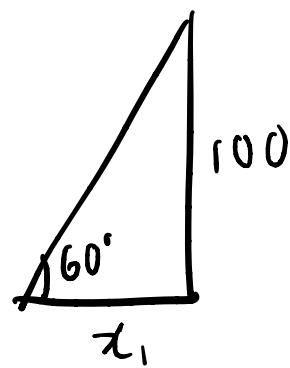
$$= \left(\frac{\sqrt{2}}{2} + \frac{1}{2} \right)^2$$

$$= \left(\frac{\sqrt{2}+1}{2} \right)^2$$

$$= \frac{2 + 2\sqrt{2} + 1}{4}$$

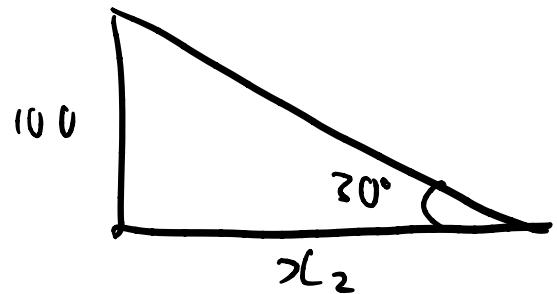
$$= \frac{3}{4} + \frac{\sqrt{2}}{2}$$

47.



$$\tan \frac{\pi}{3} = \frac{\sqrt{3}}{1} = \frac{100}{x_1}$$

$$x_1 = \frac{100}{\sqrt{3}}$$



$$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{100}{x_2}$$

[2]

$$\tan \frac{\pi}{3} = \frac{100}{x_1}$$

$$\tan \frac{\pi}{6} = \frac{100}{x_2}$$

$$x_1 = \frac{100}{\tan \frac{\pi}{3}}$$

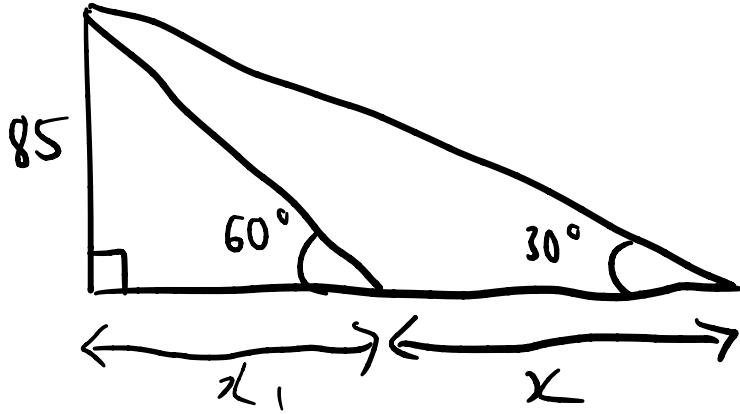
$$x_2 = \frac{100}{\tan \frac{\pi}{6}}$$

$$x = x_1 + x_2$$

$$= \frac{100}{\tan \frac{\pi}{3}} + \frac{100}{\tan \frac{\pi}{6}}$$

$$\begin{aligned} x &= x_1 + x_2 \\ &= \frac{100}{\sqrt{3}} + 100\sqrt{3} \\ &= \frac{100}{\sqrt{3}} + 100\frac{\sqrt{3}}{\sqrt{3}}(\sqrt{3}) \\ &= \frac{100}{\sqrt{3}} + \frac{300}{\sqrt{3}} \\ &= \frac{400}{\sqrt{3}} \\ &= \frac{400\sqrt{3}}{3} \\ &= 230.94^\circ \end{aligned}$$

48.



$$\tan 60^\circ = \sqrt{3} = \frac{85}{x_1}$$

$$x_1 = \frac{85}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{85}{x_2}$$

$$x_2 = 85\sqrt{3}$$

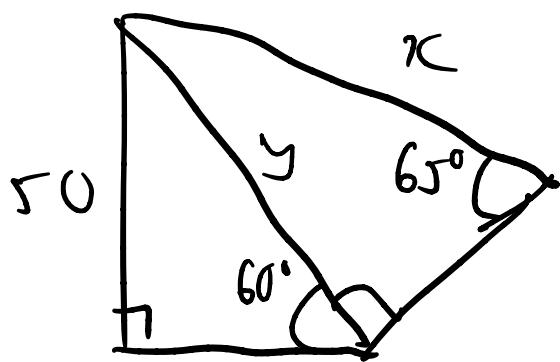
$$x_2 = x_1 + x$$

$$\begin{aligned} x &= x_2 - x_1 \\ &= 85\sqrt{3} - \frac{85}{\sqrt{3}} \end{aligned}$$

$$= \frac{85\sqrt{3}(\sqrt{3})}{\sqrt{3}} - \frac{85}{\sqrt{3}}$$

$$= \frac{170}{\sqrt{3}} \text{ or } 98.1$$

49.



$$\sin 60^\circ = \frac{50}{y}$$

$$\sin 65^\circ = \frac{100}{x}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$x = \frac{100}{\sqrt{3}}$$

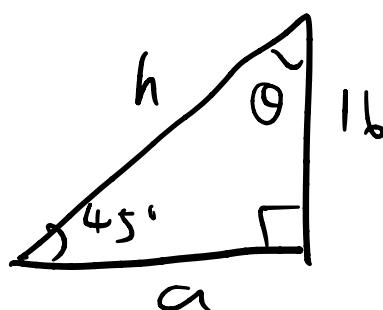
$$\frac{50}{y} = \frac{\sqrt{3}}{2}$$

$$= 5 \cdot \sin 65^\circ$$

$$y = \frac{100}{\sqrt{3}}$$

⑤ Solving a Right Triangle

37.



$$\tan 45^\circ = \frac{b}{a}$$

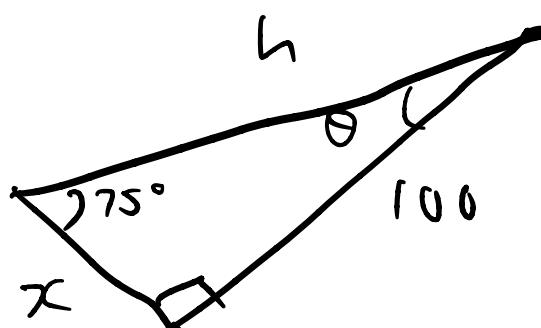
$$\begin{aligned} a &= \frac{b}{\tan 45^\circ} \\ &= \frac{b}{1} \\ &= b \end{aligned}$$

$$\sin 45^\circ = \frac{b}{h}$$

$$\begin{aligned} h &= \frac{b}{\sin 45^\circ} \\ &= \frac{b}{\frac{1}{\sqrt{2}}} \\ &= b\sqrt{2} \end{aligned}$$

$$\theta = 45^\circ$$

38.



$$\theta = 90 - 75$$

$$= 15^\circ$$

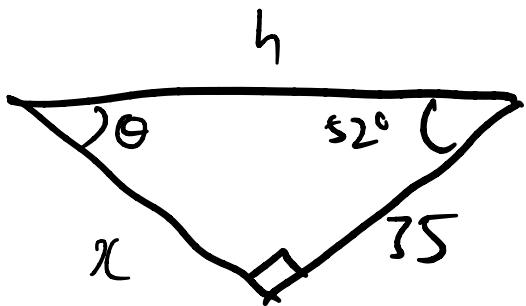
$$\sin 75^\circ = \frac{100}{h}$$

$$\begin{aligned} h &= \frac{100}{\sin 75^\circ} \\ &= 103.53 \end{aligned}$$

$$\tan 75^\circ = \frac{100}{x}$$

$$\begin{aligned} x &= \frac{100}{\tan 75^\circ} \\ &= 26.79 \end{aligned}$$

39.



$$\begin{aligned}\theta &= 90 - 52 \\ &= 38^\circ\end{aligned}$$

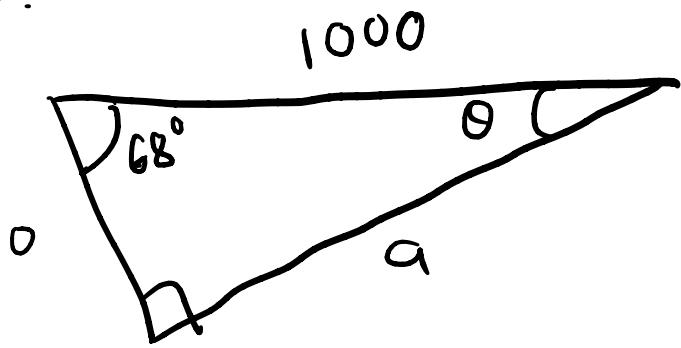
$$\tan 52^\circ = \frac{x}{35}$$

$$\begin{aligned}x &= 35 \tan 52^\circ \\ &= 44.80\end{aligned}$$

$$\cos 52^\circ = \frac{35}{h}$$

$$\begin{aligned}h &= \frac{35}{\cos 52^\circ} \\ &= 56.8\end{aligned}$$

40.



$$\begin{aligned}\theta &= 180^\circ - 68^\circ - 90^\circ \\ &= 22^\circ\end{aligned}$$

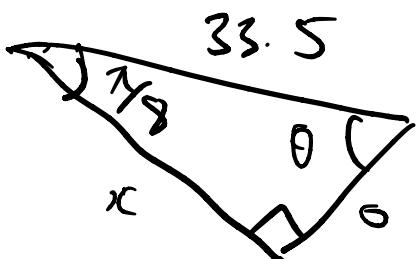
$$\sin 22^\circ = \frac{O}{100}$$

$$\begin{aligned}O &= 100 \sin 22^\circ \\ &= 374.61\end{aligned}$$

$$\cos 22^\circ = \frac{a}{100}$$

$$\begin{aligned}a &= 100 \cos 22^\circ \\ &= 927.18\end{aligned}$$

41.



$$\begin{aligned}\theta &= \frac{\pi}{2} - \frac{\pi}{8} \\ &= \frac{3\pi}{8}\end{aligned}$$

$$\sin \frac{\pi}{8} = \frac{O}{33.5}$$

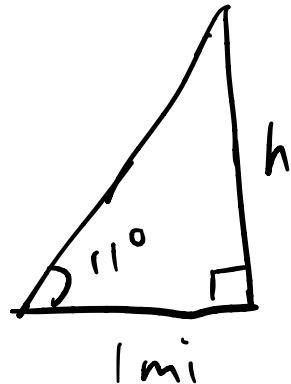
$$O = 12.82$$

$$\cos \frac{\pi}{8} = \frac{x}{33.5}$$

$$x = 30.95$$

53.

⑥ Applications



$$\tan 11^\circ = 0.19438$$

$$\tan 11^\circ = \frac{h}{1}$$

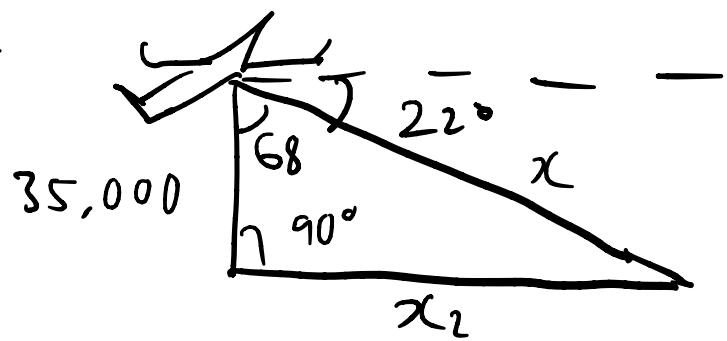
$$h = 0.19438 \text{ mi}$$

$$= 0.19438 \times 1609.344 \text{ m}$$

$$= 312.82 \text{ m}$$

$$\boxed{z} \sim 1026 \text{ ft}$$

54.



$$(a) \cos 68^\circ = \frac{35000}{x}$$

$$x = \frac{35000}{\cos 68^\circ}$$

$$= 93451.4 \text{ ft}$$

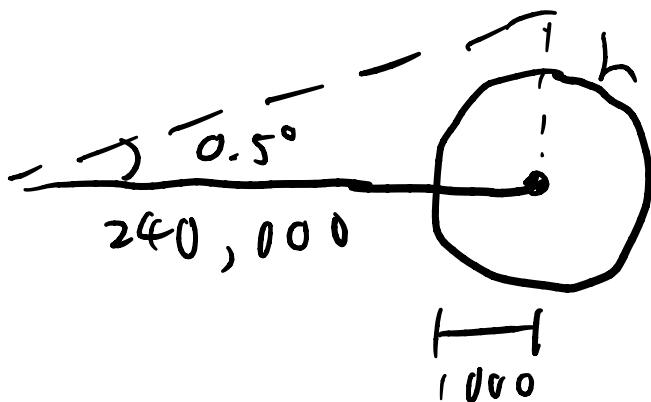
away

$$(b) \tan 68^\circ = \frac{x_2}{35000}$$

$$x_2 = 35000 \tan 68^\circ$$

$$= 86628 \text{ ft}$$

55.

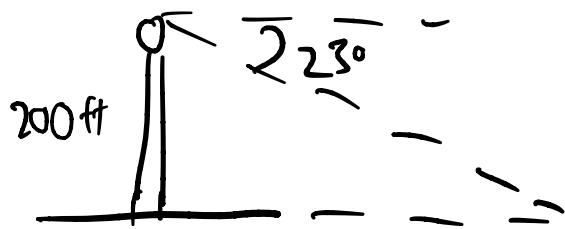


$$(a) \tan 0.5 = \frac{h}{240000}$$

$$h = 240000 \tan 0.5^\circ \\ = 2094.4 \text{ mi}$$

(b) No, because the beam diverges 2094 mi away from the center of the moon which is more than its radius

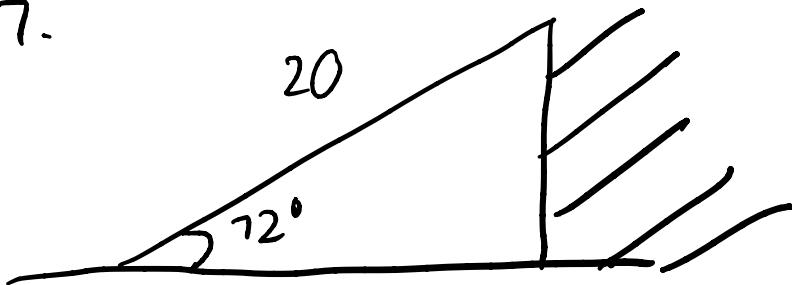
56.



$$\tan 23^\circ = \frac{200}{x}$$

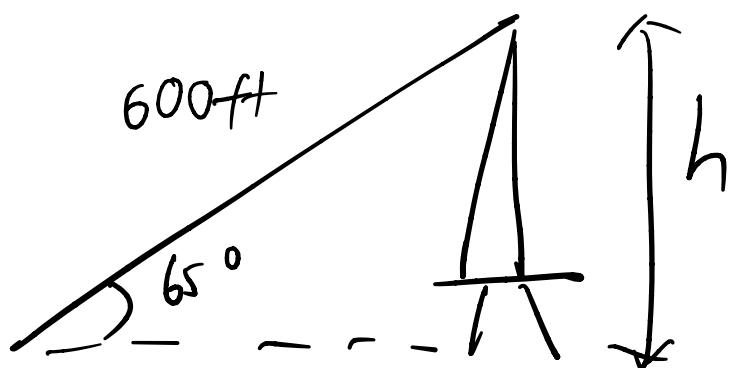
$$x = \frac{200}{\tan 23^\circ} \\ = 471.2 \text{ ft}$$

57.



$$\begin{aligned} h &= 20 \sin 72^\circ \\ &= 19.02 \text{ ft} \end{aligned}$$

58.



$$\begin{aligned} h &= 600 \sin 65^\circ \\ &= 543.78 \text{ ft} \end{aligned}$$

5.3 Trigonometric Functions of Angles

- ① Reference Angle
- ② Values of Trigonometric Functions
- ③ Expressing One Trigonometric Function
in Terms of Another
- ④ Values of Trigonometric Functions II
- ⑤ Area of a Triangle
- ⑥ Applications

① Reference Angle

5. (a) 120° (b) 200° (c) 285°
 $\bar{\theta} = 60^\circ$ $\bar{\theta} = 20^\circ$ $\bar{\theta} = 75^\circ$

6. (a) 175° (b) 310° (c) 730°

$$\begin{aligned}\bar{\theta} &= 5^\circ & \bar{\theta} &= 360^\circ - 310^\circ & = 130^\circ - 2(360^\circ) \\ & & & = 50^\circ & = 10^\circ\end{aligned}$$

7. (a) 225° (b) 810° (c) -105°
 $\bar{\theta} = 225^\circ - 180^\circ$ $\bar{\theta} = 810^\circ - 2(360^\circ)$ $\bar{\theta} = -105^\circ$
 $= 45^\circ$ $= 90^\circ$ $+ 180^\circ$
 $= 75^\circ$

8. (a) 99° (b) -199° (c) 359°

$$\begin{aligned}& = 180^\circ - 99^\circ & & 360^\circ - 199^\circ & = 360^\circ - 359^\circ \\ & = 81^\circ & & = 161^\circ & = 1^\circ \\ & & & 180^\circ - 161^\circ & \\ & & & = 19^\circ &\end{aligned}$$

9. (a) $\frac{7\pi}{10}$ (b) $\frac{9\pi}{8}$ (c) $\frac{10}{3}\pi$
 $\pi - \frac{7\pi}{10}$ $\frac{9\pi}{8} - \pi$ $\frac{10}{3}\pi - 3\pi$
 $= \frac{3}{10}\pi$ $= \frac{1}{8}\pi$ $= \frac{1}{3}\pi$

(2) Values of Trigonometric Functions

$$13. \cos 150^\circ$$

$$\begin{aligned} &= -\cos 30^\circ \\ &= -\frac{-\sqrt{3}}{2} \end{aligned}$$

$$16. \sin(-30^\circ)$$

$$\begin{aligned} &= -\sin 30^\circ \\ &= -\frac{1}{2} \end{aligned}$$

$$14. \sin 240^\circ$$

$$\begin{aligned}\bar{\theta} &= 240 - 180 \\ &= 60^\circ\end{aligned}$$

$$\sin 240^\circ = -\sin 60^\circ$$

$$= -\frac{-\sqrt{3}}{2}$$

$$15. \tan 330^\circ$$

$$\begin{aligned}\bar{\theta} &= 360 - 330 \\ &= 30^\circ\end{aligned}$$

$$\tan 330^\circ = -\tan 30^\circ$$

$$= -\frac{1}{\sqrt{3}}$$

$$16. \sin(-30^\circ)$$

$$= -\sin 30^\circ$$

$$= -\frac{1}{2}$$

$$17. \cot(-120^\circ)$$

$$= \frac{1}{\tan(-120^\circ)}$$

$$= \frac{1}{-\tan 120^\circ}$$

$$= \frac{1}{-(-\tan 60^\circ)}$$

$$= \frac{1}{\tan 60^\circ}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

37. $\sin \theta < 0$ and $\cos \theta < 0$

Quadrant 3

③ Expressing One Trigonometric Function in Terms of Another

41. $\tan \theta$, $\cos \theta$, θ in Quadrant III

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \sin^2 \theta + \cos^2 \theta &= 1 \\ &= \frac{-\sqrt{1 - \cos^2 \theta}}{\cos \theta} & \sin^2 \theta &= 1 - \cos^2 \theta \\ && \sin \theta &= \pm \sqrt{1 - \cos^2 \theta}\end{aligned}$$

42. $\cot \theta$, $\sin \theta$ in Quadrant II

$$\begin{aligned}\cot \theta &= \pm \frac{1}{\sin \theta} \sqrt{1 - \sin^2 \theta} & 1 + \cot^2 \theta &= \csc^2 \theta \\ && \cot \theta &= \pm \sqrt{\csc^2 \theta - 1} \\ \therefore \text{Quadrant II} && \cot \theta &= \pm \sqrt{\frac{1}{\sin^2 \theta} - \frac{\sin^2 \theta}{\csc^2 \theta}} \\ && &= \pm \frac{1}{\sin \theta} \sqrt{1 - \sin^2 \theta}\end{aligned}$$

43. $\cos \theta, \sin \theta$ θ in Quadrant IV

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = +\sqrt{1 - \sin^2 \theta}$$

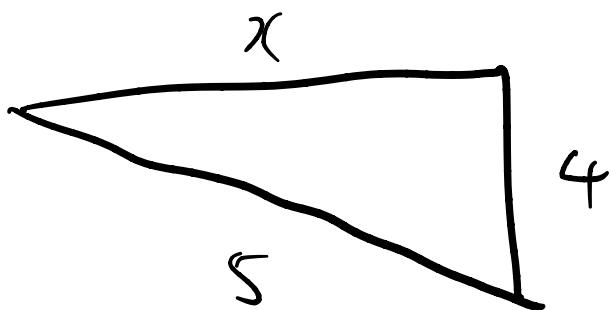
(Q4)

44. $\sec \theta, \sin \theta$ θ in Quadrant I

$$\begin{aligned}\sec \theta &= \frac{1}{\cos \theta} \\ &= \frac{1}{\sqrt{1 - \sin^2 \theta}}\end{aligned}$$

4/6 Values of Trigonometric Functions II

47. $\sin \theta = -\frac{4}{5}$, θ in Quadrant 4



$$x^2 + 4^2 = 5^2$$

$$x = \sqrt{5^2 - 4^2}$$

$$= 3$$

$$\cos \theta = \frac{3}{5}$$

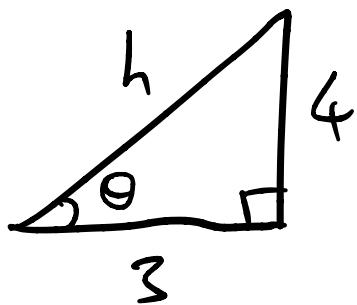
$$\tan \theta = -\frac{4}{3}$$

$$\csc \theta = -\frac{5}{4}$$

$$\sec \theta = \frac{5}{3}$$

$$\cot \theta = -\frac{3}{4}$$

$$48. \quad \tan \theta = \frac{4}{3} \rightarrow \text{Quadrant III}$$



$$\begin{aligned} h &= \sqrt{4^2 + 3^2} \\ &= 5 \end{aligned}$$

$$\sin \theta = -\frac{4}{5}$$

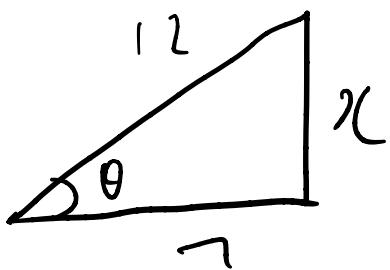
$$\cos \theta = -\frac{3}{5}$$

$$\csc \theta = -\frac{5}{4}$$

$$\sec \theta = -\frac{5}{3}$$

$$\cot \theta = \frac{3}{4}$$

$$49. \cos \theta = \frac{7}{12}, \sin \theta < 0$$



$$7^2 + x^2 = 12^2$$

$$\begin{aligned}x &= \sqrt{12^2 - 7^2} \\&= \sqrt{95}\end{aligned}$$

$$\csc \theta = -\frac{12}{\sqrt{95}}$$

$$\sec \theta = \frac{12}{7}$$

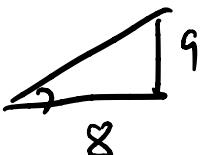
$$\sin \theta = -\frac{\sqrt{95}}{12}$$

$$\cot \theta = -\frac{7}{\sqrt{95}}$$

$$\tan \theta = -\frac{\sqrt{95}}{7}$$

$$50. \cot \theta = -\frac{8}{9}, \cos \theta > 0$$

$$\tan \theta = -\frac{9}{8}$$



$$\csc \theta = -\frac{\sqrt{145}}{9}$$

$$\sin \theta = -\frac{9}{\sqrt{145}}$$

$$\begin{aligned}h &= \sqrt{9^2 + 8^2} \\&= \sqrt{145}\end{aligned}$$

$$\sec \theta = \frac{\sqrt{145}}{8}$$

$$\cos \theta = \frac{8}{\sqrt{145}}$$

51. $\csc \theta = 2$, θ in Quadrant I

$$\sin \theta = \frac{1}{2}$$

$$\theta = \sin^{-1} \frac{1}{2}$$

$$= 30^\circ$$

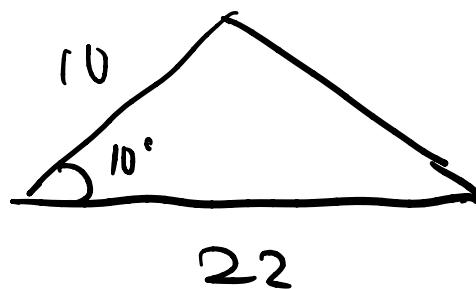
⑤ Area of a Triangle

$$57. \quad \frac{1}{2} ab \sin \theta$$

$$= \frac{1}{2}(9)(7 \sin 72^\circ)$$

$$= 29.96$$

58.

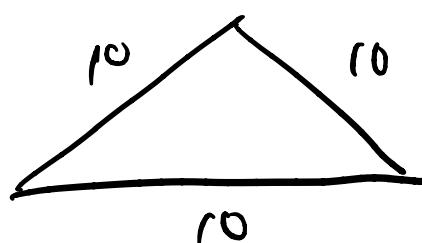


$$A = \frac{1}{2} ab \sin \theta$$

$$= \frac{1}{2} 22 (10 \sin 10^\circ)$$

$$= 19.10$$

59.



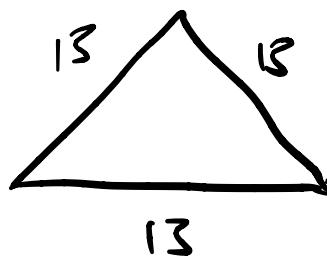
$$\theta = \frac{180}{3} = 60^\circ$$

$$A = \frac{1}{2} ab \sin \theta$$

$$= \frac{1}{2} 10 (10 \sin 60^\circ)$$

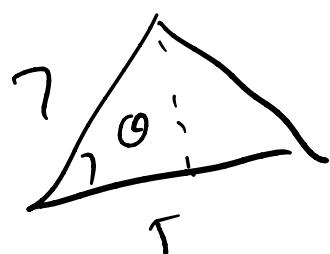
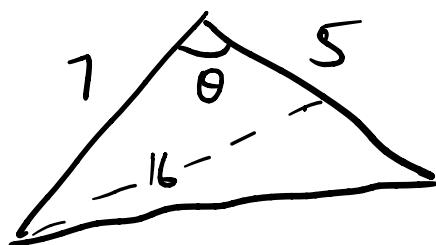
$$= \frac{1}{2} 10 \left(10 \frac{\sqrt{3}}{2}\right) = 25\sqrt{3}$$

60.



$$\begin{aligned}
 A &= \frac{1}{2} ab \sin \theta \\
 &= \frac{1}{2} (13)(13 \sin 60^\circ) \\
 &= \frac{1}{2} (13)(13) \left(\frac{\sqrt{3}}{2}\right) \\
 &= 73.179 \\
 &= 73.18
 \end{aligned}$$

61.



$$16 = \frac{1}{2} 7 (5 \sin \theta)$$

$$\sin \theta = \frac{16}{35}$$

$$\theta = \sin^{-1} \frac{16}{35}$$

$$= 66.10^\circ$$

6/6 Applications

65.



(a)

$$\tan \theta = \frac{h}{1\text{mi}}$$

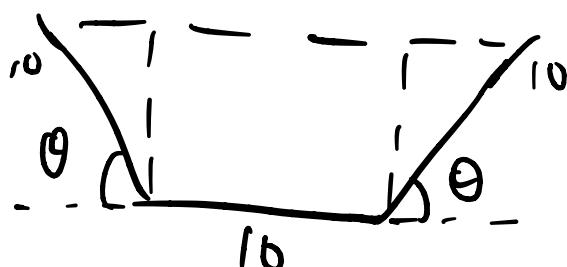
$$h = \tan \theta (1\text{mi})$$

$$= 5280 \tan \theta \text{ ft}$$

(b) θ 20° 60° 80° 85°

h 1922 9145 29,944 60,751

66.



$$\text{Area of square} = 10 \times 10 \sin \theta = 100 \sin \theta$$

$$\begin{aligned}\text{Area of each triangle} &= h \times l \times \frac{1}{2} \\ &= 10 \sin \theta \times 10 \cos \theta \times \frac{1}{2} \\ &= 50 \sin \theta \cos \theta\end{aligned}$$

Total Area $A(\theta) = \text{square} + 2 \text{ triangles}$

$$= 100 \sin \theta + 2(50 \sin \theta \cos \theta)$$

$$= 100 \sin \theta + 100 \sin \theta \cos \theta$$

\therefore shown

(b)	θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
A	0	93.3	120.71	129.90	100		

$$A\left(\frac{\pi}{6}\right) = 100\left(\frac{1}{2}\right) + 100\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= 50 + 25\sqrt{3}$$

$$= 93.3$$

$$A\left(\frac{\pi}{4}\right) = 100\left(\frac{1}{\sqrt{2}}\right) + 100\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{100\sqrt{2}}{2} + \frac{100}{2}$$

$$= 50\sqrt{2} + 50$$

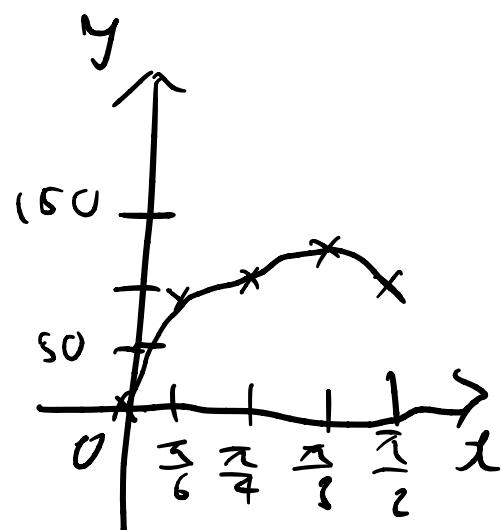
$$= 120.71$$

$$A\left(\frac{\pi}{3}\right) = 100\left(\frac{\sqrt{3}}{2}\right) + 100\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{200\sqrt{3}}{4} + \frac{100\sqrt{3}}{4}$$

$$= \frac{300\sqrt{3}}{4}$$

$$= 129.90$$



$$A\left(\frac{\pi}{2}\right)$$

$$= 100(1) + 0$$

$$= 100$$

$$(c) \theta \approx \frac{\pi}{3}$$

2 from graph, largest area is achieved

when $\theta \approx 1.047 \text{ rad} \approx 60^\circ$

$$67. d = 20 \sin \theta$$

$$w = 20 \cos \theta$$

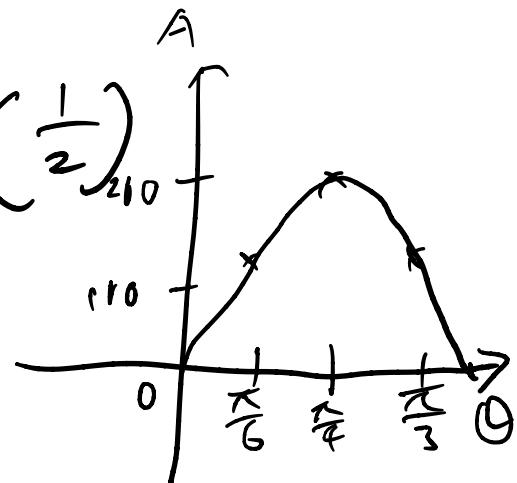
$$(a) A(\theta) = 20 \sin \theta \times 20 \cos \theta \\ = 400 \sin \theta \cos \theta$$

$$(b) A\left(\frac{\pi}{6}\right) = 20 \left(\frac{1}{2}\right) \times 20 \left(\frac{\sqrt{3}}{2}\right) \\ = 100\sqrt{3}$$

$$A\left(\frac{\pi}{4}\right) = 20 \left(\frac{1}{\sqrt{2}}\right) \times 20 \left(\frac{1}{\sqrt{2}}\right) \\ = 200$$

$$A\left(\frac{\pi}{3}\right) = 20 \left(\frac{\sqrt{2}}{2}\right) \times 20 \left(\frac{1}{2}\right) \\ = 100\sqrt{3}$$

$$A\left(\frac{\pi}{2}\right) = 20 \times 0 \\ = 0$$



$$(c) d = 20 \sin\left(\frac{\pi}{4}\right)$$

$$= 20 \left(\frac{1}{\sqrt{2}}\right)$$

$$= 10\sqrt{2}$$

$$w = 20 \sin\left(\frac{\pi}{4}\right)$$

$$= 10\sqrt{2}$$

68. Strength of a beam \propto width and square of depth

$$\text{width} = 20 \cos \theta$$

$$\text{depth} = 20 \sin \theta$$

$$\therefore \text{Strength} = k (20 \cos \theta) (20 \sin \theta)^2$$

$$= 8000 k \cos \theta \sin^2 \theta$$

$$69. R = \frac{V_0^2 \sin(2\theta)}{g}, H = \frac{V_0^2 \sin^2 \theta}{2g}$$

$$(a) R = \frac{12^2 \sin(2(\frac{\pi}{6}))}{32} \quad 3.897 \text{ ft}$$

$$= \frac{144 \left(\frac{\sqrt{3}}{2}\right)}{32} \quad \uparrow$$

$$= \frac{9 \cdot 2 \left(\frac{\sqrt{3}}{2}\right)}{2} = \frac{9\sqrt{3}}{4}$$

$$H = \frac{12^2 \left(\sin \frac{\pi}{6}\right)^2}{2 \cdot (32)} \quad \text{d. SQL 5 ft}$$

$$= \frac{144 \left(\frac{1}{2}\right)^2}{2 \cdot (32)} \quad \uparrow$$

$$= \frac{18}{32} = \frac{9}{16}$$

$$(b) R = \frac{12^2 \sin\left(2\left(\frac{\pi}{6}\right)\right)}{5.2} + = \frac{144\left(\frac{1}{2}\right)^1}{2(5.2)} \\ = 3.46 + \\ = 13.84615$$

Solution says 23.982 ft

5.4 Inverse Trigonometric Functions and Right Triangles

① Evaluating Inverse Trigonometric Functions

② Evaluating Inverse Trigonometric Functions II

7 Sub-headers

① Evaluating Inverse Trigonometric Functions

$$5. (a) \sin^{-1} 1 \quad (b) \cos^{-1} 0$$

$$= \frac{\pi}{2} \text{ rad} \quad = \frac{\pi}{2} \text{ rad}$$

$$(c) \tan^{-1} \sqrt{3}$$

$$= \frac{\pi}{3} \text{ rad}$$

$$6. (a) \sin^{-1} 0 \quad (b) \cos^{-1}(-1)$$

$$= 0 \text{ rad} \quad = \pi \text{ rad}$$

$$(c) \tan^{-1} 0$$

$$= 0 \text{ rad}$$

$$7. (a) \sin^{-1} \left(-\frac{\sqrt{2}}{2} \right) \quad (b) \cos^{-1} \left(-\frac{\sqrt{2}}{2} \right) [0, \pi]$$

\because only defined between $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= -\frac{\pi}{4} \text{ rad} \quad = \frac{3\pi}{4} \text{ rad}$$

$$[-\frac{\pi}{2}, \frac{\pi}{2}] \quad (c) \tan^{-1}(-1)$$

$$= -\frac{\pi}{4} \text{ rad}$$

$$8. (a) \sin^{-1} \left(-\frac{\sqrt{3}}{2} \right) \quad (b) \cos^{-1} \left(-\frac{1}{2} \right)$$

$$\begin{aligned} &= -\frac{\pi}{3} \text{ rad} & &= \pi - \cos^{-1} \left(\frac{1}{2} \right) \\ &&&= \pi - \frac{\pi}{3} \\ &&&= \frac{2\pi}{3} \text{ rad} \end{aligned}$$

$$(c) \tan^{-1} (-\sqrt{3})$$

$$= -\tan^{-1} (\sqrt{3})$$

$$= -\frac{\pi}{3} \text{ rad}$$

② Evaluating Inverse Trigonometric Functions II

$$9. \sin^{-1}(0.30) \\ = 0.30469$$

$$10. \cos^{-1}(-0.2) \\ = \pi - \cos^{-1}(0.2) \\ = 1.77215$$

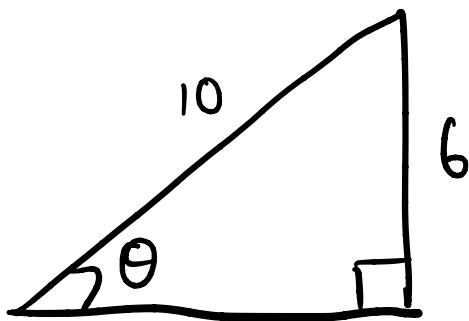
$$11. \cos^{-1}\frac{1}{3} \\ = 1.23096$$

$$12. \sin^{-1}\frac{5}{6} \\ = 0.98511$$

$$13. \tan^{-1}3 \\ = 1.24905$$

③ Finding Angles in Right Triangles

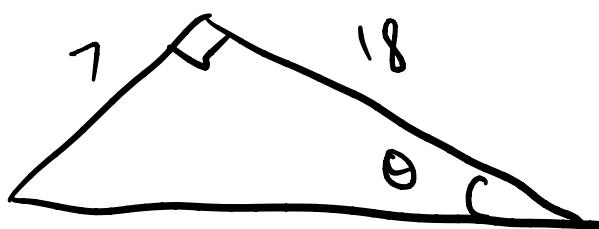
17.



$$\begin{aligned}\sin \theta &= \frac{6}{10} \\ &= \frac{3}{5}\end{aligned}$$

$$\sin^{-1} \frac{3}{5} = 36.9^\circ$$

18.

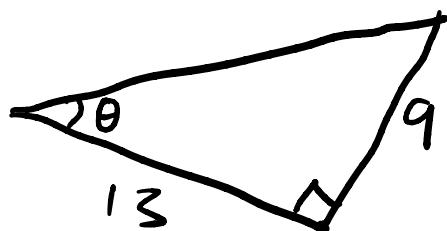


$$\tan \theta = \frac{7}{18}$$

$$\theta = \tan^{-1} \frac{7}{18}$$

$$= 21.3^\circ$$

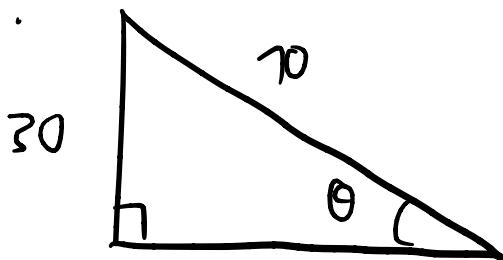
19.



$$\tan \theta = \frac{9}{13}$$

$$\begin{aligned}\theta &= \tan^{-1} \frac{9}{13} \\ &= 34.7^\circ\end{aligned}$$

20.

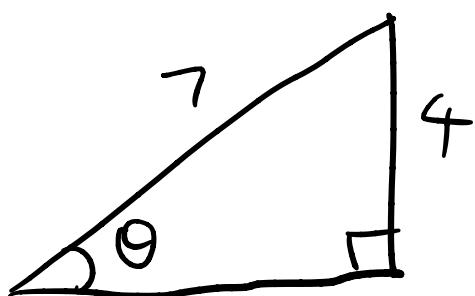


$$\sin \theta = \frac{30}{70}$$

$$\theta = \sin^{-1} \frac{3}{7}$$

$$= 25.4^\circ$$

21.

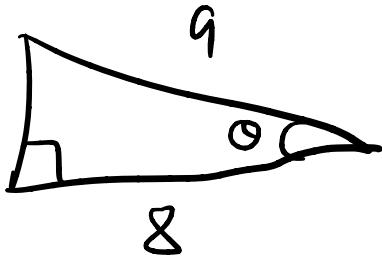


$$\sin \theta = \frac{4}{7}$$

$$\theta = \sin^{-1} \frac{4}{7}$$

$$= 34.8^\circ$$

22.



$$\cos \theta = \frac{8}{9}$$

$$\theta = \cos^{-1} \frac{8}{9}$$

$$= 27.3^\circ$$

④ Basic Trigonometric Equations

$$23. \sin \theta = \frac{2}{3}$$

$$\theta = \sin^{-1} \frac{2}{3}$$

$$= 41.8^\circ$$

$$180 - 41.8 = 138.2^\circ$$

$$\therefore \theta = 41.8^\circ, 138.2^\circ$$

$$24. \cos \theta = \frac{3}{4}$$

$$\theta = \cos^{-1} \frac{3}{4}$$

$$= 41.4^\circ$$

$$25. \cos \theta = -\frac{2}{5}$$

$$\theta = 180 - \cos^{-1} \frac{2}{5}$$

$$= 113.6^\circ$$

$$26. \tan \theta = -20$$

$$\begin{aligned}\theta &= -\tan^{-1} 20 \\ &= -87.1^\circ\end{aligned}$$

$$\theta = -87.1 + 180$$

$$= 92.9^\circ$$

$$27. \tan \theta = 5$$

$$\begin{aligned}\theta &= \tan^{-1} 5 \\ &= 28.7^\circ\end{aligned}$$

$$28. \sin \theta = \frac{4}{5}$$

$$\begin{aligned}\theta &= \sin^{-1} \frac{4}{5} \\ &= 53.1^\circ\end{aligned}$$

$$180^\circ - 53.1^\circ = 126.9^\circ$$

$$\therefore \theta = 53.1^\circ, 126.9^\circ$$

(5)

Value of an expression

$$\begin{aligned}
 29. \cos\left(\sin^{-1}\frac{4}{5}\right) &= \sqrt{1 - \sin^2\left(\sin^{-1}\frac{4}{5}\right)} \\
 &= \sqrt{1 - \left(\frac{4}{5}\right)^2} \\
 &= \sqrt{\frac{25}{25} - \frac{16}{25}} \\
 &= \frac{3}{5}
 \end{aligned}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

$\therefore \cos\theta$ is positive
between $[-\pi/2, \pi/2]$

$$\begin{aligned}
 30. \cos\left(\tan^{-1}\frac{4}{3}\right) &= \frac{1}{\sqrt{\tan^2\left(\tan^{-1}\frac{4}{3}\right) + 1}} \\
 &= \frac{1}{\sqrt{\left(\frac{4}{3}\right)^2 + \frac{9}{9}}} \\
 &= \frac{1}{\frac{5}{3}} \\
 &= \frac{3}{5}
 \end{aligned}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\frac{1}{\cos^2\theta} = \tan^2\theta + 1$$

$$\cos^2\theta = \frac{1}{\tan^2\theta + 1}$$

$$\cos\theta = \pm \sqrt{\frac{1}{\tan^2\theta + 1}}$$

(\tan^{-1} is defined
between $[-\pi/2, \pi/2]$
which $\cos\theta$ is +ve)

$$31. \sec\left(\sin^{-1}\frac{12}{13}\right)$$

$$\text{Let } u = \sin^{-1}\frac{12}{13},$$

$$\begin{aligned}\sec\theta &= \frac{1}{\cos\theta} \\ &= \frac{1}{\sqrt{1-\sin^2\theta}}\end{aligned}$$

$$\begin{aligned}\sec u &= \frac{1}{\sqrt{1-\sin^2 u}} \\ &= \frac{1}{\sqrt{1-\left(\sin\left(\sin^{-1}\frac{12}{13}\right)\right)^2}} \\ &= \frac{1}{\sqrt{1-\left(\frac{12}{13}\right)^2}} \\ &= \frac{1}{\frac{5}{13}} \\ &= \frac{13}{5}\end{aligned}$$

$$32. \csc\left(\cos^{-1}\frac{7}{25}\right)$$

$$\text{Let } \cos^{-1}\frac{7}{25} = u,$$

$$\begin{aligned}\csc\theta &= \frac{1}{\sin\theta} \\ &= \frac{1}{\sqrt{1-\cos^2\theta}}\end{aligned}$$

$$\begin{aligned}\csc(u) &= \frac{1}{\sqrt{1-\cos^2 u}} \\ &= \frac{1}{\sqrt{1-\left(\cos\left(\cos^{-1}\frac{7}{25}\right)\right)^2}} = \frac{25}{24}\end{aligned}$$

⑥ Algebraic Expressions

$$35. \cos(\sin^{-1}x)$$

$$= \sqrt{1 - \sin^2(\sin^{-1}x)}$$

$$= \sqrt{1 - (x)^2}$$

$$= \sqrt{(1+x)(1-x)}$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

$\therefore \sin^{-1} \theta$ only defined
between $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\therefore \cos \theta$$

$$= \sqrt{1 - \sin^2 \theta}$$

$$36. \sin(\tan^{-1}x)$$

$$\text{Let } u = \tan^{-1}x,$$

$$\sin(\tan^{-1}x)$$

$$= \pm \frac{\tan(\tan^{-1}x)}{\sqrt{\tan^2(\tan^{-1}x) + 1}}$$

$$= \pm \frac{x}{\sqrt{x^2 + 1}}$$

$$1 + \cot^2 u = \csc^2 u$$

$$\frac{1}{\sin^2 u} = 1 + \frac{1}{\tan^2 u}$$

$$= \frac{\tan^2 u + 1}{\tan^2 u}$$

$$\sin^2 u = \frac{\tan^2 u}{\tan^2 u + 1}$$

$$\sin u = \pm \frac{\tan u}{\sqrt{\tan^2 u + 1}}$$

$$\boxed{2} \quad \sin(\tan^{-1}x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$37. \tan(\sin^{-1}x)$$

Let $\sin^{-1}x = u$,

$$\tan u = \frac{\sin u}{\sqrt{1 - \sin^2 u}}$$

$$= \frac{\sin(\sin^{-1}x)}{\sqrt{1 - (\sin(\sin^{-1}x))^2}}$$

$$= \frac{x}{\sqrt{1 - x^2}}$$

$$38. \cos(\tan^{-1}x)$$

$$= \frac{1}{\sqrt{(\tan(\tan^{-1}x))^2 + 1}}$$

$$= \frac{1}{\sqrt{x^2 + 1}}$$

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}\end{aligned}$$

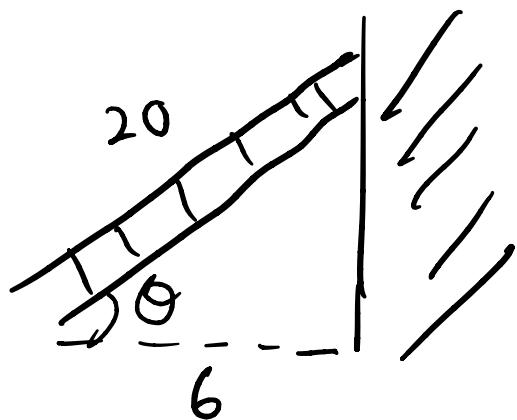
$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cos^2 \theta = \frac{1}{\tan^2 \theta + 1}$$

$$\cos \theta = \frac{1}{\sqrt{\tan^2 \theta + 1}}$$

⑦ Applications

39.



$$\cos \theta = \frac{6}{20}$$

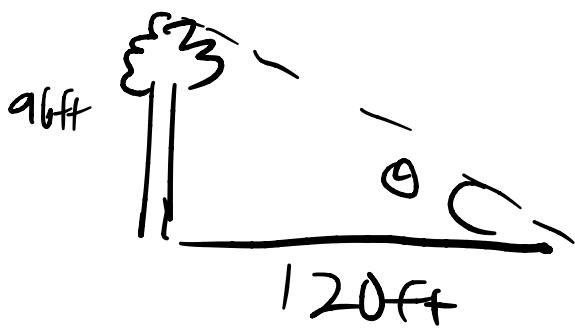
$$\therefore \theta = \cos^{-1} \frac{3}{10}$$

$$= 72.54^\circ$$

$$\therefore h = 20 \sin 72.54^\circ$$

$$= 19.08 \text{ ft}$$

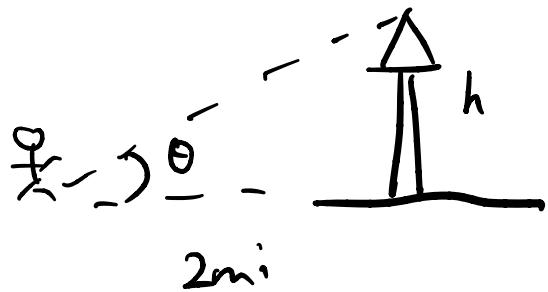
40.



$$\tan \theta = \frac{96}{120}$$

$$\theta = 38.66^\circ$$

41.

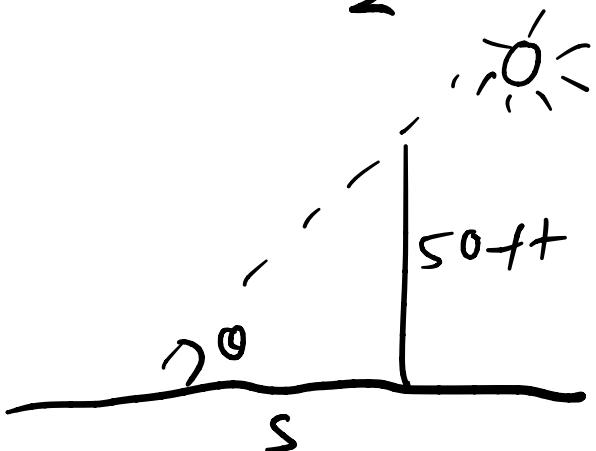


$$(a) h = 2 \tan \theta$$

$$(b) \tan \theta = \frac{h}{2}$$

$$\theta = \tan^{-1} \frac{h}{2}$$

42.



$$(a) \tan \theta = \frac{50}{s}$$

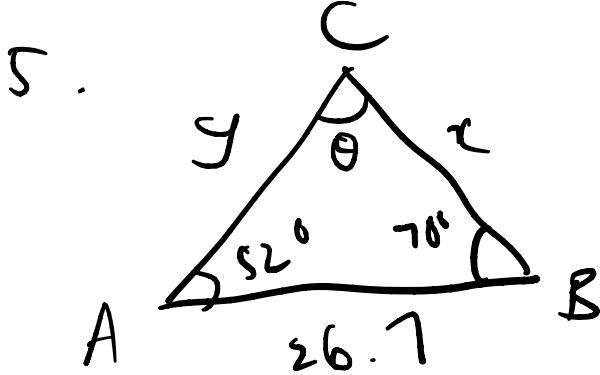
$$\theta = \tan^{-1} \frac{50}{s}$$

$$(b) s = 20 \text{ ft},$$

$$\theta = \tan^{-1} \frac{50}{20}$$

$$= 68.2 \text{ ft}$$

5.5 The Law of Sines



$$\begin{aligned}\theta &= 180 - 70 - 52^\circ \\ &= 58^\circ\end{aligned}$$

$$\frac{\sin A}{x} = \frac{\sin C}{26.7}$$

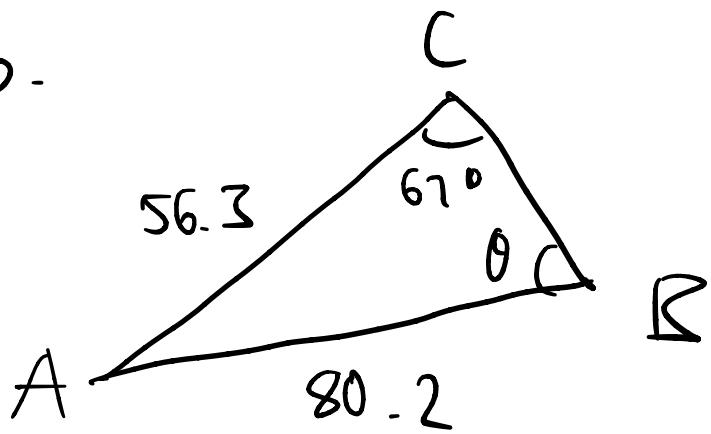
$$\frac{\sin 52^\circ}{x} = \frac{\sin \theta}{26.7}$$

$$\frac{\sin 52^\circ}{x} = \frac{\sin 58^\circ}{26.7}$$

$$x = \sin 52^\circ / \frac{\sin 58^\circ}{26.7}$$

$$= 24.8$$

6.



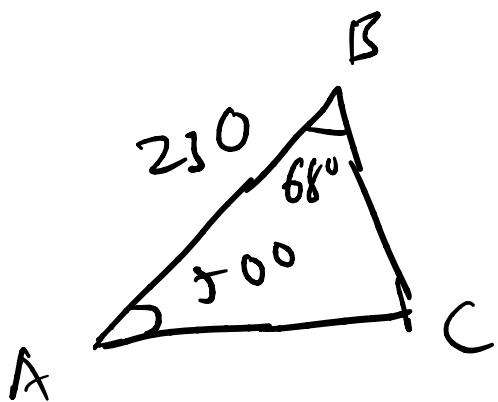
$$\frac{\sin C}{c} = \frac{\sin \theta}{56.3}$$

$$\frac{\sin 67^\circ}{80.2} = \frac{\sin \theta}{56.3}$$

$$\sin \theta = 56.3 \left(\frac{\sin 67^\circ}{80.2} \right)$$

$$\theta = 40.3^\circ$$

$$13. \angle A = 50^\circ, \angle B = 68^\circ, c = 230$$



$$\begin{aligned}\angle C &= 180 - 50 - 68 \\ &= 62^\circ\end{aligned}$$

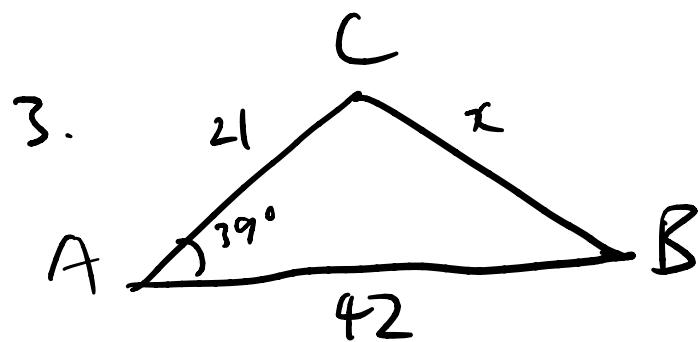
$$\frac{\sin A}{a} = \frac{\sin 62^\circ}{230}$$

$$\begin{aligned}a &= \frac{230 \sin 50^\circ}{\sin 62^\circ} \\ &= 199.5\end{aligned}$$

$$\frac{\sin B}{b} = \frac{\sin 62^\circ}{230}$$

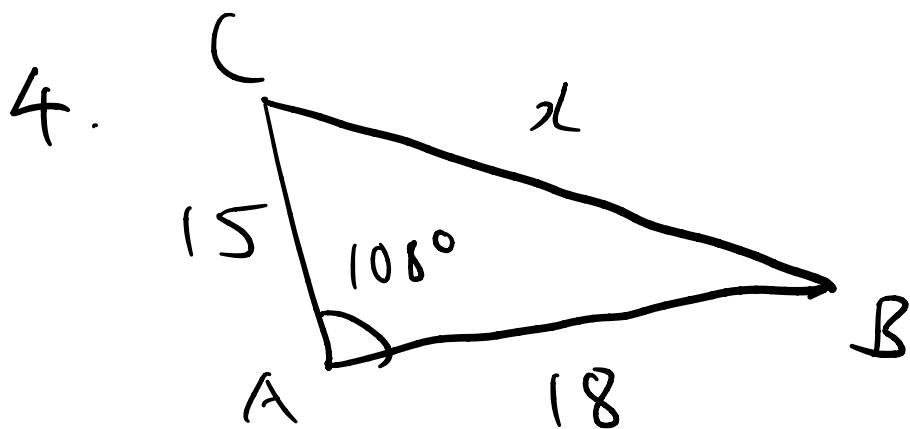
$$\begin{aligned}b &= \frac{230 \sin 68^\circ}{\sin 62^\circ} \\ &= 241.5\end{aligned}$$

5.6 The Law of Cosines



$$\begin{aligned}x^2 &= 21^2 + 42^2 - 2(21)(42)\cos 39^\circ \\&= 834.11\end{aligned}$$

$$x \approx 28.9$$



$$x^2 = 15^2 + 18^2 - 2(15)(18)\cos 108^\circ$$

$$x = 26.8$$

Chapter 9 Vectors in Two and Three Dimensions

8 Examples

Example 1 Describing Vectors in Component Form

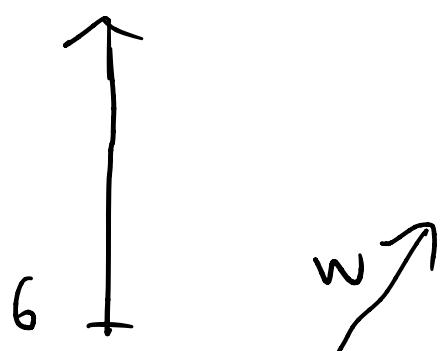
$$(a) \quad u = \langle 3 - (-2), 7 - 5 \rangle$$

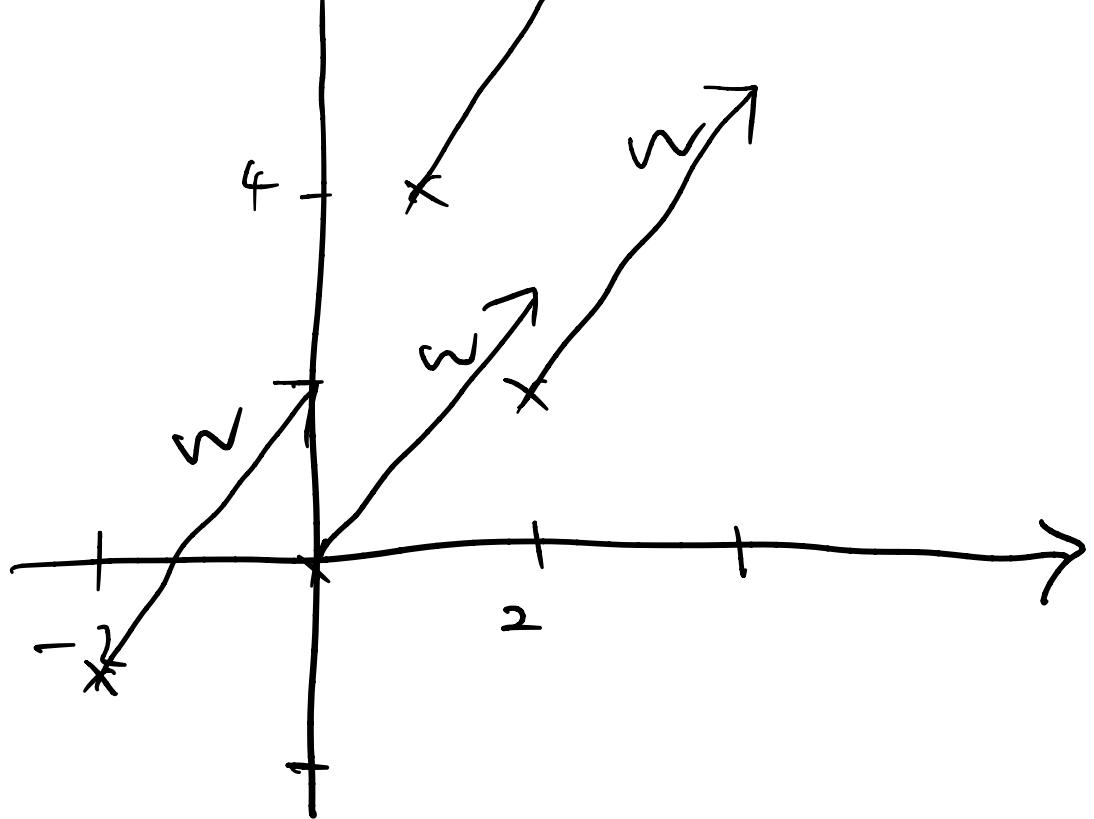
$$= \langle 5, 2 \rangle$$

$$(b) \quad v = \langle 3, 7 \rangle, \text{ initial point } (2, 4)$$

terminal point $(2+3, 4+7)$
 $= (5, 11)$

$$(c) \quad w = \langle 2, 3 \rangle$$



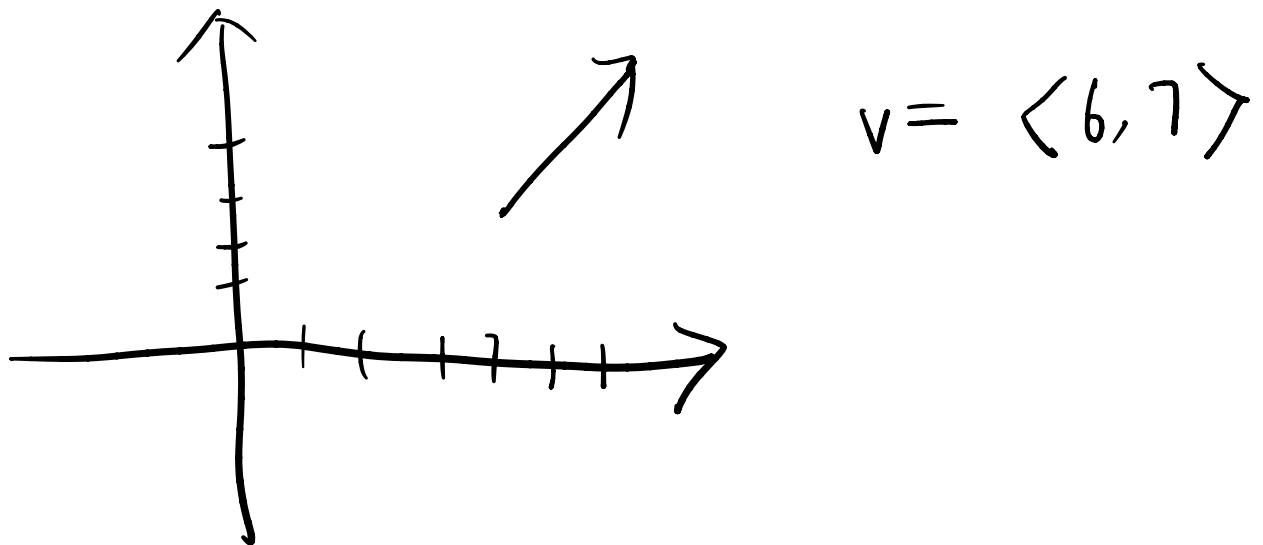


9. 1

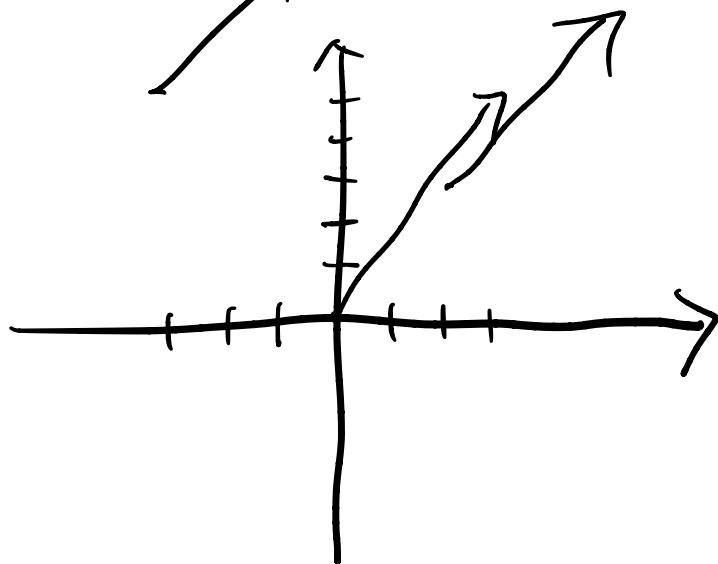
11. $P(1, 2), Q(4, 1)$

$$\vec{V} = \langle 3, 1 \rangle$$

19. $u = \langle 2, 4 \rangle$



23. $u = \langle 3, 5 \rangle$



$$37. \quad u = 2i + j, \quad v = 3i - 2j \quad 3/4/2024$$

$$\begin{aligned} |u| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5} \end{aligned}$$

$$|u+v|,$$

$$\begin{aligned} u+v &= (2i+j) + (3i-2j) \\ &= 5i - j \end{aligned}$$

$$\begin{aligned} |v| &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$|u+v|$$

$$\begin{aligned} &= \sqrt{5^2 + (-1)^2} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} 2u &= 2(2i + j) \\ &= 4i + 2j \end{aligned}$$

$$|u-v|,$$

$$\begin{aligned} |2u| &= \sqrt{4^2 + 2^2} \\ &= 2\sqrt{5} \end{aligned}$$

$$\begin{aligned} u-v &= (2i+j) - (3i-2j) \\ &= -i + 3j \end{aligned}$$

$$\begin{aligned} \left|\frac{1}{2}v\right| &= \left|\frac{1}{2}\right| |v| \\ &= \frac{1}{2} \sqrt{13} \\ &= \frac{\sqrt{13}}{2} \end{aligned}$$

$$|u-v|$$

$$\begin{aligned} &= \sqrt{(-1)^2 + (-3)^2} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned} |u| - |v| \\ = \sqrt{5} - \sqrt{13} \end{aligned}$$

$$31. \quad u = \langle 2, 7 \rangle, v = \langle 3, 1 \rangle$$

$$\begin{aligned} 2u &= \langle 2(2), 2(7) \rangle \\ &= \langle 4, 14 \rangle \end{aligned}$$

$$-3v = \langle -9, -3 \rangle$$

$$\begin{aligned} u+v &= \langle 2+3, 7+1 \rangle \\ &= \langle 5, 8 \rangle \end{aligned}$$

$$\begin{aligned} 3u-4v &= \langle 6, 21 \rangle - \langle 12, 4 \rangle \\ &= \langle 6-12, 21-4 \rangle \\ &= \langle -6, 17 \rangle \end{aligned}$$

$$27. \quad u = \langle 1, 4 \rangle$$
$$u = i + 4j$$

$$35. \quad u = 2i, \quad v = 3i - 2j$$

$$\begin{aligned} 2u &= 2(2i) \\ &= 4i \end{aligned}$$

$$\begin{aligned} -3v &= -3(3i - 2j) \\ &= -9i + 6j \end{aligned}$$

$$\begin{aligned} u+v &= 2i + 3i - 2j \\ &= 5i - 2j \end{aligned}$$

$$\begin{aligned} 3u - 4v &= 3(2i) - 4(3i - 2j) \\ &= 6i - 12i + 8j \\ &= -6i + 8j \end{aligned}$$

$$41. \quad |v| = 40, \quad \theta = 30^\circ$$

$$v = a_1 i + a_2 j$$

$$\begin{aligned} a_1 &= |v| \cos \theta \\ &= 40 \cos 30 \\ &= 40 \left(\frac{\sqrt{3}}{2} \right) \\ &= 20\sqrt{3} \end{aligned}$$

$$\begin{aligned} a_2 &= |v| \sin \theta \\ &= 40 \sin 30 \\ &= 40 \left(\frac{1}{2} \right) \\ &= 20. \end{aligned}$$

$$v = 20\sqrt{3}i + 20j$$

$$51. \quad v = i + \sqrt{3}j$$

$$|v| = \sqrt{1^2 + \sqrt{3}^2}$$

$$= 2$$

$$\theta = \tan^{-1} \frac{\sqrt{3}}{1}$$

$$= 60^\circ$$

59. Velocity v ,

$$V_x = 425 \text{ mi/h}$$

$$V_y = 40 \text{ mi/h}$$

$$(a) \quad V_y = 40j \quad \left(\theta = \tan^{-1} \frac{40}{425} \right)$$

$$(b) \quad V_x = 425i \quad = 5.38^\circ$$

$$(c) \quad V = 425i + 40j \quad \therefore \text{Speed is } 426.9 \text{ mi/h}$$

$$(d) \quad |v| = \sqrt{425^2 + 40^2}$$

$$= 426.88 \text{ mi/h}$$

x -axis

$$57. \quad V_x = -300 \text{ mi/h}$$

$$V_y = 30 \text{ mi/h}$$

$$V = -300i + 30j$$

$$\text{Ref } \theta = \tan^{-1} \frac{30}{-300}$$

$$= 5.71^\circ$$

$$\theta = 180^\circ - 5.71$$

$$= 174.29^\circ$$

. . . The airplane should head in a direction of 185.71° above the x -axis to arrive at a point due west.

$$67. \quad F_1 = \langle 2, 5 \rangle, \quad F_2 = \langle 3, -8 \rangle$$

(a)

$$\begin{aligned} F_1 + F_2 &= \langle 2, 5 \rangle + \langle 3, -8 \rangle \\ &= \langle 5, -3 \rangle \end{aligned}$$

$$\therefore \sum F = \langle 5, -3 \rangle$$

(b) If in equilibrium,

$$F_1 + F_2 + F = 0$$

$$\therefore F = \langle -5, 3 \rangle$$

9.2

5. (a)

$$u = \langle 2, 0 \rangle, v = \langle 1, 1 \rangle$$

$$\begin{aligned} u \cdot v &= a_1 b_1 + a_2 b_2 \\ &= 2 \cdot 1 + 0 \cdot 1 \\ &= 2 \end{aligned}$$

11. (a) $u = -5j, v = -i - \sqrt{3}j$

$$\begin{aligned} u \cdot v &= 0 \cdot (-1) + (-5) \cdot (-\sqrt{3}) \\ &= 0 + 5\sqrt{3} \\ &= 5\sqrt{3} \end{aligned}$$

5. (b)

$$u \cdot v = |u| |v| \cos \theta$$

$$\begin{aligned} \cos \theta &= \frac{u \cdot v}{|u| |v|} & |u| &= \sqrt{2^2 + 0^2} \\ &= \frac{2}{\sqrt{2} \sqrt{2}} & &= 2 \\ &= \frac{2}{2\sqrt{2}} & |v| &= \sqrt{1^2 + 1^2} \\ &= \frac{\sqrt{2}}{2} & &= \sqrt{2} \end{aligned}$$

$$11. (b) \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$$

$$\begin{aligned} \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} & |\mathbf{u}| &= \sqrt{0^2 + (-5)^2} \\ &= \frac{5\sqrt{3}}{(5)(2)} & &= 5 \\ &= \frac{\sqrt{3}}{2} & |\mathbf{v}| &= \sqrt{(-1)^2 + (-\sqrt{3})^2} \\ & & &= 2 \end{aligned}$$

$$15. \mathbf{u} = \langle 6, 4 \rangle, \mathbf{v} = \langle -2, 3 \rangle$$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta & |\mathbf{u}| &= \sqrt{6^2 + 4^2} \\ & & &= \sqrt{52} \\ \cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} & |\mathbf{v}| &= \sqrt{(-2)^2 + 3^2} \\ & & &= \sqrt{13} \\ &= \frac{6(-2) + 4(3)}{2\sqrt{13} \cdot \sqrt{13}} & & \end{aligned}$$

$$= 0$$

$\theta = \frac{\pi}{2} \therefore$ They are perpendicular

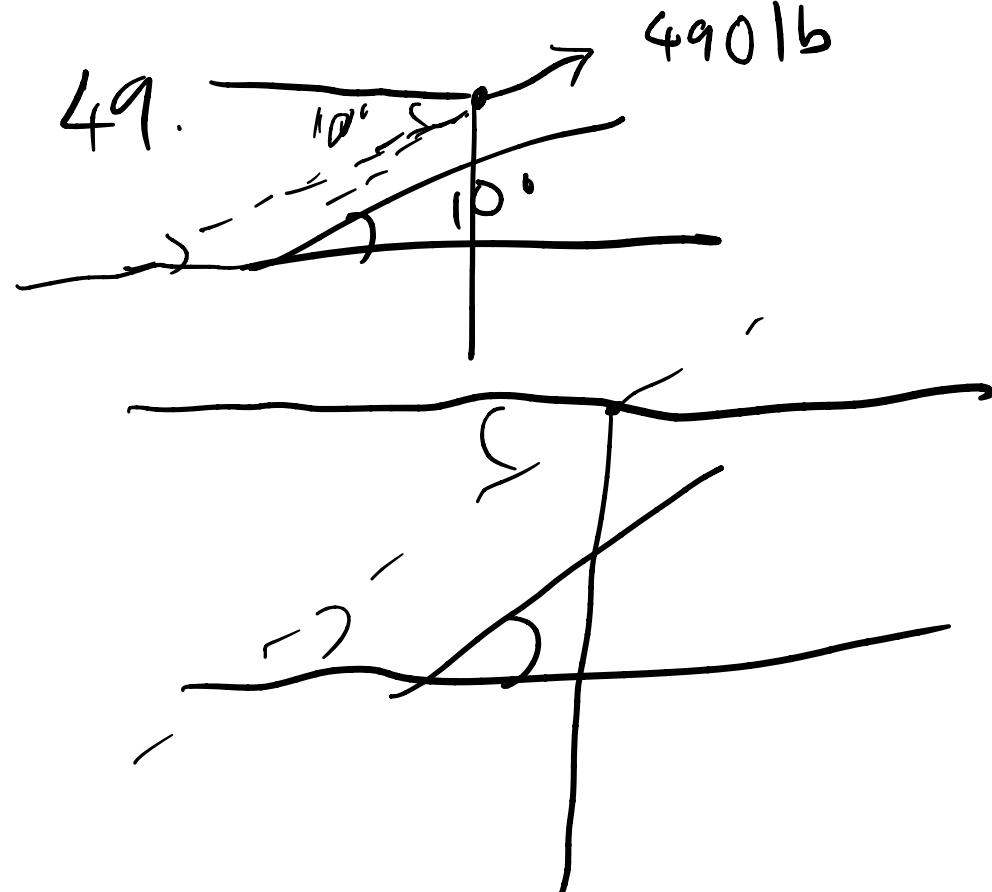
$$17. \quad u = \langle -2, -6 \rangle, \quad v = \langle 4, 2 \rangle$$

$$\cos \theta = \frac{u \cdot v}{|u| |v|}$$

$$\begin{aligned} u \cdot v &= (-2)(4) + (-6)(2) \\ &= -8 - 12 \\ &= -20 \end{aligned}$$

∴ not perpendicular

$$49. \quad \text{490 lb}$$



$$|u| = 490 \text{ lb}$$

$$W \cos \theta = |u|$$

$$\begin{aligned} W &= \frac{|u|}{\cos \theta} \\ &= \frac{490}{\cos 80} \end{aligned}$$

$$= 2821.80 \text{ lb}$$

$$W \sin \theta = F$$

$$2821.80 \sin 80^\circ = 2778.93 \text{ lb}$$

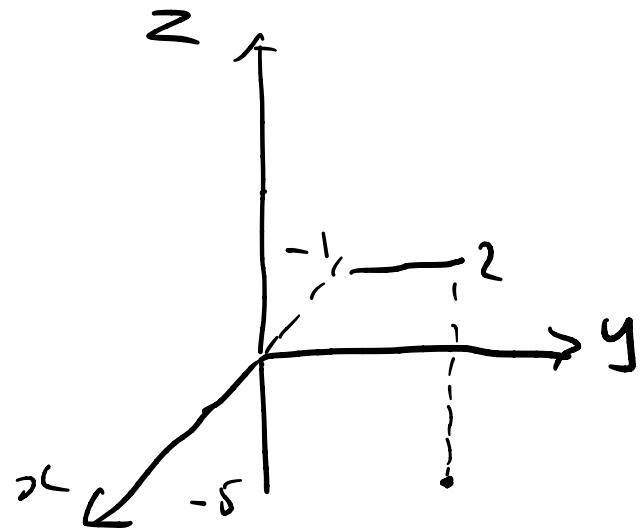
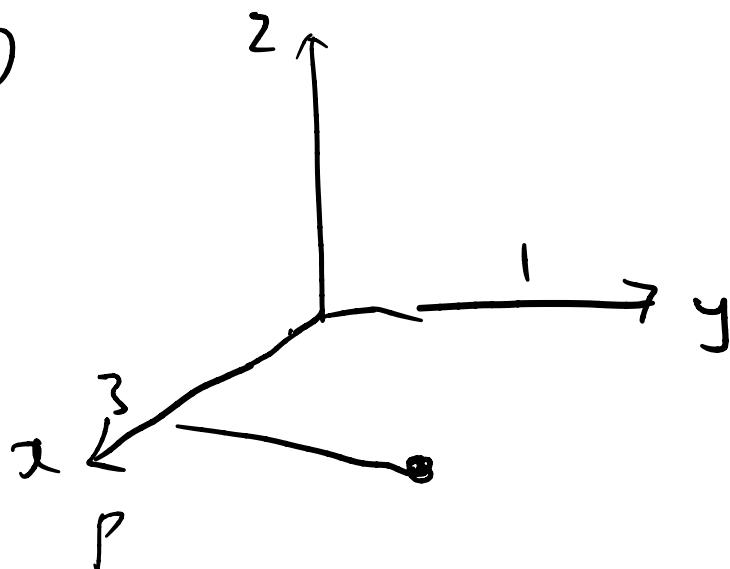
$$25. \quad u = \langle 4, 6 \rangle, v = \langle 3, -4 \rangle$$

$$\begin{aligned} |u| \cos \theta &= \frac{u \cdot v}{|v|} \\ &= \frac{4(3) + (6)(-4)}{\sqrt{3^2 + (-4)^2}} \\ &= \frac{12 - 24}{5} = -\frac{12}{5} \end{aligned}$$

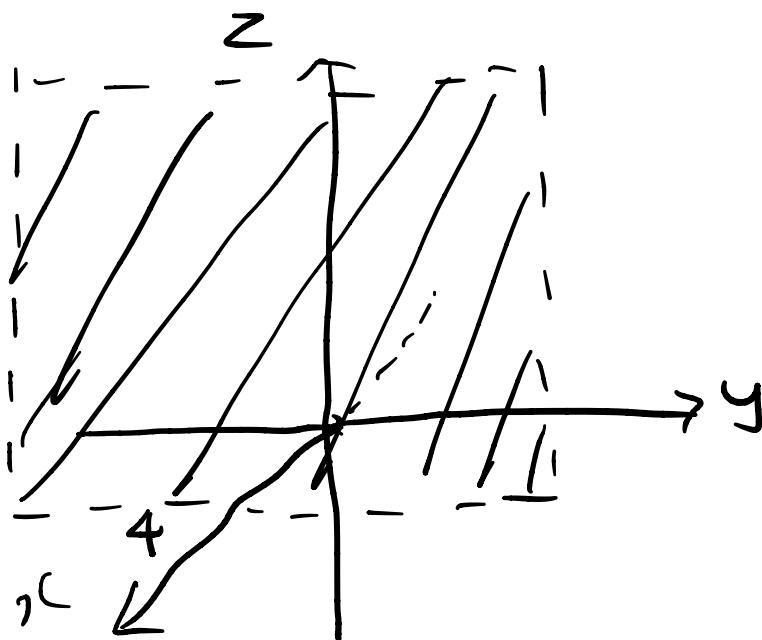
9.3

$$3. P(3, 1, 0), Q(-1, 2, -5)$$

(a)



$$7. x = 4$$



parallel to yz-plane
and 4 units in front

$$(b) d(P, Q)$$

$$\begin{aligned}
 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
 &= \sqrt{(3 - (-1))^2 + (1 - 2)^2 + (0 - (-5))^2} \\
 &= \sqrt{4^2 + (-1)^2 + 5^2} \\
 &= \sqrt{42}
 \end{aligned}$$

$$11. \quad r = 5 ; \quad C(2, -5, 3)$$

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

$$(x - 2)^2 + (y - (-5))^2 + (z - 3)^2 = 5^2$$

$$\therefore (x - 2)^2 + (y + 5)^2 + (z - 3)^2 = 25$$

$$15. \quad x^2 + y^2 + z^2 - 10x + 2y + 8z = 9$$

$$\begin{aligned} x^2 - 10x + \left(\frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 \\ + z^2 + 8z + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 = 9 \end{aligned}$$

$$(x - 5)^2 + (y + 1)^2 + (z + 4)^2 = 9 + 25 + 1 + 16$$

$$(x - 5)^2 + (y + 1)^2 + (z + 4)^2 = \sqrt{51}^2$$

\therefore Center of $(5, -1, -4)$ with radius $\sqrt{51}$

$$19. (x+1)^2 + (y-2)^2 + (z+10)^2 = 10^2$$

(a) yz plane

$$x=0,$$

$$(0+1)^2 + (y-2)^2 + (z+10)^2 = 100$$

$$(y-2)^2 + (z+10)^2 = 99$$

$$(y-2)^2 + (z+10)^2 = (3\sqrt{11})^2$$

there is a circle in the yz plane
of center $(0, 2, -10)$ and radius
 $3\sqrt{11}$

$$(b) (4+1)^2 + (y-2)^2 + (z+10)^2 = 100$$

$$(y-2)^2 + (z+10)^2 = (5\sqrt{3})^2$$

\therefore a circle in the yz plane
of center $(4, 2, -10)$ with
radius $5\sqrt{3}$

9.4

$$3. P(1, -1, 0), Q(0, -2, 5)$$

$$\begin{aligned} v &= \langle 0 - 1, -2 - (-1), 5 - 0 \rangle \\ &= \langle -1, -1, 5 \rangle \end{aligned}$$

$$7. V = \langle 3, 4, -2 \rangle, P(2, 0, 1)$$

$$\begin{aligned} Q &= (2+3, 0+4, 1+(-2)) \\ &= (5, 4, -1) \end{aligned}$$

$$11. V = \langle -2, 1, 2 \rangle$$

$$\begin{aligned} |V| &= \sqrt{(-2)^2 + (1)^2 + (2)^2} \\ &= 3 \end{aligned}$$

$$15. u = \langle 2, -7, 3 \rangle, v = \langle 0, 4, -1 \rangle$$

$$\begin{aligned} u+v &= \langle 2+0, -7+4, 3+(-1) \rangle \\ &= \langle 2, -3, 2 \rangle \end{aligned}$$

$$u - v = \langle 2 - 0, -7 - 4, 3 - (-1) \rangle \\ = \langle 2, -11, 4 \rangle$$

$$3u - \frac{1}{2}v = 3 \langle 2, -7, 3 \rangle - \frac{1}{2} \langle 0, 4, -1 \rangle \\ = \langle 6, -21, 9 \rangle - \langle 0, 2, -\frac{1}{2} \rangle \\ = \langle 6, -23, \frac{17}{2} \rangle$$

19. $\langle 12, 0, 2 \rangle$

$$v = 12i + 2k$$

23. $u = \langle 0, -2, 1 \rangle, v = \langle 1, -1, 0 \rangle$

$$\begin{aligned} -2u + 3v &= \langle -2(0) + 3(1), -2(-2) + 3(-1), \\ (a) \quad &\qquad -2(1) + 3(0) \rangle \\ &= \langle 3, 1, -2 \rangle \end{aligned}$$

$$(b) -2u + 3v = 3i + j - 2k$$

$$25. \quad u = \langle 2, 5, 0 \rangle, \quad v = \left\langle \frac{1}{2}, -1, 10 \right\rangle$$

$$\begin{aligned} u \cdot v &= 2\left(\frac{1}{2}\right) + 5(-1) + 0(10) \\ &= 1 - 5 + 0 \\ &= -4 \end{aligned}$$

$$27. \quad u = 6i - 4j - 2k, \quad v = \frac{5}{6}i + \frac{3}{2}j - k$$

$$\begin{aligned} u \cdot v &= 6\left(\frac{5}{6}\right) + (-4)\left(\frac{3}{2}\right) + (-2)(-1) \\ &= 5 - 6 + 2 \\ &= 1 \end{aligned}$$

$$29. \quad \langle 4, -2, -4 \rangle, \quad \langle 1, -2, 2 \rangle$$

$$\begin{aligned} u \cdot v &= 4 + 4 - 8 \\ &= 0 \\ \therefore & \text{ yes} \end{aligned}$$

$$37. \quad 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad |\mathbf{v}| = \sqrt{3^2 + 4^2 + 5^2}$$

$$= \sqrt{50} \\ = 5\sqrt{2}$$

$$\cos \alpha = \frac{a_1}{|\mathbf{v}|}$$

$$= \frac{3}{5\sqrt{2}} \\ = \frac{3\sqrt{2}}{10}$$

$$\cos \gamma = \frac{a_3}{|\mathbf{v}|} \\ = \frac{5}{5\sqrt{2}} \\ = \frac{\sqrt{2}}{2}$$

$$\cos \beta = \frac{a_2}{|\mathbf{v}|}$$

$$= \frac{4}{5\sqrt{2}} \\ = \frac{2\sqrt{2}}{5}$$

$$\alpha = \cos^{-1}\left(\frac{3}{10}\sqrt{2}\right)$$

$$= 64.9^\circ$$

$$\beta = 55.6^\circ$$

$$\gamma = 45^\circ$$

$$41. \quad \alpha = \frac{\pi}{3}, \quad \gamma = \frac{2}{3}\pi$$

$$\cos \beta = \pm \frac{1}{\sqrt{2}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad \beta = \frac{\pi}{4}$$

$$\cos^2 \frac{\pi}{3} + \cos^2 \beta + \cos^2 \frac{2}{3}\pi = 1$$

$$\cos^2 \beta = 1 - \frac{1}{4} - \frac{1}{4} \\ = \frac{1}{2}$$

9.5

$$3. \quad u = \langle 1, 0, -3 \rangle, \quad v = \langle 2, 3, 0 \rangle$$

$$\begin{aligned}
 u \times v &= \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix} \\
 &= (-1)^2 i \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} + (-1)^3 j \begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix} \\
 &\quad + (-1)^4 k \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\
 &= 9i - 6j + 3k
 \end{aligned}$$

$$9. \quad u = \langle 1, 1, -1 \rangle, \quad v = \langle -1, 1, -1 \rangle$$

$$\begin{aligned}
 (a) \quad u \times v &= i \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} \\
 &\quad + k \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \\
 &= 2j + 2k \quad \frac{|u \times v|}{|u \times v|} = \sqrt{0^2 + 2^2 + 2^2}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \frac{u \times v}{|u \times v|} &= \frac{1}{2\sqrt{2}} (2j + 2k) \quad = 2\sqrt{2}
 \end{aligned}$$

$$17. \quad P(0,1,0), Q(1,2,-1), R(-2,1,0)$$

$$\begin{aligned}\overrightarrow{PQ} &= \langle 1-0, 2-1, -1-0 \rangle \\ &= \langle 1, 1, -1 \rangle\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= \langle -2-0, 1-1, 0-0 \rangle \\ &= \langle -2, 0, 0 \rangle\end{aligned}$$

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ -2 & 0 & 0 \end{vmatrix} \\ &= i \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ &\quad + k \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} \\ &= 2j + 2k\end{aligned}$$

$$\therefore v = \langle 0, 2, 2 \rangle$$

$$21. \quad u = \langle 3, 2, 1 \rangle, \quad v = \langle 1, 2, 3 \rangle$$

$$|u \times v|$$

$$= |u| |v| \sin \theta$$

$$= \sqrt{14} \sqrt{14} \sin 44.4^\circ$$

$$= 9.80$$

$$\cos \theta = \frac{u \cdot v}{|u| |v|}$$

$$= \frac{3+4+3}{\sqrt{14} \sqrt{14}}$$

$$= \frac{10}{14} = \frac{5}{7}$$

$$|u| = \sqrt{3^2 + 2^2 + 1^2}$$

$$= \sqrt{14}$$

$$|v| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

$$\theta = \cos^{-1} \frac{5}{7}$$

$$= 44.4^\circ$$

$$u \times v = \langle 4, -8, 4 \rangle$$

$$|u \times v| = \sqrt{4^2 + (-8)^2 + 4^2}$$

$$= \sqrt{96}$$

$$= 4\sqrt{6}$$

$$25. P(1, 0, 1), Q(0, 1, 0), R(2, 3, 4)$$

$$\begin{aligned}\vec{PQ} &= \langle 0-1, 1-0, 0-1 \rangle \\ &= \langle -1, 1, -1 \rangle\end{aligned}$$

$$\begin{aligned}\vec{PR} &= \langle 2-1, 3-0, 4-1 \rangle \\ &= \langle 1, 3, 3 \rangle\end{aligned}$$

$$\Delta PQR = \frac{1}{2} b h$$

$$\begin{aligned}&= \frac{1}{2} |\vec{PQ}| |\vec{PR}| \sin \theta \\&= \frac{1}{2} \sqrt{6^2 + 2^2 + (-4)^2} \\&= \frac{1}{2} \sqrt{56} \\&= \frac{1}{2} \sqrt{14} \\&= \frac{1}{2} \left| \begin{vmatrix} i & j & k \\ -1 & 1 & -1 \\ 1 & 3 & 3 \end{vmatrix} \right| \\&= \frac{1}{2} \left| \begin{pmatrix} i & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 3 & 3 \end{pmatrix} \right| \\&\quad + k \left| \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} \right|\end{aligned}$$

$$\hat{C} = \frac{1}{2} |6i + 2j - 4k|$$

$$29. \quad u = \langle 1, 2, 3 \rangle, \quad v = \langle -3, 2, 1 \rangle,$$

$$w = \langle 0, 8, 10 \rangle$$

$$(a) \quad v \times w = i \begin{vmatrix} 2 & 1 \\ 8 & 10 \end{vmatrix} - j \begin{vmatrix} -3 & 1 \\ 0 & 10 \end{vmatrix} + k \begin{vmatrix} -3 & 2 \\ 0 & 8 \end{vmatrix}$$

$$= 12i + 30j - 24k$$

$$u \cdot (v \times w)$$

$$= 12 + 60 - 72$$

$$= 0$$

(b) \therefore vectors u and $v \times w$ are coplanar

9.6

3. $P(1, 0, -2)$, $\nu = \langle 3, 2, -3 \rangle$

$$r = r_0 + t\nu$$

$$r = \langle x, y, z \rangle$$

$$\begin{aligned}x &= x_0 + x_1 t \\&= 1 + 3t\end{aligned}$$

$$= \langle 1+3t, 2t, -2-3t \rangle$$

$$\begin{aligned}y &= y_0 + y_1 t \\&= 0 + 2t \\&= 2t\end{aligned}$$

$$\begin{aligned}z &= z_0 + z_1 t \\&= -2 - 3t\end{aligned}$$

$$9. \ P(1, -3, 2), Q(2, 1, -1)$$

$$\begin{aligned}\overrightarrow{PQ} &= \langle 2-1, 1-(-3), -1-2 \rangle \\ &= \langle 1, 4, -3 \rangle\end{aligned}$$

$$\begin{aligned}x &= x_0 + at \\ &= 1 + t\end{aligned}$$

$$\begin{aligned}y &= y_0 + bt \\ &= -3 + 4t\end{aligned}$$

$$\begin{aligned}z &= z_0 + ct \\ &= 2 - 3t\end{aligned}$$

$$15. \ n = \langle 1, 1, -1 \rangle, P(0, 2, -3)$$

$$(a) \quad P_0 = \langle 0, 2, -3 \rangle$$

$$r = P_0 + vt$$

$$\begin{aligned} v &= \langle x-0, y-2, z+3 \rangle \\ &= \langle x, y-2, z+3 \rangle \end{aligned}$$

$$n \cdot v = 0$$

$$\langle 1, 1, -1 \rangle \cdot \langle x, y-2, z+3 \rangle = 0$$

$$x + y - 2 - z - 3 = 0$$

$$\therefore x + y - z - 5 = 0$$

$$(b) \quad x\text{-intercept: } x = 5$$

$$y\text{-intercept: } y = 5 \quad z$$

$$z\text{-intercept: } z = -5 \uparrow$$



$$21. P(6, -2, 1), Q(5, -3, -1), R(7, 0, 0)$$

$$\begin{aligned}\overrightarrow{PQ} &= \langle 5-6, -3-(-2), -1-1 \rangle \\ &= \langle -1, -1, -2 \rangle\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= \langle 7-6, 0-(-2), 0-1 \rangle \\ &= \langle 1, 2, -1 \rangle\end{aligned}$$

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} i & j & k \\ -1 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} \\ &= i \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} - j \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} \\ &\quad + k \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} \\ &= i(5) - j(3) + k(-1) \\ &= 5i - 3j - k\end{aligned}$$

$$n = \langle 5, -3, -1 \rangle, P(6, -2, 1)$$

$$r = P_0 + vt$$

$$r - P_0 = \langle x-6, y+2, z-1 \rangle$$

$$n \cdot r = 0$$

$$\langle 5, -3, -1 \rangle \cdot \langle x-6, y+2, z-1 \rangle = 0$$

$$5(x-6) + (-3)(y+2) + (-1)(z-1) = 0$$

$$5x - 30 - 3y - 6 - z + 1 = 0$$

$$\therefore 5x - 3y - z - 35 = 0$$

$$\vec{QR} = \langle 7-5, 3, 1 \rangle = \langle 2, 3, 1 \rangle$$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} i & j & k \\ -1 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= i(-1+6) - j(-1+4) + k(-3+2)$$

$$= 5i - 3j - k$$

$\therefore \vec{QR}$ interchangeable with \vec{PR} for cross product

Chapter 10 Systems of Equations and Inequalities

10.1

5.

$$\begin{cases} x - y = 1 \\ 4x + 3y = 18 \end{cases}$$

$$x = y + 1$$

$$4(y+1) + 3y = 18$$

$$4y + 4 + 3y = 18$$

$$7y = 14$$

$$y = 2$$

$$x = (2) + 1$$

$$= 3$$

$$\therefore x = 3, y = 2$$

9.

$$\begin{cases} 3x + 4y = 10 \\ x - 4y = -2 \end{cases}$$

$$\text{Eq 1} + \text{Eq 2} : 4x = 8$$

$$x = 2$$

$$2 - 4y = -2 \quad \therefore (2, 1)$$

$$-4y = -4$$

$$y = 1$$

10.3 Partial Fractions

1. Distinct Linear Factors
2. Repeated Linear Factors
3. Irreducible Quadratic Factors
4. Repeated Irreducible Quadratic Factors

$$1. \quad r(x) = \frac{4}{x(x-2)^2}$$

$$\frac{4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$4 = A(x-2)^2 + Bx(x-2) + Cx$$

$$4 = Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx$$

$$4 = (A+B)x^2 + (-4A-2B+C)x + 4A$$

$$A = 1,$$

$$A + B = 0$$

$$B = -1,$$

$$-4A - 2B + C = 0$$

$$-4 + 2 + C = 0$$

$$C = 2$$

$$r(x) = \frac{1}{x} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

✓ (iii)

$$r(x) = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$2. \quad r(x) = \frac{2x+8}{(x-1)(x^2+4)}$$

24/12/2023

$$(ii) \quad \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$3. \quad \frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$4. \quad \frac{x}{x^2+3x-4} = \frac{x}{(x+4)(x-1)}$$

$$\frac{x}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

$$5. \quad \frac{x^2-3x+5}{(x-2)^2(x+4)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+4}$$

$$13. \quad \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x-1)$$

25/12/23

$$2 = Ax + A + Bx - B$$

$$2 = (A+B)x + A - B$$

$$A + B = 0 \quad - \textcircled{1}$$

$$A - B = 2 \quad - \textcircled{2}$$

$$\textcircled{1} + \textcircled{2}: \quad 2A = 2$$

$$A = 1$$

$$1 + B = 0$$

$$B = -1$$

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

$$14. \frac{2x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2x = A(x+1) + B(x-1)$$

$$2x = (A+B)x + A - B$$

$$A + B = 2 \quad - \textcircled{1}$$

$$A - B = 0 \quad - \textcircled{2}$$

$$\begin{aligned} \textcircled{1} + \textcircled{2}: & \quad \therefore \frac{2x}{(x-1)(x+1)} = \frac{1}{x-1} + \frac{1}{x+1} \\ 2A = 2 & \end{aligned}$$
$$A = 1$$

$$1 - B = 0$$

$$B = 1$$

26/12/23

$$15. \frac{5}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$5 = Ax + 4A + Bx - B$$

$$= (A+B)x + 4A - B$$

$$A+B=0$$

$$4A-B=5$$

$$\frac{5}{(x-1)(x+4)} = \frac{1}{x-1} - \frac{1}{x+4}$$

$$A+B=0$$

$$B=-A$$

$$4A+A=5$$

$$5A=5$$

$$A=1$$

$$B=-1$$

$$16. \frac{x+6}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$$

$$\begin{aligned}x+6 &= Ax+3A+Bx \\&= (A+B)x+3A\end{aligned}$$

$$A+B=1$$

$$3A=6$$

$$A=2$$

$$2+B=1$$

$$B=-1$$

$$\frac{x+6}{x(x+3)} = \frac{2}{x} - \frac{1}{x+3}$$

$$17. \frac{12}{x^2 - 9} = \frac{12}{(x+3)(x-3)}$$

$$\frac{12}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

$$12 = A(x-3) + B(x+3)$$

$$12 = Ax - 3A + Bx + 3B$$

$$= (A+B)x + 3B - 3A$$

$$A+B = 0 \quad \frac{12}{(x+3)(x-3)} = \frac{2}{x-3} - \frac{2}{x+3}$$

$$3B - 3A = 12$$

$$A = -B$$

$$3B - 3(-B) = 12$$

$$6B = 12$$

$$B = 2$$

$$A = -2$$

27/12/23

$$18. \quad \frac{x-12}{x^2-4x} = \frac{x-12}{x(x-4)}$$

$$\frac{x-12}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$$x-12 = Ax - 4A + Bx$$

$$x-12 = (A+B)x - 4A$$

$$\begin{aligned} A+B &= 1 \\ 12 &= 4A \\ A &= 3 \end{aligned} \quad \frac{x-12}{x(x-4)} = \frac{3}{x} - \frac{2}{x-4}$$

$$3+B=1$$

$$B = -2$$

Chapter 11 Matrices and Determinants

1. Matrices and Systems of Linear Equations
2. The Algebra of Matrices
3. Inverses of Matrices and Matrix Equations
4. Determinants and Cramer's Rule

11.1 Matrices and Systems of Linear Equations

5. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$ 3 by 2

11. $\begin{aligned} 3x + y - z &= 2 \\ 2x - y &= 1 \\ x - z &= 3 \end{aligned}$

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$$

13. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

(a) yes

(b) yes

(c) $x = -3$
 $y = 5$

21. $\begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 1 & 1 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$

$$\downarrow \quad 3R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{cccc} -1 & 1 & 2 & 0 \\ 0 & 4 & 7 & 4 \\ 1 & -2 & -1 & -1 \end{array} \right]$$

25.

$$\left[\begin{array}{cccc} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

(a) $x - 2y + 4z = 3$
 $y + 2z = 7$
 $z = 2$

(b) $y = 3$

$$x - 6 + 8 = 3$$
$$x = 1$$

$$(1, 3, 2)$$

$$29. \ x - 2y + z = 1$$

$$y + 2z = 5$$

$$x + y + 3z = 8$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 8 \end{array} \right]$$

$$R_3 - R_1 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 2 & 7 \end{array} \right]$$

$$R_3 - 3R_2 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -4 & -8 \end{array} \right]$$

$$-\frac{1}{4}R_3 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$x - 2y + z = 1$$

$$y + 2z = 5$$

$$z = 2$$

$$y + 2(2) = 5$$

$$y = 1$$

$$x - 2(1) + 2 = 1$$

$$x = 1$$

$$\therefore (1, 1, 2)$$

$$33. \quad x + 2y - z = -2$$

$$x + z = 0$$

$$2x - y - z = -3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & -3 \end{array} \right]$$

$$\downarrow \quad -\frac{1}{2}R_3 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\downarrow \quad \begin{array}{c} R_2 - R_1 \rightarrow R_2 \\ \hline R_3 - 2R_1 \rightarrow R_3 \end{array}$$

$$\downarrow \quad \begin{array}{c} R_2 + R_3 \rightarrow R_2 \\ \hline R_1 + R_3 \rightarrow R_1 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & -2 & 2 & 2 \\ 0 & -5 & 1 & 1 \end{array} \right]$$

$$\downarrow \quad -\frac{1}{2}R_2$$

$$\downarrow \quad R_1 - 2R_2 \rightarrow R_1$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -5 & 1 & 1 \end{array} \right]$$

$$\therefore \quad x = -1 \\ y = 0 \\ z = 1$$

$$\downarrow \quad R_3 + 5R_2 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -4 & -4 \end{array} \right] \uparrow$$

$$\begin{aligned}
 39. \quad x + y + z &= 2 \\
 y - 3z &= 1 \\
 2x + y + 5z &= 0
 \end{aligned}$$

$$\left[\begin{array}{ccc|c}
 1 & 1 & 1 & 2 \\
 0 & 1 & -3 & 1 \\
 2 & 1 & 5 & 0
 \end{array} \right]$$

 $R_3 - 2R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c}
 1 & 1 & 1 & 2 \\
 0 & 1 & -3 & 1 \\
 0 & -1 & 3 & -4
 \end{array} \right]$$

 $R_3 + R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c}
 1 & 1 & 1 & 2 \\
 0 & 1 & -3 & 1 \\
 0 & 0 & 0 & -3
 \end{array} \right]$$

\therefore Inconsistent
 Linear System

$$40. \quad x + 3z = 3$$

$$2x + y - 2z = 5$$

$$-y + 8z = 8$$

$$\left[\begin{array}{cccc} 1 & 0 & 3 & 3 \\ 2 & 1 & -2 & 5 \\ 0 & -1 & 8 & 8 \end{array} \right]$$

$$\downarrow \quad 2R_1 - R_2 \rightarrow R_2$$

$$\left[\begin{array}{cccc} 1 & 0 & 3 & 3 \\ 0 & -1 & 8 & 1 \\ 0 & -1 & 8 & 8 \end{array} \right]$$

$$\downarrow \quad R_2 - R_3 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & 0 & 3 & 3 \\ 0 & -1 & 8 & 1 \\ 0 & 0 & 0 & -7 \end{array} \right]$$

\therefore Inconsistent Linear System

$$41. \quad 2x - 3y - 9z = -5$$

$$x + 3z = 2$$

$$-3x + y - 4z = -3$$

$$\downarrow \begin{array}{l} R_2 - \frac{7}{3}R_3 \rightarrow R_2 \\ \hline \frac{1}{2}R_1 \end{array}$$

$$\left[\begin{array}{cccc} 2 & -3 & -9 & -5 \\ 1 & 0 & 3 & 2 \\ -3 & 1 & -4 & -3 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{9}{2} & -\frac{5}{2} \\ 0 & -7 & -35 & -21 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

\therefore Dependent

$$\downarrow \begin{array}{l} R_1 \leftrightarrow R_3 \\ \hline \frac{1}{2}R_1 \end{array}$$

$$\left[\begin{array}{cccc} 2 & -3 & -9 & -5 \\ -3 & 1 & -4 & -3 \\ 1 & 0 & 3 & 2 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{9}{2} & -\frac{5}{2} \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\downarrow \begin{array}{l} 3R_1 + 2R_2 \rightarrow R_2 \\ R_1 - 2R_3 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc} 2 & -3 & -9 & -5 \\ 0 & -1 & -35 & -21 \\ 0 & -3 & -15 & -9 \end{array} \right]$$

$$x + 3z = 2$$

$$y + 5z = 3$$

Let z be t_1

$$x = 2 - 3t_1$$

$$y = 3 - 5t_1$$

$$z = t_1$$

$$42. \begin{cases} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y + 20z = -26 \end{cases}$$

$$\left[\begin{array}{cccc} 1 & -2 & 5 & 3 \\ -2 & 6 & -11 & 1 \\ 3 & -16 & 20 & -26 \end{array} \right]$$

$\downarrow R_1 + 2R_2 \rightarrow R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 10 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x + 4z = 10$$

$$y - \frac{1}{2}z = \frac{7}{2}$$

Let z be t ,

$$\begin{aligned} x &= 10 - 4t \\ y &= \frac{7}{2} + \frac{1}{2}t \\ z &= t \end{aligned}$$

$$\left[\begin{array}{cccc} 1 & -2 & 5 & 3 \\ 0 & 2 & -1 & 7 \\ 0 & 10 & -5 & 35 \end{array} \right]$$

$\downarrow \frac{5R_2 - R_3 \rightarrow R_3}{\frac{1}{2}R_2}$

$$\left[\begin{array}{cccc} 1 & -2 & 5 & 3 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

43.

$$\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 4 & -8 & 32 & 24 \\ 2 & -3 & 11 & 4 \end{array} \right]$$

$$\downarrow \frac{R_2 - 4R_1 \rightarrow R_2}{R_1 - 2R_1 \rightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & -4 & 20 & 12 \\ 0 & -1 & 5 & -2 \end{array} \right]$$

$$\downarrow -\frac{1}{4}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & -1 & 5 & -2 \end{array} \right]$$

$$\downarrow R_3 + R_2 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & -5 \end{array} \right]$$

Inconsistent
linear system

$$49. \quad 4x - 3y + z = -8$$

$$-2x + y - 3z = -4$$

$$2x - y + 2z = 3$$

$$\left[\begin{array}{cccc} 4 & -3 & 1 & -8 \\ -2 & 1 & -3 & -4 \\ 1 & -1 & 2 & 3 \end{array} \right]$$

$$\begin{array}{c} \downarrow \frac{-1}{7} R_3 \\ \left[\begin{array}{cccc} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ 0 & 1 & 5 & 16 \\ 0 & -\frac{1}{7} & 1 & \frac{20}{7} \end{array} \right] \\ \downarrow \frac{1}{7} R_2 + R_3 \rightarrow R_3 \end{array}$$

$$\begin{array}{c} \downarrow \frac{1}{4} R_1 \\ \left[\begin{array}{cccc} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ -2 & 1 & -5 & -4 \\ 1 & -1 & 2 & 3 \end{array} \right] \\ \downarrow \begin{array}{l} R_2 + 2R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \end{array}$$

$$\begin{array}{c} \downarrow \frac{1}{7} R_2 \\ \left[\begin{array}{cccc} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ 0 & 1 & 5 & 16 \\ 0 & 0 & \frac{12}{7} & \frac{36}{7} \end{array} \right] \\ \downarrow \frac{7}{12} R_3 \end{array}$$

$$\begin{array}{c} \downarrow \\ \left[\begin{array}{cccc} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ 0 & -\frac{1}{2} & -\frac{5}{2} & -8 \\ 0 & -\frac{1}{4} & \frac{7}{4} & 5 \end{array} \right] \\ \downarrow \begin{array}{l} -2 R_2 \\ -4 R_3 \end{array} \end{array}$$

$$\begin{array}{c} \downarrow \\ \left[\begin{array}{cccc} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ 0 & 1 & 5 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \\ \downarrow \begin{array}{l} R_2 - 5R_3 \rightarrow R_2 \\ R_1 - \frac{1}{4} R_3 \rightarrow R_1 \end{array} \end{array}$$

$$\begin{aligned} \therefore x &= -2, \\ y &= 1, \\ z &= 3 \end{aligned}$$

$$\left[\begin{array}{cccc} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ 0 & 1 & 5 & 16 \\ 0 & 1 & -7 & -20 \end{array} \right]$$

$$\begin{array}{c} \downarrow \\ \left[\begin{array}{cccc} 1 & -\frac{3}{4} & 0 & -\frac{11}{4} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array} \right] \\ \downarrow \begin{array}{l} R_1 + \frac{3}{4} R_2 \rightarrow R_1 \\ R_3 - R_2 \rightarrow R_3 \end{array} \end{array}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{8} \end{array} \right]$$

11.1

Form of a Matrix

Elementary Row Operations

Back Substitution

Linear Systems with One Solution

Dependent or Inconsistent Linear Systems

Solving a Linear System

Applications

① Linear Systems 29-48

② Solving a Linear System

$$14. \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$$

(a) Yes

(b) No

$$(c) x + 3y = -3$$

$$y = 5$$

②

$$21. \begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 1 & 1 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

$$\downarrow 3R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 7 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

③ 26.

$$\begin{bmatrix} 1 & 1 & -3 & 8 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) $x + y - 3z = 8$
 $y - 3z = 5$
 $z = -1$

(b) $y - 3z = 5$
 $y - 3(-1) = 5$
 $y = 2$

$$x + y - 3z = 8$$
$$x + (2) - 3(-1) = 8$$
$$x + 5 = 8$$
$$x = 3$$

④ 30. $\begin{cases} x+y+6z=3 \\ x+y+3z=3 \\ x+2y+4z=7 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 4 & 7 \end{array} \right] \quad \begin{aligned} z &= 0 \\ y - 2z &= 4 \\ y - 0 &= 4 \\ y &= 4 \end{aligned}$$

Gaussian Elimination

$$\begin{array}{l} \xrightarrow{R_2 - R_1 \rightarrow R_2} \\ \xrightarrow{R_3 - R_1 \rightarrow R_3} \end{array} \quad \begin{aligned} x + y + 6z &= 3 \\ x + 4 + 6(0) &= 3 \\ x &= -1 \\ \therefore (-1, 4, 0) \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & -2 & 4 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{R_1 \rightarrow R_2} \\ \xrightarrow{R_2 - R_1} \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -3 & 0 \end{array} \right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \quad \left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

29. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 8 \end{array} \right]$$

$$z = 2$$

$$y + 2z = 5$$

$$y + 2(2) = 5$$

$$y = 1$$

$$\downarrow \quad R_3 - R_1 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 2 & 7 \end{array} \right]$$

$$x - 2y + z = 1$$

$$x - 2(1) + (2) = 1$$

$$x = 1$$

$$\downarrow \quad R_1 - 3R_2 \rightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -4 & -8 \end{array} \right]$$

$$\therefore (1, 1, 2)$$

$$\downarrow \quad -\frac{1}{4}R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$31. \begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -3 & 2 & 4 \\ 4 & 1 & -3 & 1 \end{array} \right]$$

$$\downarrow \frac{1}{7}R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\downarrow 2R_1 - R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 5 & 0 & 0 \\ 4 & 1 & -3 & 1 \end{array} \right]$$

$$\downarrow \begin{array}{l} R_1 - R_2 \rightarrow R_1 \\ R_1 - R_3 \rightarrow R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\downarrow \begin{array}{l} 4R_1 - R_3 \rightarrow R_3 \\ \frac{1}{5}R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 7 & 7 \end{array} \right]$$

$$\therefore x=1, y=0, z=1$$

$$\downarrow R_3 - 3R_2 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 7 \end{array} \right]$$

32.

$$\begin{cases} x+y+z = 4 \\ -x+2y+3z = 17 \\ 2x-y = -7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ -1 & 2 & 3 & 17 \\ 2 & -1 & 0 & -7 \end{array} \right]$$

$$\begin{matrix} R_1 + R_2 \rightarrow R_2 \\ 2R_1 - R_3 \rightarrow R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & 4 & 21 \\ 0 & 3 & 2 & 15 \end{array} \right]$$

$$\begin{matrix} R_2 - \frac{4}{3}R_3 \rightarrow R_2 \\ R_1 - R_3 \rightarrow R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\begin{matrix} R_1 - R_2 \rightarrow R_1 \\ \dots \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$R_2 - R_3 \rightarrow R_3$$

$$\therefore x = -2, y = 3, z = 3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & 4 & 21 \\ 0 & 0 & 2 & 6 \end{array} \right]$$

$$\begin{matrix} \frac{1}{3}R_2 \\ \frac{1}{2}R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 4/3 & 7 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

33.

$$\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & -3 \end{array} \right]$$

$\downarrow R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 2 & -1 & -1 & -3 \\ 1 & 0 & 1 & 0 \end{array} \right]$$

$R_3 - R_1 \rightarrow R_3$
 —————
 $R_2 - 2R_1 \rightarrow R_2$

$$z = 1$$

$$y - \frac{1}{5}z = -\frac{1}{5}$$

$$y = -\frac{1}{5} + \frac{1}{5} = 0$$

$$x + 2y - z = -2$$

$$x + 0 - 1 = -2$$

$$x = -1$$

$$\therefore (-1, 0, 1)$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & -5 & 1 & 1 \\ 0 & -2 & 2 & 2 \end{array} \right]$$

②

$$-\frac{1}{5}R_2 \quad \downarrow R_3 - \frac{2}{5}R_2 \rightarrow R_3 \quad -\frac{1}{2}R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{8}{5} & \frac{8}{5} \end{array} \right]$$

$\downarrow \frac{5}{8}R_3$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 1 \end{array} \right]$$

34.

$$\begin{cases} 2y+z=4 \\ x+y=4 \\ 3x+3y-z=10 \end{cases}$$

$$\left[\begin{array}{cccc} 0 & 2 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 3 & 3 & -1 & 10 \end{array} \right]$$

$$\downarrow R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 4 \\ 0 & 2 & 1 & 4 \\ 3 & 3 & -1 & 10 \end{array} \right]$$

$$\downarrow R_3 - 3R_1 \rightarrow R_3$$

$$\begin{aligned} z &= 2 \\ y + \frac{1}{2}z &= 2 \\ y + \frac{1}{2}(2) &= 2 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x+y &= 4 \\ x &= 4-1 \\ &= 3 \\ \therefore (3, 1, 2) \end{aligned}$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -1 & -2 \end{array} \right]$$

$$\downarrow \frac{\frac{1}{2}R_2}{-R_3}$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 4 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

⑤ Dependent or Inconsistent Linear Systems

40.
$$\begin{cases} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 2 & 1 & -2 & 5 \\ 0 & -1 & 8 & 8 \end{array} \right]$$

$$\downarrow R_2 - 2R_1 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -8 & -1 \\ 0 & -1 & 8 & 8 \end{array} \right]$$

$$\downarrow R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -8 & -1 \\ 0 & 0 & 0 & 7 \end{array} \right]$$

\therefore No solution, Inconsistent

⑥ Solving a Linear System

50. $\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 14 \\ 4 & -1 & -2 & -17 \\ -1 & -1 & 1 & 3 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{\frac{1}{5}R_2} \\ \xrightarrow{-\frac{2}{5}R_3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 5/2 & 7 \\ 0 & 1 & -12/5 & -9 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_2 - 2R_1 \rightarrow R_2$$

$$R_3 + \frac{1}{2}R_1 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 14 \\ 0 & 5 & -12 & -45 \\ 0 & -5/2 & 7/2 & 10 \end{array} \right]$$

$$z = 5$$

$$y - \frac{12}{5}(5) = 9$$

$$y = -9 + 12 \\ y = 3$$

$$\begin{array}{l} \xrightarrow{\frac{1}{2}R_1} \\ R_3 + \frac{1}{2}R_1 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 5/2 & 7 \\ 0 & 5 & -12 & -45 \\ 0 & 0 & -5/2 & -\frac{25}{2} \end{array} \right]$$

$$x - \frac{3}{2}(3) + \frac{5}{2}(5) = 14$$

$$x - \frac{9}{2} + \frac{25}{2} = 14$$

$$x + 8 = 14$$

$$x = -1$$

$$2x - 3y + 5z = 14$$

$$\begin{aligned} 2\left(-\frac{1}{5}\right) - 3\left(\frac{1}{5}\right) + 5(3) \\ = -\frac{2}{5} - \frac{3}{5} + \frac{15}{5} \\ = \frac{10}{5} \\ = 14 \end{aligned}$$

$$4x - y - 2z = -17$$

$$\begin{aligned} 4\left(-\frac{1}{5}\right) - \left(\frac{1}{5}\right) + 2(3) \\ = -\frac{4}{5} - \frac{1}{5} + \frac{30}{5} \\ = \frac{25}{5} \\ = 5 \end{aligned}$$

69. $5x + 10y + 15z = 50$

$$15x + 20y + 0 = 50$$

$$10x + 10y + 10z = 50$$

$$\begin{bmatrix} 5 & 10 & 15 & 50 \\ 15 & 20 & 0 & 50 \\ 10 & 10 & 10 & 50 \end{bmatrix}$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 10 \\ 3 & 4 & 0 & 10 \\ 2 & 2 & 2 & 10 \end{array} \right]$$

$$\frac{R_2 - 3R_1 \rightarrow R_2}{R_3 - 2R_1 \rightarrow R_3}$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 10 \\ 0 & -2 & -9 & -20 \\ 0 & -2 & -4 & -10 \end{array} \right]$$

$$R_3 - R_2 \rightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 10 \\ 0 & -2 & -9 & -20 \\ 0 & 0 & 5 & 10 \end{array} \right]$$

$$\frac{-\frac{1}{2}R_2}{\frac{1}{5}R_3}$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 10 \\ 0 & 1 & \frac{9}{2} & 10 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$x = 10 - 3(1) - 2(0)$$

$$z = 2 \quad y = 10 - \frac{9}{2}(2) = 2$$

$$= 1 \quad \therefore (2, 1, 2)$$

11.2 The Algebra of Matrices

- ① Equality of Matrices
- ② Matrix Operations
- ③ Matrix Equations
- ④ Linear Systems as Matrix Equations
- ⑤ Applications

②

②

Matrix Operations

8/3/2024

9. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

 $= \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

 $= \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}$

11. $3 \begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & -3 \\ 3 & 0 \end{bmatrix}$

12. $2 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

\therefore not possible, because you cannot add a 3×3 matrix and a 3×2 matrix together

$$13. \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 6 \\ -2 & 0 \end{bmatrix}$$

\therefore not possible, 3×2 matrix and a 3×2 matrix
are not allowed to be multiplied together

$$14. \begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1) + 1(3) + 2(-2) & 2(-2) + 1(6) + 2(0) \\ 6(1) + 3(3) + 4(-2) & 6(-2) + 3(6) + 4(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 7 & 6 \end{bmatrix}$$

$$15. \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & -2+4 & 3-2 \\ -1+8 & 2+8 & -3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 1 \\ 7 & 10 & -7 \end{bmatrix}$$

$$16. \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ 0 & +1 \\ 5 & +2 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ 1 \\ 7 \end{bmatrix}$$

$$32. (a) B^2 = \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

\therefore 2×3 matrix and 2×3 matrix
incompatible

$$(b) F^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$33. (a) A^2 = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6 & 24+18 \\ 4+3 & 6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 42 \\ 7 & 15 \end{bmatrix}$$

$$(b) A^3 = A^2 A$$

$$= \begin{bmatrix} 22 & 42 \\ 7 & 15 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 130 & 132+126 \\ 43 & 87 \end{bmatrix}$$

$$= \begin{bmatrix} 130 & 258 \\ 43 & 87 \end{bmatrix}$$

$$34. (a) (DA)B$$

$$= \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -14 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 21 & 28 \end{bmatrix}$$

$$(b) D(AB)$$

$$= \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 & -5 \\ 7 & -7 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 21 & 28 \end{bmatrix}$$

③ Matrix Equations

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

$$(7) \quad 2X + A = B$$

$$2X = B - A$$

$$= \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 2 \end{bmatrix}$$

$$18. \quad 3X - B = C$$

$$3X = C + B$$

∴ no solution, C and B cannot be added together as their dimensions are different

$$19. \quad 2(B - X) = D$$

$$2 \left(\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - X \right) = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

$$2 \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - 2X = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

∴ no solution because B is 2×2 ,
and D is 3×2

$$20. \quad 5(X - C) = D$$

$$5X - 5C = D$$

$$5X - 5 \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

$$5X = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix} + \begin{bmatrix} 10 & 15 \\ 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$5X = \begin{bmatrix} 20 & 35 \\ 35 & 20 \\ 10 & 10 \end{bmatrix}$$

$$X = \frac{1}{5} \begin{bmatrix} 20 & 35 \\ 35 & 20 \\ 10 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 7 \\ 7 & 4 \\ 2 & 2 \end{bmatrix}$$

④ Linear Systems as Matrix Equations

47. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$

$$\begin{bmatrix} 2 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

48. $\begin{cases} 6x - y + z = 12 \\ 2x + z = 7 \\ y - 2z = 4 \end{cases}$

$$\begin{bmatrix} 6 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \\ 4 \end{bmatrix}$$

49. $\begin{cases} 3x_1 + 2x_2 - x_3 + x_4 = 0 \\ x_1 - x_3 = 5 \\ 3x_2 + x_3 - x_4 = 4 \end{cases}$

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

⑤ Applications

$$5. A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$$

[2] ∵ matrices have different dimensions, so they cannot be equal

$$7. A = \begin{bmatrix} 3 & 4 \\ -1 & a \end{bmatrix} \quad B = \begin{bmatrix} b & 4 \\ -1 & -5 \end{bmatrix}$$

$a = -5, b = 3$

Matrix Operations

$$9. \begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + (-1) & 6 + (-3) \\ -5 + 6 & 3 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$$

Matrix Equations

$$17. 2X + A = B$$

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$2X = B - A \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$X = \frac{1}{2} (B - A)$$

$$= \frac{1}{2} \begin{bmatrix} 2-4 & 5-6 \\ 3-1 & 7-3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 2 \end{bmatrix}$$

Matrix Operations

$$23. (a) B + C$$

$$= \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5/2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 5 \\ 1 & 1 & 0 \end{bmatrix}$$

$$(b) B + F$$

\therefore cannot be performed because
the matrices have different
dimensions,

B is 2 by 3 while F is
3 by 3

27. (a) AD

$$AD = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 7 & 3 \end{bmatrix}$$

\therefore undefined,

2×2 matrix and 1×2 matrix

(b) DA

$$DA = \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 + 0 \\ -35 + 21 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ -14 \end{bmatrix}$$

Equality of Matrices

$$43. \begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$$

$$\begin{array}{l} x=2 \quad 2y = -2 \\ 4 = 2x \quad 6 = -6y \\ x=2 \quad y = -1 \end{array}$$

Linear Systems as Matrix Equations

$$47. 2x - 5y = 7$$

$$3x + 2y = 4$$

$$\begin{bmatrix} 2 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Applications

53.

$$\begin{bmatrix} 0.75 & 0.10 & 0 \\ 0.25 & 0.70 & 0.70 \\ 0 & 0.20 & 0.30 \end{bmatrix} = A$$

$$\begin{bmatrix} 4 \\ 20 \\ 10 \end{bmatrix} = B$$

(a)

$$AB = \begin{bmatrix} 0.75 & 0.10 & 0 \\ 0.25 & 0.70 & 0.70 \\ 0 & 0.20 & 0.30 \end{bmatrix} \begin{bmatrix} 4 \\ 20 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \cdot 4 + 0.10 \cdot 20 + 0 \\ 0.25 \cdot 4 + 0.70 \cdot 20 + 7 \\ 0 + 0.20 \cdot 20 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 2 + 0 \\ 1 + 14 + 7 \\ 0 + 4 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 22 \\ 7 \end{bmatrix}$$

(b) Matrix AB
 shows the
 number of
 members in
 each category
 of years of
 presecondary
 education.

5 no education,
 and so on

11.3 Inverses of Matrices and Matrix Equations

- ① Verifying the Inverse of a Matrix
- ② The Inverse of a 2×2 Matrix
- ③ Finding the Inverse of a Matrix
- ④ Solving a Linear System
- ⑤ Skills Plus

- ③ Finding the Inverse of a Matrix

① Verifying the Inverse of a Matrix

$$3. A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 8-7 & -4+4 \\ 14-14 & -7+8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$4. A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} B = \begin{bmatrix} 7/2 & -3/2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{aligned}
 AB &= \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 7/2 & -3/2 \\ 2 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 7-6 & -3+3 \\ 14-14 & -6+7 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{bmatrix} 7/2 & -3/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \\
 &= \begin{bmatrix} 7-6 & -\frac{21}{2} + \frac{21}{2} \\ 4-4 & -6+7 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

② The Inverse of a 2×2 Matrix

$$7. A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{14 - 12} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix} \\ &= \begin{bmatrix} 7 - 6 & -14 + 14 \\ 3 - 3 & -6 + 7 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$8. \quad B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$R_3 + 2R_1 \rightarrow R_3$$

↓

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{array} \right]$$

$$\frac{1}{2}R_2$$

↓

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{array} \right]$$

$$R_3 - 5R_2 \rightarrow R_3$$

↓

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 2 & -\frac{5}{2} & 1 \end{array} \right]$$

$$\begin{array}{c} -R_3 \\ \downarrow \\ \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{5}{2} & -1 \end{array} \right] \end{array}$$

$$\begin{array}{c} R_2 - R_1 \rightarrow R_2 \\ \hline R_1 - 2R_2 \rightarrow R_1 \\ \downarrow \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 5 & -5 & 2 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -2 & \frac{5}{2} & -1 \end{array} \right]$$

$$R_1 - 3R_2 \rightarrow R_1 \\ \downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -2 & \frac{5}{2} & -1 \end{array} \right]$$

$$B^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & 1 \\ -2 & \frac{5}{2} & -1 \end{bmatrix}$$

$$BB^{-1} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & \frac{5}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Finding the Inverse of a Matrix

11. $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-9 + 10} \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

12. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

$$= \frac{1}{27 - 28} \begin{bmatrix} 9 & -4 \\ -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}$$

$$13. \begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{-26+25} \begin{bmatrix} -13 & -5 \\ 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix} \end{aligned}$$

$$14. \begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{35-32} \begin{bmatrix} -5 & -4 \\ -8 & -7 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{3} & -\frac{4}{3} \\ -\frac{8}{13} & -\frac{7}{13} \end{bmatrix} \end{aligned}$$

$$15. \begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{24-24}$$

$\therefore A^{-1}$ does not exist

$$16. \begin{bmatrix} 1/2 & 1/3 \\ 5 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2 - \frac{5}{3}} \begin{bmatrix} 4 & -1/3 \\ -5 & 1/2 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 4 & -1/3 \\ -5 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -1 \\ -15 & 3/2 \end{bmatrix}$$

$$17. \begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$$

$$\begin{aligned}
 A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
 &= \frac{1}{0.24 + 0.36} \begin{bmatrix} 0.6 & 1.2 \\ -0.3 & 0.4 \end{bmatrix} \\
 &= \frac{1}{0.60} \begin{bmatrix} 0.6 & 1.2 \\ -0.3 & 0.4 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 \\ -1/2 & 2/3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \left| \begin{array}{ccc|ccc} 4 & 2 & 3 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right. &= M_{11}A_{31} + M_{32}A_{32} + M_{33}A_{33} \\
 &= (-1)^4 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} (1) + 0 \\
 &\quad + (-1)^6 \begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix} (1)
 \end{aligned}$$

$$\begin{bmatrix} 4 & 2 & 3 & | & 1 & 0 & 0 \\ 3 & 3 & 2 & | & 0 & 1 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} = -5 + 6 = 1$$

$$\downarrow \frac{R_2 - 3R_3 \rightarrow R_2}{4R_3 - R_1 \rightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & -3 \\ 0 & -2 & 1 & -1 & 0 & 4 \end{array} \right]$$

\downarrow

$\frac{1}{4}R_1, \frac{1}{3}R_2$

$3R_3 + 2R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/4 & 1/4 & 0 & 0 \\ 0 & 1 & -1/3 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 6 \end{array} \right]$$

\downarrow

$R_2 + \frac{1}{3}R_3 \rightarrow R_2$

$R_1 - \frac{3}{4}R_3 \rightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 5/2 & -3/2 & -9/2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -3 & 2 & 6 \end{array} \right]$$

\downarrow

$R_1 - \frac{1}{2}R_2 \rightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -3 & 2 & 6 \end{array} \right]$$

④ Solving a Linear System

39. $-3x - 5y = 4$

$$2x + 3y = 0$$

$$\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-9+16} \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

40. $3x + 4y = 10$

$$7x + 9y = 20$$

$$\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{27-28} \begin{bmatrix} 9 & -4 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 4 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ 10 \end{bmatrix}$$

$$47. \begin{cases} x + y - 2z = 3 \\ 2x + 5z = 11 \\ 2x + 3y = 12 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & 0 & 5 & 11 \\ 2 & 3 & 0 & 12 \end{array} \right]$$

$$\downarrow -\frac{1}{11}R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\downarrow R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & 3 & 0 & 12 \\ 2 & 0 & 5 & 11 \end{array} \right]$$

$$\begin{aligned} z &= 1 \\ y + 4z &= 6 \\ y &= 6 - 4 \\ &= 2 \end{aligned}$$

$$\downarrow R_2 - R_3 \rightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & 3 & 0 & 12 \\ 0 & 3 & -5 & 1 \end{array} \right]$$

$$\begin{aligned} x + y - 2z &= 3 \\ x &= 3 - 2 + 2 \\ &= 3 \\ \therefore (x, y, z) &= (3, 2, 1) \end{aligned}$$

$$\downarrow R_2 - 2R_1 \rightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 3 & -5 & 1 \end{array} \right]$$

$$\downarrow R_3 - 3R_2 \rightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -17 & -17 \end{array} \right]$$

11.3 Inverses of Matrices and Matrix Equations

1. (a) identity

(b) A, A

(c) Inverse matrix

$$3. \quad A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8-7 & -4+4 \\ 14-14 & -7+8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore AB = BA = I_2$$

$\therefore B$ is the inverse of A

7. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{14 - 12} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 6 & -14 + 14 \\ 3 - \frac{3}{2} & -6 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
 A^{-1}A &= \begin{bmatrix} 1 & -2 \\ -3/2 & 7/2 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 7-6 & 4-4 \\ -\frac{21}{2} + \frac{21}{2} & -6+7 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

$$\therefore AA^{-1} = A^{-1}A = I_2$$

$$19. \quad \begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1/2 & \frac{1}{2} & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$R_1 + R_2 \rightarrow R_2$$

→

$$R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 1/2 \\ 0 & 3 & -1/2 & 1/2 \\ 0 & 5 & -1 & 0 \end{array} \right]$$

$$\frac{1}{3} R_2$$

→

$$\left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 1/2 \\ 0 & 1 & -1/6 & 1/6 \\ 0 & 5 & -1 & 0 \end{array} \right]$$

$$R_3 - 5R_2 \rightarrow R_3$$

→

$$\left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 1/2 \\ 0 & 1 & -1/6 & 1/6 \\ 0 & 0 & -1/6 & -5/6 \end{array} \right]$$

$$-6R_3$$

→

$$\left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 1/2 \\ 0 & 1 & -1/6 & 1/6 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_1 - \frac{1}{2}R_3 \rightarrow R_1$$

→

$$R_2 + \frac{1}{6}R_3 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_1 - 2R_2 \rightarrow R_1$$

→

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$\therefore \text{Inverse: } \begin{bmatrix} -4 & -4 & 5 \\ 1 & 1 & -1 \\ 5 & 4 & -6 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & -1 & 0 & 1 & 0 \\ 1 & -1 & -10 & 0 & 0 & 1 \end{array} \right]$$

$R_2 - 4R_1 \rightarrow R_2$

$\xrightarrow{\hspace{1cm}}$

$R_3 - R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -13 & -4 & 1 & 0 \\ 0 & -3 & -13 & -1 & 0 & 1 \end{array} \right]$$

$R_3 - R_2 \rightarrow R_3$

$\xrightarrow{\hspace{1cm}}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -13 & -4 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{array} \right]$$

\therefore Last row all 0's, so the matrix does not have an inverse

Solving a Linear System as a Matrix Equation

39. $\begin{cases} -3x - 5y = 4 \\ 2x + 3y = 0 \end{cases}$ Let $A = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$,
 $X = \begin{bmatrix} x \\ y \end{bmatrix}$,
 $B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad AX = B$$

$$\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-9 + 10} \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \quad X = A^{-1}B$$

$$= \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0 \\ -8-0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$\therefore x = 12, y = -8$$

$$61. \text{ (a)} \begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix} = A$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 4 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_2 - \frac{4}{3}R_1 \rightarrow R_2 \\ \hline R_3 - R_1 \rightarrow R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{4}{3} & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} \frac{3}{2}R_2 \\ \hline \frac{1}{3}R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 1 & \frac{1}{15} & 0 & 0 \\ 0 & 1 & 0 & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_3 - R_2 \rightarrow R_3 \\ \hline \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 1 & \frac{1}{15} & 0 & 0 \\ 0 & 1 & 0 & -2 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 - R_3 \rightarrow R_1 \\ \hline \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 0 & -\frac{2}{15} & \frac{3}{2} & -1 \\ 0 & 1 & 0 & -2 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & 1 \end{array} \right]$$

$$R_1 - \frac{1}{3}R_2 \rightarrow R_1$$

$\xrightarrow{\hspace{1cm}}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 3/2 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1 \end{array} \right]$$

$$(b) \quad B = \begin{bmatrix} 10 \\ 14 \\ 13 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 13 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3/2 & 0 \\ 1 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 14 - 13 \\ -20 + 21 + 0 \\ 10 - 21 + 13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore x = 1, y = 1, z = 2$$

$$(c) \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{12} \\ \frac{12}{10} \\ \frac{10}{10} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3/2 & 0 \\ 1 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12} \\ \frac{12}{10} \\ \frac{10}{10} \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 12 - 10 \\ -18 + 18 + 0 \\ 9 - 18 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$(d) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3/2 & 0 \\ 1 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{4} \\ \frac{4}{4} \\ \frac{11}{11} \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 4 - 11 \\ -4 + 6 + 0 \\ 2 - 6 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 2 \\ 7 \end{bmatrix}$$

\therefore No, because $x = -7$.

11.4 Determinants and Cramer's Rule

- ① Finding Determinants
- ② Minors and Cofactors
- ③ Cramer's Rule
- ④ Skills Plus

- ① Finding Determinants
- ② Minors and Cofactors

① Finding Determinants

$$5. \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = ad - bc \\ = 6 - 0 \\ = 6$$

$$6. \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 \\ = 2$$

$$7. \begin{bmatrix} \frac{3}{2} & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}$$

$$\begin{vmatrix} \frac{3}{2} & 1 \\ -1 & -\frac{2}{3} \end{vmatrix} = ad - bc \\ = -1 + 1 \\ = 0$$

$$8. \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & -0.8 \end{bmatrix}$$

$$\begin{vmatrix} 0.2 & 0.4 \\ -0.4 & -0.8 \end{vmatrix}$$

$$= 0.2(-0.8) - 0.4(-0.4)$$

$$= -0.16 + 0.16$$

$$= 0$$

$$9. \begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & 5 \\ 0 & -1 \end{vmatrix} = ad - bc = -4 - 0 \\ = -4$$

$$10. \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} = -2 \cdot -2 - 1 \cdot 3 \\ = 1$$

$$21. \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{aligned}
 \begin{vmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{vmatrix} &= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} \\
 &= 2 \cdot (-1)^2 \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} + 1 \cdot (-1)^3 \begin{vmatrix} 0 & 4 \\ 0 & -3 \end{vmatrix} \\
 &\quad + 0 \cdot (-1)^4 \begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix} \\
 &= 2(2) + 0 + 0 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 25. \begin{vmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{vmatrix} &= a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23} \\
 &= 0 + 2(-1)^5 \begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} \\
 &\quad + 2(-1)^6 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \\
 &= 0 - 2(-6) + 2(0 - 6) \\
 &= 12 - 12 \\
 &= 0 \quad \therefore \text{no inverse}
 \end{aligned}$$

② Minors and Cofactors

$$A = \begin{bmatrix} 1 & 0 & 1/2 \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$15. M_{11} = 20 - 0 \\ = 20$$

$$A_{11} = (-1)^2 M_{11} \\ = 20$$

$$16. M_{33} = 5 - 0 \\ = 5$$

$$A_{33} = (-1)^6 M_{33} \\ = 5$$

$$17. M_{12} = -3(4) - 2(0) \\ = -12$$

$$A_{12} = (-1)^3 M_{12} \\ = -1(-12) \\ = 12$$

$$18. M_{13} = \begin{vmatrix} -3 & 5 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{13} = (-1)^4 M_{13} = 0$$

$$19. M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{23} = (-1)^5 M_{23} = 0$$

$$20. M_{32} = \begin{vmatrix} 1 & 1/2 \\ -3 & 2 \end{vmatrix} = 2 + \frac{3}{2} = \frac{7}{2}$$

$$A_{32} = (-1)^5 M_{32} = -\frac{7}{2}$$

③ Cramer's Rule

41. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

$$D = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$D_x = \begin{bmatrix} -9 & -1 \\ 8 & 2 \end{bmatrix}$$

$$D_y = \begin{bmatrix} 2 & -9 \\ 1 & 8 \end{bmatrix}$$

$$\begin{aligned} x &= \frac{|D_x|}{|D|} & y &= \frac{|D_y|}{|D|} \\ &= \frac{-18+8}{4+1} & &= \frac{16+9}{4+1} \\ &= \frac{-10}{5} & &= \frac{25}{5} \\ &= -2 & &= 5 \end{aligned}$$

$$\therefore x = -2, 5$$

$$42. \quad \begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$$

$$D = \begin{bmatrix} 6 & 12 \\ 4 & 7 \end{bmatrix} \quad D_x = \begin{bmatrix} 33 & 12 \\ 20 & 7 \end{bmatrix} \quad D_y = \begin{bmatrix} 6 & 33 \\ 4 & 20 \end{bmatrix}$$

$$\begin{aligned} x &= \frac{|D_x|}{|D|} & y &= \frac{|D_y|}{|D|} \\ &= \frac{231 - 240}{42 - 48} & &= \frac{120 - 132}{42 - 48} \\ &= \frac{-9}{-6} & &= \frac{-12}{-6} \\ &= \frac{3}{2} & &= 2 \end{aligned}$$

$$43. \quad \begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$$

$$D = \begin{bmatrix} 1 & -6 \\ 3 & 2 \end{bmatrix} \quad D_x = \begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix} \quad D_y = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\begin{aligned} x &= \frac{|D_x|}{|D|} & y &= \frac{|D_y|}{|D|} \\ &= \frac{6+6}{2+18} & &= \frac{1-9}{20} \\ &= \frac{12}{20} & &= -\frac{2}{5} \\ &= \frac{3}{5} \end{aligned}$$

$$5. \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 6 - 0$$

$$= 6$$

$$21. \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2 \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} - 1 \begin{vmatrix} 0 & 4 \\ 0 & -3 \end{vmatrix} + 0 \begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix}$$

$$= 2 (6 - 4)$$

$$= 4$$

\therefore has an inverse as

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{vmatrix} \neq 0$$

25.

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{aligned}
 \begin{vmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{vmatrix} &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\
 &= 0 + 2(-1)^5 \begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} + 2(-1)^6 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \\
 &= -2(8 - 14) + 2(0 - 6) \\
 &= -2(-6) + 2(-6) \\
 &= 12 - 12 \\
 &= 0
 \end{aligned}$$

35.

$$\begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 0 & 1 & 7 \end{vmatrix}$$

$$\text{Row 3} - \text{Row 2} \rightarrow \text{Row 3}$$

$$= \begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 7 \end{vmatrix}$$

$$\begin{aligned}
 &= 0 + 0 + a_{33} A_{33} + 0 \\
 &= 1 (-1)^6 \begin{vmatrix} 0 & 0 & 6 \\ 2 & 1 & 3 \\ 3 & 0 & 7 \end{vmatrix} \\
 &= 0 + 0 + a_{13} (-1)^4 \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \\
 &= 6 (0 - 3) \\
 &= -18
 \end{aligned}$$

41. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

$$|D| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 4 - (-1)$$

$$= 5$$

$$|D_x| = \begin{vmatrix} -9 & -1 \\ 8 & 2 \end{vmatrix}$$

$$= -18 - (-8)$$

$$= -10$$

$$|D_y| = \begin{vmatrix} 2 & -9 \\ 1 & 8 \end{vmatrix}$$

$$= 16 - (-9 \cdot 1)$$

$$= 25$$

$$x = \frac{|D_x|}{|D|} = \frac{-10}{5} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{25}{5} = 5$$

47. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$

$$|D| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$$

$$= 3(-1)^3 \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} + 0 + 1(-1)^5 \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -3(0 - 4) - 1(2 - 1)$$

$$= 12 - 1 = 11$$

$$|D_x| = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a_{32} A_{32}$$

$$= 2 \cdot (-1)^5 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -2(0 - 2)$$

$$= 44$$

$$|D_y| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= a_{31} A_{31}$$

$$= -1(-1)^4 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -1(0 - 2)$$

$$= 22$$

$$|D_z| = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix}$$

$\xrightarrow{\text{Row 1} + \text{Row 3} \rightarrow \text{Row 1}}$

$$= \begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix}$$

$$\Rightarrow = a_{12} A_{12}$$

$$= 1(-1)^3 \begin{vmatrix} -3 & 1 \\ -1 & 0 \end{vmatrix}$$

$$= -1(0 + 1)$$

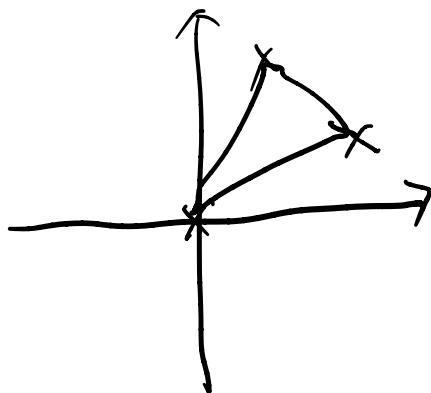
$$= -1$$

$$x = \frac{|D_x|}{|D|} \quad y = \frac{|D_y|}{|D|} \quad z = \frac{|D_z|}{|D|}$$

$$= \frac{44}{11} \quad = \frac{22}{11} \quad = -\frac{11}{11}$$

$$= 4 \quad = 2 \quad = -1$$

57.



$$\therefore A = \pm \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 6 & 2 & 1 \\ 3 & 8 & 1 \end{vmatrix}$$

$$= \pm \frac{1}{2} (a_{13} A_{13})$$

$$= \pm \frac{1}{2} (1 \cdot (-1)^4 \begin{vmatrix} 6 & 2 \\ 3 & 8 \end{vmatrix})$$

$$= \pm \frac{1}{2} (42)$$

$$= 21$$