Title: Optimization-based control in Portfolio Optimization

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Contents

1 Background

Portfolio optimization involves the allocation of a fund into different financial assets based on certain criteria, typically maximising return for a given level of risk from an investor. It is used everywhere in the world from traders automating asset allocation in the financial markets, to pension funds and endowments designing strategies that ensure stable returns over long horizons.

1.1 Aim

The is the first half of the research where we review and explain the mathematical framework of Portfolio Optimization from first principles, show how optimization plays a key role and how the problem closely resembles control problems. At the end, we apply it to empirical data to illustrate the technique.

1.2 Objectives

1. Review the literature on portfolio optimization focusing on discrete-time settings.
2. Explain and build a model of portfolio optimization using current methods.

2 Literature Review

To see how optimization and control theory are closely related to portfolio optimization, we must first look at contemporary methods of how mathematics is applied to finance, and arguably the most prominent of which is portfolio theory. This concept was pioneered by Harry Markowitz in his doctoral dissertation "Portfolio Selection" in 1952 which revolutionised the way academics and practitioners think about investment management by providing the first rigorous framework of modelling a portfolio of financial assets, now more popularly known as mean-variance analysis or **Modern Portfolio Theory (MPT)**.

The theory centers around the assumption that a rational investor intends to maximise their return while minimising the risk (potential downsides or lost) of their investments. Although it is not without criticism, Warren Buffett, who is widely regarded as the best investor in history has criticized the way of thinking of the risk of a stock as volatility in its price movement (Buffett, 1993) but it is our current best model.

(Markowitz, 1952) mathematically formalised the idea that diversification reduces portfolio risk by combining assets with low or negative correlations. He developed the concept of the **efficient frontier**, a set of a portfolios that are optimal in terms of risk and return. The idea of a risk-free interest rate from (Fisher, 1930) was used by (Tobin, 1954) to introduce the **Capital Market Line (CML)**, the straight line connecting the risk-free rate to the tangency portfolio. He demonstrated how to achieve an optimal portfolio on the efficient frontier when combining it with a risk-free asset. William Sharpe extended upon Markowitz's and Tobin's work by assuming market equilibrium (Sharpe, 1964), and developed the **Capital Asset Pricing Model (CAPM)** as a model to explain how individual assets are priced in equilibrium relative to the market. The rest of the report shows how this is formulated mathematically and illustrate the method using a real example of a portfolio of stocks.

3 Technical Progress

3.1 Basic Concepts

3.1.1 Return

Using probability theory, the return of a financial asset or stock R, can be modelled as the expected return of a random variable. Then the return of a portfolio P consisting of assets A_i with average return $\overline{R_i}$, in proportions X_i would be:

$$E(R_P) = \overline{R_P} = \sum_{i=1}^{n} X_i \overline{R_i} \tag{Eq 3.1}$$

3.1.2 Risk

According to MPT, risk can be modelled using variance or standard deviation of the return of an asset defined by:

$$Var(R) = E((R - \bar{R})^2),\ or,\ \sigma_R = Var(R)^{\frac{1}{2}} \tag{Eq 3.2}$$

3.1.3 Defining efficiency

To start out, two fundamental assumptions are made:

1. According to the CAPM, individual assets are correctly priced based on their risk relative to the market.
2. We can estimate the expected returns and risk of stocks by their historical prices.

Then determining the best portfolio out of all possible portfolios moves from picking the asset that we think will provide the highest future returns, to a conversation of comparing portfolios with varying risk and returns with each other. The following definitions from (Joshi, 2013) can then be laid out.

> **Definition 1**: The set of all possible pairs of returns and standard deviations attainable from investing in a collection of assets is called the opportunity set.
>
> **Definition 2**: A portfolio is efficient relative to a given opportunity set provided no other portfolio in that opportunity set
>
> 1. Has at least as much expected return and lower standard deviation, and
> 2. Has a higher return and an equal or smaller standard deviation

Definition 3: The subset of the opportunity set which is efficient is called the efficient frontier.

Efficiency is defined relative to the set of investment opportunities, changing the set of assets available to investors also changes the set of efficient portfolios.

3.2 Modelling Stocks

3.2.1 Two asset portfolio

We consider the simple case of the opportunity set consisting of two risky assets A and B, and attempt to construct a relationship between the risk and return of the set of portfolios of these 2 assets.

Assuming investment fractions X_A and X_B such that

$$X_A + X_B = 1$$

and from (Eq 3.1) we get

$$R_P = X_A R_A + X_B R_B \qquad \text{(Eq 3.3)}$$

Then applying the linearity of expectations to (Eq 3.3):

$$\begin{aligned} E(R_P) &= X_A E(R_A) + (1 - X_A) E(R_B) \\ &= X_A (E(R_A) - E(R_B)) + E(R_B) \end{aligned} \qquad \text{(Eq 3.4)}$$

and applying the property of variance to (Eq 3.3):

$$\sigma_P{}^2 = X_A{}^2 \sigma_A{}^2 + (1 - X_A) \sigma_B{}^2 + 2X_A(1 - X_A)\sigma_{AB} \qquad \text{(Eq 3.5)}$$

We can see then that the expected return is linear in X_A whilst the variance is quadratic and also depend on the correlation between the two assets.

We illustrate the parabola curve in the risk-return space using a numerical example for the assets A and B. Using the parameters for expected returns of 12 and 8 respectively, standard deviations of 20 and 15, and correlation of 0.3, we compute the efficient frontier using 100 weightings of X_A from 0 to 1 with increments of 0.01. The resulting plot from Python is shown in Fig. 1.

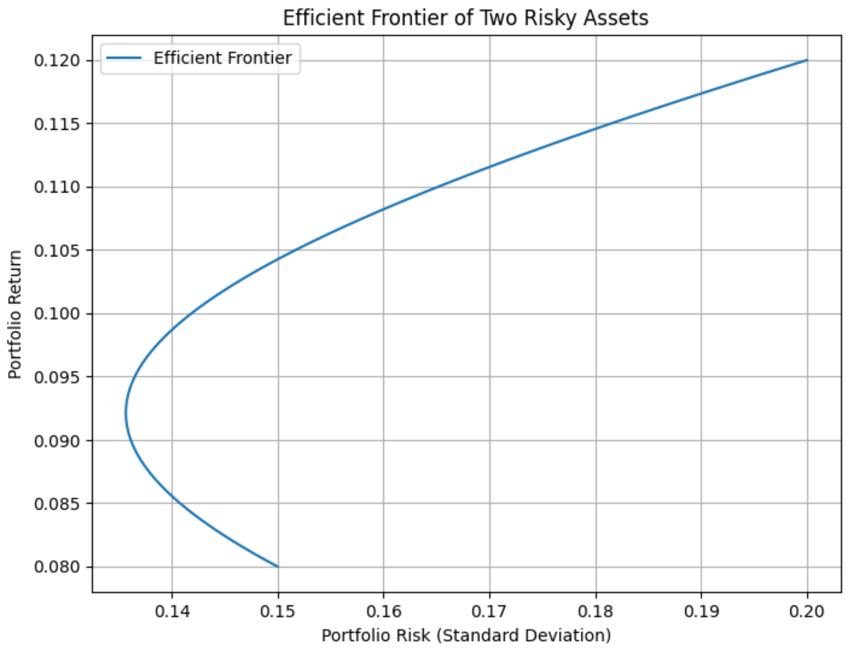

Fig 1. Efficient frontier for two risky assets

3.2.2 Risk-free asset and the Tangent Portfolio

We introduce the risk-free asset with the definition from (Joshi, 2013) as follows:

> **Definition 4**: An asset whose return is known in advance is said to be risk-free. An asset f, is risk-free if and only if:
>
> 1. The variance of returns is zero
> 2. The standard deviation of returns is zero

Suppose that a portfolio P consists of 1 – y units of the risk-free asset f with return R_f, and y units of the risky asset (or a portfolio of risky assets) A with return R_A, then expected return of P is then

$$\overline{R_P} = (1-y)R_f + y\overline{R_A} \quad \text{(Eq 3.5)}$$

applying the property of variance, and since R_f is riskless, the risk of the portfolio would be

$$Var(R_P) = y^2 Var(R_A),$$

$$\sigma_P = |y|\sigma_A$$

Restricting y ≥ 0, we have

$$y = \frac{\sigma_P}{\sigma_A}$$

Substituting y into (Eq 3.5), we have

$$\overline{R_P} = \frac{\overline{R_A} - R_f}{\sigma_A}\sigma_P + R_f$$

This shows that the new portfolio P, which is combination of a risk-free asset with a risky portfolio A produces a straight line for the opportunity set, which is called the Capital Market Line (Tobin, 1958). The gradient $\frac{\overline{R_A}-R_f}{\sigma_A}$ turns out to be the Sharpe ratio (Sharpe, 1964), which represents the ratio of return per unit of increase in risk that an investor undertakes.

The CML, the entire line through points (0, R_f) and $(\sigma_A, \overline{R_A})$ for a particular portfolio of risky assets and a risk-free asset is efficient. We state two theorems from (Joshi, 2013) omitting the proof.

> Theorem 1: If there is a risk-free asset, all efficient portfolios lie on a straight line in standard deviation/expected return space.

Even after discarding the risk-free asset, investing solely in a portfolio of risky assets A is itself efficient if the new portfolio P is efficient.

> Theorem 2: If P is efficient, then the portfolio A consisting of risky assets in P is efficient relative to investing solely in risky assets.

Reconciling the CML with the opportunity set for a portfolio of risky assets A, we summarise omitting the full proof from (Joshi, 2013) that

1. The efficient set of A is a hyperbola in risk/return space.
2. Combining the risk-free asset R_f with risky assets A produces a new portfolio P that is a straight line through points (0, R_f) and $(\sigma_A, \overline{R_A})$. And this whole line is also efficient.

3. The point of tangency of the efficient line P and the hyperbola efficient set of A is an efficient portfolio called the tangent portfolio.

3.2.3 Multi-asset case

Generalising to the multi-asset case, A is now a portfolio of risky assets with return $\overline{R_A}$ and standard deviation σ_A given by

$$\overline{R_A} = \langle x, \bar{R} \rangle, and\ \sigma_A = (x^T C x)^{\frac{1}{2}} \qquad \text{(Eq 3.6)}$$

where x is a vector or portfolio weights, C is the covariance matrix, and $\bar{R}$ is the vector of returns for the underlying assets, and notation $\langle a, b \rangle$ denotes the dot product of vectors a and b.

Let us now illustrate the efficient frontier in solely the risky case when holding a portfolio of 7 stocks. Raw public data of the closing prices for the 7 stocks of Apple, Amazon, Google, Meta, Microsoft, Nvidia and Tesla were analysed from 27th November 2023 to 22nd November 2024. The period was 252 days or roughly equivalent to a full year's worth of trading days.

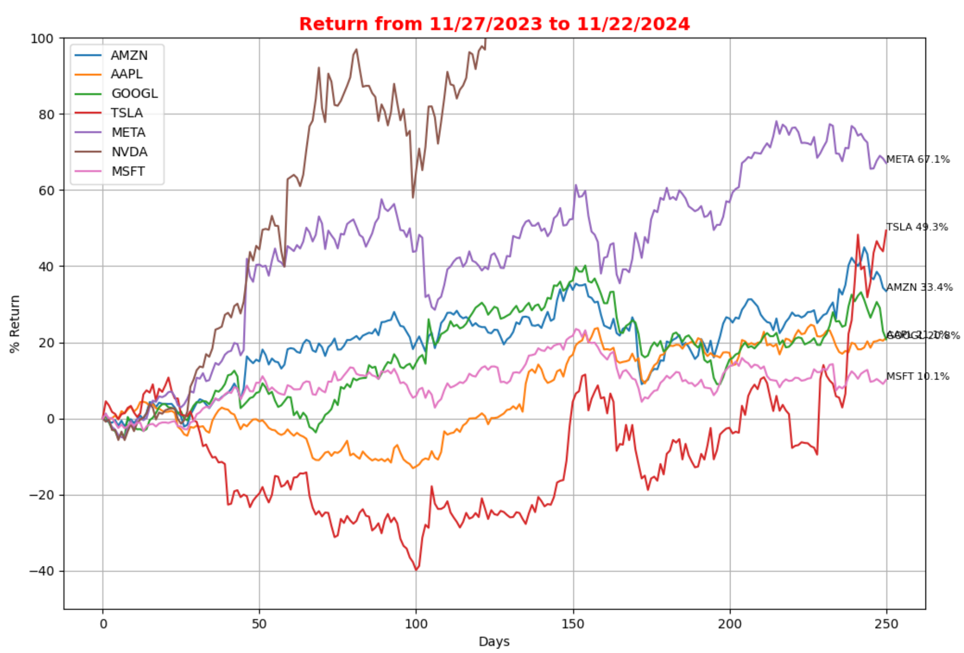

Fig. 2: The cumulative returns of the 7 tech stocks for a full year

From the daily closing prices, 3 statistics were computed for each stock. Simple returns from the start to end period, daily percentage returns, and the cumulative returns for the whole period. Cumulative returns were then plotted using Python to produce Fig. 2.

Using (Eq 3.6), we compute expected returns and the standard deviations of the 7 stocks from data of daily returns. The covariance matrix can also be determined analytically from the data but here the in-built *pandas* library from python was used. Then using Monte Carlo simulations for 10,000 random weights and limiting it for $x_i > 0$ (no short selling allowed), we compute the opportunity set and plot it as shown in Fig. 3. The full code is provided in Appendix 5.2.

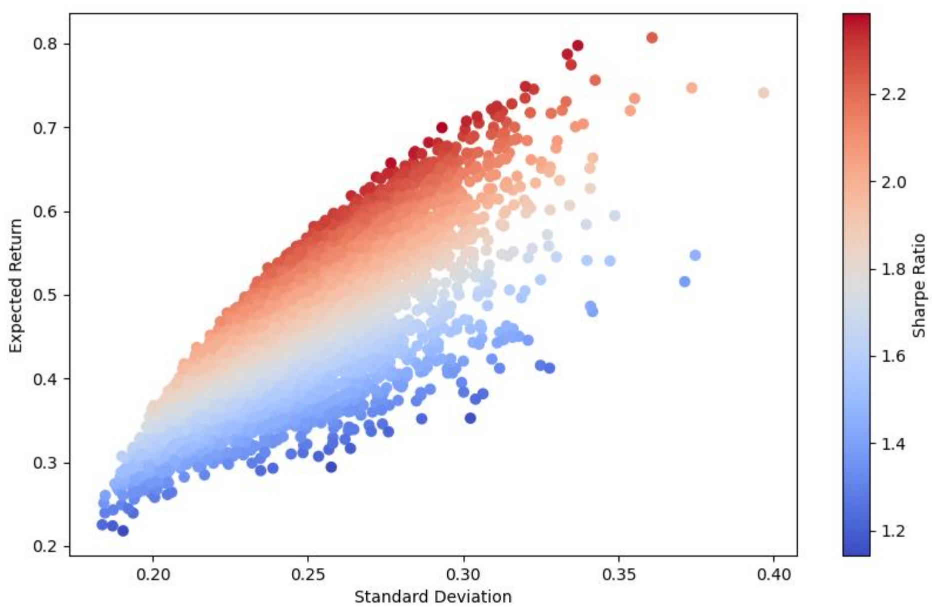

Fig. 3: The opportunity set of the "Magnificent 7" stocks with no short selling

In practice, an investor would want to take into account the return of their portfolio in relation to the return of the risk-free asset. From section 3.2.2, we know that the tangent point between the CML and the opportunity set for a portfolio of risky assets gives us the tangent portfolio, the efficient portfolio where all the funds are invested in the risky assets and none in the risk-free asset.

Hence, the problem now reduces to maximizing the slope

$$
\begin{aligned}
\theta &= \frac{\overline{R_A} - R_f}{\sigma_A} \\
&= \frac{\langle x, \bar{R} \rangle - R_f}{(x^T C x)^{\frac{1}{2}}}
\end{aligned}
$$

with the constraint $\sum_{i=1}^{n} x_i = 1$.

From (Joshi, 2013), the algorithm for computing the tangent portfolio weights of vector x is:

1. Let $\widetilde{R_i} = \overline{R_i} - R_f$
2. Solve $Cy = \tilde{R}$
3. Set $x_i = \frac{y_i}{\sum_{j=1}^{n} y_j}$

If an investor wants to determine the efficient portfolio with the minimal risk, and hence the minimal variance portfolio (MVP), we can vary the risk-free rate to get lower and lower, the slope of the CML gets steeper and steeper, and the tangent portfolio gets closer to the tip (i.e. the point of minimal variance). Omitting the full proof from (Joshi, 2013), it follows that the weights x of the MVP can be obtained by letting the risk-free rate tend to $-\infty$, and we have

$$
x = \frac{C^{-1} e}{\langle C^{-1} e, e \rangle}
$$

where *e* is a vector of ones of size n.

We use the algorithm to compute the tangent portfolio weights, and the equation for the MVP weights to compute the MVP weights. From the full code in Appendix 5.3, the expected return and standard deviations of the tangent portfolio of the “Magnificent 7” stocks are shown in Fig. 4 below.

```
Return: 3.448
Standard Deviation:  1.275
```

Fig. 4

while for the MVP they are shown in Fig. 5 below.

```
Return: 0.065
Standard Deviation:  0.175
```

Fig. 5

Assuming the November 2024 1-month Treasury Rate of 4.72% as the theoretical risk-free asset, and superimposing a straight line between the risk-free portfolio and the tangent portfolio on Fig. 3, we get a plot in Fig. 6.

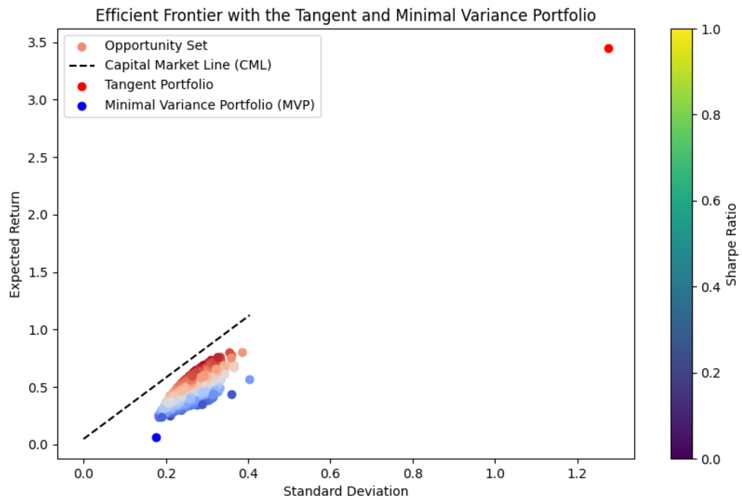

Fig. 6: The tangent portfolio and MVP of the 7 tech stocks including short selling

If done properly, the CML should cross the opportunity set of risky stocks A at the tangent portfolio with inefficient portfolios lying under the curve. We hypothesise that the irreconcilable difference between the result and the expected graph is due to two things. As the portfolios from Fig 3. were only allowed for weightings $x_i > 0$, portfolios of assets with negative weightings were left out. Secondly, the risk-free rate used might be irreconcilable with past data, and the actual theoretical risk-free rate.

4 References

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5 Appendix

5.1 Computing the cumulative returns of the "Magnificent 7" stocks

```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

stocks = ['AMZN', 'AAPL', 'GOOGL', 'TSLA', 'META', 'NVDA', 'MSFT']

data = pd.read_csv(f"data{stocks[0]}.csv", usecols=['Date', 'Close'])
data = data.rename(columns={'Close':f'{stocks[0]}'})

for i in range(1, len(stocks)):
    d = pd.read_csv(f"data{stocks[i]}.csv", usecols=['Close'])
    d.rename(columns={'Close':f'{stocks[i]}'})
    data[f"{stocks[i]}"] = d

data = data.iloc[::-1].reset_index(drop=True)
print(data)
data.to_csv("1DailyPrices.csv")
simpleReturn = ((data.iloc[-1, 1:]/data.iloc[0, 1:]) - 1) * 100
simpleReturn.to_csv("2SimpleReturn.csv")

for i in range(len(stocks)):
    name = f"{stocks[i]}"
    data[name] = (data[name]/data[name].shift())
data.to_csv("3DailyReturns.csv")

data.iloc[0, 1:] = 1
data2 = (data.iloc[:, 1:].cumprod()-1)*100
data2.to_csv("4CumulativeDailyReturns.csv")
```

5.2 Computing the efficient frontier of the "Magnificent 7" stocks

```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

data = pd.read_csv("3DailyReturns.csv", index_col=0, usecols=

data = data - 1
mean = data.mean() * 252
cov = data.cov() * 252

def random_weights(Assets):
    weights = np.random.random(len(Assets))
    weights /= np.sum(weights)
    return weights
#print(weights, weights.T)

# Expected Portfolio Annual Return
def port_returns(weights):
    return np.sum(mean * weights)

# Expected Portfolio Risk
def port_std(weights, cov):
    varP = np.dot(weights.T, np.dot(cov, weights))
    return varP**(1/2)



# Monte Carlo simulation
returnList = []
riskList = []
for i in range(10000):
    weights = random_weights(data.columns)
    returnList.append(port_returns(weights))
    riskList.append(port_std(weights, cov))
returns = np.array(returnList)
risks = np.array(riskList)

plt.figure(figsize=(10,6))
plt.scatter(risks, returns, c = returns/risks, marker='o', c
plt.xlabel('Standard Deviation')
plt.ylabel('Expected Return')
plt.colorbar(label='Sharpe Ratio')
```

5.3 Computing the tangent portfolio weights and the minimum variance portfolio weights of the “Magnificient 7” stocks

```
import pandas as pd
import numpy as np

data = pd.read_csv("3DailyReturns.csv", index_

data = data - 1
mean = data.mean() * 252
cov = data.cov() * 252

#1
mean = mean - 0.0472
mean = np.array(mean)
#2
cov = np.array(cov)
inv_cov_matrix = np.linalg.inv(cov)
weights = inv_cov_matrix @ mean

#3 Tangent Portfolio Weights
weights /= np.sum(weights)

# Minimal Variance Portfolio
vector = np.ones(7)
num = inv_cov_matrix @ vector
den = np.dot(num, vector)

minWeights = num/den
```