Part I Problems

For each of the following ODE's, draw a direction field by using about five isoclines; the picture should be square, using the intervals between -2 and 2 on both axes. Then sketch in some integral curves, using the information provided by the direction field. Finally, do whatever else is asked.

Problem 1: $y' = -\frac{y}{x}$. Solve the equation exactly and compare your integral curves with the correct ones.

Problem 2: $y' = 2x + y$. Find a solution whose graph is also an isocline, and verify this fact analytically (i.e., by calculation, and not from a picture).

Problem 3: $y' = \frac{1}{x+y}$. Use the interval -3 to 3 on both axes; draw in the integral curves that pass respectively through $(0,0)$, $(-1,1)$, $(0,-2)$. Will these curves cross the line $y = -x - 1$? Explain by using the Intersection Principle.

Part II Problems

Problem 1: [Direction fields, isoclines] In this problem you will study solutions of the differential equation

$$\frac{dy}{dx} = y^2 - x\,.$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) Draw a large pair of axes and mark off units from -4 to $+4$ on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -1$, $m = 0$, $m = 1$, and $m = 2$. On this same graph, sketch, as best you can, a couple of solutions, using just the information given by these four isoclines.

Having done this, you will continue to investigate this equation using one of the Mathlets. So invoke `http://math.mit.edu/mathlets/mathlets` in a web browser and select `Isoclines` from the menu. (To run the applet from this window, click the little black box with a white triangle inside.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with a brief description of the applet's functionalities.

Select from the pull-down menu our differential equation $y' = y^2 - x$. Move the m slider to $m = -2$ and release it; the $m = -2$ isocline is drawn. Do the same for $m = 0$, $m = 1$, and $m = 2$. Compare with your sketches. Then depress the mousekey over the graphing window and drag it around; you see a variety of solutions. How do they compare with what you drew earlier?

(b) A separatrix is a curve such that above it solutions behave (as x increases) in one way, while below it solutions behave (as x increases) in quite a different way. There is a separatrix for this equation such that solutions above it grow without bound (as x increases) while solutions below it eventually decrease (as x increases). Use the applet to find its graph, and submit a sketch of your result.

(c) Suppose $y(x)$ is a solution to this differential equation whose graph is tangent to the $m = -1$ isocline: it touches the $m = -1$ isocline at a point (a, b), and the two curves have the same slope at that point. Find this point on the applet, and then calculuate the values of a and b.

(d) Now suppose that $y(x)$ is a solution to the equation for which $y(a) < b$, where (a, b) is the point you found in **(c)**. What happens to it as $x \to \infty$? I claim that its graph is asymptotic to the graph of $f(x) = -\sqrt{x}$. Explain why this is so. For large x, is $y(x) > f(x)$, $y(x) < f(x)$, or does the answer depend on the value of $y(a)$?

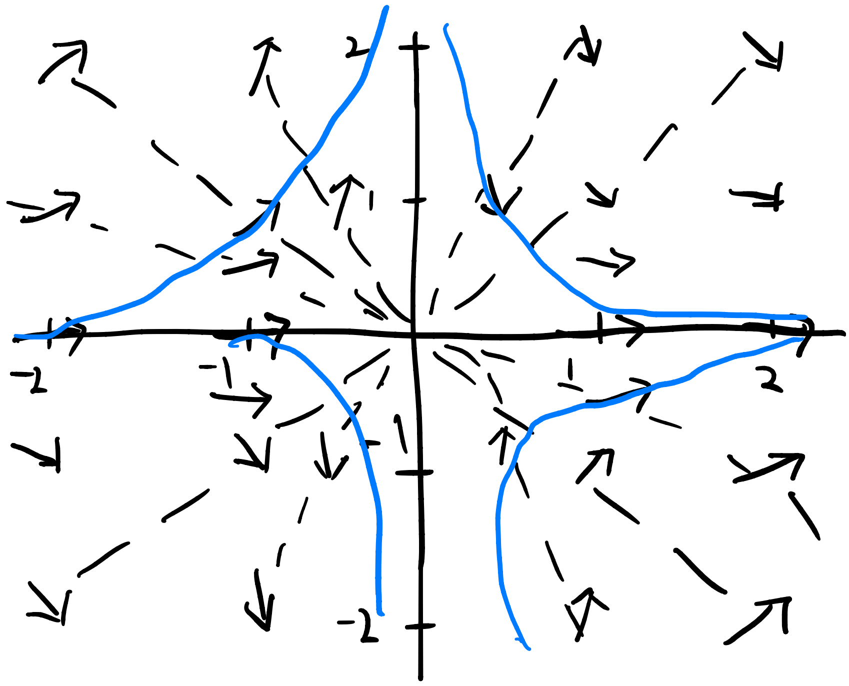
The following observations will be useful in justifying your claims. Please explain as clearly as you can why each is true.
(i) The graph of $y(x)$ can't cross the $m = -1$ isocline at a point (x, y) with $x > a$.
(ii) If $c > a$ and $y(c)$ lies above the nullcline, then the graph of $y(x)$ continues to lie above the nullcline for all $x > c$.

(iii) If $c > a$ and $y(c)$ lies below the nullcline, then the graph of $y(x)$ will cross the nullcline for some $x > c$.

(e) Suppose a solution $y(x)$ has a critical point at (c, d)—that is, $y'(c) = 0$ and $y(c) = d$. What can you say about the relationship between c and d? The applet can be very helpful here, but verify your answer.

(f) It appears from the applet that all critical points are local maxima. Is that true?

1. $\frac{dy}{dx} = -\frac{y}{x}$



$$\int \frac{1}{y}\, dy = \int -\frac{1}{x}\, dx$$

$$\ln|y| = -\ln|x| + C$$

$$|y| = e^{-\ln|x|} \cdot e^{C}$$

$$y = \frac{1}{x} \cdot C$$

$$= \frac{C}{x}$$

2. $\frac{dy}{dx} = 2x + y$ $\Rightarrow m = 2x + y \Rightarrow y = m - 2x$

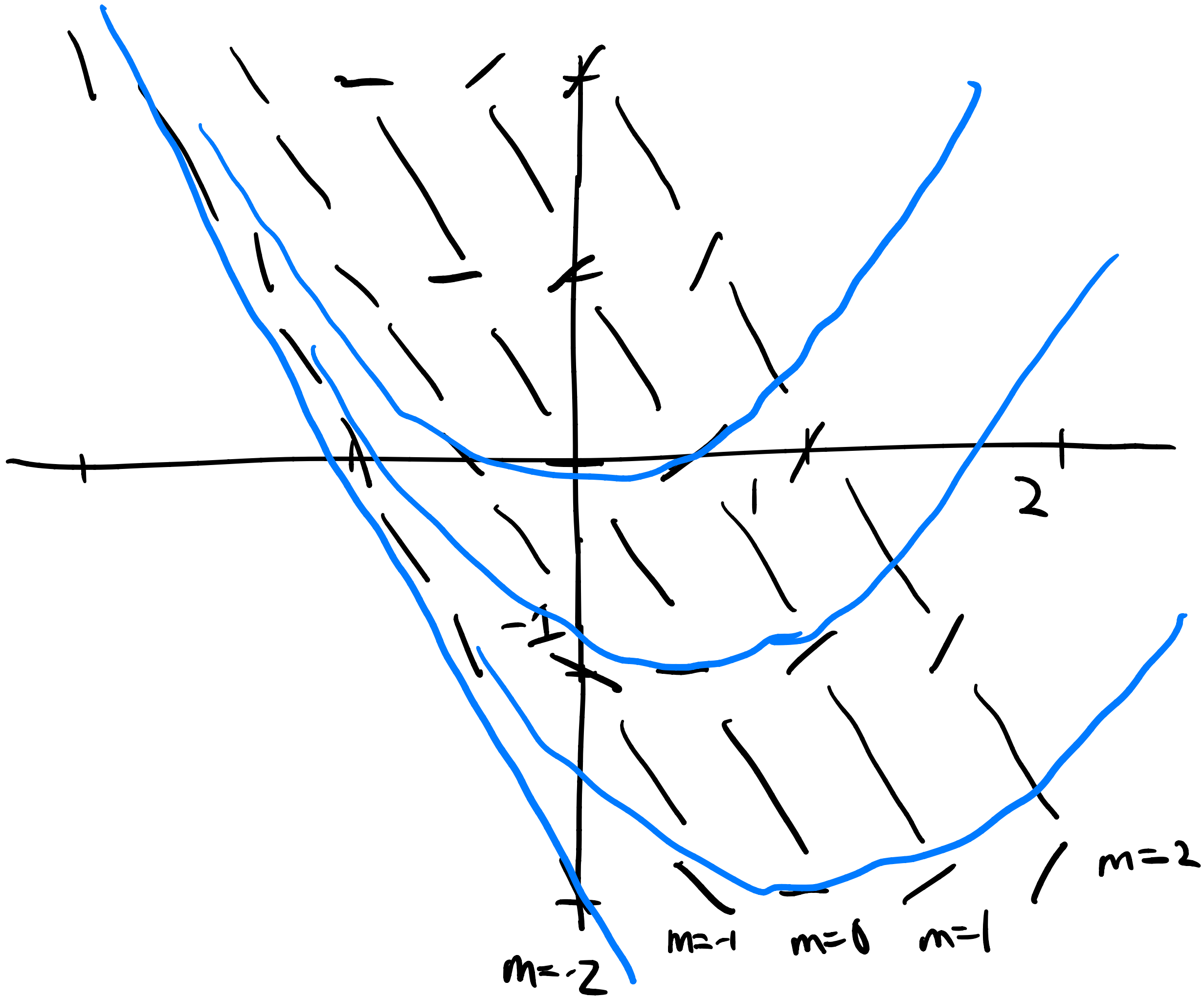
$m = 2, \quad y = 2 - 2x$

$m = 1, \quad y = 1 - 2x$

$m = 0, \quad y = -2x$

$m = -1, \quad y = -1 - 2x$

$m = -2, \quad y = -2 - 2x$



$\therefore y = -2 - 2x$

LHS: -2

RHS: $2x + (-2 - 2x)$
$= -2$

3. $\dfrac{dy}{dx} = \dfrac{1}{x+y}$ $\Rightarrow m = \dfrac{1}{x+y}$

$x+y = \dfrac{1}{m}$

$y = \dfrac{1}{m} - x$

$m=1,\ y = 1-x$

$m=2,\ y = \frac{1}{2} - x$

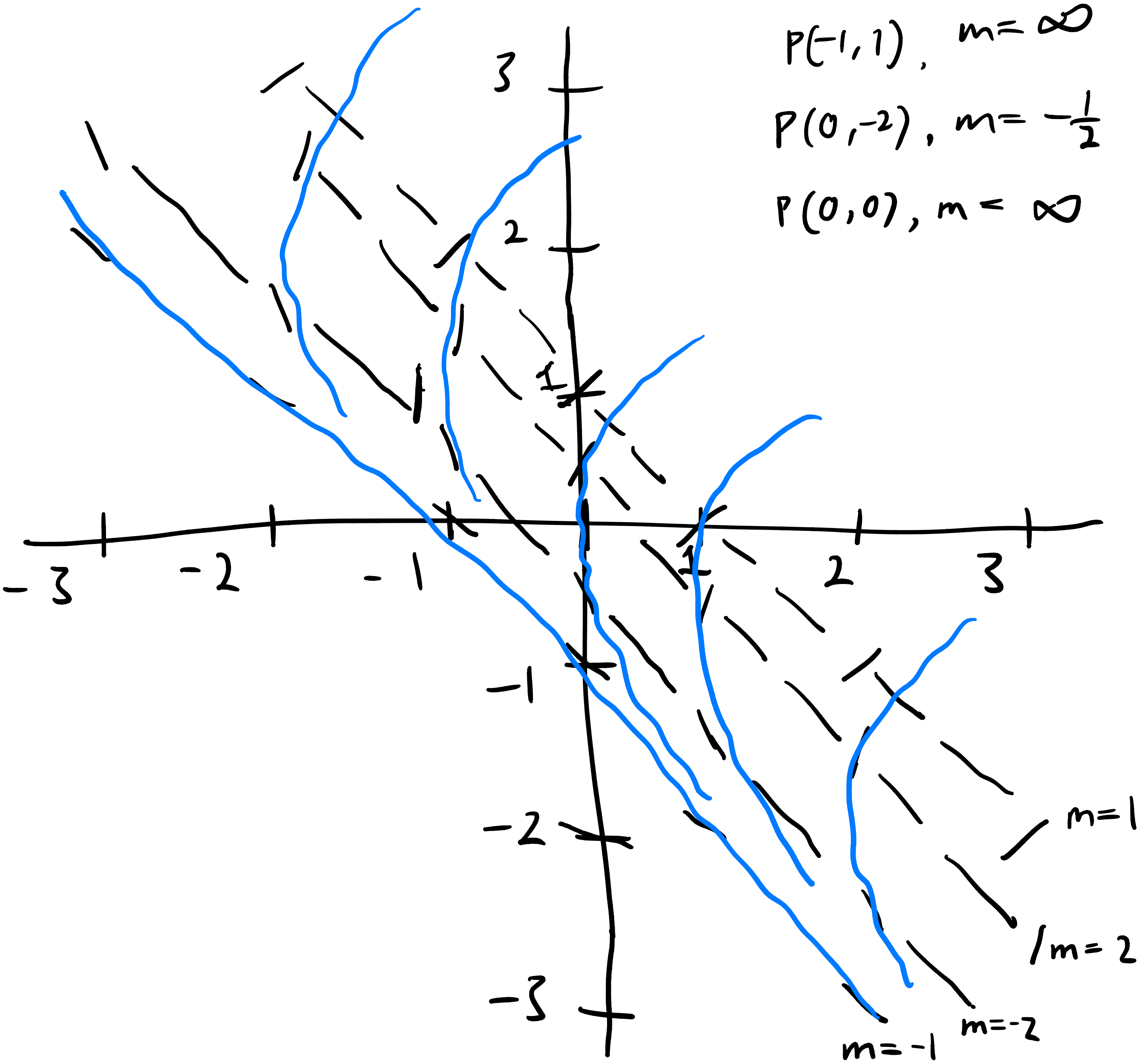
$m=-1,\ y = -1-x$

$m=-2,\ y = -\frac{1}{2} - x$

$P(-1,1),\ m = \infty$

$P(0,-2),\ m = -\frac{1}{2}$

$P(0,0),\ m = \infty$



$\therefore$ yes

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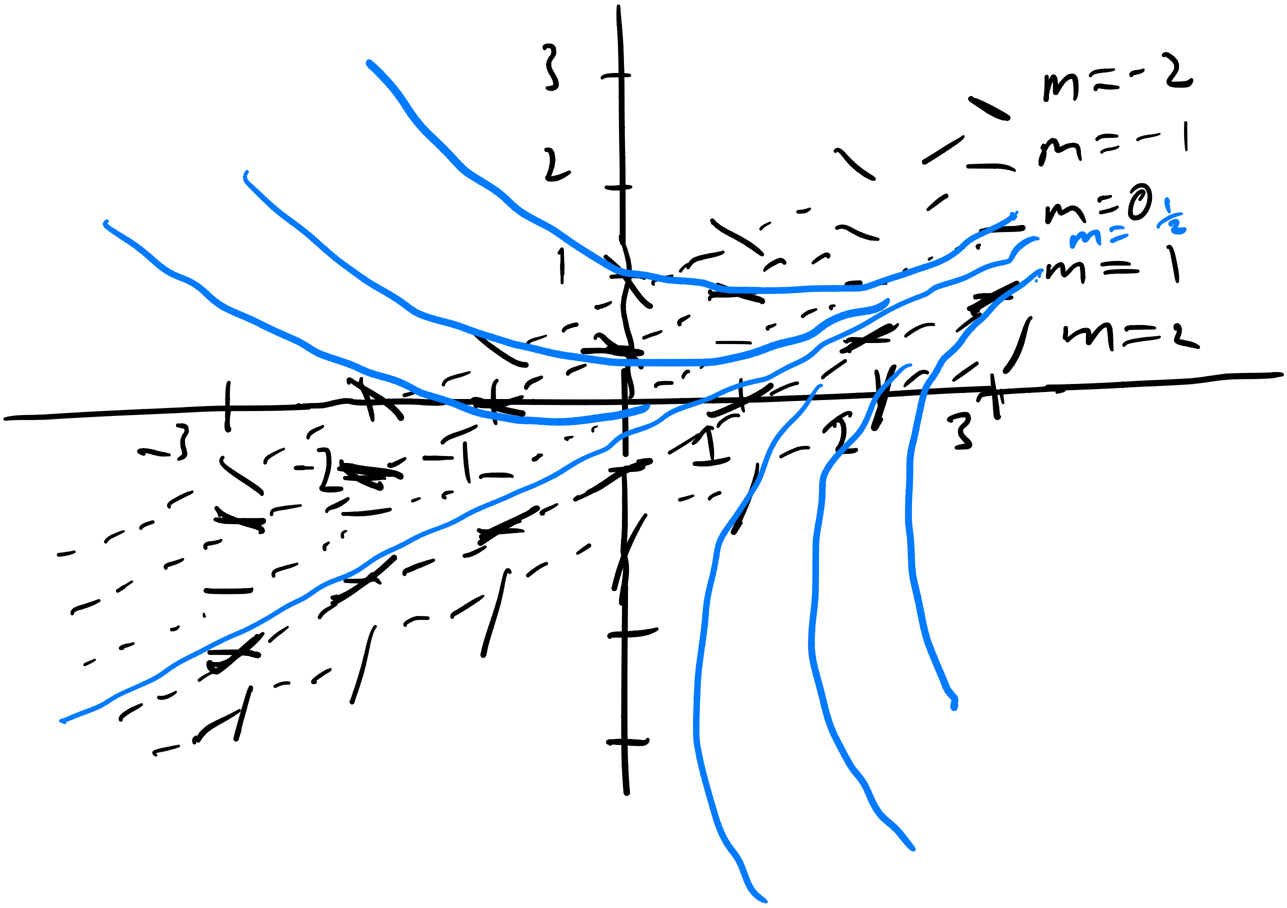
$$\frac{dy}{dx} = x - 2y$$

1. $m = x - 2y$

$2y = x - m$

$y = \frac{1}{2}x - \frac{m}{2}$



2. Yes, $m = -\frac{1}{2}$. $y = \frac{1}{2}\left(x - \frac{1}{2}\right)$

$\therefore m = \frac{1}{2},\ b = -\frac{1}{4}$

3. $\frac{dy}{dx} = m$ would be equal with the gradient of the solution.

4. Because by the Intersection Principle theorem, the direction fields cannot intersect each other.

5. 0 or 1 critical points.

6.

$$\frac{dy}{dx} = y^2 - x^2$$

$\Rightarrow m = y^2 - x^2$

$y^2 = m + x^2$

$m = 0, \quad x^2 - y^2 = 0$

$m = 1, \quad x^2 - y^2 = -1$