

When does a function equal its Taylor series?

We have computed the Taylor series for a differentiable function, and earlier in the course, we explored how to use their partial sums, i.e. Taylor polynomials, to approximate the function. So we know that the Taylor series can be quite useful. But we haven't addressed the question of when a function is equal to its Taylor series. We do this now.

Recall that when we write down an infinite series with upper bound " ∞ ," we mean the following:

$$\sum_{n=0}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n$$

In the context of Taylor series for a function f , this means that the Taylor series $T_f(x)$ is expressible as a limit of Taylor polynomials $P_N(x)$ as $N \rightarrow \infty$. That is, the notation

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \quad \text{means} \quad T_f(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{f^{(n)}(c)}{n!} (x-c)^n = \lim_{N \rightarrow \infty} P_N(x),$$

where $P_N(x)$ is the N -th degree Taylor polynomial to $f(x)$ at $x = c$.

Hence $f(x)$ will equal $T(x)$ provided

$$f(x) - \lim_{N \rightarrow \infty} P_N(x) = \lim_{N \rightarrow \infty} (f(x) - P_N(x)) = 0.$$

For each N , the function $(f(x) - P_N(x))$ is often called the "remainder" (i.e., what remains after taking the N -th Taylor polynomial from f) and denoted $R_N(x)$. Thus we must find all x for which

$$\lim_{N \rightarrow \infty} R_N(x) = 0.$$

A very useful bound on $R_N(x)$ is given by Taylor's inequality:

Taylor's Inequality. *If $|f^{(N+1)}(x)| \leq B$ for all x in the interval $[c-d, c+d]$, then the remainder $R_N(x)$ (for the Taylor polynomial to $f(x)$ at $x = c$) satisfies the inequality*

$$|R_N(x)| \leq \frac{B}{(N+1)!} |x-c|^{N+1} \quad \text{for all } x \text{ in } [c-d, c+d].$$

If the right-hand side of Taylor's inequality goes to 0 as $N \rightarrow \infty$, then the remainder must go to 0 as well, and hence for those x values, the function matches its Taylor series.

Question: Use Taylor's inequality to show that e^x converges to its Taylor series at 0 for all real x .

10/9/25

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If the right-hand side of Taylor's inequality goes to 0 as $N \rightarrow \infty$, then the remainder must go to 0 as well, and hence for those x values, the function matches its Taylor series.

Question: Use Taylor's inequality to show that e^x converges to its Taylor series at 0 for all real x .

$$\text{At } c = 0, \quad |R_N(x)| \leq \frac{B}{(N+1)!} |x|^{N+1} \quad \text{for all } x \text{ in } [-d, d]$$

$$f(x) = e^x \Rightarrow f^{(N+1)}(x) = e^x \Rightarrow |f^{(N+1)}(x)| \leq e^x.$$

$$|R_N(x)| \leq \frac{e^x}{(N+1)!} |x|^{N+1} \leq \frac{e^d}{(N+1)!} d^{N+1} \Rightarrow \lim_{N \rightarrow \infty} \frac{d^{N+1}}{(N+1)!} e^d = 0 \Rightarrow \lim_{N \rightarrow \infty} |R_N(x)| = 0$$

$\therefore e^x$ converges to its Taylor series at 0 for all real x .