1. Compounding

Def 1.1 (Present and Future Values)

Define the following:

- Discrete time $t \in \{0, 1, 2, \ldots\}$,
- One period compounding - the interest is compounded every year,
- PV - present value,
- FV - future value,
- r - interest rate (e.g. 5%).
  Then for $t = 1$:

  $$FV = PV + rPV = PV(1 + r),$$

  for $t = 2$:

  $$FV = (1 + r)(1 + r)PV = PV(1 + r)^2,$$

  for $t = n$:

  $$FV = PV(1 + r)^n.$$

Def 1.2 (Frequent compounding)

Let f be the number of times that interest rate is calculated within a unit time. For example, if we do it every third month, then $f = 4$. We have

$$FV = PV\left(1 + \frac{r}{f}\right)^{nf}.$$

If we let $f \to \infty$, so that $nf- > t$, we get the continuous compounding formula

$$FV = PVe^{rt}.$$

Def 1.3 (Discounting)

Discounting works the other way around:

$$PV = FV\left(1 + \frac{r}{f}\right)^{-nf},$$

and for the continuous case:

$$PV = FVe^{-rt}.$$

Def 1.4 (Risk-Free Instrument)

A risk-free instrument is defined by

$$B_t = B_0 e^{rt},$$

where r is a risk-free interest rate. For discrete time we have

$$B_n = B_0 \left(1 + \frac{r}{f}\right)^{nf}.$$

The goal

We want to find the fair price of some financial instrument/derivative, which is often defined with a function (called a payout function) of the asset price. For example, in the european call option

$$C_T = (S_T - K)^+ = f(S_T),$$

where T is called the maturity date, K is given and called the strike price, and $(x)^+ = \max\{x, 0\}$
.

Def 1.5 (Hedging)

A replication/hedging strategy is given by the following

$$\varphi_t = (\alpha_t, \beta_t),$$

where α_t is the amount of assets existing in the portfolio at time t, and β_t is the amount of risk-free instruments B_t in the portfolio at time t.
Note: α_t and β_t can be negative, which corresponds to borrowing.

Example

Let X be a derivative, for example

$$X = (S_1 - K)^+ = \begin{cases} (S^u - K)^+, & \omega = \omega_1, \\ (S^d - K)^+, & \omega = \omega_2. \end{cases}$$

Let $x = V_1(\varphi)$ be the value of the portfolio. Let $\beta_0 = 1$. We have

$$x = \alpha_1 S_1 + \beta_1(1 + r).$$

Let $\alpha = \alpha_1$, and $\beta = \beta_1$. Looking for replication strategy $\varphi = (\alpha, \beta)$, we obtain the following equations:

$$\begin{aligned} \alpha S^u + \beta(1 + r) = x^u = (S^u - K)^+ \\ \alpha S^d + \beta(1 + r) = x^d = (S^d - K)^+. \end{aligned}$$

Then, solving for α and β, we have

$$\alpha = \frac{x^u - x^d}{S^u - S^d}, \quad \beta = \frac{x^d S^u - x^u S^d}{(1+r)(S^u - S^d)}.$$

Hence the price equals

$$\Pi(X) = \Pi_0(x) = \alpha S_0 + \beta.$$

Def 1.6 (Arbitrage)

We say that there is an arbitrage opportunity if there exists a portfolio φ such that

$$V_0(\varphi)(\omega) = 0, \quad V_T(\varphi)(\omega) \geq 0, \quad \forall \omega \in \Omega,$$

and there exists $\omega \in \Omega$ such that $V_T(\varphi)(\omega) > 0$.
We want to consider only the markets without arbitrage opportunities.

Theorem 1.1 (Two stage market)

The two stage market

$$S_1(\omega) = \begin{cases} S^u = S_0 u, & \omega = \omega_1, \\ S^d = S_0 d, & \omega = \omega_2 \end{cases}$$

is arbitrage free iff

$$S^d < (1+r)S_0 < S^u,$$

or, in other words

$$d < 1 + r < u.$$

Theorem 1.2 (Risk-neutral pricing formula)

The price of derivative X at time $t = 0$ is

$$\Pi_0(X) = E^Q\left[\frac{X}{(1+r)^T}\right],$$

where Q is the martingale/risk-neutral measure.

Def 1.7 (Cox-Rubinstein (CRR) model)

Let $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, so that we have $u \cdot d = 1$, where Δt is a fixed unit time and σ is so-called volatility. The volatility can be estimated by

$$\hat{\sigma} = \frac{1}{N-1}\sum_{i=1}^{N}(s_i - \bar{s})^2.$$

2. General theory

Def 2.1 (Market)

The market is a continuous-time stochastic process

$$S_t = \left(S_t^0 = B_t, S_t^1, S_t^2, \ldots, S_t^d\right),$$

comprised of d assets (S_t^i), and one risk-free instrument B_t.

Def 2.2 (Investment strategy)

The investment strategy is a stochastic process

$$\varphi_t = \left(\varphi_t^0, \varphi_t^1, \ldots, \varphi_t^d\right),$$

where $\varphi_t^i \in \mathbb{R}$ (negative values are meant as borrowing of assets) are predictable (generated by left-continuous processes) processes satisfying

$$\int_0^T E\left[|\varphi_t^0|\right] dt < \infty, \quad \int_0^T E\left[|\varphi_t^i|^2\right] dt < \infty, \quad 1 \leq i \leq d,$$

where T, the maturity date, can also be infinity.

Def 2.3 (Value and gain process)

Define the value process $V_t(\varphi)$:

$$V_t(\varphi) = \sum_{i=0}^{d} \varphi_t^i S_t^i = \varphi_t \bullet S_t,$$

and the gain process $G_t(\varphi)$:

$$G_t(\varphi) = \sum_{i=0}^{d} \int_0^t \varphi_t^i dS_t^i.$$

Def 2.4 (Self-financing)

We say that the strategy φ is self-financing if

$$V_t(\varphi) = V_0(\varphi) + G_t(\varphi).$$

Def 2.5 (Measure equivalence)

We say that measures Q and P are equivalent if

$$P(A) = 0 \iff Q(A) = 0$$

for every $A \in \mathbb{F}$.
By Radon-Nikodym theorem there exists a so-called density (non-negative martingale with mean 1) process L_t

$$\left.\frac{dQ}{dP}\right|_{F_t} = L_t.$$

Def 2.6 (Martingale measure)

Define the discounted asset price as $\tilde{S}^i_t = e^{-rt} S^i_t$.
Q is a martingale (risk-neutral) measure if under Q, $\tilde{S}^i_t$ are martingales. Very often it is written as EMM (equivalent martingale measure).

Def 2.7 (Arbitrage-free market)

The market is arbitrage-free if there are no arbitrage opportunities.

Theorem 2.1 (First fundamental theorem)

Let $\mathcal{P}$ be a family of all risk-neutral measures. If $\mathcal{P} \neq \emptyset$ then the market is arbitrage-free.

Def 2.8 (NFLVR - no free lunch with vanishing risk)

The strategy $\varphi_t = \varphi \cdot \mathbb{1}_{[t_1,t_2]}(t)$ is called a simple strategy where t_1, t_2 are stopping times. We say that simple strategy is δ-admissible if

$$P(\tilde{V}_t(\varphi) \geq -\delta, \forall_{t \in [0,T]}) = 1.$$

NFLVR holds if

$$V_T(\varphi) \xrightarrow{P} 0, \text{ as } \delta \to 0.$$

Theorem 2.2 (NFLVR)

If NFLVR holds, then $\mathcal{P} \neq \emptyset$.

Def 2.9 (Derivative)

X if a derivative (claim), if

$$X = F(S_T).$$

Def 2.10 (Hedging strategy)

φ is a replicating/hedging strategy if

$$V_T(\varphi)(\omega) = X(\omega), \quad \forall_{\omega \in \Omega}.$$

Theorem 2.3 (Risk-neutral (martingale) pricing formula)

The fair price of X at time t equals

$$\Pi_t(X) = S_t^0 E^Q \left[\frac{X}{S_T} \middle| \mathcal{F}_t \right],$$

in particular, for $t = 0$:

$$\Pi_0(X) = e^{-rT} E^Q[X]. \text{ - W chuj ważne}$$

Def 2.11 (Complete market)

We say that a market is complete if there exists a strategy φ such that $V_T(\varphi) = X$ for any derivative X.

Theorem 2.4 (Second fundamental theorem)

If there exists a unique martingale measure, then the market is complete.

3. Black-Scholes model

Def 3.1 (Black-Scholes model)

Let $S_t^0 = B_t = e^{rt}$, and let S_t^i be such that they solve the following SDEs

$$dS_t^i = S_t^i \left(b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t) dB_t^i \right),$$

where B_t^i are independent Brownian motions.
From now on we will assume that $d = 1$, and that b and σ are constant, so that

$$dS_t = bS_t dt + \sigma S_t dB_t.$$

Note that

$$d\tilde{S}_t = (b - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t.$$

Theorem 3.1 (Girsanov theorem)

If $L_t = e^{\gamma B_t - \frac{1}{2}\gamma^2 t} \in \mathcal{M}$ and

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = L_t,$$

then $\tilde{B}_t = B_t + \gamma t$ is a brownian motion under measure Q. Hence, $B_t = \tilde{B}_t - \gamma t$.
Now, if we take $\gamma = \frac{b-r}{\sigma}$, then

$$d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{B}_t,$$

hence there exists a unique martingale measure and there is no arbitrage and the market is complete.

Under Q we have

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}_t,$$

which is solved by

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma}{2}\right)^2 t + \sigma \tilde{B}_t\right).$$

Theorem 3.2 (Black-Scholes equation)

Let $X = H(S_T)$, and $\Pi_t(X) = F(t, S_t)$ for $t \in [0, T]$. Then $F(t, s)$ solves the following differential equation

$$F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0,$$

with the terminal condition

$$F(T, s) = H(s).$$

Theorem 3.3 (Hedging in a B-S market)

The hedging strategy in a B-S market is given. by $\varphi_t = (\varphi_t^0, \varphi_t^1)$, where

$$\begin{aligned}
&\varphi_t^0 = e^{-rt}\left(F(t, S_t) - S_t F_s(t, S_t)\right), \\
&\varphi_t^1 = F_s(t, S_t).
\end{aligned}$$

Def 3.2 (European option)

The european call option is characterised by the payout function

$$H(x) = (x - K)^+ = \max(\{x - K, 0\}),$$

the european put option on the other hand has the following

$$H(x) = (K - x)^+.$$

K is called the strike price.

Theorem 3.4 (B-S formula for european call option)

The price of the european call option at time t in B-S market is

$$C(t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where $\Phi(x)$ is the CDF of the standard gaussian distribution, and

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(\gamma + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

4. Greeks (Sensitivity analysis)

Def 4.1 (Delta)

Define

$$\Delta = \frac{\partial V}{\partial S}.$$

It describes how big the change in the price of some derivative is in relation to small changes if initial price of the underlying.