1. Metric spaces and Banach's FPT (Fixed Point Theorem)

Def 1.1 (Metric and Metric Space)

Let X be a nonempty set, and $d : X^2 \to \mathbb{R}$ be a function satisfying:

- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$
  Then the function d is called the metric, the pair (X, d) is called the metric space, and the number $d(x, y)$ is called the distance between x and y in X.

Examples

- $(\mathbb{R}^n, d)$, with $d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$
- $(\mathbb{R}^n, d)$, with $d(x, y) = \max\limits_{i \leq n} |x_i - y_i|$
- $(C[a, b], d)$, with $d(f, g) = \left(\int_a^b |f(x) - g(x)|^2 dx\right)^{\frac{1}{2}}$
- $(C[a, b], d)$, with $d(f, g) = \sup\limits_{a \leq x \leq b} |f(x) - g(x)|$
- $(L^p[a, b], d)$, with $d(f, g) = \left(\int_a^b |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}$

Def 1.2 (Fixed Point)

A fixed point of the mapping $T : X \to X$ is the point $x^* \in X$ such that $T(x^*) = x^*$.

Def 1.3 (Contraction)

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X of there exists a constant $0 < k < 1$ such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for all $x, y \in X$.

Theorem 1.1 (Banach's FPT)

Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x^* \in X$.

Corollary 1.1 (Banach's FPT)

The iterative sequence $x_{n+1} = T(x_n)$ for $n = 1, 2, \dots$ with arbitrary starting point $x_0 \in X$ converges, under assumptions of Banach's FPT, to the unique fixed point of T. Moreover, the following estimates hold:

- $d(x_m, x^*) \leq \frac{k^m}{1-k} d(x_1, x_0)$ - the prior estimate,
- $d(x_m, x^*) \leq \frac{k}{1-k} d(x_{m-1}, x_m)$ - the posterior estimate.

2. Applications of Banach's FPT

2.1 Applications to real-valued functions

Let $g \in C^1[a, b]$, and suppose we want to find the solution to the equation $g(x) = 0$ on $[a, b]$. We note that we can always rewrite this equation as $x = g(x) + x$, and then out problem is equivalent with finding a fixed point of the function $f(x) = x + g(x)$.

Theorem 2.1 (Differentiable Contraction)

Let $(\mathbb{R}, d)$ be a metric space of real numbers with the metric $d(x, y) = |x - y|$ and let $[a, b]$ be a closed interval in $\mathbb{R}$. Moreover, let $f : [a, b] \to [a, b]$ be a continuous and differentiable function such that $\sup\limits_{x \in [a,b]} |f'(x)| \leq k < 1$. Then there exists a unique fixed point $x^* \in [a, b]$ of f.

Example 2.1

We want to find the solution to the equation $cos(x) - 2x = 0$ on $[0, \pi]$. Then we can write this equation as $x = \frac{1}{2}cos(x)$, and try to find the fixed point of the function $f(x) = \frac{1}{2}cos(x)$ on $[0, \pi]$. We have to show that f is a contraction on $[0, \pi]$. To do so, we apply the theorem 2.1. We have

$$\sup_{x \in [0,\pi]} |f'(x)| = \sup_{x \in [0,\pi]} \left| -\frac{1}{2} sin(x) \right| = \frac{1}{2} < 1.$$

We have shown that f is a contraction and, by the Banach's FPT, it has a fixed point x^* that is the limit of the sequence $\{x_n\}$ generated by the scheme $x_{n+1} = f(x_n)$ with any starting point $x_0 \in [0, \pi]$.
Note that to show that f is a contraction we could also directly apply the definition:

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{2}cos(x) - \frac{1}{2}sin(x) \right| = \left| sin\left(\frac{x+y}{2}\right) sin\left(\frac{x-y}{2}\right) \right| \\ &\leq \sup_{x,y \in [0,\pi]} \left| sin\left(\frac{x+y}{2}\right) \frac{1}{2}|x-y| \right| = \frac{1}{2}|x-y| \leq |x-y|. \end{aligned}$$

2.2 Applications to integral equations

We consider integral equations in the following form

$$f(x) = g(x) + \mu \int_a^b k(x,y) f(y) dy,$$

where $f : [a,b] \to \mathbb{R}$ is an unknown function, $g : [a,b] \to \mathbb{R}$, and $k : [a,b]^2 \to \mathbb{R}$ are given functions, and μ is a parameter.
The above integral equation can be considered in various function spaces. Here we consider this equation only in $(C[a,b], d)$ with $d(f,g) = \sup\limits_{x \in [a,b]} |f(x) - g(x)|$.
We assume that $g \in C[a,b]$, and that the kernel k is continuous on the square $[a,b]^2$, which implies that k is bounded on $[a,b]^2$, meaning that there exists a constant c, such that $|k(x,y)| \leq c$ for all $(x,y) \in [a,b]^2$.

Theorem 2.2

The metric space $(C[a,b], d)$ is complete

Note that our integral equation can be rewritten as $T(f) = f$, where

$$T(f)(x) = g(x) + \mu \int_a^b k(x,y) f(y) dy.$$

First we have to show that the mapping $T : C[a,b] \to C[a,b]$ is well-defined, but this is obvious, as g and k are both continuous on their domains. Let us now determine for which values of μ the map T is a contraction. We have

$$\begin{aligned} d(T(f_1), T(f_2)) = \sup_{x \in [a,b]} |T(f_1)(x) - T(f_2)(x)| = \sup_{x \in [a,b]} |\mu| \left| \int_a^b k(x,y)(f_1(y) - f_2(y)) dy \right| \leq \\ \leq |\mu| \sup_{x \in [a,b]} \int_a^b |k(x,y)||f_1(y) - f_2(y)| dy \leq c|\mu| \sup_{x \in [a,b]} |f_1(x) - f_2(x)| \int_a^b dy = \\ = c|\mu|(b-a) d(f_1, f_2). \end{aligned}$$

It is now required that $c|\mu|(b-a) < 1$, or $|\mu| < \frac{1}{c(b-a)}$, for T to be a contraction. Applying the Banach's FPT, we see that the map T has a unique fixed point $f^* \in C[a,b]$.

Theorem 2.3 (Integral equation)

Consider the integral equation

$$f(x) = g(x) + \mu \int_a^b k(x,y) f(y) dy.$$

Suppose that k and g are continuous on $[a,b]^2$ and $[a,b]$ respectively, and assume that the parameter μ satisfies $|\mu| < \frac{1}{c(b-a)}$ with the constant c such that $|k(x,y)| < c$ for all $(x,y) \in [a,b]^2$.

Then the integral equation has a unique solution $f \in C[a,b]$. Moreover, this solution is a limit of the sequence $\{f_n\}$ where f_0 is a continuous function on $[a,b]$, and

$$f_{n+1} = g(x) + \mu \int_a^b k(x,y) f_n(y) dy.$$

2.3 Applications to differential equations

Let's consider the initial value problem

$$\begin{aligned} x'(t) &= f(t, x(t)) \\ x(t_0) &= x_0 \end{aligned}$$

where $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is a given function and $x(t)$ is an unknown function that we want to find.

Theorem 2.4 (Picard-Lindelöf)

Let f be continuous on the rectangle

$$R = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : |t - t_0| \leq a, |x - x_0| \leq b\}$$

and thus bounded on R, say $|f(t,x)| \leq c$ for all $(t,x) \in R$. Suppose that f satisfies the Lipschitz condition on R with respect to the second argument, i.e., there exists a constant k such that

$$|f(t,x) - f(t,y)| \leq k|x - y|$$

for all $(t,x), (t,y) \in R$.
Then the initial value problem has a unique solution, which exists on the interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta = \min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}.$$

Corollary 2.1 (Picard-Lindelöf)

Under the assumptions of Picard-Lindelöf theorem, the sequence given by

$$\begin{aligned} x_0(t) &= x_0 \\ x_{n+1}(t) &= T(x_n)(t) = x_0 + \int_{t_0}^t f(s, x_n(s)) ds \end{aligned}$$

converges uniformly to the unique solution $x(t)$ on $J = [t_0 - \beta, t_0 + \beta]$.

Example 2.2

Consider the differential equation

$$x'(t) = \sqrt{x(t)} + x^3(t), \quad x(1) = 2.$$

We have

$$x_1(t) = 2 + \int_1^t \left(\sqrt{2} + 2^2\right) ds = 2 + \left(\sqrt{2} + 8\right)(t-1)$$
$$x_2(t) = 2 + \int_1^t \left(\sqrt{x_1(s)} + x_1(s)^3\right) ds = *\text{hot mess}*$$

2.4 Applications to matrix equations

Suppose we want to find a solution of the matrix equation

$$Ax = B$$

where $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^n$.
We note that this equation can be rewritten as

$$x = (I - A)x + b$$

where I is the identity matrix.
Let's define the map $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = (I - A)x + b.$$

Then the problem of solving the matrix equation $Ax = b$ is equivalent with finding a fixed point of T.
Let's define $\alpha_{ij} = \delta_{ij} - a_{ij}$ where a_{ij} are elements of the matrix A, and δ_{ij} is the Kronecker delta. Using this notation we have

$$(Tx)_i = \sum_{j=1}^n \alpha_{ij} x_j + b_i$$

We will show that $Ax = b$ has a unique solution if

$$\sum_{j=1}^n |\alpha_{ij}| \le \alpha < 1.$$

for all $i = 1, 2, \dots, n$. Consider the matrix space $(\mathbb{R}^n, d)$, with $d(x, y) = \max\limits_{1\le i\le n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$
. We have

$$
\begin{aligned}
d(Tx, Ty) = \max_i |(Tx)_i - (Ty)_i| = \\
&= \max_i \left| \sum_{j=1}^{n} \alpha_{ij}(x_j - y_j) \right| \leq \\
&\leq \max_i \sum_{j=1}^{n} |\alpha_{ij}||x_j - y_j| \leq \\
&\leq \max_i \sum_{j=1}^{n} |\alpha_{ij}| \cdot \max_j |x_j - y_j| = \\
&= \max_i \sum_{j=1}^{n} |\alpha_{ij}| \cdot d(x, y).
\end{aligned}
$$

We notice that if $\sum_{j=1}^{n} |\alpha_{ij}| < 1$, for all $i = 1, 2, \dots, n$, then $\max_i \sum_{j=1}^{n} |\alpha_{ij}| < 1$. We have

$$
\sum_{j=1}^{n} |\alpha_{ij}| = |a_{i1}| + |a_{i2}| + \dots + |1 - a_{ii}| + \dots + |a_{in}| < 1,
$$

so

$$
\sum_{j=1, j \neq i}^{n} |a_{ij}| < 1 - |1 - a_{ii}| < |a_{ii}|.
$$

We get the condition for the matrix A for which T is a contraction]. This condition is given by

$$
|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|,
$$

or, in other words, matrix A should be strictly diagonally dominant.

Theorem 2.5 (Matrix equation)

The matrix equation $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ an $b \in \mathbb{R}^n$ has a unique solution $x \in \mathbb{R}^n$ if A is strictly diagonally dominant. The iteration method is as follows

$$
x_{n+1} = (I - A)x_n + b, \quad x_0 \in \mathbb{R}^n.
$$

In general, we can rewrite the equation $Ax = b$ as $Qx = (Q - A)x + b$, where $Q \in \mathbb{R}^{n \times n}$. We then have the following iterative scheme

$$
Qx_{n+1} = (Q - A)x_n + b.
$$

Examples:

- $Q = I$ - Richardson method,
- Q diagonal, with $q_{ii} = a_{ii}$ - Jacobi method,
- $Q = D - L$ with D diagonal, and L lower triangular - Gauss-Seidel method.

3. Normed spaces

Def 3.1 (Norm and normed space)

A norm on a vector space X is a real-valued function denoted by $||\cdot||$ which satisfies the following conditions:

- $||x|| \geq 0$ for all x. $||x|| = 0$ iff $x = 0$,
- $||\alpha x|| = |\alpha|||x||$ for any α, and $x \in X$,
- $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$.
  A normed space is a vector space equipped with a norm, depicted by $(X, ||\cdot||)$, or with a shorthand X.

Remark 3.1.1

A norm on X defines the metric $d(\cdot, \cdot)$ on $X \times X$, which is defined by $d(x, y) = ||x - y||$, and is called the metric induced by the norm $||\cdot||$.

Remark 3.1.2

Every normed space X is a metric space, converse might not be true.
For example, a metric defined by

$$d(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}$$

then $||\alpha(x - y)|| = d(\alpha x, \alpha y) \neq |\alpha| d(x, y) = |\alpha|||x - y||$.

Lemma 3.1 (Norm continuity)

The norm $||\cdot||$ defined on X is a continuous mapping of X into $\mathbb{R}$.

Examples of normed spaces

- $(\mathbb{R}^n, ||\cdot||_2)$, with $||x||_2 = \left(\sum_{m=1}^{n} x_i^2\right)^{\frac{1}{2}}$,
- $(C[a, b], ||\cdot||)$, with $||f|| = \max\limits_{x \in [a,b]} |f(x)|$,
- $(L^p(\Omega), ||\cdot||_{L^p(\Omega)})$, with $\Omega \subset \mathbb{R}$, $p \geq 1$ and

$$||f||_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega f(x)^p dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ ess \sup\limits_{x \in \Omega} |f(x)|, & p = \infty. \end{cases}$$

Def 3.2 (Norm equivalence)

Two normed spaces $(X, ||\cdot||_1)$, $(X, ||\cdot||_2)$ are called topologically equivalent, or two norms $||\cdot||_1$, and $||\cdot||_2$ are called equivalent if there exist positive constants C_1, and C_2, such that

$$C_1||x||_2 \leq ||x||_1 \leq C_2||x||_2$$

for all $x \in X$.

Theorem 3.1 (Equivalence of norms in finite dimensional spaces)

All norms of finite dimensional space X are equivalent.

Def 3.3 (Convergence in normal spaces)

A sequence $\{x_n\}$ in a normed space $(X, ||\cdot||)$ is convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} ||x_n - x|| = 0$.

Def 3.4 (Cauchy sequence)

A sequence $\{x_n\}$ in a normed space $(X, ||\cdot||)$ is a Cauchy sequence if

$$\lim_{m,n\to\infty} ||x_n - x_m|| = 0.$$

Def 3.5 (Complete space)

We say that a normal space $(X, ||\cdot||)$ is complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some $x \in X$.

Def 3.6 (Banach space)

A complete normed space is called a Banach space.

Theorem 3.2 (Euclidean space is complete)

The space $(\mathbb{R}^N, ||\cdot||_2)$ with the standard Euclidean norm is complete.

Theorem 3.3 ()

Let Ω be a compact subset of $\mathbb{R}^n$. Then the set $C(\Omega)$ of all continuous functions on Ω equipped with the norm $||f|| = \max\limits_{x\in\Omega} |f(x)|$ is a Banach space.

4. Hilbert spaces

Def 4.1 (Inner product and inner product space)

Let X be a vector space over the field $\mathbb{F}$ over the real or complex numbers. A mapping $\langle\cdot,\cdot\rangle : X^2 \to \mathbb{F}$ is called an inner product if for all $x, y \in X$ the following conditions are satisfied

1. $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0 \iff x = 0$,
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
3. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for $\alpha \in \mathbb{F}$,
4. $\langle x + x', y\rangle = \langle x, y\rangle + \langle x', y\rangle$.
   The vector space X together with an inner product $\langle\cdot,\cdot\rangle$ is called an inner product space or pre-Hilbert space and is denoted $(X, \langle\cdot,\cdot\rangle)$.

Remark 4.1

- $\overline{\langle x, y\rangle}$ denoted the complex conjugate of $\langle x, y\rangle$,
- The condition 2 implies that $\langle x, x\rangle$ must be a real number,
- If $\mathbb{F} = \mathbb{R}$ then $\langle x, y\rangle = \langle y, x\rangle$,
- Conditions 3 and 4 imply that the function $\langle\cdot,\cdot\rangle$ is linear in the first variable. It is easy to see that $\langle\cdot,\cdot\rangle$ is also linear in the second variable if $\mathbb{F} = \mathbb{R}$,

Examples

- $\mathbb{R}^N$, with $\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$,
- $C(\Omega)$, with $\langle f, g\rangle = \int_\Omega f(x)\overline{g(x)}dx$,
- $L_2(\Omega)$, with $\langle f, g\rangle = \int_\Omega f(x)g(x)dx$.

Theorem 4.1 (Cauchy-Schwarz-Bunyakowski inequality)

For all $x, y \in (X, \langle\cdot,\cdot\rangle)$ we have

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle.$$

Theorem 4.2 (Inner product space is normed)

Every inner product space $(X, \langle\cdot,\cdot\rangle)$ is a normed space with respect to the norm $||x|| = \sqrt{|\langle x, x\rangle|}$
.

Def 4.2 (Hilbert space)

An inner product space $(X, \langle\cdot,\cdot\rangle)$ is called a Hilbert space if the normed space $(X, ||\cdot||)$ with the norm induced by the inner product is a Banach space.