

# Applied Functional Analysis - Exercise sheet 2

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## Exercise 1

We have an equation

$$\pi - x + \frac{1}{2} \sin\left(\frac{x}{2}\right) = 0$$

for  $x \in [0, 2\pi]$ . We can rewrite it as

$$x = \pi + \frac{1}{2} \sin\left(\frac{x}{2}\right) = f(x),$$

so our fixed point problem takes the form

$$x_{n+1} = \pi + \frac{1}{2} \sin\left(\frac{x_n}{2}\right).$$

Note that  $f$  has values in  $[\pi, \pi + \frac{1}{2}] \subset [0, 2\pi]$  on  $[0, 2\pi]$ , so we can write  $f : [0, 2\pi] \rightarrow [0, 2\pi]$ , just not a surjection. Now

$$|f'(x)| = \left| \frac{1}{4} \cos(x) \right| \leq \frac{1}{4} = k \leq 1,$$

hence, by the theorem 2.1 from the lecture, the fixed point  $x^*$  of  $f$  exists, is unique, and  $x^* \in [0, 2\pi]$ .

Let's find the maximum of  $g(x) = |f(x) - x| = \left| \pi + \frac{1}{2} \sin\left(\frac{x}{2}\right) - x \right|$  on  $[0, 2\pi]$ . We know that  $h(x) = \pi + \frac{1}{2} \sin\left(\frac{x}{2}\right) - x$  is decreasing, so  $g$  is decreasing on  $[0, x^*]$ . On  $[x^*, 2\pi]$  the sign gets flipped and  $g$  is increasing. So the maximum lies at 0 or  $2\pi$ . We have

- $g(0) = |\pi + 0 - 0| = \pi,$
  - $g(2\pi) = |\pi + 0 - 2\pi| = \pi,$
- so  $\max_{0 \leq x \leq 2\pi} |x_1 - x_0| = \max_{0 \leq x \leq 2\pi} g(x) = \pi.$

We want

$$\frac{k^m}{1 - k} |x_1 - x_0| < 0.01,$$

plugging in the values we have

$$\begin{aligned}\frac{\left(\frac{1}{4}\right)^m}{\frac{3}{4}}\pi &< 0.01 \\ 4^m &> \frac{400}{3}\pi \\ m &> \log_4\left(\frac{400}{3}\pi\right) \approx 4.355,\end{aligned}$$

so you need at least 5 iterations to guarantee  $|x_m - x^*| < 0.01$ .

Now, let  $x_0 = 0$ . We have

$$\begin{aligned}x_1 &= f(x_0) = \pi + \frac{1}{2}\sin\left(\frac{x_0}{2}\right) = \pi && \approx 3.14159, \\ x_2 &= \pi + \frac{1}{2} && \approx 3.64159, \\ x_3 &= \pi + \frac{1}{2}\cos\left(\frac{1}{4}\right) && \approx 3.62605, \\ x_4 &= \pi + \frac{1}{2}\cos\left(\frac{1}{4}\cos\left(\frac{1}{4}\right)\right) && \approx 3.62699, \\ x_5 &= \pi + \frac{1}{2}\cos\left(\frac{1}{4}\cos\left(\frac{1}{4}\cos\left(\frac{1}{4}\right)\right)\right) && \approx 3.62693,\end{aligned}$$

where the actual limit is  $x^* \approx 3.62694$ , so we see that actually in less than 5 iterations we were able to get an accuracy of 0.01.

## Exercise 2

We have an equation

$$x^2 - p = 0.$$

The Newton's method gives us the following iteration scheme

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)},$$

with  $g(x) = x^2 - p$  which is a particular case of the fixed point iteration method. To see this, let  $f(x) = x - \frac{g(x)}{g'(x)}$ , we have  $x_{n+1} = f(x_n)$ .

Let's write  $f(x)$  explicitly, we have

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2 - p}{2x} = \frac{1}{2}\left(x + \frac{p}{x}\right).$$

Now let's prove that starting at any point  $x_0 > 0$ , the sequence generated by  $x_{n+1} = f(x_n)$  will converge to  $\sqrt{p}$ .

Assume that  $x_0 > 0$ . We have

$$x_{n+1}^2 - p = \frac{1}{4}\left(x_n + \frac{p}{x_n}\right)^2 - p = \frac{1}{4}x_n^2 + \frac{1}{2}p + \frac{1}{4}\frac{p^2}{x_n^2} - p = \frac{1}{4}\left(x_n - \frac{p}{x_n}\right)^2 > 0,$$

hence  $x_n > \sqrt{p}$  for all  $n > 0$ . Also

$$x_n > \sqrt{p} > \frac{p}{x_n}.$$

Now we can show that  $x_n$  is decreasing for all  $n > 0$ . We have

$$x_{n+1} - x_n = \frac{1}{2} \left( x_n + \frac{p}{x_n} \right) - x_n = \frac{1}{2} \left( \frac{p}{x_n} - x_n \right) < 0,$$

which proves that  $x_n$  is, in fact, decreasing for  $n > 0$ . From this, and from the fact that  $x_n > \sqrt{p}$ , we know that  $x_n$  has a limit. Let's compute it. We have

$$\begin{aligned} x^* &= \frac{1}{2} \left( x^* + \frac{p}{x^*} \right) \\ \frac{1}{2} x^* &= \frac{1}{2} \frac{p}{x^*} \\ x^* &= \frac{p}{x^*} \\ (x^*)^2 &= p, \end{aligned}$$

and, since  $x_n > 0$ , we have  $x^* = \sqrt{p}$ . If, on the other hand, we assumed  $x_0 < 0$ , it would be obvious from the definition of  $x_n$ , that  $x_n < 0$  for every  $n$ , which would prevent the sequence from converging to  $\sqrt{p}$ .

So the interval that guarantees convergence to  $\sqrt{p}$  is  $(0, \infty)$ .

Let us use the contraction condition to determine the interval that guarantees convergence to  $\sqrt{p}$ . We want

$$|f'(x)| = \left| \frac{1}{2} \left( 1 - \frac{p}{x^2} \right) \right| < 1.$$

We have three cases there, let's go over them one by one:

- case  $x > \sqrt{p}$ :

$$\begin{aligned} \frac{1}{2} \left( 1 - \frac{p}{x^2} \right) &< 1 \\ 1 - \frac{p}{x^2} &< 2 \\ -\frac{p}{x^2} &< 1 \\ -p &< x^2, \end{aligned}$$

and that is true always, because  $p$  is positive, so from this case we get  $x \in (\sqrt{p}, \infty)$ ,

- case  $x = \sqrt{p}$ :  
we get  $0 < 1$ , also, we're already at the limit, so  $x \in \{\sqrt{p}\}$ ,
- case  $x < \sqrt{p}$ :

$$\begin{aligned}\frac{1}{2} \left( \frac{p}{x^2} - 1 \right) &< 1 \\ \frac{p}{x^2} - 1 &< 2 \\ \frac{p}{x^2} &< 3 \\ x &> \sqrt{\frac{p}{3}}\end{aligned}$$

so  $x \in \left( \sqrt{\frac{p}{3}}, \sqrt{p} \right)$ .

Combining these three cases we get the condition  $x \in \left( \sqrt{\frac{p}{3}}, \infty \right)$ . So the method using the contraction of  $f$  gave us a worse result than the direct analysis of the sequence.

## Exercise 3

We consider the following integral equation

$$f(x) = x + \frac{1}{4} \int_0^{\frac{\pi}{2}} f(y) \cos(x) dy.$$

Relating this to the equation from the lecture we have  $g(x) = x$ ,  $k(x, y) = \cos(x)$ , and  $\mu = \frac{1}{4}$ . Furthermore,  $|k(x, y)| \leq 1 = c$ , so

$$\frac{1}{4} = \mu < \frac{1}{c(b-a)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} < \frac{2}{3}.$$

This proves that the functional  $T$  defined by

$$T(f)(x) = x + \frac{1}{4} \int_0^{\frac{\pi}{2}} f(y) \cos(x) dy$$

is a contraction, and hence, by the Banach's FPT, has a unique fixed point in  $C \left[ 0, \frac{\pi}{2} \right]$ . Let's rewrite the operator  $T$  as

$$T(f)(x) = x + \frac{1}{4} \cos(x) \int_0^{\frac{\pi}{2}} f(y) dy.$$

We can make this into a function sequence in the following way

$$f_{n+1} = T(f_n).$$

Let's take  $f_0(x) = x$ . We can write a few terms of the sequence by applying  $T$ :

$$\begin{aligned}
f_0(x) &= x \\
f_1(x) &= x + \frac{1}{4}\cos(x) \int_0^{\frac{\pi}{2}} x dx = x + \frac{1}{4}\cos(x) \cdot \frac{\pi^2}{8} \\
f_2(x) &= x + \frac{1}{4}\cos(x) \cdot \frac{5\pi^2}{32} \\
f_3(x) &= x + \frac{1}{4}\cos(x) \cdot \frac{21\pi^2}{128}.
\end{aligned}$$

We can see that the functions  $f_n$  can be characterised by just the one number, that being the integral of the previous one in the sequence,  $f_{n-1}$ . We can write

$$\begin{aligned}
f_C(x) &= x + \frac{C}{4}\cos(x), \\
C_{n+1} &= \int_0^{\frac{\pi}{2}} f_{C_n}(x) dx = \int_0^{\frac{\pi}{2}} \left( x + \frac{C_n}{4}\cos(x) \right) dx = \frac{\pi^2}{8} + \frac{C_n}{4},
\end{aligned}$$

with  $C_0 = 0$ . Omitting the formal proof of the convergence of  $C_n$ , we can find it by looking for solutions to

$$C = \frac{\pi^2}{8} + \frac{C}{4},$$

which yields  $C = \frac{\pi^2}{6}$ , which, if we plug that in, in fact gives us  $T(f_C) = f_C$ .

## Exercise 4

We are presented with an initial value problem:

$$\begin{aligned}
x'(t) &= 2x(t) + 2 \\
x(0) &= 0
\end{aligned}$$

Solving this problem is trivial, we obtain  $x(t) = e^{2t} - 1$ .

The successive approximation scheme looks as follows

$$x_{n+1}(t) = x(0) = \int_0^t x'_n(s) ds = \int_0^t (2x_n(s) + 2) ds = 2t + 2 \int_0^t x_n(s) ds.$$

Let us also choose  $x_0(t) = 0$ . We will prove that  $x_n(t) = \sum_{m=1}^n \frac{(2t)^m}{m!}$ . To that end, let's check that equality for an initial  $n = 0$ . We have

$$x_0(t) = 0 = \sum_{m=1}^0 \frac{(2t)^m}{m!}.$$

Now let's assume that this holds for  $n$ , and look at  $n + 1$ . We have

$$\begin{aligned}
x_{n+1}(t) &= 2t + 2 \int_0^t x_n(s) ds \\
&= 2t + 2 \int_0^t \sum_{m=1}^n \frac{(2s)^m}{m!} ds \\
&= 2t + \sum_{m=1}^n \int_0^t \frac{(2s)^m}{m!} 2 ds \\
&= 2t + \sum_{m=1}^n \int_0^{2t} \frac{u^m}{m!} du \\
&= 2t + \sum_{m=1}^n \frac{(2t)^{m+1}}{(m+1)!} \\
&= \frac{(2t)^1}{1!} + \sum_{m=2}^{n+1} \frac{(2t)^m}{m!} \\
&= \sum_{m=1}^{n+1} \frac{(2t)^m}{m!}.
\end{aligned}$$

We can also write  $x_n(t) = \sum_{m=0}^n \frac{(2t)^m}{m!} - 1$ . It is now trivial to note that

$$\lim_{n \rightarrow \infty} x_n(t) = \sum_{m=1}^{\infty} \frac{(2t)^m}{m!} - 1 = e^{2t} - 1,$$

which we obtained from solving the equation analytically.