# 1. Summary of net and gross premiums

## Def 1.1 (Age of death)

Define $X$, a random variable, to be the age of death of a newborn. $X$ is assumed to be a continuous, non-negative random variable.
We have:

- $F_X(x) = P(X \leq x)$,
- $S(x) = 1 - F_X(x) = P(X \geq x)$ - survival function, probability that a newborn will survive to $x$,
- $f_x(x) = F'_X(x)$
  Let us also define $(x)$ to be a life aged $x$, usually meant as a person aged $x$ years.

## Def 1.2 (Future lifetime)

Define $T(x)$ to be the future lifetime of $(x)$, the amount of time that a person aged $x$ will live starting now. $T(x)$ is also a continuous, non-negative random variable, similar to $X$.
We have:

- $G(t) = G_{T(x)}(t) = P(T(x) \leq t)$

## Def 1.3 (Probability symbols)

Define the following:

- ${}_tp_x = P(T(x) \geq t) = P(X - x \geq t | X \geq x)$
- ${}_tq_x = 1 - {}_t p_x = G(t)$
- $K(x)$ - curtate future lifetime of $(x)$, $k(x) = \lfloor T(x) \rfloor$,
- ${}_{t|u}q_x = P(t \leq T(x) \leq t + u) = {}_{t+u} q_x - {}_t q_x$ - probability that $(x)$ will survive $t$ years, and die within the following $u$ years.
  Let us also define a convention regarding $p$ and $q$ functions:
- ${}_1p_x = p_x$,
- ${}_1q_x = q_x$.

## Def 1.4 (Force of mortality)

Define $\mu_x$ to be the force of mortality at age $x$. $\mu_x = -\frac{S'(x)}{S(x)}, \mu_x \geq 0.$

## Theorem 1.1 (Relationships)

The following equalities are true:

- $F(x) = \int_0^\infty f(s)ds = 1 - s(x) = 1 - \exp(-\int_0^x \mu_s ds)$,
- $f(x) = F'(x) = -S'(x) = \mu_x \exp(-\int_0^x \mu_s ds)$,
- $S(x) = 1 - F(x) = 1 - \int_0^\infty f(s)ds = \exp(-\int_0^x \mu_s ds)$,
- $\mu_x = \frac{F'(x)}{1-F(x)} = \frac{f(x)}{\int_x^\infty f(s)ds} = -\frac{S'(x)}{S(x)}$.

## Def 1.5 (UDD)

The uniform distribution of deaths (UDD) assumption assumes the following:

$$S(x+1) = (1-t)S(x) + tS(x+1), 0 \le t \le 1.$$

It also implies that

$$_t q_x = t q_x, 0 \le t \le 1,$$

and that $K(x)$ and $S(x)$ are independent.

## Def 1.6 (Life insurance products)

Typical life insurance products:

1. Whole life insurance,
2. $n$-year term insurance,
3. $n$-year pure endearment,
4. $n$ - year endearment,
5. annuities,
6. unit - united life insurance.
   We have:

- $T = T(x)$ - insurances payable at the moment of death,
- $K = K(x)$ - insurances payable at the end of year of death.

## Def 1.7 (Net single premium)

Let $A_x$, and $\overline{A}_x$ indicate the life insurance benefit of 1 payable at the end of the year of death, and at the time of death, respectively. These two cases correspond to discrete, and continuous models. Let $b_t$ be the benefit function, $v_t$ be the discount function. Let $Z = Z_t = b_t v_t$. Then the net single premium is defined as

$$NSP = E[Z] = \overline{A}_x = \int_0^\infty v_t g(t)dt.$$

Usually we take $b_t = 1$, and $v_t = v^t$, with a fixed $v$. Then the $NSP$ takes the following form

$$NSP = \int_0^\infty v^t\,_tp_x\mu_{x+t}dt,$$

with $v = e^{-\delta}$.

In the discrete case, substitute $t = k + 1$, and $v = \frac{1}{1+i}$, then we have

$$NSP = \sum_{k=0}^\infty v^{k+1}\,_kp_xq_{x+k}.$$

Now let's explicitly define the life insurance benefits:

$$\overline{A}_x = \int_0^\infty v^t\,_tp_x\mu_{x+t}dt,$$

$$\overline{A}_{x:\overline{n}|} = \int_0^n v^t\,_tp_x\mu_{x+t}dt$$

## Def 1.8 (Life annuity)

A life annuity is a series of periodic or continuous payments provided that the insured is alive. We have $Y$ - present value of future whole life annuity

$$Y = \int_0^T v^t dt = \overline{a}_{\overline{T}|},$$

$$v = e^{-\delta},$$

$$\overline{a}_{\overline{T}|} = \frac{1 - v^T}{\delta}.$$

# 2. Whole life insurance

## Def 2.1 (L)

Let $L$ be the loss of insurance company, so $L = $ the present value of venefits-present value of premiums.

## Def 2.2 (Equivalence principle)

Net premium is calculated in such a way that $E[L] = 0$. So if

$$L = v^{k+1} - p\ddot{a}_{\overline{k+1}|},$$

then

$$p_x = \frac{A_x}{\ddot{a}_x}.$$

# Def 2.3 (Endowment)

Since $p_x = \frac{A_x}{\ddot{a}_x}$ describes the whole life premium, then let's define

$$p^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \text{ - term (?)},$$

and

$$p_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \text{ - endowment.}$$

# Def 2.4 (Fully continuous case)

Let

$$\bar{a}_t = \int_0^t v^s ds,$$

then

$$L = v^T - \bar{p}_{\bar{a}_{\overline{T}|}},$$

and

$$\bar{p}_x = \frac{\overline{A}_x}{\bar{a}_x}.$$

# Def 2.5 (gross premiums)

The continuous-discrete case:

$$p(\overline{A}_x) = \frac{\overline{A}_x}{\ddot{a}_x},$$

$$p(\overline{A}^1_{x:\overline{n}|}) = \frac{\overline{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}},$$

$$p(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

The discrete-continuous case:

$$\bar{p} = \frac{A_x}{\bar{a}_x},$$

$$\bar{p}^1_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{a_{x:\overline{n}|}},$$

$$\bar{p}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}.$$

# Def 2.6 (Gross premiums costs)

The costs of gross premiums consist of:

- Acquisition costs,
- Agent's common,
- Collection expenses,
- Administration costs,
- Claim hedging expenses.

# Def 2.7 (Equivalence principle)

Let $p_a = A$ (net premium), and $b_a = A + \cos As$ (gross premium).
Then $_tL$ - financial loss of an insurance company,

$$_tL = Z - PY,$$

where $Y$ - present value of future payments, $P$ - net premium, $Z$ - present value of future payments of the benefits.
Then the net reserve equals

$$E[_tL] = {}_tV.$$

# Def 2.8 (recursive formulas for net reserves)

Let:

- $b_k$ - sum insured in $k$-th year of the policy,
- $\pi_0, \pi_1, \ldots, \pi_k$ - annual premiums paid up to the moment $k$,
- $L = Z - \sum_{m=0}^{k} z_m v^m = b_{k+1} v^{k+1} - \sum_{m=0}^{k} z_m v^m.$
  The recursive formula:

$$_kV + \pi_k = V(b_{k+1}qx + k + {}_{k+1}Vp_{x+k}).$$

Another form:

$$_kV + \pi_k = v({}_{k+1}V + (b_{k+1} - {}_{k+1}V)q_{x+k}).$$

This gives us a division of the premium $\pi_k$:

$$\pi_k = \pi_k^s + \pi_k^r,$$
$$\pi_k^s = {}_{k+1}V_v - {}_kV,$$
$$\pi_k^r = (b_{k+1} - {}_{k+1}V)Vq_{x+k}.$$