3 skipped lectures/labs allowed

1. What are inverse problems?

Mathematically, an inverse problem is formalised as solving an operator equation of the form

$$f = A(u) + e,$$

where:

- $f \in Y$ is the measured data
- $u \in X$ is the model parameter we aim to reconstruct
- The mapping $A : X \to Y$ is the forward operator, which describes how the model parameters give rise to the data in the absence of noise and measurement errors
- e is the noise and observation errors

2. Examples of inverse problems

2.1 Computed tomography (CT)

The mathematical foundation of CT is based on the Radon transform, which describes how X-ray projections are formed as they pass through a body and is represented by the integral:

$$A(u)(\omega, x) = \int_{-\infty}^{\infty} u(x + s\omega) ds.$$

The model parameter is a real-valued function $u : \Omega \to \mathbb{R}$, $\Omega \in \mathbb{R}^d$, which represents an image of a cross-section of the body. Here, the unit vector ω and x, which is orthogonal to ω represent the line $l : s \mapsto x + s\omega$, along which X-rays travel. $A(u)(\omega, x)$ is the recovered projection.

2.2 Electrical Impedance Tomography (EIT)

$$\begin{aligned} \nabla(a(x)\nabla u) &= 0, \text{ in } \Omega, \\ u &= f, \text{ on } \partial\Omega \end{aligned}$$

u is the electric potential, a is the conductivity. The measured currents over the boundary for a specific voltage f are given by

$$g_f = a\frac{\partial u}{\partial n}.$$

Then in EIT the data consists of the Dirichlet-to-Neumann operator

$$\Lambda_a : f \mapsto g_f.$$

2.3 Groundwater filtration

The groundwater filtration problem is often modelled by the Darcy's law and the following elliptic equation

$$-\nabla(a(x)\nabla u) = f, \text{ in } \mathbb{R}^d.$$

Here, u is the hydraulic head (pressure potential of groundwater), a is the permability (hydraulic conductivity), and f is the source term.

2.4 Earthquake source location

Here the inverse problem is: given observed seismic wave data, estimate the location and magnitude of the earthquake.

2.5 Engineering

Common inverse problem are

- given temperature measurements at time $t = T$, determine the initial temperature distribution at $t = 0$
- given temperature measurements over time, determine the unknown heat source

$$\frac{\partial u}{\partial t} = \Delta u + f$$