Organisational stuff

Consultations: Wed. 17-19, Thu. 13-15, room C-19 3.25
Attendance obligatory, skip at most 3
Test: 2025-01-24 09:15:00+01:00
2nd term: 2025-02-03 09:15:00+01:00

Literature:

- E.J. Hinch, "Perturbation Methods", Cambridge, 1992
- M.H. Holmes, "Introduction to Perturbation Methods", Springer, 2013
- R.S. Johnson, "Singular Perturbation Theory", Springer, 2009
- J.D. Logan, "Applied Mathematics", Wiley, 2013

Plan of the lecture:

1. Introduction
2. Asymptotic solutions of algebraic equations
3. Regular perturbations of ODEs and PDEs
4. Singular perturbations
5. Lindstedt expansion
6. Multiple scales
7. Asymptotic expansion of integrals
8. Asymptotic expansion in homogenisation

1. Introduction

Say we have a problem

$$P(u_\varepsilon, \varepsilon) = 0,$$

and this problem is "difficult" to solve. We call this problem a **perturbed** problem. Then the problem

$$P(u_0, 0) = 0,$$

which is easier to solve, is an unperturbed problem, that is an approximation to the first one. Then we can apply corrections to the unperturbed problem by e.g.

$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

Example

Consider the following ODE:

$$u_\varepsilon'' + (u_\varepsilon')^2 + \varepsilon u_\varepsilon = 0.$$

We can consider the unperturbed problem:

$$u_0'' + (u_0')^2 = 0,$$

which is easier to solve.

Def 1.1 Big O and small O

We write $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ if there exists a constant $M > 0$ such that the following inequality holds for all ε in some neighbourhood of ε_0:

$$|f(\varepsilon)| \leq M|g(\varepsilon)|.$$

We write $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ if

$$\lim_{\varepsilon \to \varepsilon_0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0.$$

Remarks

- If $\lim\limits_{\varepsilon \to \varepsilon_0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right|$ exists and is finite, then $f(\varepsilon) = O(g(\varepsilon))$
- The notation $f(\varepsilon) = O(g(\varepsilon))$ means that functions f and g are of the same size or order.
- The notation $f(\varepsilon) = o(g(\varepsilon))$ means that f is much smaller than g, or f goes to 0 faster than g. We can also write $f \ll g$.
- The notation $f(\varepsilon) = O(1)$ means that f is bounded in the neighbourhood of ε_0.
- The notation $f(\varepsilon) = o(1)$ means that $f(\varepsilon) \to 0$ as $\varepsilon \to \varepsilon_0$.

Def 1.2 Asymptotic approximation

We call a function g an asymptotic approximation of f for $\varepsilon \to \varepsilon_0$ if

$$f(\varepsilon) = g(\varepsilon) + o(g(\varepsilon))$$

and then we write $f(\varepsilon) \sim g(\varepsilon)$.

Example 1

Verify that $\varepsilon^2 \ln(\varepsilon) = o(\varepsilon)$ as $\varepsilon \to 0$. I'm not writing all of that, just compute the limit, use L'Hopital.

Example 2

Verify that $\sin(\varepsilon) = O(\varepsilon)$. We know that

$$\lim_{\varepsilon \to 0} \frac{\sin(\varepsilon)}{\varepsilon} = 1,$$

so we get the result by the first remark.

Def 1.3 O/o uniform

We write $f(x;\varepsilon) = O(g(x;\varepsilon))$ as $\varepsilon \to \varepsilon_0$ and $x \in \mathbb{I}$ if there exists a positive function $M(x)$ on $\mathbb{I}$ such that

$$f(x;\varepsilon) \leq M(x)|g(x;\varepsilon)|,$$

for all ε in some neighbourhood of ε_0 and all $x \in \mathbb{I}$.
If $M(x)$ is a bounded function on $\mathbb{I}$, then we write $f(x;\varepsilon) = O(g(x;\varepsilon))$ as $\varepsilon \to \varepsilon_0$ **uniformly** on $\mathbb{I}$.

We write $f(x;\varepsilon) = o(g(x;\varepsilon))$ as $\varepsilon \to \varepsilon_0$ and $x \in \mathbb{I}$ if

$$\lim_{\varepsilon \to \varepsilon_0} \left| \frac{f(x;\varepsilon)}{g(x;\varepsilon)} \right| = 0$$

pointwise on $\mathbb{I}$.
If the limit is uniform on $\mathbb{I}$, we write $f(x;\varepsilon) = o(g(x;\varepsilon))$ as $\varepsilon \to \varepsilon_0$ **uniformly** on $\mathbb{I}$.

Def 1.4 Uniformly valid asymptotic approximation

We say that a function g is a **uniformly valid asymptotic approximation** to a function f on $\mathbb{I}$ as $\varepsilon \to \varepsilon_0$ if the error $e(x;\varepsilon) = f(x;\varepsilon) - g(x;\varepsilon)$ converges to 0 as $\varepsilon \to \varepsilon_0$ uniformly.

Example

Let $f(x;\varepsilon) = \exp(-\varepsilon x)$ for $x > 0$ and $0 < \varepsilon \ll 1$. The first terms of the Taylor expansion in powers of ε provide an approximation $g(x;\varepsilon) = 1 - \varepsilon x + \frac{1}{2}x^2\varepsilon^2$

I'm not writing all of that, just compute the error (it's not uniform, just pointwise).

Def 1.5 Asymptotic sequence

A sequence of function $\{\phi_i\}$ is an asymptotic sequence as $\varepsilon \to \varepsilon_0$ and $x \in \mathbb{I}$ if

$$\phi_{i+1}(x;\varepsilon) = o(\phi_i(x;\varepsilon)), \quad \text{as } \varepsilon \to \varepsilon_0.$$

Def 1.6 Asymptotic expansion

Given an asymptotic sequence $\{\phi_i\}$, we say that a function f has an asymptotic expansion to n terms with respect to the sequence $\{\phi_i\}$ if

$$f(x;\varepsilon) = \sum_{i=1}^{k} a_i \phi_i(x;\varepsilon) + o(\phi_k(x;\varepsilon))$$

for $k = 1, \ldots, n$, and $\varepsilon \to \varepsilon_0$, where the coefficients a_i are independent on ε. In this case we write

$$f(x;\varepsilon) \sim \sum_{i=1}^{n} a_i \phi_i(x,\varepsilon).$$

The functions ϕ_i are called scale or basis functions.

Remark

Frequently we use the power functions $\phi_i(\varepsilon) = \varepsilon^{\alpha i}$ where $\alpha_i < \alpha_{i+1}$, as basis functions. An asymptotic expansion using such functions is called Poincare expansion.

Example

$$\sin(\varepsilon) \sim \sum_{i=0}^{n} a_i \varepsilon^{2i+1}, \text{ with } a_i = \frac{(-1)^i}{(2i+1)!}.$$

Example

Let's consider the error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

How do we approximate this function?

The first idea is to expand the integrand into its Taylor series:

$$e^{-t^2}=\sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!},$$

and then integrate:

$$E_n(x)=\frac{2x}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{-x^{2k}}{k!(2k+1)}.$$

We may expand that $E_n(x)\to erf(x)$ as $n\to\infty$, but the convergence rate is shit.

The second idea consists on constructing an asymptotic expansion for $x\gg 1$. To do so we write

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt-\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt=1-\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

Then we integrate with $u=t^2-x^2$

$$\int_x^\infty e^{-t^2}dt=\int_0^\infty\frac{1}{2x}e^{-x^2}e^{-u}\left(1+\frac{u}{x^2}\right)^{-\frac{1}{2}}du.$$

Next we use the generalised binomial $(1+y)^\alpha=\sum_{k=0}^\infty\binom{\alpha}{k}y^k$, with $\alpha=-\frac{1}{2}$. We note that

$$\binom{-\frac{1}{2}}{k}=\frac{(-1)^k}{\sqrt{\pi}k!}\Gamma\left(\frac{1}{2}+k\right),$$

then we get

$$\int_0^\infty\frac{e^{-x^2}}{2\sqrt{\pi}x}e^{-u}\sum_{k=0}^{n}\frac{1}{k!}\Gamma\left(\frac{1}{2}+k\right)\left(-\frac{u}{x^2}\right)^k du=\frac{e^{-x^2}}{2\sqrt{\pi}x}\sum_{k=0}^{n}\Gamma\left(\frac{1}{2}+k\right)(-x)^{2k}\Gamma(k+1).$$

Then the second expansion of erf is

$$E_n(x)=1-\frac{e^{-x^2}}{\pi x}\sum_{k=0}^{n}\Gamma\left(\frac{1}{2}+k\right)(-x)^{2k}k!.$$

Remarks

- Asymptotic expansions need not to be convergent with an increasing number of terms ($n\to\infty$).
- Asymptotic expansions can be added and multiplied, if they are obtained in a special way.
- In general, we can not differentiate asymptotic expansions, but if

  $$f(x;\varepsilon)\sim\sum_{i=1}^{n}a_i\phi_i(x;\varepsilon),$$

  and

$$\frac{\partial f}{\partial x}(x;\varepsilon) \sim \sum_{i=1}^{n} b_i(x)\phi(x;\varepsilon)$$

then $b_i = \frac{\partial}{\partial x} a_i$.

- Asymptotic expansions can be integrated

$$\int_a^b f(x;\varepsilon)dx \sim \sum_{i=1}^{n} \int_a^b a_i(x)\phi_i(x;\varepsilon).$$

2. Asymptotic solutions of algebraic equations

Example

Suppose we want to solve the following equation

$$x^2 - 3.99x + 3.02 = 0.$$

Then we may introduce $\varepsilon = 0.01$, and rewrite this equation to

$$x^2 + (\varepsilon - 4)x + (3 + 2\varepsilon) = 0.$$

We want to find the asymptotic approximation of its solution.
The unperturbed problem is

$$x^2 - 4x + 3 = 0,$$

and it has solutions $x_0 = 1$, and $x_0 = 3$.
Using power functions as basis, we consider the following approximation

$$x_\varepsilon \sim x_0 + a_1\varepsilon^{\alpha_1} + a_2\varepsilon^{\alpha_2}.$$

I'm not writing all of that, insert x_ε approximation above into the rewritten equation, should be $\sim$ to 0