Applied Functional Analysis - Exercise sheet 5

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Exercise 1

We are presented with the following function

$$u(x) = \int_0^1 K(x, y) f(y) dy,$$

for $x \in (0, 1)$, and

$$K(x, y) = \begin{cases} \frac{1}{T} y(1-x), y \in [0, x], \\ \frac{1}{T} x(1-y), y \in [x, 1]. \end{cases}$$

We have

$$\begin{aligned} u(x) &= \frac{1}{T}\left((1-x)\int_0^x yf(y)dy + x\int_x^1 (1-y)f(y)dy\right), \\ u'(x) &= \frac{1}{T}\left(\int_0^1 yf(y)dy + \int_x^1 f(y)dy\right), \\ u''(x) &= -\frac{1}{T} f(x), \end{aligned}$$

rearranging, we get

$$Tu''(x) + f(x) = 0,$$

or

$$f(x) = -Tu''(x).$$

moreover, it is apparent from the definition of u, that $u(0) = u(1) = 0$.
Now let $u(x) = (x-1)\sin(x)$, and let a small perturbation be defined as $n_\delta(x) = \delta(x-1)\sin\left(\frac{x}{\delta}\right)$, and define $u_\delta(x) = u(x) + n_\delta(x)$. We have

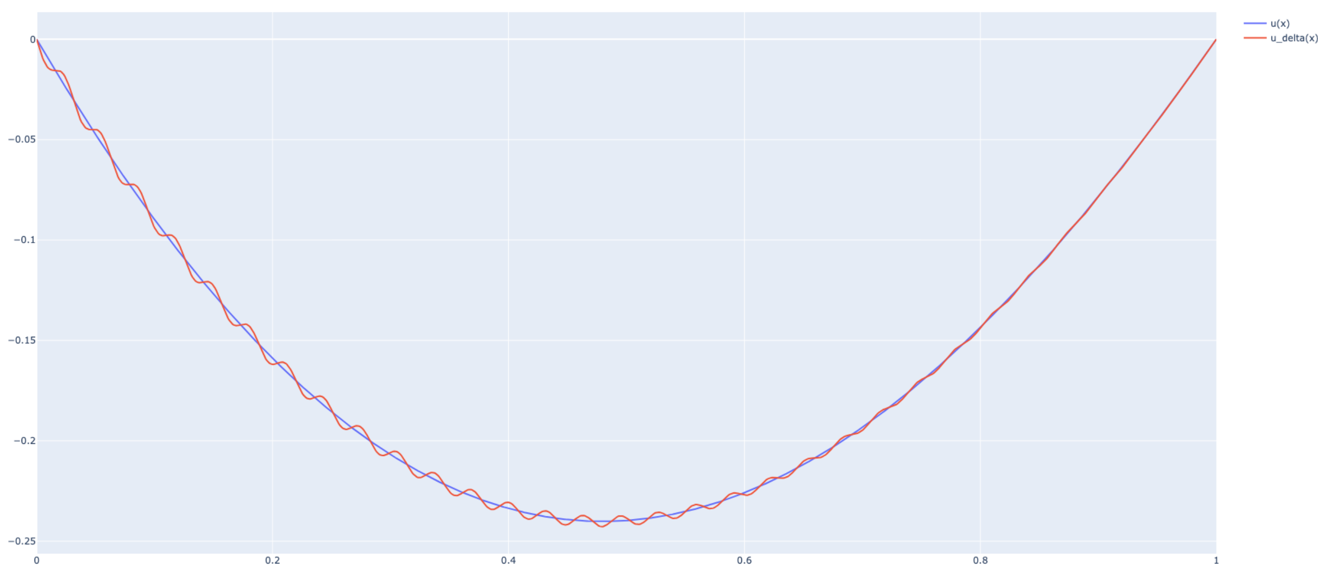
$$\begin{aligned} ||u - u_\delta|| &= ||n_\delta||, \\ ||f - f_\delta|| &= T||n''_\delta||. \end{aligned}$$

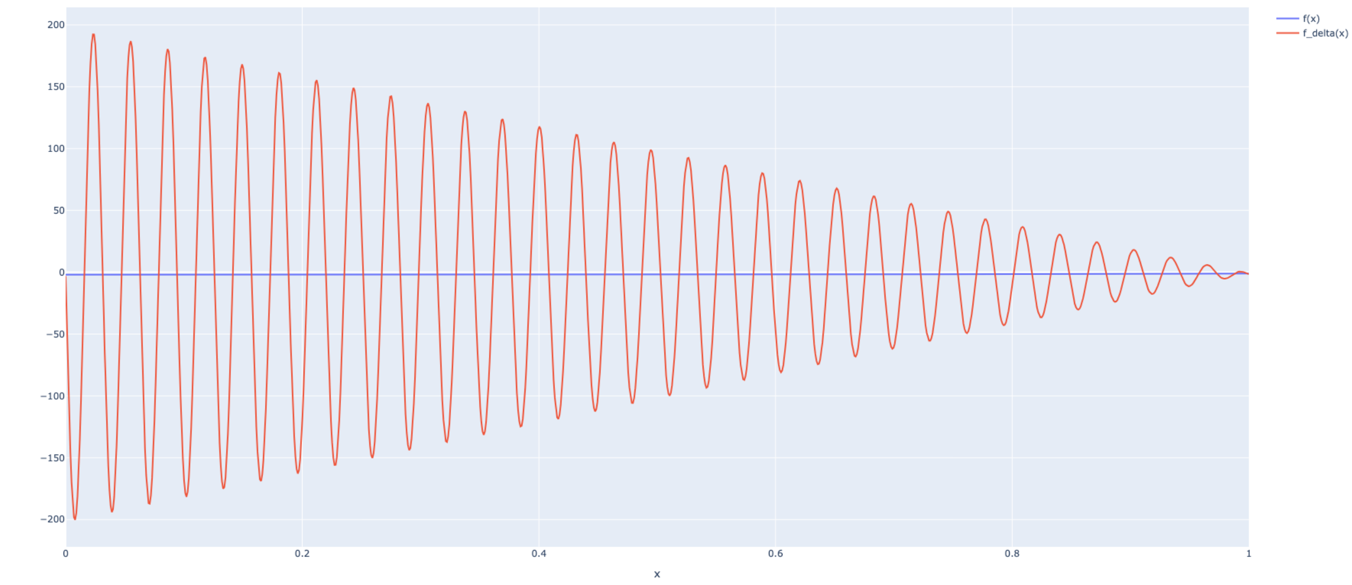
Let's now compute the L^2 and L^∞ norms of both of these:

$$
\begin{aligned}
||u-u_\delta||_{L^2}^2 &= \int_0^1 \left|\delta \sin\left(\frac{x}{\delta}\right)\right|^2 dx = \frac{\delta^2}{24}\left(3\delta^3 \sin\left(\frac{2}{\delta}\right) - 6\delta^2 + 4\right),\\
||f-f_\delta||_{L^2}^2 &= \int_0^1 \left|T\left(\frac{1}{\delta}(x-1)\sin\left(\frac{x}{\delta}\right) - \cos\left(\frac{x}{\delta}\right)\right)\right|^2 dx = T^2\left(\frac{1}{6\delta^2} + \frac{1}{8}\sin\left(\frac{2}{\delta}\right) + \frac{3}{4}\right),\\
||u-u_\delta||_{L^\infty} &= \max_{0\le x\le 1}\left|\delta(x-1)\sin\left(\frac{x}{\delta}\right)\right| \le \delta,\\
||f-f_\delta||_{L^\infty} &= \max_{0\le x\le 1}\left|T\left(\frac{1}{\delta}(x-1)\sin\left(\frac{x}{\delta}\right) - \cos\left(\frac{x}{\delta}\right)\right)\right| \ge\\
&\ge \left|\max_{0\le x\le 1}\left(\frac{T}{\delta}(x-1)\sin\left(\frac{x}{\delta}\right)\right) - \max_{0\le x\le 1}\left(T\cos\left(\frac{x}{\delta}\right)\right)\right| \ge T\left|-\frac{1}{\delta}-1\right| =\\
&= T\left(\frac{1}{\delta}+1\right).
\end{aligned}
$$

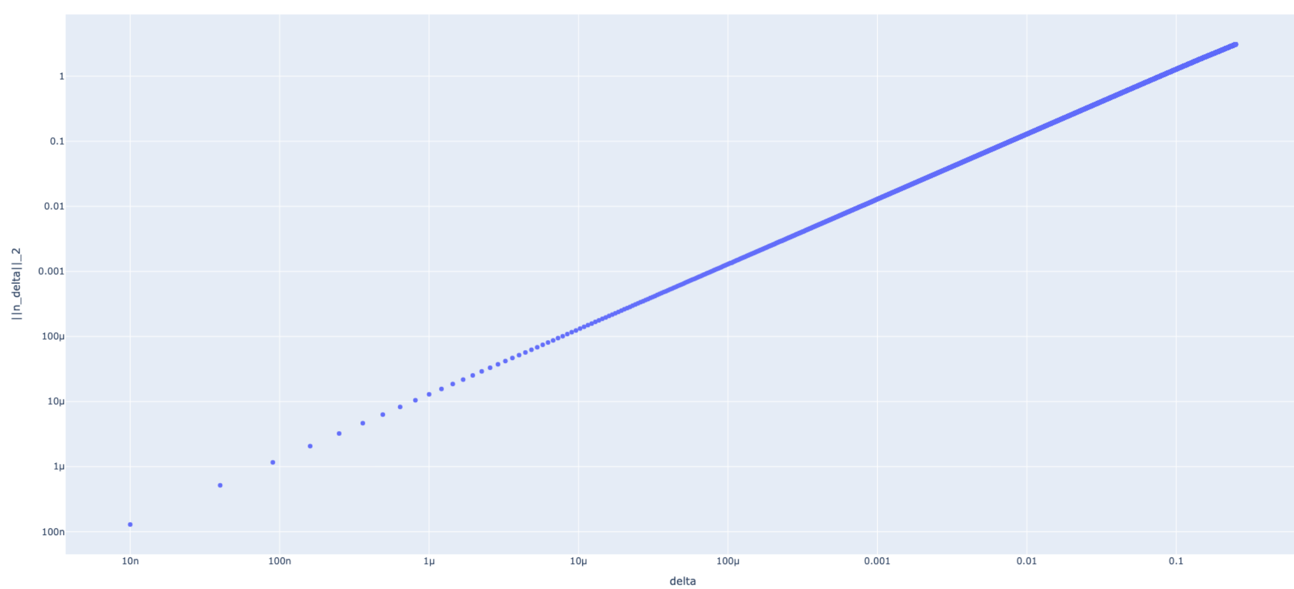
As we can see, in both cases, L^2, and L^∞, the norm of $u-u_\delta$ approaches zero when δ approaches zero, on the other hand the norm of $f-f_\delta$ gets arbitrarily large when δ approaches zero, this means that the problem of finding f when we have perturbations in initial data u is ill-posed.

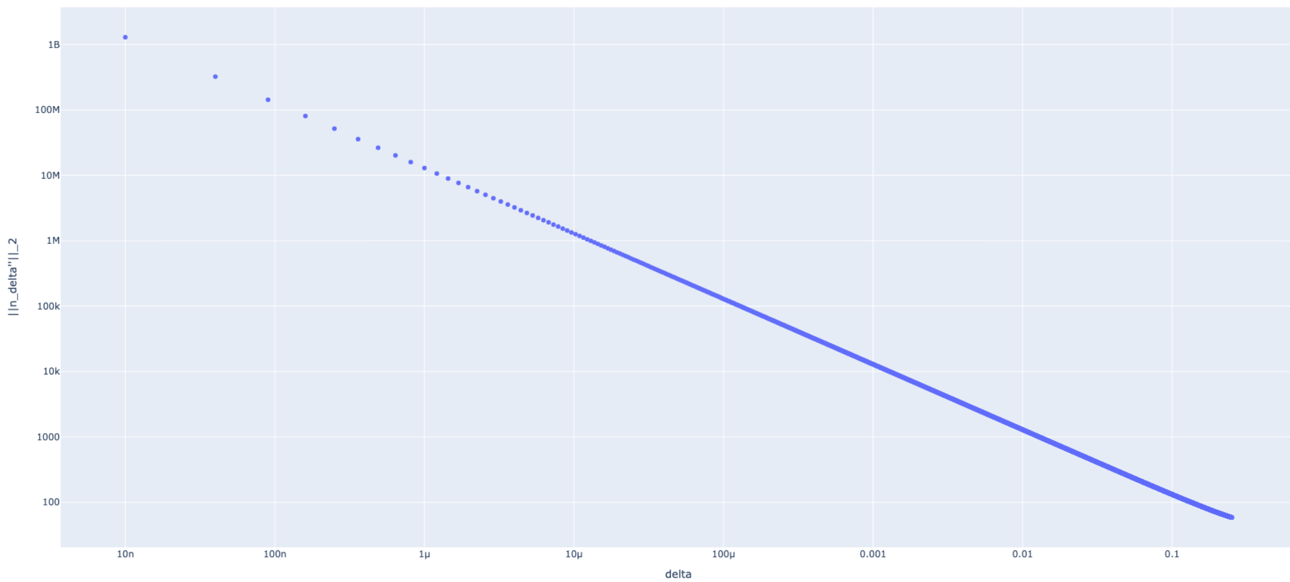
Now, we shall see some graphs illustrating the problem at hand. First, we can see all the functions of interest: u, u_δ, f, and f_δ. For simplicity we take $T=1$, and we plot the functions with $\delta = 0.005$, where applicable. For all plots and computations we use discretisation of the $[0,1]$ interval into $N = 1000$ equally spaced points.



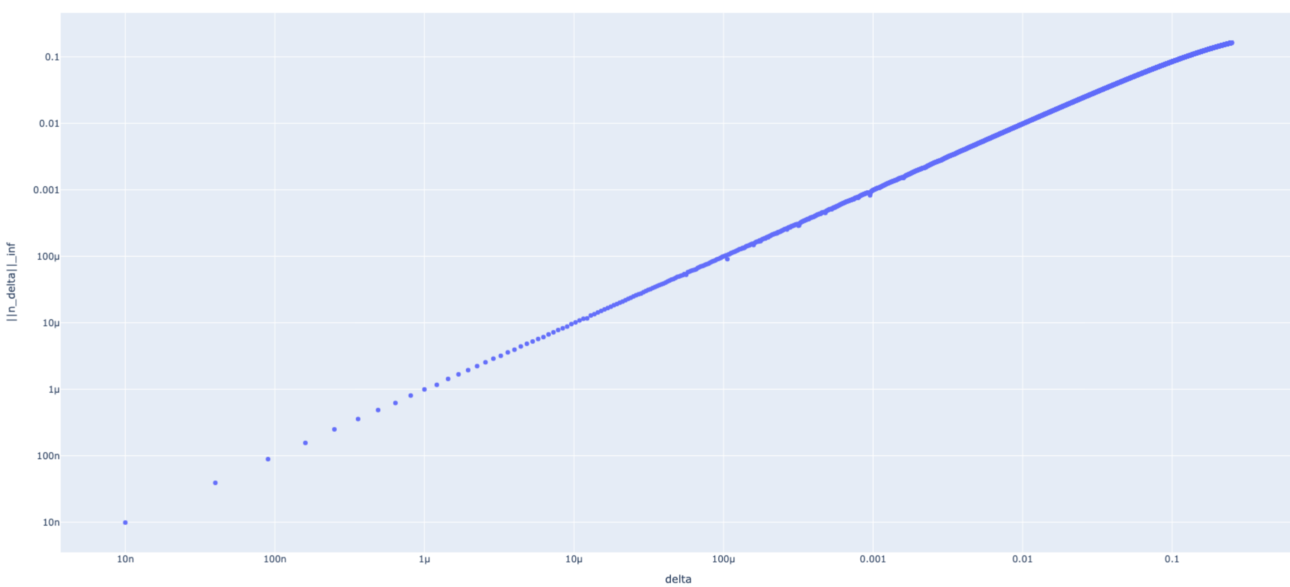


Now, to show the convergence and divergence of norms, we calculate the L^2 and L^∞ norms numerically for various values of δ:

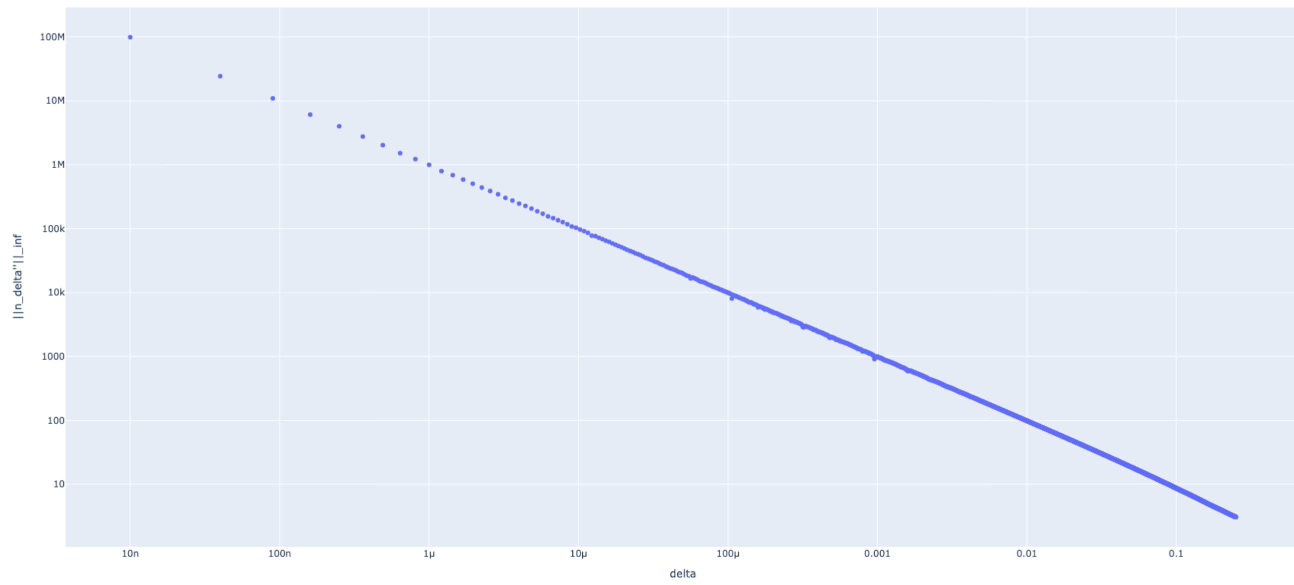




1B
100M
10M
1M
100k
10k
1000
100
||n_delta''||_2
10n
100n
1µ
10µ
100µ
0.001
0.01
0.1
delta



0.1
0.01
0.001
100µ
10µ
1µ
100n
10n
||n_delta||_inf
10n
100n
1µ
10µ
100µ
0.001
0.01
0.1
delta



As we can see, $||u - u_\delta|| = ||n_\delta||$ stay small when δ approaches zero, on the other hand $||f - f_\delta|| = ||n''_\delta||$ explode as δ gets smaller.

Exercise 2

We are presented with a problem of calculating the derivative of noisy data, that is

$$f_\delta(x) = f(x) + n_\delta(x),$$

for $x \in (0, 1)$, and $f_\delta(0) = f(0) = 0 = f_\delta(1) = f(1) = 0$, with

$$n_\delta(x) = \sqrt{2}\delta \sin(2\pi k x)$$

with a fixed, small δ. Obviously we have

$$\begin{aligned} ||f - f_\delta|| &= ||n_\delta||, \\ ||f' - f'_\delta|| &= ||n'_\delta||, \end{aligned}$$

so, calculating the L^2 and L^∞ norms we get (omitting the messy details this time)

$$\begin{aligned} ||f - f_\delta||_2^2 = ||n_\delta||_2^2 &= \delta^2 \left(1 - \frac{\sin(4\pi k)}{4\pi k}\right), \\ ||f' - f'_\delta||_2^2 = ||n'_\delta||_2^2 &= \pi^2 k^2 \delta^2, \\ ||f - f_\delta||_\infty = ||n_\delta||_\infty &= \sqrt{2}\delta, \\ ||f' - f'_\delta||_\infty = ||n'_\delta||_\infty &= 2\sqrt{2}\pi k \delta. \end{aligned}$$

Again, we have the same situation as in the first exercise, where the norm of the difference $f - f_\delta$ stays small, due to δ being small, even when k goes to infinity. On the other hand, the norm of the difference $f' - f'_\delta$ diverges to infinity when k goes to infinity. This proves that the problem of differentiating noisy data is ill-posed.

Let's estimate the error that we introduce when calculating the derivatives numerically using Euler central difference scheme, that is, let's estimate

$$E_f = \left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right|.$$

We know that $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi_+)$, where $\xi_+ \in [x, x+h]$, and that $f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(\xi_-)$, where $\xi_- \in [x-h, x]$. From that we get

$$\begin{aligned} \frac{1}{2}\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| &\leq \left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{h}{2} |f''(\xi_+)|, \\ \frac{1}{2}\left| f'(x) - \frac{f(x) - f(x-h)}{h} \right| &\leq \left| f'(x) - \frac{f(x) - f(x-h)}{h} \right| \leq \frac{h}{2} |f''(\xi_-)|, \end{aligned}$$

and, using the triangle equality we obtain a rough estimate

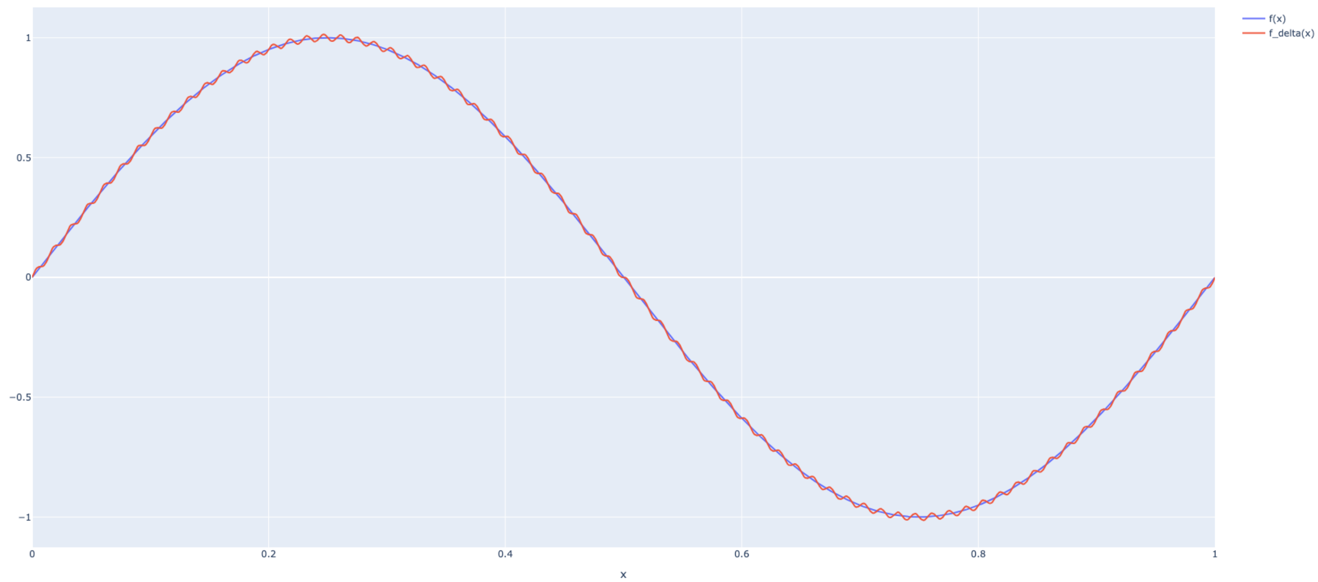
$$\begin{aligned} E_f = \left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right| = \left| \frac{f'(x)}{2} - \frac{f(x+h) - f(x)}{2h} + \frac{f'(x)}{2} - \frac{f(x) - f(x-h)}{2h} \right| \leq \\ \leq \frac{h}{2}|f''(\xi_+)| + \frac{h}{2}|f''(\xi_-)| \leq h|f''(\xi)|, \end{aligned}$$

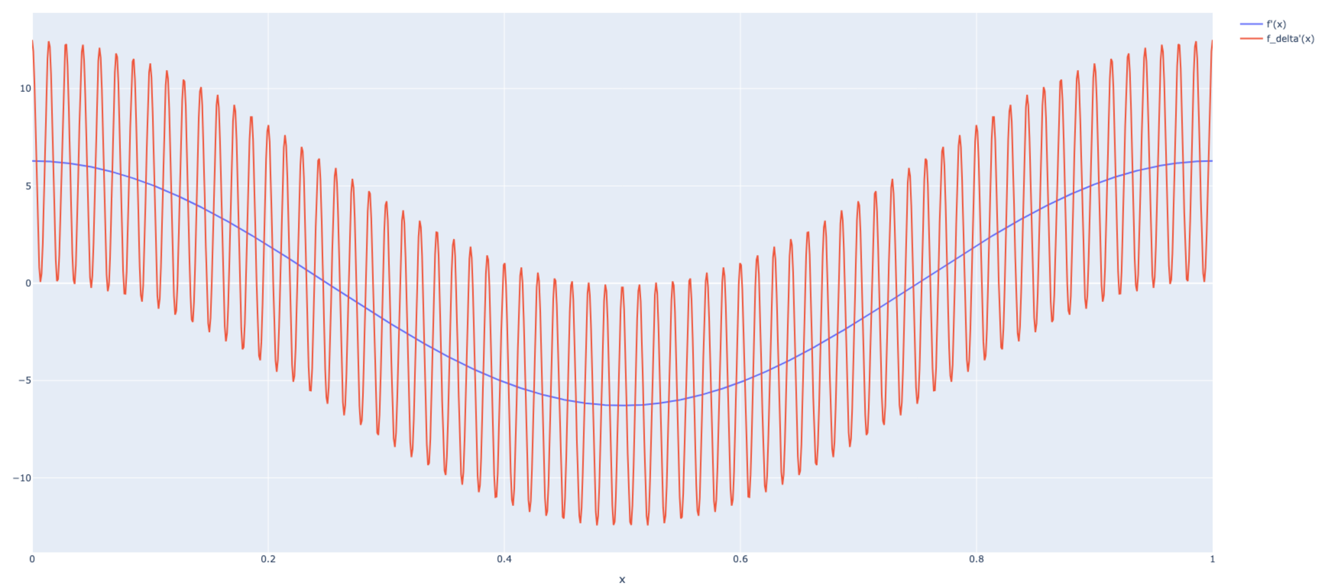
where $\xi \in [x-h, x+h]$. Directly substituting our f and f_δ into this inequality we get

$$\begin{aligned} &E_f \leq h \left| -4\pi^2 \sin(2\pi\xi) \right| \leq 4\pi^2 h, \\ &E_{f_\delta} \leq h \left| -4\pi^2 sin(2\pi\xi) - 4\sqrt{2}\pi^2 \delta k^2 \sin(2\pi k\xi) \right| \leq h \left(4\pi^2 + 4\sqrt{2}\pi^2\delta k^2 \right). \end{aligned}$$

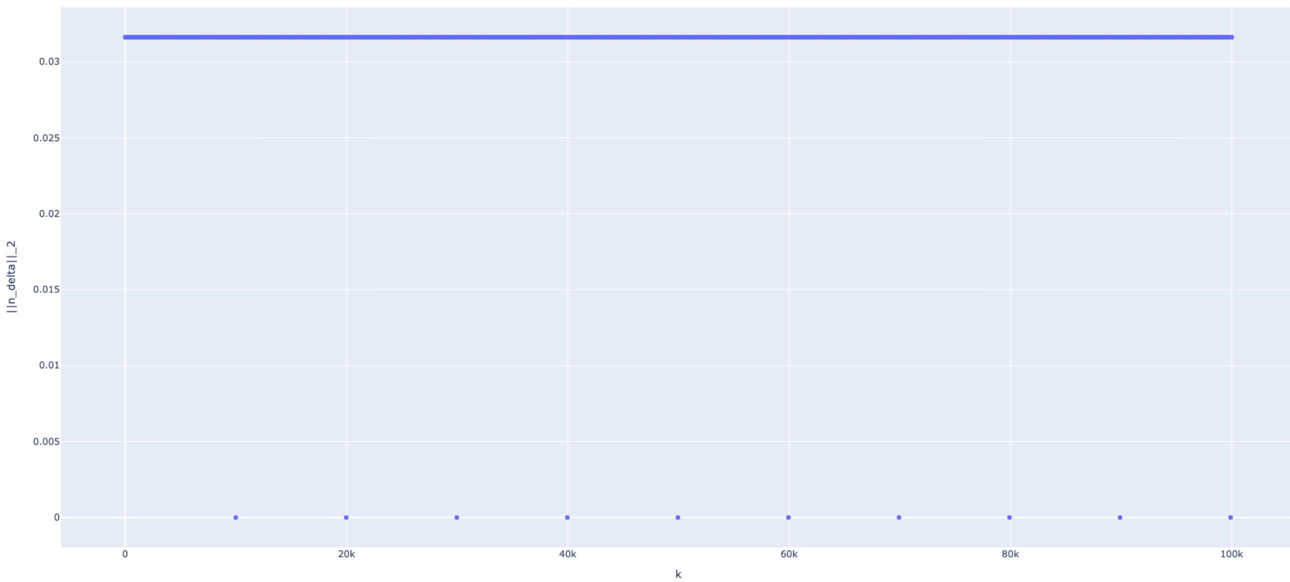
We can thus see from this, that the first error stays low, and gets lower with more fine-grained discretisation. On the other hand, the error E_{f_δ} grows with increasing k, and so diverges to infinity when k diverges to infinity.

Again, we leave the graphs for the end of the exercise. We will again use discretisation with $N = 1000$ equally spaced points. All of the following plots use $\delta = 0.01$. First we plot f with f_δ, and f' with f'_δ to illustrate the issue. We use $K = 70$ here.

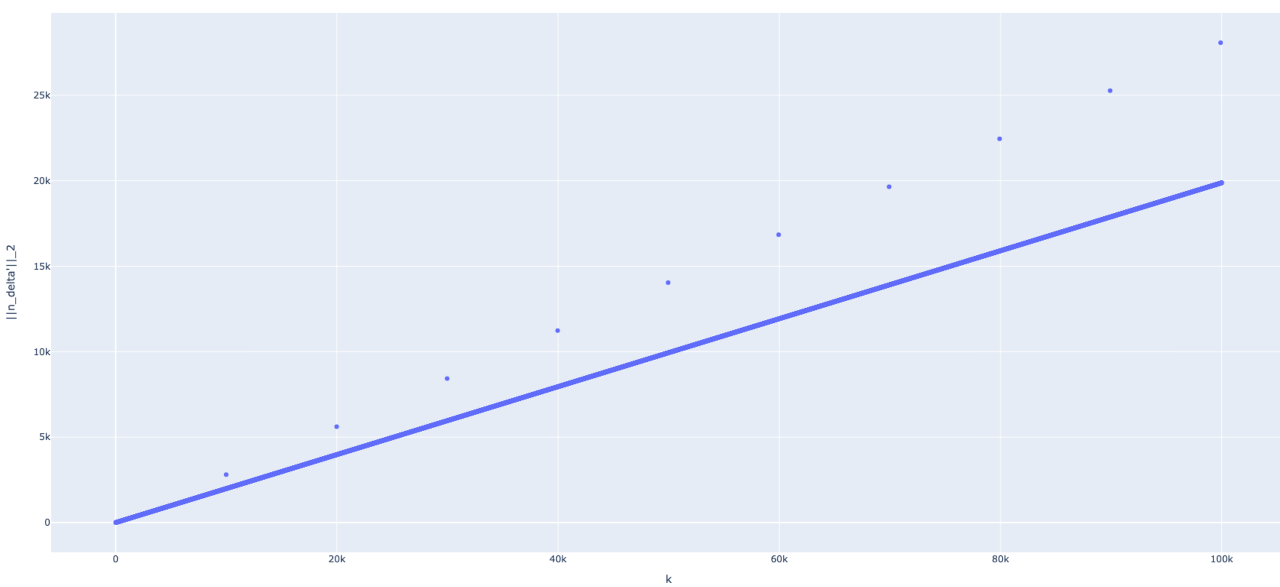




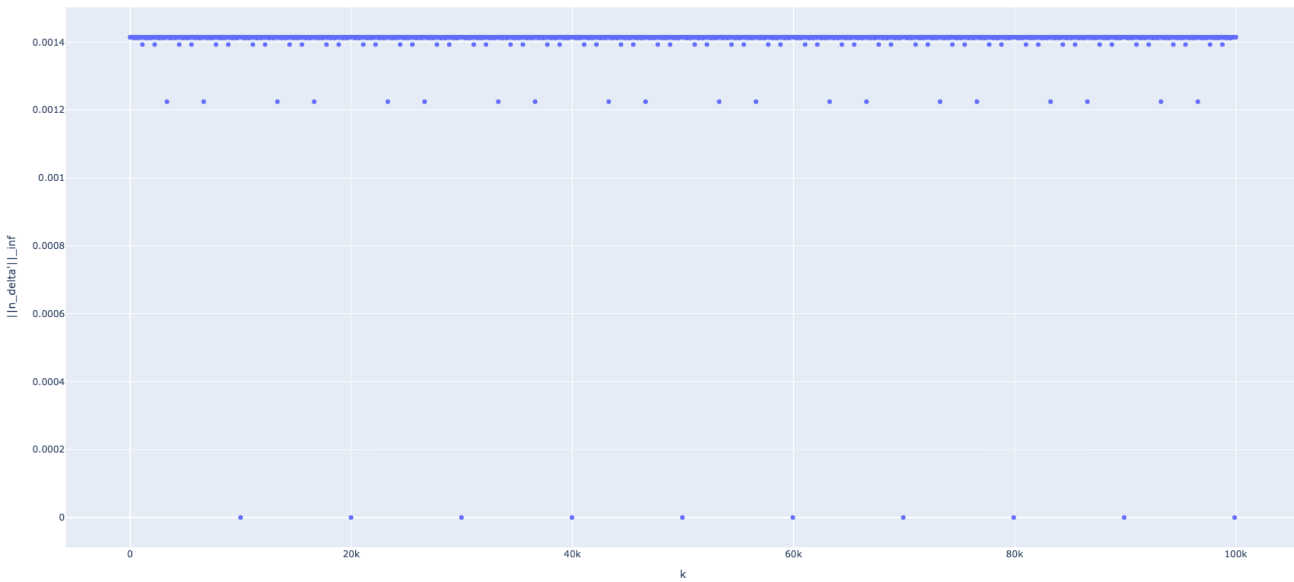
We will also show the norms of $f - f_\delta$, and $f' - f'_\delta$ with respect to increasing k.

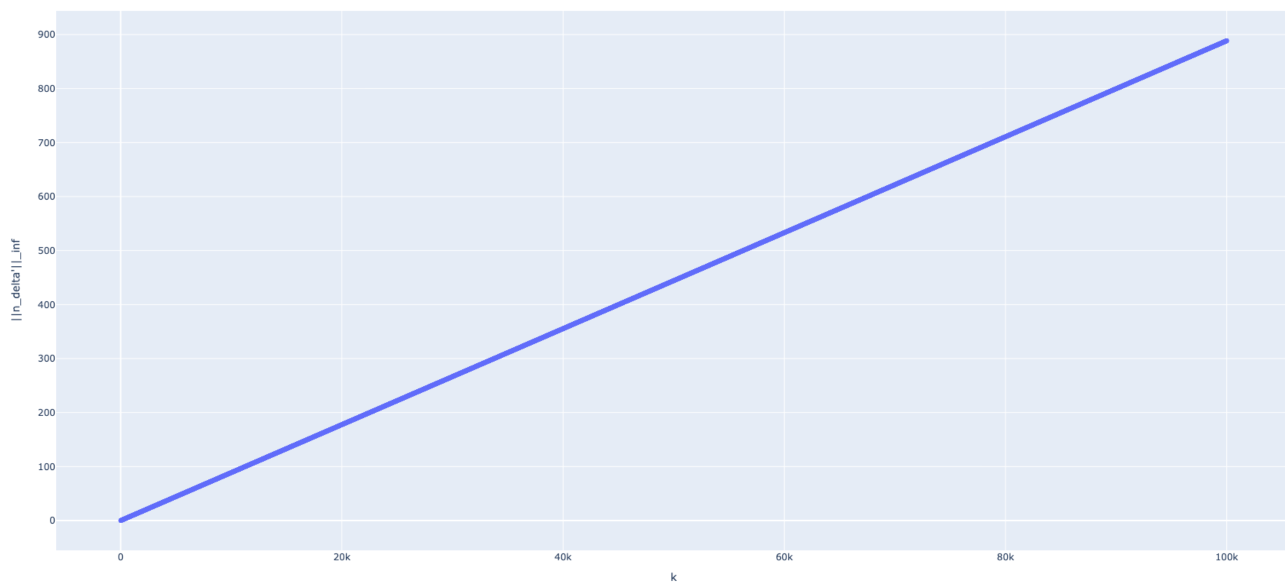


0.03
0.025
0.02
0.015
0.01
0.005
0
||n_delta||_2
0
20k
40k
60k
80k
100k
k



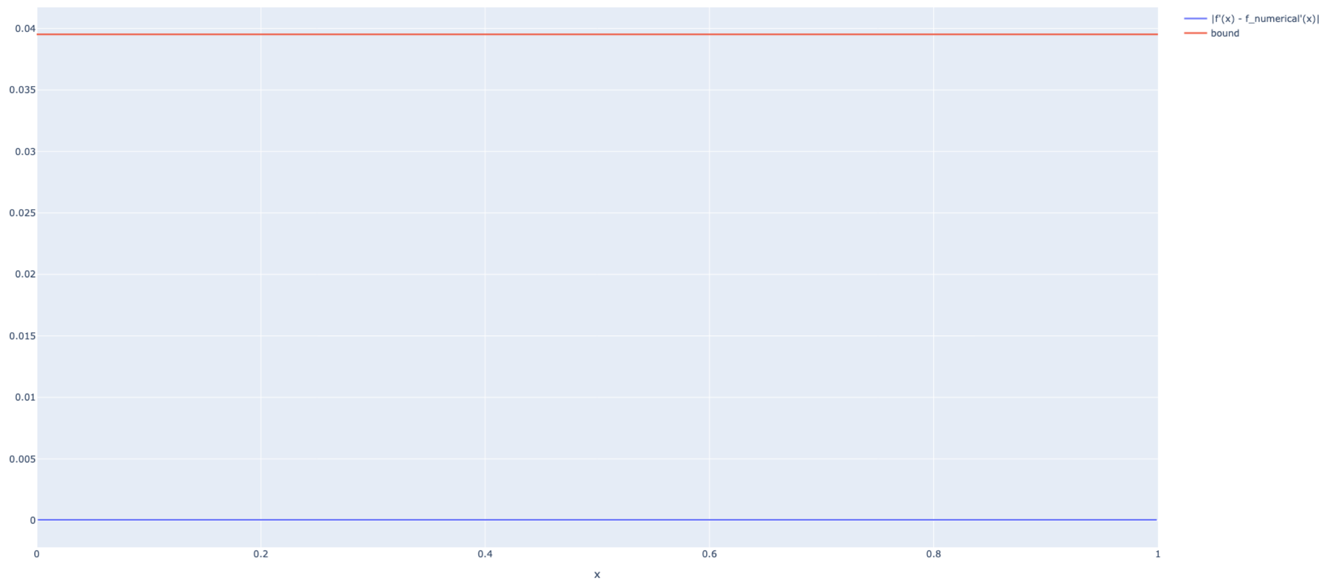
25k
20k
15k
10k
5k
0
||n_delta'||_2
0
20k
40k
60k
80k
100k
k

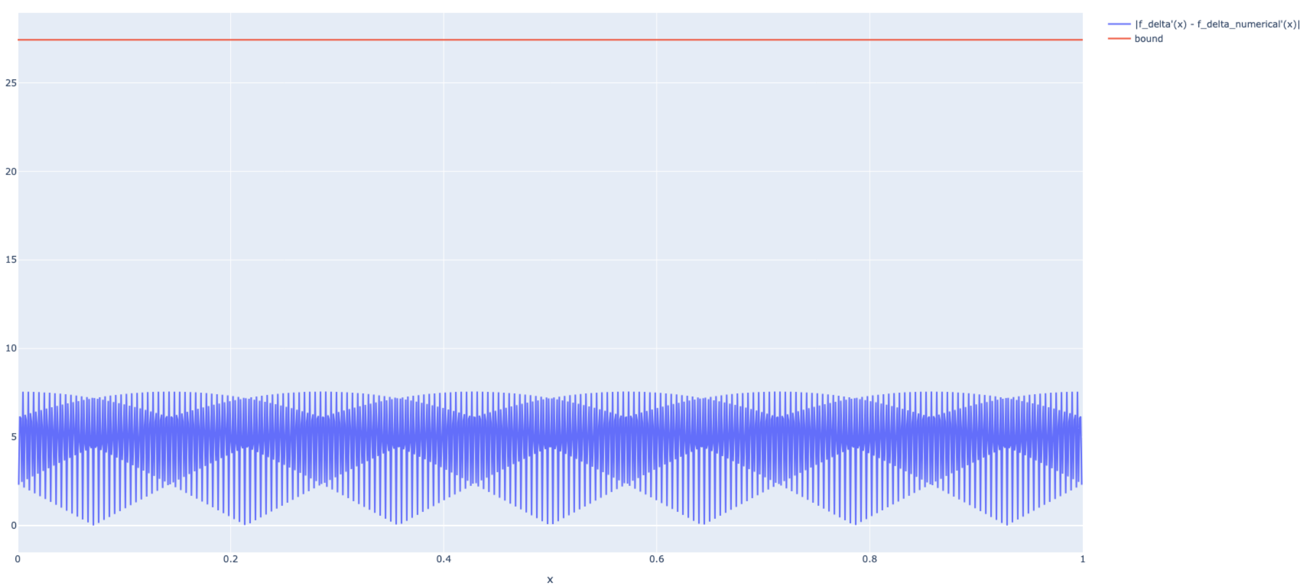




As we can see, the norms of $f - f_\delta$ stay about constant, with some numerical anomalies here and there, but the norms of $f' - f_\delta'$ grow linearly, as predicted by the computations, with k.

We will also illustrate the bounds on the errors that we get from numerical differentiation. Here we arbitrarily take $k = 70$:





Exercise 3

We are presented with a Fredholm integral equation of the form

$$u(x) = \int_0^1 K(x,y)f(y)dy,$$

for $x \in (0,1)$.The function f represents the true image, kernel K characterises the blurring effect, and u is the blurred image. We wish to recover f from a previously blurred image u. Assume that the kernel K is a gaussian kernel, that is

$$K(x, y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-y)^2\right),$$

where $\sigma > 0$ is a parameter. Let's approximate the equation by constructing vectors

$$\begin{aligned} \vec{x} &= [x_1, x_2, \ldots, x_M]^T, \\ \vec{y} &= [y_1, y_2, \ldots, y_N]^T, \\ \vec{u} &= [u(x_1), u(x_2), \ldots, u(x_M)]^T, \\ \vec{f} &= [f(y_1), f(y_2), \ldots, f(y_N)]^T, \end{aligned}$$

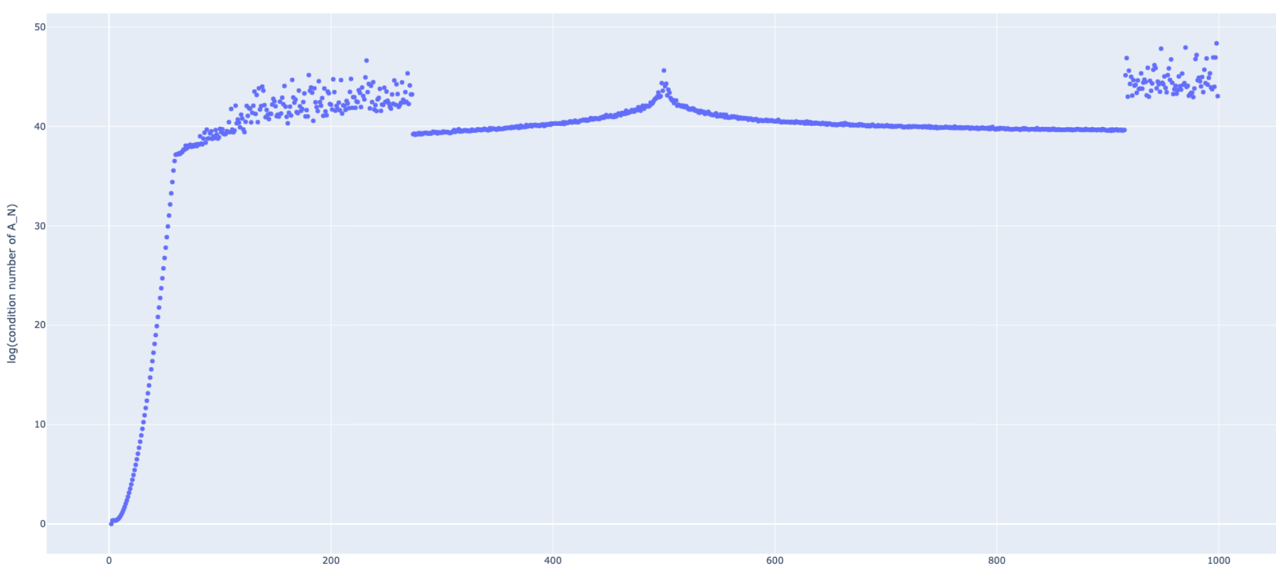
and the matrix

$$A = [w_j K(x_i, y_j)]_{M \times K}.$$

The equation now becomes

$$A\vec{f} = \vec{u}.$$

Let's calculate the condition number of some matrices A for various numbers of N. Let's fix $\sigma = 0.05$, and $M = 500$. The plots of $\log(C(A_N))$, where $C(A)$ is the condition number of A, are displayed below.



Ns are varying from 2 to 1000, we can see a peak at $N = 500$, where the matrix becomes a square matrix. Also the curve changes regime at $N = 60$, $N = 270$, and $N = 915$.

We can see that the matrix is ill-conditioned due to its enormous condition number, even for small N, so we will not be able to find the vector $\vec{f}$ from our linear system.

We will use a method of truncated singular value decomposition. We factorise the matrix A as

$$A = U\Sigma V,$$

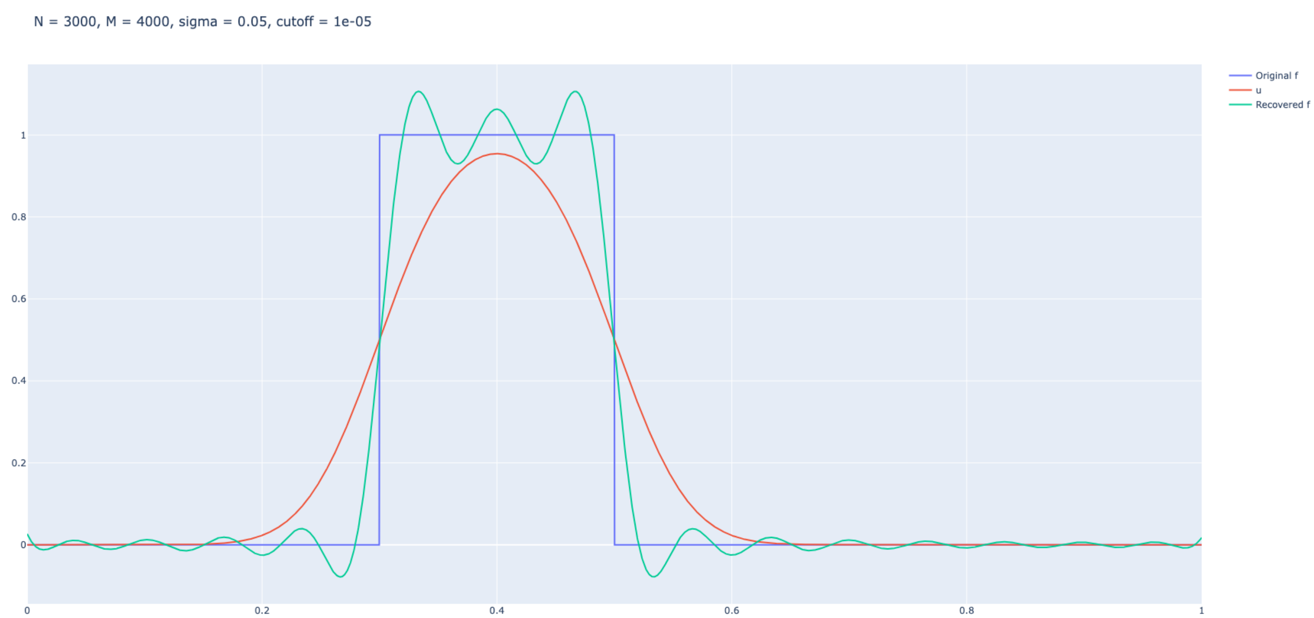
where U and V are square unitary matrices, and Σ is a rectangular diagonal matrix. Due to the fact that Σ contains very small numbers we choose a cutoff point a. We then "invert" the matrix A as follows

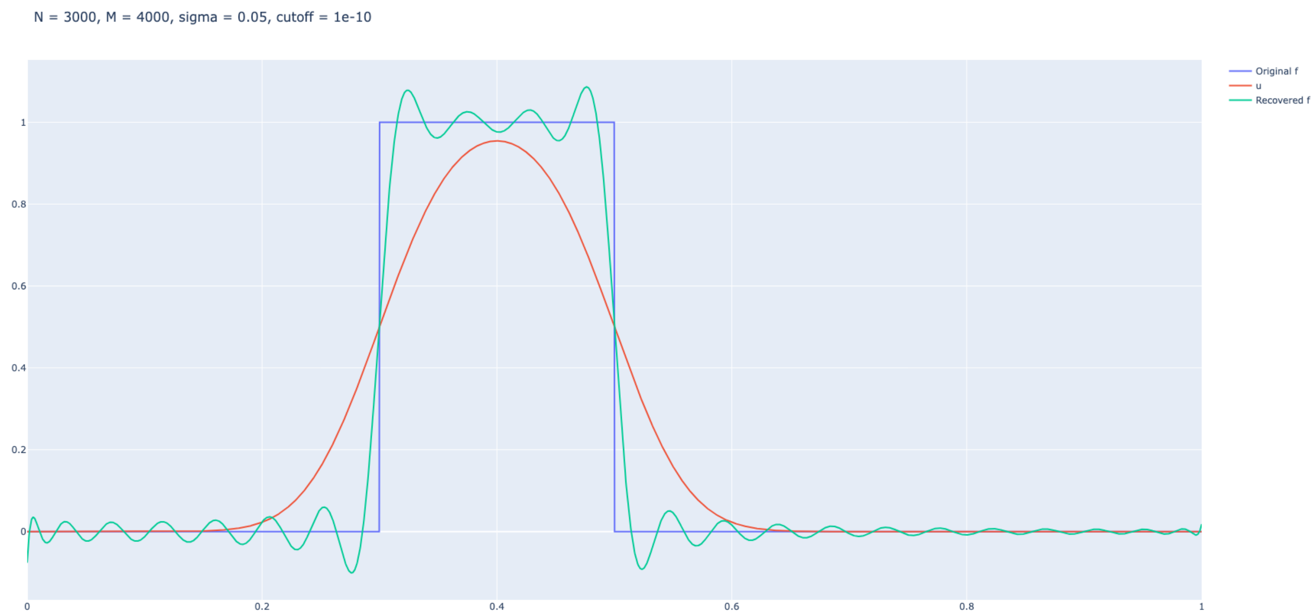
$$A^{-1} = (U\Sigma V)^{-1} = V^{-1}\Sigma^{-1}U^{-1} = V^T\Sigma^{-1}U^T,$$

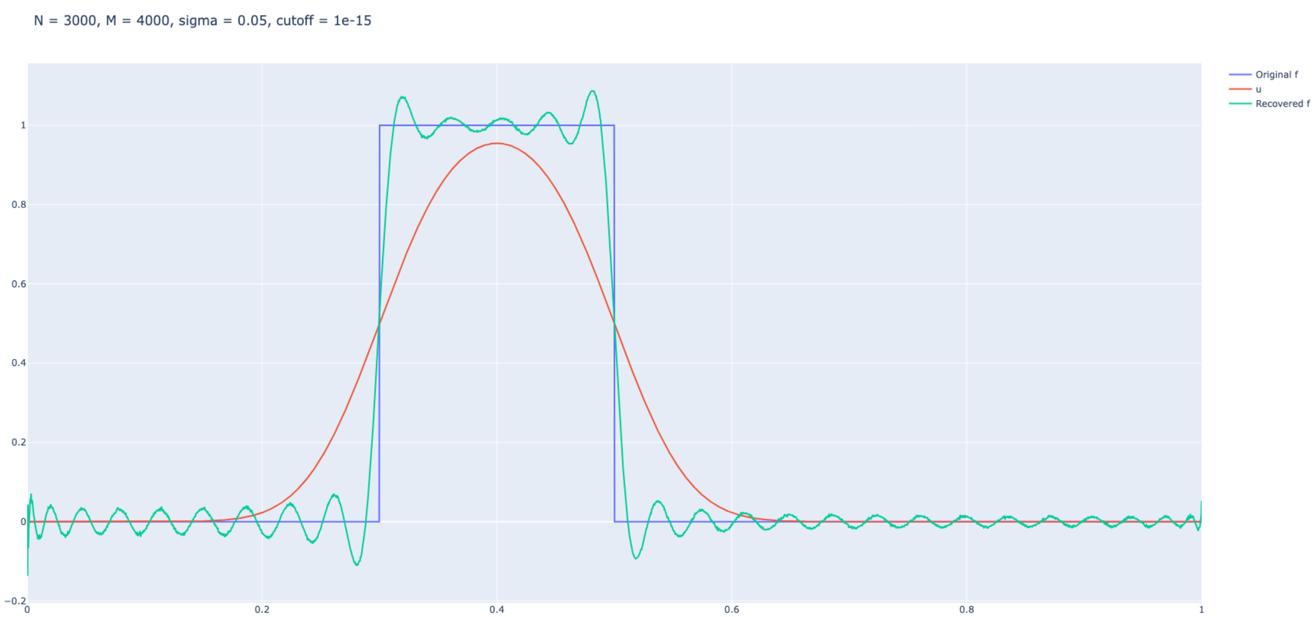
since the transpose of a unitary matrix is its inverse. To "invert" Σ we decide on a cutoff point a, and we set $\Sigma^{-1}_{m,n}$ to be $\frac{1}{\Sigma_{m,n}}$ if $\Sigma_{m,n} > a$, and 0 otherwise. We can then find $\vec{f}$ with

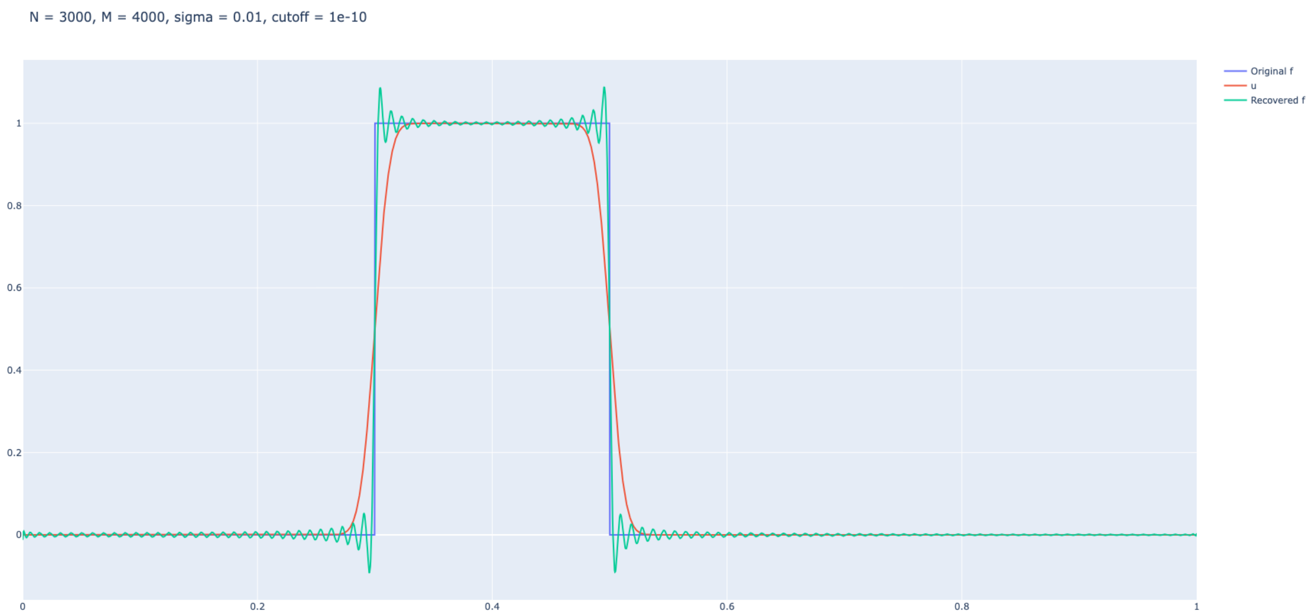
$$\vec{f} = A^{-1}\vec{u}.$$

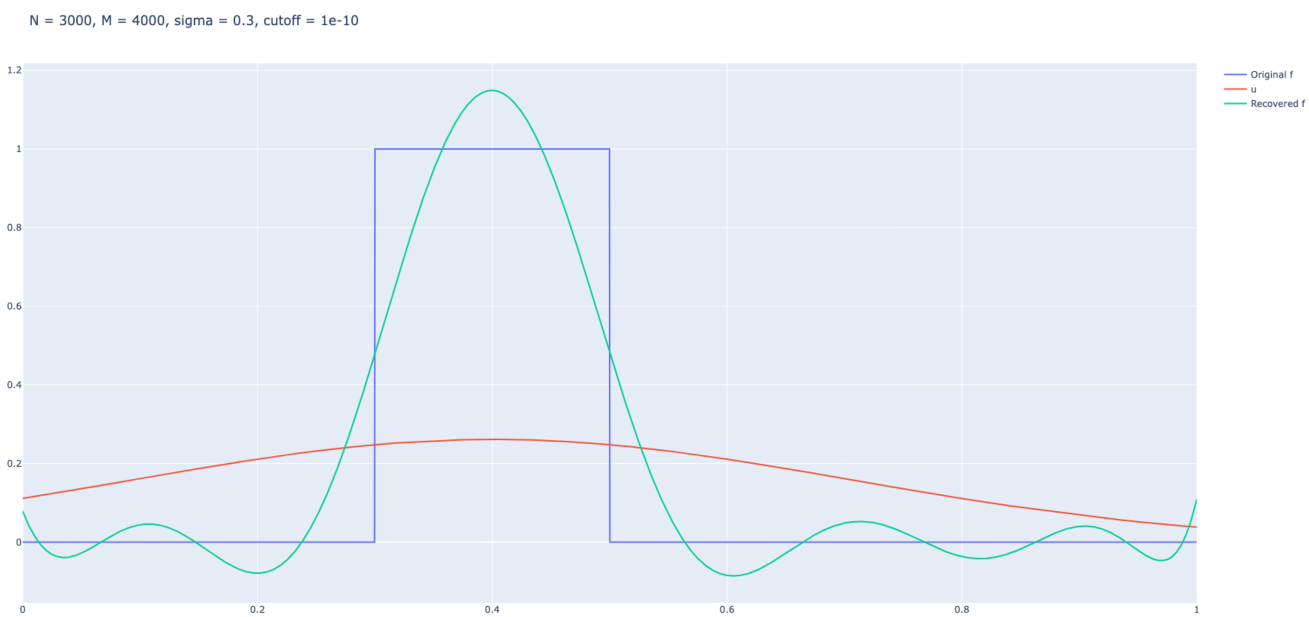
Assume that $f(x) = H(x-0.3) - H(x-0.5)$, where H is the Heavyside function. We will obtain u by directly convolving f and K, then we will get f back using the described discretisation scheme. Below are the plots of original f, calculated u, and f recovered by solving the inverse problem numerically, for various parameters.




N = 3000, M = 4000, sigma = 0.05, cutoff = 1e-10
Original f
u
Recovered f

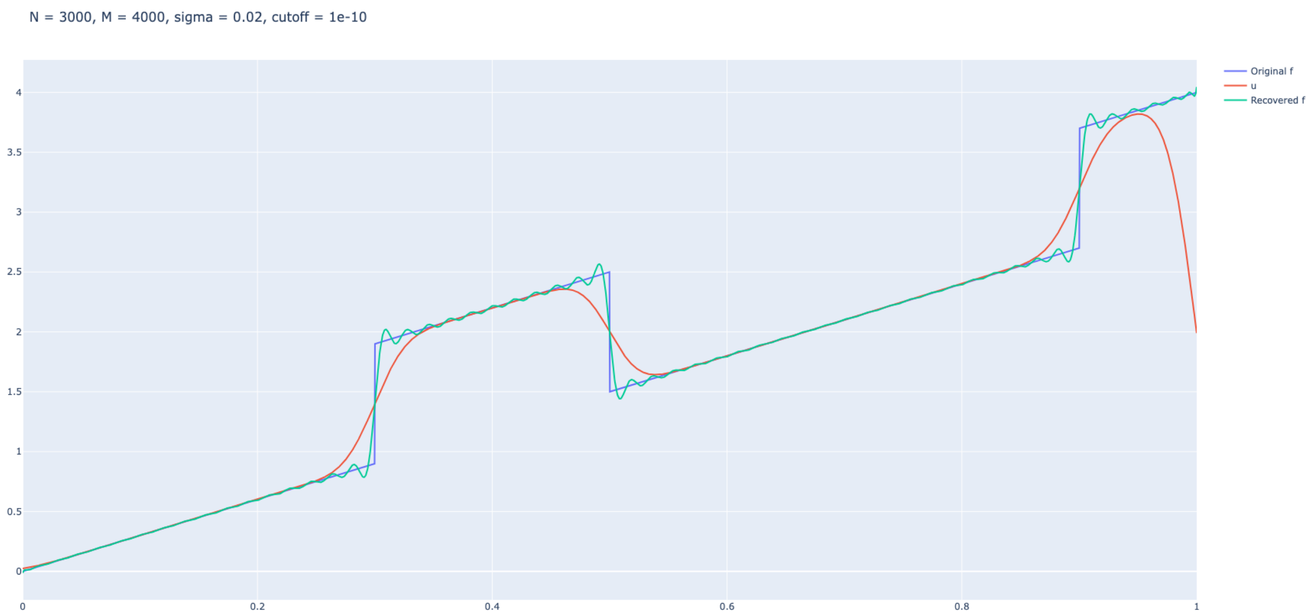

N = 3000, M = 4000, sigma = 0.05, cutoff = 1e-15
Original f
u
Recovered f



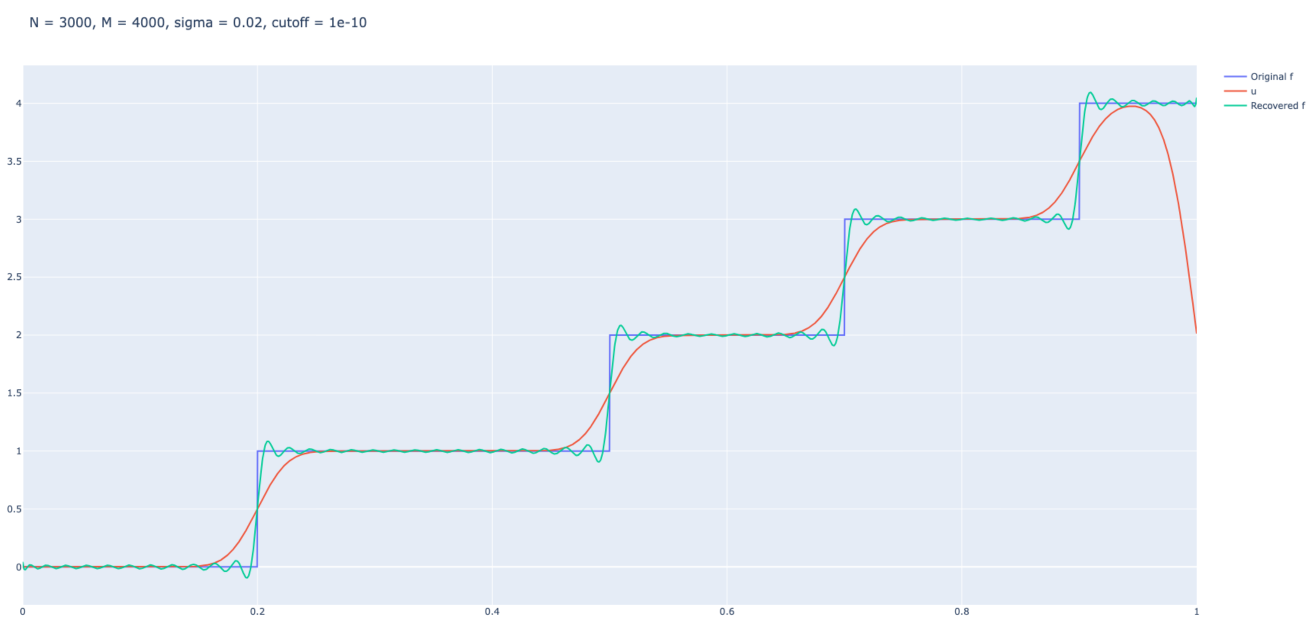


Also, just for fun, I included a few cases where I modified f a little bit:

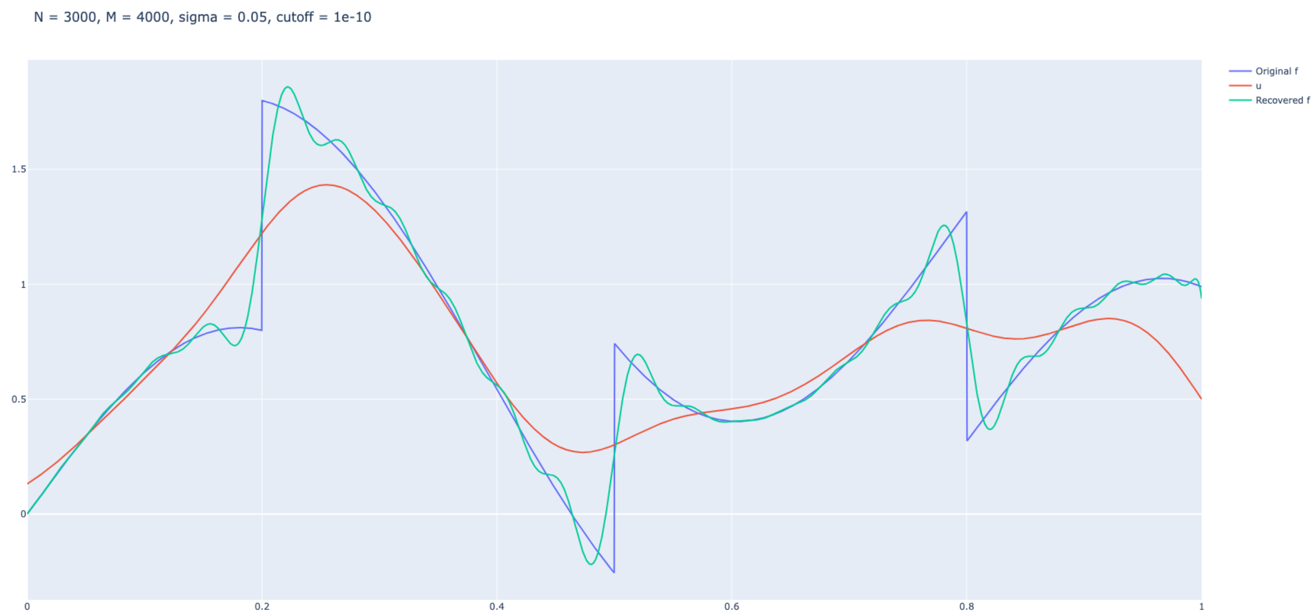
$$f(x) = H(x - 0.3) - H(x - 0.5) + H(x - 0.9) + 3x$$



$$f(x) = H(x - 0.2) + H(x - 0.5) + H(x - 0.7) + H(x - 0.9)$$



$$f(x) = -x + \sin(8x) + H(x - 0.2) + H(x - 0.5) - H(x - 0.8)$$



As we can see the discretisation and the singular value decomposition do a really good job at deblurring the images.