1. Metric spaces and Banach's FPT (Fixed Point Theorem)

Def 1.1 (Metric and Metric Space)

Let X be a nonempty set, and $d : X^2 \to \mathbb{R}$ be a function satisfying:

- $(d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$
  Then the function d is called the metric, the pair (X, d) is called the metric space, and the number $d(x, y)$ is called the distance between x and y in X.

Examples

- $(\mathbb{R}^n, d)$, with $d(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$
- $(\mathbb{R}^n, d)$, with $d(x, y) = \max\limits_{i \leq n} |x_i - y_i|$
- $(C[a, b], d)$, with $d(f, g) = \left(\int_a^b |f(x) - g(x)|^2 dx\right)^{\frac{1}{2}}$
- $(C[a, b], d)$, with $d(f, g) = \sup\limits_{a \leq x \leq b} |f(x) - g(x)|$
- $(L^p[a, b], d)$, with $d(f, g) = \left(\int_a^b |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}$

Def 1.2 (Fixed Point)

A fixed point of the mapping $T : X \to X$ is the point $x^* \in X$ such that $T(x^*) = x^*$.

Def 1.3 (Contraction)

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X of there exists a constant $0 < k < 1$ such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for all $x, y \in X$.

Theorem 1.1 (Banach's FPT)

Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x^* \in X$.

Corollary

The iterative sequence $x_{n+1} = T(x_n)$ for $n = 1, 2, \ldots$ with arbitrary starting point $x_0 \in X$ converges, under assumptions of Banach's FPT, to the unique fixed point of T. Moreover, the following estimates hold:

- $d(x_m, x^*) \leq \frac{k^m}{1-k} d(x_1, x_0)$ - the prior estimate,
- $d(x_m, x^*) \leq \frac{k}{1-k} d(x_{m-1}, x_m)$ - the posterior estimate.

2. Applications of Banach's FPT

Applications to real-valued functions

Let $g \in C^1[a, b]$, and suppose we want to find the solution to the equation $g(x) = 0$ on $[a, b]$. We note that we can always rewrite this equation as $x = g(x) + x$, and then out problem is equivalent with finding a fixed point of the function $f(x) = x + g(x)$.

Theorem 2.1 (Differentiable Contraction)

Let $(\mathbb{R}, d)$ be a metric space of real numbers with the metric $d(x, y) = |x - y|$ and let $[a, b]$ be a closed interval in $\mathbb{R}$. Moreover, let $f : [a, b] \to [a, b]$ be a continuous and differentiable function such that $\sup\limits_{x \in [a,b]} |f'(x)| \leq k < 1$. Then there exists a unique fixed point $x^* \in [a, b]$ of f.

Example

We want to find the solution to the equation $cos(x) - 2x = 0$ on $[0, \pi]$. Then we can write this equation as $x = \frac{1}{2} cos(x)$, and try to find the fixed point of the function $f(x) = \frac{1}{2} cos(x)$ on $[0, \pi]$. We have to show that f is a contraction on $[0, \pi]$. To do so, we apply the theorem 2.1. We have

$$\sup_{x \in [0,\pi]} |f'(x)| = \sup_{x \in [0,\pi]} \left| -\frac{1}{2} sin(x) \right| = \frac{1}{2} < 1.$$

We have shown that f is a contraction and, by the Banach's FPT, it has a fixed point x^* that is the limit of the sequence $\{x_n\}$ generated by the scheme $x_{n+1} = f(x_n)$ with any starting point $x_0 \in [0, \pi]$.
Note that to show that f is a contraction we could also directly apply the definition:

$$\begin{aligned} |f(x) - f(y)| = \left| \frac{1}{2} cos(x) - \frac{1}{2} sin(x) \right| &= \left| sin\left(\frac{x+y}{2}\right) sin\left(\frac{x-y}{2}\right) \right| \\ &\leq \sup_{x,y \in [0,\pi]} \left| sin\left(\frac{x+y}{2}\right) \frac{1}{2} |x-y| \right| = \frac{1}{2}|x-y| \leq |x-y|. \end{aligned}$$

Applications to integral equations

We consider integral equations in the following form

$$f(x) = g(x) + \mu \int_a^b k(x,y) f(y) dy,$$

where $f : [a,b] \to \mathbb{R}$ is an unknown function, $g : [a,b] \to \mathbb{R}$, and $k : [a,b]^2 \to \mathbb{R}$ are given functions, and μ is a parameter.
The above integral equation can be considered in various function spaces. Here we consider this equation only in $(C[a,b], d)$ with $d(f,g) = \sup\limits_{x \in [a,b]} |f(x) - g(x)|$.

We assume that $g \in C[a,b]$, and that the kernel k is continuous on the square $[a,b]^2$, which implies that k is bounded on $[a,b]^2$, meaning that there exists a constant c, such that $|k(x,y)| \leq c$ for all $(x,y) \in [a,b]^2$.

Theorem 2.2

The metric space $(C[a,b], d)$ is complete

Note that our integral equation can be rewritten as $T(f) = f$, where

$$T(f)(x) = g(x) + \mu \int_a^b k(x,y) f(y) dy.$$

First we have to show that the mapping $T : C[a,b] \to C[a,b]$ is well-defined, but this is obvious, as g and k are both continuous on their domains. Let us now determine for which values of μ the map T is a contraction. We have

$$\begin{aligned} d(T(f_1), T(f_2)) = \sup_{x \in [a,b]} |T(f_1)(x) - T(f_2)(x)| = \sup_{x \in [a,b]} |\mu| \left| \int_a^b k(x,y)(f_1(y) - f_2(y)) dy \right| \leq \\ \leq |\mu| \sup_{x \in [a,b]} \int_a^b |k(x,y)| |f_1(y) - f_2(y)| dy \leq c|\mu| \sup_{x \in [a,b]} |f_1(x) - f_2(x)| \int_a^b dy = \\ = c|\mu|(b-a) d(f_1, f_2). \end{aligned}$$

It is now required that $c|\mu|(b-a) < 1$, or $|\mu| < \frac{1}{c(b-a)}$, for T to be a contraction. Applying the Banach's FPT, we see that the map T has a unique fixed point $f^* \in C[a,b]$.