# 1. PDEs

## Def 1.1 (PDE)

A PDE is a relation that involves the unknown function $u(x_1, x_2, \ldots, x_n, t)$, where $(x_1, x_2, \ldots, x_n)$ is the spacial coordinate, and $t$ is the time, with its derivatives.
Symbolically we have

$$F(x_1, x_2, \ldots, x_n, t, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots) = 0,$$

for example

$$u_t + u_x + u = 0,$$

$$u_t = u_{xx},$$

etc.

# 2. Method of characteristics

## Def 2.1 (Quasilinear 1st order PDE)

A quasilinear first order PDE is

$$u_t + c(x, t, u)u_x = f(x, t, u),$$

where $u = u(x, t)$, $x \in \mathbb{R}$, $t > 0$. Moreover, $c$ and $f$ are known.
If $c = c(x, t)$, we call the above equation a semilinear equation, and additionally, if $f = f(x, t)$, we call it linear.

### Remarks

- Quasilinear means "linear" in the derivatives. So that the nonlinearity involves only $u$
- We can consider a more general equation $d(x, t, u)u_t + c(x, t, u) = f(x, t, u)$, but we assume it is always possible to divide by $d$.
- A general balance law looks like the following

$$u_t + q_x(x, t, u) = g(x, t, u).$$

- Function $c$ is actually the velocity of the wave, and we will see that later.

## Def 2.2 (Method Of Characteristics)

Let $X_\xi(t)$ be an arbitrary curve on the $x-t$ plane. Define $U_\xi(t) = u(X_\xi(t), t)$, where $u$ is the solution of our equation. Now differentiate:

$$U'_\xi(t) = u_t + X'_\xi(t) u_x.$$

Notice that this is exactly the left hand side of our equation, provided we have $X'_\xi(t) = c(X_\xi(t), t, U_\xi(t))$. Then we have

$$U'_\xi(t) = f(X_\xi(t), t, U_\xi(t)).$$

We have thus reduced our equation into an ODE

$$\begin{aligned} X'_\xi(t) &= c(X_\xi(t), t, U_\xi(t)), \\ U'_\xi(t) &= f(X_\xi(t), t, U_\xi(t)). \end{aligned}$$

These are the so-called characteristics equation, and $X_\xi(t)$ are called the characteristics.
We have to impose initial conditions. Assume that $u(x,0) = \phi(x)$, then let $X_\xi(0) = \xi$, and obtain $U_\xi(0) = u(X_\xi(0), 0) = \phi(X_\xi(0)) = \phi(\xi)$.
In order to find the solution at any $(x,t)$ we find a unique characteristic $X_\xi(t)$ passing through $(x,t)$, go back to $t=0$, and use the initial condition, then read the solution from $U_\xi(t)$.

# 3. Heat equation

Consider a homogenous initial-boundary value problem (IBV):

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in (0,L) \times (0,T], \\ u(x,0) = \varphi(x), & x \in [0,L], \\ u(0,t) = u(L,t) = 0, & t \in [0,T], \end{cases}$$

assume that $\varphi$ is continuous on $[0,L]$, and that $\varphi(0) = \varphi(L) = 0$.
The Fourier's method gives

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-\frac{\alpha^2 n^2 \pi^2}{L^2} t\right),$$

where

$$A_n = \frac{2}{L} \int_0^L \varphi(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

## Theorem 3.1 (Classical solution)

Let $\varphi \in C([0,L])$, and $\varphi(0) = \varphi(L) = 0$. Then $u(x,t)$ defined by the above is a classical solution of homogenous IBV problem. Moreover $u \in C^\infty([0,L] \times [0,T])$.

## Definition 3.2 (Green's function)

Notice that

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\int_0^L \frac{2}{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right)\exp\left(-\frac{\alpha^2n^2\pi^2}{L^2}t\right)\varphi(y)dy = \\
&= \int_0^L \sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right)\exp\left(-\frac{\alpha^2n^2\pi^2}{L^2}t\right)\varphi(y)dy = \\
&= \int_0^L G(x,y,t)\varphi(y)dy.
\end{aligned}
$$

We say that $G$, defined by $G(x,y,t) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right)\exp\left(-\frac{\alpha^2n^2\pi^2}{L^2}t\right)$ is the Green's function for homogenous IBV for the heat equation.

Now let's consider a non-homogenous heat equation:

$$
\begin{cases}
u_t = \alpha^2 u_{xx} + f(x,t), & (x,t)\in(0,L)\times(0,T], \\
u(x,0) = \varphi(x), & x\in[0,L], \\
u(0,t) = u(L,t) = 0, & t\in[0,T],
\end{cases}
$$

Suppose that

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi}{L}x\right), \\
f(x,t) &= \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi}{L}x\right).
\end{aligned}
$$

Plugging into the equation, and integrating yields

$$
u(x,t) = \int_0^t\int_0^L G(x,y,t-\tau)f(y,\tau)dyd\tau.
$$

## Theorem 3.3 (Green's function for heat equation in $\mathbb{R}$)

The Green's function given by

$$
G(x,y,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x-y)^2}{4t}\right)
$$

is:

- the Green's function for heat equation in $\mathbb{R}$,
- the fundamental solution of the heat equation,
- the Gauss characteristic kernel,
- the heat kernel.