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1.1

$$n^{\delta}(x) = \sqrt{2}\delta \sin(2\pi k x).$$

$$\left\| f - f^{\delta} \right\| = ||n^{\delta}||$$

$$\left\| f' - (f^{\delta})' \right\| = ||(n^{\delta})'||$$

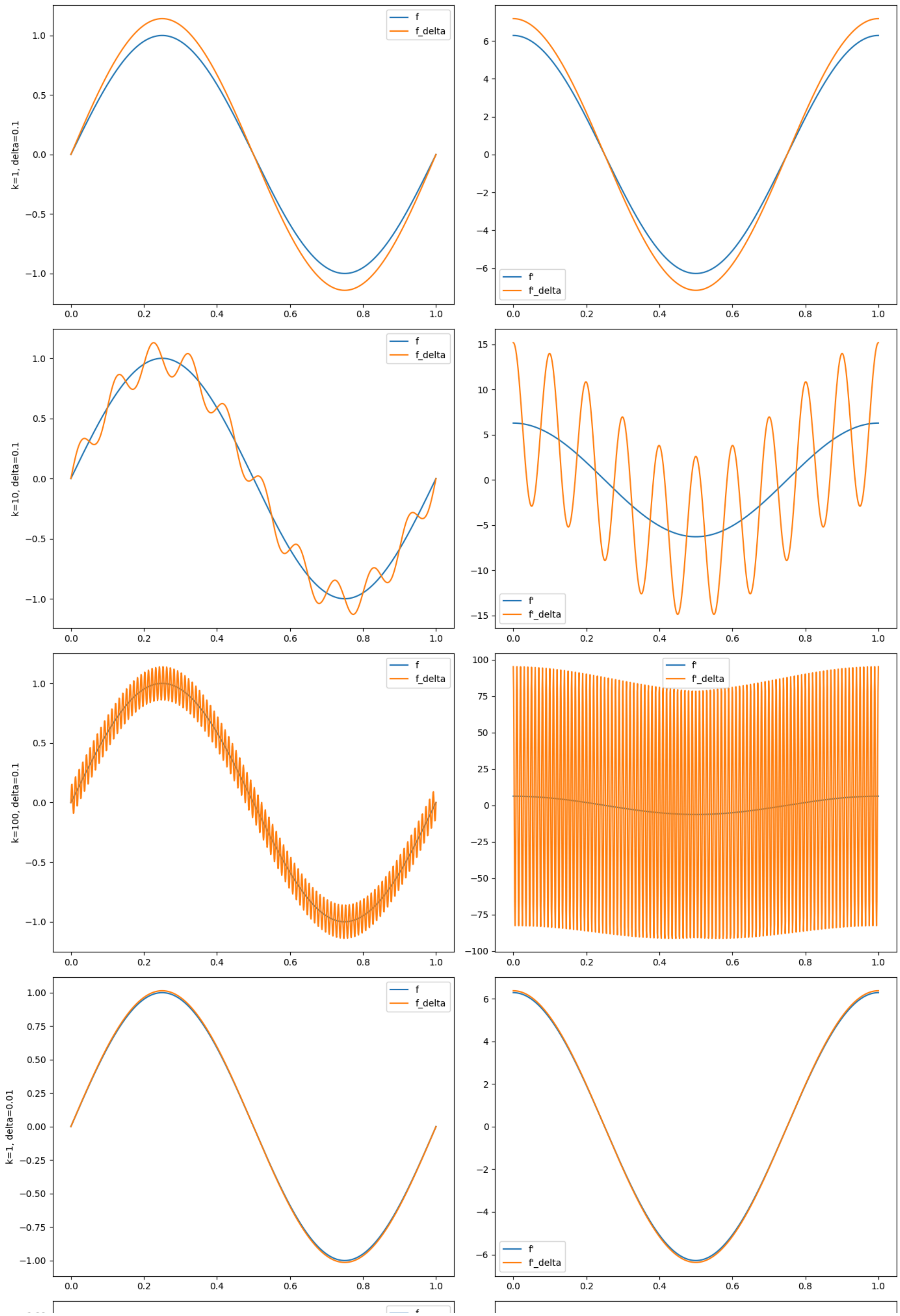
$$||f - f^{\delta}||_{L^2}^2 = \int_0^1 2\delta^2 \sin(2\pi k x)^2 dx = \delta^2 \left(1 - \frac{1}{4\pi k}\sin(4\pi k)\right) \to 0.$$

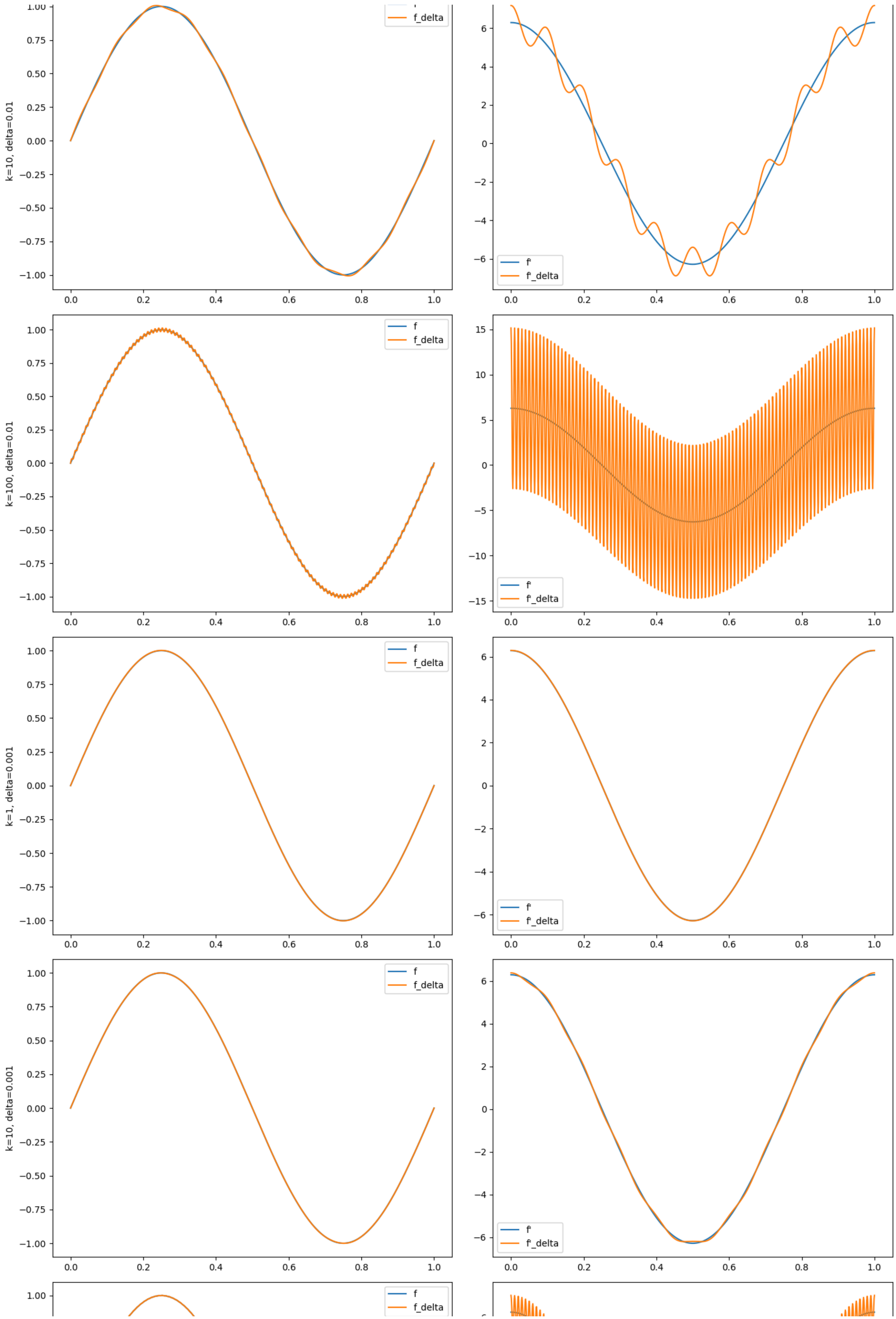
$$||f' - (f^{\delta})'||_{L^2}^2 = \int_0^1 8\delta^2 \pi^2 k^2 \cos(2\pi k x) dx = \pi\delta^2 k\left(4\pi k + \sin(4\pi k)\right) \to \infty.$$

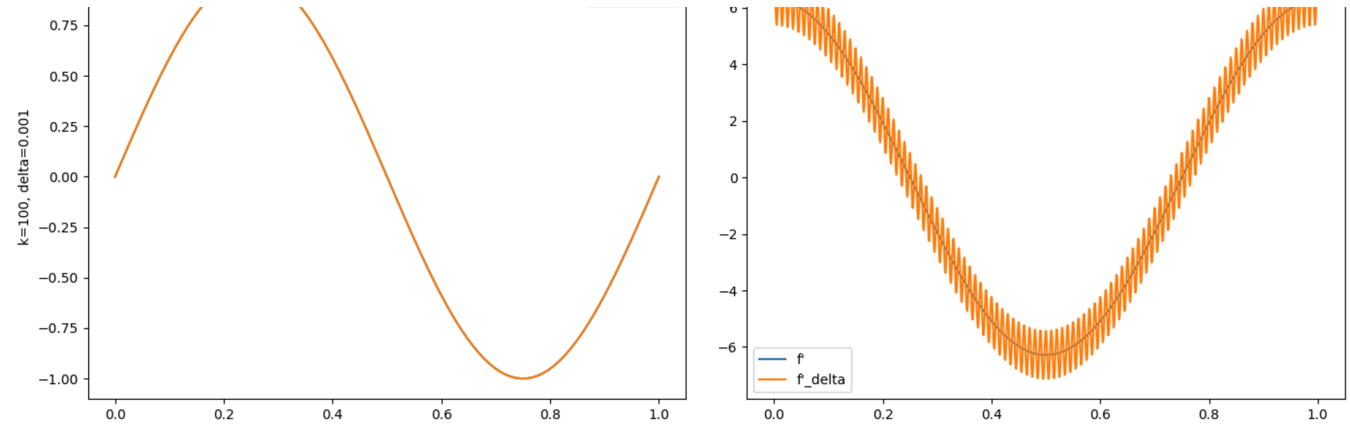
$$||f - f^{\delta}||_{L^\infty} = \max_{x\in(0,1)} \left\{\sqrt{2}\delta \sin(2\pi k x)\right\} = \sqrt{2}\delta \to 0.$$

$$||f' - (f^{\delta})'||_{L^\infty} = \max_{x\in(0,1)} \left\{2\sqrt{2}\delta\pi k \cos(2\pi k x)\right\} = 2\sqrt{2}\delta\pi k \to \infty.$$

1.2


k=1, delta=0.1
k=10, delta=0.1
k=100, delta=0.1
k=1, delta=0.01
f
f_delta
f'
f'_delta


k=10, delta=0.01
k=100, delta=0.01
k=1, delta=0.001
k=10, delta=0.001
f
f_delta
f'
f'_delta



1.3

Let's estimate the error that we introduce when calculating the derivatives numerically using Euler central difference scheme, that is, let's estimate

$$E_f = \left\Vert f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right\Vert_{L^\infty}.$$

We know that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi^x_+), \text{ where } \xi^x_+ \in [x, x+h],$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(\xi^x_-), \text{ where } \xi^x_- \in [x-h, x]. \$\$ Fromthatweget$$

\begin{aligned}
\frac{1}{2}\left\Vert f'(x) - \frac{f(x+h) - f(x)}{h}\right\Vert{L^\infty} &\le \left\Vert f'(x) - \frac{f(x+h) - f(x)}{h}\right\Vert{L^\infty} \le \frac{h}{2}|| f''(\xi+)||{L^\infty} \le \frac{h}{2}||f''||{L^\infty} , \
\frac{1}{2}\left\Vert f'(x) - \frac{f(x) - f(x-h)}{h}\right\Vert{L^\infty} &\le \left\Vert f'(x) - \frac{f(x) - f(x-h)}{h}\right\Vert{L^\infty} \le \frac{h}{2}|| f''(\xi-)||{L^\infty} \le \frac{h}{2}||f''||{L^\infty} ,
\end{aligned}

$$and, usingthetriangleequalityweobtainanestimate$$

\begin{aligned}
Ef = \left\Vert f'(x) - \frac{f(x+h) - f(x-h)}{2h}\right\Vert{L^\infty} &= \left\Vert \frac{f'(x)}{2} - \frac{f(x+h) - f(x)}{2h} + \frac{f'(x)}{2} - \frac{f(x) - f(x-h)}{2h}\right\Vert{L^\infty} \le \
&\le \frac{h}{2}||f''||{L^\infty} + \frac{h}{2}||f''||{L^\infty} = h||f''||{L^\infty}.
\end{aligned}

$$Now, let's estimate the error when we differentiate data with noise, that is$$

\begin{aligned}
&\left\Vert f'(x) - \frac{f^\delta(x + h) - f^\delta(x - h)}{2h} \right\Vert{L^\infty} = \=&\left\Vert f'(x) -

\frac{f(x + h) - f(x - h)}{2h} - \frac{n^\delta(x + h) - n^\delta(x - h)}{2h} \right\Vert{L^\infty} \le \
\le &\left\Vert f'(x) - \frac{f(x + h) - f(x - h)}{2h}\right\Vert{L^\infty} + \left\Vert\frac{n^\delta(x + h) - n^\delta(x - h)}{2h} \right\Vert{L^\infty} \le \
\le & h||f''||{L^\infty} + \frac{1}{2h}\left\Vert n^\delta(x + h) - n^\delta(x - h) \right\Vert{L^\infty}.
\end{aligned}

Now, ifwedidn'tknowanythingaboutthenoisen^δ, wewouldestimatethisbyusingtheMinkowskiinequa

n^\delta(x+h) - n^\delta(x-h) = 2\sqrt{2}\delta\cos(2 \pi k x)\sin(2 \pi k h),

hence

\left\Vert n^\delta(x+h) - n^\delta(x-h) \right\Vert_{L^\infty} = 4\sqrt{2}\pi\delta kh,

since$sin(x) \approx x$forsmall$x$. NowtheL^∞normoftheerrorcanbeestimatedby

\left\Vert f'(x) - \frac{f^\delta(x + h) - f^\delta(x - h)}{2h} \right\Vert{L^\infty} \le h||f''||{L^\infty} + 2\sqrt{2}\pi\delta k,

which, inourconcreteexample, with

f(x) = \sin(2\pi x),

becomes

4\pi^2h + 2\sqrt{2}\pi\delta k.

1.4 ![[sub_04.png]] As we can see, no matter how small h we take, the differentiation error will not go dow

\alpha = \frac{\delta}{\int_0^1\left|f''(x)\right|^2dx}.

Inourcase$f(x) = \sin(2\pi x)$, so$f''(x) = -4\pi^2\sin(2\pi x)$, hence

\int_0^1|f''(x)|^2dx = 4\pi^2\int_0^1|\sin(2\pi x)|^2dx = 2\pi^2.

Inourcase, $\delta = 0.01$, so

\alpha_{opt} = \frac{0.01}{2\pi^2} \approx 0.0005066.

Thenumericalexperimentyielded0.0007939698492462312, whichisinthesameorderofmagnitude.