Applied Functional Analysis - Exercise sheet 6

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Exercise 1

Let's prove that the functional J given by

$$J(u) = \frac{1}{2}\int_0^1 (u(x) - f_\delta(x))^2 dx + \frac{\alpha}{2}\int_0^1 |u'(x)|^2 dx$$

has a unique minimum, which is achieved at the function $u_\alpha \in H_0^1(0,1)$, which is a weak solution to the problem

$$\begin{cases} u - \alpha u'' = f_\delta, \\ u(0) = u(1) = 0. \end{cases}$$

To that end notice that

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^1 u(x)^2 dx + \frac{\alpha}{2}\int_0^1 |u'(x)|^2 dx - \int_0^1 u(x) f_\delta(x) dx + \frac{1}{2}\int_0^1 f_\delta(x)^2 dx = \\ &= J^*(u) + \frac{1}{2}\int_0^1 f_\delta(x)^2 dx = J^*(u) + C, \end{aligned}$$

where C is independent of u. This means that the u that minimises J is the same u that minimises J^*. Let's take a look at the differential equation, we have

$$u - \alpha u'' = f_\delta.$$

Now multiply the equation by a function $v \in H_0^1(0,1)$:

$$uv - \alpha u'' v = f_\delta v,$$

and integrate both sides:

$$\int_0^1 uv dx - \alpha \int_0^1 u'' v dx = \int_0^1 f_\delta v dx.$$

Using integration by parts we get

$$\int_0^1 uv dx + \alpha \int_0^1 u' v' dx = \int_0^1 f_\delta v dx,$$

or

$$a(u, v) = l(v),$$

with

$$a(u,v) = \int_0^1 uv dx + \alpha \int_0^1 u'v' dx, \quad l(v) = \int_0^1 f_\delta v dx.$$

Notice, that $J(u) = \frac{1}{2}a(u,u) - l(u)$. We can also see that

$$a(u,v) = \int_0^1 uv dx + \alpha \int_0^1 u'v' dx = \int_0^1 vu dx + \alpha \int_0^1 v'u' dx = a(v,u),$$

so a is symmetric, and

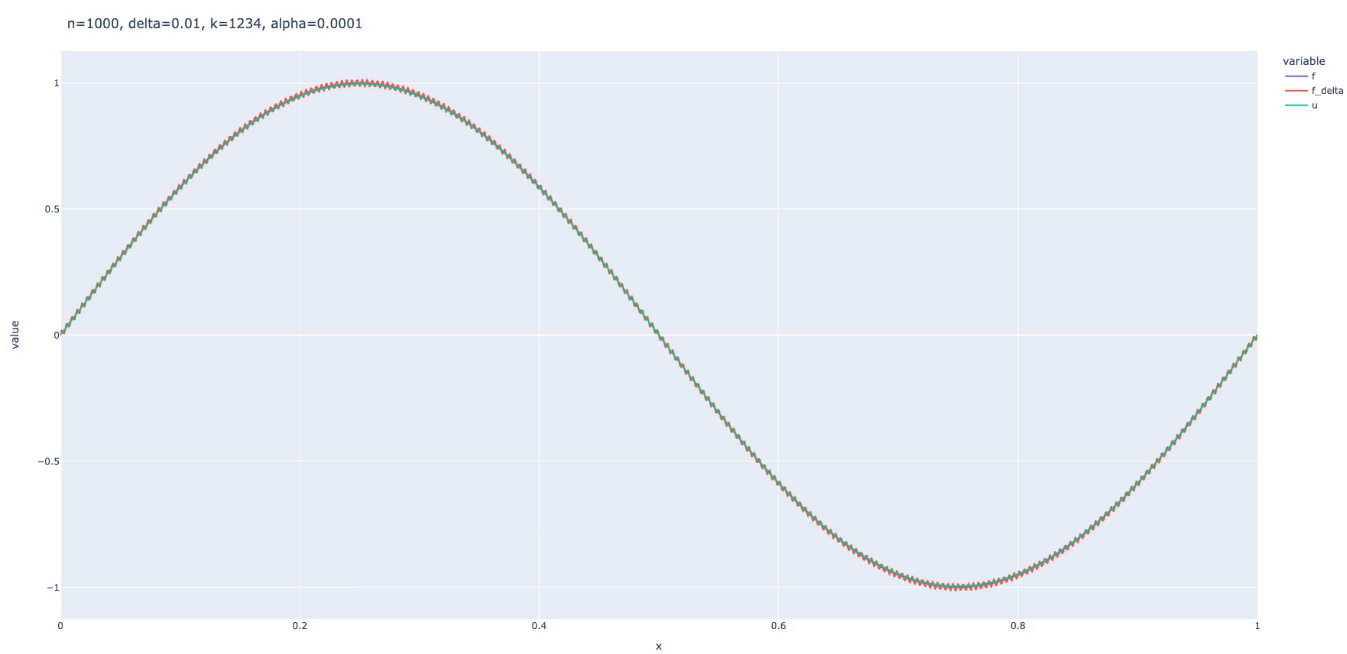
$$a(u,u) = \int_0^1 u^2 dx + \alpha \int_0^1 u'^2 \geq 0,$$

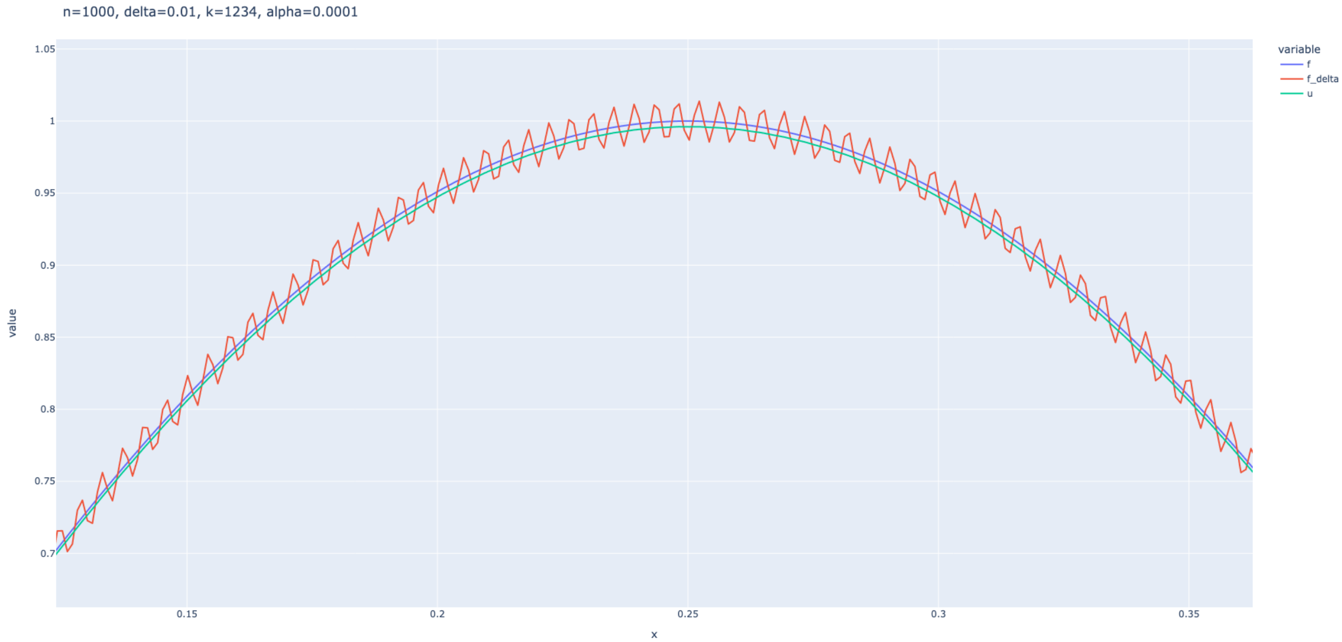
so a is also positive. Using one of the theorems presented at the lecture we can conclude that J^*, and hence J, has a unique minimum, which is a weak solution to

$$\begin{cases} u - \alpha u'' = f_\delta, \\ u(0) = u(1) = 0. \end{cases}$$

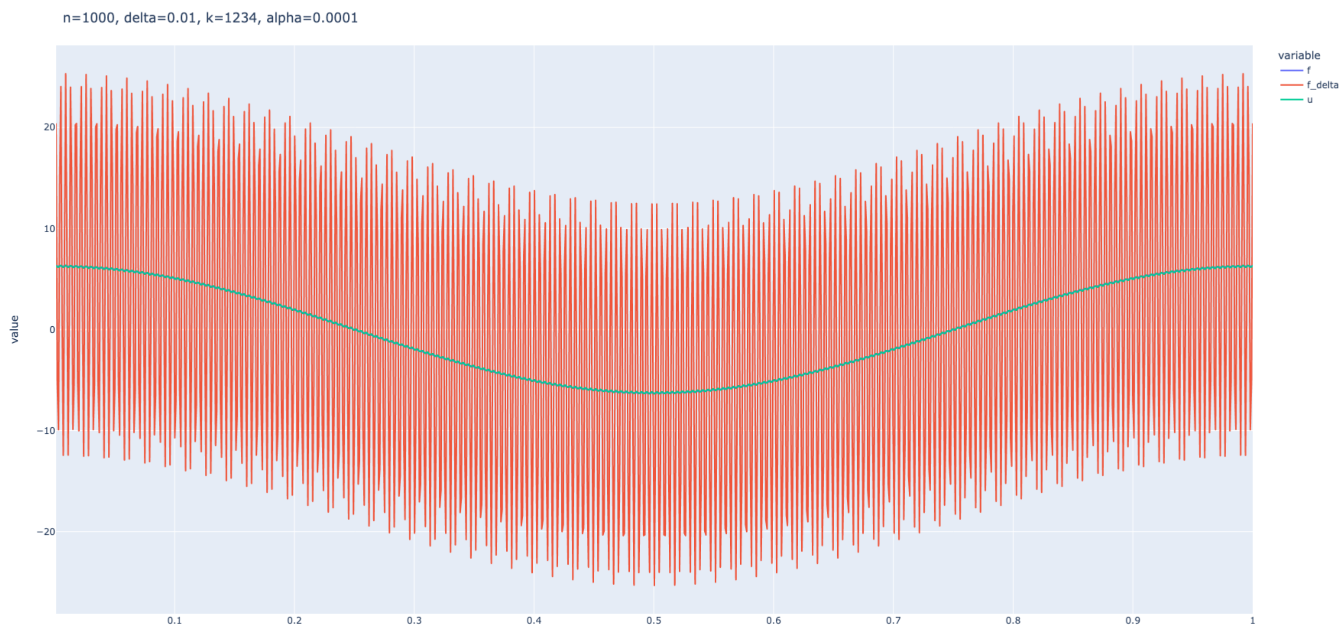
Now let's solve the above equation numerically. Let $f(x) = \sin(2\pi x)$, $n_\delta(x) = \sqrt{2}\delta \sin(2\pi k x)$, and $f_\delta(x) = f(x) + n_\delta(x)$, as in the last exercise sheet.

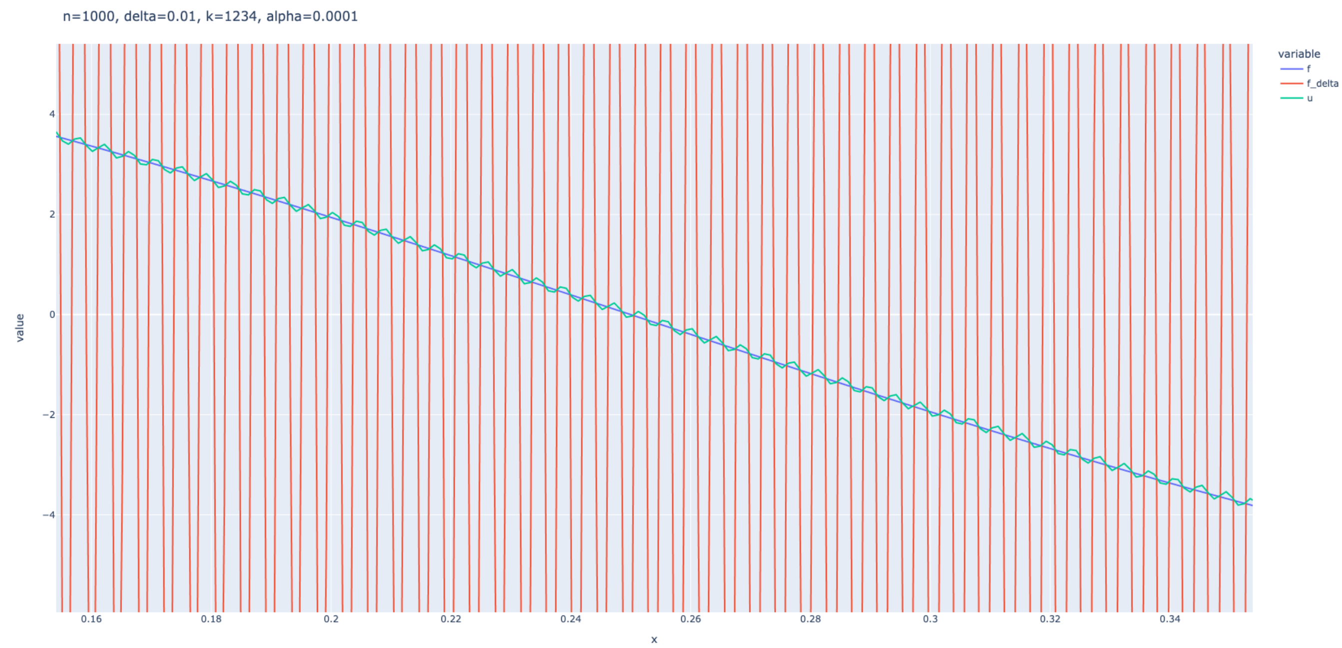
We are going to use the finite difference method described in the notes. Below we can see the plots of the solution



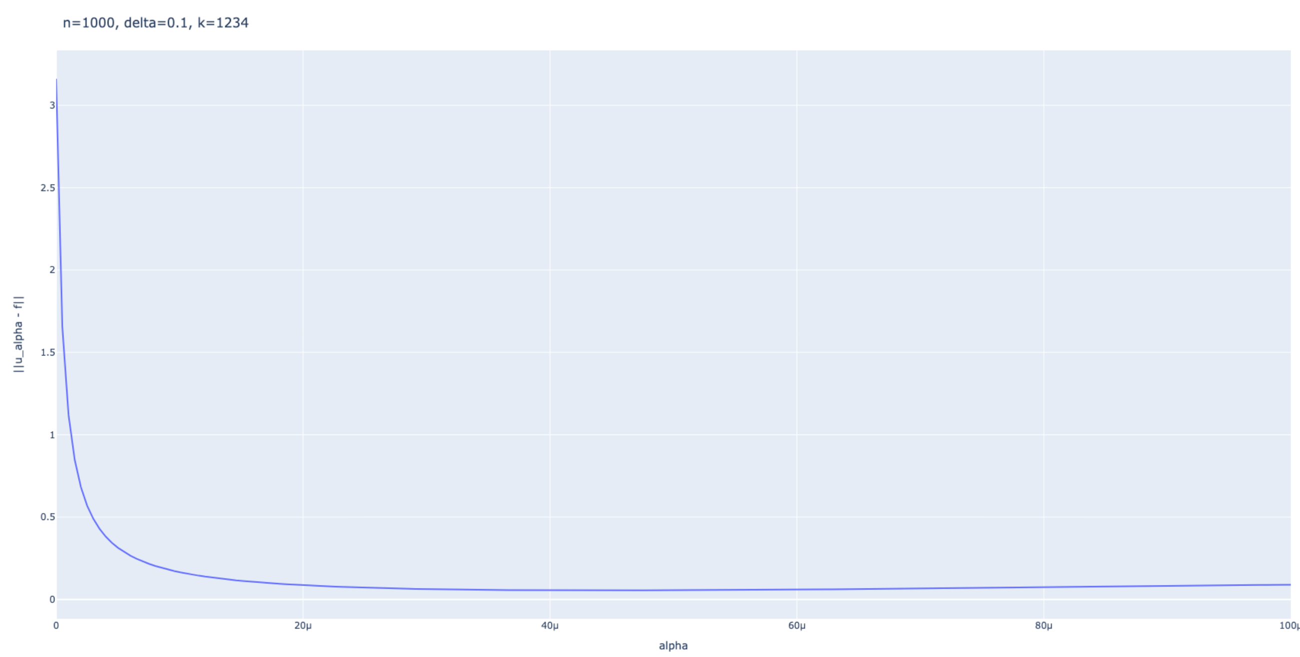


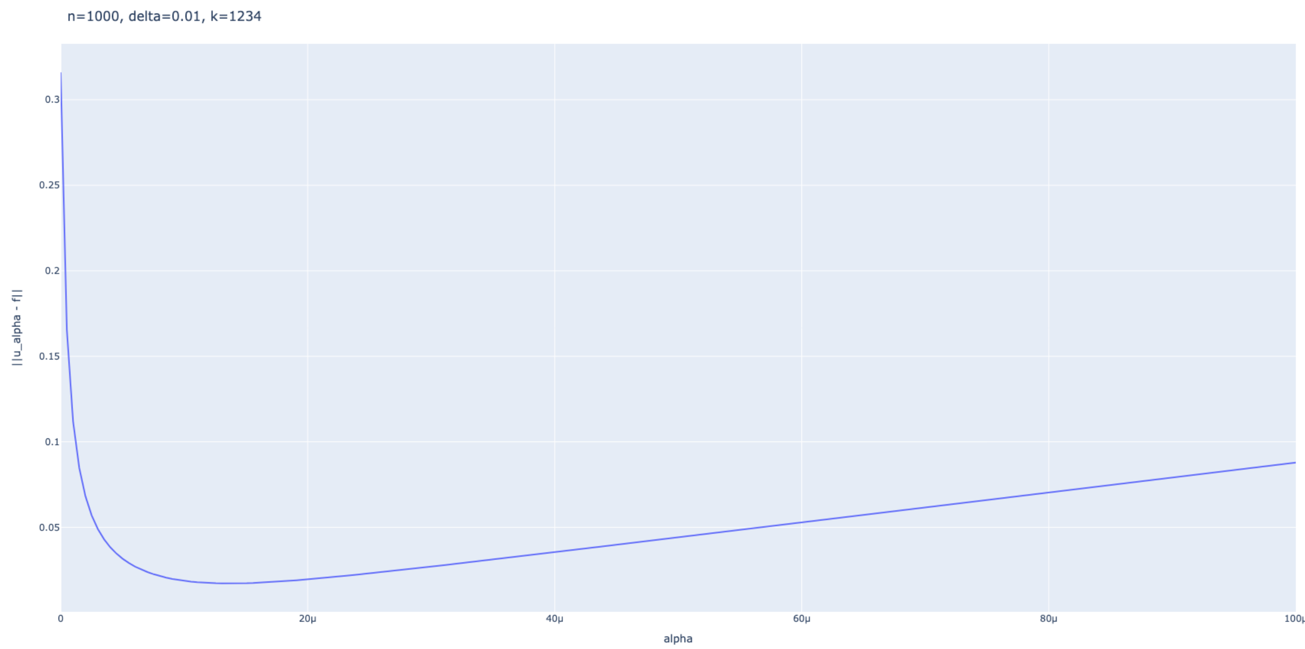
and the derivatives of the solution u, f, and f_δ:

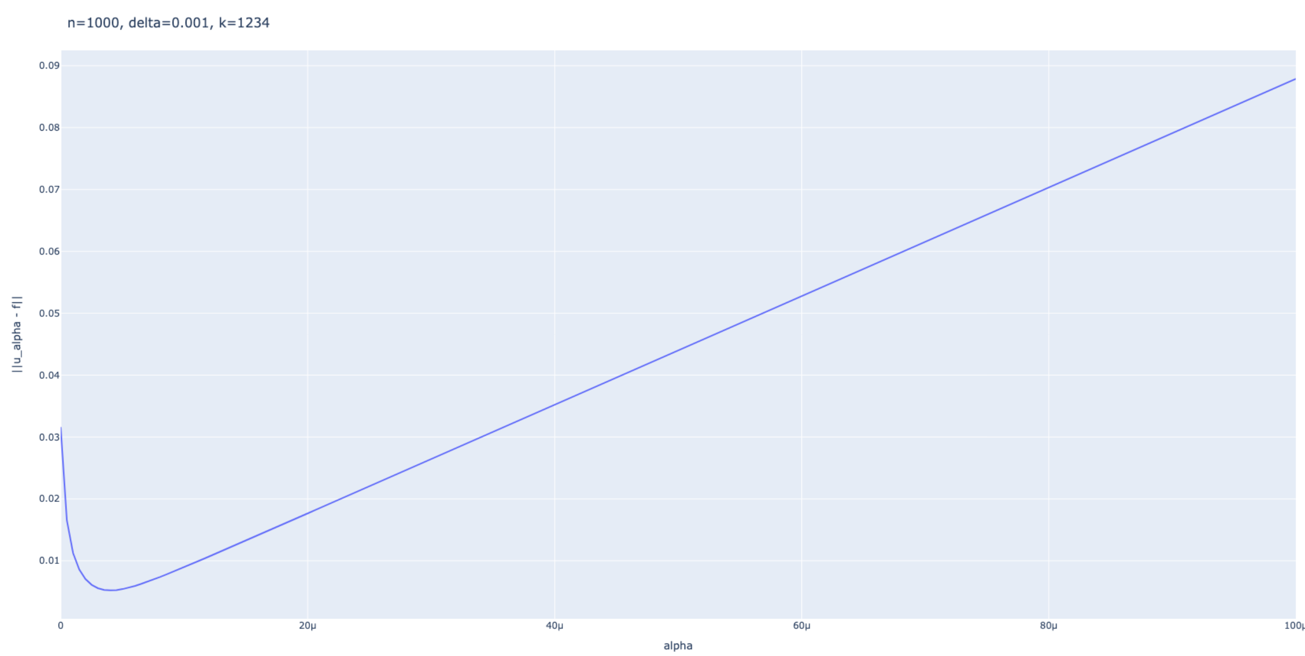




As we can see, the Tikhonov regularisation works very well in this example. We can also investigate the errors $||u_\alpha - f||$ of the numerical solution. Below we can see three plots with varying δ, where the x axis is α.







As we can see, the minimum is achieved at a very small alpha.
Unfortunately, due to time constraints, and the fact that I couldn't easily derive the bound for the norm of the difference $u'_\alpha - f'$, I will not be adding my solution here, I understand that I will get some points deducted, and I'm fine with that.

Exercise 2

Let's solve the following ordinary differential equation:

$$-\big(a(x)u'(x)\big)' = f(x),$$

for a. To that end, multiply by -1, and integrate both sides of the equation:

$$a(x)u'(x) = C - \int_0^x f(y)dy,$$

and then divide by $u'(x)$:

$$a(x) = \frac{1}{u'(x)}\left(C - \int_0^x f(y)dy\right).$$

We can see that the instability in determining a from the measurements of u might arise in a case where u is approximately constant on some interval. It would, in fact, be impossible, if u was exactly constant on a some interval.

Assume that $f(x) = -1$, $u(x) = x$, and the boundary conditions are $a(0)u'(0) = 0$, $a(1)u'(1) = 1$. The solution to that particular problem becomes

$$a(x) = x.$$

Now assume that u is perturbed to

$$u_\delta(x) = x + \delta \sin\left(\frac{x}{\delta^2}\right).$$

We have

$$u'_\delta(x) = 1 + \frac{1}{\delta}\cos\left(\frac{x}{\delta^2}\right).$$

Solution a then becomes

$$a(x) = \frac{x}{1 + \frac{1}{\delta}\cos\left(\frac{x}{\delta^2}\right)}.$$

The function u_δ approaches $u(x) = x$ as δ approaches zero, but a doesn't approach x. The $\frac{1}{\delta}\cos\left(\frac{x}{\delta^2}\right)$ part will approach infinity everywhere but at the points where $\cos\left(\frac{x}{\delta^2}\right)$ is close to zero, and so $a(x)$ will have sudden spikes wherever $\cos\left(\frac{x}{\delta^2}\right) \approx -\delta$. We can see that even small perturbations in u can have a big effect on a. Hence, the coefficient identification problem is not stable with respect to perturbations in u.

Now let's assume that functions u_1, and u_2 are solutions to the ODE at hand, with coefficients a_1, and a_2 respectively, so that

$$\begin{aligned}
-\left(a_1(x)u_1'(x)\right)' &= f(x) \\
-\left(a_2(x)u_2'(x)\right)' &= f(x).
\end{aligned}$$

Let's subtract the first equation from the other:

$$\left(a_1(x)u_1'(x) - a_2(x)u_2'(x)\right)' = 0,$$

integrating, and assuming that $a_1(0)u_1'(0) = a_2(0)u_2'(0) = 0$, we get

$$a_1(x)u_1'(x) = a_2(x)u_2'(x),$$

or

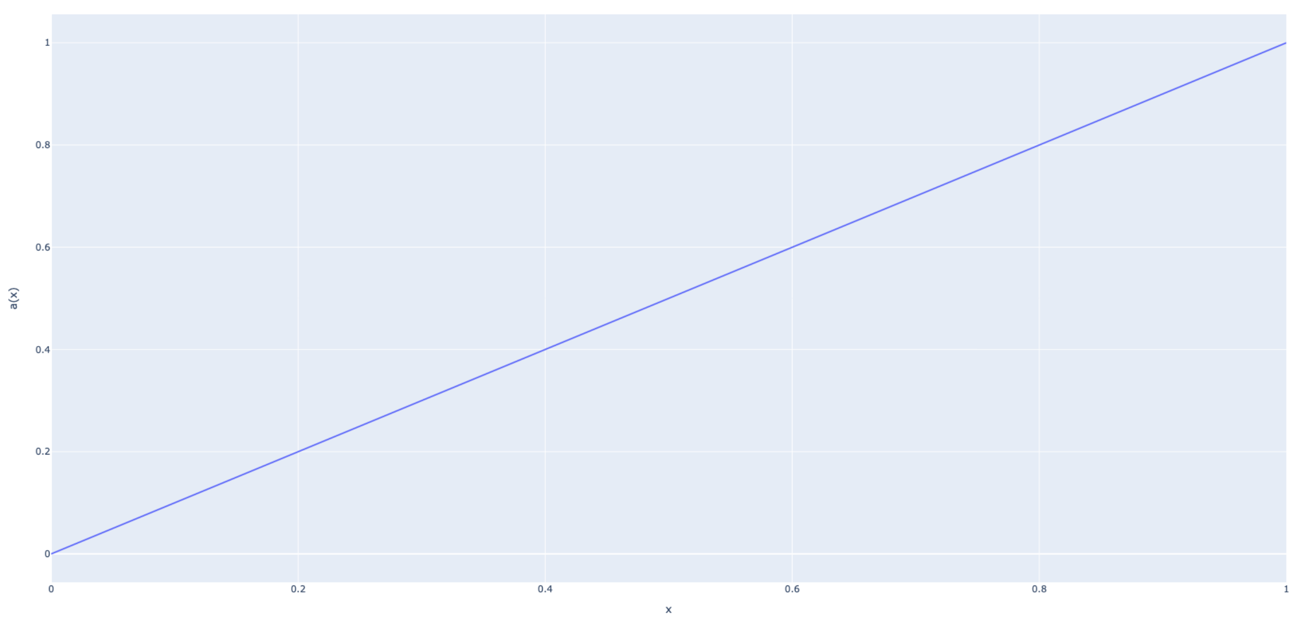
$$a_2(x) = a_1(x)\frac{u_1'(x)}{u_2'(x)},$$

so that

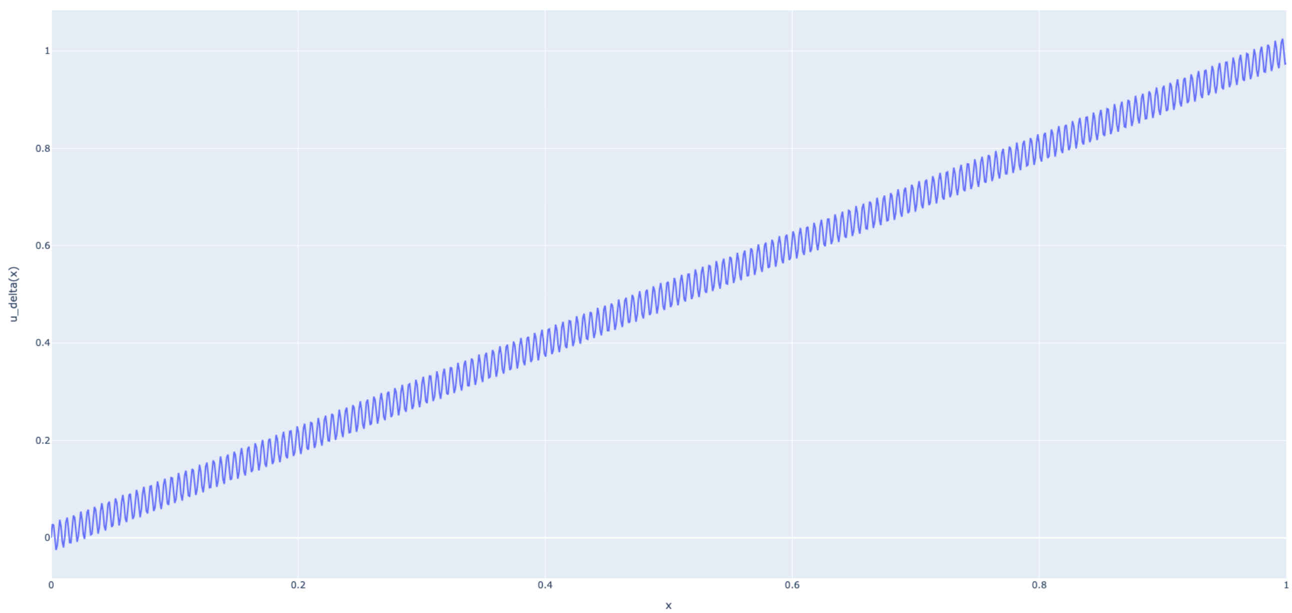
$$||a_1 - a_2|| = \left\| a_1(x)\left(1 - \frac{u_1'(x)}{u_2'(x)}\right)\right\|.$$

Now I don't know what to do with this, so I'm gonna give up right there knowing that I'll get points deducted for that.

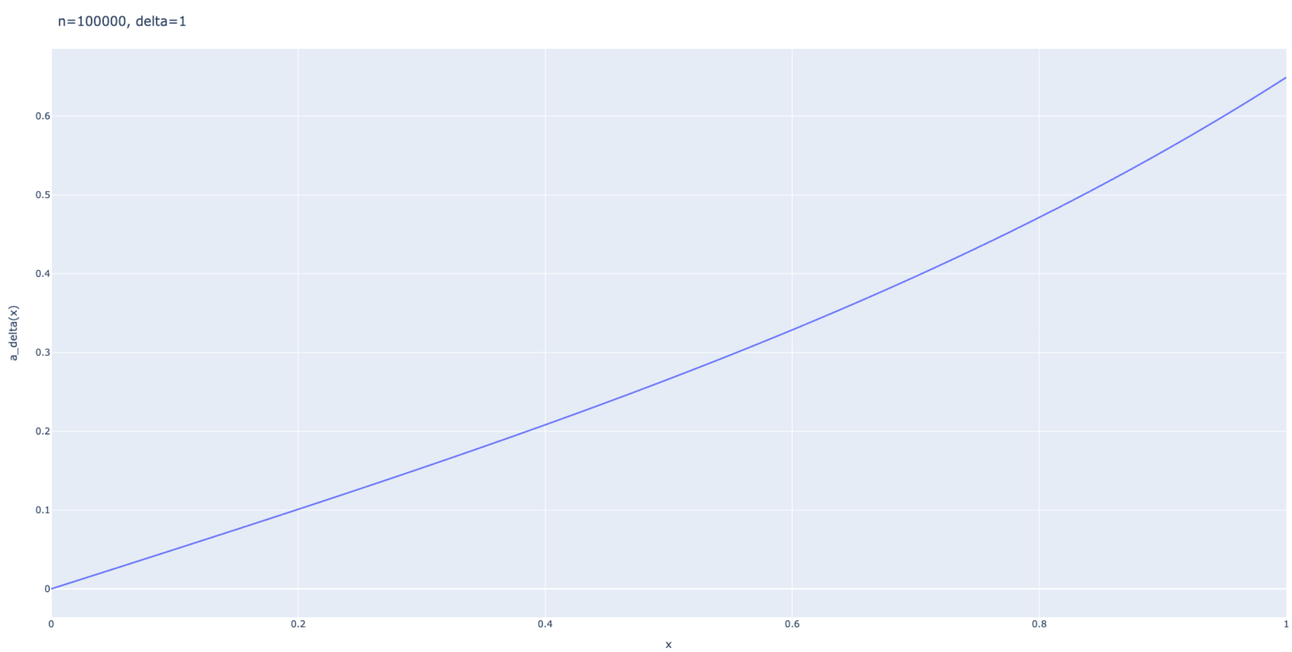
To illustrate the problem we can plot the solutions a, and a_δ. We discretise the problem on the unit interval with n equally spaced points. First, the normal solution without perturbation:

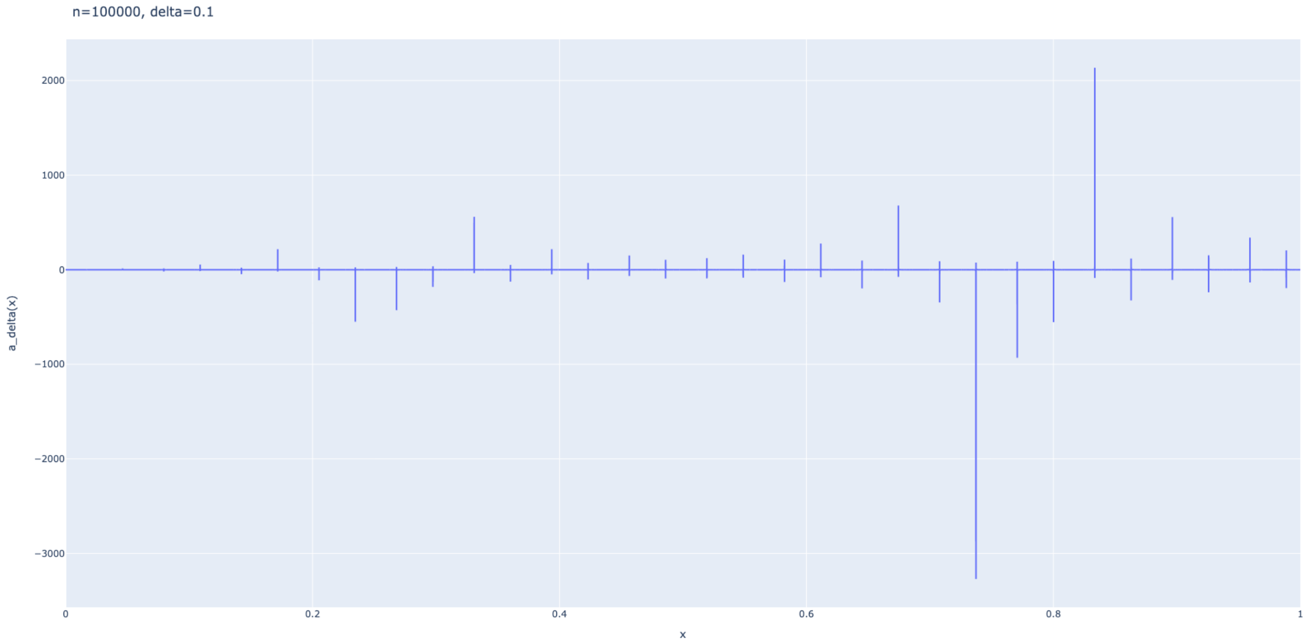


Now, let's add a perturbation. The perturbed function u_δ looks as follows:

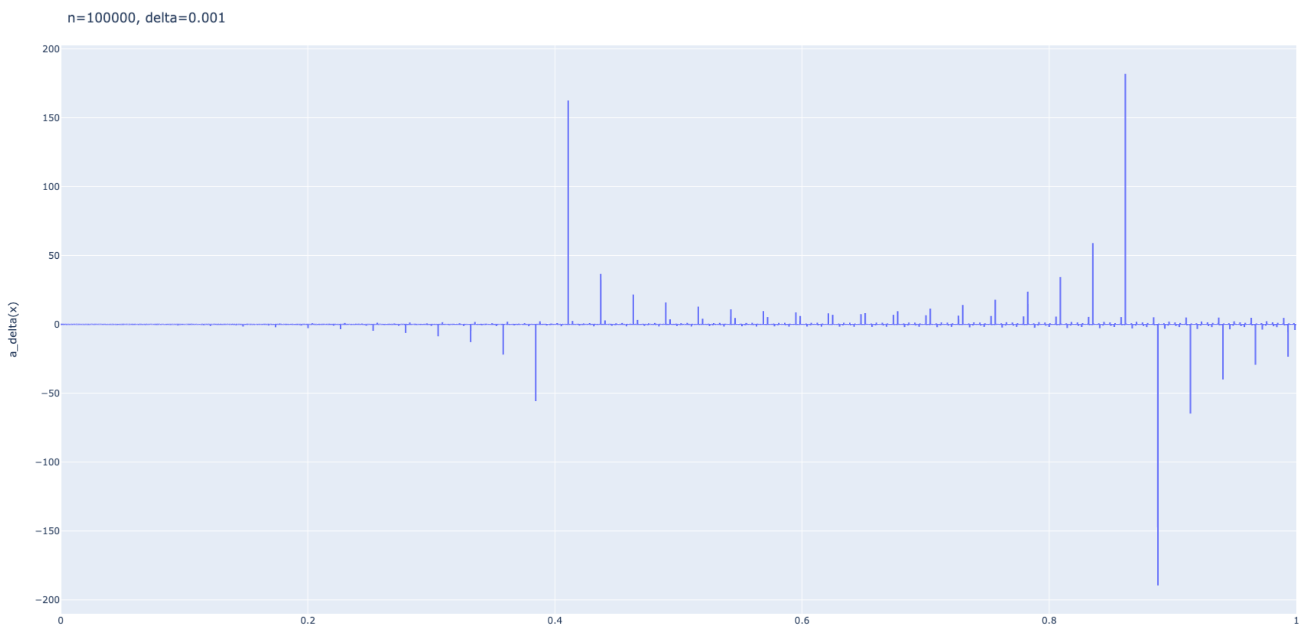


And the solution for different values of δ:

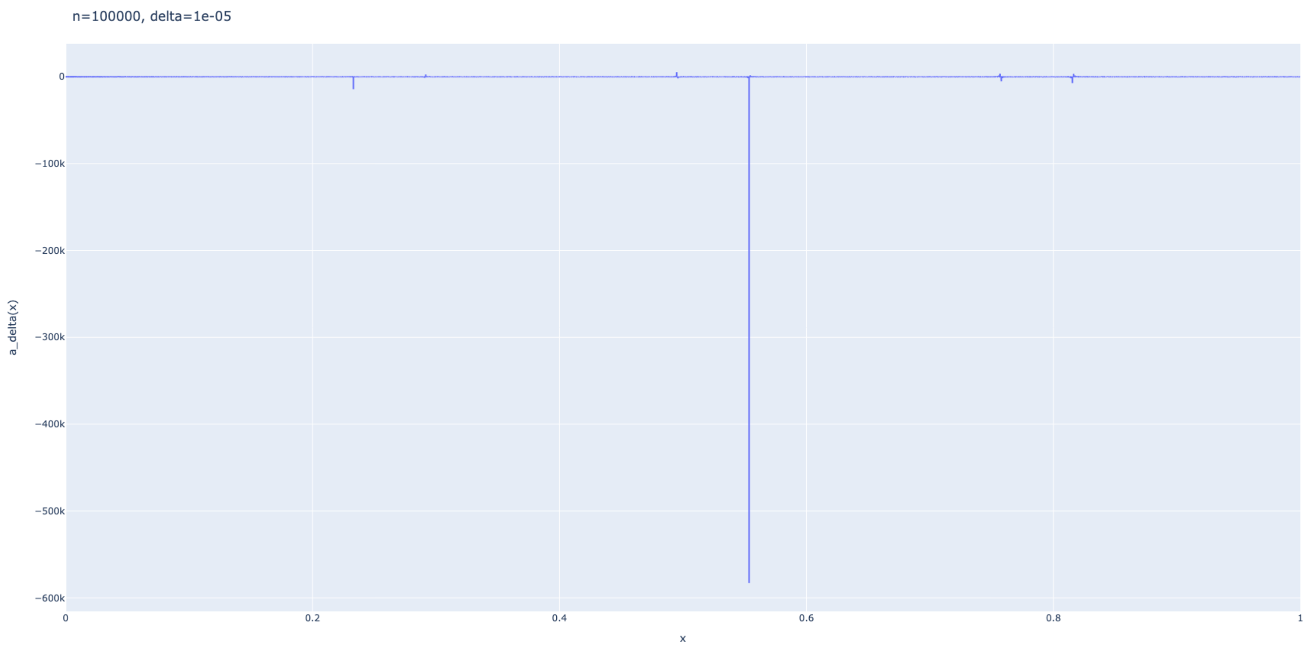




n=100000, delta=0.1
2000
1000
0
−1000
−2000
−3000
a_delta(x)
0
0.2
0.4
0.6
0.8
1
x



n=100000, delta=0.001
200
150
100
50
0
−50
−100
−150
−200
a_delta(x)
0
0.2
0.4
0.6
0.8
1
x


n=100000, delta=1e-05
a_delta(x)
0
−100k
−200k
−300k
−400k
−500k
−600k
0
0.2
0.4
0.6
0.8
1
x