1. PDEs

Def 1.1 (PDE)

A PDE is a relation that involves the unknown function $u(x_1, x_2, \ldots, x_n, t)$, where $(x_1, x_2, \ldots, x_n)$ is the spacial coordinate, and t is the time, with its derivatives.
Symbolically we have

$$F(x_1, x_2, \ldots, x_n, t, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots) = 0,$$

for example

$$u_t + u_x + u = 0,$$
$$u_t = u_{xx},$$

etc.

2. Method of characteristics

Def 2.1 (Quasilinear 1st order PDE)

A quasilinear first order PDE is

$$u_t + c(x,t,u)u_x = f(x,t,u),$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$. Moreover, c and f are known.
If $c = c(x,t)$, we call the above equation a semilinear equation, and additionally, if $f = f(x,t)$, we call it linear.

Remarks

- Quasilinear means "linear" in the derivatives. So that the nonlinearity involves only u
- We can consider a more general equation $d(x,t,u)u_t + c(x,t,u) = f(x,t,u)$, but we assume it is always possible to divide by d.
- A general balance law looks like the following

$$u_t + q_x(x,t,u) = g(x,t,u).$$

- Function c is actually the velocity of the wave, and we will see that later.

Def 2.2 (Method Of Characteristics)

Let $X_\xi(t)$ be an arbitrary curve on the $x-t$ plane. Define $U_\xi(t) = u(X_\xi(t), t)$, where u is the solution of our equation. Now differentiate:

$$U'_\xi(t) = u_t + X'_\xi(t) u_x.$$

Notice that this is exactly the left hand side of our equation, provided we have $X'_\xi(t) = c(X_\xi(t), t, U_\xi(t))$. Then we have

$$U'_\xi(t) = f(X_\xi(t), t, U_\xi(t)).$$

We have thus reduced our equation into an ODE

$$\begin{aligned} X'_\xi(t) &= c(X_\xi(t), t, U_\xi(t)), \\ U'_\xi(t) &= f(X_\xi(t), t, U_\xi(t)). \end{aligned}$$

These are the so-called characteristics equation, and $X_\xi(t)$ are called the characteristics.
We have to impose initial conditions. Assume that $u(x, 0) = \phi(x)$, then let $X_\xi(0) = \xi$, and obtain $U_\xi(0) = u(X_\xi(0), 0) = \phi(X_\xi(0)) = \phi(\xi)$.
In order to find the solution at any (x, t) we find a unique characteristic $X_\xi(t)$ passing through (x, t), go back to $t = 0$, and use the initial condition, then read the solution from $U_\xi(t)$.