1. Necessary and sufficient optimality conditions

Theorem 1.1 (Necessary optimality condition)

Let $x^* \in \mathbb{R}^n$ be an unconstrained local minimum (maximum) of $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$. If f is continuously differentiable on an open set S containing x^*, then

$$\nabla f(x^*) = 0.$$

If, in addition, f is twice continuously differentiable on S, then $\nabla^2 f(x^*)$ is positive semidefinite (negative semidefinite).

Theorem 1.2 (Sufficient optimality condition)

Let the function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$, be twice continuously differentiable on an open set S. Suppose that $x^* \in S$ satisfies

$$\nabla f(x^*) = 0, \quad \nabla^2 f(x^*) \text{ is positive (negative) definite.}$$

Then x^* is a strict unconstrained local minimum (maximum) of f.

Def 1.1 Definite matrices

An $n \times n$ symmetric real matrix A is called:

- Positive definite iff $x^T A x > 0$ for any $x \neq 0$.
- Positive semidefinite iff $x^T A x \geq 0$ for any x.
- Negative definite iff $x^T A x < 0$ for any $x \neq 0$.
- Negative semidefinite iff $x^T A x \leq 0$ for any x.

Remark

In practice we check the above characteristics via:

- Sylvester's criterion:
  - A positive definite iff all its leading principal minors are positive.
  - A negative definite iff all its leading principal minors of odd size are negative and all of even size are positive.
  - A positive semidefinite iff all its principal minors are nonnegative.

    - A negative semidefinite iff all its principal minors of odd size are nonpositive and all of even size are nonnegative.
- Eigenvalue criterion:
    - A positive (semi)definite iff all its eigenvalues are positive (nonnegative).
    - A negative (semi)definite iff all its eigenvalues are negative (nonpositive).

Def 1.2 Convexity and concavity

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, if for any $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If for any $x, y \in \mathbb{R}^n$ the inequality above is strict, we say that f is strictly convex.

Flip the inequality sign, and you get the definition of concave and strictly concave.

Theorem 1.3

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. Then:

- f is convex iff for any $x \in \mathbb{R}^n$ $\nabla^2 f(x)$ is positive semidefinite
- f is strictly convex iff for any $x \in \mathbb{R}^n$ $\nabla^2 f(x)$ is positive definite
- f is concave iff for any $x \in \mathbb{R}^n$ $\nabla^2 f(x)$ is negative semidefinite
- f is strictly concave iff for any $x \in \mathbb{R}^n$ $\nabla^2 f(x)$ is negative definite

Theorem 1.4

Suppose $f : C \to \mathbb{R}$, where $C \subset \mathbb{R}^n$ is convex, is a convex function. Then the following statements are true:

- Any local minimum of f is its global minimum.
- If f has continuous first-order derivatives on C, each stationary point of f is a global minimum.
- If, in addition, f is strictly convex, there exists at most one global minimum of f.

Remark

Similar theorem goes for concave functions and maxima.

Remark

Because of these properties problems of minimisation of convex functions over convex sets (convex problems) are of special interest to optimisation theory.

Def 1.3 Coercive function

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive if

$$\lim_{||x|| \to \infty} f(x) = \infty.$$

Theorem 1.5

Suppose $f : \mathbb{R}^n - \mathbb{R}$ is coercive. Then it has a global minimum.

Remark

In practice, this means that for a coercive function one of the local minimizers is a global minimiser, so we only need to search through them and compare their values.