Applied Functional Analysis - Exercise sheet 2

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Exercise 1

We have an equation

$$\pi - x + \frac{1}{2}\sin\left(\frac{x}{2}\right) = 0$$

for $x \in [0, 2\pi]$. We can rewrite it as

$$x = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) = f(x),$$

so our fixed point problem takes the form

$$x_{n+1} = \pi + \frac{1}{2}\sin\left(\frac{x_n}{2}\right).$$

Note that f has values in $\left[\pi, \pi + \frac{1}{2}\right] \subset [0, 2\pi]$ on $[0, 2\pi]$, so we can write $f : [0, 2\pi] \to [0, 2\pi]$, just not a surjection. Now

$$|f'(x)| = \left|\frac{1}{4}\cos(x)\right| \le \frac{1}{4} = k < 1,$$

hence, by the theorem 2.1 from the lecture, the fixed point x^* of f exists, is unique, and $x^* \in [0, 2\pi]$.
Let's find the maximum of $g(x) = |f(x) - x| = \left|\pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) - x\right|$ on $[0, 2\pi]$. We know that $h(x) = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) - x$ is decreasing, so g is decreasing on $[0, x^*]$. On $[x^*, 2\pi]$ the sign gets flipped and g is increasing. So the maximum lies at 0 or 2π. We have

- $g(0) = |\pi + 0 - 0| = \pi$,
- $g(2\pi) = |\pi + 0 - 2\pi| = \pi$,
  so $\max\limits_{0 \le x \le 2\pi} |x_1 - x_0| = \max\limits_{0 \le x \le 2\pi} g(x) = \pi$.
  We want

  $$\frac{k^m}{1-k}|x_1 - x_0| < 0.01,$$

  plugging in the values we have

$$\frac{\left(\frac{1}{4}\right)^m}{\frac{3}{4}}\pi < 0.01$$

$$4^m > \frac{400}{3}\pi$$

$$m > log_4\left(\frac{400}{3}\pi\right) \approx 4.355,$$

so you need at least 5 iterations to guarantee $|x_m - x^*| < 0.01$.
Now, let $x_0 = 0$. We have

$$\begin{aligned}
x_1 &= f(x_0) = \pi + \frac{1}{2}\sin\left(\frac{x_0}{2}\right) = \pi && \approx 3.14159,\\
x_2 &= \pi + \frac{1}{2} && \approx 3.64159,\\
x_3 &= \pi + \frac{1}{2}\cos\left(\frac{1}{4}\right) && \approx 3.62605,\\
x_4 &= \pi + \frac{1}{2}\cos\left(\frac{1}{4}\cos\left(\frac{1}{4}\right)\right) && \approx 3.62699,\\
x_5 &= \pi + \frac{1}{2}\cos\left(\frac{1}{4}\cos\left(\frac{1}{4}\cos\left(\frac{1}{4}\right)\right)\right) && \approx 3.62693,
\end{aligned}$$

where the actual limit is $x^* \approx 3.62694$, so we see that actually in less than 5 iterations we were able to get an accuracy of 0.01.

Exercise 2

We have an equation

$$x^2 - p = 0.$$

The Newton's method gives us the following iteration scheme

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)},$$

with $g(x) = x^2 - p$ which is a particular case of the fixed point iteration method. To see this, let $f(x) = x - \frac{g(x)}{g'(x)}$, we have $x_{n+1} = f(x_n)$.
Let's write $f(x)$ explicitly, we have

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2 - p}{2x} = \frac{1}{2}\left(x + \frac{p}{x}\right).$$

Now let's prove that starting at any point $x_0 > 0$, the sequence generated by $x_{n+1} = f(x_n)$ will converge to $\sqrt{p}$.
Assume that $x_0 > 0$. It is obvious that $x_n > 0$ for any n. We have

$$x_{n+1}^2 - p = \frac{1}{4}\left(x_n + \frac{p}{x_n}\right)^2 - p = \frac{1}{4}x_n^2 + \frac{1}{2}p + \frac{1}{4}\frac{p^2}{x_n^2} - p = \frac{1}{4}\left(x_n - \frac{p}{x_n}\right)^2 > 0,$$

hence $x_n > \sqrt{p}$ for all $n > 0$. Also

$$x_n > \sqrt{p} > \sqrt{p} \cdot \frac{\sqrt{p}}{x_n} > \frac{p}{x_n}.$$

Now we can show that x_n is decreasing for all $n > 0$. We have

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{p}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{p}{x_n} - x_n\right) < 0,$$

which proves that x_n is, in fact, decreasing for $n > 0$. From this, and from the fact that $x_n > \sqrt{p}$, we know that x_n has a limit. Let's compute it. We have

$$\begin{aligned} x^* &= \frac{1}{2}\left(x^* + \frac{p}{x^*}\right) \\ \frac{1}{2}x^* &= \frac{1}{2}\frac{p}{x^*} \\ x^* &= \frac{p}{x^*} \\ (x^*)^2 &= p, \end{aligned}$$

and, since $x_n > 0$, we have $x^* = \sqrt{p}$. If, on the other hand, we assumed $x_0 < 0$, it would be obvious from the definition of x_n, that $x_n < 0$ for every n, which would prevent the sequence from converging to $\sqrt{p}$.
So the interval that guarantees convergence to $\sqrt{p}$ is $(0, \infty)$.

Let us use the contraction condition to determine the interval that guarantees convergence to $\sqrt{p}$. We want

$$|f'(x)| = \left|\frac{1}{2}\left(1 - \frac{p}{x^2}\right)\right| < 1.$$

We have three cases there, let's go over them one by one:

- case $x > \sqrt{p}$:

  $$\begin{aligned} \frac{1}{2}\left(1 - \frac{p}{x^2}\right) &< 1 \\ 1 - \frac{p}{x^2} &< 2 \\ -\frac{p}{x^2} &< 1 \\ -p &< x^2, \end{aligned}$$

  and that is true always, because p is positive, so from this case we get $x \in (\sqrt{p}, \infty)$,
- case $x = \sqrt{p}$:
  we get $0 < 1$, also, we're already at the limit, so $x \in \{\sqrt{p}\}$,
- case $x < \sqrt{p}$:

$$\begin{aligned} \frac{1}{2}\left(\frac{p}{x^2}-1\right) &< 1 \\ \frac{p}{x^2}-1 &< 2 \\ \frac{p}{x^2} &< 3 \\ x &> \sqrt{\frac{p}{3}} \end{aligned}$$

so $x \in \left(\sqrt{\frac{p}{3}}, \sqrt{p}\right)$.

Combining these three cases we get the condition $x \in \left(\sqrt{\frac{p}{3}}, \infty\right)$. So the method using the contraction of f gave us a less precise result than the direct analysis of the sequence.

Exercise 3

We consider the following integral equation

$$f(x) = x + \frac{1}{4}\int_0^{\frac{\pi}{2}} f(y)\cos(x)dy.$$

Relating this to the equation from the lecture we have $g(x) = x$, $k(x,y) = \cos(x)$, and $\mu = \frac{1}{4}$. Furthermore, $|k(x,y)| \leq 1 = c$, so

$$\frac{1}{4} = \mu < \frac{1}{c(b-a)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} < \frac{2}{3}.$$

This proves that the functional T defined by

$$T(f)(x) = x + \frac{1}{4}\int_0^{\frac{\pi}{2}} f(y)\cos(x)dy$$

is a contraction, and hence, by the Banach's FPT, has a unique fixed point in $C\left[0, \frac{\pi}{2}\right]$.

Let's rewrite the operator T as

$$T(f)(x) = x + \frac{1}{4}\cos(x)\int_0^{\frac{\pi}{2}} f(y)dy.$$

We can make this into a function sequence in the following way

$$f_{n+1} = T(f_n).$$

Let's take $f_0(x) = x$. We can write a few terms of the sequence by applying T:

$$\begin{aligned}
f_0(x) &= x \\
f_1(x) &= x + \frac{1}{4}\cos(x)\int_0^{\frac{\pi}{2}} x dx = x + \frac{1}{4}\cos(x)\cdot\frac{\pi^2}{8} \\
f_2(x) &= x + \frac{1}{4}\cos(x)\cdot\frac{5\pi^2}{32} \\
f_3(x) &= x + \frac{1}{4}\cos(x)\cdot\frac{21\pi^2}{128}.
\end{aligned}$$

We can see that the functions f_n can be characterised by just the one number, that being the integral of the previous one in the sequence, f_{n-1}. We can write

$$\begin{aligned}
f_C(x) &= x + \frac{C}{4}\cos(x), \\
C_{n+1} &= \int_0^{\frac{\pi}{2}} f_{C_n}(x)dx = \int_0^{\frac{\pi}{2}} \left(x + \frac{C_n}{4}\cos(x)\right)dx = \frac{\pi^2}{8} + \frac{C_n}{4},
\end{aligned}$$

with $C_0 = 0$. Omitting the formal proof of the convergence of C_n, we can find it by looking for solutions to

$$C = \frac{\pi^2}{8} + \frac{C}{4},$$

which yields $C = \frac{\pi^2}{6}$, which, if we plug that in, in fact gives us $T(f_C) = f_C$.

Exercise 4

We are presented with an initial value problem:

$$\begin{aligned}
x'(t) &= 2x(t) + 2 \\
x(0) &= 0
\end{aligned}.$$

Solving this problem analytically is trivial, we obtain $x(t) = e^{2t} - 1$.
The successive approximation scheme looks as follows

$$x_{n+1}(t) = x(0) = \int_0^t x_n'(s)ds = \int_0^t (2x_n(s) + 2)ds = 2t + 2\int_0^t x_n(s)ds.$$

Let us also choose $x_0(t) = 0$. We will prove that $x_n(t) = \sum_{m=1}^n \frac{(2t)^m}{m!}$. To that end, let's check that equality for an initial $n = 0$. We have

$$x_0(t) = 0 = \sum_{m=1}^0 \frac{(2t)^m}{m!},$$

where we used a conventional notion of $\sum_{m=k}^n a_m = 0$ for $n < k$.
Now let's assume that this holds for n, and look at $n+1$. We have

$$
\begin{aligned}
x_{n+1}(t) &= 2t + 2\int_0^t x_n(s)ds \\
&= 2t + 2\int_0^t \sum_{m=1}^{n} \frac{(2s)^m}{m!} ds \\
&= 2t + \sum_{m=1}^{n} \int_0^t \frac{(2s)^m}{m!} 2ds \\
&= 2t + \sum_{m=1}^{n} \int_0^{2t} \frac{u^m}{m!} du \\
&= 2t + \sum_{m=1}^{n} \frac{(2t)^{m+1}}{(m+1)!} \\
&= \frac{(2t)^1}{1!} + \sum_{m=2}^{n+1} \frac{(2t)^m}{m!} \\
&= \sum_{m=1}^{n+1} \frac{(2t)^m}{m!}.
\end{aligned}
$$

We can also write $x_n(t) = \sum_{m=0}^{n} \frac{(2t)^m}{m!} - 1$. It is now trivial to note that

$$
\lim_{n\to\infty} x_n(t) = \sum_{m=1}^{\infty} \frac{(2t)^m}{m!} - 1 = e^{2t} - 1,
$$

which we obtained from solving the equation analytically.