1. Compounding

Def 1.1 (Present and Future Values)

Define the following:

- Discrete time $t \in \{0, 1, 2, \ldots\}$,
- One period compounding - the interest is compounded every year,
- PV - present value,
- FV - future value,
- r - interest rate (e.g. 5%).
  Then for $t = 1$:

$$FV = PV + rPV = PV(1 + r),$$

for $t = 2$:

$$FV = (1 + r)(1 + r)PV = PV(1 + r)^2,$$

for $t = n$:

$$FV = PV(1 + r)^n.$$

Def 1.2 (Frequent compounding)

Let f be the number of times that interest rate is calculated within a unit time. For example, if we do it every third month, then $f = 4$. We have

$$FV = PV\left(1 + \frac{r}{f}\right)^{nf}.$$

If we let $f \to \infty$, so that $nf- > t$, we get the continuous compounding formula

$$FV = PVe^{rt}.$$

Def 1.3 (Discounting)

Discounting works the other way around:

$$PV = FV\left(1 + \frac{r}{f}\right)^{-nf},$$

and for the continuous case:

$$PV = FVe^{-rt}.$$

Def 1.4 (Risk-Free Instrument)

A risk-free instrument is defined by

$$B_t = B_0 e^{rt},$$

where r is a risk-free interest rate. For discrete time we have

$$B_n = B_0 \left(1 + \frac{r}{f}\right)^{nf}.$$

The goal

We want to find the fair price of some financial instrument/derivative, which is often defined with a function (called a payout function) of the asset price. For example, in the european call option

$$C_T = (S_T - K)^+ = f(S_T),$$

where T is called the maturity date, K is given and called the strike price, and $(x)^+ = \max\{x, 0\}$
.

Def 1.5 (Hedging)

A replication/hedging strategy is given by the following

$$\varphi_t = (\alpha_t, \beta_t),$$

where α_t is the amount of assets existing in the portfolio at time t, and β_t is the amount of risk-free instruments B_t in the portfolio at time t.
Note: α_t and β_t can be negative, which corresponds to borrowing.

Example

Let X be a derivative, for example

$$X = (S_1 - K)^+ = \begin{cases} (S^u - K)^+, & \omega = \omega_1, \\ (S^d - K)^+, & \omega = \omega_2. \end{cases}$$

Let $x = V_1(\varphi)$ be the value of the portfolio. Let $B_0 = 1$. We have

$$x = \alpha_1 S_1 + B_1(1 + r).$$

Let $\alpha = \alpha_1$, and $\beta = \beta_1$. Looking for replication strategy $\varphi = (\alpha, \beta)$, we obtain the following equations:

$$\begin{aligned} \alpha S^u + B(1 + r) &= x^u = (S^u - K)^+ \\ \alpha S^d + B(1 + r) &= x^d = (S^d - K)^+. \end{aligned}$$

Then, solving for α and β, we have

$$\alpha = \frac{x^u - x^d}{S^u - S^d}, \quad \beta = \frac{x^d S^u - x^u S^d}{(1+r)(S^u - S^d)}.$$

Hence the price equals (bruh, how?)

$$\Pi(X) = \Pi_0(x) = \alpha S_0 + \beta.$$