1. Summary of net and gross premiums

Def 1.1 (Age of death)

Define X, a random variable, to be the age of death of a newborn. X is assumed to be a continuous, non-negative random variable.
We have:

- $F_X(x) = P(X \leq x)$,
- $S(x) = 1 - F_X(x) = P(X \geq x)$ - survival function, probability that a newborn will survive to x,
- $f_x(x) = F'_X(x)$
  Let us also define (x) to be a life aged x, usually meant as a person aged x years.

Def 1.2 (Future lifetime)

Define $T(x)$ to be the future lifetime of (x), the amount of time that a person aged x will live starting now. $T(x)$ is also a continuous, non-negative random variable, similar to X.
We have:

- $G(t) = G_{T(x)}(t) = P(T(x) \leq t)$

Def 1.3 (Probability symbols)

Define the following:

- ${}_tp_x = P(T(x) \geq t) = P(X - x \geq t | X \geq x)$
- ${}_tq_x = 1 -_t p_x = G(t)$
- $K(x)$ - curtate future lifetime of (x), $k(x) = \lfloor T(x) \rfloor$,
- ${}_{t|u}q_x = P(t \leq T(x) \leq t + u) =_{t+u} q_x -_t q_x$ - probability that (x) will survive t years, and die within the following u years.
  Let us also define a convention regarding p and q functions:
- ${}_1p_x = p_x$,
- ${}_1q_x = q_x$.

Def 1.4 (Force of mortality)

Define μ_x to be the force of mortality at age x. $\mu_x = -\frac{S'(x)}{S(x)}, \mu_x \geq 0$.

Theorem 1.1 (Relationships)

The following equalities are true:

- $F(x) = \int_0^\infty f(s)ds = 1 - s(x) = 1 - \exp(-\int_0^x \mu_s ds)$,
- $f(x) = F'(x) = -S'(x) = \mu_x \exp(-\int_0^x \mu_s ds)$,
- $S(x) = 1 - F(x) = 1 - \int_0^\infty f(s)ds = \exp(-\int_0^x \mu_s ds)$,
- $\mu_x = \frac{F'(x)}{1-F(x)} = \frac{f(x)}{\int_x^\infty f(s)ds} = -\frac{S'(x)}{S(x)}$.

Def 1.5 (UDD)

The uniform distribution of deaths (UDD) assumption assumes the following:

$$S(x+1) = (1-t)S(x) + tS(x+1), 0 \leq t \leq 1.$$

It also implies that

$${}_tq_x = tq_x, 0 \leq t \leq 1,$$

and that $K(x)$ and $S(x)$ are independent.

Def 1.6 (Life insurance products)

Typical life insurance products:

1. Whole life insurance,
2. n-year term insurance,
3. n-year pure endearment,
4. n - year endearment,
5. annuities,
6. unit - united life insurance.
   We have:

- $T = T(x)$ - insurances payable at the moment of death,
- $K = K(x)$ - insurances payable at the end of year of death.

Def 1.7 (Net single premium)

Let A_x, and $\overline{A}_x$ indicate the life insurance benefit of 1 payable at the end of the year of death, and at the time of death, respectively. These two cases correspond to discrete, and continuous models. Let b_t be the benefit function, v_t be the discount function. Let $Z = Z_t = b_t v_t$.
Then the net single premium is defined as

$$NSP = E[Z] = \overline{A}_x = \int_0^\infty v_t g(t)dt.$$

Usually we take $b_t = 1$, and $v_t = v^t$, with a fixed v. Then the NSP takes the following form

$$NSP = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt,$$

with $v = e^{-\delta}$.
In the discrete case, substitute $t = k + 1$, and $v = \frac{1}{1+i}$, then we have

$$NSP = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k}.$$

Now let's explicitly define the life insurance benefits:

$$\begin{aligned} \overline{A}_x &= \int_0^\infty v^t {}_tp_x \mu_{x+t} dt, \\ \overline{A}_{x:\overline{n}|} &= \int_0^n v^t {}_tp_x \mu_{x+t} dt \end{aligned}$$

Def 1.8 (Life annuity)

A life annuity is a series of periodic or continuous payments provided that the insured is alive.
We have Y - present value of future whole life annuity

$$\begin{aligned} Y &= \int_0^T v^t dt = \bar{a}_{\overline{T}|}, \\ v &= e^{-\delta}, \\ \bar{a}_{\overline{T}|} &= \frac{1 - v^T}{\delta}. \end{aligned}$$

2. Whole life insurance

Def 2.1 (L)

Let L be the loss of insurance company, so $L =$ the present value of venefits-present value of premiums.

Def 2.2 (Equivalence principle)

Net premium is calculated in such a way that $E[L] = 0$. So if

$$L = v^{k+1} - p\ddot{a}_{\overline{k+1}|},$$

then

$$p_x = \frac{A_x}{\ddot{a}_x}.$$

Def 2.3 (Endowment)

Since $p_x = \frac{A_x}{\ddot{a}_x}$ describes the whole life premium, then let's define

$$p^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \text{ - term (?)},$$

and

$$p_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \text{ - endowment}.$$

Def 2.4 (Fully continuous case)

Let

$$\bar{a}_t = \int_0^t v^s ds,$$

then

$$L = v^T - \bar{p}_{\bar{a}_{\overline{T}|}},$$

and

$$\bar{p}_x = \frac{\overline{A}_x}{\bar{a}_x}.$$

Def 2.5 (gross premiums)

The continuous-discrete case:

$$\begin{aligned} p(\overline{A}_x) &= \frac{\overline{A}_x}{\ddot{a}_x}, \\ p(\overline{A}^1_{x:\overline{n}|}) &= \frac{\overline{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}, \\ p(\overline{A}_{x:\overline{n}|}) &= \frac{\overline{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}. \end{aligned}$$

The discrete-continuous case:

$$\bar{p} = \frac{A_x}{\bar{a}_x},$$
$$\bar{p}^1_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{a_{x:\overline{n}|}},$$
$$\bar{p}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}.$$

Def 2.6 (Gross premiums costs)

The costs of gross premiums consist of:

- Acquisition costs,
- Agent's common,
- Collection expenses,
- Administration costs,
- Claim hedging expenses.

Def 2.7 (Equivalence principle)

Let $p_a = A$ (net premium), and $b_a = A + \cos As$ (gross premium).
Then ${}_tL$ - financial loss of an insurance company,

$${}_tL = Z - PY,$$

where Y - present value of future payments, P - net premium, Z - present value of future payments of the benefits.
Then the net reserve equals

$$E[{}_tL] = {}_tV.$$

Def 2.8 (recursive formulas for net reserves)

Let:

- b_k - sum insured in k-th year of the policy,
- $\pi_0, \pi_1, \ldots, \pi_k$ - annual premiums paid up to the moment k,
- $L = Z - \sum_{m=0}^{k} z_m v^m = b_{k+1} v^{k+1} - \sum_{m=0}^{k} z_m v^m$.
  The recursive formula:

$${}_kV + \pi_k = V(b_{k+1} qx + k + {}_{k+1}V p_{x+k}).$$

  Another form:

$${}_kV + \pi_k = v({}_{k+1}V + (b_{k+1} - {}_{k+1}V) q_{x+k}).$$

  This gives us a division of the premium π_k:

$$\begin{aligned}\pi_k &= \pi_k^s + \pi_k^r,\\ \pi_k^s &= {}_{k+1}V_v - {}_kV,\\ \pi_k^r &= (b_{k+1} - {}_{k+1}V)Vq_{x+k}.\end{aligned}$$

Def 2.9 (Zillmer's reserve)

$${}_kV^z = {}_kV_x - \alpha(1 - {}_kV_x), \quad k = 0, 1, 2 \dots.$$

3 Multiple decrement model

Def 3.1

Let $T(x) = T$ be the time to leave the initial status, $J(x) = J$ - the cause of leaving the status, $J = j, j = 1, 2, 3, \dots, m$. $T(x) \sim g(t)$, with

$$g(t, j)dt = P(t < T < t + dt | J = j),$$

and

$$g(t) = \sum_{j=1}^{m} g(t, j).$$

Def 3.2

Let:

- ${}_tp_x^{(\tau)}$ - probability of staying in the status up to time t,
- ${}_tq_x^{(\tau)} = 1 - {}_tp_x^{(\tau)}$ - probability of leaving the status up to time t,
- ${}_tq_x^{(j)}$ - probability of leaving the status due to cause j.
  We have:

$$\begin{aligned}{}_tq_x^{(\tau)} &= G(t) = \int_0^t g(s)ds\\ {}_tq_x^{(j)} &= \int_0^t g(s, j)ds\\ \mu_{x+t}^{(\tau)} &= \frac{g(t)}{1 - G(t)} \text{ - force of decrement}\\ {}_tp_x^{(\tau)} &= \exp\left(-\int_0^t \mu_{x+s}^{(\tau)}ds\right)\\ \mu_{x+t}^{(j)} &= \frac{g(t, j)}{P(T > t)} = \frac{g(t, j)}{1 - G(t)} \text{ - force of decremrnt due to j.}\end{aligned}$$

Theorem 3.3

Properties of the functions defined above:

$$
\begin{aligned}
{}_tq_x^{(\tau)} &= \sum_{j=1}^{m} {}_tq_x^{(j)} \\
\mu_{x+t}^{(\tau)} &= \sum_{j=1}^{m} \mu_{x+t}^{(j)} \\
g(t,j) &= {}_tp_x^{(\tau)}\mu_{x+t}^{(j)} \\
g(t) &= {}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)} \\
{}_tq_x^{(j)} &= \int_0^t {}_sp_x^{(\tau)}\mu_{x+s}^{(j)} ds
\end{aligned}
$$

Def 3.4

$K(x)$ - the number of years before leaving the status. We have:

$$P(K=k, J=j) = P(k < T \leq k+1, J=j).$$