

1. Metric spaces and Banach's FPT (Fixed Point Theorem)

Def 1.1 (Metric and Metric Space)

Let X be a nonempty set, and $d : X^2 \rightarrow \mathbb{R}$ be a function satisfying:

- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Then the function d is called the metric, the pair (X, d) is called the metric space, and the number $d(x, y)$ is called the distance between x and y in X .

Examples

- (\mathbb{R}^n, d) , with $d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}$
- (\mathbb{R}^n, d) , with $d(x, y) = \max_{i \leq n} |x_i - y_i|$
- $(C[a, b], d)$, with $d(f, g) = \left(\int_a^b |f(x) - g(x)|^2 dx\right)^{\frac{1}{2}}$
- $(C[a, b], d)$, with $d(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)|$
- $(L^p[a, b], d)$, with $d(f, g) = \left(\int_a^b |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}$

Def 1.2 (Fixed Point)

A fixed point of the mapping $T : X \rightarrow X$ is the point $x^* \in X$ such that $T(x^*) = x^*$.

Def 1.3 (Contraction)

Let (X, d) be a [metric space](#). A mapping $T : X \rightarrow X$ is called a contraction on X if there exists a constant $0 < k < 1$ such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for all $x, y \in X$.

Theorem 1.1 (Banach's FPT)

Let (X, d) be a complete [metric space](#) and let $T : X \rightarrow X$ be a [contraction](#) on X . Then T has a unique [fixed point](#) $x^* \in X$.

Corollary 1.1 (Banach's FPT)

The iterative sequence $x_{n+1} = T(x_n)$ for $n = 1, 2, \dots$ with arbitrary starting point $x_0 \in X$ converges, under assumptions of [Banach's FPT](#), to the unique [fixed point](#) of T . Moreover, the following estimates hold:

- $d(x_m, x^*) \leq \frac{k^m}{1-k} d(x_1, x_0)$ - the prior estimate,
- $d(x_m, x^*) \leq \frac{k}{1-k} d(x_{m-1}, x_m)$ - the posterior estimate.

2. Applications of [Banach's FPT](#)

2.1 Applications to real-valued functions

Let $g \in C^1[a, b]$, and suppose we want to find the solution to the equation $g(x) = 0$ on $[a, b]$. We note that we can always rewrite this equation as $x = g(x) + x$, and then our problem is equivalent with finding a fixed point of the function $f(x) = x + g(x)$.

Theorem 2.1 (Differentiable Contraction)

Let (\mathbb{R}, d) be a metric space of real numbers with the [metric](#) $d(x, y) = |x - y|$ and let $[a, b]$ be a closed interval in \mathbb{R} . Moreover, let $f : [a, b] \rightarrow [a, b]$ be a continuous and differentiable function such that $\sup_{x \in [a, b]} |f'(x)| \leq k < 1$. Then there exists a unique [fixed point](#) $x^* \in [a, b]$ of f .

Example 2.1

We want to find the solution to the equation $\cos(x) - 2x = 0$ on $[0, \pi]$. Then we can write this equation as $x = \frac{1}{2}\cos(x)$, and try to find the fixed point of the function $f(x) = \frac{1}{2}\cos(x)$ on $[0, \pi]$. We have to show that f is a [contraction](#) on $[0, \pi]$. To do so, we apply the [theorem 2.1](#). We have

$$\sup_{x \in [0, \pi]} |f'(x)| = \sup_{x \in [0, \pi]} \left| -\frac{1}{2}\sin(x) \right| = \frac{1}{2} < 1.$$

We have shown that f is a [contraction](#) and, by the [Banach's FPT](#), it has a [fixed point](#) x^* that is the limit of the sequence $\{x_n\}$ generated by the scheme $x_{n+1} = f(x_n)$ with any starting point $x_0 \in [0, \pi]$.

Note that to show that f is a contraction we could also directly apply the definition:

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{2}\cos(x) - \frac{1}{2}\cos(y) \right| = \left| \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \right| \\ &\leq \sup_{x, y \in [0, \pi]} \left| \sin\left(\frac{x+y}{2}\right) \right| \frac{1}{2}|x-y| = \frac{1}{2}|x-y| \leq |x-y|. \end{aligned}$$

2.2 Applications to integral equations

We consider integral equations in the following form

$$f(x) = g(x) + \mu \int_a^b k(x, y) f(y) dy,$$

where $f : [a, b] \rightarrow \mathbb{R}$ is an unknown function, $g : [a, b] \rightarrow \mathbb{R}$, and $k : [a, b]^2 \rightarrow \mathbb{R}$ are given functions, and μ is a parameter.

The above integral equation can be considered in various function spaces. Here we consider this equation only in $(C[a, b], d)$ with $d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$.

We assume that $g \in C[a, b]$, and that the kernel k is continuous on the square $[a, b]^2$, which implies that k is bounded on $[a, b]^2$, meaning that there exists a constant c , such that $|k(x, y)| \leq c$ for all $(x, y) \in [a, b]^2$.

Theorem 2.2

The metric space $(C[a, b], d)$ is complete

Note that our integral equation can be rewritten as $T(f) = f$, where

$$T(f)(x) = g(x) + \mu \int_a^b k(x, y) f(y) dy.$$

First we have to show that the mapping $T : C[a, b] \rightarrow C[a, b]$ is well-defined, but this is obvious, as g and k are both continuous on their domains. Let us now determine for which values of μ the map T is a [contraction](#). We have

$$\begin{aligned} d(T(f_1), T(f_2)) &= \sup_{x \in [a, b]} |T(f_1)(x) - T(f_2)(x)| = \sup_{x \in [a, b]} \left| \mu \int_a^b k(x, y) (f_1(y) - f_2(y)) dy \right| \leq \\ &\leq |\mu| \sup_{x \in [a, b]} \int_a^b |k(x, y)| |f_1(y) - f_2(y)| dy \leq c |\mu| \sup_{x \in [a, b]} |f_1(x) - f_2(x)| \int_a^b dy = \\ &= c |\mu| (b - a) d(f_1, f_2). \end{aligned}$$

It is now required that $c |\mu| (b - a) < 1$, or $|\mu| < \frac{1}{c(b-a)}$, for T to be a contraction. Applying the [Banach's FPT](#), we see that the map T has a unique [fixed point](#) $f^* \in C[a, b]$.

Theorem 2.3 (Integral equation)

Consider the integral equation

$$f(x) = g(x) + \mu \int_a^b k(x, y) f(y) dy.$$

Suppose that k and g are continuous on $[a, b]^2$ and $[a, b]$ respectively, and assume that the parameter μ satisfies $|\mu| < \frac{1}{c(b-a)}$ with the constant c such that $|k(x, y)| < c$ for all $(x, y) \in [a, b]^2$.

Then the integral equation has a unique solution $f \in C[a, b]$. Moreover, this solution is a limit of the sequence $\{f_n\}$ where f_0 is a continuous function on $[a, b]$, and

$$f_{n+1} = g(x) + \mu \int_a^b k(x, y) f_n(y) dy.$$

2.3 Applications to differential equations

Let's consider the initial value problem

$$\begin{aligned} x'(t) &= f(t, x(t)) \\ x(t_0) &= x_0 \end{aligned}$$

where $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a given function and $x(t)$ is an unknown function that we want to find.

Theorem 2.4 (Picard-Lindelöf)

Let f be continuous on the rectangle

$$R = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : |t - t_0| \leq a, |x - x_0| \leq b\}$$

and thus bounded on R , say $|f(t, x)| \leq c$ for all $(t, x) \in R$. Suppose that f satisfies the Lipschitz condition on R with respect to the second argument, i.e., there exists a constant k such that

$$|f(t, x) - f(t, y)| \leq k|x - y|$$

for all $(t, x), (t, y) \in R$.

Then the initial value problem has a unique solution, which exists on the interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta = \min \left\{ a, \frac{b}{c}, \frac{1}{k} \right\}.$$

Corollary 2.1 (Picard-Lindelöf)

Under the assumptions of [Picard-Lindelöf theorem](#), the sequence given by

$$\begin{aligned} x_0(t) &= x_0 \\ x_{n+1}(t) &= T(x_n)(t) = x_0 + \int_{t_0}^t f(s, x_n(s)) ds \end{aligned}$$

converges uniformly to the unique solution $x(t)$ on $J = [t_0 - \beta, t_0 + \beta]$.

Example 2.2

Consider the differential equation

$$x'(t) = \sqrt{x(t)} + x^3(t), \quad x(1) = 2.$$

We have

$$x_1(t) = 2 + \int_1^t (\sqrt{2} + 2^2) ds = 2 + (\sqrt{2} + 8)(t - 1)$$
$$x_2(t) = 2 + \int_1^t (\sqrt{x_1(s)} + x_1(s)^3) ds = \text{*hot mess*}$$

2.4 Applications to matrix equations

Suppose we want to find a solution of the matrix equation

$$Ax = B$$

where $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$.

We note that this equation can be rewritten as

$$x = (I - A)x + b$$

where I is the identity matrix.

Let's define the map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$Tx = (I - A)x + b.$$

Then the problem of solving the matrix equation $Ax = b$ is equivalent with finding a [fixed point](#) of T .

Let's define $\alpha_{ij} = \delta_{ij} - a_{ij}$ where a_{ij} are elements of the matrix A , and δ_{ij} is the Kronecker delta. Using this notation we have

$$(Tx)_i = \sum_{j=1}^n \alpha_{ij} x_j + b_i$$

We will show that $Ax = b$ has a unique solution if

$$\sum_{j=1}^n |\alpha_{ij}| \leq \alpha < 1.$$

for all $i = 1, 2, \dots, n$. Consider the matrix space (\mathbb{R}^n, d) , with $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ for $x, y \in \mathbb{R}^n$.

. We have

$$\begin{aligned}
d(Tx, Ty) &= \max_i |(Tx)_i - (Ty)_i| = \\
&= \max_i \left| \sum_{j=1}^n \alpha_{ij}(x_j - y_j) \right| \leq \\
&\leq \max_i \sum_{j=1}^n |\alpha_{ij}| |x_j - y_j| \leq \\
&\leq \max_i \sum_{j=1}^n |\alpha_{ij}| \cdot \max_j |x_j - y_j| = \\
&= \max_i \sum_{j=1}^n |\alpha_{ij}| \cdot d(x, y).
\end{aligned}$$

We notice that if $\sum_{j=1}^n |\alpha_{ij}| < 1$, for all $i = 1, 2, \dots, n$, then $\max_i \sum_{j=1}^n |\alpha_{ij}| < 1$. We have

$$\sum_{j=1}^n |\alpha_{ij}| = |a_{i1}| + |a_{i2}| + \dots + |1 - a_{ii}| + \dots + |a_{in}| < 1,$$

so

$$\sum_{j=1, j \neq i}^n |a_{ij}| < 1 - |1 - a_{ii}| < |a_{ii}|.$$

We get the condition for the matrix A for which T is a [contraction](#). This condition is given by

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|,$$

or, in other words, matrix A should be strictly diagonally dominant.

Theorem 2.5 (Matrix equation)

The matrix equation $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ has a unique solution $x \in \mathbb{R}^n$ if A is strictly diagonally dominant. The iteration method is as follows

$$x_{n+1} = (I - A)x_n + b, \quad x_0 \in \mathbb{R}^n.$$

In general, we can rewrite the equation $Ax = b$ as $Qx = (Q - A)x + b$, where $Q \in \mathbb{R}^{n \times n}$. We then have the following iterative scheme

$$Qx_{n+1} = (Q - A)x_n + b.$$

Examples:

- $Q = I$ - Richardson method,
- Q diagonal, with $q_{ii} = a_{ii}$ - Jacobi method,
- $Q = D - L$ with D diagonal, and L lower triangular - Gauss-Seidel method.

3. Normed spaces

Def 3.1 (Norm and normed space)

A norm on a vector space X is a real-valued function denoted by $\|\cdot\|$ which satisfies the following conditions:

- $\|x\| \geq 0$ for all x . $\|x\| = 0$ iff $x = 0$,
- $\|\alpha x\| = |\alpha|\|x\|$ for any α , and $x \in X$,
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A normed space is a vector space equipped with a norm, depicted by $(X, \|\cdot\|)$, or with a shorthand X .

Remark 3.1.1

A [norm](#) on X defines the [metric](#) $d(\cdot, \cdot)$ on $X \times X$, which is defined by $d(x, y) = \|x - y\|$, and is called the metric induced by the norm $\|\cdot\|$.

Remark 3.1.2

Every [normed space](#) X is a [metric space](#), converse might not be true.

For example, a [metric](#) defined by

$$d(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}$$

then $\|\alpha(x - y)\| = d(\alpha x, \alpha y) \neq |\alpha|d(x, y) = |\alpha|\|x - y\|$.

Lemma 3.1 (Norm continuity)

The [norm](#) $\|\cdot\|$ defined on X is a continuous mapping of X into \mathbb{R} .

Examples of normed spaces

- $(\mathbb{R}^n, \|\cdot\|_2)$, with $\|x\|_2 = (\sum_{m=1}^n x_i^2)^{\frac{1}{2}}$,
- $(C[a, b], \|\cdot\|)$, with $\|f\| = \max_{x \in [a, b]} |f(x)|$,
- $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$, with $\Omega \subset \mathbb{R}$, $p \geq 1$ and

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_{\Omega} f(x)^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in \Omega} |f(x)|, & p = \infty. \end{cases}$$

Def 3.2 (Norm equivalence)

Two [normed spaces](#) $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ are called topologically equivalent, or two [norms](#) $\|\cdot\|_1$, and $\|\cdot\|_2$ are called equivalent if there exist positive constants C_1 , and C_2 , such that

$$C_1\|x\|_2 \leq \|x\|_1 \leq C_2\|x\|_2$$

for all $x \in X$.

Theorem 3.1 (Equivalence of norms in finite dimensional spaces)

All [norms](#) of finite dimensional space X are equivalent.

Def 3.3 (Convergence in normed spaces)

A sequence $\{x_n\}$ in a [normed space](#) $(X, \|\cdot\|)$ is convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$.

Def 3.4 (Cauchy sequence)

A sequence $\{x_n\}$ in a [normed space](#) $(X, \|\cdot\|)$ is a Cauchy sequence if

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m\| = 0.$$

Def 3.5 (Complete space)

We say that a [normed space](#) $(X, \|\cdot\|)$ is complete if every [Cauchy sequence](#) $\{x_n\}$ in X is convergent to some $x \in X$.

Def 3.6 (Banach space)

A [complete normed](#) space is called a Banach space.

Theorem 3.2 (Euclidean space is complete)

The space $(\mathbb{R}^N, \|\cdot\|_2)$ with the standard Euclidean norm is [complete](#).

Theorem 3.3 ()

Let Ω be a compact subset of \mathbb{R}^n . Then the set $C(\Omega)$ of all continuous functions on Ω equipped with the norm $\|f\| = \max_{x \in \Omega} |f(x)|$ is a [Banach space](#).

4. Hilbert spaces

Def 4.1 (Inner product and inner product space)

Let X be a vector space over the field \mathbb{F} over the real or complex numbers. A mapping $\langle \cdot, \cdot \rangle : X^2 \rightarrow \mathbb{F}$ is called an inner product if for all $x, y \in X$ the following conditions are satisfied

1. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0 \iff x = 0$,
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for $\alpha \in \mathbb{F}$,
4. $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$.

The vector space X together with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space or pre-Hilbert space and is denoted $(X, \langle \cdot, \cdot \rangle)$.

Remark 4.1

- $\overline{\langle x, y \rangle}$ denoted the complex conjugate of $\langle x, y \rangle$,
- The condition 2 implies that $\langle x, x \rangle$ must be a real number,
- If $\mathbb{F} = \mathbb{R}$ then $\langle x, y \rangle = \langle y, x \rangle$,
- Conditions 3 and 4 imply that the function $\langle \cdot, \cdot \rangle$ is linear in the first variable. It is easy to see that $\langle \cdot, \cdot \rangle$ is also linear in the second variable if $\mathbb{F} = \mathbb{R}$,

Examples

- \mathbb{R}^N , with $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$,
- $C(\Omega)$, with $\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} dx$,
- $L_2(\Omega)$, with $\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} dx$.

Theorem 4.1 (Cauchy-Schwarz-Bunyakowski inequality)

For all $x, y \in (X, \langle \cdot, \cdot \rangle)$ we have

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Theorem 4.2 (Inner product space is normed)

Every [inner product space](#) $(X, \langle \cdot, \cdot \rangle)$ is a [normed space](#) with respect to the norm $\|x\| = \sqrt{|\langle x, x \rangle|}$.

Def 4.2 (Hilbert space)

An [inner product space](#) $(X, \langle \cdot, \cdot \rangle)$ is called a Hilbert space if the normed space $(X, \|\cdot\|)$ with the [norm](#) induced by the inner product is a [Banach space](#).

Theorem 4.2 (Parallelogram law)

Let $(X, \langle \cdot, \cdot \rangle)$ be an [inner product space](#). Then for all $x, y \in X$ we have

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

Remark 4.2

The parallelogram law is not valid for an arbitrary norm on a vector space.

Theorem 4.4 (Polarisation identity)

For any two elements x, y in an [inner product space](#) we have

$$\langle x, y \rangle = \frac{1}{4} (||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2).$$

Theorem 4.5 (Normed space is inner product space sometimes)

A [normed space](#) is an [inner product space](#) if and only if the norm of the normed space satisfies the parallelogram law.

5. Linear operators

Def 5.1 (Linear operator)

Let $(X, || \cdot ||_X)$ and $(Y, || \cdot ||_Y)$ be [Banach spaces](#), and let $A : X \rightarrow Y$ be a map. We say that A is linear if $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $x, y \in X$, and $\alpha, \beta \in \mathbb{R}$.

Def 5.2 (Bounded operator)

A [linear operator](#) is bounded if there exists a constant $M > 0$ such that

$$||Ax||_Y \leq M||x||_X$$

for all $x \in X$.

We denote the set of all linear and bounded operators as

$$\mathcal{L}(X, Y) = \{ A : X \rightarrow Y | A \text{ is linear and bounded} \}.$$

Def 5.3 (Operator norm)

A set $\mathcal{L}(X, Y)$ can be equipped with the operator norm

$$||A||_{op} = \inf \{ M | ||Ax||_Y \leq M||x||_X \} = \sup_{x \neq 0} \frac{||Ax||_Y}{||x||_X} = \sup_{||x||_X=1} ||Ax||_Y.$$

Theorem 5.1 (Operator set is a Banach space)

The set $\mathcal{L}(X, Y)$ equipped with $|| \cdot ||_{op}$ norm is a [Banach space](#).

Theorem 5.2 ()

Let $A : X \rightarrow Y$ be a [linear operator](#), then A is [bounded](#) if and only if A is continuous.

Example 5.1

Consider a matrix $A \in \mathbb{R}^{m \times n}$. The matrix A is a linear operator from $(\mathbb{R}^n, \|\cdot\|_\alpha)$ to $(\mathbb{R}^m, \|\cdot\|_\beta)$ and the corresponding induced norm (or the operator norm) on the space $\mathbb{R}^{m \times n}$ is defined by

$$\|A\|_{op} = \sup \{ \|Ax\|_\beta \mid \|x\|_\alpha = 1 \}.$$