Exponentially many 3-colorings of planar graphs with no short separating cycles

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Based on Carsten Thomassen's article [Tho21]

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Motivation

Conjecture [Thomassen 2007]

Every planar triangle-free graph G has exponentially many 3-colorings.

Main Theorem 1.

For infinitely many n, there exists a triangle-free planar graph with separating cycles of length 4 and 5 whose number of proper vertex-3-colorings is $< 2^{15n/log_2(n)}$.

Results

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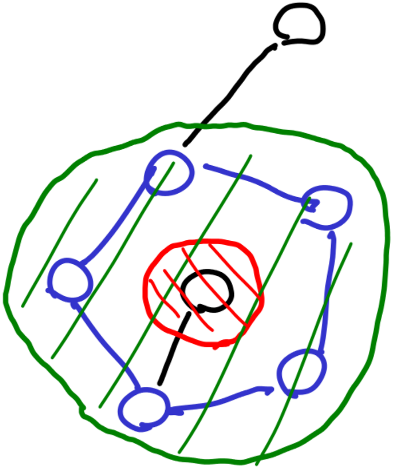
For infinitely many n, there exists a triangle-free planar graph with separating cycles of length 4 and 5 whose number of proper vertex-3-colorings is $< 2^{15n/\log_2(n)}$.

Main Theorem 2.

The number of proper vertex-3-colorings of every triangle-free planar graph with n vertices and with no separating cycle of length 4 or 5 is at least $2^{n/17700000}$.

Definitions

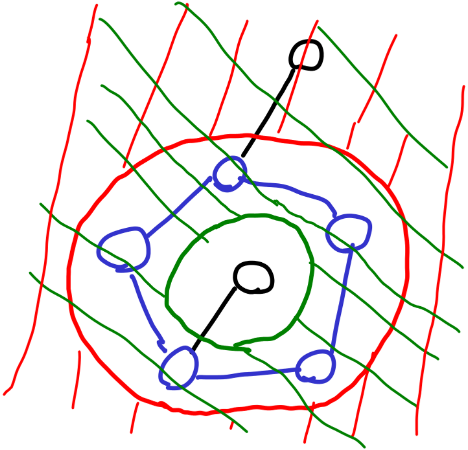


C - cycle

int(C)

Int(C)

Definitions



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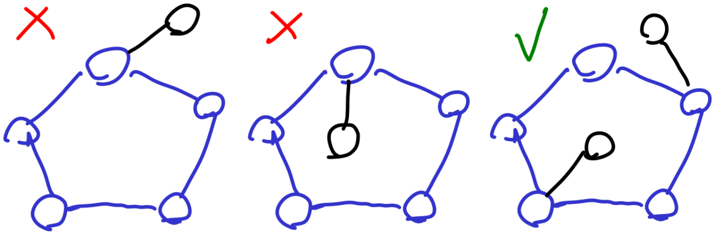
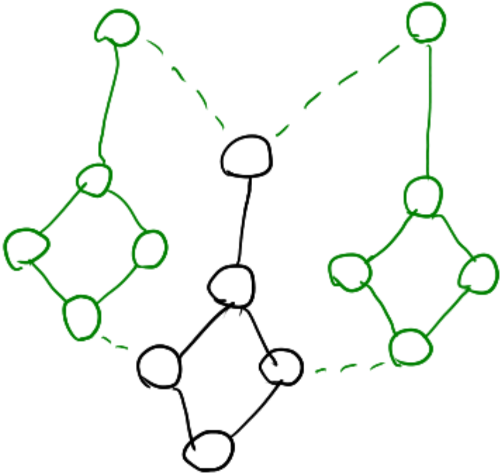


Figure: Seperating cycle

Definition

If $int(C)$ is empty or $ext(C)$ is empty, then C is facial.

Planar triangle-free graphs with only subexponentially many 3-colorings



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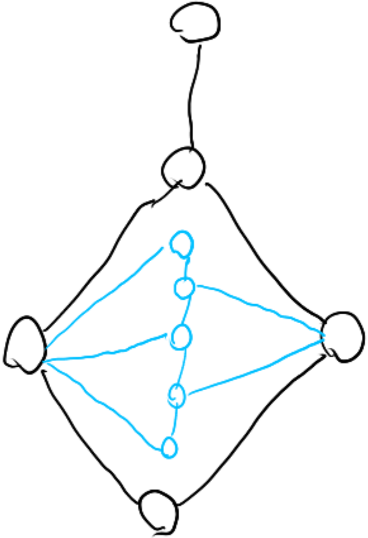


Theorem 3.

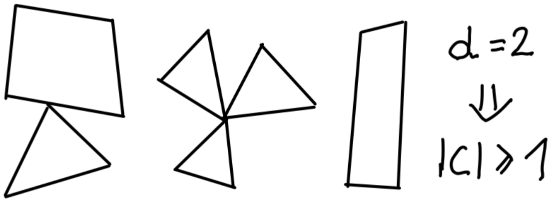
Let G be a connected plane graph of girth at least 4 with an outer cycle C of length 4 or 5. Let q denote the number of facial 5-cycles inside C. Let c be a 3-coloring of C.
If C has length 4, then the number of distinct 3-colorings of G extending c is at least $2^{3q/20000}$.
If C has length 5, then the number of distinct 3-colorings of G extending c is at least $2^{3(q-1)/20000}$.

Theorem 4.

Let d be a natural number ≥ 2, and let $\mathcal{C}$ be a collection of m simple closed curves in the Euclidean plane such that

1. the interior of a curve in $\mathcal{C}$ contains no point of any curve in $\mathcal{C}$,
2. any two curves in $\mathcal{C}$ have at most one point common, and
3. any curve in $\mathcal{C}$ has at most d points of intersection with other curves in $\mathcal{C}$.

Then $\mathcal{C}$ contains a subset $\mathcal{C}'$ of at least $2^{3-2d}3^{1-d}m$ curves such that each connected component $\mathcal{C}''$ of the union of curves in $\mathcal{C}'$ contains a point which belongs to all curves in $\mathcal{C}''$.



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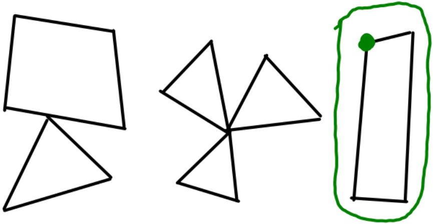

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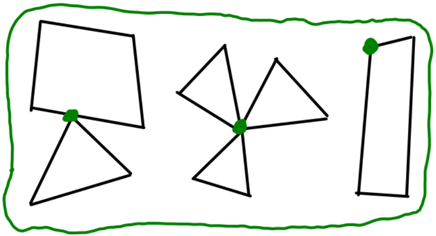

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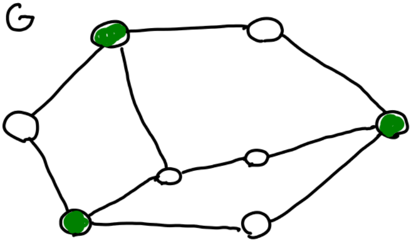
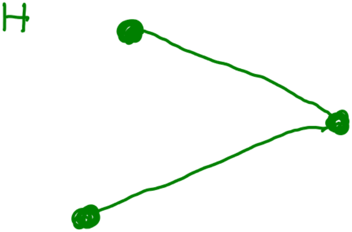
Theorem 5.

Let G be a graph of girth at least 4, and let A be a set of n pairwise nonadjacent vertices on the outer face boundary. Let H denote the graph with vertex set A such that two vertices in H are neighbors if and only if they are joined by a path of length (precisely) 3 in G. Then H is 16-colorable and has therefore an independent set of at least $n/16$ vertices.





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Theorem 6.

Let G be a connected plane graph of girth at least 4 with an outer cycle C of length 4 or 5. Let n denote the number of vertices inside C. Assume G has no separating cycle of length 4 or 5. Assume also that no vertex inside C is joined to precisely two vertices on C. Let c be a 3-coloring of C. Then G has at least $2^{n/17700000}$ distinct 3-colorings extending c.

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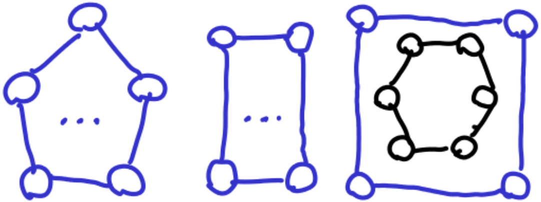

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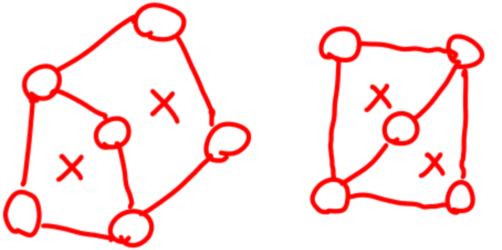

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Assume (reductio ad absurdum) that G is a smallest counterexample for $n > 1$.

Proof of Main Theorem 2.

(1): The number of 5-cycles in G is $< n/100$.

Proof of (1): All 5-cycles in G are facial. If the number of 5-cycles in G is at least $n/100$, then Theorem 3. implies that G is not a counterexample to Theorem 6, a contradiction.

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(2): The number of vertices in $int(C)$ which have degree at most 4 and which are not contained in any 5-cycle is at least $n/20$.

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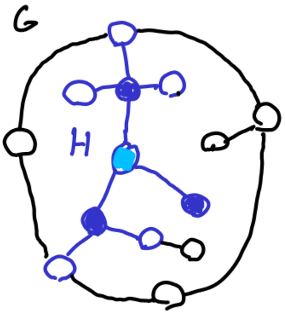
By (1), at most $n/20$ vertices are contained in 5-cycles.
So, the number described in (2) is at least $n - 9n/10 - n/20 = n/20$.

(3): G contains an induced subgraph H such that each component H' of H is a tree containing a vertex v (which we call the root of H') such that
(i) each H-neighbor x of v is in $int(C)$ and satisfies $d_G(x) = d_H(x) \leq 4$ and is not contained in a 5-cycle, and
(ii) each vertex in H' has distance (in both G and H) at most 2 to v.
Moreover, at least $n/1105920$ vertices x in H satisfy $d_G(x) = d_H(x) \leq 4$ and are not contained in a 5-cycle.

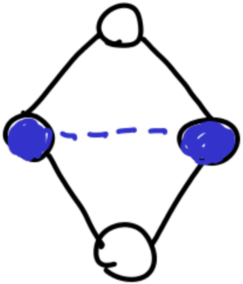


Proof of (3): By (2) and the 4-Color Theorem, G contains an independent set I_1 of at least $n/80$ vertices, all in $int(C)$, all of degree at most 4, and all not contained in any 5-cycle.

I_1 contains a subset I_2 with at least $n/320$ vertices such that no two are in a common 4-cycle. (4-Color)

We apply Theorem 4. with $d = 4$ to the family of curves around neighborhood of I_2 vertices and let G' be the subgraph of G consisting of the edges and vertices which are in the resulting subfamily. Then G' contains at least $n/276480$ vertices x that satisfy $d_G(x) = d_{G'}(x) \leq 4$ and are not contained in a 5-cycle.

G' may be not induced. Then some edges with ends in G' connect components of G'. Using the 4-Color Theorem we find a subgraph H of G consisting of pairwise nonadjacent components of G' such that H contains at least $n/1105920$ vertices x in H that satisfy $d_G(x) = d_H(x) \leq 4$ and are not contained in a 5-cycle.



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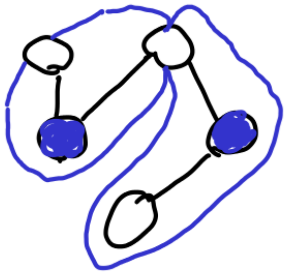

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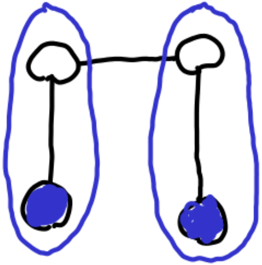

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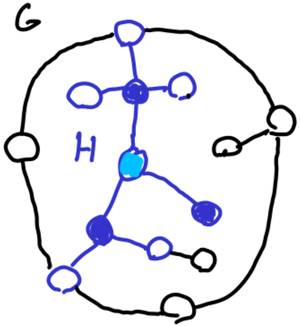

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(4): The subgraph H defined in (3) contains an induced subgraph Q with at least $n/17694720$ vertices x satisfying $d_G(x) = d_Q(x) \leq 4$ and x not contained in a 5-cycle, such that the contraction of each component Q' of Q into a vertex q results in a triangle-free planar graph.

Proof of (4): H could work, but triangle may occur after contraction.

Such triangle have only one contracted vertex (H is induced).

Such triangle corresponds to path P of length 3 in G.

P must join two vertices y_1, y_2 which are neighbors of distinct vertices x_1, x_2 which are both neighbors of the root v.

We delete v, and for each neighbor x of v we contract x and all (at most 3) neighbors of x in $G \setminus v$ into a single vertex.

Theorem 5 on the graph with A being the set of contracted vertices results in a subset A' of A with at least $|A|/16$ vertices. Each vertex in A corresponds to a vertex x (joined to a root v that was deleted). Q is the subgraph induced by x's with neighborhood.

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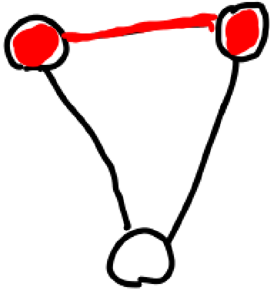

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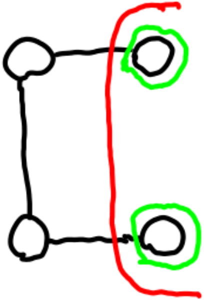

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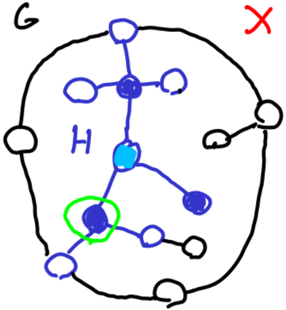

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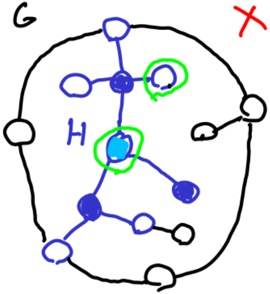

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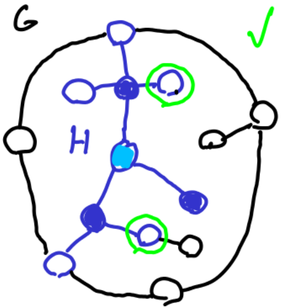

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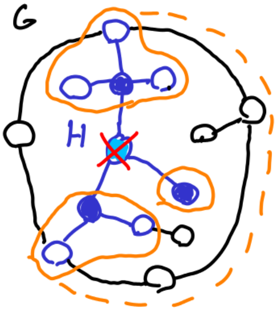
Theorem 5 on the graph with A being the set of contracted vertices results in a subset A' of A with at least $|A|/16$ vertices. Each vertex in A corresponds to a vertex x (joined to a root v that was deleted). Q is the subgraph induced by x's with neighborhood.



(4): The subgraph H defined in (3) contains an induced subgraph Q with at least $n/17694720$ vertices x satisfying $d_G(x) = d_Q(x) \leq 4$ and x not contained in a 5-cycle, such that the contraction of each component Q' of Q into a vertex q results in a triangle-free planar graph.

Proof of (4): H could work, but triangle may occur after contraction.

Such triangle have only one contracted vertex (H is induced).

Such triangle corresponds to path P of length 3 in G.

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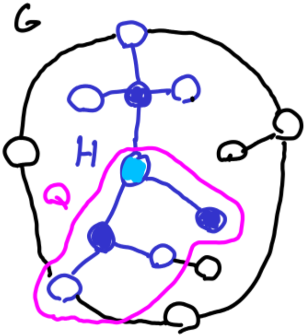
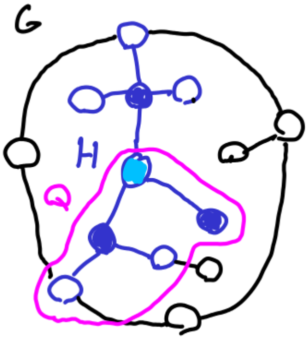


We contract each component Q' of Q (defined in (4)) into a vertex q. By Grötzsch's theorem, the resulting graph has a 3-coloring extending the coloring of C.

We color roots and grandchildren in Q' components to the $c(q')$.

Now, there are at least $n/17694720$ uncolored vertices in G with same color neighbors.

There are at least $2^{n/17694720}$ color extensions of the coloring of C. Contradiction, G was not a counterexample.



Theorem 6.

Let G be a connected plane graph of girth at least 4 with an outer cycle C of length 4 or 5. Let n denote the number of vertices inside C. Assume G has no separating cycle of length 4 or 5. Assume also that no vertex inside C is joined to precisely two vertices on C. Let c be a 3-coloring of C. Then G has at least $2^{n/17700000}$ distinct 3-colorings extending c.

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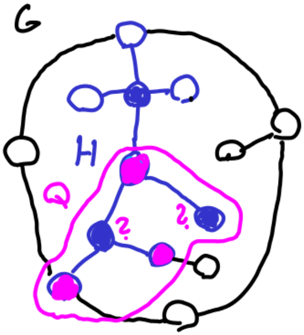

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$20 \cdot 4 \cdot 4 \cdot 864 \cdot 4 \cdot 16 = 17694720$	Subgraph Q

References

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