Besselove funkcije. Gre za rešitve diferencialne enačbe

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0$$

Za $\nu \in \mathbb{R}$ je

$$J_\nu(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}$$

Zanje velja:

1. Ko rešujemo parcialne diferencialne enačbe z metodo ločitve spremenljivk, pogosto dobimo probleme, ki privedejo do te enačbe.
2. Za $\nu \notin \mathbb{Z}$ sta J_ν in $J_{-\nu}$linearno neodvisni.
3. Za $\nu \in \mathbb{Z}$ je $J_\nu = (-1)^\nu J_\nu$
4. Von Neumannova funkcija:

   $$Y_\nu(z) = \frac{J_\nu(z)\sin(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

   Za $\nu \in \mathbb{N}$ pa

   $$Y_n = \lim_{\nu \to n} Y_\nu(z)$$

Naloga. Velja $J_{1/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{\sin z}{z}$. Pokaži, da velja

$$J_{n+1/2} = (-1)^n z^n \sqrt{\frac{2z}{\pi}} \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left(\frac{\sin z}{z}\right)$$

Reševanje. $n = 0$: $J_{1/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{\sin z}{z}$

Želimo ugotoviti, kako se $\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n$ spreminja z n.

$$n = 0: \quad \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^0 f(z) = f(z)$$

$$n = 1: \quad \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^1 f(z) = \frac{1}{z}f'(z)$$

$$n = 2: \quad \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 f(z) = \frac{1}{z}\left(\frac{1}{z}f''(z) - \frac{1}{z^2}f'(z)\right)$$

Za dokaz bomo uporabili indukcijo. Preverili smo že, da velja za $n = 0$. Predpostavimo torej, da velja do nekega n, in poglejmo, kako je z $n+1$.

$$J_{n+3/2} = (-1)^{n+1} z^{n+1} \sqrt{\frac{2\pi}{z}} \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+1} \left(\frac{\sin z}{z}\right)$$

$$= (-1)^{n+1} z^{n+1} \sqrt{\frac{2\pi}{z}} \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right) \left(\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left(\frac{\sin z}{z}\right)\right)$$

$\left(\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left(\frac{\sin z}{z}\right)\right)$ izrazimo iz $J_{n+1/2}$

$$J_{n+3/2}(z) = (-1)^{n+1} z^{n+1} \sqrt{\frac{2\pi}{z}} \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right) \left(J_{n+1/2}(z) \frac{1}{z^n} (-1)^n \sqrt{\frac{2\pi}{z}}\right)$$

$$= -z^n \sqrt{\frac{2\pi}{z}} \frac{\pi}{2} \frac{\mathrm{d}}{\mathrm{d}z} \left(J_{n+1/2}(z)\, z^{-n-1/2}\right)$$

Za Besselove funkcije velja $2J_\nu' = J_{\nu-1} - J_{\nu+1}$, $\frac{2\nu}{z}J_\nu = J_{\nu-1} + J_{\nu+1}$. Od faktorja pred odvodom ostane le še $-z^{n+1/2}$

$$= -\frac{J_{n-1/2}(z)}{2} + \frac{J_{n+3/2}(z)}{2} + \frac{(n+1/2)}{z}J_{n+1/2}(z) = -\frac{J_{n-1/2}(z)}{2} + \frac{J_{n+3/2}(z)}{2} + \frac{J_{n-1/2}(z)}{2} + \frac{J_{n+3/2}(z)}{2} = J_{n+3/2}(z)$$