Naloga. Reši diferencialno enačbo

$$4z^2y'' - 4zy' + (1+2z)y = 0$$

okoli $z = 0$

$$y'' - y' + \frac{1-2z}{4z^2} = 0$$

$$p = -1 \quad p_0 = 0$$

$$q = \frac{1-2z}{4z^2} \quad q_0 = \frac{1}{4}$$

$$\mu(\mu - 1) + \frac{1}{4} = 0$$

$$\mu_1 = \mu_2 = \frac{1}{2}$$

$$Y_1 = z^{1/2}\sum_{n=0}^{\infty} c^n z^n = \sum_{n=0}^{\infty} c_n z^{n+1/2}$$

Vstavimo v originalno enačbo, pogledamo člene pri z^n in dobimo

$$c_n = \frac{1}{n}c_{n-1}$$

$$Y_1 = z^{1/2}e^z$$

Ker je $\mu_1 - \mu_2 = 0$ in imamo linearno odvisni rešitvi, vzamemo nastavek

$$Y_2 = Y_1 \ln z + \sum_{n=0}^{\infty} c_n z^{n+1/2}$$

Vstavimo to v originalno enačbo. Upoštevamo, da je $4z^2y_1'' - 4z^2y_1' + (1-2z)y_1 = 0$

$$4\left(2\sum_{n=0}^{\infty}\frac{(n+1/2)}{n!}z^{n-1/2} + \sum_{n=0}^{\infty}\frac{z^n+1/2}{n!} + \sum_{n=0}^{\infty} c_n(n-1/2)(n+1/2)z^{n-3/2}\right) -$$

$$-4\left(\sum_{n=0}^{\infty}\frac{z^{n-3/2}}{n!} + \sum_{n=0}^{\infty} c_n(n+1/2)z^{n-1/2}\right) + \sum_{n=0}^{\infty} c_n z^n - 2\sum_{n=0}^{\infty} c_n z^{n+3/2} = 0$$

Pri $n = 0$ $(z^{1/2})$:

$$8\cdot\frac{1}{2} - 4 + 4c_0\cdot\left(-\frac{1}{4}\right) = 0 \Rightarrow c_0 = 0$$

Pri $n = 1$ $(z^{3/2})$:

$$4 - 4c_1 - 4c_0 = 0 \Rightarrow c_1 = -1$$

In tako naprej za poljuben n.

Besselove funkcije. So funkcije, ki rešijo enačbo

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0$$

$$J_\nu(z) = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)}$$

Velja:

- Če $\nu \notin \mathbb{Z}$, sta J_ν in $J_{-\nu}$ linearno neodvisni, sicer je $J_\nu = (-1)^\nu J_{-\nu}$.
- Von Neumannova funkcija:
$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$
  Za $n \in \mathbb{N}$:
$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z)$$

- $\sum_{n=0}^{\infty} J_n(z)t^n = e^{\frac{z}{2}\left(1-\frac{1}{t}\right)}$
- $2J_\nu'(z) = J_{\nu-1}(z) - J_{\nu+1}(z)$
- $\dfrac{2\nu}{z} J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$

Naloga. Za $a \neq b$:

$$u(x) = J_\nu(ax)$$
$$v(x) = J_\nu(bx)$$

Za Besselovo funkcijo velja

$$x^2 J_\nu''(x) + xJ_\nu(x) + (x^2\nu^2)J_\nu(x) = 0$$
$$x^2 v''(x) + xv'(x) + (b^2x^2 - \nu^2)v(x) = 0$$
$$x^2 u''(x) + xu'(x) + (a^2x^2 - \nu^2)u(x) = 0$$

Prvo enačbo množimo z u, drugo z v, nato ju odštejemo.

$$x^2(u''v - uv'') + x(v'u - uv') + x^2(a^2 + b^2)uv = 0$$

$$\frac{-\left(x(u'v - uv')\right)'}{a^2 + b^2} = xuv$$

To integriramo (npr. z mejama 0 in 1):

$$\int_0^1 xu(x)v(x)\,\mathrm{d}x = \int_0^1 xJ_\nu(ax) + J_\nu(bx)\,\mathrm{d}x = 0$$

$$\frac{-1}{a^2 + b^2}(u'v - v'u)\Big|_0^1 = \frac{u(a)v'(b) - u'(a)v(b) - u(0)v'(0) + u'(0)v(0)}{a^2 + b^2} = 0$$