Kinetično energijo smo pri prejšnjem predavanju zapisala kot

$$\sum_{i=1}^{N}\frac{1}{2}m_i v_i^2 = \sum_{j,k=1}^{n}\frac{1}{2}w_{jk}\dot{q}_j\dot{q}_k$$

$$w_{jk}(\underline{q}) = \sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k} = w_{kj}$$

Uvedemo posplošeni impulz (gibalna količina, moment; tudi kanonični impulz).

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}$$

če je L neodvisna od q_j.

$$\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j} = \sum_{k=1}^{n} w_{jk}\dot{q}_k - \frac{\partial V}{\partial \dot{q}_j}$$

1 Hamiltonova funkcija

Ne tisti Hamilton. Neki Irec, eden bolj priznanih angleško govorečih matematikov.

$$H = \sum_j p_j\dot{q}_j - L = H(\underline{q}, \underline{p}, t)$$

Če V ni odvisen od $\dot{q}$, velja tudi:

$$H = \sum_{j,k} w_{jk}\dot{q}_k\dot{q}_j - L$$

$$H = 2T - L = T + V$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_j\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right] - \sum_j\left[\frac{\partial L}{\partial q_j} + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right] - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

Izrek Emmy Noethen: Koordinato $q_j(t)$ zamenjajmo s $Q_j(t,s)$, pri čemer velja $\lim_{s\to 0} Q_j = q_j$. Razen tega naj bo L neodvisen od s.

$$\frac{\partial L}{\partial s} = \frac{\partial}{\partial s}L\left(\underline{Q}(t,s), \underline{\dot{Q}}(t,s), t\right) = 0$$

$$\frac{\partial L}{\partial s} = \sum_j\left[\frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial s} + \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial s}\right] = 0$$

Vzeli bomo limito $s \to 0$. Tedaj je Q_j kar enak q_j.

$$= \frac{\partial L}{\partial q_j}\frac{\partial Q_j}{\partial s}\Big|_{s=0} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial Q_j}{\partial s}\Big|_{s=0}$$

V prvi vsoti upoštevamo $\frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}$:

$$= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\frac{\partial Q_j}{\partial s}\Big|_{s=0} \Rightarrow \sum_{j=1}^{n} p_j\frac{\partial Q_j}{\partial s}\Big|_{s=0} = \text{konst.}$$

Fermatov princip: Oglejmo si lomni količnik. Ta je definiran kot $n(\overrightarrow{r}) = \dfrac{c}{v(\overrightarrow{r})}$.

$$t_0 = \int_{t_1}^{t_2} \frac{1}{c}\frac{c}{v}\frac{\mathrm{d}s}{\mathrm{d}t} dt = \int_0^{s_{AB}} n(s) ds$$

Hamiltonov princip. Definiramo akcijo:

$$\int_{t_1}^{t_2} L(\underline{q}(t), \underline{\dot{q}}(t), t)\, dt = S$$

Oznalimu tudi $A : \underline{q}(t) \to S$

Recimo, da imamo majhen odmik od začetnih koordinat:

$$q_j(t) \to q_j(t) + \delta q_j(t)$$

$$S = \int_{t_1}^{t_2} L\, dt$$

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \left(\sum_j \frac{\partial L}{\partial q_j} \delta q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt \\
&= \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} \delta q_j - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) dt \\
&= \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j\, dt
\end{aligned}$$

To pa mora biti enako 0, da je zadoščeno E-L pogoju.

Primer. $L' = L + \dfrac{\mathrm{d}}{\mathrm{d}t} F(\underline{q}(t), t)$

$$S' = S + F\Big|_{t_1}^{t_2}$$

$$0 = \delta S' = \delta S + \delta F \Rightarrow \delta F = 0$$

Enodimenzionalni problemi. Zanima nas $q(t)$, poznamo $q(0)$ in $\dot{q}(0)$.

$$L = \frac{1}{2} w(q) \dot{q}^2 - V(q)$$

$$H = T + V = E = \frac{1}{2} w(q) \dot{q}^2 + V(q) = \text{konst.}$$

Očitno je $0 \le \dot{q}^2 = \dfrac{2(E - V)}{w}$

$$\dot{q} = \pm \sqrt{\frac{2(E - V(q))}{w(q)}} = \frac{\mathrm{d}q}{\mathrm{d}t}$$

Stvar lahko integriramo po času in dobimo $q(t)$. Integral ni vedno trivialen, je pa uporabno vedeti, da vsak eno-dimenzionalen problem lahko rešimo s takim integralom.