Naloga: Vrtimo čašo vode, imamo hitrostno polje trdega telesa. Zanima nas $p(r)$. Pomagamo si lahko z Eulerjevo enačbo:

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla p + \vec{g}$$

Reševanje: Izrazimo hitrostno polje:

$$\vec{v} = \vec{w} \times \vec{r} = rw\hat{e}_\varphi$$

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = rw\dot{\hat{e}}_\varphi = -rw^2\hat{e}_r$$

$$\nabla p = -\rho g\hat{e}_z + w^2 r\hat{e}_r$$

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow p_z = -\rho g z$$

$$\frac{\partial p}{\partial r} = \rho w^2 r \Rightarrow p_r = \rho\frac{w^2r^2}{2}$$

$$p = p_z + p_r = \rho\frac{w^2r^2}{2} - \rho g z + C$$

Mimogrede omenimo, da je smer tega vektorja pravokotna na gladino vode. Izračunajmo obliko le-te:

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \partial_z p/\partial_r p = \frac{w^2 r}{g}$$

To pomeni, da ima površina obliko parabole:

$$z = \frac{w^2r^2}{2g} + z_0$$

Naloga: Sukanje v viskozni tekočini v 2D, velja Navier-Stokesov zakon:

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla p + \eta\nabla^2\vec{v}$$

Reševanje:

$$\rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \eta\nabla^2\vec{v}$$

$$\vec{v} = v(r)\hat{e}_\varphi$$

$$\vec{v\nabla}\vec{v} = v(r)\frac{1}{r}\frac{\partial\vec{v}}{\partial\varphi} = -\frac{v(r)}{r}\hat{e}_r$$

∇^2 v 2D poiščemo med formulami (in izpustimo člene, ki so enaki 0):

$$\nabla^2\vec{v} = \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}\right)\hat{e}_r + \left(\nabla^2 v\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}\right)\hat{e}_\varphi + \nabla^2 v_z\hat{e}_z$$

Vemo: $v_r = v_z = 0$, $v_\varphi = v(r)$ Če to vstavimo v Navier-Stokesov zakon, dobimo:

$$-\rho\frac{v^2}{r}\hat{e}_r = -\nabla p + \eta(\nabla^2 v - \frac{v}{r^2})\hat{e}_\varphi$$

V smeri $\hat{e}_r$: $\rho\frac{v^2}{r} = \frac{\partial p}{\partial r}$

V smeri $\hat{e}_\varphi$: $\nabla^2 v = \frac{v^2}{r^2}$

$$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2}$$

Dobimo Eulerjevo diferencialno enačbo:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = 0$$

$$v(r_1) = v_1$$

$$v(r_0) = 0$$

Rešujemo z nastavkom $v = Ar^l$, $\frac{\partial v}{\partial r} = lAr^{l-1}$, $\frac{\partial v}{\partial r} = l(l-1)Ar^{l-2}$ Ko to vstavimo v enačbo, dobimo:

$$l(l-1)Ar^{l-2} + lAr^{l-2} + Ar^{l-2} = 0$$

$$l(l-1) + l - 1 = 0$$

$$l^2 - 1 = 0 \Rightarrow l = \pm 1$$

$$v(r) = ar + \frac{b}{r}$$

Nazadnje v rešitev vstavimo robne pogoje:

$$ar_0 + \frac{b}{r_0} = 0$$

$$ar_1 + \frac{b}{r_1} = v_1$$

$$a = -\frac{b}{r_0^2}$$

$$\frac{b}{r_1} - \frac{br_1}{r_0^2} = v_1$$

$$br_1\left(\frac{1}{r_1^2} - \frac{1}{r_0^2}\right) = v_0$$

$$b = \frac{v_0 r_1}{\left(\frac{1}{r_1^2} - \frac{1}{r_0^2}\right)}$$