Zadnjič smo dobili

$$S = \begin{pmatrix} r & t \\ t & r \end{pmatrix}$$

Iz lastnosti $\psi_1(0) = \psi_2(0)$ in $[\psi_2'(0) - \psi_1'(0)] = -2\kappa_0\psi(0)$ izrazimo:

$$\frac{1}{\sqrt{V}} + \frac{r}{\sqrt{V}} = \frac{t}{\sqrt{V}}$$

$$1 + r = t$$

Poleg tega:

$$\psi_1' = \frac{ik}{\sqrt{V}}e^{ikx} - \frac{rik}{\sqrt{V}}e^{ikx}$$

$$\psi_2' = \frac{tik}{\sqrt{V}}e^{ikx}$$

To dvoje izrazimo v točki 0 in dobimo:

$$\frac{ik}{\sqrt{V}}[t - 1 + r] = -2\frac{\kappa_0}{\sqrt{V}}$$

$$ik2r = -2\kappa_0(1 + r)$$

Sledi:

$$r = \frac{-\kappa_0}{ik + \kappa_0}$$

$$t = \frac{ik}{ik + \kappa_0}$$

Tako lahko izrazimo S kot:

$$S = \frac{1}{ik + \kappa_0}\begin{pmatrix} -\kappa_0 & ik \\ ik & -\kappa_0 \end{pmatrix}$$

Transmitivnost:

$$T = |t|^2 = \left|\frac{ik}{ik + \kappa_0}\right|^2 = ... = \frac{E}{E + \frac{\hbar^2\kappa_0^2}{2m}}$$

Označimo $E_0 = \frac{\hbar^2\kappa_0^2}{2m}$. Tako je

$$T(E) = \frac{E}{E + E_0}$$

$$R(E) = \frac{E_0}{E + E_0}$$

In imamo $T(E) + R(E) = 1$, kar je tudi prav.

Naloga. Heisenbergova nedoločenost pravi, da je

$$\delta x \cdot \delta p \geq \frac{\hbar}{2}$$

Imejmo zdaj dva hermitska, sebi adjungirana operatorja ($A^\dagger = A$ in $B^\dagger = B$) in izračunajmo $\delta A \cdot \delta B$. Ker sta A in B hermitsko sebi adjungurana, sta njuni pričakovani vrednosti realni.

$$\delta^2 A = \langle (A - \langle A \rangle)^2 \rangle$$

$$\delta^2 B = \langle (B - \langle B \rangle)^2 \rangle$$

Konstruiramo nova operatorja $\tilde{A}$ in $\tilde{B}$:

$$\tilde{A}^2 = (A - \langle A \rangle)^2$$

$$\tilde{B}^2 = (B - \langle B \rangle)^2$$

Sledi

$$\delta^2 A \delta^2 B = \langle \tilde{A}^2 \rangle \langle \tilde{B}^2 \rangle$$

Ker sta $\tilde{A}$ in $\tilde{B}$ hermitsko sebi adjungirana, je $\langle\psi|\tilde{A}^2|\psi\rangle = \langle\tilde{A}\psi|\tilde{A}\psi\rangle$ in $\langle\psi|\tilde{B}^2|\psi\rangle = \langle\tilde{B}\psi|\tilde{B}\psi\rangle$. Zaradi Cauchy-Schwarzove neenakosti velja:

$$\langle\tilde{A}\psi|\tilde{A}\psi\rangle\langle\tilde{B}\psi|\tilde{B}\psi\rangle \geq \left|\langle\tilde{A}\psi|\tilde{B}\psi\rangle\right|$$

To je (spet, zaradi hermitskih operatorjev) enako $\langle\psi|\tilde{A}\tilde{B}\psi\rangle$. Naredimo pomožni izračun:

$$\tilde{A}\tilde{B} = \frac{\tilde{A}\tilde{B}+\tilde{B}\tilde{A}}{2} + \frac{\tilde{A}\tilde{B}-\tilde{B}\tilde{A}}{2}$$

Izrazu v števcu prvega odlokma rečemo antikomutator ($\{\tilde{A},\tilde{B}\}$), izrazu v števcu drugega odlomka pa komutator ($[\tilde{A},\tilde{B}]$). Izračunamo lahko $[\tilde{A},\tilde{B}]^\dagger = [\tilde{B},\tilde{A}] = -[\tilde{A},\tilde{B}]$ in $\{\tilde{A},\tilde{B}\} = \{\tilde{A},\tilde{B}\}$ Vidimo, da je $\{\tilde{A},\tilde{B}\}$ hermitski operator, $[\tilde{A},\tilde{B}]$ pa antihermitski. Sledi:

$$\delta^2 A\delta^2 B = \left|\langle\psi|\left(\frac{\{\tilde{A},\tilde{B}\}}{2} + \frac{[\tilde{A},\tilde{B}]}{2}\right)\psi\rangle\right|^2 =$$

$$= \langle\psi|\frac{\{\tilde{A},\tilde{B}\}}{2}\psi\rangle^2 + \left|\langle\psi|\frac{[\tilde{A},\tilde{B}]}{2}\psi\rangle\right|^2$$

Obe vrednosti sta pozitivni, zato je rezultat gotovo večji od le ene od njiju. Izberemo komutator (izbira je do neke mere arbitrarna).

$$\langle\psi|\frac{\{\tilde{A},\tilde{B}\}}{2}\psi\rangle^2 + \left|\langle\psi|\frac{[\tilde{A},\tilde{B}]}{2}\psi\rangle\right|^2 \geq \left|\langle\psi|\frac{[\tilde{A},\tilde{B}]}{2}\psi\rangle\right|^2$$

S pomožnim izračunom pokažemo, da je $[\tilde{A}, tildeB] = [A,B]$. Sledi:

$$\delta A\cdot\delta B = \left|\langle\frac{[A,B]}{2}\rangle\right|$$

Primer: x in $p = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$.

$$[x,p]\psi(x) = (xp-px)\psi = (-xi\hbar\frac{\mathrm{d}}{\mathrm{d}x} + \left(i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)x)\psi(x) =$$

$$-i\hbar\psi'(x) + i\hbar\left(\psi(x)+x\psi'(x)\right) = i\hbar\psi(x)$$

$$\delta x\cdot\delta p \geq |\langle i\hbar/2\rangle|$$

Če se želimo znebiti znaka $\geq$, imamo dve zahtevi:

Da Cauchy-Schwarzova neenakost postane enakost, mora veljati

$$\tilde{B}|\psi\rangle = \lambda\tilde{A}|\psi\rangle$$

Poleg tega smo izpustili člen z antikomutatorjem, torej

$$\langle\psi|\{\tilde{A},\tilde{B}\}\psi\rangle = 0 = \langle\psi|\tilde{A}\tilde{B}\psi\rangle + \langle\psi|\tilde{B}\tilde{A}\psi\rangle$$

$$= ... = (\lambda+\lambda^*)\langle\tilde{A}\psi|\tilde{A}\psi\rangle$$

Torej bo enakost veljala, če je $\lambda = \mu i,\ \mu\in\mathbb{R}$, ali pa, če je $\langle\tilde{A}\psi|\tilde{A}\psi\rangle = 0$ (kar pa je nek poseben primer, v katerem je $\delta B = 0$, zato se s tem ne bomo ukvarjali). Katere funkcije imajo torej minimalni produkt nedoločenosti x in p? Dobimo diferencialno enačbo:

$$\left(-ih\frac{\mathrm{d}}{\mathrm{d}x} - \langle p\rangle\right)\psi(x) = -\mu\left(x-\langle x\rangle\right)\psi(x)$$

Rešujemo s separacijo:

$$\frac{\mathrm{d}\psi}{\psi} = \frac{1}{-i\hbar}\left[i\mu(x-\langle x\rangle)+\langle p\rangle\right]\mathrm{d}x$$

$$\ln\psi = \frac{1}{-ih}\left[i\mu x\langle x\rangle + \langle p\rangle\right] - i\mu\frac{x^2}{2} + C$$

$$\psi(x) = C\exp\left(\frac{\mu}{\hbar}\langle x\rangle x + i\langle p\rangle x + \frac{\mu x^2}{2\hbar}\right)$$

Stvar je malo podobna Gaussovemu valovnemu paketu.