Dobili smo Fourierovo transformiranko funkcije, ki opisuje potencial (U) v okolici točkastega naboja:

$$U(\vec{k}) = \frac{e}{(2\pi)^3\varepsilon_0}\frac{1}{k^2}$$

Transformiramo nazaj:

$$U(\vec{r}) = \int U(\vec{k})e^{i\vec{k}\cdot\vec{r}}\,\mathrm{d}^3\vec{k}$$

$$= \frac{e}{(2\pi)^3\varepsilon_0}\,2\pi\int_{-1}^{1}\int_0^\infty k^2\frac{e^{ikr\cos\theta}}{k^2}\,\mathrm{d}k\,\mathrm{d}(\cos\theta)$$

$$= \frac{e}{(2\pi)^2\varepsilon_0}\int_0^\infty \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)\,\mathrm{d}k$$

$$= \frac{2e}{(2\pi\varepsilon_0)r}\int_0^\infty \frac{\sin(kr)}{k}\,\mathrm{d}k$$

Vzamemo novo spremenljivko $u = kr$

$$= \frac{2e}{(2\pi)^2}\varepsilon_0 r\int_0^\infty \frac{\sin u}{u}\,\mathrm{d}u$$

To je znan integral z vrednostjo $\pi/2$. Sledi:

$$U(\vec{r}) = \frac{e}{4\pi\varepsilon_0 r}$$

Poljubno porazdelitev sestavimo iz točkastih nabojev.

$$e(\vec{r}) = \int \rho(\vec{r'})\delta(\vec{r} - \vec{r'})\,\mathrm{d}^3\vec{r}$$

$$U(\vec{r}) = \int \frac{\rho(\vec{r'})\,\mathrm{d}^3\vec{r'}}{4\pi\varepsilon_0|\vec{r} - \vec{r'}|}$$

$$\vec{E}(\vec{r}) = -\nabla U(\vec{r}) = -\frac{\partial}{\partial\vec{r}}U(\vec{r})$$

$$E(\vec{r}) = \int \frac{\rho(\vec{r'})(\vec{r} - \vec{r'})\,\mathrm{d}^3\vec{r}}{4\pi\varepsilon_0|\vec{r} - \vec{r'}|^3}$$

Gostota naboja v vodikovem atomu (osnovno stanje)

$$U(\vec{r}) = \frac{e}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)$$

Zanima nas $\rho(r)$.

V sferičnih koordinatah s sferično simetrijo velja:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

Vemo:

$$\nabla^2 U = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\rho(\vec{r}) = -\frac{e}{4\pi}\nabla^2\left[\frac{e}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)\right]$$

$$= -\frac{e}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left\{\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)\right\}\right]$$

$$= -\frac{e}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{-\alpha r e^{-\alpha r} - e^{-\alpha r}}{r^2} - \alpha^2\frac{e^{-\alpha^r}}{2}\right)\right]$$

$$= -\frac{e}{4\pi r^2}\frac{\partial}{\partial r}\left(-\alpha r e^{-\alpha r} - e^{-\alpha r} - \frac{\alpha^2}{2}r^2 e^{-\alpha r}\right)$$

$$= -\frac{e}{4\pi r^2}\left[-\alpha e^{-\alpha r}\left(1+\alpha r+\frac{\alpha^2 r^2}{2}\right) + e^{-\alpha r}\left(\alpha+\alpha^2 r\right)\right]$$

$$= -\frac{e}{4\pi r^2}\left[e^{-\alpha r}\frac{\alpha^3 r^2}{2}\right] = -\frac{e_0 e^{-\alpha r}\alpha^3}{8\pi}$$

Ampak! V $r = 0$ ima atom vodika singularnost (proton). Izračunali smo le porazdelitev negativnega naboja. Skupna porazdelitev je torej enaka:

$$\rho(\overrightarrow{r}) = e_0\delta(\overrightarrow{r}) - \frac{e_0\alpha^2}{8\pi}\, e^{\alpha r}$$

Laplaceova enačba Bodi $\rho(\overrightarrow{r}) = 0$ skoraj povsod (kjer ni, dobimo robne pogoje). Rešujemo torej Laplaceovo enačbo:

$$\nabla^2 U(\overrightarrow{r}) = 0$$

Ploščat kondenzator s prečnim trakom Imamo dve ozemljeni plošči, med njima je potencial enak 0. V ta potencial postavimo nabit trak širine a - ki je enaka razdalji med ploščama kondenzatorja. Vidimo, da os z ne igra nobene vloge, torej je naš problem omejen le na osi x in y. Rešujemo Laplaceov problem:

$$\nabla^2 U(x, y) = 0$$

Z robnimi pogoji:

$$U(x,0) = 0 \quad \text{(spodnja plošča)}$$

$$U(x,a) = 0 \quad \text{(zgornja plošča)}$$

$$U(0,y) = U_0 \quad \text{(trak)}$$

$$U(\infty, y) < \infty$$