1 Spin

V dveh dimenzijah imamo delec s spinom $S = 1/2$. Zapišemo Hamiltonian

$$H = \frac{p^2}{2m} + \lambda \left(p_x S_y - p_y S_x\right), \quad \overrightarrow{p} = (p_x, p_y)$$

Rešujemo Schrödingerjevo enačbo

$$H\psi = E\psi$$

V nadaljnjih izračunih nam bo koristil komutator $[H, \overrightarrow{p}]$. Izračunamo ga po komponentah.

$$[H, p_x] = \left[\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \lambda \left(p_x S_y - p_y S_x\right),\ p_x\right] = 0$$

Količini p_x in $p_x S_y$ sta v različnih podprostorih, torej gotovo komutirata. Z ostalimi nimamo težav. Ker komutirata, lahko lastna stanje H izrazimo z lastnimi stanji $\overrightarrow{p}$. Za krajevni del vemo:

$$\overrightarrow{p}\left|\overrightarrow{k}\right\rangle = \hbar \overrightarrow{k}\left|\overrightarrow{k}\right\rangle$$

$$\overrightarrow{p} = -i\hbar\nabla$$

Takšna enačba nam da rešitev

$$\psi_{\overrightarrow{k}}(\overrightarrow{r}) \propto e^{i\overrightarrow{k}\cdot\overrightarrow{r}}$$

Ker lahko rešimo krajevni del in ker operatorja H in $\overrightarrow{p}$ komutirata, lahko uporabimo nastavek

$$|\psi\rangle = \left|\overrightarrow{k}\right\rangle |\chi\rangle$$

Kjer $\left|\overrightarrow{k}\right\rangle$ pomeni krajevni del, $|\chi\rangle$ spinski del.

$$\left[\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \lambda \left(p_x S_y - p_y S_x\right)\right]\left|\overrightarrow{k}\right\rangle |\chi\rangle = E\left|\overrightarrow{k}\right\rangle |\chi\rangle$$

$$\left(\frac{p^2}{2m}\left|\overrightarrow{k}\right\rangle\right)|\chi\rangle + \lambda\left(p_x\left|\overrightarrow{k}\right\rangle\right)\left(S_y|\chi\rangle\right) - \lambda\left(p_y\left|\overrightarrow{k}\right\rangle\right)\left(S_x|\chi\rangle\right)$$

Vstavimo lastne vrednosti krajevnega dela:

$$\frac{\hbar^2 k^2}{2m}\left|\overrightarrow{k}\right\rangle|\chi\rangle + \lambda\hbar k_x\left|\overrightarrow{k}\right\rangle S_y|\chi\rangle - \lambda\hbar k_y\left|\overrightarrow{k}\right\rangle S_x|\chi\rangle = E\left|\overrightarrow{k}\right\rangle|\chi\rangle$$

Pokrajšamo $\overrightarrow{k}$:

$$\frac{\hbar^2 k^2}{2m}|\chi\rangle + \lambda\hbar k_x S_y|\chi\rangle - \lambda\hbar k_y S_x|\chi\rangle = E|\chi\rangle$$

Dobili smo dvidimenzionalen problem v spinskem prostoru. Za $S = 1/2$ imamo dve možnosti. Označimo:

$$S = \frac{1}{2} \qquad \begin{array}{l}\left|\frac{1}{2}, +\frac{1}{2}\right\rangle = |\uparrow\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = |\downarrow\rangle\end{array} \qquad \begin{array}{l} S_x = \frac{S_+ + S_-}{2} \\ S_y = \frac{S_+ - S_-}{2i}\end{array}$$

Veljajo sledeče zveze:

$$\begin{aligned} S_+|\uparrow\rangle &= 0 \\ S_+|\downarrow\rangle &= \hbar|\uparrow\rangle \\ S_-|\uparrow\rangle &= \hbar|\downarrow\rangle \\ S_-|\downarrow\rangle &= 0 \end{aligned}$$

Tako izpeljemo zveze za S_x in S_y:

$$S_x \left|\uparrow\right\rangle = \frac{\hbar}{2} \left|\downarrow\right\rangle$$
$$S_x \left|\downarrow\right\rangle = \frac{\hbar}{2} \left|\uparrow\right\rangle$$
$$S_y \left|\uparrow\right\rangle = \frac{\hbar}{2i} \left|\downarrow\right\rangle$$
$$S_y \left|\downarrow\right\rangle = \frac{\hbar}{2} \left|\uparrow\right\rangle$$

Za nastavek $\left|\chi\right\rangle$ uporabimo linearno kombinacijo:

$$\left|\chi\right\rangle = \alpha \left|\uparrow\right\rangle + \beta \left|\downarrow\right\rangle$$
$$\frac{\hbar^2 k^2}{2m} \left(\alpha \left|\uparrow\right\rangle + \beta \left|\downarrow\right\rangle\right) + \lambda k_x \frac{\hbar^2}{2i} \left(-\alpha \left|\downarrow\right\rangle + \beta \left|\uparrow\right\rangle\right) - \lambda k_y \frac{\hbar}{2i} \left(\alpha \left|\downarrow\right\rangle + \beta \left|\uparrow\right\rangle\right) = E \left(\alpha \left|\uparrow\right\rangle + \beta \left|\downarrow\right\rangle\right)$$

Iz ujemanja koeficientov dobimo dve linearni enačbi:

$$\frac{\hbar^2 k^2}{2m}\alpha + \lambda\hbar\beta \left(k_x \frac{\hbar}{2i} - k_y \frac{\hbar}{2}\right) = E\alpha \tag{1}$$
$$\frac{\hbar^2 k^2}{2m}\beta + \lambda\hbar\alpha \left(-k_x \frac{\hbar}{2i} - k_y \frac{\hbar}{2}\right) = E\beta \tag{2}$$

Sistem prepišemo v problem lastnih vrednosti:

$$\begin{bmatrix} \frac{\hbar^2 k^2}{2m} & \frac{\lambda\hbar^2}{2}\left(\frac{k_x}{i} - k_y\right) \\ \frac{\lambda\hbar^2}{2}\left(-\frac{k_x}{i} - k_y\right) & \frac{\hbar^2 k^2}{2m} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = E \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$
$$\begin{vmatrix} \frac{\hbar^2 k^2}{2m} - E & \frac{\lambda\hbar^2}{2}\left(\frac{k_x}{i} - k_y\right) \\ \frac{\lambda\hbar^2}{2}\left(-\frac{k_x}{i} - k_y\right) & \frac{\hbar^2 k^2}{2m} - E \end{vmatrix} = 0 = \left(\frac{\hbar^2 k^2}{2m} - E\right) + \frac{\lambda^2\hbar^4}{4}\left(-k^2\right)$$
$$= \left(\frac{\hbar^2 k^2}{2m} - E - \frac{\lambda\hbar^2 k}{2}\right)\left(\frac{\hbar^2 k^2}{2m} - E + \frac{\lambda\hbar^2 k}{2}\right)$$

Dobili smo dve lastni vrednosti E_1 in E_2:

$$E_1 = \frac{\hbar^2 k^2}{2m} - \frac{\lambda\hbar^2 k}{2}$$
$$E_2 = \frac{\hbar^2 k^2}{2m} + \frac{\lambda\hbar^2 k}{2}$$

Da izračunamo lastni funkciji, nam bo koristila menjava koordinatnega sistema:

$$k_x + ik_y = ke^{i\varphi}$$
$$\frac{k_x}{i} - k_y = \frac{k}{i}\, e^{-i\varphi}$$
$$-\frac{k_x}{i} - k_y = -\frac{k}{i}\, e^{i\varphi}$$

Tako za E_1 dobimo:

$$\lambda\hbar^2 \left(\frac{\hbar}{2}\alpha + \frac{\hbar}{2i}\, e^{-i\varphi}\beta\right) = 0 \quad \rightarrow \beta = -i\alpha e^{-i\varphi}$$

Nato dobimo α iz normalizacije $|\alpha|^2 + |\beta|^2 = 1$: dobimo $2|\alpha|^2 = 1$

$$\chi_1 = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle - ie^{i\varphi}\left|\downarrow\right\rangle\right)$$

Podobno:

$$\chi_2 = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle + ie^{i\varphi}\left|\downarrow\right\rangle\right)$$

Bolj splošno, za sfero:

$$\alpha \left|\uparrow\right\rangle + \beta \left|\downarrow\right\rangle = \cos\frac{\vartheta}{2} \pm \sin\frac{\vartheta}{2} e^{i\varphi} \left|\downarrow\right\rangle$$

V našem primeru je $\vartheta = \pi/2$