Harmonični oscilator. (Teorija) Opravka imamo s Hamiltonianom oblike

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \hbar\omega(a^\dagger a + \frac{1}{2})$$

Pri tem je:

$$\omega = \sqrt{\frac{k}{m}}$$
$$a \equiv \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right)$$
$$a^\dagger \equiv \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right)$$

Dobimo sledeče:

$$x = \frac{x_0}{\sqrt{2}}\left(a + a^\dagger\right)$$
$$p = \frac{p_0}{\sqrt{2}}\left(a - a^\dagger\right)$$
$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad n = 1, 2, 3...$$
$$a|n\rangle - \sqrt{n}|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a^\dagger a|n\rangle = n|n\rangle$$
$$[a, a^\dagger] = 1$$

Naloga. Opravka imamo z valovno funkcijo

$$|\psi, 0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

Zanima nas $\langle x, t\rangle$, $\langle p, t\rangle$, $\delta x(t)$ in $\delta p(t)$.
Schrödingerjeva slika:

$$|\psi, t\rangle = \frac{1}{\sqrt{2}}e^{-i\frac{\hbar\omega}{2\hbar}t}|0\rangle + \frac{i}{\sqrt{2}}e^{-i\frac{3\hbar\omega}{2\hbar}t}|1\rangle$$
$$\langle x, t\rangle = \langle\psi, t|x|\psi, t\rangle = \langle\psi, t|\frac{x_0}{\sqrt{2}}\left(a + a^\dagger\right)|\psi, t\rangle = \frac{x_0}{\sqrt{2}}\,2\,\mathfrak{Re}\langle a, t\rangle$$
$$\langle p, t\rangle = \langle\psi, t|\frac{p_0}{\sqrt{2}i}\left(a - a^\dagger\right)\langle\psi, t| = \frac{p_0}{\sqrt{2}}\,2\,\mathfrak{Im}\langle a, t\rangle$$

Stvar se poenostavi na iskanje časoavnega razvoja $\langle a, t\rangle$:

$$\langle a, t\rangle = \langle\psi, t|a|\psi, t\rangle = \langle\psi, t||\left(\frac{1}{\sqrt{2}}e^{-i\frac{\omega t}{2}}a|0\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{3\omega t}{2}}a|1\rangle\right)\rangle$$

Vemo: $a|0\rangle = 0$ in $a|1\rangle = |0\rangle$.

$$= \left(\frac{1}{\sqrt{2}}e^{i\frac{1}{2}\omega t}\langle 0| - \frac{i}{\sqrt{2}}e^{i\frac{3}{2}\omega t}\langle 1|\right)\left(\frac{i}{2}e^{-i\frac{3}{2}\omega t}|0\rangle\right) = \frac{i}{2}e^{i\frac{1}{2}\omega t - i\frac{3}{2}\omega t} = \frac{i}{2}e^{-i\omega t} = \frac{i}{2}\cos\omega t + \frac{1}{2}\sin\omega t$$

Če pri časovnem razvoju vzamemo le realni in imaginarni del, so to kosinusni in sinusi, torej dejansko dobimo nihanje.

Ernfestov teorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A, t\rangle = \frac{i}{\hbar}\langle[H, A], t\rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x,t\rangle = \frac{\langle p,t\rangle}{m}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p,t\rangle = \frac{i}{\hbar}\langle[\frac{1}{2}kx^2,p],t\rangle = \frac{i}{\hbar}\frac{k}{2}\langle[x^2,p],t\rangle = \frac{i}{\hbar}\frac{k}{2}\langle(x[x,p]-[p,x]x)\,,t\rangle = \frac{i}{\hbar}\frac{k}{2}\langle ki\hbar x,t\rangle = -k\langle x,t\rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p,t\rangle = -k\langle x,t\rangle$$

Preverimo, ali to res velja tudi za ta sistem: Odvedemo prej izračunana časovna razvoja za $\langle x,t\rangle$ in $\langle p,t\rangle$:

$$\langle x,t\rangle = \frac{x_0}{\sqrt{2}}\sin\omega t$$

$$\langle p,t\rangle = \frac{p_0}{\sqrt{2}}\cos\omega t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_0}{\sqrt{2}}\sin\omega t\right) = \frac{x_0}{\sqrt{2}}\omega\cos\omega t = \frac{x_0}{\sqrt{2}}\sqrt{\frac{k}{m}}\cos\omega t = \frac{1}{\sqrt{m}}\sqrt{\frac{kx_0^2}{2}}\cos\omega t = \frac{1}{\sqrt{m}}\sqrt{\frac{p_0^2}{2m}}\cos\omega t = \frac{\langle p,t\rangle}{m}\cos\omega t$$

Podobno za $\langle p,t\rangle$

Heisenbergova slika: Začnemo z Ernfestovim teoremom in rešujemo problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = \frac{i}{\hbar}[H,a](t) = \frac{i}{\hbar}(-\omega)a(t)$$

Začetni pogoj: $a(0) = a_0$ Enačbo hitro rešimo s separacijo in dobimo

$$a(t) = a_0\,e^{-i\omega t}$$

Dobili smo $\langle a,t\rangle$, kar lahko uporabimo v rezultatu, dobljenem v Schrödingerejvi sliki in si prihranimo nekaj računanja.

$$\langle x,t\rangle = \sqrt{2}x_0\,\mathfrak{Re}\langle a,t\rangle = \sqrt{2}x_0\,\mathfrak{Re}\langle a_0e^{-i\omega t},0\rangle$$
$$\langle p,t\rangle = \sqrt{2}p_0\,\mathfrak{Im}\langle a,t\rangle = \sqrt{2}p_0\,\mathfrak{Im}\langle a_0e^{-i\omega t},0\rangle$$

In tako naprej. Prednost takega zapisa je v tem, da moramo izračunati le časovni razvoj operatorja, ostalo pa lahko izpeljemo direktno iz tega.

$$\delta^2x(t) = \langle x^2,t\rangle - \langle x,t\rangle^2$$

$$\delta^2x(t) = \langle p^2,t\rangle - \langle p,t\rangle^2$$

$$\langle x^2,t\rangle = \frac{x_0^2}{2}\langle(a+a^\dagger)^2,t\rangle = \frac{x_0^2}{2}\langle a^2+aa^\dagger+a^\dagger a+{a^\dagger}^2,\ t\rangle$$

$$\langle p^2,t\rangle = \frac{p_0^2}{2}\langle(a-a^\dagger)^2,t\rangle = \frac{p_0^2}{2}\langle a^2-aa^\dagger-a^\dagger a+{a^\dagger}^2,\ t\rangle$$

Opomba: Ker je $[a,a^\dagger]=1$, lahko izrazimo $aa^\dagger = 1+a^\dagger a$.

$$\langle x^2,t\rangle = \frac{x_0}{2}\left(2\mathfrak{Re}\langle a^2,t\rangle + \langle a^\dagger a,t\rangle + 1\right)$$

$$\langle p^2,t\rangle = \frac{p_0}{2}\left(2\mathfrak{Re}\langle a^2,t\rangle - \langle a^\dagger a,t\rangle - 1\right)$$

Zdaj izračunamo $\langle a^2,t\rangle$ in $\langle a^\dagger a,t\rangle$. Spomnimo se:

$$(AB)(t) = A(t)B(t)$$

$$A^\dagger(t) = (A(t))^\dagger$$

Tako je $a^2(t) = (a(t))^2$ in $(a^\dagger a)(t) = (a(t))^\dagger\,a(t)$

$$a^2(t) = a_0^2e^{-2i\omega t}$$

$$(a^\dagger a)(t) = \left(a_0e^{-i\omega t}\right)^\dagger a_0e^{-i\omega t} = a_0^\dagger e^{i\omega t}a_0e^{-i\omega t} = a_0^\dagger a_0$$

To vstavimo v prej pridobljeni izraz in dobimo:

$$\langle x^2, t\rangle = \frac{x_0^2}{2}\left(2\mathfrak{Re}\left(e^{-2i\omega t}\langle a^2, 0\rangle\right) + 2\langle a_0^\dagger a_0, 0\rangle\right)$$

$$\langle p^2, t\rangle = \frac{p_0^2}{2}\left(2\mathfrak{Im}\left(e^{-2i\omega t}\langle a^2, 0\rangle\right) - 2\langle a_0^\dagger a_0, 0\rangle\right)$$

Na hitro izpeljemo, da je $\langle a_0^\dagger a_0, 0\rangle = \frac{1}{2}$ in dobimo končni rezultat:

$$\delta^2 x(t) = x_0^2\left(1 - \frac{1}{2}\sin^2 \omega t\right)$$

$$\delta^2 p(t) = p_0^2\left(1 - \frac{1}{2}\cos^2 \omega t\right)$$