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The complexity of computing minimum separating polygons

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Abstract

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Suppose that S and T are two sets of points in the plane. A *separating polygon* separating S from T is a simple polygonal circuit P for which every point of S is interior or on the boundary of P, and every point of T is exterior or on the boundary of P. Let $D(P)$ denote the perimeter of a polygon P, that is, the sum of the Euclidean lengths of the edges of P. If P is a separating polygon for S and T with minimum value of $D(P)$ then P is called a *minimum separating polygon*. The problem of computing a minimum separating polygon has been studied in various forms in connection with applications to computer vision and collision avoidance. In this note we show that the version of the problem where S and T are finite sets of points is NP-hard.

1. Introduction

Suppose that S and T are two sets of points in the plane. A *separating polygon* separating S from T is a simple polygonal circuit P for which every point of S is interior or on the boundary of P, and every point of T is exterior or on the boundary of P. Let $D(P)$ denote the perimeter of a polygon P, that is, the sum of the Euclidean lengths of the edges of P. If P is a separating polygon for S and T with minimum value of $D(P)$ then P is called a *minimum separating polygon*. See Figure 1. In this note we examine the problem of efficiently computing minimum separating polygons.

Various forms of this problem have been studied in connection with applications to computer vision and collision avoidance; see [4] for a bibliography.

Consider, for example, the case where T is a simple polygon containing another simple polygon S. A polygon that separates S from T with minimum perimeter is referred to by Toussaint as the *convex hull of S relative to T* and is denoted by $CH(S \mid T)$. The results

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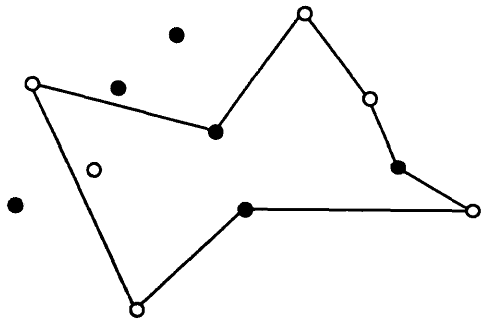

Figure 1. The minimum separating polygon of S and T. The solid points represent the set T.

of [4] can be used to give a linear time algorithm to compute $CH(S \mid T)$.

Another example is the case where S is a finite set of points enclosed within a polygon T. Here the relative convex hull of S with respect to T is a *weakly simple polygon*, that is, a polygonal circuit where some vertices and edges are permitted to overlap but not cross [4]. In Figure 2, the convex hull of the points S relative to the polygon T illustrates this concept. An $O(n \log n)$ algorithm based on triangulation and shortest path methods is given in [4] to find $CH(S \mid T)$.

In this note we consider the case where S and T are finite sets of points. In contrast to the examples above, an efficient algorithm for this case is not known. In fact, we show that the problem is NP-hard.

A standard reference on the theory of NP-completeness is [1].

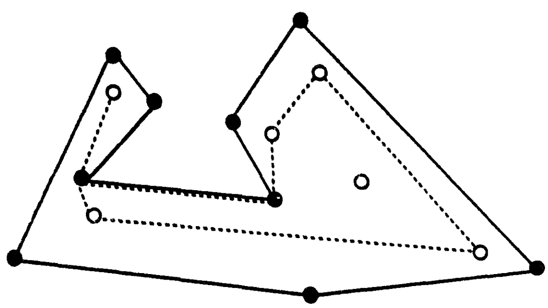

Figure 2. $CH(S \mid T)$ is denoted by dashed lines.

2. Computing a minimum separating polygon is NP-hard

We show that computing the minimum separating polygon is NP-hard by presenting a polynomial time transformation from the following NP-complete variant of the travelling salesman problem [3].

TSP

Instance. A set S of n points with integer coordinates.

Question. Is there a travelling salesman tour of the set S whose Euclidean length is n?

We will prove that the following decision problem is NP-hard.

MSP

Instance. Finite sets S and T of points, and a bound K.

Question. Is there a separating polygon P for the sets S and T such that $D(P) \leqslant K$?

Theorem. MSP *is NP-hard.*

Proof. Given an instance S of TSP we will construct a second set of points T and a bound K so that any algorithm for MSP can be used to solve TSP.

For any point p, denote the x- and y-coordinates of p by $x(p)$ and $y(p)$ respectively. For each point $s \in S$ define $t(s)$ to be the point $(x(s), y(s)+\varepsilon)$, for some positive value ε. Let T denote the set $\{t(s) : s \in S\}$. This construction is illustrated in Figure 3.

Choose K to be $n(1+2\varepsilon)$.

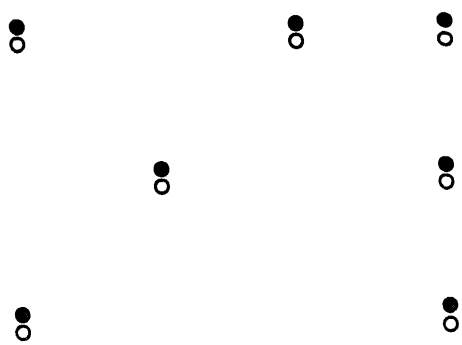

Figure 3. The construction of T.

Suppose that there is a separating polygon P for the sets S and T with $D(P) \leqslant K$.

For each point s, denote the line segment between s and $t(s)$ by $l(s)$. Note that any separating polygon must cross $l(s)$ for each point $s \in S$. We can construct a polygon $\bar{Q}$ through all of the points in S by connecting the points s to s' if and only if P crosses $l(s)$ then crosses $l(s')$. Observe that if the polygon P crosses the line segment $l(s)$ at p and crosses the line segment $l(s')$ at p', then

$$d(s, s') \leqslant d(p, p') + \varepsilon ,$$

where d denotes Euclidean distance. Summing this inequality over all edges (s, s') of the polygon $\bar{Q}$ we obtain:

$$D(\bar{Q}) \leqslant D(P) + n\varepsilon < n(1+3\varepsilon) ,$$

since $D(P) \leqslant K$. Now choose $\varepsilon < 1/(9n)$; then

$$D(\bar{Q}) < n + 1/3 . \qquad (*)$$

We claim further that $D(\bar{Q}) = n$. Note that every edge of a travelling salesman tour of the set S has length at least one, and so $D(\bar{Q}) \geqslant n$. If $D(\bar{Q}) > n$, then at least one of the edges of $\bar{Q}$ has length at least $\sqrt{2}$, thus $D(\bar{Q}) \geqslant n - 1 + \sqrt{2}$. This contradicts $(*)$, so $D(\bar{Q}) = n$.

Conversely, suppose that there is a travelling salesman tour Q of the set S of length n. To construct a simple polygon separating the set S from the set T, add the segment $(s, t(s))$ and replace (s, s') by $(t(s), s')$ whenever a point $t(s)$ is interior to Q. The perimeter of the separating polygon $\bar{P}$ constructed this way has the property:

$$D(\bar{P}) \leqslant D(Q) + 2n\varepsilon ,$$

because $d(s, t(s)) = \varepsilon$ and $d(t(s), s') \leqslant d(s, s') + \varepsilon$. Hence $D(\bar{P}) < K$, since $D(Q) = n$.

For each instance S of TSP we have constructed an instance S, T, K of MSP such that a tour of length n of S exists if and only if there is a separating polygon for S and T of length at most K. Therefore, MSP is NP-hard. □

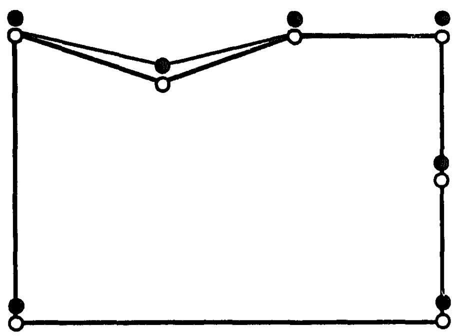

Figure 4. We replace two edges in the polygon P to create the travelling salesman tour $\bar{Q}$ and increase the total length by no more than 2ε.

3. Discussion

We have shown that computing a minimum separating polygon is NP-hard. Therefore heuristic methods are justified. Although we do not know of any provably good approximation methods, some promising heuristics have been tested [2].

We can suggest an approximation algorithm for a simplified version of our problem. Let MSPCON denote a restricted version of MSP where the set S in the instance is restricted to the vertices of a convex polygon, as in Figure 5. We can obtain a provably good approximate solution to MSPCON in time $O(n \log n)$ as follows.

The approximation scheme distinguishes those points of the set T that are inside the convex hull of S from those that are outside or on the boundary of the convex hull of S. We take the convex hull H of those points of T inside and a single edge e of the convex hull of S. Note that the edge e occurs both on H and on the convex hull of S; thus we merge H with the convex hull of S by deleting the edge e. This gives a separating polygon whose perimeter is no more that

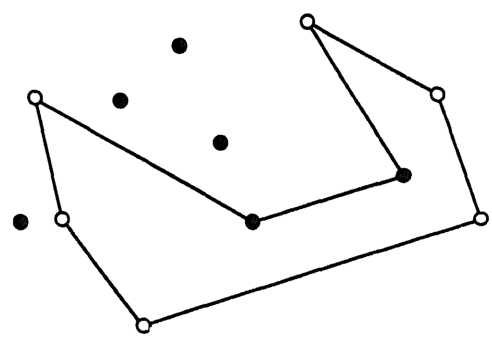

Figure 5. An instance of MSPCON and the separating polygon as obtained by our approximation algorithm.

twice the perimeter of the convex hull of S. Since the minimum length of a separating polygon is at least as large as the perimeter of the convex hull of S, the result of our approximation algorithm is at most twice the minimum.

An unanswered question is whether there is an efficient approximation algorithm for the general minimum separating polygon problem. It would also be interesting to know whether there are any special cases of this problem where efficient exact algorithms can be found.

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