Classical and Neoclassical Transport in Fusion Plasmas

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PHD in Fusion Science and Engineering

Advanced course on plasma physics & diagnostics

Why Transport?

- Fusion on Earth requires a really hot plasma ($\geq 10keV$).
- For a commercial use of fusion energy we need to *keep* that plasma hot, few meters away from the room temperature.
- Heat flows from hot to cold regions.
- Time and space scales of energy/particle transport contribute to determine the reactor dimension.

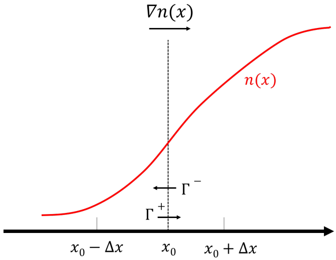
Transport

These slides deal with the question of how well a magnetically confined plasma keeps itself hot.
This subject is called plasma transport.

Outline

Diffusion

- 1D example with density gradient $\nabla n(x)$ along the x-axis.
- Random walk of particle with step $\pm\Delta x$ every time step Δt.
- Flux Γ^+ at x_0 from 1/2 particles in the region $[x_0 - \Delta x, x_0]$.
- Expand $n(x) = n(x_0) + n'(x_0)(x - x_0)$.



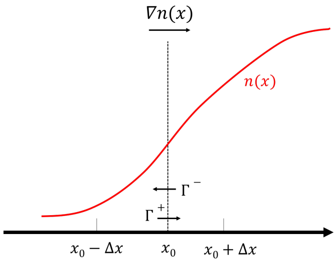
$$\Gamma^+ = \frac{1}{2}\frac{1}{\Delta t}\int_{x_0-\Delta x}^{x_0} n(x)dx =$$

$$\frac{1}{2\Delta t}\left[n(x_0)x + \left(\frac{\partial n}{\partial x}\right)_{x_0}\frac{(x-x_0)^2}{2}\right]_{x_0-\Delta x}^{x_0} =$$

$$\frac{1}{2\Delta t}\left[n(x_0)\Delta x - \left(\frac{\partial n}{\partial x}\right)_{x_0}\frac{(\Delta x)^2}{2}\right]$$

Diffusion

- 1D example with density gradient $\nabla n(x)$ along the x-axis.
- Random walk of particle with step $\pm\Delta x$ every time step Δt.
- Flux Γ^- at x_0 from 1/2 particles in the region $[x_0, x_0 + \Delta x]$.
- Expand $n(x) = n(x_0) + n'(x_0)(x - x_0)$.



$$\Gamma^- = \frac{1}{2}\frac{1}{\Delta t}\int_{x_0}^{x_0+\Delta x} n(x)dx =$$

$$\frac{1}{2\Delta t}\left[n(x_0)x + \left(\frac{\partial n}{\partial x}\right)_{x_0}\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_0+\Delta x} =$$

$$\frac{1}{2\Delta t}\left[n(x_0)\Delta x + \left(\frac{\partial n}{\partial x}\right)_{x_0}\frac{(\Delta x)^2}{2}\right]$$

$$\Gamma = \Gamma^+ - \Gamma^-$$

Diffusion equation

- Total flux: $\Gamma = \Gamma^{+} - \Gamma^{-} = -\frac{(\Delta x)^2}{2\Delta t} n'(x_0) \rightarrow \Gamma = -\frac{(\Delta x)^2}{2\Delta t}\nabla n.$
- Diffusion coefficient $D = \frac{(\Delta x)^2}{2\Delta t}$
- Fick Law: $\Gamma = -D\nabla n$
- Conservation of particles number: variation in the volume V implies a flux $\Gamma = n\mathbf{u}$ across the surface S:

$$-\frac{\partial}{\partial t}\int_V n dV = \int_S \Gamma dS = \int_V \nabla \cdot \Gamma dV \longrightarrow \frac{\partial n}{\partial t} + \nabla \cdot \Gamma = 0$$

- Using the Fick Law: $\frac{\partial n}{\partial t} = \nabla \cdot (D\nabla n).$

Diffusion equation: a solution in 1D

- In 1D and ∇n along x: $\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}(D\frac{\partial}{\partial x}n)$.
- Solution depends on the initial conditions, take $n(x,0) = \delta(x)N$:

$$n(x,t) = \frac{N}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$$

- note that $< x >= \int xn(x,t)dx = 0$ but

$$< x^2 >= \int n(x,t)x^2dx = 2Dt \longrightarrow d = \sqrt{< x^2 >} = \sqrt{2Dt}$$

Classical Transport

Classical transport theory aims to calculate the diffusion coefficient $D \approx \frac{(\Delta x)^2}{\Delta t}$ for particle and energy transport in magnetized plasmas due to collisions assuming a cylindrical geometry.

Parallel diffusion

Consider a plasma with no magnetic field ($\mathbf{B} = 0$).

- Δx is proportional to the mean free path λ_m between two collisions.
- The collision frequency is ν (i.e. $1/\Delta t$).
- If $v_{th} = \sqrt{T/m}$ is the thermal velocity: $\lambda_m \sim v_{th}\,\Delta t$, then:

$$D \sim \frac{(\Delta x)^2}{\Delta t} = \nu \lambda_m^2 \sim \frac{v_{th}^2}{\nu} = \frac{T}{m\nu}$$

- Transport along **B** (i.e. *parallel*) is also characterized by the same D.

Parallel transport

- Collisions reduce parallel transport.
- In a closed system with conserved flux surfaces parallel transport only smooths density variation along **B** (no losses).

Perpendicular diffusion

Without collisions:

- Particles orbit around the field lines, no losses.
- In axisymmetric toroidal systems like Tokamaks there are drifts ($\nabla \mathbf{B}$, R_c etc) but the resulting orbits are closed, no losses.

With Collisions:

- Collisions modify the gyration phase discontinuously.
- Guiding center position changes by random walk with step Δr_{gc}:

$$D_\perp \sim \nu (\Delta r_{gc})^2$$

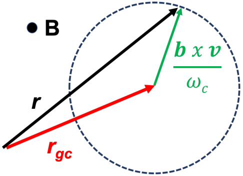
Perpendicular transport

- Collisions enhance perpendicular transport and relative losses.
- Apparently $D_{\perp,i} > D_{\perp,e}$ (for the same ν and with $\Delta r_{gc} \sim r_L$).

Which are the collisions relevant for transport?

Collisions

Only unlike particle collisions contribute to particle transport.



$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \longrightarrow \mathbf{r} = \mathbf{r}_{gc} + \frac{m}{qB}\mathbf{b} \times \mathbf{v}$$
$$\mathbf{r}_{gc} = \mathbf{r} + \frac{m}{qB}\mathbf{v} \times \mathbf{b}$$

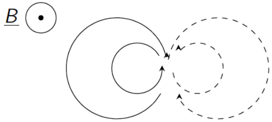
- Conservation of momentum: $(m_1\Delta\mathbf{v}_1 + m_2\Delta\mathbf{v}_2) = 0$;
- thus also: $(m_1\Delta\mathbf{v}_1 + m_2\Delta\mathbf{v}_2) \times \mathbf{b} = 0$
- collision is fast: $\Delta\mathbf{r} = 0$, so $\Delta\mathbf{r}_{gc} = \frac{m}{qB}\Delta\mathbf{v} \times \mathbf{b}$;
- replace in the momentum equation:

$$q_1\Delta\mathbf{r}_{gc1} + q_2\Delta\mathbf{r}_{gc2} = 0$$

Like and unlike particle collisions

- Unlike Particles:

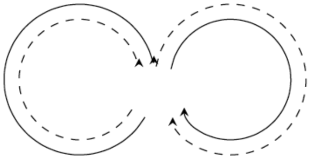
$q_1 = -q_2 \longrightarrow \Delta\mathbf{r}_{gc2} = \Delta\mathbf{r}_{gc1}$



the g.c. is moved of the same quantity and in the same direction!

- Like Particles:

$q_1 = q_2 \longrightarrow \Delta\mathbf{r}_{gc2} = -\Delta\mathbf{r}_{gc1}$



just a position shift, no net diffusion!

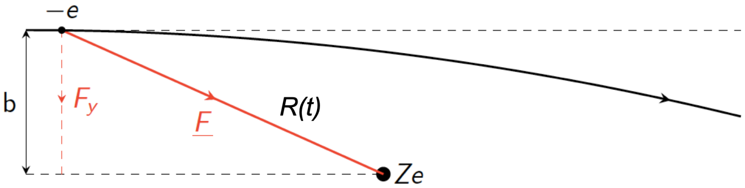
- Only e-i collisions contribute to particle diffusion;
- since $m_e \ll m_i$ then: $\Delta v_{e,\perp} \approx v_{e,\perp}$ and $\Delta v_{i,\perp} \sim 0$;
- this gives: $\Delta r_{gc} \approx \frac{m v_{e,\perp}}{eB} = r_{Le}$

D evaluation still requires the e-i collision frequency ν_{ei} .

Coulomb collisions

In a fully ionized plasma charged particles have long range Coulomb interactions which dominate over collisions with neutrals.

- Consider an electron with initial velocity $v_x = v$:



- the force on the electron at the time t is:

$$F = \frac{Ze^2}{4\pi\epsilon_0 R^2(t)} \longrightarrow F_y = \frac{Ze^2}{4\pi\epsilon_0 R^2(t)} \frac{b}{R(t)} = m_e \frac{dv_y}{dt}$$

- for small deflections: $R^2 \approx b^2 + (vt)^2$.

Coulomb collisions - evaluation of $\Delta v_\perp$

- The velocity deflection along y is given by:

$$\Delta v_y = \frac{Ze^2 b}{4\pi\epsilon_0 m_e} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{(b^2+v^2t^2)^3}} dt$$

- replace $vt = b \cdot sinh(\tau) \rightarrow dt = (b/v)cosh(\tau)d\tau$ and use $cosh^2(\tau) - sinh^2(\tau) = 1$:

$$\Delta v_y = \frac{Ze^2 b}{4\pi\epsilon_0 v m_e} \int_{-\infty}^{+\infty} \frac{d\tau}{cosh^2(\tau)} = \frac{Ze^2}{4\pi\epsilon_0 v m_e b} \left[tanh(\tau)\right]_{-\infty}^{+\infty}$$

$$= \frac{Ze^2}{4\pi\epsilon_0 v m_e b} \left[\frac{e^{2\tau}-1}{e^{2\tau}+1}\right]_{-\infty}^{+\infty} = \frac{Ze^2}{4\pi\epsilon_0 v m_e} \frac{2}{b}$$

- this result can be generalized to the perpendicular component of velocity variation $\Delta v_\perp$:

$$\Delta v_\perp = \frac{Ze^2}{4\pi\epsilon_0 v m_e} \frac{2}{b}$$

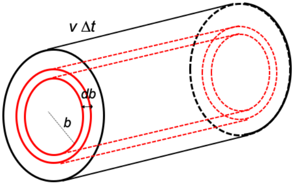
Parallel force on the electron due to many collisions

- From energy conservation and small deflections $(\Delta v_{\parallel})^2 \sim 0$:

$$(v + \Delta v_{\parallel})^2 + (\Delta v_{\perp})^2 = v^2 \longrightarrow (\Delta v_{\parallel})^2 + 2v\Delta v_{\parallel} + (\Delta v_{\perp})^2 = 0$$

$$\Delta v_{\parallel} \approx -\frac{(\Delta v_{\perp})^2}{2v} \longrightarrow \Delta v_{\parallel} \approx -\frac{Z^2 e^4}{(4\pi\epsilon_0)^2 m_e^2 v^3} \frac{2}{b^2}$$

- the parallel force on the electron is $F_{\parallel} = m_e \Delta v_{\parallel} / \Delta t$; sum up over all the collisions in the time Δt with ions of density n_i at the distance b i.e. $dN_i = 2\pi b(db) n_i v \Delta t$.



$$dF_{\parallel} = dN_i \frac{m_e \Delta v_{\parallel}}{\Delta t} = 2\pi m_e b n_i v \Delta v_{\parallel} db$$

Parallel force on the electron due to many collisions

- From energy conservation and small deflections $(\Delta v_{\|})^2 \sim 0$:

$$(v + \Delta v_{\|})^2 + (\Delta v_{\perp})^2 = v^2 \longrightarrow (\Delta v_{\|})^2 + 2v\Delta v_{\|} + (\Delta v_{\perp})^2 = 0$$

$$\Delta v_{\|} \approx -\frac{(\Delta v_{\perp})^2}{2v} \longrightarrow \Delta v_{\|} \approx -\frac{Z^2 e^4}{(4\pi\epsilon_0)^2 m_e^2 v^3}\frac{2}{b^2}$$

- the parallel force on the electron is $F_{\|} = m_e \Delta v_{\|}/\Delta t$; sum up over all the collisions in the time Δt with ions of density n_i at the distance b i.e. $dN_i = 2\pi b(db)n_i v\Delta t$. Thus:

$$F_{\|} = -2\pi m_e n_i v \frac{2Z^2 e^4}{(4\pi\epsilon_0)^2 m_e^2 v^3}\int_{b_{min}}^{b_{max}} \frac{1}{b}db$$

- $b_{max} \sim \lambda_D$ (Debye shielding)
- b_{min} from small deflections condition i.e. $|\Delta v_{\|}| \leq |\Delta v_{\perp}|$:

$$|\Delta v_{\|}| = \frac{(\Delta v_{\perp})^2}{2v} \leq \Delta v_{\perp} \longrightarrow \frac{Ze^2}{4v\pi\epsilon_0}\frac{2}{b} \leq 2v \longrightarrow b \geq \frac{Ze^2}{4\pi\epsilon_0 v^2 m_e} = b_{min}$$

The collision frequency ν_{ei}

- The parallel force from the interaction with the ions is:

$$F_{\|} = -2\pi m_e n_i v \frac{2Z^2 e^4}{(4\pi\epsilon_0)^2 m_e^2 v^3} ln \frac{\lambda_D}{b_{min}} = -\frac{4\pi n_i Z^2 e^4 ln(\Lambda)}{(4\pi\epsilon_0)^2 m_e^2 v^3} m_e v$$

- $ln(\Lambda) = ln(\frac{\lambda_D}{b_{min}})$ is the Coulomb logarithm ($\sim 15 - 20$).
- Define $\nu_{ei} = \frac{n_i Z^2 e^4 ln(\Lambda)}{4\pi\epsilon_0^2 m_e^2 v^3} \propto \frac{1}{T_e^{3/2}}$ so:

$$F_{\|} = m_e \frac{\Delta v_{\|}}{\Delta t} = -\nu_{ei} v m_e \rightarrow \Delta v_{\|} = -\nu_{ei} v \Delta t$$

in $\Delta t = 1/\nu_{ei}$, $\Delta v_{\|} = -v$ i.e. a deflection of 90°.

ν_{ei}

The inverse of ν_{ei} represents the time an electron takes to scatter 90° in velocity space due to the cumulative effect of small deflections.

Neutral and charged particle trajectories

Neutral particle in a ionized gas:

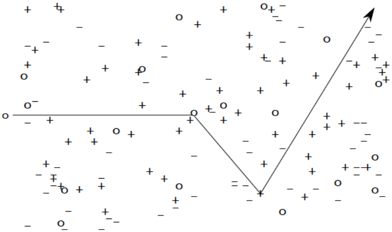


Figure 2.1: The trajectory of a neutral particle in a partially ionized gas exhibits "straight-line" motion between abrupt atomic collisions. In this and the next figure, the (assumed stationary) random positions of "background" particles in the partially ionized plasma are indicated as follows: neutral particles (circles), electrons (minus signs) and ions (plus signs). The typical distance between neutral particle collisions is called the "collision mean free path."

Electron trajectory in a ionized gas:

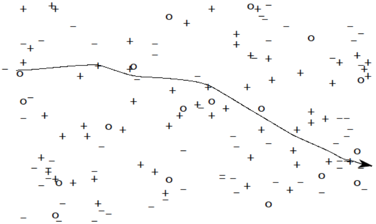


Figure 2.2: The trajectory of a "test" charged particle (electron) in a partially ionized gas exhibits continuous small-angle deflections or scatterings of its direction of motion. The largest deflections occur when it passes close to another charged particle. The "collision length" of a charged particle in a plasma is defined to be the average distance it moves in being deflected through one radian.

from *J.D.Callen, Fundamentals of physics*

Classical Diffusion coefficient

The classical diffusion coefficient D_{class} can be estimated as:

$$D_{class} = D_{\perp,i} = D_{\perp,e} \sim \frac{(\Delta x)^2}{\Delta t} \approx r_{Le}^2 \nu_{ei} = \left(\frac{mv}{eB}\right)^2 \frac{n_i Z^2 e^4 ln(\Lambda)}{4\pi\epsilon_0^2 m_e^2 v^3} \propto \frac{1}{B^2\sqrt{T_e}}$$

- ions and electrons diffuse at the same rate i.e. the transport is *intrinsically ambipolar*: $\Gamma_i = \Gamma_e$;
- ambipolarity from momentum conservation during collisions;
- in other contexts $D_i \neq D_e$ and a radial field E_r arises to balance the fluxes; the final D_{amb} is of the order of $min(D_e, D_i)$;

Examples

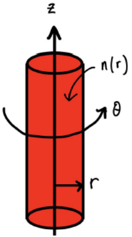
$B = 1T$, $n_e = 10^{20} m^{-3}$ and $T_e = 10keV$, $t = 1s$

- $r_{Le} \approx 2.5 \cdot 10^{-4} m$, $\nu_{ei} \approx 5kHz$
- $d \sim \sqrt{D_{class} t} = r_{Le}\sqrt{\nu_{ei} t} \approx 1.8cm$ good, only few cm!

MHD approach (1/2)

A more rigorous method to compute D_{class} by single fluid MHD.

- Fully ionized plasmas, cyl. geometry $\partial_\theta = 0$, $\partial_z = 0$;
- stationary equilibrium $\partial_t = 0$;
- constant and uniform B along z;

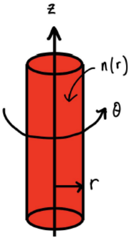


$$\mathbf{J} \times \mathbf{B} = \nabla p$$
$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$

- $\eta = \frac{m_e}{ne^2} < \nu_{ei} >$, with $< ... >$ an average on velocity distribution ($< \nu_{ei} > \approx 0.3 \nu_{ei}$);
- the components along r and θ are:

$$\frac{dp}{dr} = J_\theta B$$
$$E_\theta - u_r B = \eta J_\theta$$

MHD approach (2/2)



- note that $E_\theta = 0$ ($\nabla \times \mathbf{E} = 0$) thus:

$$\frac{dp}{dr} = J_\theta B = -\frac{u_r B^2}{\eta}$$

- since $p = n(r)(T_e + T_i)$, assuming constant $T_{i,e}$:

$$\nabla n(T_i + T_e) = -\frac{u_r B^2}{m_e < \nu_{ei} >} e^2 n$$

- the radial flux is:

$$\Gamma_r = n u_r = - < \nu_{ei} > \left(\frac{T_e m_e}{B^2 e^2}\right)\left(1 + \frac{T_i}{T_e}\right) \nabla n = -D_{class} \nabla n$$

- which gives: $\boxed{D_{class} = r_{Le}^2 < \nu_{ei} > \left(1 + \frac{T_i}{T_e}\right)}$

Energy Transport

Like-particle (e-e, i-i) collisions do not contribute to net particle diffusion:

$$< \nu_{ee} > \approx \frac{< \nu_{ei} >}{n_i Z^2 / n_e} \qquad < \nu_{ii} > \approx < \nu_{ei} > \sqrt{\frac{m_e}{m_i}}$$

They contribute to the heat flux $Q = -\chi n \nabla T$ since their energy centroid ($r_{cE} \propto (r_1 v_1^2 + r_2 v_2^2)/(v_1^2 + v_2^2)$) moves with a random walk:

electron dominates, notably at the edge: divertor

$$\chi_{\parallel} \approx T/m\nu : T_e/m_e\nu_{ee} > T_i/m_i\nu_{ii} \rightarrow \chi_{e,\parallel} \sim \sqrt{m_i/m_e}\chi_{i,\parallel}$$

$$\chi_{\perp} \approx r_L^2 \nu : r_{Li}^2 \nu_{ii} > r_{Le}^2 \nu_{ee} \rightarrow \chi_{i,\perp} \sim \sqrt{m_i/m_e}\chi_{\perp,e}$$

ions dominate

Examples

$B = 1T$, $n_e = 10^{20} m^{-3}$ and $T_e = 10keV \rightarrow r_{Li} \approx 1cm$, $\nu_{ii} \approx 100Hz$

$$D_{i,e} \approx \chi_{e,\perp} \sim 10^{-4} m^2/s,\ \chi_{i,\perp} \sim 10^{-2} m^2/s$$

- $t = 1s \rightarrow d_E = \sqrt{< x^2 >_E} \sim \sqrt{\chi_{\perp,i} t} = r_{Li}\sqrt{\nu_{ii} t} = 10cm$

Good, we need a reactor with r of only $\sim 50 - 100cm$ radius! Or not?

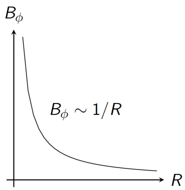
Summary on classical transport

- Classical transport deals with particle/energy diffusion relative to a fully ionized plasmas in cylindrical geometry due to collisions.
- Only e-i Coulomb collisions contribute to the net particle diffusion.
- Intrinsic ambipolarity: $\Gamma_e = \Gamma_i$, $D_i = D_e =< \nu_{ei} > r_{Le}^2 \left(1 + \frac{T_i}{T_e}\right)$.
- Perpendicular heat transport dominated by ions $\chi_{i,\perp} \approx \nu_{ii} r_{Li}^2$ ($\chi_{e,\perp} \approx \chi_{i,\perp} \sqrt{m_e/m_i}$) while the parallel one by electrons.
- Parallel transport in closed systems does not lead to losses.
- Efficient scaling with T and B: $D, \chi \propto 1/\left(B^2\sqrt{T}\right)$.
- Classical diffusion is slow.
- Experimental measurements of diffusivities are quite larger!

Neoclassical Transport

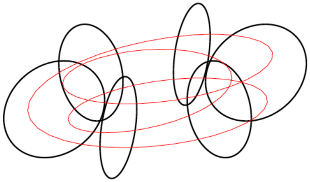
Real fusion devices are toroidal!

- Axisymmetry in Tokamaks: $\partial_\phi = 0$.
- The magnetic field depends on $R = R_0 + rcos(\theta)$.
- Vertical drift due to R_c and ∇B:



$$v_D = \frac{(v_\parallel^2 + v_\perp^2/2)}{R_0 \omega_c}$$

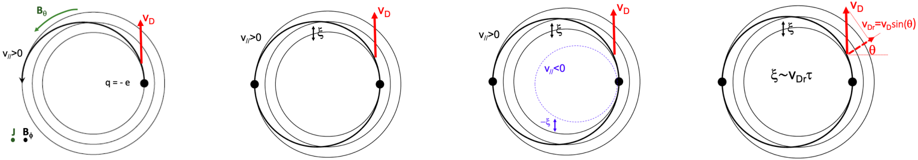
Vertical drift leads to charge separation and $\mathbf{E} \times \mathbf{B}$ losses.



- add B_θ (by a toroidal current);
- particles follow a helix;
- $q = \frac{d\phi}{d\theta} = \frac{rB_\phi}{R_0 B_\theta}$;
- q:number of toroidal turns for $\Delta\theta = 2\pi$;
- distance along a field line: $R_0 d\phi = qR_0\Delta\theta$.

Passing particle trajectories ($v_\| \gg v_\perp$)

- With $B_\theta \neq 0$ a particle still drifts vertically but the parallel motion along the helix moves it poloidally.
- This results in a closed shifted orbit with average displacement $\sim \xi$.



- suppose the particle collides at some point during the orbit: the g.c. is displaced with respect to $\theta = 0$ of $\sim \pm\xi$;
- the radial drift velocity is: $v_{Dr} = v_D sin(\theta) \approx v_D$;
- distance along ϕ from $\theta = 0$ to $\pi/2$: $d = qR_0\pi/2 = \frac{\pi R_0 q}{2}$;
- corresponding time : $\tau \approx \frac{\pi R_0 q}{2v_\|}$.

Neoclassical Diffusion coefficient

- The radial drift is $\xi \approx v_D \tau \approx \frac{v_{\|}^2}{R_0 \omega_c} \frac{\pi R_0 q}{2 v_{\|}} = \frac{v_{th} q m \pi}{2eB}$ with $v_{\|} \sim v_{th}$;
- Coloumb collisions between un-like particles: $\nu = \nu_{ei}$, $\xi = \xi_e$;
- $\xi_e = \frac{\pi}{2} \frac{v_{th,e} m_e}{eB} q \approx r_{Le} q$;
- Evaluate D with $\Delta x \sim \xi_e$ and $\Delta t = 1/\nu_{ei}$:

Same as classical, but multiplied by q^2

$$\boxed{D_{NEO} = \frac{(\Delta x)^2}{\Delta t} \approx \nu_{ei} \xi_e^2 = q^2 r_{Le}^2 \nu_{ei} = D_{class} q^2}$$

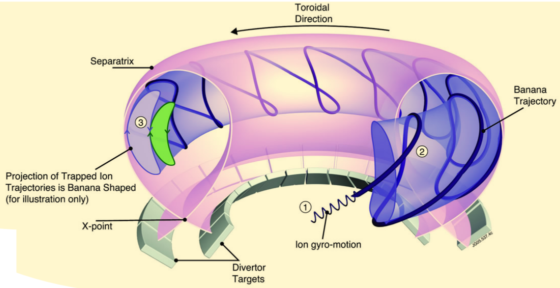
Note that:

- in Tokamaks $q > 1$, diffusion is higher than classical!
- Full derivation with MHD using $B(r, \theta)$ gives:

$$D = < \nu_{ei} > r_{Le}^2 (1 + q^2) \left(1 + \frac{T_i}{T_e}\right) = D_{class} + D_{NEO} \left(1 + \frac{T_i}{T_e}\right)$$

Trapped particles ($v_\perp \gg v_\parallel$)

- D_{NEO} ok for high collisionality but in fusion plasmas $\nu_{ei} \propto 1/T_e^{3/2}$;
- particles are not all passing, a fraction might have $v_\parallel = 0$ at some point and be reflected back (*trapped*);
- trapping condition + vertical drift $v_D \rightarrow$ closed orbits with banana shape on the poloidal plane.
- By collisions a trapped particle might become passing and its g.c. is displaced of $\sim$ banana width size.



Fraction of trapped particles (1/2)

The fraction of trapped particles can be evaluated from magnetic moment and energy conservation:

$$\mu = \frac{mv_\perp^2}{2B}, \quad v^2 = v_\parallel^2 + v_\perp^2 = v_{\parallel,0}^2 + v_{\perp,0}^2, \quad v_{\parallel,0}, v_{\perp,0}(\theta = 0)$$

Taylor

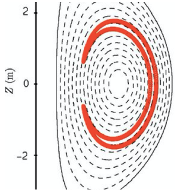
- Explicit $B_\phi = \frac{R_0 B_0}{R} = \frac{R_0 B_0}{R_0 + r cos\theta} = \frac{B_0}{1+\epsilon cos\theta} \approx B_0(1 - \epsilon cos\theta)$, $\epsilon = r/R_0$
- Conservation of magnetic moment, $\mu =$ constant:

$$\frac{v_{\perp,0}^2}{1-\epsilon} = \frac{v_\perp^2}{1-\epsilon cos\theta} = \frac{v^2 - v_\parallel^2}{1-\epsilon cos\theta} \rightarrow v_\parallel^2 = v^2\left[1 - \frac{v_{\perp,0}^2(1-\epsilon cos\theta)}{v^2(1-\epsilon)}\right]$$

$$\approx v^2\left[1 - \frac{v_{\perp,0}^2}{v^2}(1-\epsilon cos\theta)(1+\epsilon)\right] \approx v^2\left[1 - \frac{v_{\perp,0}^2}{v^2}\left(1 + \epsilon(1 - cos\theta) + o(\epsilon^2)\right)\right]$$

Fraction of trapped particles (2/2)

- since $1 - cos\theta = 2sin^2\frac{\theta}{2}$:



Parallel velocity of the particle along the orbit

$$v_{\parallel}^2 = v^2\left[1 - \frac{v_{\perp,0}^2}{v^2}\left(1 + 2\epsilon sin^2\frac{\theta}{2}\right)\right]$$

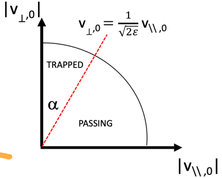
- consider a barely trapped particle $v_{\parallel}^2 \leq 0$ at $\theta = \pi$:

Condition for trapping

$$1 - \frac{v_{\perp,0}^2}{v^2}(1+2\epsilon) \leq 0 \rightarrow \frac{v_{\parallel,0}^2 + v_{\perp,0}^2}{v_{\perp,0}^2} \leq 1 + 2\epsilon \rightarrow \frac{v_{\parallel,0}^2}{v_{\perp,0}^2} \leq 2\epsilon \rightarrow \left|\frac{v_{\parallel,0}}{v_{\perp,0}}\right| \leq \sqrt{2\epsilon}$$

- isotrope velocity, trapped fraction is $f_T \sim \alpha/(\pi/2)$;
- $cotg\alpha = 1/\sqrt{2\epsilon} \rightarrow tan\alpha = \sqrt{2\epsilon}$ and for $\epsilon \ll 1$:

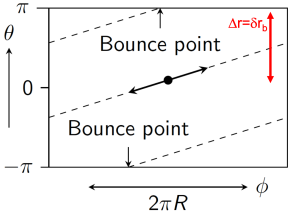
$$tan(\alpha) \sim \alpha \sim \sqrt{2\epsilon} \rightarrow \boxed{f_T} \approx \frac{\sqrt{2\epsilon}}{\pi/2} \propto \boxed{\sqrt{\epsilon}}$$



Only a small part of particles are trapped

Bounce time and banana width

A trapped particle moves along the field with $v_{\parallel} \approx v_{\perp}\sqrt{2\epsilon} \approx v_{th}\sqrt{2\epsilon}$ with also a vertical drift due to v_D.



- A barely trapped particle moves from $\theta = 0$ to π and back ($\Delta\theta = 2\pi$);
- from the distance in the toroidal direction $2\pi R_0 q$ with velocity $v_{\parallel} \approx v_{th}\sqrt{2\epsilon}$:

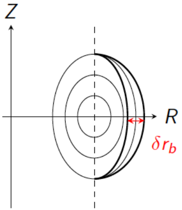
$$\boxed{t_b = \frac{2\pi R_0 q}{v_{th}\sqrt{2\epsilon}}} \sim \frac{R_0 q}{v_{th}\sqrt{\epsilon}}$$

Velocity drift * bounce time

- banana width $\delta r_b \approx v_D t_b$:

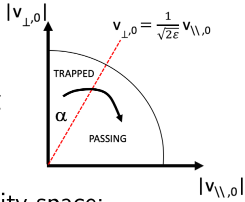
$$\boxed{\delta r_b} = \frac{v_{\parallel}^2 + v_{\perp}^2/2}{R_0\omega_c}\frac{2\pi R_0 q}{v_{th}\sqrt{2\epsilon}} \approx \frac{m\pi q v_{th}^2}{eBv_{th}\sqrt{2\epsilon}}$$

$$= \frac{\pi}{\sqrt{2}}\frac{q}{\sqrt{\epsilon}}\frac{v_{th}m}{eB} \approx \boxed{\frac{qr_L}{\sqrt{\epsilon}}}$$



Trapped particles

Effective collision frequency



During the time t_b the particle drifts to a new surface at a distance δr_b from the original one.

- if at some point a collision occurs a trapped-passing conversion might happen with a step δr_b in g.c.;
- to become passing the particle doesn't need a $\pi/2$ deflection but only to diffuse of $\sim \alpha \sim \sqrt{2\epsilon}$ in velocity space;
- time for random walk in angular displacement $t \propto < \alpha^2 >$ (in analogy with $t \propto < x^2 >$);
- the ratio for the respective collision times is thus:

$$\frac{t_{trapped}}{t_{\pi/2}} \propto \frac{< \alpha^2 >}{(\pi/2)^2} \approx \frac{(\sqrt{2\epsilon})^2}{(\pi/2)^2} \approx \epsilon \rightarrow \boxed{\frac{\nu_{eff}}{\nu_{ei}} = \frac{1}{\epsilon}}$$

effective collision width

- larger collision frequency for trapped particles with respect to passing by a factor $1/\epsilon$.

Banana particle and heat diffusion coefficients

The banana particle diffusion coefficient is:

Dban > D passing, epslion^3/2

$$D_{ban} = f_T \frac{(\Delta x)^2}{\Delta t} \approx f_T \delta r_b^2 \nu_{eff} = \sqrt{\epsilon} \frac{q^2 r_L^2}{\epsilon} \frac{\nu}{\epsilon} \approx \frac{\nu_{ei} q^2 r_L^2}{\epsilon^{3/2}} \gg D_{NEO}.$$

- Axisymmetry ensures ambipolarity $D_{ban,i} \approx D_{ban,e} = \frac{\nu_{ei} q^2 r_{Le}^2}{\epsilon^{3/2}}$;
- This is not true for energy where ions dominate:

$$\chi_{i,\perp} \sim \sqrt{\epsilon} \frac{\delta r_{b,i}^2 \nu_{ii}}{\epsilon} = \frac{q^2 r_{Li}^2 \nu_{ii}}{\epsilon^{3/2}}, \chi_{e,\perp} \sim \sqrt{\epsilon} \frac{\delta r_{b,e}^2 \nu_{ei}}{\epsilon} = \frac{q^2 r_{Le}^2 \nu_{ee}}{\epsilon^{3/2}} = \chi_{i,\perp} \sqrt{\frac{m_e}{m_i}}$$

Examples

For $B = 1T, n_e = 10^{20} m^{-3}, T = 10keV, q = 2, \epsilon = 0.1$:

$$\delta r_{b,e} \approx 1.5mm,\ \delta r_{b,i} \approx 6cm$$

$$D_{i,e} \approx \chi_{e,\perp} \sim 10^{-2} m^2/s,\ \chi_{i,\perp} \sim 1m^2/s$$

$$t = 1s \rightarrow d_E \sim \sqrt{t\chi_i} \sim \sqrt{t\nu_{eff} f_T} \delta r_{b,i} = 100cm \geq 10 d_{E,class}$$

10xclassical

Definition of ν^*

A useful quantity to characterize transport regimes in magnetic fusion devices is the collisionality ν^* defined as:

$$\nu^* = \frac{t_b}{1/\nu_{eff}} \approx \frac{\nu_{ei} R_0 q}{\epsilon^{3/2} v_{th,e}} \propto \frac{n_e R_0 q}{\epsilon^{3/2} T_e^2}$$

which compares the banana orbit bounce time with the collision time.

- for $\nu^* \ll 1 \rightarrow \nu_{eff} \ll 1/t_b$ trapped particles can complete many banana orbits before collide; the characteristic D is :

$$D_{ban} = \frac{\nu_{ei} r_{Le}^2 q^2}{\epsilon^{3/2}} = r_{Le}^2 \frac{\nu_{ei} R_0 q}{v_{th,e} \epsilon^{3/2}} \frac{q v_{th,e}}{R_0} = \frac{\nu^* v_{th,e} q r_{Le}^2}{R_0}$$

- at high collisionality both trapped/passing particles cannot complete their orbits before scattering since the time for a poloidal turn $\tau_p \sim \frac{R_0 q}{v_{th}}$ is much longer of the collision time $1/\nu_{ei}$:

Neoclassical transport dominates in high collisionality

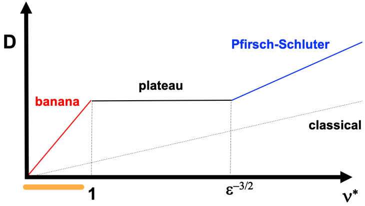
$$\nu_{ei} \tau_p \gg 1 \rightarrow \frac{\nu_{ei} R_0 q}{v_{th,e}} \gg 1 \rightarrow \nu^* \epsilon^{3/2} \gg 1 \rightarrow \nu^* \gg \epsilon^{-3/2}$$

Diffusion regimes

- $\nu^* \gg \epsilon^{-3/2}$ corresponds to high collisional regime (*Pfirsch-Schluter*):

$$D_{PS} = D_{NEO} = \nu_{ei} r_{Le}^2 q^2 = \frac{\nu_{ei} q R_0}{v_{th,e}\epsilon^{3/2}} \frac{r_{Le}^2 q}{R_0} v_{th,e}\epsilon^{3/2} = \nu^* v_{th,e}\epsilon^{3/2} \frac{r_{Le}^2 q}{R_0}$$

- $\nu^* = 1 \rightarrow D_{ban} = r_{Le}^2 \frac{v_{th,e} q}{R_0}$
- $\nu^* = \epsilon^{-3/2} \rightarrow D_{PS} = r_{Le}^2 \frac{v_{th,e} q}{R_0}$
- *Plateau* regime for $1 \leq \nu^* \leq \epsilon^{-3/2}$: only passing particles can complete their orbits.

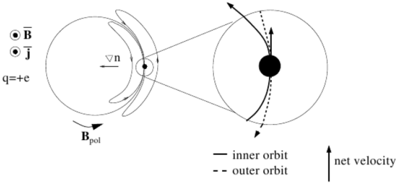


$$v_{th} \propto \sqrt{T} \qquad \nu_{ei} \propto n/T^{3/2} \rightarrow \boxed{\nu^* \propto \frac{n}{T^2}}$$

Trapped particles and banana regime dominant in hot Tokamaks!

Banana current

Trapped particles are important in Tokamak also because they generate a toroidal current which contributes to the poloidal magnetic field.



- $n(r)$ decreases from the core $\rightarrow$ on a given flux surface more trapped particles circulate in the inner side;
- net flow in the toroidal direction; Banana density current
- parallel current along the field lines $J_{ban} \approx v_{\parallel} e \Delta n$

Since $v_{\parallel} \sim v_{th}\sqrt{\epsilon}$ and $\Delta n = \delta r_b \frac{d(n\sqrt{\epsilon})}{dr}$:

$$J_{ban} = \delta r_b e v_{th}\sqrt{\epsilon}\frac{dn}{dr}\sqrt{\epsilon} = \frac{q r_L}{\sqrt{\epsilon}} e v_{th}\frac{dn}{dr}\epsilon = \frac{dn}{dr}\sqrt{\epsilon} v_{th} e \frac{rB_\phi}{R_0 B_\theta}\frac{m v_{th}}{e B_\phi} =$$

$$\frac{dn}{dr}\sqrt{\epsilon} v_{th}^2 \frac{rm}{R_0 B_\theta} = \frac{dn}{dr}\sqrt{\epsilon}\frac{T}{m}\frac{r}{R_0}\frac{m}{B_\theta} = \epsilon^{3/2}\frac{dn}{dr}\frac{T}{B_\theta}$$

Small current usually

Bootstrap current

Trapped particles are in collisional equilibrium with the passing ones to which they transfer their momentum.

- The rate of transfer of parallel momentum by trapped particles for unit of time and volume is: $\Delta P_{\parallel,t} = mv_{\parallel,t}(\Delta n)\nu_{eff} = mJ_{ban}/e$ *Veff
- Collisions between passing particles with density n_p redistribute the momentum $mv_{\parallel,p}$ with frequency ν_{ei} : $\Delta P_{\parallel,p} = mv_{\parallel,p}n_p\nu_{ei}$.
- At the equilibrium: $\Delta P_{\parallel,t} = \Delta P_{\parallel,p}$:

$$mv_{\parallel,p}n_p\nu_{ei} = mv_{\parallel,t}\Delta n\nu_{eff} \rightarrow v_{\parallel,p}n_p = \frac{J_{ban}\nu_{eff}}{e\nu_{ei}}$$

$$J_{BS} = v_{\parallel,p}n_p e = J_{ban}\nu_{eff}/\nu_{ei} \rightarrow J_{BS} = J_{ban}/\epsilon$$

Bootstrap current

- Replacing the expression for J_{ban}: $J_{BS} = \epsilon^{3/2}\frac{dn}{dr}\frac{T}{B_\theta}\frac{1}{\epsilon} = \epsilon^{1/2}\frac{dn}{dr}\frac{T}{B_\theta}$
- Collisions have a crucial role but do not appear in the result for J_{BS}.
- J_{BS} mostly carried by passing particles but caused by those trapped.

Bootstrap current in Tokamaks

Also the temperature gradient contributes to the banana current: if particles move faster on the inner than on the outer banana orbit, this generates a further toroidal current. The full computation leads to:

$$< \mathbf{J}_{BS} \cdot \mathbf{B} >= \sqrt{2\epsilon} f(\psi) p(\psi) \left[\frac{a_1}{n} \frac{\partial n}{\partial \psi} + \frac{a_2}{T_e} \frac{\partial T_e}{\partial \psi} + \frac{a_3}{T_i} \frac{\partial T_i}{\partial \psi} \right]$$

with ψ a flux function (e.g. toroidal flux) and $a_{1,2,3}$ coefficients depending on geometry and collisionality. Advantages of J_{BS}:

- provides poloidal field;
- allows steady state operation reducing the current externally driven;

Bootstrap/ohmic

$$\frac{J_{BS}}{J_{OH}} \sim \frac{\sqrt{\epsilon} p}{B_\theta a} \frac{\pi a^2}{I_p} = \frac{\sqrt{\epsilon} p}{B_\theta r} \frac{\pi a^2 \mu_0}{B_\theta 2 \pi a} = \sqrt{\epsilon} \frac{p}{4 B_\theta^2 / (2\mu_0)} \sim \sqrt{\epsilon} \beta_p$$

- $\beta_p \sim 1$, $\epsilon = 0.3 - 0.4 \rightarrow J_{BS}/J_{OH} \sim 1/2$; in advanced scenario (JT-60,ITER) it can provide most of the current $(60\% - 80\%)$.

Summary of Neoclassical transport

- Neoclassical transport is the minimum achievable in fusion devices with toroidal geometry.
- Passing particles have a diffusion coefficient $D_{NEO} \approx \nu_{ei} r_{Le}^2 q^2$.
- Trapped particles move on banana shape orbits and increase the transport by a factor $\sim (r/R_0)^{-3/2}$.
- Intrinsic ambipolarity for axisymmetric systems and $D \sim D_e$.
- Heat transport dominated by ions with $\chi_{i,\perp} \sim \delta r_{bi}^2 \nu_{ii}/\sqrt{\epsilon}$.
- The values of n and T determine ν^* and the particle diffusion regime: the banana regime is dominant at high temperature.
- Trapped particles are at the origin of the bootstrap current which can provide a significative fraction of the poloidal magnetic field.
- Higher diffusivities than neoclassical predictions are experimentally measured $\rightarrow$ anomalous transport.

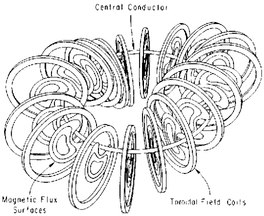
Stellarators

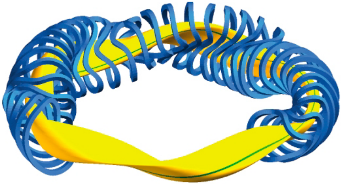
The poloidal field is a key element in axisymmetric devices to avoid the losses due to the vertical drift but is generated by driving a current: this is a *free energy* source which can trigger several kind of instabilities (current driven modes).

How can we twist the field lines without driving a current?

- A deformation (torsion) in 3D of the magnetic axis;
- Poloidally elongated and rotated magnetic surfaces in the toroidal direction;

These systems are named Stellarators.

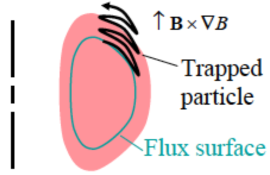




Stellarators

Advantages:

- steady state (no current needs to be driven);
- no instabilities like sawteeth, ELMS, disruption
- no density limit;



Drawbacks:

- more complex geometry and coils design;
- no conservation of canonical momentum $p_\phi = mv_\phi + e\psi$ $(\partial_\phi \neq 0)$: particles trajectories are not closed and can radially drift outward, also without collisions.

$\rightarrow$ high level of neoclassical transport!

$\rightarrow$ optimization required to reduce the particles drift!

Toroidal and Helical ripple

In flux coordinates, the particle guiding center motion is determined by $B(s,\theta,\zeta) = |\mathbf{B}|$, with s a flux label (e.g. $\propto \sqrt{\psi}$).

In general: $B = \sum_{m,n} B_{m,n}(s)cos(m\theta - n\zeta)$ or:

$$B = B_{00} + \sum_{m\neq 0, n=0} B_{m,0}cos(m\theta) + \sum_{m\neq 0, n\neq 0} B_{m,n}cos(m\theta - n\zeta) =$$

$$B_{00}\left[1 + \sum_{m\neq 0, n=0} \frac{B_{m,0}}{B_{00}}cos(m\theta) + \sum_{m\neq 0, n\neq 0} \frac{B_{m,n}}{B_{00}}cos(m\theta - n\zeta)\right]$$

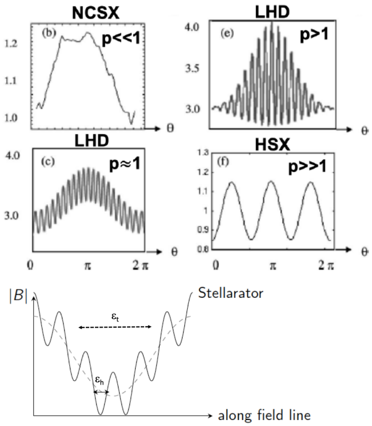
In many cases: $B \sim B_{00}\left[1 - \epsilon_t(s)c(\theta) - \epsilon_h(s)f(M\theta - N\zeta)\right]$

- ϵ_t,ϵ_h are flux surface averages;
- $\epsilon_t(s) \sim B_{m,0}$: toroidal ripple ($n = 0, m \neq 0$);
  in a Tokamak $\rightarrow \epsilon_t = \epsilon = r/R_0$, $c(\theta) = cos(\theta), B_{00} = B(R_0)$;
- $\epsilon_h(s) \sim B_{m,n}$: helical ripple ($n \neq 0, m \neq 0$).

Examples

The ratio $p = \epsilon_h/\epsilon_t$ estimates the distance from axisymmetry.

B along a field line:



from *H.E. Mynick, Phys.Plasmas 13 (2006)*

- $p << 1 \rightarrow$ quasi-axisymmetric stellarator;
- $p >> 1 \rightarrow$ quasi-helical stellarator;
- $p = 0 \rightarrow$ Tokamak;
- $p = \infty \rightarrow$ helical stellarator.

- trapped particles like in tokamaks due to toroidicity;
- in stellarator other harmonics arise; particles get trapped also in local minima $\rightarrow$ *superbananas*.

Neoclassical Transport in Stellarators: $1/\nu$ regime

Superbanana drifts are important at low collisionality; the diffusion coefficient can be estimated replacing $\epsilon_t = \epsilon$ with ϵ_h. For electrons:

Epsilon -> helical ripple

- effective collision frequency $\nu_{eff} \approx \nu_e/\epsilon_h$;
- fraction of trapped particles in local minima : $f_T \sim \sqrt{\epsilon_h}$;
- no closed orbits! in $\Delta t = 1/\nu_{eff}$ the radial drift is $\Delta r = v_D/\nu_{eff}$;

$$\boxed{D_e} = f_T \frac{(\Delta r)^2}{\Delta t} \approx \sqrt{\epsilon_h} v_D^2 \frac{\epsilon_h}{\nu_e} = \frac{\epsilon_h^{3/2}}{\nu_e} \left(\frac{v_{th,e}^2}{\omega_c R_0} \right)^2 = \boxed{\frac{\epsilon_h^{3/2}}{\nu_e} \frac{v_{th,e}^4 m_e^2}{e^2 B^2 R_0^2}}$$

$$\propto \epsilon_h^{3/2} \frac{\sqrt{m_e} T_e^2}{e^2 B^2 R_0^2} \frac{T_e^{3/2}}{n_e} \approx \epsilon_h^{3/2} \frac{\sqrt{m_e} T_e^{7/2}}{n_e B^2 R_0^2}$$

important for low collisionality

Note:

- $1/\nu_e$ regime and $\Delta r \sim$ system scale length;
- strong T_e scaling: losses dominate at high temperature!
- higher values for ions: ambipolarity requires a radial electric field E_r!

Neoclassical Transport in Stellarators: $\sqrt{\nu}$ regime

The ambipolar electric field E_r is negative since must prevent ions from diffusing to the wall and generates a poloidal $\mathbf{E} \times \mathbf{B}/B^2$ drift.

- ions drift poloidally with a frequency $\Omega_E \sim \frac{E_r}{rB}$;
- the radial distance traveled in the time $1/\Omega_E$ is $\Delta r \sim v_D/\Omega_E$;
- collisions make the ions move randomly out and in the local minima in a time $t \propto (\alpha)^2$ with α the distance from the trapping boundary in velocity space so that $\nu_{eff} = \nu_i/(\alpha)^2$, $f_T \sim \alpha$;
- $D_i \sim f_T(\Delta r)^2 \nu_{eff} = (\alpha)\frac{v_D^2}{\Omega_E^2}\frac{\nu_i}{(\alpha)^2} = \frac{v_D^2}{\Omega_E^2}\frac{\nu_i}{(\alpha)}$;
- condition for low collisionality: $\nu_{eff} < \Omega_E$ gives $\alpha \geq \left(\frac{\nu_i}{\Omega_E}\right)^{1/2}$;
- replacing in D_i:

$$D_i = \frac{v_D^2 \nu_i}{\Omega_E^{3/2} \nu_i^{1/2}} = \frac{v_D^2 \sqrt{\nu_i}}{\Omega_E^{3/2}}$$

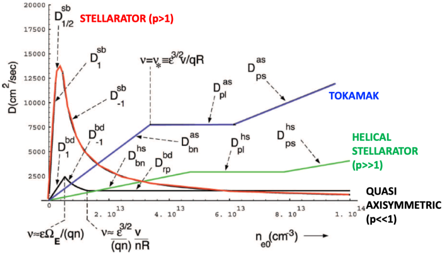
$E_r < 0 \rightarrow$ electrons in $1/\nu$ regime while ions in $\sqrt{\nu}$.

Very different from tokamaks

Optimization

Optimization reducing the drift:

- in presence of a symmetry a canonical momentum is conserved $\rightarrow$ limit the deviation of particles from flux surfaces;
- make the radial drift average-out over a bounce orbit (*omnigeneous systems*).



from *H.E. Mynick, Phys.Plasmas 13 (2006)*

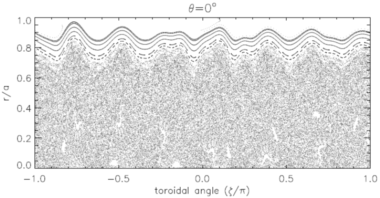
Ambipolarity $\sum q_s\Gamma_s(r) = 0$ determines three solutions for $E_r(r)$:

- one unstable;
- $E_r < 0$: ion root;
- $E_r > 0$: electron root, large Ω_E and transport reduction;

Poloidal drift

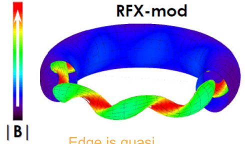
$\longrightarrow$ optimization forcing the system in the electron root (ECRH, NBI).

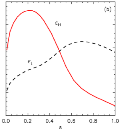
Transport in the Reversed Field Pinch (RFP)



- $B_T \approx B_P$, $q << 1$;
- $+$ resonant kink tearing modes $\tilde{b}$;
- at low I_p: Multiple Helicity, no conserved flux surfaces $\rightarrow$ transport dominated by magnetic chaos $D, \chi \propto (\tilde{b}/B)^{1.5}$ $(D \sim 10 - 20 m^2/s)$;

Toroidal ripple
black, helical red





Edge is quasi
symmetric

Core is helical

- at high I_p: helical topology in the core;
- reduction of magnetic chaos and eITB formation;
- no superbanana $1/\nu$ regime;
- neoclassical estimates: $D_{ITB} < 0.5 m^2/s$, $\chi_{e,ITB} \sim 1 m^2/s$;

Electron thermal
diffusivity

Experimentally residual magnetic chaos still enhances particle/energy transport: $D_{ITB} \sim 1 m^2/s$, $\chi_{e,ITB} \sim 5 m^2/s$

The kinetic theory

Kinetic theory provides a rigorous description of transport, commonly used to study several phenomena in plasma physics.

- A distribution function $f(\mathbf{x}, \mathbf{v}, t)$ describes the particle density in a $6D$ space with coordinates (x, y, z, v_x, v_y, v_z) at the time t;
- the number of particles dN in the small $6D$ volume $dV = d\mathbf{x}d\mathbf{v}$ at $(\mathbf{x}, \mathbf{v})$ at the time t is $dN = f(\mathbf{x}, \mathbf{v}, t)d\mathbf{x}d\mathbf{v}$;
- every species α has its own distribution f_α;
- $dV = d\mathbf{x}d\mathbf{v}$ contains a large number of particles.

Consider a volume V with surface S in $6D$ space; without collisions, a variation of the number of particles in the volume V corresponds to a flux $\Gamma = f\mathbf{U}dS$ across the surface S, where $\mathbf{U} = (\dot{\mathbf{x}}, \dot{\mathbf{v}})$:

$$\frac{\partial N}{\partial t} = -\int_S \Gamma ds \rightarrow \frac{\partial}{\partial t}\int_V f(\mathbf{x}, \mathbf{v}, t)d\mathbf{x}d\mathbf{v} \overset{\text{Divergence theorem}}{=} -\int_S f\mathbf{U}dS$$

The Vaslov equation

The divergence theorem gives:

$$\int_V \left[\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{U}) \right] d\mathbf{x} d\mathbf{v} = 0$$

Note that:

- $\nabla \cdot (f\mathbf{U}) = \nabla f \cdot \mathbf{U} + f \nabla \cdot \mathbf{U}$
- $\nabla f \cdot \mathbf{U} = \nabla_{\mathbf{x}} f \cdot \mathbf{v} + \nabla_{\mathbf{v}} f \cdot \dot{\mathbf{v}}$
- $f \nabla \cdot \mathbf{U} = f \nabla_{\mathbf{x}} \cdot \mathbf{v} + f \nabla_{\mathbf{v}} \cdot \dot{\mathbf{v}} = 0 + f \nabla_{\mathbf{v}} \cdot \dot{\mathbf{v}}, \qquad \dot{\mathbf{v}} = \mathbf{F}/m$
- $f \nabla_{\mathbf{v}} \cdot (\mathbf{F}/m) = 0$ if $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

Thus we obtain the Vaslov equation:

$$\boxed{\frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} f \cdot \mathbf{v} + \nabla_{\mathbf{v}} f \cdot \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0}$$

Solving this gives the distribution function

and, once known f, the density $n(\mathbf{x}, t)$ is given by:

$$n(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$$

Collision operator

The Vaslov equation must be solved together with Maxwell equations using the density charge ρ_q and the density current $\mathbf{J}$ self-consistently computed from f:

$$\rho_q = \sum_\alpha q_\alpha \int f_\alpha d\mathbf{v}, \quad \mathbf{J(x,t)} = \sum_\alpha q_\alpha \int \mathbf{v} f_\alpha d\mathbf{v}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \cdot \mathbf{E} = \rho_q/\epsilon_0 \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

- Vaslov equation is correct when studying phenomena evolving more rapidly than typical collision times.
- Collisions are included with a term $(\partial f/\partial t)_c$ i.e. the particles variation per unit of time due to collisions with other species:

Collisions

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} f \cdot \mathbf{v} + \nabla_{\mathbf{v}} f \cdot \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \left(\frac{\partial f}{\partial t}\right)_c$$

Macroscopic quantities

Macroscopic parameters can be obtained by averages, like done for n:

- mean flow $\mathbf{V}(\mathbf{x}, t) = \frac{1}{n} \int f \mathbf{v} d\mathbf{v}$ and particle flux $\Gamma = \int f \mathbf{v} d\mathbf{v}$;
- pressure $p(\mathbf{x}, t) = nT = \int f \frac{m v_r^2}{3} d\mathbf{v}$, $\quad \mathbf{v}_r = \mathbf{v} - \mathbf{V}$;
- stress tensor: $\pi(\mathbf{x}, t) = \int f m(\mathbf{v}_r \mathbf{v}_r - v_r^2 \mathbf{I}) d\mathbf{v}$, $\quad \mathbf{I}$=identity tensor
- conductive heat flux: $\mathbf{q}(\mathbf{x}, t) = \int f \mathbf{v}_r \frac{m v_r^2}{2} d\mathbf{v}$;
- the frictional force density: $\mathbf{R} = \int d\mathbf{v} m \mathbf{v} \left(\frac{\partial f}{\partial t}\right)_c$;
- the collisions energy exchange: $Q = \int d\mathbf{v} \frac{m v_r^2}{2} \left(\frac{\partial f}{\partial t}\right)_c$;

with their time evolution given by:

$$\int d\mathbf{v} g(\mathbf{v}) \left[\frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} f \cdot \mathbf{v} + \nabla_{\mathbf{v}} f \cdot \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \left(\frac{\partial f}{\partial t}\right)_c \right] = 0$$

using different expression for $g(\mathbf{v}) \propto \mathbf{v}^k$. g(v) moments

Moments

- $g(\mathbf{v}) = 1$, continuity equation:

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{V}) = 0$$

- $g(\mathbf{v}) = m\mathbf{v}$, motion equation:

$$mn\frac{\partial \mathbf{V}}{\partial t} = nq\left[\mathbf{E} + \mathbf{V} \times \mathbf{B}\right] - \nabla p - \nabla \cdot \pi + \mathbf{R}$$

- $g(\mathbf{v}) = mv^2/2$, heat transport:

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{5}{2}p\nabla \cdot \mathbf{V} = -\nabla \cdot \mathbf{q} - \pi : \nabla \mathbf{V} + Q$$

- higher moments have not a clear physical meaning;
- the system is not closed; at zero order n and $\mathbf{V}$ are unknown, at first order also $p = nT$ and π , etc...

Moments closure

- A closure can be obtained assuming a relation between $\pi, \mathbf{q}, \mathbf{R}, Q, \Gamma ...$ and $n, T, \mathbf{V}$ phenomenologically.
- If variations are slow also kinetic methods can be used with $f = f_0 + f_1$, $f_0 = f_M(n, \mathbf{V}, T)$ a Maxwellian, and $f_1 << f_0$;
- replacing $f_0 + f_1$ in the K.E. and neglecting terms $o(f_1^2)$ a new equation for f_1 is obtained depending on spatial derivates of flow and temperature which describe the deviation from the Maxwellian;
- f_1 is used to calculate $\pi, \mathbf{q}, \mathbf{R}, Q, \Gamma ...$ which result proportional to the effects causing the deviations from f_0 ($e.g. Q \propto \nabla T, \Gamma \propto \nabla n$);
- the corresponding coefficients of proportionality are called the *transport coefficients* and their determination is the basic goal of kinetic theory.
- An example of solution obtained by Braginskii (see Review of Plasma Physics 1, 205 (1965)).

The Bhatnagar, Gross and Krook model (BGK)

A simple model for the collision operator:

- *Lorentz gas*: electrons scatter off infinitely heavy and stationary ions;
- at the equilibrium $f_0 = n\left(\frac{m}{2\pi T}\right)^{3/2} e^{-m\frac{(\mathbf{v}-\mathbf{V})^2}{2T}}$;
- deviation from equilibrium by small external perturbations;
- collisions drive f towards the equilibrium (BGK):

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nu_c(f - f_0)$$

An example: the determination of resistivity.

- Apply an electric field $\mathbf{E}$ in the x direction;
- Linearize the steady state K.E. with $f = f_0 + f_1$, assuming isotropy in space and $\mathbf{V} = 0$; only terms $\sim \mathbf{E} \sim f_1$ and note: $\nabla_{\mathbf{v}} f_0 = -m\frac{\mathbf{v}}{T} f_0$
Mean flow

$$\frac{q\mathbf{E}}{m} \cdot \nabla_{\mathbf{v}} f_0 = -\nu_c f_1 \rightarrow \frac{q\mathbf{E}}{m} \cdot \frac{m\mathbf{v}}{T} f_0 = \nu_c f_1 \rightarrow f_1 = \frac{q\mathbf{E}}{T\nu_c} \cdot \mathbf{v} f_0$$

Resistivity

Ohm law: $\mathbf{E} = \eta \mathbf{J}$ used to determine the value of η.

- $\mathbf{J} = q \int f_1 \mathbf{v} d\mathbf{v} = \frac{q^2 \mathbf{E}}{T \nu_c} \cdot \int f_0 \mathbf{v}\mathbf{v} d\mathbf{v}$ (dyadic dot product);
- $\mathbf{E} \cdot \mathbf{v}\mathbf{v} = \sum_k E_k v_k \mathbf{v} = E v_x^2 \mathbf{e}_x + E v_x v_y \mathbf{e}_y + E v_x v_z \mathbf{e}_z \; (E_{y,z} = 0)$;
- the integral in space velocity is:

$$\int f_0 v_x v_y d\mathbf{v} = \int f_0 v_x v_z d\mathbf{v} = 0$$

$$\int f_0 v_x^2 dv_x dv_y dv_z = n \left(\frac{m}{2\pi T} \right)^{3/2} \int e^{-m \frac{v_x^2}{2T}} v_x^2 dv_x \int e^{-m \frac{v_y^2}{2T}} dv_y \int e^{-m \frac{v_z^2}{2T}} dv_z$$

- with $v_k = s\sqrt{2T/m}$:

$$n \left(\frac{m}{2\pi T} \right)^{3/2} \left(\frac{2T}{m} \right)^{3/2} \frac{2T}{m} \left(\int e^{-s^2} ds \right)^2 \int e^{-s^2} s^2 ds = \frac{2Tn\pi}{m\pi^{3/2}} \int e^{-s^2} s^2 ds$$

$$= \frac{2Tn}{m} \frac{1}{\sqrt{\pi}} \int (-2s) e^{-s^2} \frac{s}{2} ds = \frac{2Tn}{m} \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{Tn}{m}$$

- $\mathbf{J} = \frac{q^2 \mathbf{E}}{T \nu_c} \frac{Tn}{m} \rightarrow \eta = \frac{m \nu_c}{n q^2}$

Diffusion coefficient in a plasma with $\mathbf{B}=0$

To study particle diffusion we assume a spatial profile for the density in the initial Maxwellian so that:

$$f_0 = n(\mathbf{r}) \left(\frac{m}{2\pi T} \right)^{3/2} e^{-m \frac{\mathbf{v}^2}{2T}}$$

Assuming $\mathbf{F} = 0$ the linearized kinetic equation becomes:

$$\mathbf{v} \cdot \nabla_{\mathbf{r}} f_0 = -\nu_c f_1 \rightarrow \frac{f_0}{n} \mathbf{v} \cdot \nabla_{\mathbf{r}} n = -\nu_c f_1 \rightarrow f_1 = -\frac{f_0}{n\nu_c} \mathbf{v} \cdot \nabla_{\mathbf{r}} n$$

and the flux of particles is given by:

$$\mathbf{\Gamma} = \int f_1 \mathbf{v} d\mathbf{v} = -\frac{1}{n\nu_c} \nabla_{\mathbf{r}} n \int f_0 \mathbf{v}\mathbf{v} d\mathbf{v}$$

The integral in space velocity gives Tn/m so :

$$\mathbf{\Gamma} = -\frac{T}{m\nu_c} \nabla_{\mathbf{r}} n \rightarrow \boxed{D = \frac{T}{m\nu_c}}$$

Same result as before, but this is more rigorous

Fokker-Planck collision operator

Used in the papers

A general formulation for treating variations in the distribution function due to many collisional events producing small changes in the velocity.

- The starting point is the assumption that the distribution function at the time t depends on its value at the time $t - \Delta t$ i.e.:

$$f(\mathbf{v}, t) = \int f(\mathbf{v} - \Delta\mathbf{v}, t - \Delta t)\phi(\mathbf{v} - \Delta\mathbf{v}, \Delta\mathbf{v})d^3\Delta v$$

- $\phi(\mathbf{v}, \Delta\mathbf{v})$ is the probability that a particle with velocity $\mathbf{v}$ acquires an increment of $\Delta\mathbf{v}$ in the time Δt.
- From this definition the rate of change for f due to collisions can be derived and is called the Fokker-Planck collisional operator:

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{f(\mathbf{v}, t) - f(\mathbf{v}, t - \Delta t)}{\Delta t} =$$

$$-\frac{\partial}{\partial \mathbf{v}}\left(\frac{d < \Delta\mathbf{v} >}{dt} f\right) + \frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}} : \left(\frac{d < \Delta\mathbf{v}\Delta\mathbf{v} >}{dt} f\right)$$

Drift-kinetic equation

Alternative to kinetic eq

When the gyro-radius is much smaller than the system scale length ($r_L << L$), a change of variable can be performed considering the g.c. motion: $(\mathbf{x}, \mathbf{v}) \rightarrow (\mathbf{x}_{gc}, \mu, \epsilon_{gc}, \varphi)$; by a gyro-averaging procedure the evolution for the g.c. distribution f_{gc} is given by:

$$\frac{\partial f_{gc}}{\partial t} + (v_{\|}\mathbf{b} + \mathbf{v}_{d\perp}) \cdot \nabla f_{gc} + \frac{d\epsilon_{gc}}{dt}\frac{\partial f_{gc}}{\partial \epsilon_{gc}} = < \left(\frac{\partial f_{gc}}{\partial t}\right)_c >$$

- $\mathbf{v}_{d\perp}$ is the drift velocity (including $\mathbf{E} \times \mathbf{B}$ drift etc).
- with $d\epsilon_{gc}/dt \approx q\partial\Phi/\partial t + \mu\partial B/\partial t - qv_{\|}\mathbf{b} \cdot \partial\mathbf{A}/\partial t$
- $< \left(\frac{\partial f}{\partial t}\right)_c >$ is the collision operator averaged over gyrophase.

In many applications the drift-kinetic equation is solved numerically to evaluate the transport coefficients in fusion plasmas.

More general and accurate gyrokinetic equations that include finite gyroradius effects have been also derived and are used when $r_L/L \sim 1$.

Summary on non-axisymmetric devices and kinetic theory

- Stellarators as alternative to Tokamak: helical winding of the field lines obtained without recurring to the toroidal current.
- Lack of axisymmetry: orbits are not closed, higher level of neoclassical transport.
- If the system is not optimized often: electrons in the $1/\nu$ regime while ions in $\sqrt{\nu}$ regime.
- Optimization introducing a symmetry in the system or forcing it in the electron root regime.

- Kinetic equation to describe the particles distribution of a plasma.
- Macroscopic quantities and their time evolution obtained by integration over the velocity space.
- Closure of the equations system by assuming small deviations from the equilibrium and the linearization of the K.E.
- Examples of collision operators: BGK and Fokker-Planck.

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Some figures in the slides have been adapted from these papers/books.

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EXTRA

Transport in a weakly ionized gas

An example of a system not-intrinsically ambipolar: a weakly ionized gas.

- ∇n in the x direction, $\mathbf{B}$ along z, focus on perpendicular transport.
- Collisions mainly with neutral atoms at frequency $\nu_0 \propto \sqrt{m}$;
- In steady state $(d/dt \sim 0)$ the motion equation for each species is:

$$qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - T\nabla n - mn\nu_0 \mathbf{u} = 0$$

- along the x and y directions this becomes:

$$qnE_x + qnu_y B - T\frac{dn}{dx} - mn\nu_0 u_x = 0$$

$$-qnu_x B - mn\nu_0 u_y = 0 \rightarrow u_y = -\frac{u_x B}{m\nu_0}$$

- which gives for u_x:

$$u_x = \frac{1}{\frac{q^2B^2}{m\nu_0} + m\nu_0}\left[qE_x - T\frac{\nabla n}{n}\right] = \frac{\nu_0}{m(\omega_c^2 + \nu_0^2)}\left[qE_x - T\frac{\nabla n}{n}\right]$$

Ambipolar Electric field

The effect of the magnetic field becomes important when $\omega_c >> \nu_0$:

$$u_x \approx \frac{\nu_0}{m\omega_c^2}\left[qE_x - T\frac{\nabla n}{n}\right] \propto \sqrt{m}\left[qE_x - T\frac{\nabla n}{n}\right]$$

- apply to ions, which diffusion is dominant across the field ($\propto r_{L,i}^2$);
- an electric field E_x arises to retard ions and largely reduces $u_{x,i}$ by a factor $\sqrt{m_i/m_e}$, at zero order: $E_x \approx \frac{T_i}{ne}\nabla n$;
- replacing for electrons: $nu_{x,e} = -\frac{\nu_{0e}}{m_e\omega_{ce}^2}(T_i + T_e)\nabla n = -D_a\nabla n$
- so that the ambipolar diffusion coefficient is :

$$D_a = \frac{\nu_{e0}(T_e + T_i)}{m_e\omega_{ce}^2} = \nu_{0e} r_{Le}^2\left(1 + \frac{T_i}{T_e}\right)$$

Note:

- Diffusion rate is controlled by the species that diffuses more slowly!
- For diffusion along the field or in unmagnetized plasmas, $D_a \approx D_i$.

Diffusion coefficient in a plasma with $\mathbf{B} \neq 0$ (1/3)

Consider a steady state plasma with $\mathbf{B} = B_0 \mathbf{e}_z$ and density profile $n(x, z)$ with gradients in the direction parallel (z) and perpendicular (x) to $\mathbf{B}$.

- The corresponding f_0 is a bi-Maxwellian since in a plasma the temperature can be different along and across $\mathbf{B}$:

$$f_0 = n(x,z) \left(\frac{m}{2\pi T_\perp} \right) \left(\frac{m}{2\pi T_\parallel} \right)^{1/2} exp \left[-\frac{m(v_x^2 + v_y^2)}{2T_\perp} - \frac{mv_z^2}{2T_\parallel} \right]$$

- the linearized KE is:

$$\mathbf{v} \cdot \nabla_\mathbf{r} f_0 + \frac{q}{m}(\mathbf{v} \times \mathbf{B}) \cdot \nabla_\mathbf{v} f_1 = -\nu_c f_1 \rightarrow$$

$$\left(v_x \frac{\partial n}{\partial x} + v_z \frac{\partial n}{\partial z} \right) \frac{f_0}{n} + \frac{qB_0}{m} \left(v_y \frac{\partial f_1}{\partial v_x} - v_x \frac{\partial f_1}{\partial v_y} \right) = -\nu_c f_1$$

- Take the v_x, v_y and v_z moments. For instance with v_x:

$$\int \left[\left(v_x^2 \frac{\partial n}{\partial x} + v_z v_x \frac{\partial n}{\partial z} \right) \frac{f_0}{n} + \omega_c \left(v_y v_x \frac{\partial f_1}{\partial v_x} - v_x^2 \frac{\partial f_1}{\partial v_y} \right) - \nu_c v_x f_1 \right] dv_x dv_y dv_z = 0$$

Diffusion coefficient in a plasma with $\mathbf{B} \neq 0$ (2/3)

- note that $\int f_0 v_x v_z dv_x dv_z dv_y = 0$ and $\int v_x^2 \frac{f_0}{n} dv_x dv_y dv_z = \frac{T_\perp}{n}$;
- $\int v_x^2 \frac{\partial f_1}{\partial v_y} dv_x dv_y dv_z \propto \int \frac{\partial f_1}{\partial v_y} dv_y = 0$ since $f_1 \to 0$ at $\pm\infty$;
- $-\int v_x \nu_c f_1 dv_x dv_y dv_z = -\nu_c \int v_x f_1 dv_x dv_y dv_z = -\nu_c \Gamma_x$;
- the term in $\nabla_\mathbf{v}$ includes the particle flux along y:

$$\int v_x v_y \frac{\partial f_1}{\partial v_x} dv_x dv_y dv_z = \int v_y dv_y dv_z \left[(v_x f_1)_{-\infty}^{+\infty} - \int f_1 dv_x \right] = -\int f_1 v_y dv_x dv_y dv_z = -\Gamma_y$$

Repeating the same procedure for v_y and v_z, the moments are:

$$\frac{T_\perp}{m} \frac{\partial n}{\partial x} - \omega_c \Gamma_y = -\nu_c \Gamma_x$$

$$\omega_c \Gamma_x = -\nu_c \Gamma_y$$

$$\frac{T_\parallel}{m} \frac{\partial n}{\partial z} = -\nu_c \Gamma_z$$

Diffusion coefficient in a plasma with $\mathbf{B} \neq 0$ (3/3)

Solving for Γ_x and Γ_z:

$$\Gamma_x = -\frac{T_\perp}{m}\frac{\nu_c}{\nu_c^2 + \omega_c^2}\frac{\partial n}{\partial x} \qquad \Gamma_z = -\frac{T_\parallel}{m\nu_c}\frac{\partial n}{\partial z}$$

From the definition of the parallel and perpendicular diffusion coefficients:

$$\Gamma_x = -D_\perp \frac{\partial n}{\partial x} \qquad \Gamma_z = -D_\parallel \frac{\partial n}{\partial z}$$

their values are:

$$D_\perp = \frac{T_\perp}{m}\frac{\nu_c}{\nu_c^2 + \omega_c^2} \qquad D_\parallel = \frac{T_\parallel}{m\nu_c}$$

- $D_\parallel$ is the same obtained for transport in an unmagnetized plasma;
- for $\omega_c >> \nu_c \rightarrow D_\perp \approx \frac{T_\perp}{m}\frac{\nu_c}{\omega_c^2} = r_L^2 \nu_c$ (classical result).