CONSORZIO RFX
Ricerca Formazione Innovazione

MHD equilibrium & stability of magnetically confined plasmas

Plasma physics & diagnostics (AC1)

Padova (& zoom), November 26th 2024

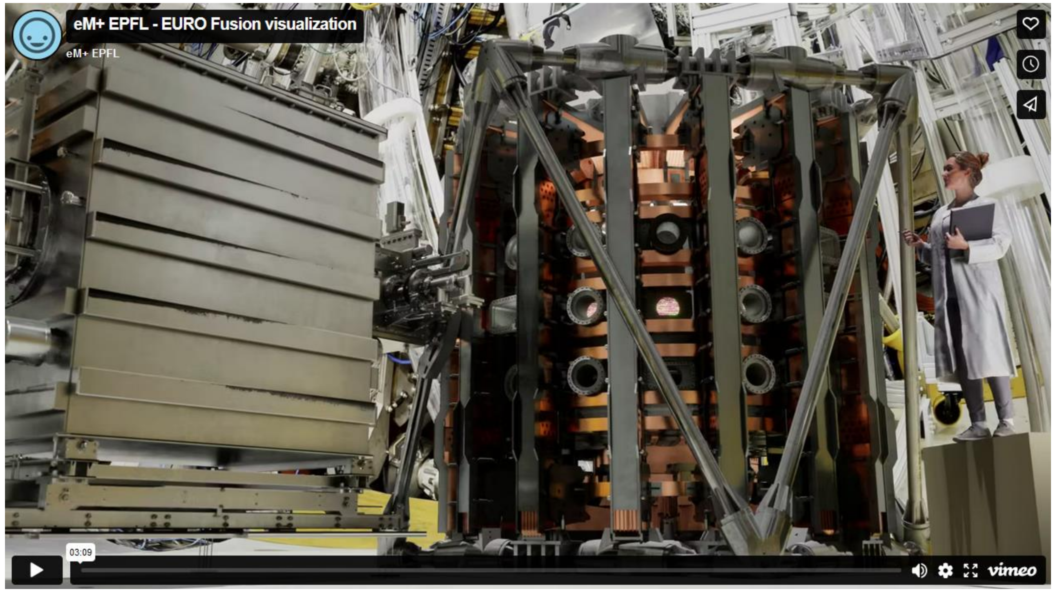
By Leonardo Pigatto (presented by Tommaso Bolzonella)

Consorzio RFX (CNR, ENEA, INFN, Università di Padova, Acciaierie Venete SpA) Corso Stati Uniti, 4 – 35127 Padova (Italy)

leonardo.pigatto@igi.cnr.it *tommaso.Bolzonella@igi.cnr.it*

0039 049 829 5606 *0039 049 829 5083*

Video: 3D visualization brings nuclear fusion to life



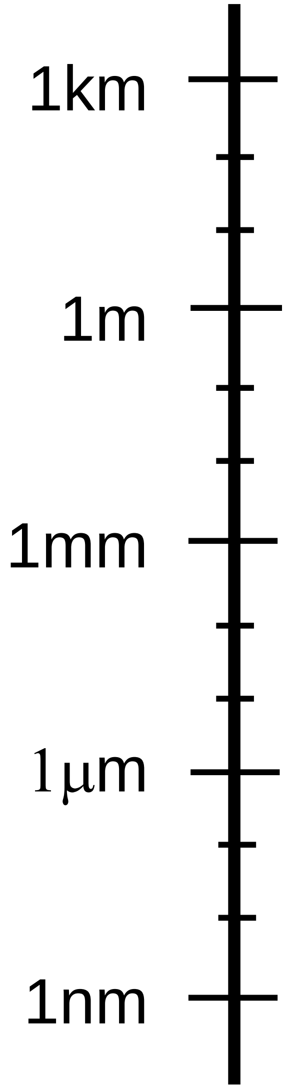
Aknowledgement:
Render from the realtime application developed by the Laboratory for Experimental Museology and the Swiss Plasma Centre as part of EPFL's Advanced Computing Hub of the EUROfusion Consortium, 2023.

►**Video**: https://player.vimeo.com/video/986450227?share=copy

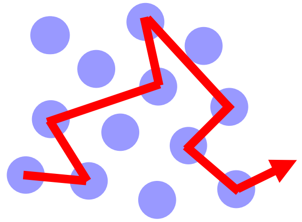
►**Description**: https://www.epfl.ch/labs/emplus/advanced-computing-hub-eurofusion-2021-2025/

► **EUROfusion news**: https://euro-fusion.org/member-news/3d-visualization-brings-nuclear-fusion-to-life/

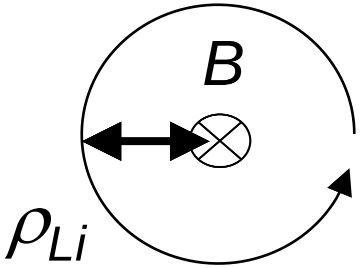
Scales in plasma physics: lengths



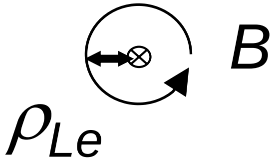
- Mean free path

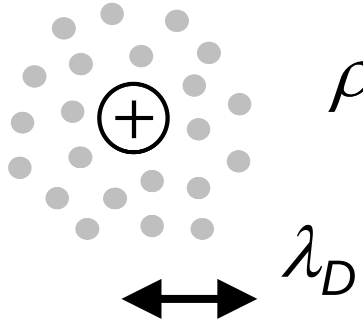


- Major radius *R*, minor radius *a*
- Gradient length of n and T
- Ion Lamour radius ρ_{Li}

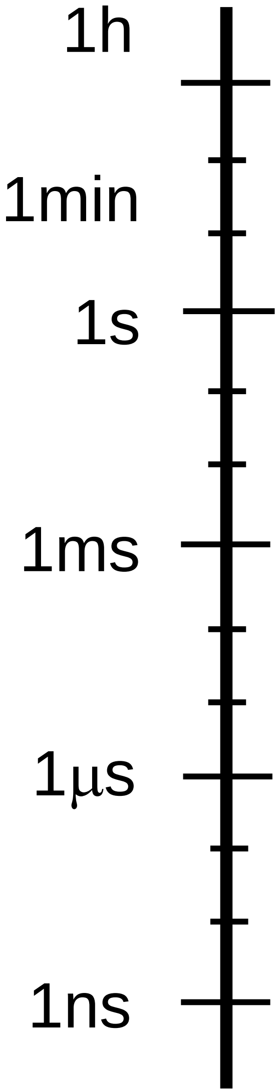


- Electron Lamour radius ρ_{Le}
- Debye length λ_D
- Visible light
- X-ray





Scales in plasma physics: times



- Current diffusion time
- Plasma discharge time
- Energy confinement time
- Energy exchange time
- Ion collision time
- Electron collision time

Outline

1. Recap on models for plasma description

2. Magnetohydrodynamic model
   - Basic properties
   - Ideal MHD
   - Equilibrium

3. Toroidal equilibrium: the Grad-Shafranov equation

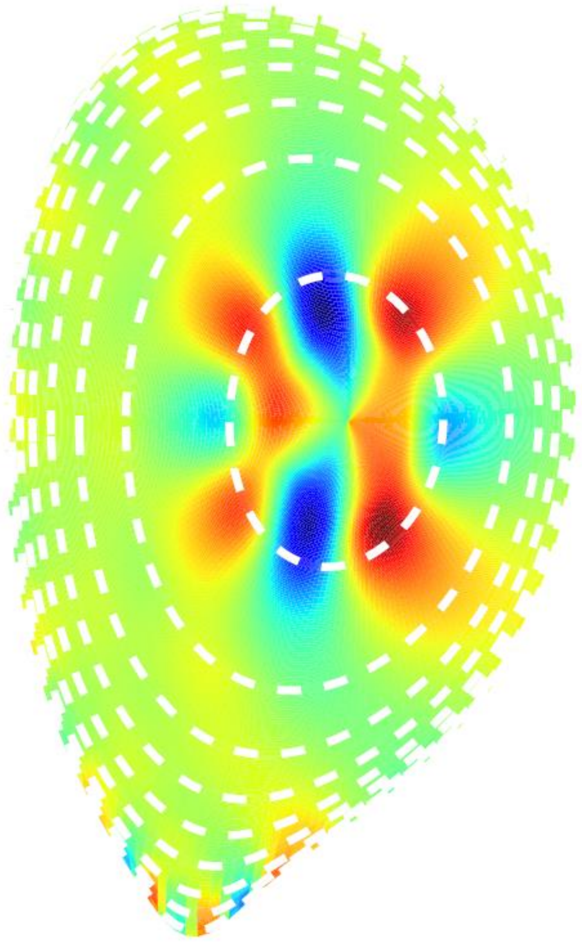
4. Figures of merit and “straight tokamak” limit

5. Linearized equations and force operator

6. Stability
   - Normal modes
   - Energy principle and intuitive form of W

7. Current driven instabilities

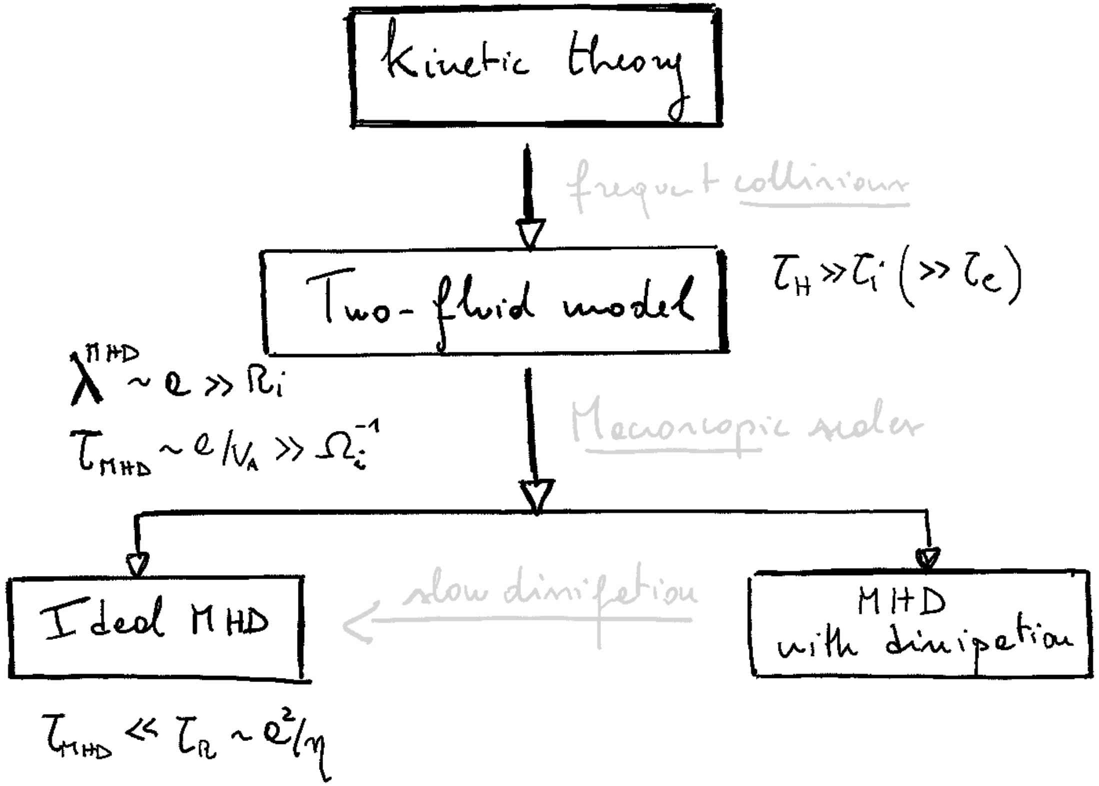
8. Global MHD instabilities in Tokamaks



PART 1: MHD model and equilibrium

Different ways to describe charged particles in EM field depending on scales & interactions...

- **Single particle** description: equation of motion for a charged particle in EM field
- **Kinetic** plasma theory: a statistical approach describing collective behavior with distribution functions



Frequent collisions for separation of e- & ion fluids

$$\tau_H \gg \tau_i \, (\gg \tau_e)$$

Macroscopic scales for **single fluid** description of plasma in terms of averaged macroscopic quantities

Slow (no) dissipation: **ideal MHD**

The MHD model

Covers a wide range of plasma dynamics relevant for equilibrium, waves and instabilities with $\tau \sim [10^{-6} \div 10^{-3}]s$

$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$	(1) continuity
$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) + \nabla p - \mathbf{j} \times \mathbf{B} = 0$	(2) momentum
$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \Gamma p \nabla \cdot \mathbf{v} = (\Gamma - 1)\eta\|\mathbf{j}\|^2$	(3) internal energy
$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$	(4) Faraday
$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$	(5) Ampere
$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$	(6) Ohm
$\nabla \cdot \mathbf{B} = 0$	(7) No magnetic monopoles

Assumptions in MHD

Typical system **length** much **larger than single particle orbit** extension (i.e. Larmor radius)

$$r_{Li} = \frac{\sqrt{m_i k T_i}}{eB} \ll L$$

This requirement can be violated when applying the model to some instabilities, and finite Larmor radius effects need to be taken into account.

Time scales longer than collision time (i.e. many collisions equilibrate temperature in the MHD characteristic time). This also translates to the mean free path being small w.r.t. characteristic length

$$\tau_{coll} \ll \tau \qquad \lambda_{mfp} \ll L$$

Can be violated in fusion plasmas, dynamics parallel to the magnetic field, can have large *mfp*. Nevertheless, perpendicular macroscopic phenomena are well described by the model

Remarks on MHD model

- ►The MHD model **can be rigorously derived from two-fluid equations** or using an intuitive approach involving conservation macroscopic quantities (see Goedbloed & Freidberg in bibliography)

- ►The MHD equations **involve conservation** of mass, momentum, energy and magnetic flux. Can be written in conservation form (some algebra required, see references)

- ►The momentum equation (2) only shows the Lorentz force ($\perp$ **B**) and the pressure gradient. Other forces can be added and contribute to acceleration (or damping) $\parallel$ **B**, e.g. gravity, viscous drag etc.

Remarks on MHD model

► $\mathbf{j}$ & $\mathbf{E}$ can be ruled out by using (5), (6) in Faraday's law (4). This yields the **induction equation**

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{v} \times \mathbf{B})}_{\textit{Convective term}} - \underbrace{\frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B})}_{\textit{Diffusion term}} \qquad (8)$$

► **Ideal MHD** case: when the plasma is a perfect conductor (or can be treat as one) a closure of the MHD system is obtained ($\eta = 0$) with the equation of state:

$$\frac{d}{dt}\left(\frac{p}{\rho^{\Gamma}}\right) = 0 \quad \rightarrow \quad \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\Gamma p \nabla \cdot \mathbf{v} \qquad (9)$$

$\eta = 0$ → only convective term → variation of B only with fluid → magnetic flux conservation

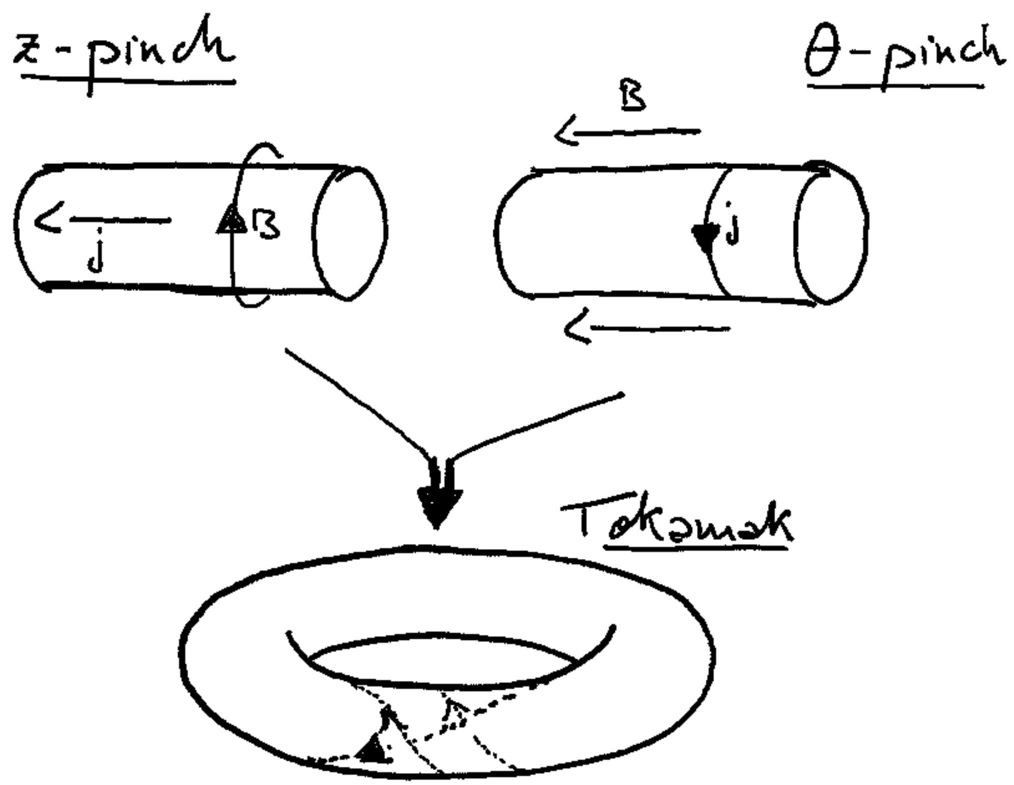
"In ideal MHD the field lines are frozen in the plasma"

Plasma equilibrium
In the beginning there was the cylinder

In cylindrical geometry plasma equilibria are either very **unstable** and/or suffer of **end losses** that are detrimental for confinement

Closing the cylinder into a **torus**:

- Cures losses ∥ B_z
- Instabilities can be avoided within certain operational boundaries



Note: although the Tokamak configuration is one of the most advanced, it is not the only toroidal confinement scheme!
e.g. RFP (also pinch-like) or Stellarator

Equilibrium is balance between confining magnetic field and plasma pressure

Important **assumptions** are often made in studying toroidal equilibrium configurations (mostly fine for Tokamak cases):

1. Take the **static** case as fist approx.
   - $\mathbf{v} = 0,\ \frac{\partial}{\partial t}\{\rho, p, \mathbf{B}\} = 0$

2. **Axisymmetry** (used later)
   - $\partial/_{\partial\Phi} = 0$

Static equilibrium for toroidal plasmas

$$\mathbf{v} = 0, \qquad \frac{\partial}{\partial t}\{\rho, p, \mathbf{B}\} = 0$$

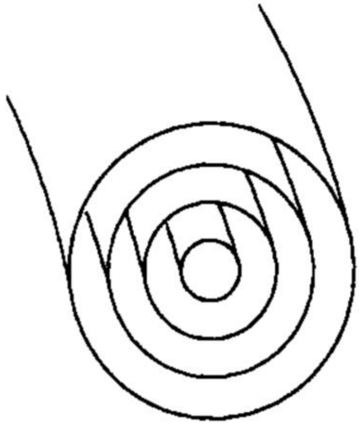
With these assumptions the ideal MHD equations reduce to:

$$\mathbf{j} \times \mathbf{B} = \nabla p$$ Force balance

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$ Ampere's law

$$\nabla \cdot \mathbf{B} = 0$$ Magnetic flux law

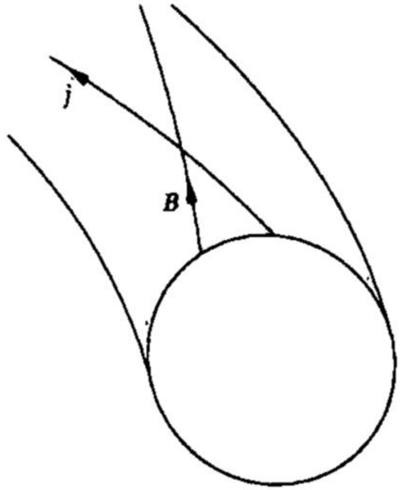
Focus on the pressure balance equation



If the magnetic force perfectly balances the plasma pressure

$$\mathbf{j} \times \mathbf{B} = \nabla \mathrm{p}$$

This implies that $\mathbf{B}$ is perpendicular to both $\mathbf{j}$ and $\nabla \mathbf{p}$:



► $\mathbf{B} \cdot \nabla \mathrm{p} = 0$

There is no pressure gradient along the field lines and the magnetic surfaces are surfaces of constant pressure

► $\mathbf{j} \cdot \nabla \mathrm{p} = 0$

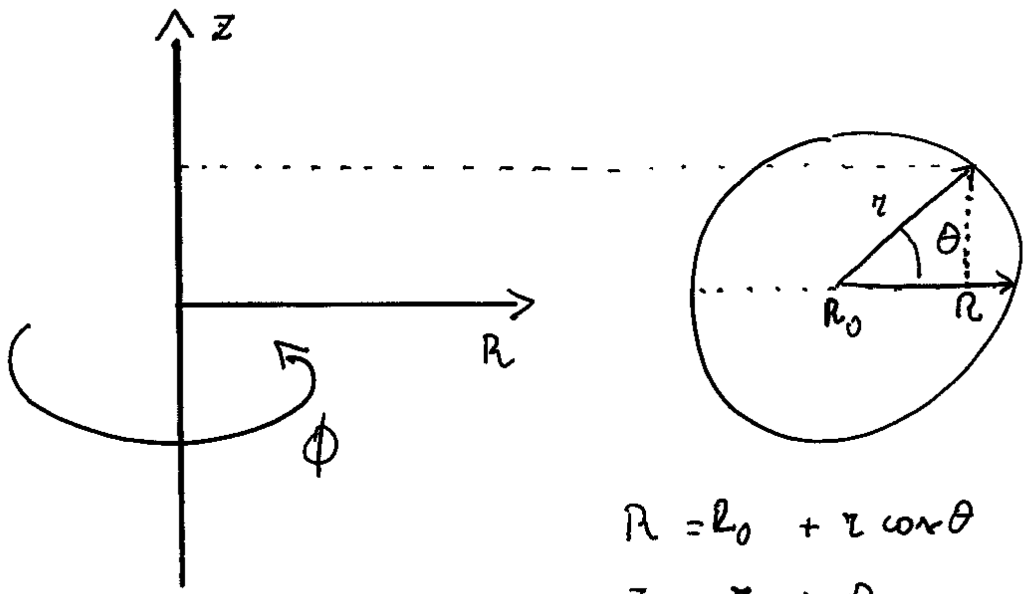
Current lines also lie on isobaric surfaces

How can we describe a toroidal equilibrium?

To do so we can work in a cylindrical coordinate system (R, ϕ, z) where we can write the magnetic field as:

$$\mathbf{B} = B_\phi \hat{e}_\phi + \mathbf{B}_p$$

Where the poloidal field lies on (R,Z)



$R = R_0 + r \cos\theta$

$z = r \sin\theta$

How can we describe a toroidal equilibrium?

To do so we can work in a cylindrical coordinate system (R, ϕ, z) where we can write the magnetic field as:

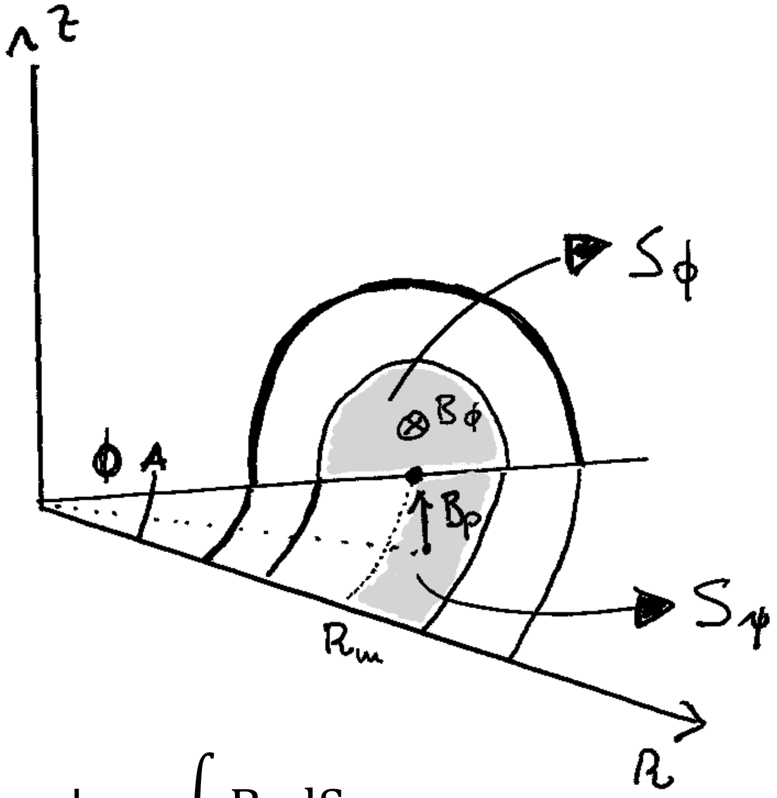
$$\mathbf{B} = B_{\phi}\hat{e}_{\phi} + \mathbf{B}_p$$

Where the poloidal field lies on (R,Z)

We introduce a function ($\boldsymbol{\psi}$) defined as the **poloidal flux within each magnetic surface**

And a function $\boldsymbol{\Phi}$: the toroidal flux inside a given magnetic surface.

Both are constant on a given surface



$$\psi = \int B_p dS_{\psi}$$

$$\Phi = \int B_{\phi} dS_{\phi}$$

How can we describe a toroidal equilibrium?

We introduce a function (**ψ**) define as the **poloidal flux within each magnetic surface**, therefore this acts as a radial coordinate and is constant on a given surface

$$\begin{cases} B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z} \\ B_z = \frac{1}{R}\frac{\partial\psi}{\partial R} \end{cases}$$

This satisfies the divergence requirement, which in **axisymmetric** assumption becomes:

$$\frac{1}{R}\frac{\partial(RB_R)}{\partial R} + \frac{\partial B_z}{\partial z} = 0$$

The flux (**ψ**) is found to be **constant along the magnetic surfaces**:

$$\mathbf{B}\cdot\nabla\psi = B_R\frac{\partial\psi}{\partial R} + B_z\frac{\partial\psi}{\partial z} = RB_RB_z - RB_zB_R = 0$$

ψ *can be used to label the surfaces where B lays, finding these surfaces is equivalent to finding the* ***ψ****=const surfaces*

How can we describe a toroidal equilibrium?

The same can be done for the current, defining a **current flux function** (**f**) so that the components of the poloidal current density become:

$$\begin{cases} j_R = -\frac{1}{R}\frac{\partial f}{\partial z} \\ j_z = \frac{1}{R}\frac{\partial f}{\partial R} \end{cases}$$

Using Ampere's law we get an expression for **f**:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad \Longrightarrow \quad \begin{cases} j_R = -\frac{1}{\mu_0}\frac{\partial B_\phi}{\partial z} \\ j_z = \frac{1}{\mu_0}\frac{1}{R}\frac{\partial (R B_\phi)}{\partial R} \end{cases} \quad \Longrightarrow \quad f = \frac{R B_\phi}{\mu_0}$$

Again the flux function (**f**) is constant of the current density surfaces:

$$\mathbf{j} \cdot \nabla f = 0$$

How can we describe a toroidal equilibrium?

We can now take the force balance equation and write it in the new notation, splitting toroidal and poloidal parts

$$\begin{aligned}\nabla p &= \mathbf{j}_p \times \hat{e}_\phi B_\phi + \hat{e}_\phi j_\phi \times \mathbf{B}_p \\ &= \frac{1}{R}(\nabla f \times \hat{e}_\phi) \times \hat{e}_\phi B_\phi + \hat{e}_\phi j_\phi \times \frac{1}{R}(\nabla \psi \times \hat{e}_\phi) \\ &= -\frac{B_\phi}{R}\nabla f + \frac{j_\phi}{R}\nabla\psi\end{aligned}$$

How can we describe a toroidal equilibrium?

We can now take the force balance equation and write it in the new notation, splitting toroidal and poloidal parts

$$
\begin{aligned}
\nabla p &= \mathbf{j}_p \times \hat{e}_\phi B_\phi + \hat{e}_\phi j_\phi \times \mathbf{B}_p \\
&= \frac{1}{R}(\nabla f \times \hat{e}_\phi) \times \hat{e}_\phi B_\phi + \hat{e}_\phi j_\phi \times \frac{1}{R}(\nabla \psi \times \hat{e}_\phi) \\
&= -\frac{B_\phi}{R}\nabla f + \frac{j_\phi}{R}\nabla \psi
\end{aligned}
$$

$$\mathbf{j}_p = \frac{1}{R}(\nabla f \times \hat{e}_\phi)$$ Poloidal current density

$$\mathbf{B}_p = \frac{1}{R}(\nabla \psi \times \hat{e}_\phi)$$ Poloidal field

How can we describe a toroidal equilibrium?

The next step is writing pressure (**p**) and current flux function (**f**) in terms of poloidal flux

$$\nabla f(\psi) = \frac{df}{d\psi}\nabla\psi \qquad \nabla p(\psi) = \frac{dp}{d\psi}\nabla\psi$$

And by substitution in the force balance we get (recall $f = RB_\phi/\mu_0$!!):

$$j_\phi = R\frac{dp}{d\psi} + B_\phi\frac{df}{d\psi} = R\frac{dp}{d\psi} + \frac{\mu_0}{R}f\frac{df}{d\psi}$$

Named p' & f'

Ampere's law gives us the current density in terms of poloidal flux:

$$\begin{aligned}\mu_0 j_\phi &= \frac{dB_R}{dz} - \frac{dB_z}{dR} \\ &= -\frac{1}{R}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)\end{aligned}$$

How can we describe a toroidal equilibrium?

Putting everything together we obtain the **Grad-Shafranov** equation

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial z^2}=-\mu_0 R^2 p'(\psi)-\mu_0^2 f(\psi)f'(\psi)$$

p & **f** are arbitrary flux functions, by specifying which everything else is determined by the solution of the G-S equation !

Solutions of the G-S equation can be obtained analytically (e.g. Solov'ev equilibria, circular cross section case) or numerically with a variety of equilibrium codes (e.g. CHEASE, HELENA …)

Ingredients of toroidal axisymmetric equilibrium: current & pressure

$$\underbrace{R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial z^2}}_{\text{Usually written as } \Delta^*\psi \text{. Where } \Delta^* \text{ is a Laplacian-like operator with inverted R and 1/R}} = -\mu_0 R^2 \underbrace{p'(\psi)}_{\text{Flux function linked to } \textbf{pressure}} - \mu_0^2 \underbrace{f(\psi)f'(\psi)}_{\text{Flux function linked to } \textbf{current}}$$

Usually written as $\Delta^*\psi$
Where Δ^* is a Laplacian-like operator with inverted R and 1/R

Flux function linked to **pressure**

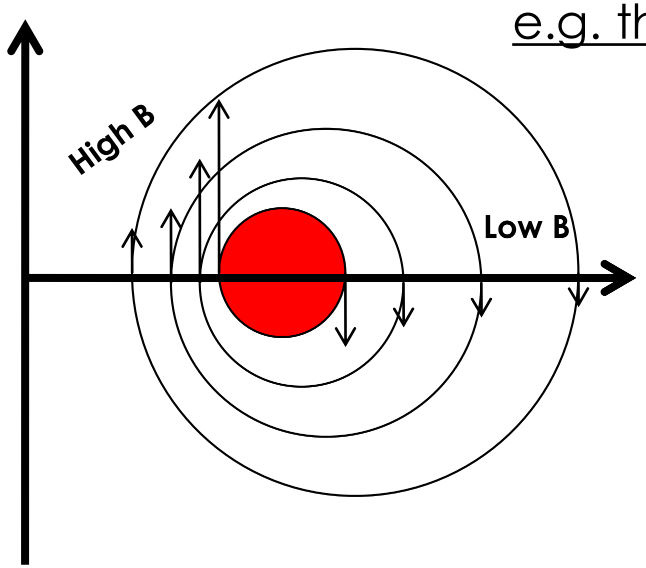
Flux function linked to **current**

Defining current and pressure (or the source profiles p' & f') is the starting point to solve the G-S equation
The solution yields ψ → B → equilibrium

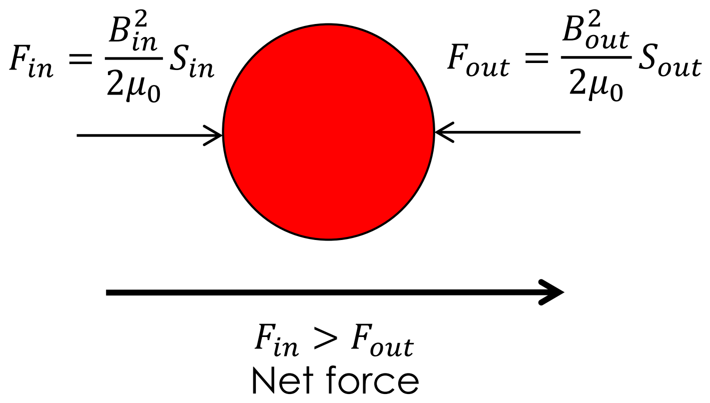
Remarks on equilibrium for toroidal plasmas

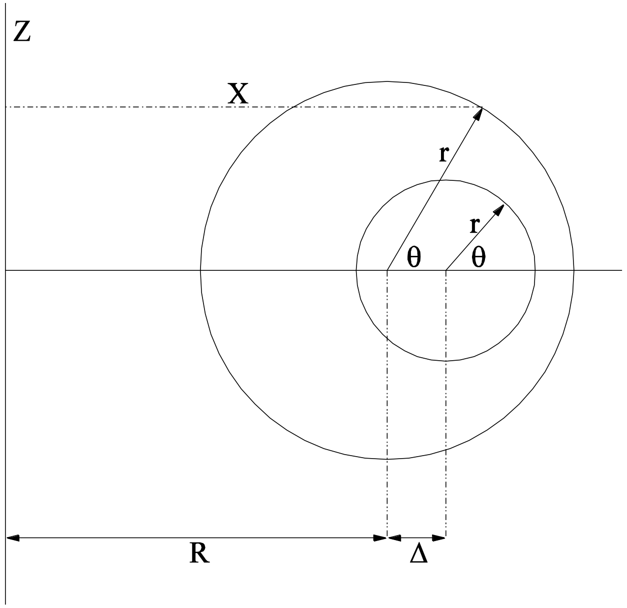
Toroidal geometry introduces **outward directed forces** which cause radial shift of the GS solution: flux surfaces whose centers are no longer concentric but **shifted** outward along R with respect to one another

<u>e.g. the hoop force</u>



Same flux inside and outside
→ inside is squeezed in smaller area

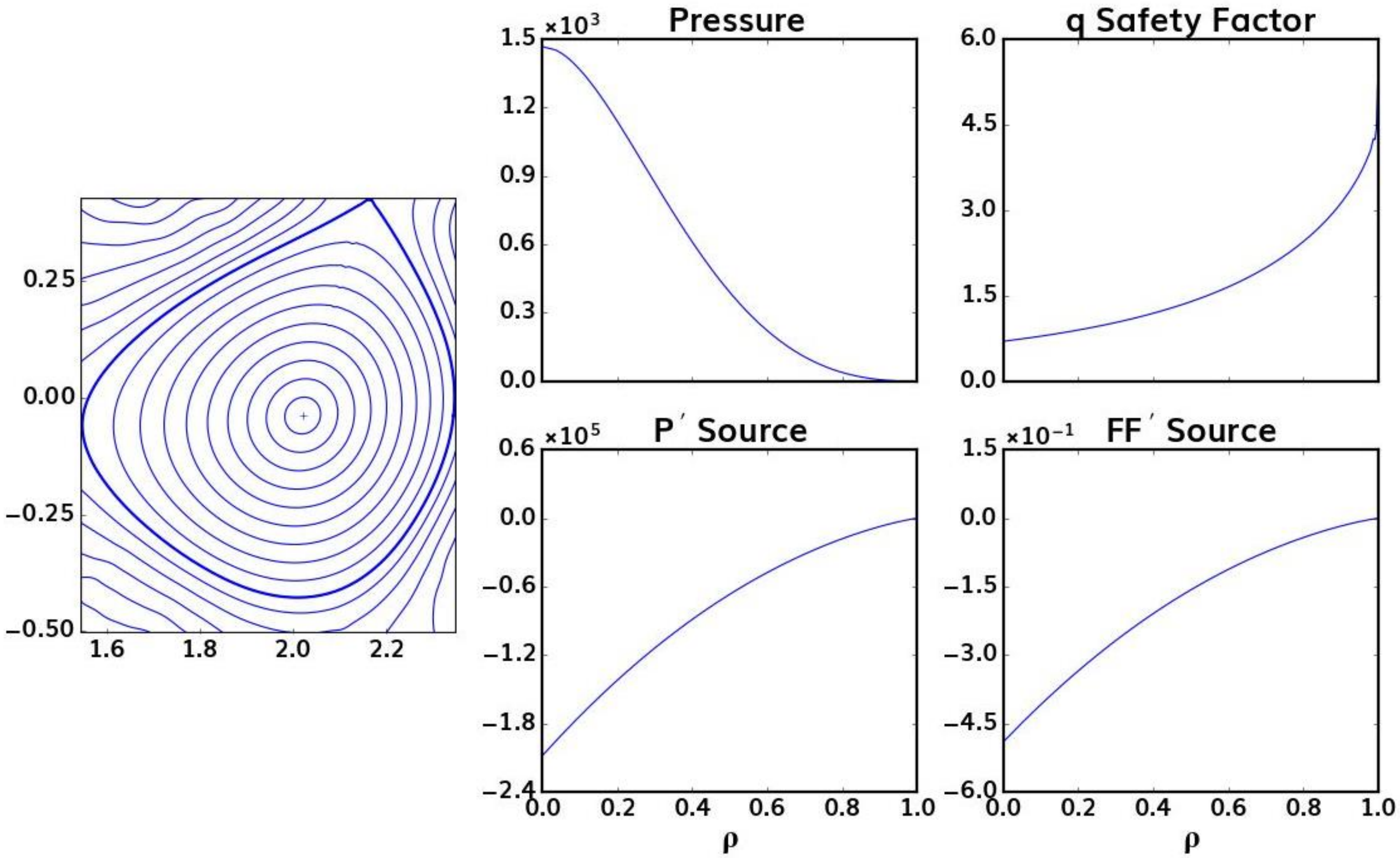




"Shafranov shift"

Net outward forces (like hoop force, from poloidal field, or tyre force, due to plasma pressure) require vertical external field for compensation

Numerical solution of the Grad-Shafranov equation with the fixed-boundary code CHEASE



In this example a numerical solution of the G-S equation is shown

The solver takes p’ and ff’ as input profiles and calculates the poloidal flux function (ψ) within a computational boundary defined by the plasma last closed surface

Figures of merit: safety factor (**q**)

Represents the **number of toroidal turns after which the field line returns to a given poloidal position**. In axisymmetric equilibria each field line has a specific value of *q*:

$$q = \frac{\Delta\Phi}{2\pi}$$

In infinitesimal terms, the toroidal angle that a field line runs for given poloidal angle, translating into flux we get the definition:

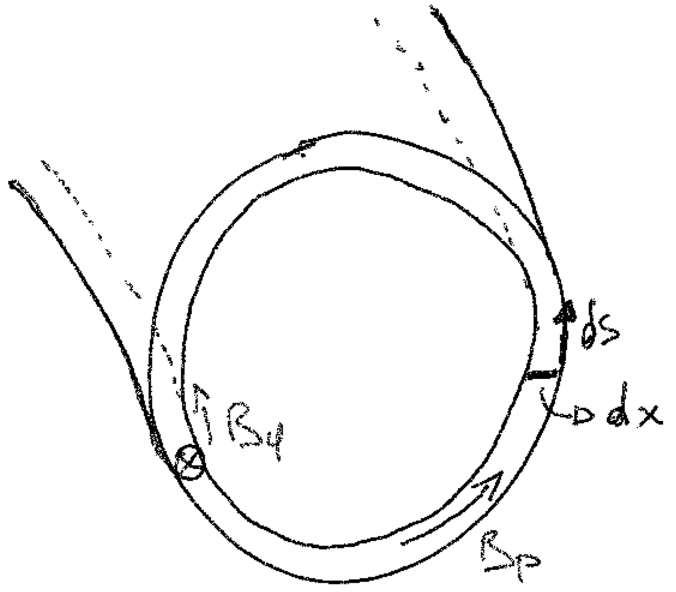
$$q = \frac{d\Phi}{d\Psi}$$

Recalling the definitions of toroidal and poloidal fluxes, **for an infinitesimal ring between two surfaces** the poloidal flux is:

$$d\psi = 2\pi R\, B_p dx$$

While the toroidal flux:

$$d\Phi = \oint \left(B_\phi dx\right) ds \xrightarrow{\text{yields}} q(\psi) = \frac{1}{2\pi}\oint_\psi \frac{B_\phi}{RB_p} ds$$



Figures of merit: **β**

The ratio between **kinetic** and **magnetic pressure**: telling us how efficiently the magnetic field is confining plasma pressure

$$\beta = \frac{p}{B^2/2\mu_0}$$

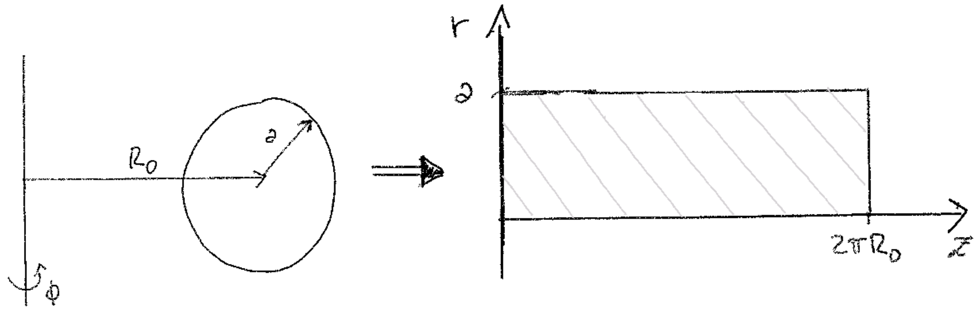
Several forms and definitions of this parameters are used: average β, poloidal β_p, toroidal β_t, normalized β_N…

$$\langle\beta\rangle = \frac{\int p\, dV/\int dV}{B_0^2/2\mu_0},\ \beta_p = \frac{\langle p\rangle}{B_\theta^2(a)/2\mu_0},\ \beta_t = \frac{\langle p\rangle}{B_\phi^2/2\mu_0},\ \beta_N = \beta_t \frac{a\,[m]\,B_\phi\,[T]}{I\,[MA]}$$

The "straight tokamak" approximation

We define the **inverse aspect ratio** of a torus as:

$$^{a}/_{R_0} = \varepsilon$$



When $\boldsymbol{\varepsilon} \ll \mathbf{1}$ (i.e. large aspect ratio) a *slender* torus can be approximated by a periodic cylinder of length $L = 2\pi R_0$. The toroidal angle becomes the 'z' axis: $\boldsymbol{\phi} \rightarrow \boldsymbol{z}$

Using differential operators expressed in cylindrical coordinates (r, θ, z) and with symmetries: $^{\partial}/_{\partial\theta} = 0$, $^{\partial}/_{\partial z} = 0$

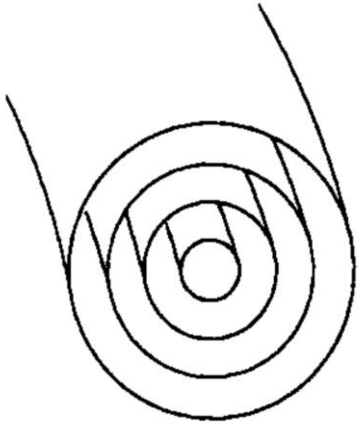
This is called the **straight tokamak** limit and gives a first approximation of toroidal equilibrium. Leading order expansions can be obtained in ε for each quantity.

$$\begin{cases} \mathbf{j} \times \mathbf{B} = \nabla p \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \\ \nabla \cdot \mathbf{B} = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \frac{dp}{dr} = j_\theta B_z - j_z B_\theta \\ j_\theta = -\frac{1}{\mu_0}\frac{dB_z}{dr} \\ j_z = \frac{1}{\mu_0 r}\frac{d(rB_\theta)}{dr} \end{cases}$$

Equilibrium equations are greatly simplified in this limit, and figures of merit can aid understanding physical properties of the system.

Figures of merit: safety factor (**q**) in straight tokamak limit

In large aspect ratio approx. with circular cross section the flux surfaced become nested cylinders with radius $r \leq a$



The q-profile integral can be carried out over a single poloidal circuit of radius r:

$$q(\psi) = \frac{1}{2\pi}\oint_{\psi} \frac{B_\phi}{RB_p}\,ds \xrightarrow{\text{yields}} \boxed{q(r) = \frac{rB_\phi}{R_0 B_p}}$$

Using Ampère's law we get a useful expression:

$$2\pi r B_p = \mu_0 I(r) \rightarrow q(r)\,\frac{2\pi r^2 B_\phi}{\mu_0 I(r) R_0}$$

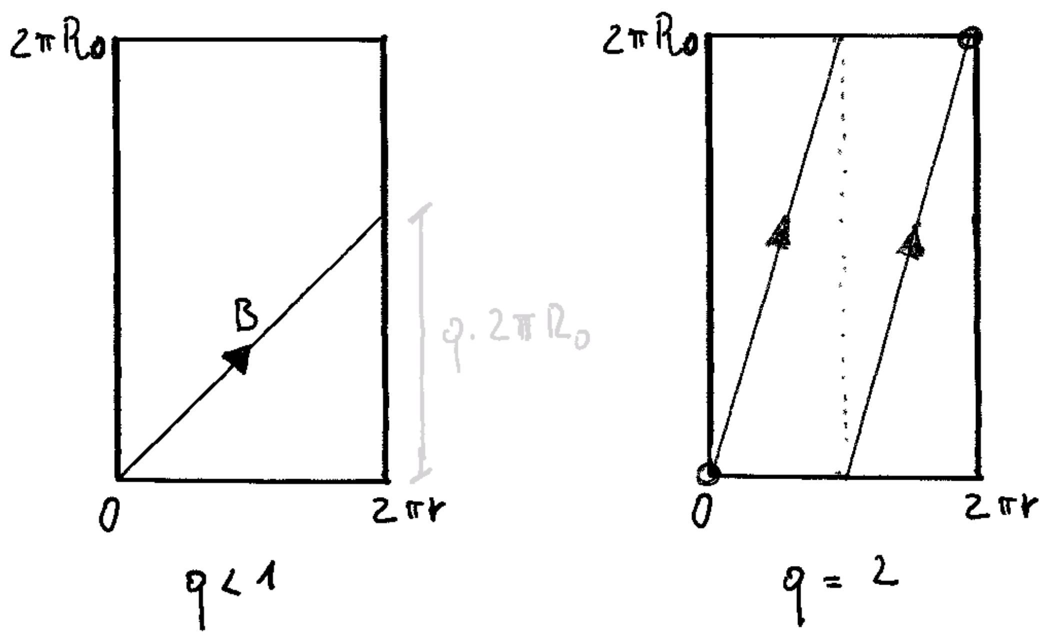
At plasma edge: $\boxed{q_a = \frac{2\pi a^2 B_\phi}{\mu_0 I_p R_0}}$

Figures of merit: safety factor (**q**) in straight tokamak limit

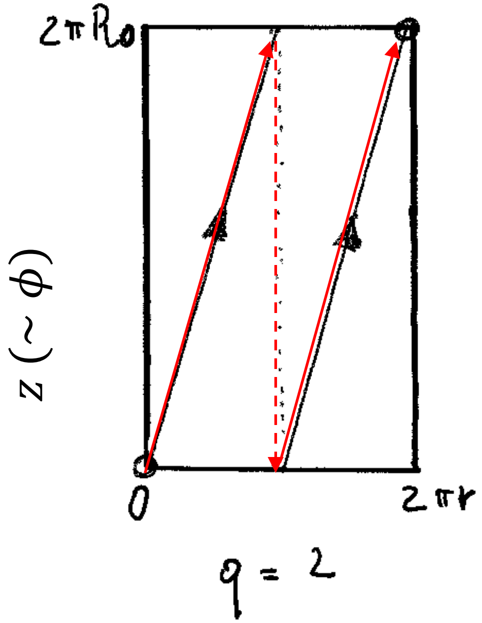
Each magnetic surface has a specific safety factor value (i.e. a specific pitch of magnetic field lines) that can be visualized by **unrolling the cylinder** !

An important concept is that of **rational magnetic surfaces**: those where the field lines close upon themselves after M poloidal turns and N toroidal revolutions

$$q_{rat} = \frac{N}{M}$$



Figures of merit: safety factor (**q**) in straight tokamak limit

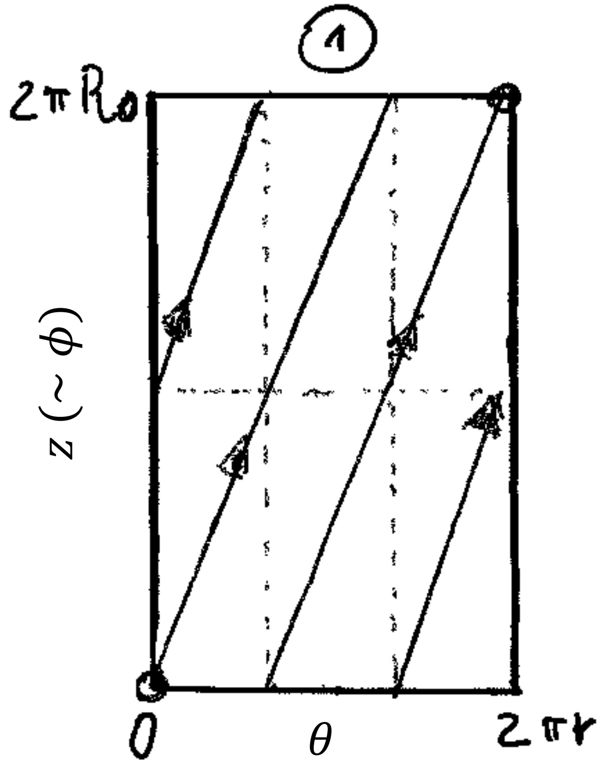


θ

Example: following the field line from the origin (lower left angle) we see that it runs along the 'z' axis twice (N=2) in order to make a single complete poloidal turn (M=1). Note the periodic boundary conditions needed to properly handle the straight&unrolled tokamak model.

$$\rightarrow q = 2$$

Figures of merit: safety factor (**q**) in straight tokamak limit



$$q = ?$$

A useful model for toroidal current density

With **large aspect-ratio** and **circular cross section**, q-profile is determined by toroidal current density $j(r)$

Useful class of profiles: $\mathrm{j} = \mathrm{j}_0\left(1-\frac{\mathrm{r}^2}{\mathrm{a}^2}\right)^{\nu}$

Use Ampère to get B_θ:

$$\mu_0 \mathrm{j} = \frac{1}{\mathrm{r}}\frac{\mathrm{d}(\mathrm{r}\mathrm{B}_\theta)}{\mathrm{dr}}$$

$$\mathrm{I}(\mathrm{r}) = 2\pi\int_0^{\mathrm{r}} \mathrm{j}(\mathrm{r}')\mathrm{r}'\mathrm{dr}' = \mathrm{j}_0\frac{\pi \mathrm{a}^2}{\nu+1}\left[1-\left(1-\frac{\mathrm{r}^2}{\mathrm{a}^2}\right)^{\nu+1}\right]$$

$$\mathrm{B}_\theta(\mathrm{r}) = \frac{\mu_0 \mathrm{j}_0 \mathrm{a}^2}{2\mathrm{r}(\nu+1)}\left[1-\left(1-\frac{\mathrm{r}^2}{\mathrm{a}^2}\right)^{\nu+1}\right]$$

For the q-profile we get the expression:

$$q(r) = \frac{2(\nu+1)}{\mu_0 j_0}\frac{B_\phi}{R}\frac{r^2/a^2}{[1-\ (1-r^2/a^2)^{\nu+1}]}$$

$$\frac{q(a)}{q(0)} = \nu+1$$

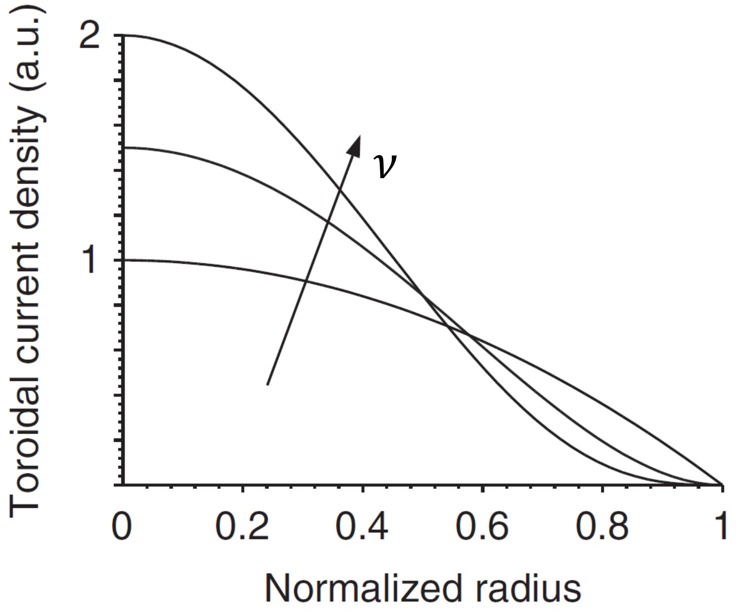
Use approximation for q:

$$\mathrm{q}(\mathrm{r}) = \frac{\mathrm{r}\mathrm{B}_\phi}{\mathrm{R}_0\mathrm{B}_\mathrm{p}}$$

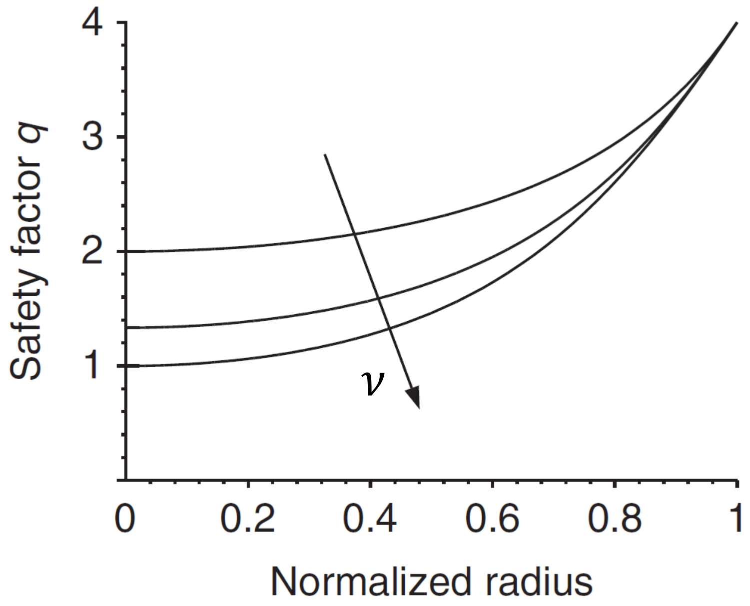
A useful model for toroidal current density

With large aspect-ratio and circular cross section, q-profile is determined by toroidal current density $j(r)$

$$j = j_0\left(1 - \frac{r^2}{a^2}\right)^{\nu}$$

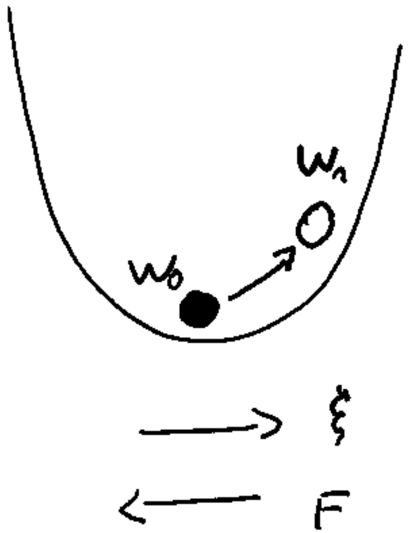


$$q(r)$$

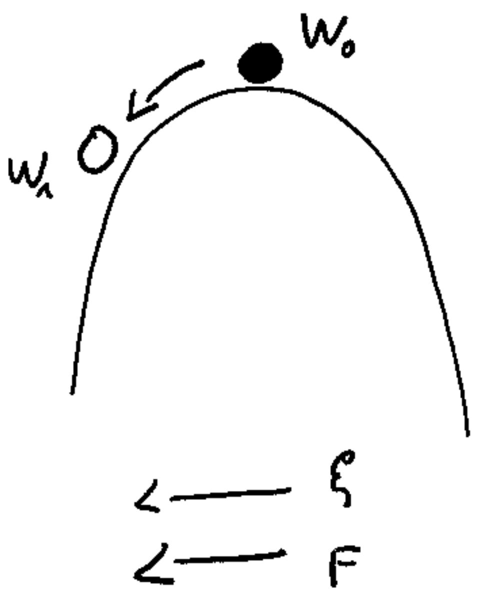


PART 2: linear MHD stability

Intuitive stability



$W_0 < W_1$



$W_0 > W_1$

e.g. potential energy W due to gravity

} eq. of motion

} energy principle (variational methods)

Linear stability concerns the behavior of a dynamical system w.r.t. **small** (infinitesimal) **perturbations**

The original non-linear problem is split into **equilibrium** and **perturbation**, we will start again from time-independent static equilibrium defined by $\mathbf{B}_0, \mathbf{j}_0, p_0, \rho_0, \mathbf{v}_0$:

$$\mathbf{j}_0 \times \mathbf{B}_0 = \nabla p_0$$
$$\nabla \times \mathbf{B}_0 = \mu_0 \mathbf{j}_0$$
$$\nabla \cdot \mathbf{B}_0 = 0$$
$$\mathbf{v}_0 = 0$$

Time dependence is kept in the perturbed quantities X_1. The assumption here is that perturbations are small w.r.t equilibrium: $|X_1| \ll |X_0|$

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_1(\mathbf{r},t)$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r},t) \qquad \rho(\mathbf{r},t) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r},t)$$
$$\mathbf{j}(\mathbf{r},t) = \mathbf{j}_0(\mathbf{r}) + \mathbf{j}_1(\mathbf{r},t) \qquad p(\mathbf{r},t) = p_0(\mathbf{r}) + p_1(\mathbf{r},t)$$

Perturbed momentum equation

By substitution into the momentum equation, and retaining just the first order terms, we obtain the first piece of linearized (ideal) MHD

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\nabla\mathbf{v}\right)=\mathbf{j}\times\vec{B}-\nabla p$$

$$(\rho_0+\rho_1)\left(\frac{\partial \mathbf{v}_1}{\partial t}+\mathbf{v}_1\nabla\mathbf{v}_1\right)=(\mathbf{j}_0+\mathbf{j}_1)\times(\mathbf{B}_0+\mathbf{B}_1)-\nabla(p_0+p_1)$$

$$\rho_0\frac{\partial \mathbf{v}_1}{\partial t}=-\nabla p_1+\mathbf{j}_1\times\mathbf{B}_0+\mathbf{j}_0\times\mathbf{B}_1$$

Linearized ideal MHD equations

The same is done for all the MHD equations and with some easy algebra the linearized system is obtained:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1$$

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \Gamma p_0 \nabla \cdot \mathbf{v}_1$$

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1)$$

$$\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j}_1$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$$

$$\nabla \cdot \mathbf{B}_1 = 0$$

Linearized ideal MHD equations: introducing plasma displacement

These can be simplified by introducing the *Lagrangian displacement vector* $\boldsymbol{\xi}(\mathbf{r},t)$ of a plasma element from its equilibrium state.

The fluid velocity is then the Lagrangian time derivative of this new variable: the variation in time in a coordinate system co-moving with the fluid.

$$\mathbf{v} = \frac{D\boldsymbol{\xi}}{Dt} \equiv \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{v}\cdot\nabla\boldsymbol{\xi}$$

NOTE: the displacement $\boldsymbol{\xi}$ is a first order quantity by definition!

For the first order linearized problem this becomes:

$$\mathbf{v} \approx \mathbf{v}_1 = \frac{\partial\boldsymbol{\xi}}{\partial t}$$

Using $\boldsymbol{\xi}$ in the linearized equations leads to the **force operator**

The definition of plasma displacement helps with time integration:

$$\begin{cases} \frac{\partial p_1}{\partial t} = -\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla p_0 - \Gamma p_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ \frac{\partial \rho_1}{\partial t} = -\nabla \cdot \left(\rho_0 \frac{\partial \boldsymbol{\xi}}{\partial t} \right) \\ \frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \right) \end{cases} \Longrightarrow \begin{cases} p_1 = -\boldsymbol{\xi} \cdot \nabla p_0 + \Gamma p_0 \nabla \cdot \boldsymbol{\xi} \\ \rho_1 = -\nabla \cdot (\rho_0 \boldsymbol{\xi}) \\ \mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \\ \Rightarrow \mathbf{j}_1 = \frac{1}{\mu_0} \nabla \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)) \end{cases}$$

This leads to a very useful expression of the momentum equation, which now depends solely on $\boldsymbol{\xi}$!

$$\begin{aligned} \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\nabla \left(-\boldsymbol{\xi} \nabla p_0 - \Gamma p_0 \nabla \boldsymbol{\xi} \right) + \mathbf{j}_0 \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)] + \frac{1}{\mu_0} \nabla \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)] \times \mathbf{B}_0 \\ &= \mathbf{F}(\boldsymbol{\xi}) \end{aligned}$$

The right hand side of this expression is called force operator

Using $\boldsymbol{\xi}$ in the linearized equations leads to the **force operator**

The definition of plasma displacement helps with time integration:

$$\begin{cases} \frac{\partial p_1}{\partial t} = -\frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\nabla p_0 - \Gamma p_0 \nabla\cdot\frac{\partial \boldsymbol{\xi}}{\partial t} \\ \frac{\partial \rho_1}{\partial t} = -\nabla\cdot\left(\rho_0 \frac{\partial \boldsymbol{\xi}}{\partial t}\right) \\ \frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\times\mathbf{B}_0\right) \end{cases} \Longrightarrow \begin{cases} p_1 = -\boldsymbol{\xi}\cdot\nabla p_0 + \Gamma p_0 \nabla\cdot\boldsymbol{\xi} \\ \rho_1 = -\nabla\cdot(\rho_0\boldsymbol{\xi}) \\ \mathbf{B}_1 = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) \\ \Rightarrow \mathbf{j}_1 = \frac{1}{\mu_0}\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)) \end{cases}$$

This leads to a very useful expression of the momentum equation, which now depends solely on $\boldsymbol{\xi}$!

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla\left(-\boldsymbol{\xi}\nabla p_0 - \Gamma p_0 \nabla\boldsymbol{\xi}\right) + \mathbf{j}_0 \times \underbrace{\left[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\right]}_{\text{perturbed field } \equiv \mathbf{Q}} + \frac{1}{\mu_0}\nabla\times\underbrace{\left[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\right]}_{\text{perturbed field } \equiv \mathbf{Q}}\times\mathbf{B}_0$$
$$= \mathbf{F}(\boldsymbol{\xi})$$

The right hand side of this expression is called force operator

Using ξ in the linearized equations leads to the **force operator**

We can rearrange the terms to highlight the physical meaning:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}) = \underbrace{\nabla\left(\Gamma p_0 \nabla\cdot\boldsymbol{\xi}\right)}_{\text{Isotropic force due to plasma compressibility}} - \underbrace{\frac{1}{\mu_0}\mathbf{B}_0 \times \left(\nabla \times \mathbf{Q}\right)}_{\perp\,\mathbf{B}\text{ force due to field line bending}} + \underbrace{\nabla\left(\boldsymbol{\xi}\cdot\nabla p_0\right)}_{\text{force due to equilibrium pressure gradient}} + \underbrace{\mathbf{j}_0 \times \mathbf{Q}}_{\text{force due to equilibrium current}}$$

Isotropic force due to plasma ***compressibility***

$\perp$ **B** force due to ***field line bending***

force due to equilibrium ***pressure gradient***

force due to equilibrium ***current***

Present even in homogeneous plasma, responsible for stable perturbations such as Alfvén waves

For inhomogeneous plasmas (such as fusion relevant ones!) pressure gradients and currents can lead to instability

Normal modes

Equilibrium quantities are time independent, we can separate variables and write solutions in the form of **normal modes**:

$$\boldsymbol{\xi}(\mathbf{r},t) = \hat{\boldsymbol{\xi}}(\mathbf{r})e^{-i\omega t}$$

We can then write the momentum equation as an **eigenvalue problem**!

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho_0 \omega^2 \hat{\boldsymbol{\xi}}(\mathbf{r}) = \mathbf{F}(\hat{\boldsymbol{\xi}})$$

Where $\mathbf{F}$ is again the linear force operator and ω^2 its eigenvalues

-> full picture of system stability!

Aside few analytical solutions, a numerical approach is usually needed.

These eigenvalues have interesting properties

1. The eigenvalues ω^2 are **purely real**

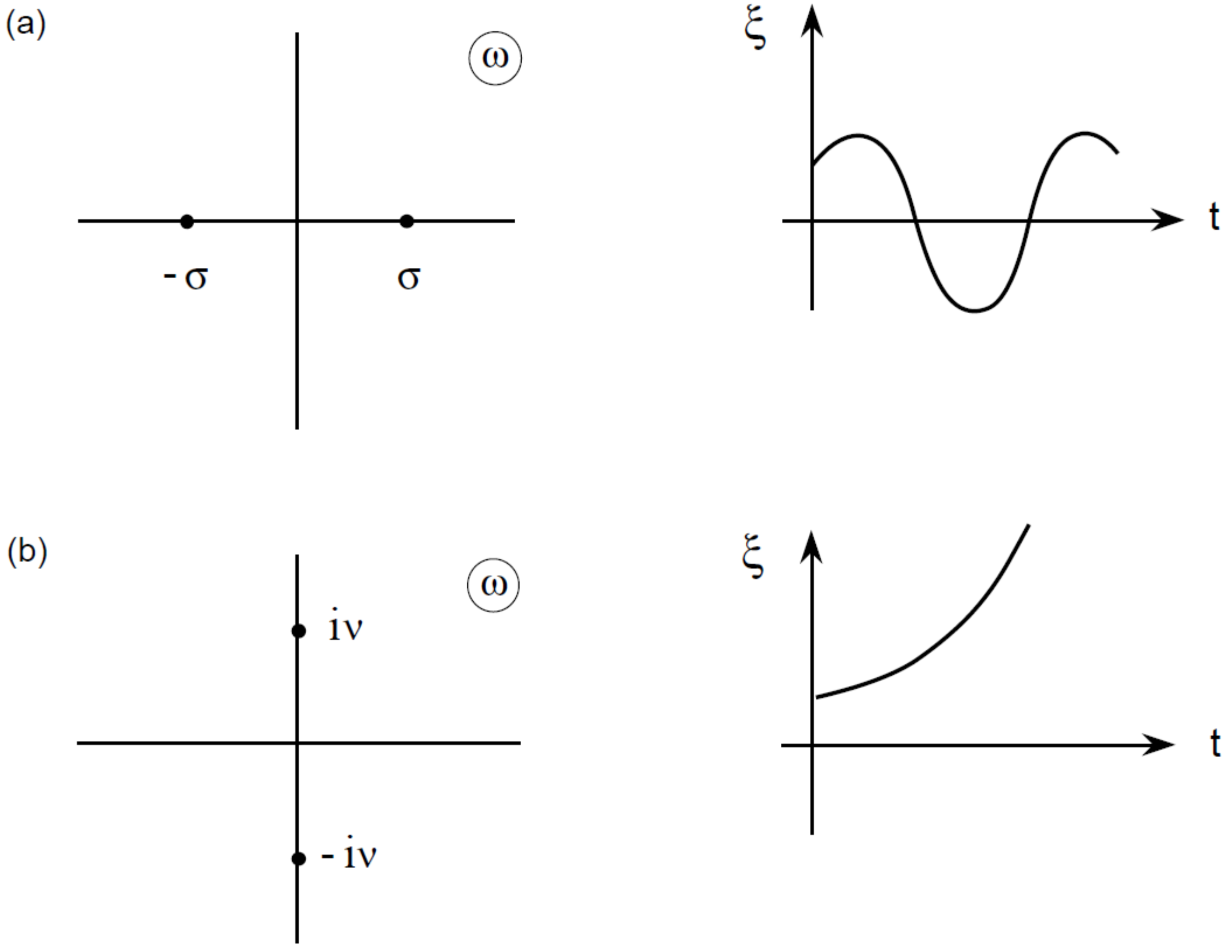
   → Consequence of the operator **F** being **self-adjoint** (for ideal MHD) in the Hilbert space of displacement vectors (quite lengthy calculation for proof):
   $\int \boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi})\mathrm{dV} = \int \boldsymbol{\xi} \cdot \mathrm{F}(\boldsymbol{\eta}^*)\mathrm{dV}$

2. Given the form of perturbations $\boldsymbol{\xi}(\mathbf{r}, \mathrm{t}) = \hat{\boldsymbol{\xi}}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}$, we can have:

   1. $\omega^2 > 0$ then ω is real and the perturbations $\boldsymbol{\xi}$ are stable oscillations
   2. $\omega^2 < 0$ then ω is imaginary and $\boldsymbol{\xi} \sim \mathrm{e}^{+\omega\mathrm{t}}$ is exponentially unstable!

In **ideal MHD** we only have **stable waves** or **exponential instabilities**

$\omega = \sigma + i\nu$



$\omega^2 > 0$

$\omega^2 < 0$

[Goedbloed, H. *Lectures*]

The meaning of $\mathrm{Re}(\omega)$

Assuming an **m=1 external kink** in a "**straight tokamak**" ($\xi_z = 0$)

➢ constant eigenfunction $\xi_r = \xi_a = const.$

Assuming **incompressibility**:

➢ $\nabla \cdot \xi = 0 \rightarrow \xi_\theta = \frac{i}{m}\frac{d(r\xi_r)}{dr} = i\xi_a$

The complete eigenfunction in time is described by:

$$\xi_r(r,\theta,\phi,t) = \mathrm{Re}\{\xi_a e^{i\theta - in\phi - i\omega t}\} = \xi_a \cdot \cos(\theta - n\phi - \omega t)$$

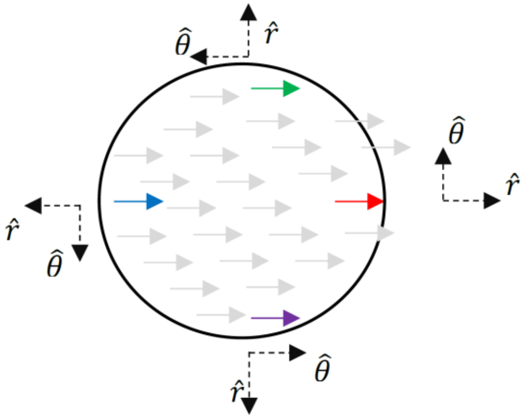
$$\xi_\theta(r,\theta,\phi,t) = \mathrm{Re}\{i\xi_a e^{i\theta - in\phi - i\omega t}\} = -\xi_a \cdot \sin(\theta - n\phi - \omega t)$$

The components evolve coherently with a **rotation** on the poloidal plane

$\phi = 0,\ t = 0$

$\theta = 0: \begin{cases} \xi_r = \xi_a \\ \xi_\theta = 0 \end{cases}$ $\theta = \frac{\pi}{2}: \begin{cases} \xi_r = 0 \\ \xi_\theta = -\xi_a \end{cases}$

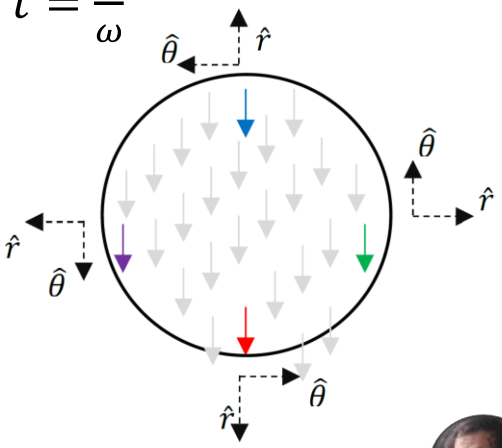
$\theta = \pi: \begin{cases} \xi_r = -\xi_a \\ \xi_\theta = 0 \end{cases}$ $\theta = \frac{3}{2}\pi: \begin{cases} \xi_r = 0 \\ \xi_\theta = \xi_a \end{cases}$



$\phi = 0,\ t = \frac{\pi}{\omega}$

$\theta = 0: \begin{cases} \xi_r = 0 \\ \xi_\theta = -\xi_a \end{cases}$ $\theta = \frac{\pi}{2}: \begin{cases} \xi_r = -\xi_a \\ \xi_\theta = 0 \end{cases}$

$\theta = \pi: \begin{cases} \xi_r = 0 \\ \xi_\theta = \xi_a \end{cases}$ $\theta = \frac{3}{2}\pi: \begin{cases} \xi_r = \xi_a \\ \xi_\theta = 0 \end{cases}$



Courtesy of L. Spinicci

Energy principle: insight on stability without solving the eigenvalue problem

Task: write an expression for **plasma potential energy** and **minimize** it using plasma displacements $\boldsymbol{\xi}$ as test functions.

A step back to plasma displacement: it can be used to define a linear expression for plasma **kinetic energy**!

$$K \equiv \frac{1}{2}\int \rho \mathbf{v}^2 dV \approx \frac{1}{2}\int \rho_0 \left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)^2 dV = \frac{1}{2}\int \rho_0 \dot{\boldsymbol{\xi}}^2 dV$$

The variation in time of the total energy $\mathbf{H}$, with $\mathbf{W}$ the potential energy, is zero for energy conservation $\mathbf{H} = \mathbf{K} + \mathbf{W} = \mathbf{const}$

$$\frac{d}{dt}(K + W) = 0$$

Energy principle: insight on stability without solving the eigenvalue problem

Using the expression for K and the equation of motion:

$$\frac{dK}{dt} = \frac{d}{dt}\left[\frac{1}{2}\int \rho_0 \dot{\boldsymbol{\xi}}^2 dV\right] = \int \rho_0 \dot{\boldsymbol{\xi}}^\star \cdot \ddot{\boldsymbol{\xi}} dV = \int \dot{\boldsymbol{\xi}}^\star \cdot \mathbf{F}(\boldsymbol{\xi}) dV$$

Using self-adjointness and energy conservation:

$$\begin{aligned}\frac{dW}{dt} = -\frac{dK}{dt} &= -\int \dot{\boldsymbol{\xi}}^\star \cdot \mathbf{F}(\boldsymbol{\xi}) dV \\ &= -\frac{1}{2}\left[\int \dot{\boldsymbol{\xi}}^\star \cdot \mathbf{F}(\boldsymbol{\xi}) dV + \int \boldsymbol{\xi}^\star \cdot \mathbf{F}(\dot{\boldsymbol{\xi}}) dV\right] \\ &= \frac{d}{dt}\left[-\frac{1}{2}\int \boldsymbol{\xi}^\star \cdot \mathbf{F}(\boldsymbol{\xi}) dV\right]\end{aligned}$$

$$\mathrm{W} = -\frac{1}{2}\int \boldsymbol{\xi}^\star \cdot \mathbf{F}(\boldsymbol{\xi}) \mathrm{dV}$$

Expression for **linearized potential energy**

* Indicates the complex conjugate and integrals are carried out over the plasma volume

Energy principle: insight on stability without solving the eigenvalue problem

Finally using energy conservation we find:

$$\frac{d}{dt}\left[K \underbrace{- \frac{1}{2}\int \boldsymbol{\xi}^{\star} \cdot \boldsymbol{F}(\boldsymbol{\xi})\, dV}_{= W}\right] = \frac{d}{dt}(K + \delta W) = 0$$

If we only have discrete eigenvalues $\{\omega_n\}$ with eigenfunctions $\{\boldsymbol{\xi}_n\}$:

$$-\rho_0 \omega_n^2 \boldsymbol{\xi}_n = F(\boldsymbol{\xi}_n) \xrightarrow{\text{yields}} \delta W = -\frac{1}{2}\int \boldsymbol{\xi}_n \cdot \boldsymbol{F}(\boldsymbol{\xi}_n)\, dV = \frac{1}{2}\omega_n^2 \int \rho_0 \xi_n^2 dV$$

$$\boxed{\omega_n^2 = \frac{\delta W}{\frac{1}{2}\int \rho_0 \xi_n^2 dV}}$$

The sign if δW defines stability !!

The meaning of W

The perturbed potential energy can be put into the so-called *intuitive form* to highlight the underlying physics

$$W = \frac{1}{2\mu_0}\int_F dV \, |\mathbf{Q}_\perp|^2$$

$$+ B_0^2 \left|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2$$

$$+ \Gamma\mu_0 p_0 \left|\nabla\cdot\boldsymbol{\xi}\right|^2$$

Positive terms: stabilizing

Associated with stable waves (Alfvèn waves, sound waves)

$$- 2\mu_0 \left(\boldsymbol{\xi}_\perp\cdot\nabla p_0\right)\left(\boldsymbol{\xi}_\perp^\star\cdot\boldsymbol{\kappa}\right)$$

$$- \mu_0 \frac{j_{0\parallel}}{B_0}\left(\boldsymbol{\xi}_\perp^\star \times \mathbf{B}_0\right)\cdot\mathbf{Q}_\perp$$

Can be either positive or negative! These terms describe the main sources of instability

The meaning of W: pressure driven term

$$(\boldsymbol{\xi}_\perp \cdot \nabla p_0)\,(\boldsymbol{\xi}_\perp^\star \cdot \boldsymbol{\kappa})$$

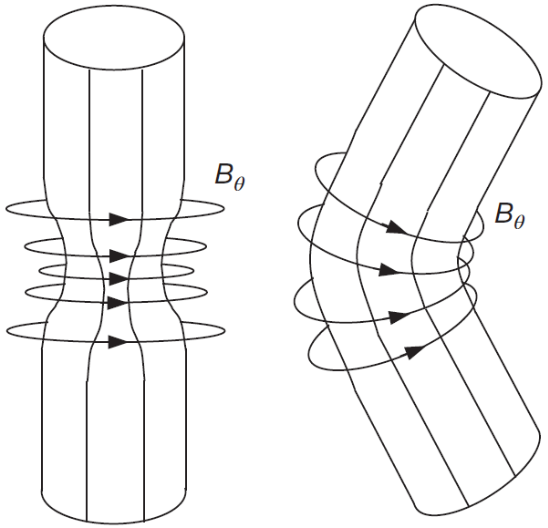
- Depends on pressure gradient ($\nabla \mathbf{p}_0$) and magnetic field curvature ($\boldsymbol{\kappa}$)
- In a generic configuration the confining magnetic field will have regions with *good* and *bad* curvature
- Overall stability depends on both contributions
- In a toroidal system the inner part has good curvature, the outer part bad: pressure driven instabilities will tend to go that way!

The meaning of W: current driven term

$$\frac{j_{0\|}}{B_0}\left(\boldsymbol{\xi}_\perp^\star \times \mathbf{B}_0\right) \cdot \mathbf{Q}_\perp$$

- Energy source is current density parallel to the equilibrium field
- Examples can be the sausage instability and the current driven kink

e.g. z-pinch with no axial field



[Zohm, H. *Magnetohydrodynamic stability of tokamaks*]

In general a real-life instability will be driven by a mix of both pressure and current

Current driven kink for large aspect ratio with low β

! Use perturbations of the form

$$\xi \sim e^{i(m\theta - n\phi)}$$

m = poloidal mode number
n = toroidal mode number

Consider the ordering

$$W \sim \varepsilon^2$$
$$\beta \sim \varepsilon^2$$

We can describe one of the strongest ideal MHD instabilities. The potential energy can be written as:

$$W = \frac{2\pi^2 B_z^2}{\mu_0 R_0} \int_0^a \left[\left(r\frac{d\xi}{dr} \right)^2 + (m^2-1)\xi^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2 r dr$$
$$+ \frac{2\pi^2 B_z^2}{\mu_0 R_0} \xi_a^2 a^2 \left(\frac{n^2}{m^2} - \frac{1}{q_a^2} + \Lambda m \left(\frac{1}{q_a} - \frac{n}{m} \right)^2 \right)$$

$W = W_P + W_B$ = plasma + boundary (with vacuum)

Current driven kink for large aspect ratio with low β

! Use perturbations of the form

$\xi \sim e^{i(m\theta - n\phi)}$

m = poloidal mode number

n = toroidal mode number

Consider the ordering

$$W \sim \varepsilon^2$$
$$\beta \sim \varepsilon^2$$

We can describe one of the strongest ideal MHD instabilities. The potential energy can be written as:

$$W = \frac{2\pi^2 B_z^2}{\mu_0 R_0} \int_0^a \left[\left(r\frac{d\xi}{dr} \right)^2 + (m^2 - 1)\xi^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2 r dr$$
$$+ \frac{2\pi^2 B_z^2}{\mu_0 R_0} \xi_a^2 a^2 \left(\frac{n^2}{m^2} - \frac{1}{q_a^2} + \Lambda m \left(\frac{1}{q_a} - \frac{n}{m} \right)^2 \right)$$

$\xi_a = \xi(a)$ is the plasma displacement at $r = a$, i.e. at the plasma boundary!

$W = W_P + W_B$ = plasma + boundary (with vacuum)

Current driven kink for large aspect ratio with low β

Consider the ordering

$$W \sim \varepsilon^2$$
$$\beta \sim \varepsilon^2$$

! Use perturbations of the form

$$\xi \sim e^{i(m\theta - n\phi)}$$

m = poloidal mode number
n = toroidal mode number

We can describe one of the strongest ideal MHD instabilities. The potential energy can be written as:

$$W = \frac{2\pi^2 B_z^2}{\mu_0 R_0} \int_0^a \left[\left(r \frac{d\xi}{dr} \right)^2 + (m^2 - 1)\xi^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2 r dr$$
$$+ \frac{2\pi^2 B_z^2}{\mu_0 R_0} \xi_a^2 a^2 \left(\frac{n^2}{m^2} - \frac{1}{q_a^2} + \Lambda m \left(\frac{1}{q_a} - \frac{n}{m} \right)^2 \right)$$

$\xi_a = \xi(a)$ is the plasma displacement at $r = a$, i.e. at the plasma boundary!

Effect of a perfectly conducting (ideal) wall is stabilizing and represented by Λ

$W = W_P + W_B$ = plasma + boundary (with vacuum)

Current driven kink for large aspect ratio with low β

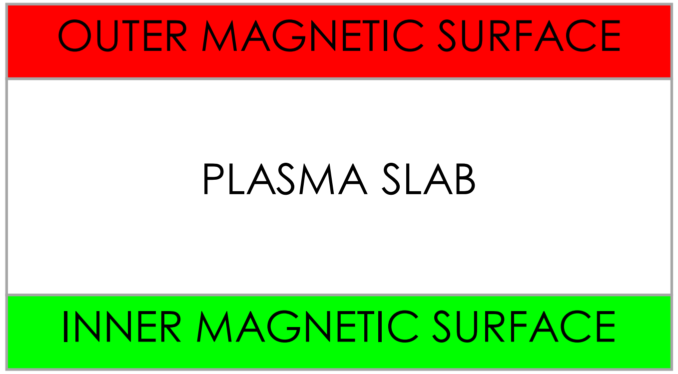
- For $\xi_a = 0$ the boundary term is zero and the integral is positive: called **internal kink** modes, stable in $\sim \varepsilon^2$! Need expansion to $\sim \varepsilon^4$

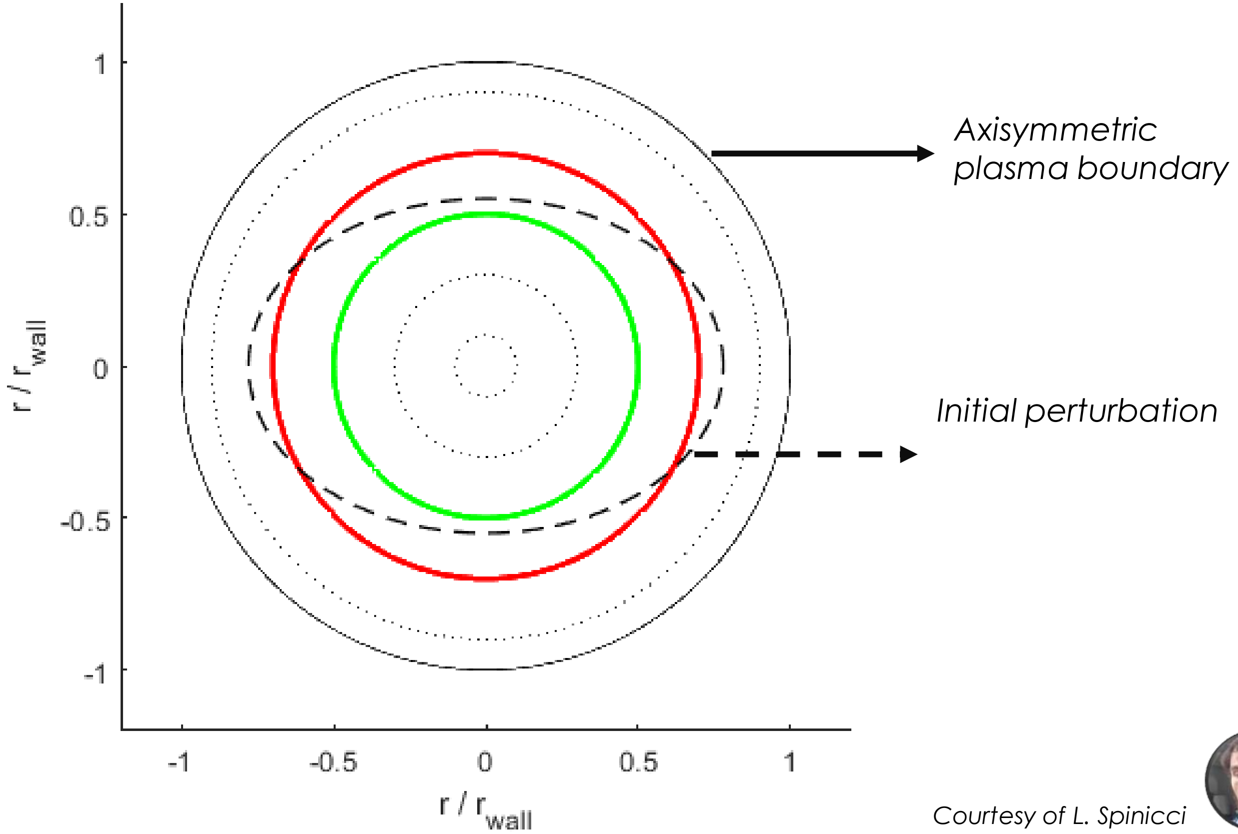
- When $\xi_a \neq 0$ the perturbation affects the plasma boundary: called **external kink** mode

Let's consider the case $\xi_a > 0$ and $\boldsymbol{m} = \mathbf{1}$, with wall at infinity $(\Lambda = 1)$ -> the integral part of W is positive, we only need to discuss the boundary term (W_B)!

Toy model for **current driven kink**

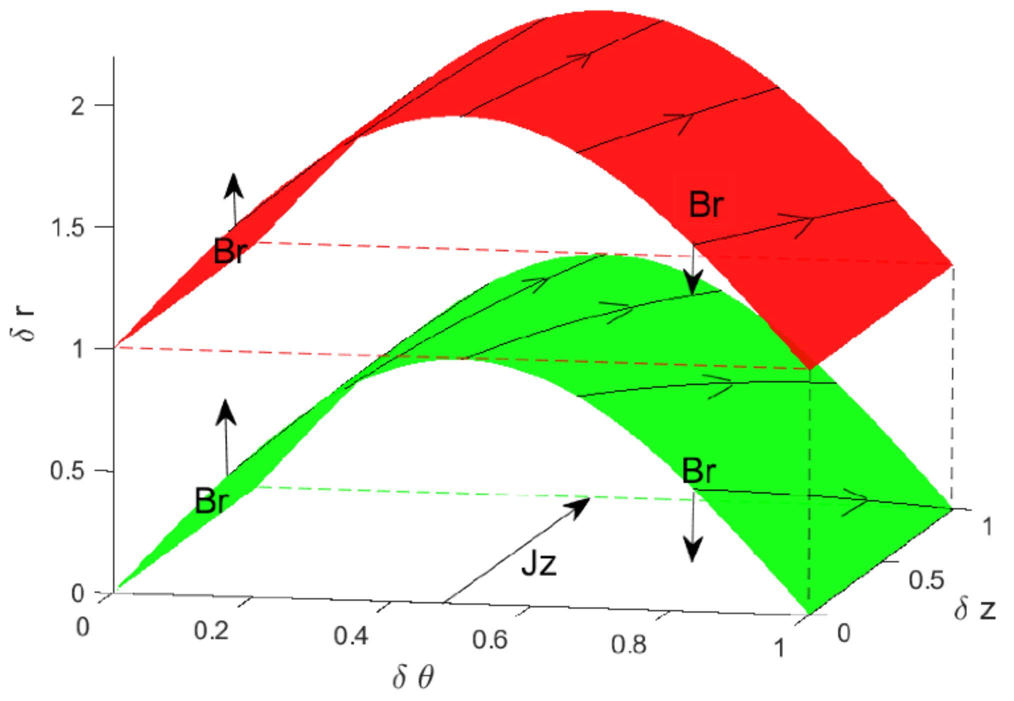
- Large aspect ratio $R/a \gg 1$
- Monotonically decreasing current density towards edge (e.g. Wesson's parametrization)

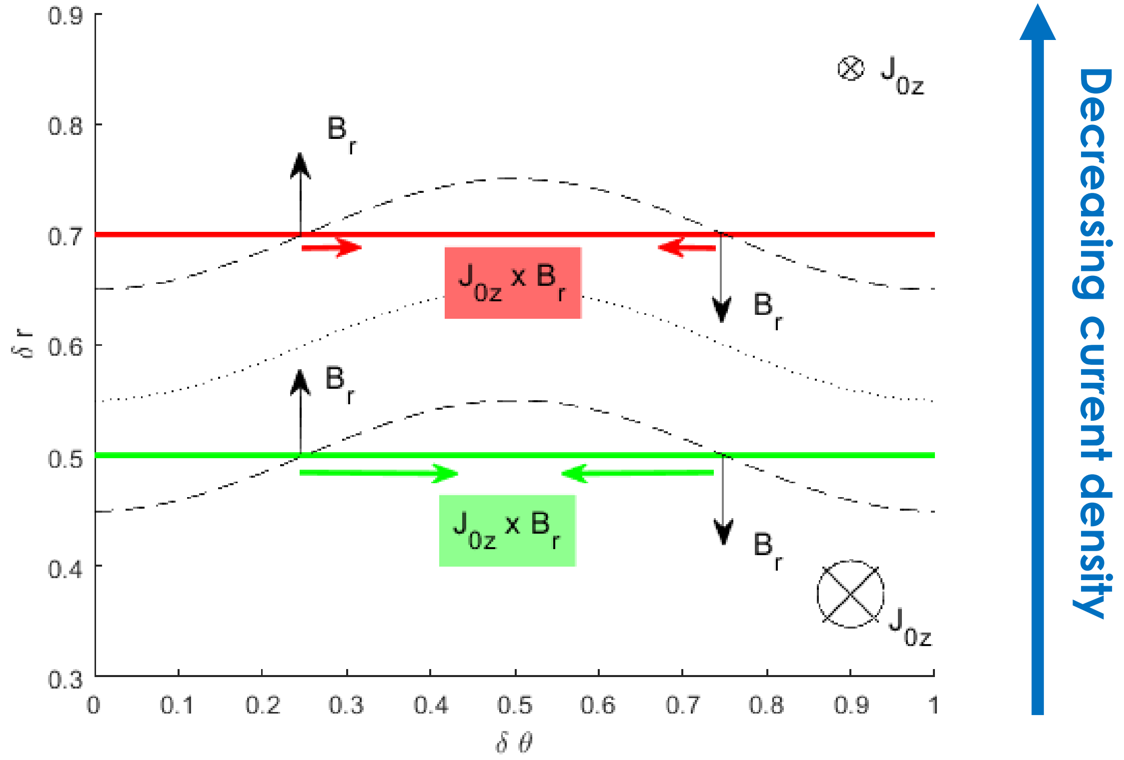




Courtesy of L. Spinicci

Toy model for **current driven kink**





On bent surfaces the magnetic field gets a radial dependence: B_r component

B_r interacts with toroidal current into $J \times B$ force-couples, **stronger where J_z is larger**

These forces enhance the initial perturbation → instability

Courtesy of L. Spinicci

Current driven **external kink**

$$W_B = \frac{4\pi^2 B_z^2}{\mu_0 R_0} n \left(n - \frac{1}{q_a} \right) a^2 \xi_a^2$$

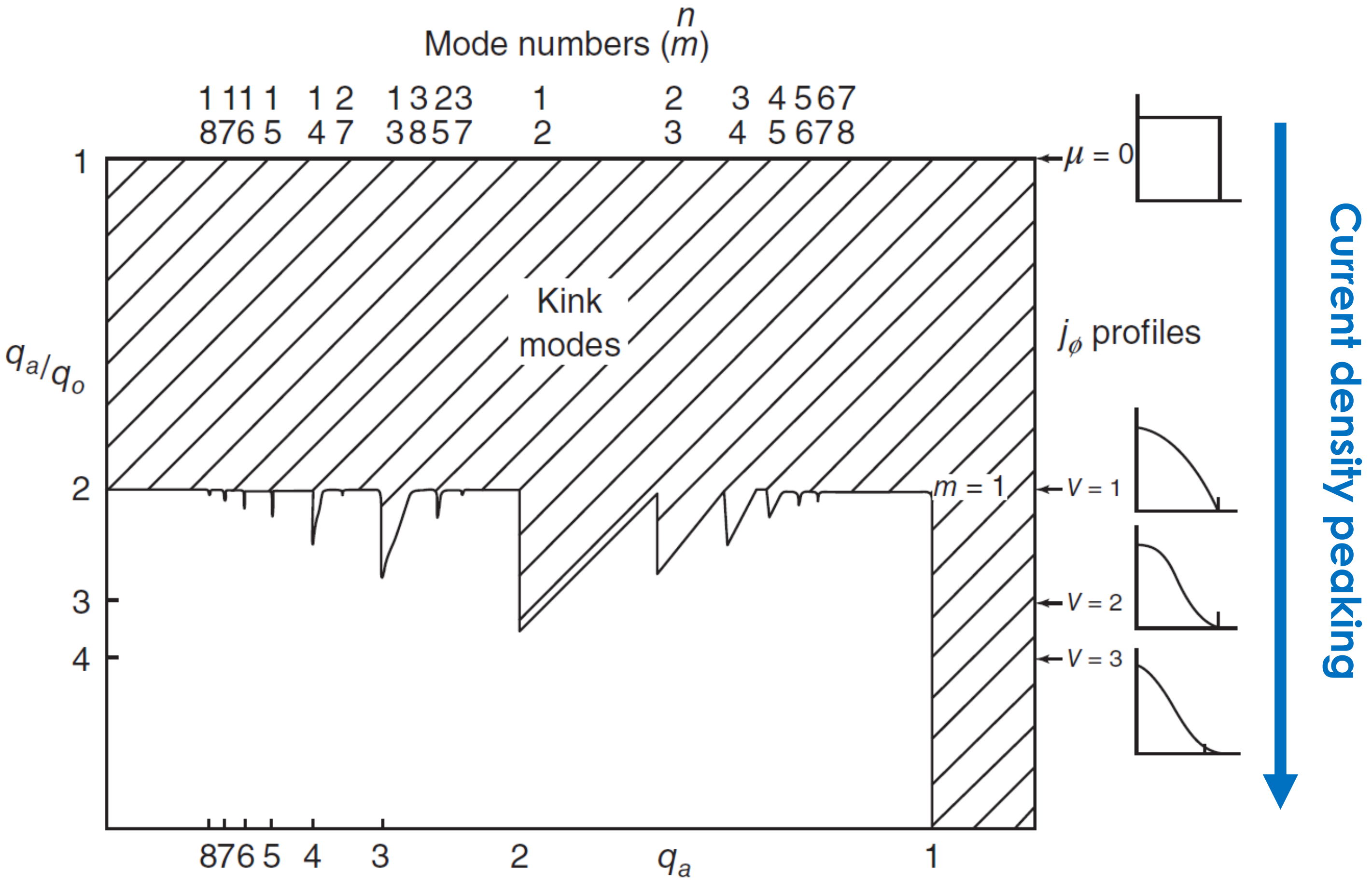
This implies stability if W_B:

$$q_a > \frac{1}{n} \xrightarrow{\text{restrictive}} q_a > 1$$

This translates into a limit on the plasma current

$$q_a = \frac{2\pi a^2 B_\phi}{\mu_0 I_p R_0} \xrightarrow{\text{yields}} \mathbf{I_p < \frac{2\pi a^2}{\mu_0 R_0} B_\phi}$$

Kruskal-Shafranov limit



Stability plot for equilibria with $j = j_0 \left(1 - \frac{r^2}{a^2}\right)^\nu$

Remember that $\frac{q_a}{q_0} = \nu + 1$!!

Comments on the Kruskal-Shafranov limit

This operational boundary is actually not so limiting in present day machines, other instabilities are triggered first!

The (**m=2,n=1**) external kink (current driven) kicks in when $q_a \sim 2$

The presence of a **perfectly conducting wall** also stabilizes the external kink

- Real world shells are not ideal though -> partial stabilization leads the **Resistive Wall Mode**

For $\mathbf{m \geq 2}$ stability depends on balance of terms in W_B. For given (m,n) the mode can be unstable if $\mathbf{q_a} < {}^{\mathbf{m}}/_{\mathbf{n}}$ i.e. if the mode resonant surface lies outside the plasma

Stability diagram of the current driven kink

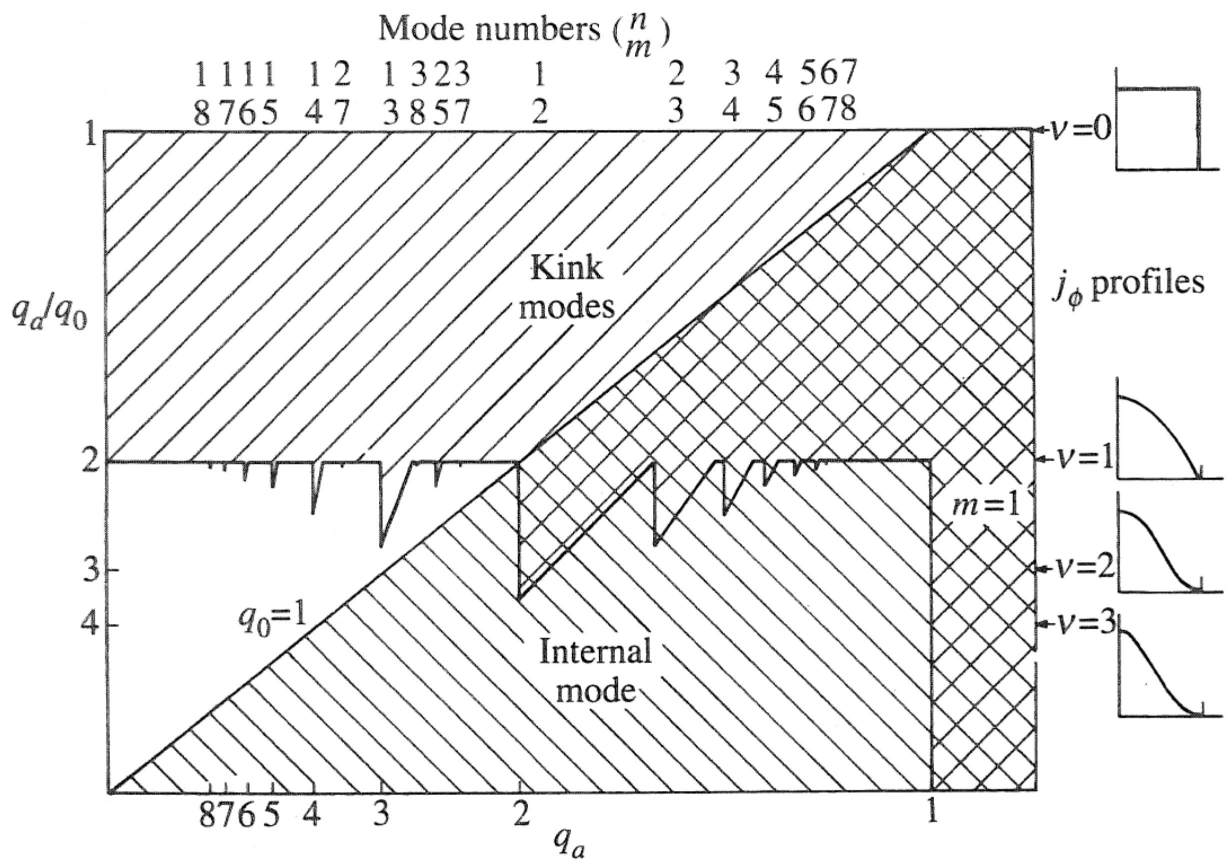
Internal kink stability (for m=1) leads to a further requirement*

$$\mathbf{q_0 > 1}$$

An effect of pressure is also possible here (finite β effect): usually destabilizing

Ideal kink modes limit **achievable plasma current**

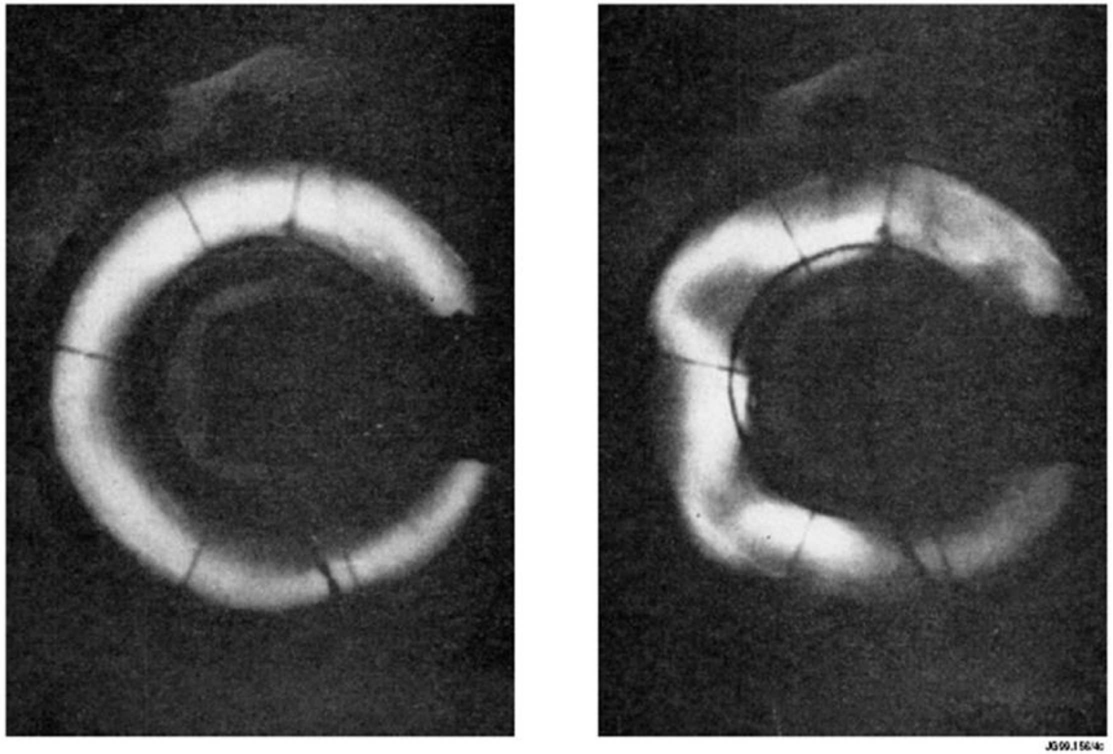
Current density peaking important for kink stability (and for other modes as well!)



Current density peaking

*see references: Zohm, Wesson

Early kinks



Kink instability in an early pinch experiment

From: Carruthers, PhysSoc. London (1957)

Global MHD instabilities

Instabilities with a global structure causing non-negligible displacement at the plasma surface.

These instabilities are appropriately known as e**X**ternal **K**ink modes (**XK**).

- Predicted for long time *[D. Pfirsh and H. Tasso, Nucl. Fusion **11**, 259 1971]*

In **tokamaks** this led to the prediction of the Troyon β limit (i.e. maximum achievable pressure with no conducting wall)

► Instability linked to **pressure**

► Strongly depends on shape of plasma radial profiles!

Intrinsic in **Reversed Field Pinch** even at $\beta \rightarrow 0$

► Instability linked to **current**

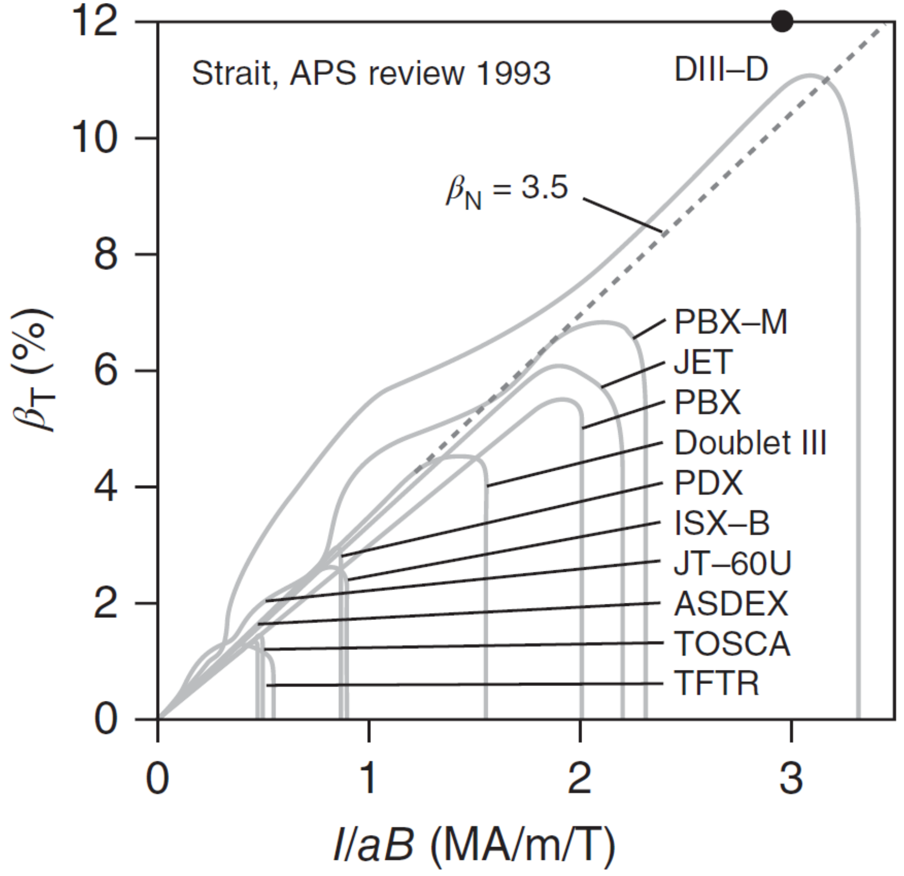
The **Troyon limit**

Obtained **numerically** with a combination of all ideal MHD limits, gives an estimate of the **maximum achievable β** for given machine parameters (with no stabilizing wall)

$$\beta_{t,max}[\%] = [2.8 \div 5.6] \frac{I_p\ [MA]}{a\ [m]\ B_t\ [T]}$$

Leads to definition of **normalized β**

$$\beta_N = \beta_{t,max}[\%] \frac{a\ [m]\ B_t\ [T]}{I_p\ [MA]}$$



The **A**dvanced **T**okamak (**AT**) scenario for steady state operation

Steady state achieved through high bootstrap current

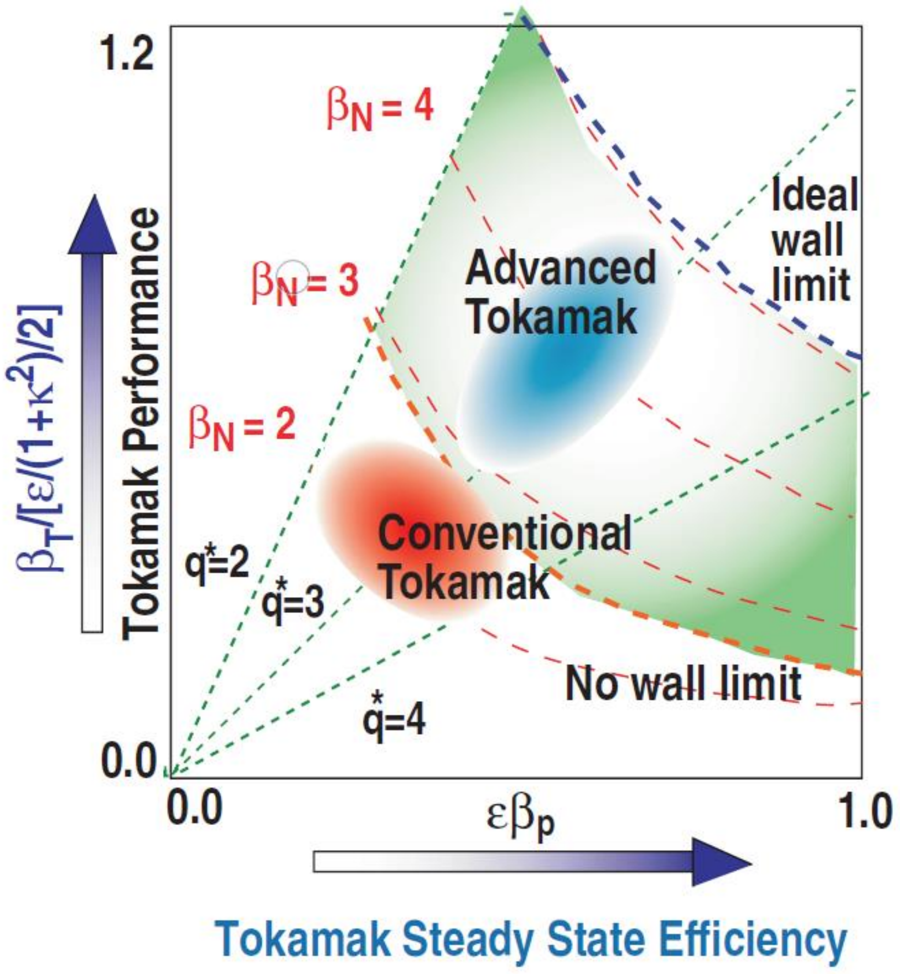
-> proportionality with $\beta_p = 2\mu_0\langle p\rangle/\langle B_p^2\rangle$

Sustained fusion reaction

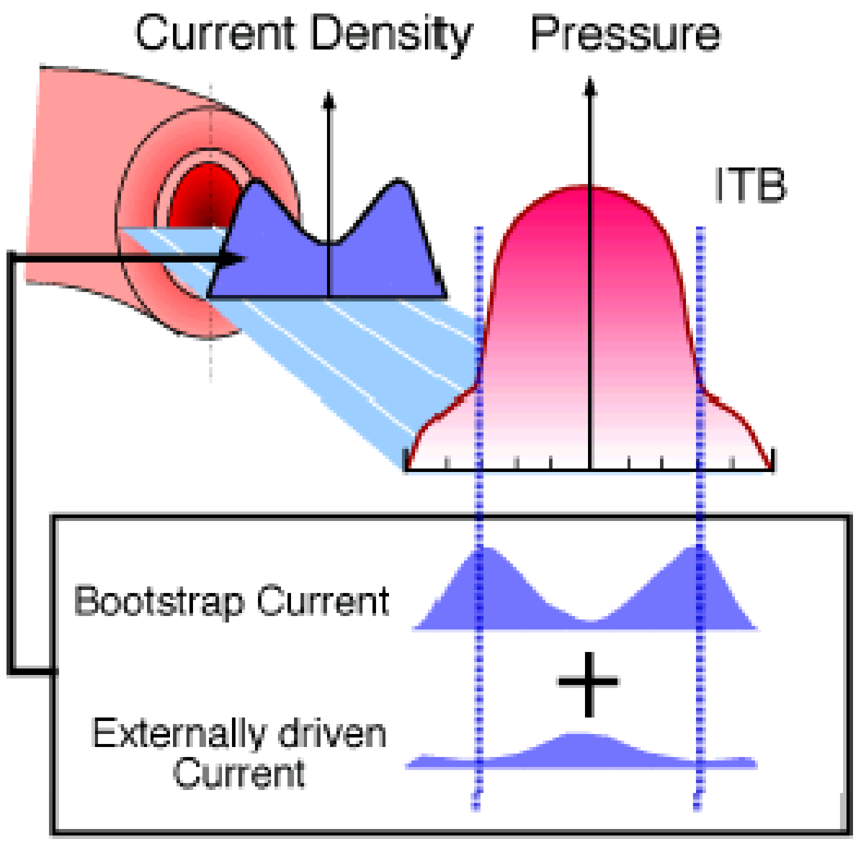
-> high enough $\beta_T = 2\mu_0\langle p\rangle/B_T^2$

$$\beta_p\beta_T \sim 25\left[\frac{1+E^2}{2}\right]\left(\frac{\beta_N}{100}\right)^2$$

Raising β_N will allow the simultaneous achievement of high β_p for high **bootstrap current fraction** and high β_T for **high fusion power** density.



The many faces of having high bootstrap fraction



High boostrap current fraction for steady state operation (75-90%)

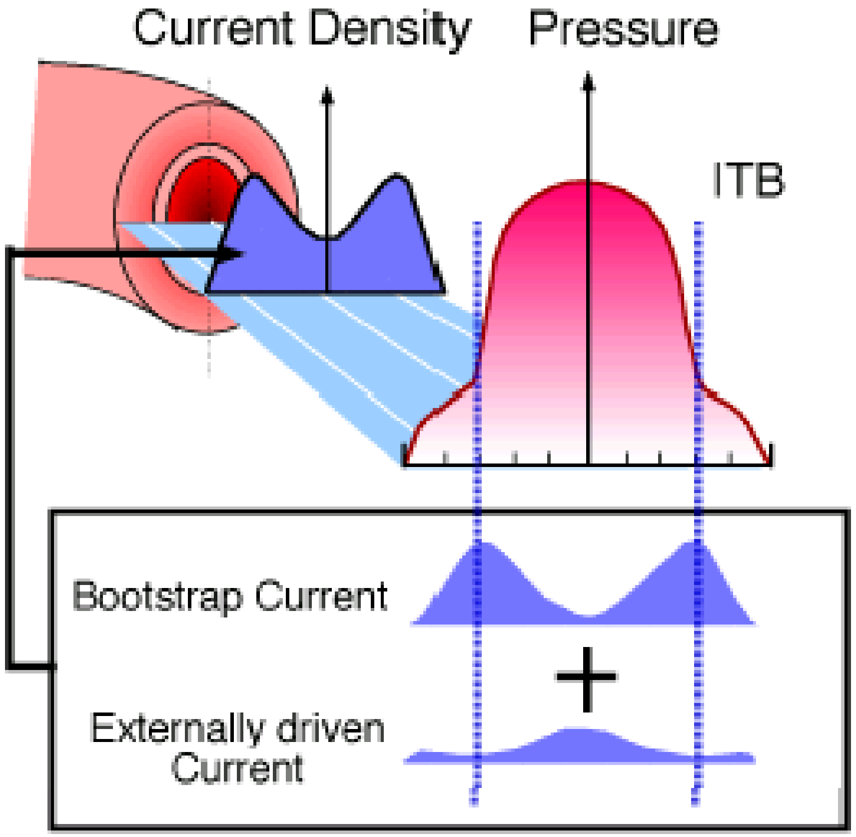
$$j_b \sim \nabla p$$

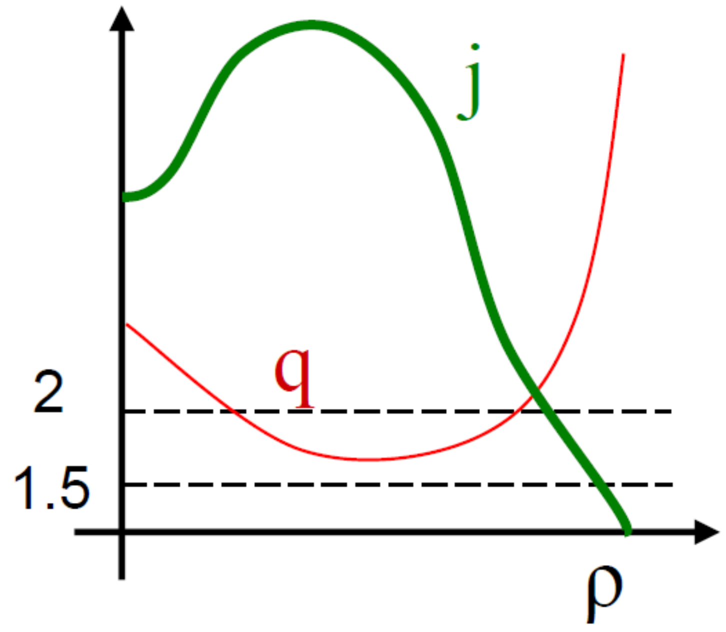
-> high β_N is required to achieve the target bootstrap current
-> usually exceeds Troyon pressure limit: **XK unstable**

Current profiles in AT are usually hollow:

-> Turbulence suppression & Internal Transport Barriers (**ITBs**)
-> Reversed q-profiles: low-shear combined with pressure gradient produces new MHD (Infernal modes)

The many faces of having high bootstrap fraction





A close fitting conducting wall is required to slow down the XK → Resistive Wall Mode (**RWM**)

RWM stabilization methods are needed for Advanced Tokamak operation

Current-driven kinks in Reversed Field Pinch devices

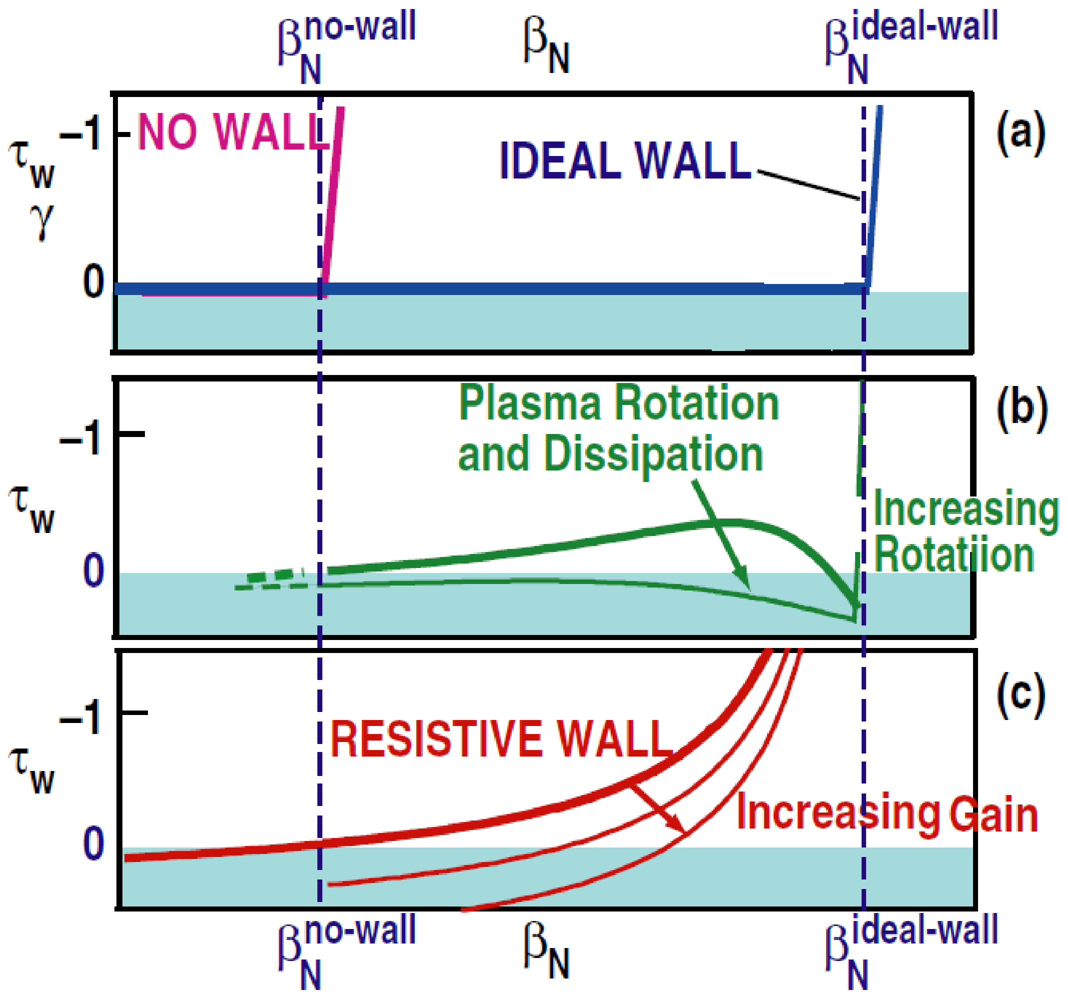
The RFP relies on an **ideally conducting wall** to maintain plasma stability with respect to the **current-driven** kink modes.

- A **resistive shell** can reduce the mode growth rates to values comparable to the vertical field penetration time of the wall
- Resistive Wall Modes predicted by theory and found by experiments. The main RWMs in the RFP are intrinsic, non-resonant, current-driven kink modes that are largely unaffected by sub Alfvénic plasma rotation

[C. G. Gimblett, Nucl. Fusion ***26****, 617 1986]*
[R. Paccagnella, Nucl. Fusion ***38****, 1067 1998]*
[S.C. Guo, J.P. Freidberg and R. Nachtrieb Phys. Plasmas ***6*** *3868 1999]*

- RFPs have different RWM **spectrum** w.r.t tokamaks: a range of modes are unstable. **Simultaneous** stabilization of multiple RWMs is required.

What is a Resistive Wall Mode?



Focusing on **pressure-driven** instabilities in tokamaks

The Resistive Wall Mode (**RWM**) is an external kink (**XK**) mode which interacts with external structures having finite conductivity

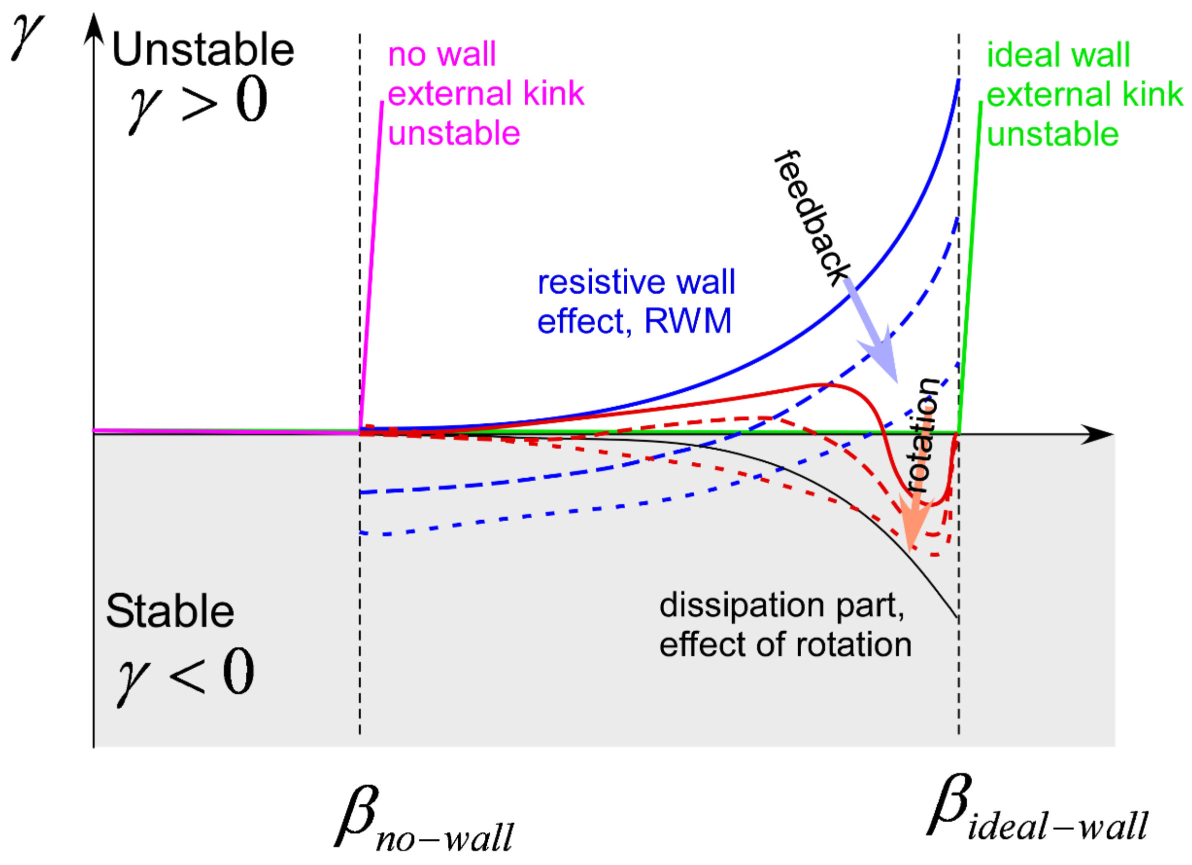
Schematic diagram of XK and RWM growth rates as a function of normalized β_N

(a) Stabilization of the XK mode by an ideal wall and of the RWM by a wall with finite conductivity.
(b) Stabilization of the RWM by increasing plasma rotation and dissipation.
(c) Stabilization of the RWM with magnetic feedback.

Effects of resistive wall & plasma rotation on the XK: the Chu et al. dispersion relation

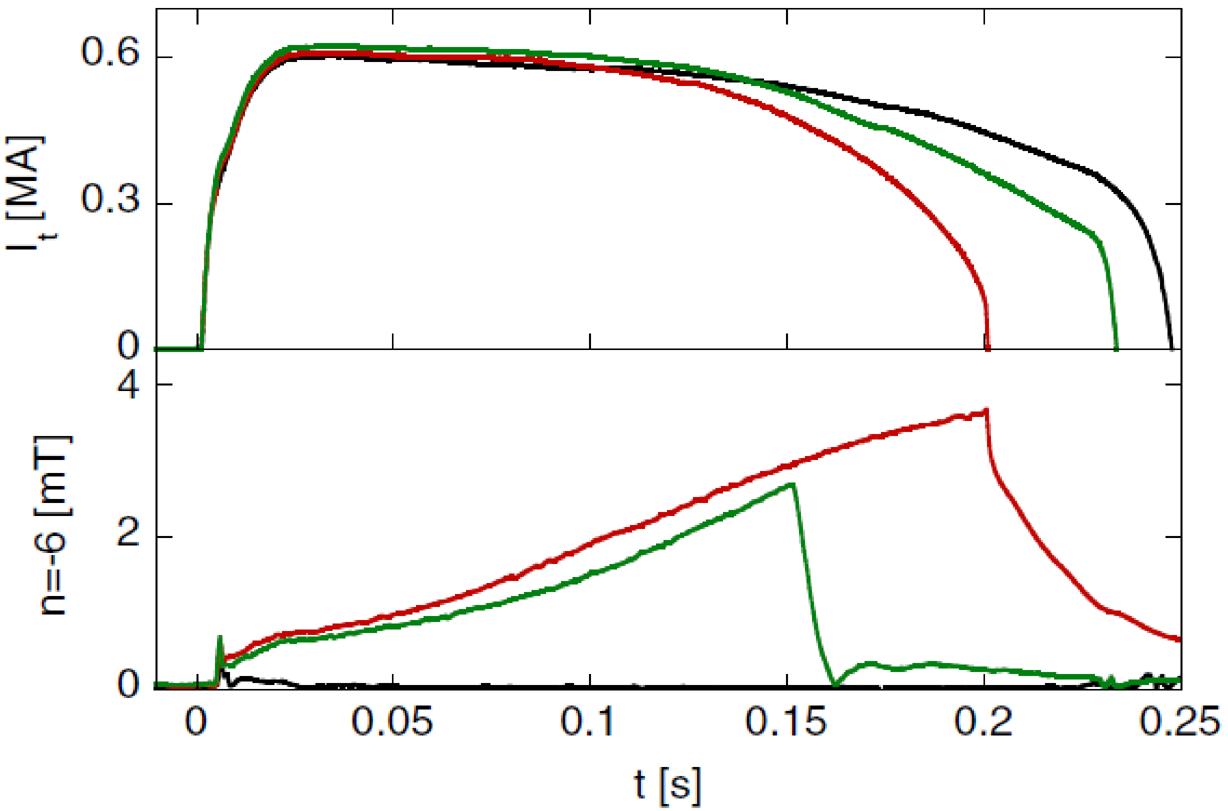
$$(\gamma + \imath n\Omega)D + \delta W_p + \frac{\delta W_{vac}^{b}\,\gamma\tau_w + \delta W_{vac}^{\infty}}{\gamma\tau_w + 1} = 0$$

- D : dissipation integral
- δW_p : Plasma potential energy integral
- δW_{vac}^{b} : vacuum energy with ideal wall at r=b
- δW_{vac}^{∞} : vacuum energy with ideal wall at r=∞

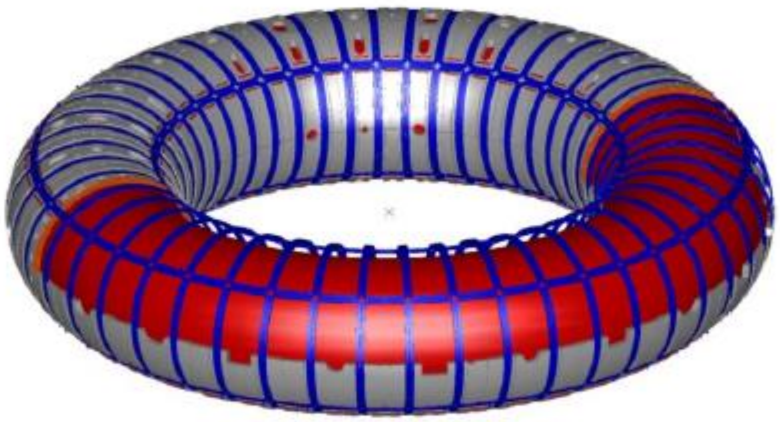


[Chu et al. Phys. Plasmas, 2, 2236 (1995)]

Active control of the Resistive Wall Mode: **RFX-mod**



192 independent active coils arranged in 4 toroidal arrays and 48 poloidal sections



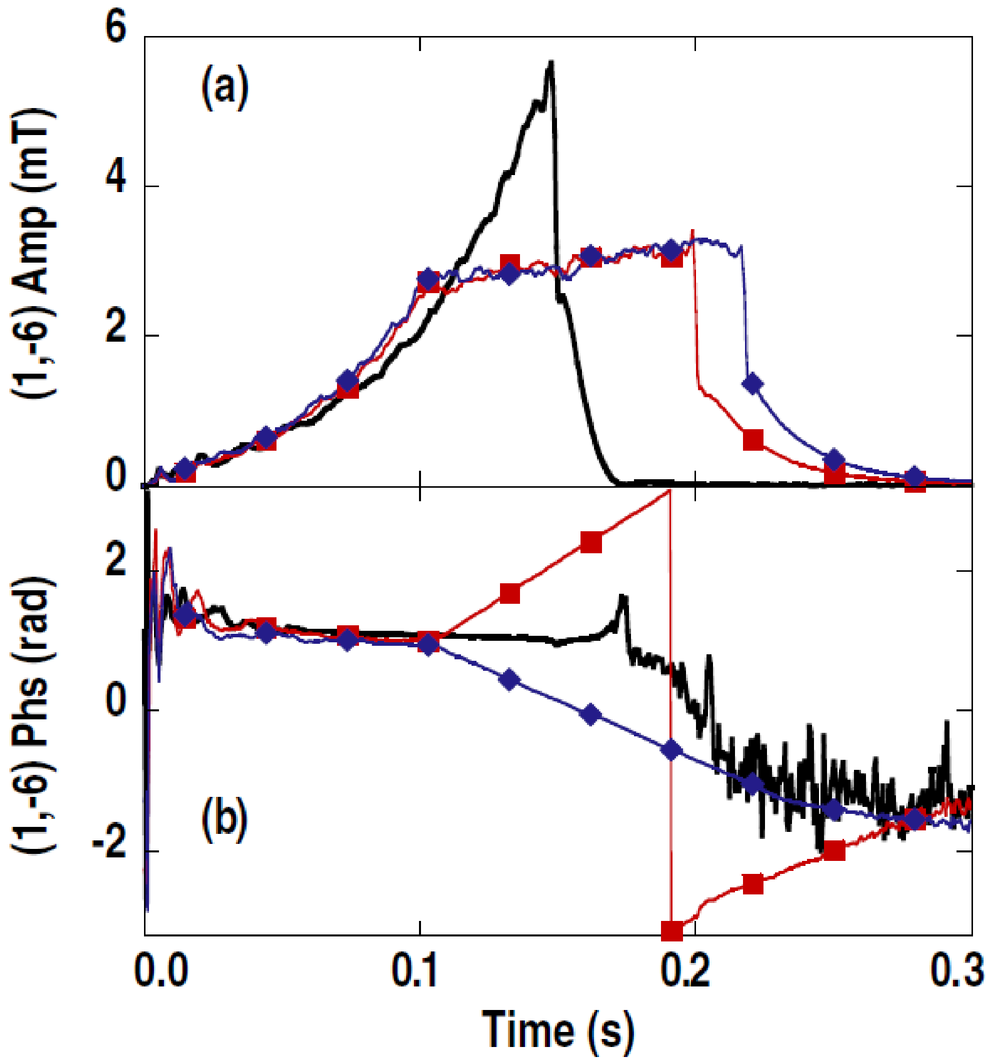
[S. Ortolani and the RFX team 2006 Plasma Phys. Control. Fusion ***48*** *B371]*

Multi-modal RWM feedback on RFX-mod. **Black** (#17287) full virtual shell. **Red** (#17301) $m = 1$, $n = -3$ to -6 are excluded from the control: early pulse termination. **Green** (#17304) the control on the $m = 1$, $n = -6$ mode is switched on and rapid stabilization is achieved.

Active control of the Resistive Wall Mode: **RFX-mod**

A complex proportional gain can be used to **rotate** a selected RWM in a given direction.

Reversing the phase of feedback rotates the RWM in the opposite direction.



[Bolzonella T., et al., 2008 Phys. Rev. Lett. **101** 165003]

Useful bibliography

This lecture is based on a number of books treating the subject of MHD at different levels. The following list is by no means intended to be complete, but rather provides a starting point for diving into the subject.

- Wesson, John, and David J. Campbell. *Tokamaks*. Vol. 149. Oxford university press, 2011.
- Zohm, Hartmut. *Magnetohydrodynamic stability of tokamaks*. John Wiley & Sons, 2015.
- Freidberg, Jeffrey P. *ideal MHD*. Cambridge University Press, 2014.
- Schnack, Dalton D. *Lectures in magnetohydrodynamics: with an appendix on extended MHD*. Vol. 780. Springer, 2009.
- Goedbloed, Hans, Rony Keppens, and Stefaan Poedts. *Magnetohydrodynamics: Of Laboratory and Astrophysical Plasmas*. Cambridge University Press, 2019.

Feedback is welcome !

leonardo.pigatto@igi.cnr.it

tommaso.Bolzonella@igi.cnr.it