

Superdiffusive central limit theorems for geodesic flows on nonpositively curved surfaces

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Abstract

We prove a nonstandard central limit theorem and weak invariance principle, with superdiffusive normalisation $(t \log t)^{1/2}$, for geodesic flows on a class of nonpositively curved surfaces with flat cylinder. We also prove that correlations decay at rate t^{-1} . An important ingredient of the proof, which is of independent interest, is an improved results on the regularity of the stable/unstable foliations induced by the Green bundles.

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1 Introduction

It is well-known that geodesic flows on negatively curved closed manifolds have very strong statistical properties. Ergodicity with respect to volume was proved for surfaces in [25] and for general dimension in [1]. The Bernoulli property was shown in [36, 40]. Statistical limit laws such as the central limit theorem (CLT), the weak invariance principle (WIP), also known as the functional CLT, and the almost sure invariance principle quickly followed [13, 39]. In major breakthroughs, exponential decay of correlations was established in [16] for surfaces and [31] in general dimension.

In contrast, there are few results on statistical properties beyond the Bernoulli property for geodesic flows on nonpositively curved manifolds when there exist points of zero curvature. In [29], we initiated the systematic study of such properties for geodesic flows. In particular, we considered a class of closed surfaces with one flat geodesic and proved polynomial decay of correlations and various statistical limit laws including the CLT and WIP (with standard normalisation). In this paper, we provide examples of geodesic flows on nonpositively curved surfaces that satisfy a CLT and WIP with *nonstandard normalisation*.

Our approach to studying geodesic flows on nonpositively curved manifolds resembles that for (semi)dispersing billiards [10]. General results on statistical limit laws and decay of correlations seem infeasible, but it is possible to analyse interesting classes of examples. Moreover, whereas rigorous results for dispersing billiards have been restricted to planar billiards due to the complicated structure of singularities under iteration, there is in principle no such restriction for geodesic flows since they are smooth. On the other hand, the prerequisites for the study of dispersing billiards are firmly established. For geodesic flows on nonpositively curved manifolds, the corresponding prerequisites are still under development, so for the moment we focus on surfaces too. A case in point is the smoothness of the foliations induced by the Green bundles and the necessity to extend existing results such as those in [19, 20], see Section 4.

The class of geodesic flows that we study in this paper is as follows. Let $r \in [5, \infty)$ and fix $0 < L < \varepsilon_0$. Define \mathcal{R} to be the surface of revolution with profile ξ given by $\xi(s) = 1$ for $|s| \leq L$ and $\xi(s) = 1 + (|s| - L)^r$ for $L \leq |s| \leq \varepsilon_0$. We form a closed C^4 Riemannian surface S of nonpositive curvature by isometrically gluing two negatively curved surfaces with boundary to the circles β_{\pm} bounding \mathcal{R} , see Figure 1. We call S a *surface with flat cylinder* $\mathcal{C} = [-L, L] \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Away from \mathcal{C} the surface S has negative curvature.

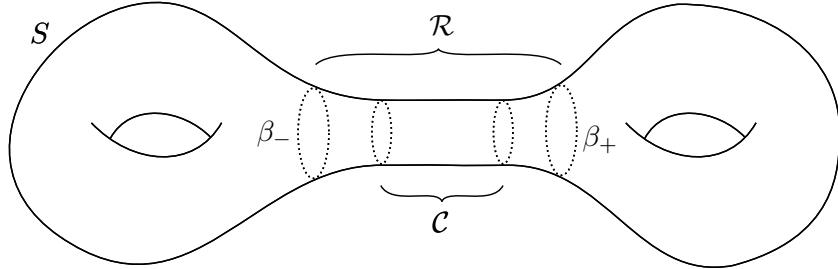


Figure 1: Surface with flat cylinder \mathcal{C} . The region \mathcal{R} between the two curves β_{\pm} is a surface of revolution with profile ξ .

Let $g_t : M \rightarrow M$ denote the geodesic flow on the unit tangent bundle $M = T^1S$, and let μ denote the normalised Riemannian volume on the three-dimensional manifold M . Then μ is an ergodic g_t -invariant probability measure [38].

The flat cylinder \mathcal{C} is foliated by two families of periodic orbits with clockwise/counterclockwise orientation. Let $v : M \rightarrow \mathbb{R}$ be Hölder with $\int_M v d\mu = 0$. We define $I_v = \alpha_0^2 + \alpha_\pi^2$ where α_0 and α_π are the averages of v over these two families of periodic orbits. (The notation derives from the explicit formulas (7.1) given in Section 7.) For $t > 0$, define $v_t = \int_0^t v \circ g_s ds : M \rightarrow \mathbb{R}$. Our first main result is a CLT for v_t with standard normalisation $t^{1/2}$ or nonstandard normalisation $(t \log t)^{1/2}$ depending on whether $I_v = 0$ or $I_v > 0$.

Theorem A (Nonstandard/Standard CLT) *If $I_v > 0$, then v satisfies a nonstandard CLT. That is, $(t \log t)^{-1/2} v_t \rightarrow_d N(0, \sigma_v^2)$ as $t \rightarrow \infty$, where $\sigma_v^2 = b_S I_v > 0$ and b_S is a positive constant depending only on the surface S .*

If $I_v = 0$, then there exists $\sigma^2 \geq 0$, typically nonzero, such that v satisfies a standard CLT. That is, $t^{-1/2} v_t \rightarrow_d N(0, \sigma^2)$ as $t \rightarrow \infty$.

Remark 1.1 As in [30, Remark 4.6], the word ‘‘typically’’ is interpreted here in the very strong sense that $\sigma^2 = 0$ only within a closed subspace of infinite codimension amongst Hölder observables v with $\int_M v d\mu = I_v = 0$. See for example the discussion in [24, End of Section 4].

We note that the vanishing of $\int_M v d\mu$ is ‘‘without loss of generality’’ since one can always centre the observable v by considering $v - \int_M v d\mu$. The vanishing of I_v is ‘‘codimension two’’ since $\alpha_0 = \alpha_\pi = 0$ may occur in generic two-parameter families. In contrast, the degenerate case $\sigma^2 = 0$ requires infinitely many coincidences beyond the constraint $I_v = 0$.

In the case $I_v > 0$, we also prove a nonstandard WIP (which implies the CLT). Define the sequence of continuous functions

$$W_n \in C[0, 1], \quad W_n(t) = (n \log n)^{-1/2} v_{nt}, \quad n \geq 1.$$

Since each function W_n depends on the initial condition in the probability space (M, μ) , we can regard W_n as a random element in the metric space $(C[0, 1], \| \cdot \|_\infty)$.

Theorem B (Nonstandard WIP) *If $I_v > 0$, then v satisfies a nonstandard WIP. That is, $W_n \rightarrow_w W$ in $C[0, 1]$ as $n \rightarrow \infty$, where W is a Brownian motion with variance σ_v^2 .*

There are only a few situations in deterministic dynamical systems where the nonstandard CLT/WIP is known to hold. The first example was for intermittent maps f satisfying $f(x) = x + bx^{3/2}$ at a neutral fixed point $x = 0$; see [21] for the nonstandard CLT and [12] for the nonstandard WIP. For various billiard examples, namely the infinite horizon Lorentz gas, Bunimovich stadium and billiards with cusps, we refer to [3, 4, 30, 41].

Our proofs of Theorems A and B closely follows abstract arguments in [3] and [30], using the latter to reduce to a nonstandard CLT for a simpler class of observables, and using the former to establish this simplified nonstandard CLT.

Our final main result concerns the rate of decay of correlations for the flow g_t .

Theorem C *Let $v, w : M \rightarrow \mathbb{R}$ be sufficiently smooth observables. Then there is a constant $C > 0$ such that*

$$\left| \int_M v \cdot (w \circ g_t) d\mu - \int_M v d\mu \int_M w d\mu \right| \leq Ct^{-1} \quad \text{for all } t > 0.$$

Remark 1.2 An observable is “sufficiently smooth” if it is C^k in the flow direction for some $k \geq 0$ that depends only on the surface S . In particular, if we choose the surface S to be C^∞ away from \mathcal{C} , then any observable that is C^∞ on M and flat near $\{s = \pm L\}$ is sufficiently smooth.

The analogous result for the infinite horizon Lorentz gas and Bunimovich stadia was obtained in [2].

Remark 1.3 The upper bound in Theorem C is sharp in the sense that if v is Hölder with $\int_M v d\mu = 0$ and $I_v \neq 0$, then

$$t \int_M v \cdot (v \circ g_t) d\mu \not\rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This is a consequence of the nonstandard limit law in Theorem A. See [4, Corollary 1.3] or [2, Proposition 9.14].

The remainder of this paper is organised as follows. In Section 2, we review known facts about the geometry and dynamics of geodesic flows on nonpositively curved surfaces, with special attention to surfaces with flat cylinder. In Section 3, we make a systematic study of the dynamics of the geodesic flow on the surface of revolution \mathcal{R} . In Section 4, we prove sufficient regularity of the stable/unstable foliations induced by the Green bundles,

partly extending [20]. In Section 5, we construct a uniformly hyperbolic first return map $f : \Sigma_0 \rightarrow \Sigma_0$ satisfying the Chernov axioms [9], showing that f is modelled by a Young tower with exponential tails [42].

In Section 6, we recall and slightly extend an abstract result on the nonstandard CLT due to [3]. In Section 7, we prove Theorems A and B. Finally, in Section 8 we prove Theorem C as well as a related result, Theorem 8.2, for a suitable global Poincaré map g . The Chernov axioms are stated for convenience in Appendix A.

Notation We use “big O” and \ll notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $a_n \leq Cb_n$ for all $n \geq n_0$. We write $a_n \approx b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$, and $a_n \sim b_n$ if $a_n/b_n \rightarrow 1$.

2 Nonpositively curved surfaces

In this section, we recall some known facts from differential geometry, especially for geodesic flows on nonpositively curved surfaces and for surfaces of revolution. Standard references are [6, 18]. We also give a precise description of the class of surfaces considered in this article, and then describe their properties that will be used in the sequel.

2.1 Geodesic flows

Let S be a C^4 closed Riemannian surface. Let $M = T^1S$ be its unit tangent bundle, which is a closed three dimensional Riemannian manifold. The *geodesic flow* on S is the flow $g_t : M \rightarrow M$ defined by $g_t(x) = \gamma'_x(t)$, where $\gamma_x : \mathbb{R} \rightarrow S$ is the unique geodesic such that $\gamma'_x(0) = x$. The volume form on S induces a smooth g_t -invariant probability measure μ on M .

For $p \in S$, let $K(p)$ be the Riemannian curvature at p . We assume that S has nonpositive Riemannian curvature: $K(p) \leq 0$ for all $p \in S$. Then we have a partition $M = \text{Deg} \cup \text{Reg}$ into the *degenerate set*¹ Deg and the *regular set* Reg defined by

$$\begin{aligned}\text{Deg} &= \{x \in M : K(\gamma_x(t)) = 0 \text{ for all } t \in \mathbb{R}\}, \\ \text{Reg} &= \{x \in M : K(\gamma_x(t)) < 0 \text{ for some } t \in \mathbb{R}\}.\end{aligned}$$

The dynamical properties of g_t are usually studied via Jacobi fields. We refer the reader to [29, Sect. 2.1] for their definition and an exposition of their basic properties, including how they define invariant subspaces and foliations. Below, we just recall some of their properties.

Using Jacobi fields that remain bounded in positive/negative time, we can define dg_t -invariant one-dimensional subspaces $\widehat{E}_x^{s/u} \subset T_x M$ for each $x \in M$.² These are the *Green bundles*, introduced in [23]. As in [29], we refer to $\widehat{E}_x^{s/u}$ as the stable/unstable subspaces of $x \in M$. These subspaces are orthogonal to the one-dimensional subspace Z_x tangent to

¹The classical literature uses the terminology “singular set” instead of “degenerate set”, but here we reserve the term “singular” for the dynamical setting of the Chernov axioms.

²We reserve the notation $E_x^{s/u}$ for the stronger notion of stable/unstable subspace in the sense of hyperbolic dynamics, as described in Section 5.

the geodesic flow. Moreover, $\widehat{E}_x^s = \widehat{E}_x^u$ if and only if $x \in \text{Deg}$; hence $\widehat{E}_x^s \oplus Z_x \oplus \widehat{E}_x^u = T_x M$ if and only if $x \in \text{Reg}$.

Using Busemann functions and horospheres [29, Sect. 2.1], for each $x \in M$ we define g_t -invariant C^1 curves $\widehat{W}_x^{s/u}$ tangent to $\widehat{E}_x^{s/u}$, called the *stable/unstable manifolds* for the flow.

There is a natural metric on M , called the *Sasaki metric*, which is the product of horizontal and vertical vectors, see e.g. [14, Chapter 3, Exercise 2]. For $\delta > 0$ small, there is an equivalent version of the Sasaki metric [26, §17.6], which we call the δ -*Sasaki metric*. In our calculations, we will fix a δ -Sasaki metric for δ small enough and denote it simply by $\|\cdot\|$.

2.2 Surfaces of revolution

Let $I \subset \mathbb{R}$ be a compact interval, and let $\xi : I \rightarrow \mathbb{R}$ be a positive C^4 function. The *surface of revolution* defined by ξ around the x axis is the surface \mathcal{R} with global chart $\Xi : I \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ given by $\Xi(s, \theta) = (s, \xi(s) \cos \theta, \xi(s) \sin \theta)$. We collect some known facts about these surfaces, referring to [15] for the details.

By [15, Example 4, p. 161], the curvature at $p = \Xi(s, \theta)$ is equal to

$$K(p) = -\frac{\xi''(s)}{\xi(s)[1 + (\xi'(s))^2]^2}. \quad (2.1)$$

Geodesics on surfaces of revolution satisfy the so-called *Clairaut relation*. Let $\gamma(t) = \Xi(s(t), \theta(t))$ be a geodesic, and let $\psi(t) \in \mathbb{S}^1$ be the angle that the circle $s = s(t)$, more precisely its image under Ξ , makes with γ at $\gamma(t)$. The Clairaut relation states that the value

$$c = \xi(s(t)) \cos \psi(t) = \xi(s(t))^2 \theta'(t) \quad (2.2)$$

is constant along γ . We call c the *Clairaut constant* of γ and of all of its tangent vectors.

Let $\gamma(t) = \Xi(s(t), \theta(t))$ be a geodesic with Clairaut constant c . Then $s(t)$ satisfies the *equation of geodesics* [15, Example 5, p. 255],

$$[1 + \xi'(s)^2](s')^2 + \frac{c^2}{\xi(s)^2} = 1.$$

For a fixed $s_0 \in \mathbb{R}$, the curve $\Xi(s_0, \theta)$ is called a *meridian*. We fix an orientation of \mathbb{S}^1 and induce it on the meridians. Observe that \mathcal{R} is diffeomorphic to $I \times \mathbb{S}^1$ and $T^1 \mathcal{R}$ is diffeomorphic to $\mathcal{R} \times \mathbb{S}^1 \cong I \times \mathbb{S}^1 \times \mathbb{S}^1$, where $(p, \psi) \in \mathcal{R} \times \mathbb{S}^1$ is identified to the unit tangent vector with basepoint p that makes an angle ψ with the oriented meridian passing through p . Writing $p = \Xi(s, \theta)$, we refer to the coordinates $(s, \theta, \psi) \in I \times \mathbb{S}^1 \times \mathbb{S}^1$ as *Clairaut coordinates* on $T^1 \mathcal{R}$.

Using the Clairaut relation (2.2), we define the *Clairaut function* $c : T^1 \mathcal{R} \rightarrow \mathbb{R}$ by $c(s, \theta, \psi) = \xi(s) \cos \psi$.

The *Clairaut metric* on $T^1 \mathcal{R}$ is the Riemannian metric given by the canonical product on $I \times \mathbb{S}^1 \times \mathbb{S}^1$; it is equivalent to the δ -Sasaki metric $\|\cdot\|$.

2.3 Surfaces with flat cylinder

We now define the class of surfaces that we are interested in in this paper. They exhibit two special features: the only region of zero curvature is a flat cylinder, and on a neighbourhood of this cylinder the surface is a particular surface of revolution.

We begin by defining the surface of revolution. Fix $r > 4$ and $0 < L < \varepsilon_0$. Consider the profile function $\xi : [-\varepsilon_0, \varepsilon_0] \rightarrow \mathbb{R}$ given by

$$\xi(s) = \begin{cases} 1, & |s| \leq L \\ 1 + (|s| - L)^r, & |s| \geq L. \end{cases}$$

This defines a surface of revolution \mathcal{R} with boundary meridians $\beta_{\pm} = \Xi(\pm\varepsilon_0, \theta)$. We call $\mathcal{C} = [-L, L] \times \mathbb{S}^1$ the *flat cylinder* and $\mathcal{N} = \mathcal{R} \setminus \mathcal{C}$ the *neck*.

Definition 2.1 A C^4 surface of nonpositive curvature S is a *surface with flat cylinder* if S contains a copy of \mathcal{R} and $K|_{S \setminus \mathcal{C}} < 0$. See Figure 2.

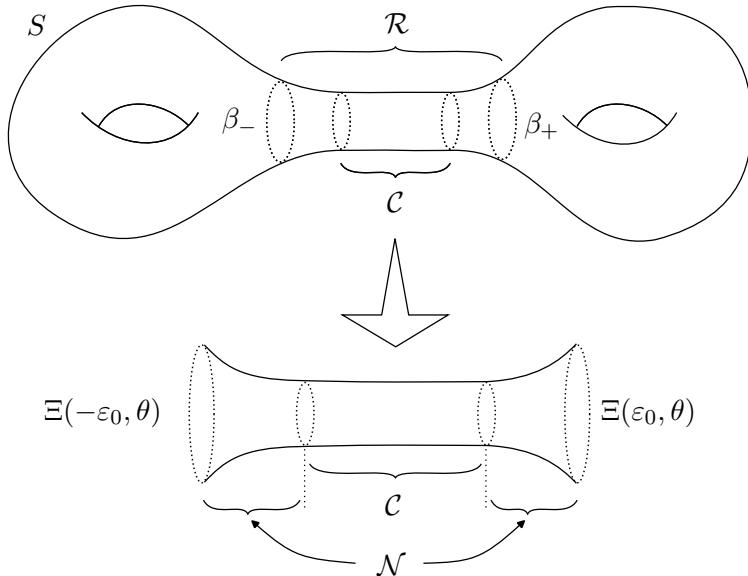


Figure 2: An example of a surface S with flat cylinder.

Such surfaces indeed exist, and can be obtained by interpolating the neck with a hyperbolic surface ($K \equiv -1$) with one cusp on each side. Since near the cusp a hyperbolic surface is a surface of revolution, it is enough to interpolate its profile function with the function ξ , in a way that the resulting function is strictly convex for $s > L$. This can be done as in [17, Appendix A.2], using a partition of unity.

In the sequel, we fix a surface S with flat cylinder and $r > 4$. Eventually, we specialise to $r \geq 5$ as in the introduction since this is required in Theorem 4.6 below. All other arguments hold also for $r \in (4, 5)$.

In the Clairaut coordinates, Deg is the union of $[-L, L] \times \mathbb{S}^1 \times \{0\}$ and $[-L, L] \times \mathbb{S}^1 \times \{\pi\}$. In particular, $\mu(\text{Deg}) = 0$, so [38] implies that the flow g_t is ergodic. In fact, g_t is Bernoulli;

see [37] and [7, Thm. 12.2.13] for the classical proofs, and [28] for a proof using symbolic dynamics.

2.4 Asymptotic, bouncing and crossing geodesics/vectors

As in [29], we use the Clairaut function to distinguish some vectors in $T^1\mathcal{N}$ that will play a key role in the next sections. Let $x = (s, \theta, \psi) \in T^1\mathcal{N}$ such that the geodesic starting at x points towards \mathcal{C} . The vector x , and the corresponding geodesic with initial condition x , is called:

Asymptotic if $c(x) = \pm 1$: the geodesic path $g_{[0,\infty)}(x)$ is asymptotic to \mathcal{C} .

Bouncing if $|c(x)| > 1$: there is $t > 0$ such that $\psi(t) = 0$ or π , i.e. the geodesic path $g_{[0,t]}(x)$ spirals towards \mathcal{C} , $g_t(x)$ is tangent to a meridian, and after that the geodesic path spirals away from \mathcal{C} in the reverse direction. In such cases, the geodesic does not reach \mathcal{C} .

Crossing if $|c(x)| < 1$: there is $t > 0$ such that $s(t) = 0$, i.e. the geodesic path $g_t(x)$ spirals into \mathcal{C} , crosses \mathcal{C} from one side to the other, and then spirals away from \mathcal{C} .

Figure 3 depicts the three possibilities.

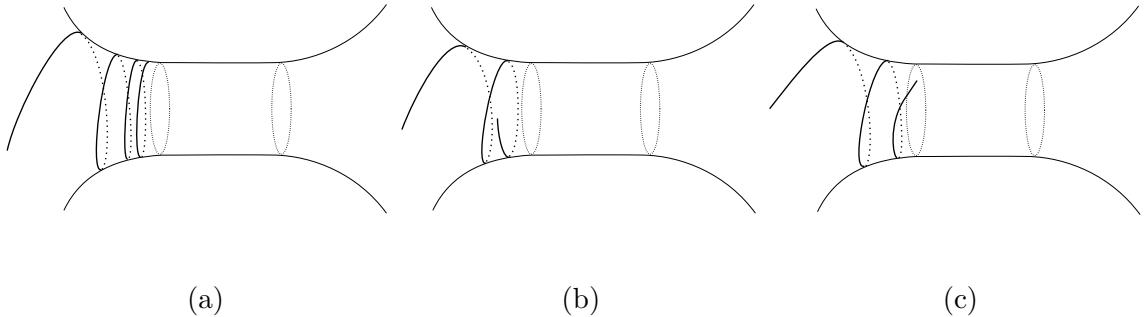


Figure 3: Geodesics that are (a) asymptotic, (b) bouncing, (c) crossing.

3 Dynamics of transitions in \mathcal{R}

In this section, we initiate the study of the geodesic flow on a surface with flat cylinder as defined in Section 2.3. In particular, we describe in detail the transitions in the surface of revolution \mathcal{R} .

In Subsection 3.1, we construct a two-dimensional section Ω , the *transition section*, situated at (or near) the ends of \mathcal{R} . We also recall an explicit formula from [29] for the associated *transition map* f_0 . In Subsection 3.2, we estimate transition times and derivatives of f_0 . The calculations in this section use the Clairaut coordinates $x = (s, \theta, \psi)$ on $T^1\mathcal{R} \cong [-\varepsilon_0, \varepsilon_0] \times \mathbb{S}^1 \times \mathbb{S}^1$.

3.1 The transition map $f_0 = g_{2\Upsilon_0} : \Omega_{\text{in}} \rightarrow \Omega_{\text{out}}$

The transition section Ω we construct³ allows a very simple description of the transitions of geodesics in \mathcal{R} . Let $\varepsilon_1 = (L + \varepsilon_0)/2$. As detailed below, we define $\Omega = \Omega_{\text{in}} \cup \Omega_{\text{out}}$ where:

- $\Omega_{\text{in}} \subset \{\pm\varepsilon_1\} \times \mathbb{S}^1 \times \mathbb{S}^1$ is a neighbourhood of four families of geodesics entering \mathcal{R} and asymptotic to \mathcal{C} ;
- $\Omega_{\text{out}} \subset \{\pm\varepsilon_1\} \times \mathbb{S}^1 \times \mathbb{S}^1$ is a neighbourhood of four families of geodesics exiting \mathcal{R} and asymptotic to \mathcal{C} in backwards time;
- Each of Ω_{in} and Ω_{out} is a disjoint union of four annular regions (diffeomorphic to $\mathbb{S}^1 \times (-1, 1)$).

To construct $\Omega_{\text{in/out}}$, we use the Clairaut function c from Section 2.2. There is a unique $\psi_0 \in (0, \frac{\pi}{2})$ such that $\xi(\pm\varepsilon_1) \cos \psi_0 = 1$. Recalling the notation of asymptotic vector introduced in Section 2.4, the vectors $(\pm\varepsilon_1, \theta, \pm\psi_0)$, $\theta \in \mathbb{S}^1$, constitute four families of asymptotic vectors with Clairaut constant $c = 1$ and asymptotic to \mathcal{C} . Similarly, $(\pm\varepsilon_1, \theta, \pm(\pi - \psi_0))$, $\theta \in \mathbb{S}^1$, constitute four families of asymptotic vectors with $c = -1$ and asymptotic to \mathcal{C} . Of these, $(-\varepsilon_1, \theta, \psi_0)$, $(\varepsilon_1, \theta, -\psi_0)$, $(-\varepsilon_1, \theta, \pi - \psi_0)$, $(\varepsilon_1, \theta, -(\pi - \psi_0))$ correspond to the four families of vectors that enter \mathcal{R} and are asymptotic to \mathcal{C} as $t \rightarrow \infty$. The remaining four families of vectors exit \mathcal{R} and are asymptotic to \mathcal{C} as $t \rightarrow -\infty$.

Focusing momentarily on $x = (-\varepsilon_1, \theta, \psi_0)$, we define

$$\Omega_1 = \{x = (-\varepsilon_1, \theta, \psi) \in T^1 \mathcal{R} : |c(x) - 1| < \chi\}.$$

Shrinking $\chi > 0$, we can ensure that $\Omega_1 = \{-\varepsilon_1\} \times \mathbb{S}^1 \times I$ where I is an open interval containing ψ_0 with $\bar{I} \subset (0, \frac{\pi}{2})$. Treating the other three families of entering asymptotic vectors similarly, we obtain Ω_{in} as the union of four sets isomorphic to Ω_1 . Similarly, the set Ω_{out} , isomorphic to Ω_{in} , is obtained by considering the four families of exiting asymptotic vectors. It is easy to see that Ω is transverse to the flow direction.

Denote geodesics in \mathcal{R} by $\mathbf{x} = \mathbf{x}(t) = (s(t), \theta(t), \psi(t))$ with Clairaut constant $c = c(\mathbf{x})$. We parametrise bouncing geodesics \mathbf{x} taking $s'(0) = 0$ and $\psi(0) = 0$ or π , i.e. \mathbf{x} bounces back exactly at time $t = 0$. Similarly, we parametrise crossing geodesics \mathbf{x} taking $s(0) = 0$.

Let $\mathbf{x} = (s(t), \theta(t), \psi(t))$ be a bouncing/crossing geodesic parametrised in this way. We define the *transition time* $\Upsilon_0(\mathbf{x}) = \min\{t > 0 : |s(t)| = \varepsilon_1\}$. By symmetry, the transition time of \mathbf{x} from Ω_{in} to Ω_{out} is actually equal to $2\Upsilon_0(\mathbf{x})$ (but it is convenient to call Υ_0 the transition time).

Next, we introduce the *transition map*

$$f_0 : \Omega_{\text{in}} \rightarrow \Omega_{\text{out}}, \quad f_0(x) = g_{2\Upsilon_0(x)}(x)$$

³The construction is the same as in [29] but the notation is different. The sections Ω_{\pm} there are called $\Omega_{\text{in}/\text{out}}$ here (the \pm is not related to stable/unstable or to $\pm\varepsilon_0$, so was perhaps overused). The section Ω_0 in [29] is not part of the definition of f_0 and hence is no longer part of Ω . It reappears in Section 7.1.

In addition, Ω is situated inside the neck at $\pm\varepsilon_1$ instead of at $\pm\varepsilon_0$ in [29]. This ensures that the perturbed cross-sections $\Sigma_{\text{in}/\text{out}}$ in Section 5 lie inside \mathcal{R} so that the Clairaut function c is defined.

(where defined). By [29, Proposition 4.2],

$$f_0(-\varepsilon_1, \theta, \psi) = \begin{cases} (-\varepsilon_1, \theta \pm \zeta(\psi), -\psi) & \text{for bouncing vectors} \\ (\varepsilon_1, \theta \pm \zeta(\psi), \psi) & \text{for crossing vectors} \end{cases}$$

$$\text{where } \zeta(\psi) = 2 \int_{|s(0)|}^{\varepsilon_1} \frac{|c(\mathbf{x})|}{\xi(s)} \left[\frac{1 + \xi'(s)^2}{\xi(s)^2 - c(\mathbf{x})^2} \right]^{\frac{1}{2}} ds.$$

3.2 Transition times and derivatives of f_0

As in [29], the following partition of Ω_{in} near the set of asymptotic vectors is helpful.

Definition 3.1 We define *homogeneity bands* $\mathcal{C}_n = \mathcal{C}_n^> \cup \mathcal{C}_n^<$, $n \geq 1$, in Ω_{in} by

$$\begin{aligned} \mathcal{C}_n^> &= \left\{ x \in \Omega_{\text{in}} : 1 + \frac{1}{(n+1)^2} < |c(x)| < 1 + \frac{1}{n^2} \right\} \quad (\text{bouncing}); \\ \mathcal{C}_n^< &= \left\{ x \in \Omega_{\text{in}} : 1 - \frac{1}{n^2} < |c(x)| < 1 - \frac{1}{(n+1)^2} \right\} \quad (\text{crossing}). \end{aligned}$$

The next result estimates Υ_0 and ζ in the homogeneity bands. Write $\Upsilon_0 = \Upsilon_1 + \Upsilon_2$ where Υ_1 is the transition time in the portion of the neck between $s = -\varepsilon_1$ and $s = -L$ and Υ_2 is the time spent in \mathcal{C} between $s = -L$ and $s = 0$.

Lemma 3.2 *The following are true.*

- (1) *Let \mathbf{x} be a geodesic with entry vector in \mathcal{C}_n . Then $\Upsilon_1(\mathbf{x}) \approx n^{\frac{r-2}{r}}$. If \mathbf{x} is bouncing then $\Upsilon_2(\mathbf{x}) = 0$; if \mathbf{x} is crossing then $\Upsilon_2(\mathbf{x}) \approx n$.*
- (2) *If $\mathbf{x}, \bar{\mathbf{x}}$ are both bouncing or both crossing geodesics, then*

$$|\psi(\Upsilon_0(\mathbf{x})) - \bar{\psi}(\Upsilon_0(\bar{\mathbf{x}}))| \approx |c(\mathbf{x}) - c(\bar{\mathbf{x}})|.$$

In particular, if the entry vectors of $\mathbf{x}, \bar{\mathbf{x}}$ are both in the same connected component of $\mathcal{C}_n^<$ or of $\mathcal{C}_n^>$ then

$$|\psi(\Upsilon_0(\mathbf{x})) - \bar{\psi}(\Upsilon_0(\bar{\mathbf{x}}))| \leq 2(\varepsilon_1^{2r} + 2\varepsilon_1^r)^{-1/2} n^{-3} + O(n^{-4}).$$

- (3) *The following estimates on ζ hold.*

- (a) *If $(-\varepsilon_1, \theta, \psi) \in \mathcal{C}_n^<$ then $\zeta'(\psi) \approx -n^3$ and $\zeta''(\psi) \approx n^5$.*
- (b) *If $(-\varepsilon_1, \theta, \psi) \in \mathcal{C}_n^>$ then $\zeta'(\psi) \approx n^{3-\frac{2}{r}}$ and $|\zeta''(\psi)| \ll n^{5-\frac{2}{r}}$.*

Proof (1) By [29, Lemma 4.4], $\Upsilon_1 \approx n^{\frac{r-2}{r}}$. Clearly $\Upsilon_2 = 0$ for bouncing geodesics. For crossing geodesics, $c = \cos \psi(-\Upsilon_2)$, so

$$\Upsilon_2 = \frac{L}{|\sin \psi(-\Upsilon_2)|} = \frac{L}{\sqrt{1 - c^2}} \sim \frac{L}{\sqrt{2}} n.$$

- (2) The proof is the same as for [29, Lemma 4.5].

(3) The estimates for bouncing geodesics are unchanged from [29]. We consider crossing geodesics with $c(\mathbf{x}) > 0$ (the case $c(\mathbf{x}) < 0$ is treated analogously). Let $a = 1 + \varepsilon_1^r$. We have $c(\mathbf{x}) = a \cos \psi$, hence $\cos \psi \sim a^{-1} \approx 1$ and $\sin \psi \sim (1 - a^{-2})^{\frac{1}{2}} \approx 1$. Also, $s(0) = 0$, so

$$\zeta(\psi) = 2a \cos \psi \int_0^{\varepsilon_1} A(s) [\xi(s)^2 - a^2 \cos^2 \psi]^{-\frac{1}{2}} ds$$

where $A(s) = \frac{[1+\xi'(s)^2]^{\frac{1}{2}}}{\xi(s)} \approx 1$. By direct calculation,

$$\begin{aligned} \zeta'(\psi) &= -2a \sin \psi \int_0^{\varepsilon_1} A(s) \xi(s)^2 [\xi(s)^2 - a^2 \cos^2 \psi]^{-\frac{3}{2}} ds \\ &\approx - \int_0^{\varepsilon_1} [\xi(s) - c(\mathbf{x})]^{-\frac{3}{2}} ds = -L[1 - c(\mathbf{x})]^{-\frac{3}{2}} - \int_L^{\varepsilon_1} [\xi(s) - c(\mathbf{x})]^{-\frac{3}{2}} ds \end{aligned}$$

and

$$\begin{aligned} \zeta''(\psi) &= 2a \cos \psi \int_0^{\varepsilon_1} A(s) \xi(s)^2 [3a^2 \sin^2 \psi + a^2 \cos^2 \psi - \xi(s)^2] [\xi(s)^2 - a^2 \cos^2 \psi]^{-\frac{5}{2}} ds \\ &\approx \int_0^{\varepsilon_1} [\xi(s) - c(\mathbf{x})]^{-\frac{5}{2}} ds = L[1 - c(\mathbf{x})]^{-\frac{5}{2}} + \int_L^{\varepsilon_1} [\xi(s) - c(\mathbf{x})]^{-\frac{5}{2}} ds. \end{aligned}$$

We note that $1 - c(\mathbf{x}) \approx n^{-2}$ and $\xi(s) - c(\mathbf{x}) \approx |s - L|^r + n^{-2}$ for $s \in [L, \varepsilon_1]$. Applying [29, Proposition 4.3(1)] with $\alpha = \frac{3}{2}$ and $b = n^{-2}$ gives that

$$\int_L^{\varepsilon_1} [\xi(s) - c(\mathbf{x})]^{-\frac{3}{2}} ds \approx (n^{-2})^{-\frac{3}{2} + \frac{1}{r}} = n^{3-\frac{2}{r}}$$

and so $\zeta'(\psi) \approx -(n^{-2})^{-\frac{3}{2}} - n^{3-\frac{2}{r}} = -n^3$.

Similarly, applying [29, Proposition 4.3(1)] with $\alpha = \frac{5}{2}$ and $b = n^{-2}$ gives that

$$\int_L^{\varepsilon_1} [\xi(s) - c(\mathbf{x})]^{-\frac{5}{2}} ds \approx (n^{-2})^{-\frac{5}{2} + \frac{1}{r}} = n^{5-\frac{2}{r}}$$

and so $\zeta''(\psi) \approx (n^{-2})^{-\frac{5}{2}} + n^{5-\frac{2}{r}} \approx n^5$. ■

4 Regularity of the stable and unstable foliations

For general geodesic flows in nonpositive curvature, the stable/unstable manifolds $\widehat{W}_x^{s/u}$ are C^1 and the bundles $\widehat{E}_x^{s/u}$ are continuous. For C^5 surfaces, the stable/unstable manifolds $\widehat{W}_x^{s/u}$ are uniformly $C^{1+\frac{1}{2}}$, see [20, Proposition III]. However, for the Chernov axioms (A5) and (A7) from Appendix A, we require extra regularity.

In some situations, it is possible to increase the regularity to $C^{1+\text{Lip}}$ and Hölder respectively, see [20, 29]. But certain key estimates in these references do not hold for surfaces with flat cylinder. (See the discussion in [20, Example at end of Section 3].)

In this section, we show that for surfaces with flat cylinder with $r \geq 5$, the stable/unstable manifolds $\widehat{W}_x^{s/u}$ are indeed uniformly $C^{1+\text{Lip}}$ as required for Chernov axiom (A5). We also prove for all $r > 4$ that the bundles $\widehat{E}_x^{s/u}$ have sufficient regularity (even if not Hölder continuous) to enable verification of Chernov axiom (A7).

4.1 Curvature estimates

For general $x \in M = T^1S$, let $p \in S$ denote its footprint and $k_{\pm}(x) \geq 0$ the corresponding geodesic curvatures, equal to the curvatures of the curves $\widehat{W}_x^{s/u}$ at x . On $T^1\mathcal{R}$, we use Clairaut coordinates $x = (p, \psi) = (s, \theta, \psi) \in [-\varepsilon_0, \varepsilon_0] \times \mathbb{S}^1 \times \mathbb{S}^1$. For $p \in \mathcal{N}$, write $p = (\pm(L + a), \theta)$ where $a \in [0, \varepsilon_0 - L]$. Throughout this subsection, we only assume $r > 4$. Also, $C > 1$ is a constant depending only on the surface S .

To deal with surfaces with flat cylinder, we formulate the following key lemma with is a suitably weakened analogue of [20, Lemma 3.3].

Lemma 4.1 *The following hold:*

- (1) $k_{\pm}(x) \leq C|\psi|^{(r-2)/r}$ for all $x \in T^1\mathcal{C}$;
- (2) $k_{\pm}(x) \leq C \max\{a^{(r-2)/2}, |\psi|^{(r-2)/r}\}$ for all $x \in T^1\mathcal{N}$;
- (3) $k_{\pm}(x) \geq C^{-1}|\psi|$ for all $x \in T^1\mathcal{R}$;
- (4) $k_{\pm}(x) \geq C^{-1}a^{(r-1)/2}$ for all $x \in T^1\mathcal{N}$.

Proof Our proof largely follows [20, Lemma 4.3].⁴

The estimates in (1) and (2) were proved in [19, Theorem 3.1].

Next, we consider (3) and (4) simultaneously with the convention that $a = 0$ for $x \in T^1\mathcal{C}$. It suffices to consider k_+ since the situation for k_- is identical.

Let \tilde{S} be the universal cover of S . Lift x to a point in \tilde{S} , which we will also denote by x , and let γ be the geodesic on \tilde{S} with $\gamma'(0) = x$. Let u be the unstable Riccati solution along γ , so that $u \geq 0$ with $u' = -u^2 - K \circ \gamma$ and $k_+(x) = u(0)$.

Suppose first that $a^{r-1} \leq \psi(0)^2$. Then $a^r \leq a\psi(0)^2 \leq \frac{1}{4}\psi(0)^2$ for a small. Without loss of generality, we can assume that c is close to 1 and $\psi(0)$ is close to zero. By (2.2), the Clairaut constant c satisfies

$$\begin{aligned} c &= \xi(L + a) \cos \psi(0) = (1 + a^r) \cos \psi(0) \\ &\leq (1 + \frac{1}{4}\psi(0)^2)(1 - \frac{1}{2}\psi(0)^2 + O(\psi(0)^4)) = 1 - \frac{1}{4}\psi(0)^2 + O(\psi(0)^4) < 1. \end{aligned}$$

This means that we are in the crossing case. Supposing that the geodesic γ has entry vector in $\mathcal{C}_n^<$, the definition of homogeneity bands in Definition 3.1 gives that $c \sim 1 - n^{-2}$, so $|\psi(0)| \ll n^{-1}$. We choose $T > 0$ such that $\gamma(-T)$ is leaving \mathcal{R} , with $u(-T) \approx 1$. By Lemma 3.2(1), $T \leq 2\Upsilon_0 \ll n$. Hence by [20, Lemma 3.1(v)],

$$k_+(x) = u(0) \geq \frac{u(-T)}{Tu(-T) + 1} \approx \frac{1}{T+1} \gg n^{-1} \gg |\psi(0)| \geq a^{(r-1)/2},$$

yielding the desired lower bounds.

It remains to consider the case $\psi(0)^2 \leq a^{r-1}$. Let $\gamma(t)$, $t \in [-T, 0]$, be a maximal geodesic segment lying in $[L + \frac{1}{2}a, L + 2a] \times \mathbb{S}^1$ with $\gamma(0) \in \{L + a\} \times \mathbb{S}^1$ and $\gamma(-T) \in$

⁴We note that m in [20, Lemma 4.3] and our r are related by $r = m + 2$. The boundary case is $|\psi(0)| = a^{(r-1)/2}$ compared with $|\psi(0)| = a^{r/2}$ in [20, Lemma 4.3].

$\{L + \frac{1}{2}a, L + 2a\} \times \mathbb{S}^1$. We claim that $T \gg a^{-(r-3)/2}$. To see this, note that $-K(\gamma(t)) \approx a^{r-2}$ for $t \in [-T, 0]$ by (2.1), so by [20, Lemma 4.2(ii)], $|\psi'(t)| \ll a^{r-1}$ for $t \in [-T, 0]$. Hence $|\psi(t)| \leq |\psi(0)| + Ta^{r-1} \ll a^{(r-1)/2} + Ta^{r-1}$. If $T \leq a^{-(r-3)/2}$ then $|\psi(t)| \ll a^{(r-1)/2}$ and so we have the trigonometric estimate

$$T \geq \frac{1}{2}a / \max \sin |\psi(t)| \gg a^{-(r-3)/2}.$$

Hence $T \gg a^{-(r-3)/2}$.

On the interval $[-T, 0]$, $u(t)$ satisfies the Riccati equation $u' = -u^2 + \alpha$ where $\alpha = -K \circ \gamma$. Let $\alpha_* = \min_{[-T, 0]} \alpha \approx a^{r-2}$. If $u(0) \leq \frac{1}{2}\alpha_*^{1/2}$, then $u'(0) \geq \frac{3}{4}\alpha_*$ and working backwards in time it follows that on $[-T, 0]$ we have u non-decreasing, $u(t) \leq \frac{1}{2}\alpha_*^{1/2} \approx a^{(r-2)/2}$ and $u'(t) \geq \frac{3}{4}\alpha_* \approx a^{r-2}$. Hence

$$u(0) \gg u(-T) + Ta^{r-2} \geq Ta^{r-2} \gg a^{(r-1)/2}.$$

Otherwise, $u(0) \geq \frac{1}{2}\alpha_*^{1/2} \approx a^{(r-2)/2}$. Either way, $k_+(x) = u(0) \gg a^{(r-1)/2} \geq |\psi(0)|$ as required. \blacksquare

Corollary 4.2 $k_-(x)^{r/(r-2)} \leq Ck_+(x)$ and $k_+(x)^{r/(r-2)} \leq Ck_-(x)$ for all $x \in T^1 S$.

Proof It suffices to prove the first inequality. For $p \in \mathcal{R}$, this follows from Lemma 4.1. The same argument as in [20, Lemma 3.4] deals with $p \in S \setminus \mathcal{R}$. Indeed, $k_+(x) = 0$ only for x tangent to a geodesic along which K vanishes, so $k_+(x) > 0$ for $p \in S \setminus \mathcal{C}$. By the continuity of k_\pm ,

$$k_-(x)^{r/(r-2)} \ll k_+(x) \quad \text{for } p \notin \mathcal{R},$$

completing the proof. \blacksquare

Corollary 4.3 *The following hold:*

- (1) $|K(p)| \leq Ck_+(x)^2$ for $p \in S \setminus \mathcal{N}$.
- (2) $|K(p)| \leq Ck_+(x)^{2(r-2)/(r-1)}$ for $p \in \mathcal{N}$.

Proof Part (1) is trivial for $p \in \mathcal{C}$ since $K|_{\mathcal{C}} = 0$. The same argument as in [20, Lemma 3.4] deals with $p \in S \setminus \mathcal{R}$: again, $k_+(x) > 0$ on $S \setminus \mathcal{C}$, so part (1) follows by continuity of k_+ and K .

In addition, $K(p) \approx -a^{r-2}$ on \mathcal{N} , so part (2) follows from Lemma 4.1(4). \blacksquare

Lemma 4.4 *The following hold:*

- (1) $|K(p_0) - K(p_1)| \leq C(|K(p_0)|^{1/2}d(p_0, p_1) + d(p_0, p_1)^2)$ for all $p_0, p_1 \in S$.
- (2) $|K(p_0) - K(p_1)| \leq C \max\{|K(p_0)|, |K(p_1)|\}^{(r-3)/(r-2)}d(p_0, p_1)$ for all $p_0, p_1 \in \mathcal{N}$.

Proof Part (1) is [20, Lemma 3.2]. To prove part (2), suppose without loss of generality that p_0, p_1 lie in the right-hand portion of \mathcal{N} and write $p_i = (L+a_i, \theta_i)$ where $0 \leq a_i \leq \varepsilon_0 - L$, $\theta_i \in \mathbb{S}^1$. By (2.1), $-K = \xi''\xi^{-1}(1+(\xi')^2)^{-2}$ so $-K(p_i) \approx a_i^{r-2}$ and $-K'(p_i) \approx a_i^{r-3}$. By the mean value theorem, $|K(p_0) - K(p_1)| \ll \max\{|K'(p_0)|, |K'(p_1)|\}|a_0 - a_1|$ and the result follows. \blacksquare

Corollary 4.5 Set $q = \min\{1, 2(r-3)/(r-1)\}$. Let $x_0, x_1 \in T^1 S$ with footprints $p_0, p_1 \in S$ and corresponding geodesic curvatures $k_+(x_0), k_+(x_1)$. Then

$$|K(p_0) - K(p_1)| \leq C\{(k_+(x_0)^q + k_+(x_1)^q)d(p_0, p_1) + d(p_0, p_1)^2\}.$$

Proof If $p_0 \notin \mathcal{N}$, then it follows from Corollary 4.3(1) and Lemma 4.4(1) that

$$|K(p_0) - K(p_1)| \ll |K(p_0)|^{1/2}d(p_0, p_1) + d(p_0, p_1)^2 \ll k_+(x_0)d(p_0, p_1) + d(p_0, p_1)^2,$$

which implies the required estimate. The same argument applies if $p_1 \notin \mathcal{N}$.

If $p_0, p_1 \in \mathcal{N}$, then it follows from Corollary 4.3(2) and Lemma 4.4(2) that

$$\begin{aligned} |K(p_0) - K(p_1)| &\ll \max\{|K(p_0)|, |K(p_1)|\}^{(r-3)/(r-2)}d(p_0, p_1) \\ &\ll \max\{k_+(x_0), k_+(x_1)\}^{2(r-3)/(r-1)}d(p_0, p_1) \end{aligned}$$

which again implies the required estimate. \blacksquare

4.2 Two regularity theorems

Theorem 4.6 Assume $r \geq 5$. The curves $\widehat{W}_x^{s/u}$ are uniformly $C^{1+\text{Lip}}$.

Proof As in [20], we focus on the unstable leaves \widehat{W}_x^u . The stable case is identical. Let γ_0, γ_1 be two geodesics on \widetilde{S} such that $\text{dist}_{\widetilde{S}}(\gamma_0(t), \gamma_1(t))$ is bounded for $t \leq 0$ and set $\epsilon = \text{dist}_{\widetilde{S}}(\gamma_0(0), \gamma_1(0))$. We show that the geodesic curvatures $k_+(\gamma'_i(0))$ satisfy $|k_+(\gamma'_0(0)) - k_+(\gamma'_1(0))| \ll \epsilon$.

By convexity of $\text{dist}_{\widetilde{S}}$, we have

$$\text{dist}_{\widetilde{S}}(\gamma_0(t), \gamma_1(t)) \leq \text{dist}_{\widetilde{S}}(\gamma_0(0), \gamma_1(0)) = \epsilon \quad \text{for } t \leq 0. \quad (4.1)$$

Let u_i be the unstable Riccati solutions along γ_i , so that $u_i \geq 0$ and $u'_i = -u_i^2 - K_i$ where $K_i = K \circ \gamma_i$. Then $k_+(\gamma'_i(t)) = u_i(t)$ and we obtain by [20, Lemma 3.1(ii)] that

$$\begin{aligned} |k_+(\gamma'_0(0)) - k_+(\gamma'_1(0))| &= |u_0(0) - u_1(0)| \\ &\leq \int_{-A}^0 |K_0(t) - K_1(t)| \hat{j}_0(t) \hat{j}_1(t) dt + |u_0(-A) - u_1(-A)| \hat{j}_0(-A) \hat{j}_1(-A), \end{aligned} \quad (4.2)$$

where $\hat{j}_i(t) = \exp\left\{-\int_t^0 u_i(t') dt'\right\} \in [0, 1]$. Since $r \geq 5$, we have $q = 1$ in Corollary 4.5. Taking $A = \epsilon^{-1}$, by Corollary 4.5 and (4.1)

$$\begin{aligned} \int_{-A}^0 |K_0 - K_1| \hat{j}_0 \hat{j}_1 &\ll A\epsilon^2 + \epsilon \int_{-A}^0 (k_+(\gamma'_0) + k_+(\gamma'_1)) \hat{j}_0 \hat{j}_1 = \epsilon + \epsilon \int_{-A}^0 (u_0 + u_1) \hat{j}_0 \hat{j}_1 \\ &\leq \epsilon + \epsilon \int_{-A}^0 (u_0 \hat{j}_0 + u_1 \hat{j}_1) = \epsilon + \epsilon \int_{-A}^0 (\hat{j}'_0 + \hat{j}'_1) \\ &= \epsilon + \epsilon (\hat{j}_0(0) + \hat{j}_1(0) - \hat{j}_0(-A) - \hat{j}_1(-A)) \leq 3\epsilon, \end{aligned}$$

which deals with the first term on the right-hand side of (4.2). To estimate the remaining term, we can suppose without loss of generality that $u_0(-A) \geq u_1(-A)$. By [20, Lemma 3.1(iii)], $\hat{j}'_0(-A) \leq 1/A$. Hence

$$|u_0(-A) - u_1(-A)| \hat{j}_0(-A) \hat{j}_1(-A) \leq u_0(-A) \hat{j}_0(-A) = \hat{j}'_0(-A) \leq 1/A = \epsilon,$$

completing the proof. \blacksquare

Remark 4.7 We also obtain a regularity result for $r \in (4, 5)$. In the first term in (4.2), we have to estimate expressions like $\int_{-A}^0 u_0^{1/p} \hat{j}_0 \leq \int_{-A}^0 (u_0 \hat{j}_0)^{1/p} = \int_{-A}^0 (\hat{j}'_0)^{1/p}$ where $p = (r-1)/(2(r-3)) > 1$. Setting $q = (r-1)/(5-r)$ and applying Hölder's inequality,

$$\int_{-A}^0 (\hat{j}'_0)^{1/p} \leq \left(\int_{-A}^0 \hat{j}'_0 dt \right)^{1/p} A^{1/q} \leq A^{1/q} = A^{(5-r)/(r-1)}.$$

Taking $A = \epsilon^{-(r-1)/4}$, we obtain

$$|k_+(\gamma'_0(0)) - k_+(\gamma'_1(0))| \ll A\epsilon^2 + A^{(5-r)/(r-1)}\epsilon + A^{-1} \leq 3\epsilon^{(r-1)/4}.$$

Hence, the curves $\widehat{W}_x^{s/u}$ are uniformly $C^{1+\frac{1}{4}(r-1)}$ which improves the estimate $C^{1+\frac{1}{2}}$ in [20, Proposition III] for all $r \in (4, 5)$. However, we do not obtain $C^{1+\text{Lip}}$.

We will say that a function $g : M_1 \rightarrow M_2$ between two metric spaces (M_1, d_1) and (M_2, d_2) has ω -logarithmic modulus of continuity ($\omega > 0$) if there is a constant $C > 0$ such that $d_2(g(x), g(y)) \leq C|\log d_1(x, y)|^{-\omega}$ for all $x, y \in M_1$ with $d_1(x, y) \leq C$.

Theorem 4.8 Assume $r > 4$. Then $x \mapsto \widehat{E}_x^{s/u}$ has ω -logarithmic modulus of continuity for all $\omega \in (0, r/2)$.

Proof As in [20], we reduce to the case where $x_0, x_1 \in T^1 S$ have the same footprint. Let $\theta \geq 0$ be the angle between them; without loss of generality $\theta < 1$. Define $T \in (0, \theta^{-1/2}]$ as in [20, Eq. (3.14)]. Without loss of generality $T > 1$.

Let $u_0(t), u_1(t)$ be the corresponding unstable Riccati solutions. As in the proof of Theorem 4.6,

$$\begin{aligned} |k_+(x_0) - k_+(x_1)| &= |u_0(0) - u_1(0)| \\ &\leq \int_{-T}^0 |K_0(t) - K_1(t)| \hat{j}_0(t) \hat{j}_1(t) dt + |u_0(-T) - u_1(-T)| \hat{j}_0(-T) \hat{j}_1(-T). \end{aligned}$$

By Corollary 4.5,

$$\begin{aligned} \int_{-T}^0 |K_0 - K_1| \hat{j}_0 \hat{j}_1 &\ll T\theta + \theta^{1/2} \int_{-T}^0 (k_+(x_0)^q + k_+(x_1)^q) \hat{j}_0 \hat{j}_1 \\ &\ll \theta^{1/2} \left\{ 1 + \int_{-T}^0 (\hat{j}'_0)^q + \int_{-T}^0 (\hat{j}'_1)^q \right\}. \end{aligned}$$

By Hölder's inequality, $\int_{-T}^0 (\hat{j}'_i)^q \leq \left(\int_{-T}^0 j'_i\right)^q T^{1-q} \ll T^{1-q} \ll \theta^{-(1-q)/2}$. Hence, $\int_{-T}^0 |K_0 - K_1| \hat{j}_0 \hat{j}_1 \ll \theta^{q/2}$ and

$$|k_+(x_0) - k_+(x_1)| \leq \theta^{q/2} + |u_0(-T) - u_1(-T)| \hat{j}_0(-T) \hat{j}_1(-T). \quad (4.3)$$

As in the proof of Theorem 4.6, we thereby obtain $|k_+(x_0) - k_+(x_1)| \ll \theta^{q/2} + T^{-1}$. If $T \geq |\log \theta|^\omega$, then we are finished. Hence we can suppose from now on that

$$1 < T < |\log \theta|^\omega. \quad (4.4)$$

If $k_+(x_i) \leq \theta^{q/4}$ for $i = 0, 1$, then again there is nothing to prove, so we can suppose without loss of generality that

$$k_+(x_0) \geq \theta^{q/4}. \quad (4.5)$$

Continuing from (4.3), we obtain

$$|k_+(x_0) - k_+(x_1)| \ll \theta^{q/2} + \exp\left\{-\int_{-T}^0 u_0\right\}.$$

By Corollary 4.2, $u_0(-t) \gg u_0(t)^p$ where $p = r/(r-2)$. Hence, our analogue of [20, Eq. (3.17)] is that there exists $\beta > 0$ such that

$$|k_+(x_0) - k_+(x_1)| \ll \theta^{q/2} + \exp\left\{-\beta \int_0^T u_0^p\right\} \leq \theta^{q/2} + \exp\left\{-\beta \int_1^T u_0^p\right\}.$$

The remainder of the argument in [20] shows that

$$\exp\left\{-\int_1^T u_0\right\} \ll k_+(x_0)^{-1} \theta^{q/2} \ll \theta^{q/4},$$

where in the second inequality we used (4.5). (The arguments in [20] are unchanged except for the derivation of [20, Eq. (3.22)]. Instead of applying [20, Lemma 3.5], the same argument used to obtain (4.3) above shows that, in their notation, $|\tilde{y}(t)| \ll \theta^{q/2} + |\tilde{y}(1)|$.) Hence

$$\int_1^T u_0 \gg |\log \theta|. \quad (4.6)$$

Let $p' = r/2$. By Hölder's inequality, $\int_1^T u_0 \leq \|u_0\|_p T^{1/p'} \leq \theta^{q/2} + \exp\left\{-\beta' |\log \theta|^{p(1-\omega/p')}\right\} = \theta^{q/2} + \exp\left\{-\beta' |\log \theta|^s\right\}$

$$\int_1^T u_0^p \geq \left(\int_1^T u_0\right)^p T^{-p/p'} \gg |\log \theta|^{p(1-\omega/p')}.$$

This implies that there is a constant $\beta' > 0$ such that

$$|k_+(x_0) - k_+(x_1)| \ll \theta^{q/2} + \exp\left\{-\beta' |\log \theta|^{p(1-\omega/p')}\right\} = \theta^{q/2} + \exp\left\{-\beta' |\log \theta|^s\right\}$$

where $s = p(1-\omega/p') > 0$ since $\omega < p' = r/2$. Using the inequality $e^{-x} \ll x^{-m}$ which holds for any fixed $m > 0$, we see (taking $m > \omega/s$) that

$$\exp\left\{-\beta' |\log \theta|^s\right\} \ll |\log \theta|^{-sm} \leq |\log \theta|^{-\omega}$$

completing the proof. ■

5 The first return map $f = g_\Upsilon : \Sigma_0 \rightarrow \Sigma_0$

In this section, we construct a first hit Poincaré map $f = g_\Upsilon : \Sigma_0 \rightarrow \Sigma_0$. The map f has unbounded excursions through the surface of revolution \mathcal{R} but is bounded elsewhere. We then obtain some hyperbolicity properties of f and prove that it satisfies the Chernov axioms listed in Appendix A, thereby showing that f is modelled by a Young tower with exponential tails [42].

5.1 The cross-section Σ_0 and flux measure μ_{Σ_0}

Roughly speaking, the section Σ_0 consists of Ω_{in} and Ω_{out} together with various local sections situated in the negatively curved region $T^1(S \setminus \mathcal{R})$. However, for technical reasons explained in [29], the cross-sections $\Omega_{\text{in/out}}$ are not suitable to be part of Σ_0 ; instead nearby cross-sections (called $\Sigma_{\text{in,out}}$ below) contained in $g_{[-\chi, \chi]} \Omega_{\text{in/out}} \subset T^1 \mathcal{R}$ (for $\chi > 0$ fixed sufficiently small) are used in the definition of Σ_0 . The cross-section Σ_0 is constructed in such a way that $\inf_{\Sigma_0} \Upsilon > 0$ and Υ is bounded on Σ_0 except in a neighbourhood of asymptotic vectors.

The precise construction of the section Σ_0 works as in [29], since here we also have that the only source of unboundedness of excursions in \mathcal{R} occurs near the subset $\{|c| = 1\} \cap \Omega_{\text{in}}$. The main properties are that (i) Σ_0 is almost perpendicular to the flow direction, and (ii) a large number of iterates of the singular set under the first return map f to Σ_0 do not have triple intersections.

Recall that μ is the smooth probability measure on M induced by the Riemannian metric on S . Locally, it is the product of the area form on S and Lebesgue measure on \mathbb{S}^1 . The *flux measure*⁵ is defined on Σ_0 by the equality $A(X) = \lim_{t \rightarrow 0} t^{-1} \mu(g_{[0,t]} X)$ for $X \subset \Sigma_0$, and thus we obtain a smooth probability measure $\mu_{\Sigma_0} = (A(\Sigma_0))^{-1} A$ on Σ_0 . By definition, given two sections transverse to the flow, the holonomy map in the flow direction from the first section to the second sends the flux measure of the first to the flux measure of the second. Therefore, μ_{Σ_0} is invariant under the first return map to Σ_0 . It is for this measure that we will check the Chernov axioms⁶.

Given a compact interval $J \subset (0, \pi)$, the section $\{-L\} \times \mathbb{S}^1 \times J$ in Clairaut coordinates is transverse to the flow. Note that $g_t(-L, \theta, \psi) = (-L + t \sin \psi, \theta + t \cos \psi, \psi)$ for $t \geq 0$ sufficiently small. Hence

$$g_{[0,t]}(B \times C) = \bigcup_{\psi \in C} B_{\psi,t} \times \{\psi\} \quad \text{for } B \subset \mathbb{S}^1 \text{ and } C \subset J,$$

where $B_{\psi,t}$ is a parallelogram of base length $|B|$ and height $t \sin \psi$. It follows by Fubini that the flux measure on $\{-L\} \times \mathbb{S}^1 \times J$ is equal to $\sin \psi d\theta d\psi$.

5.2 The first return map $f = g_\Upsilon$

The *first return time function* of Σ_0 is

$$\Upsilon : \Sigma_0 \rightarrow (0, \infty], \quad \Upsilon(x) = \inf\{t > 0 : g_t x \in \Sigma_0\}.$$

⁵This measure is usually not equal to the measure induced by the restriction of the Riemannian metric to Σ_0 .

⁶Although not explicitly stated, it was for μ_{Σ_0} that we verified the Chernov axioms in [29].

By construction, all flow trajectories intersect Σ_0 infinitely often except those forward and backward asymptotic to the cylinder \mathcal{C} . We have $\Upsilon(x) = \infty$ if and only if x is an asymptotic vector, hence $\{\Upsilon = \infty\}$ is a finite union of compact curves, each contained in $g_{[-\chi, \chi]}(\{|c| = 1\} \cap \Omega_{\text{in}})$.

The *first return map* $f = g_\Upsilon : \Sigma_0 \rightarrow \Sigma_0$ inherits the regularity of the flow g_t ; hence it is C^2 on $\Sigma_0 \setminus \{\Upsilon = \infty\}$ and it has uniformly bounded derivatives away from $\{\Upsilon = \infty\}$. Since Σ_0 is almost perpendicular to the flow direction, the hyperbolicity properties of f away from $\{\Upsilon = \infty\}$ are almost the same as those of the flow.

To maintain control near $\{\Upsilon = \infty\}$, we partition into homogeneity bands, similarly to Definition 3.1. For this, fix a sufficiently large integer n_0 (how large n_0 is will depend on a finite number of conditions, which include the validity of Lemma 5.4(4) and the verification of Chernov axiom (A8) in Theorem 5.5).

Definition 5.1 We define *homogeneity bands on Σ_0* given by $\mathcal{D}_n = \mathcal{D}_n^> \cup \mathcal{D}_n^<$, $n \geq n_0$, where

$$\begin{aligned}\mathcal{D}_n^> &= \left\{x \in \text{int}(\Sigma_0) \cap g_{[-\chi, \chi]}\Omega_{\text{in}} : 1 + \frac{1}{(n+1)^2} < |c(x)| < 1 + \frac{1}{n^2}\right\}; \\ \mathcal{D}_n^< &= \left\{x \in \text{int}(\Sigma_0) \cap g_{[-\chi, \chi]}\Omega_{\text{in}} : 1 - \frac{1}{n^2} < |c(x)| < 1 - \frac{1}{(n+1)^2}\right\}.\end{aligned}$$

Definition 5.2 The *primary singular sets* \mathcal{S}_P^\pm are defined as

$$\begin{aligned}\mathcal{S}_P^+ &= \{x \in \Sigma_0 : \Upsilon(x) < \infty \text{ and } g_{\Upsilon(x)}(x) \in \partial\Sigma_0\} \cup \{\Upsilon = \infty\}; \\ \mathcal{S}_P^- &= \{x \in \Sigma_0 : \Upsilon_-(x) > -\infty \text{ and } g_{\Upsilon_-(x)}(x) \in \partial\Sigma_0\} \cup \{\Upsilon_- = -\infty\};\end{aligned}$$

where $\Upsilon_-(x) = \sup\{t < 0 : g_t(x) \in \Sigma_0\}$. The *secondary singular sets* \mathcal{S}_S^\pm are

$$\mathcal{S}_S^+ = \bigcup_{n \geq n_0} \partial\mathcal{D}_n \quad \text{and} \quad \mathcal{S}_S^- = \{g_{\Upsilon(x)}(x) : x \in \mathcal{S}_S^+\}.$$

Let $\mathcal{S}^+ = \mathcal{S}_P^+ \cup \mathcal{S}_S^+$, and $\mathcal{S}^- = \mathcal{S}_P^- \cup \mathcal{S}_S^-$. Then $f(\text{int}(\Sigma_0) \setminus \mathcal{S}^+) = \text{int}(\Sigma_0) \setminus \mathcal{S}^-$.

Define

$$\begin{aligned}\Sigma_{\text{in}} &= \left\{x \in \Sigma_0 \cap g_{[-\chi, \chi]}\Omega_{\text{in}} : ||c(x)| - 1| < \frac{1}{n_0^2} \text{ and } fx \in g_{[-\chi, \chi]}\Omega_{\text{out}}\right\}; \\ \Sigma_{\text{out}} &= \left\{x \in \Sigma_0 \cap g_{[-\chi, \chi]}\Omega_{\text{out}} : ||c(x)| - 1| < \frac{1}{n_0^2} \text{ and } f^{-1}x \in g_{[-\chi, \chi]}\Omega_{\text{in}}\right\}.\end{aligned}$$

Then $f(\Sigma_{\text{in}}) = \Sigma_{\text{out}}$.

The maps $\mathbf{p}_{\text{in}} : \Sigma_{\text{in}} \rightarrow \Omega_{\text{in}}$ and $\mathbf{t}_{\text{in}} : \Sigma_{\text{in}} \rightarrow [-\chi, \chi]$ are defined by the equality $z = g_{\mathbf{t}_{\text{in}}(z)}[\mathbf{p}_{\text{in}}(z)]$ for $z \in \Sigma_{\text{in}}$. The maps $\mathbf{p}_{\text{out}} : \Sigma_{\text{out}} \rightarrow \Omega_{\text{out}}$ and $\mathbf{t}_{\text{out}} : \Sigma_{\text{out}} \rightarrow [-\chi, \chi]$ are defined analogously. They have the same regularity of g_t , hence they are C^2 . The map \mathbf{p}_{in} is surjective. It is also injective, because if $x \in \Sigma_{\text{in}}$ then x, fx are uniquely characterised as being the starting/ending point of the transition in \mathcal{R} . Therefore, \mathbf{p}_{in} is a bijection. Since Σ_0 and Ω are uniformly transverse to the flow direction, $\|d\mathbf{p}_{\text{in}}^{\pm 1}\| \approx 1$. By symmetry, the same properties hold for \mathbf{p}_{out} .

Recalling the definition of \mathcal{C}_n in Definition 3.1, we note that $\mathcal{C}_n = \mathbf{p}_{\text{in}}(\mathcal{D}_n)$ for all $n \geq n_0$ and $f|_{\Sigma_{\text{in}}} = \mathbf{p}_{\text{out}}^{-1} \circ f_0 \circ \mathbf{p}_{\text{in}}$.

The subspaces $E^{s/u}$ are defined in $\Sigma_{\text{in}/\text{out}}$, $\Omega_{\text{in}/\text{out}}$ as the projections of $\widehat{E}^{s/u}$ onto the respective tangent spaces, and the maps $\mathfrak{p}_{\text{in}/\text{out}}$ preserve these subspaces. Since $\|d\mathfrak{p}_{\text{in}/\text{out}}^{\pm 1}\| \approx 1$, we obtain that $\|df|_{E_x^{s/u}}\| \approx \|df_0|_{E_{\mathfrak{p}_{\text{in}}(x)}^{s/u}}\|$ for $x \in \Sigma_{\text{in}}$.

5.3 Excursion times

The first return time $\Upsilon : \Sigma_0 \rightarrow \mathbb{R}^+$ is bounded on $\Sigma_0 \setminus \Sigma_{\text{in}}$ and inherits the smoothness of the underlying flow. Here, we are interested in the regularity of the time $\Upsilon|_{\Sigma_{\text{in}}}$ taken to pass from Σ_{in} to Σ_{out} .

Recall that $2\Upsilon_0$ is the transition time from Ω_{in} to Ω_{out} . Let $\mathbf{x} = \mathbf{x}(t)$ be a bouncing/crossing geodesic undergoing an excursion in the surface of revolution \mathcal{R} . Since $f|_{\Sigma_{\text{in}}} = \mathfrak{p}_{\text{out}}^{-1} \circ f_0 \circ \mathfrak{p}_{\text{in}}$, we have the relation $\Upsilon(\mathbf{x}) = -\mathfrak{t}_{\text{in}}(x) + 2\Upsilon_0(\mathbf{x}) + \mathfrak{t}_{\text{out}}(fx)$, where x is the starting point of \mathbf{x} in Σ_{in} .

Lemma 5.3 *Let $\mathbf{x}, \bar{\mathbf{x}}$ be both bouncing or both crossing geodesics with entry vectors $x, \bar{x} \in \Sigma_{\text{in}}$ such that $\bar{x} \in W_x^{s/u}$. Then*

$$|\Upsilon(\mathbf{x}) - \Upsilon(\bar{\mathbf{x}})| \ll d(x, \bar{x}) + d(fx, f\bar{x}).$$

Proof The proof is the same as for [29, Lemma 5.11]. ■

5.4 Hyperbolicity properties of f on Σ_{in}

We now establish some hyperbolicity properties of $f|_{\Sigma_{\text{in}}}$ with respect to the δ -Sasaki metric $\|\cdot\|$ for a small $\delta > 0$. (Recall that this metric is equivalent to the Sasaki metric and also to the Clairaut metric.)

A *local unstable manifold (LUM)* is a curve $W \subset \Sigma_0$ such that f^{-n} is well-defined and smooth on W for all $n \geq 0$, and $d(f^{-n}x, f^{-n}y) \rightarrow 0$ exponentially quickly as $n \rightarrow \infty$ for all $x, y \in W$. We write W_x^u to represent an LUM containing x . Similarly, we define the notion of *local stable manifold (LSM)* and write W_x^s to represent an LSM containing x .

Lemma 5.4 *Let $r \geq 5$, $\omega \in (1, r/2)$. The following are true:*

- (1) *Lebesgue almost every $x \in \Sigma_0$ has an LSM/LUM $W_x^{s/u}$ for f .*
- (2) *For all $x, \bar{x} \in \Sigma_{\text{in}}$,*
 - (a) *There is a continuous function $a : \Sigma_{\text{in}} \rightarrow \mathbb{R}$ such that E_x^u is spanned by $\begin{bmatrix} a(x) \\ 1 \end{bmatrix}$. Moreover, $|a(x) - a(\bar{x})| \ll |\log d(x, \bar{x})|^{-\omega}$.*
 - (b) *There is a $C^{1+\text{Lip}}$ function Θ such that W_x^u is locally the graph $\{(\Theta(\psi), \psi)\}$ of Θ .*
- (3) **Growth bounds:**
 - (a) *If $x \in \mathcal{D}_n^>$, then $\|df|_{E_x^u}\| \approx n^{3-\frac{2}{r}}$.*

- (b) If $x \in \mathcal{D}_n^<$, then $\|df|_{E_x^u}\| \approx n^3$.
- (4) **Distortion bounds:** If $x, \bar{x} \in \mathcal{D}_n^>$ or $x, \bar{x} \in \mathcal{D}_n^<$, with $\bar{x} \in W_x^u$, then

$$|\log \|df|_{E_x^u}\| - \log \|df|_{E_{\bar{x}}^u}\| | \ll d(fx, f\bar{x})^{\frac{1}{3}}.$$

- (5) **Jacobian of holonomies:** If $x, \bar{x} \in \mathcal{D}_n^>$ or $x, \bar{x} \in \mathcal{D}_n^<$, with $\bar{x} \in W_x^s$, then

$$|\log \|df|_{E_x^u}\| - \log \|df|_{E_{\bar{x}}^u}\| | \ll |\log d(x, \bar{x})|^{-\omega}.$$

- Proof** (1) This is [29, Lemma 5.6], which works equally for surfaces with flat cylinder.
(2) This is [29, Lemma 5.7], which is almost unchanged here. The only difference is the regularity of a which follows from Theorem 4.8 in place of [29, Theorem 2.6].
(3) The proof is the same of [29, Lemma 5.8], noting that the estimates of ζ' are given by Lemma 3.2(3). Regarding regularity of a , only continuity is used for this estimate.
(4) This is a version of [29, Lemma 5.9], which uses Lemmas 3.2(2,3) and part (2) above. While Lemma 3.2(3) is different here, it still gives that $\zeta' \rightarrow \infty$ as $n \rightarrow \infty$ and $|\zeta''|_\infty / |\zeta'| \ll n^2$ on \mathcal{D}_n , which are the necessary ingredients to prove the distortion bounds.
(5) This is similar to [29, Lemma 5.10] but using the different regularity property of the function a in part (2) instead of Hölder continuity in [29, Lemma 5.7(1)]. ■

We now have all the ingredients to prove that f satisfies the Chernov axioms (A1)–(A8) listed in Appendix A.

Theorem 5.5 *The first return map f satisfies the Chernov axioms (A1)–(A8).*

Proof The verifications are essentially identical to those in [29, Section 6]. The main difference is that the last line of the argument for (A7) becomes

$$\begin{aligned} |\log JH(x)| &\ll \sum_{i=0}^{\infty} |\log d(f^{-i}x, f^{-i}H(x))|^{-\omega} \leq \sum_{i=0}^{\infty} |\log \{\Lambda^{-i}d(x, H(x))\}|^{-\omega} \\ &= \sum_{i=0}^{\infty} (i \log \Lambda + \log d(x, H(x))^{-1})^{-\omega} \ll 1 + \sum_{i=1}^{\infty} i^{-\omega} < \infty. \end{aligned}$$

6 Nonstandard CLT for exponential Young towers

As shown in [30], the key step in proving the nonstandard weak invariance principle in Theorem B is to establish a nonstandard CLT for certain piecewise constant observables (called JR_0 below) related to a discrete first return time function. In this section we recall, and extend slightly, an abstract argument due to [3] for proving such a nonstandard CLT.

Let (Z, μ_Z) be a probability space with an at most countable measurable partition $\{Z_j : j \geq 1\}$ and let $F : Z \rightarrow Z$ be an ergodic measure-preserving map. Define the separation time $s(z, z')$ to be the least integer $n \geq 0$ such that $F^n z$ and $F^n z'$ lie in distinct

partition elements. We assume that $s(z, z') = \infty$ if and only if $z = z'$; then $d_\theta(z, z') = \theta^{s(z, z')}$ is a metric for $\theta \in (0, 1)$.

We say that $F : Z \rightarrow Z$ is a (*full-branch*) *Gibbs-Markov map* if (i) $F : Z_j \rightarrow Z$ is a measure-theoretic bijection for each $j \geq 1$; (ii) there exists $\theta \in (0, 1)$ such that $\log \xi$ is d_θ -Lipschitz, where $\xi = d\mu_Z/(d\mu_Z \circ F)$.

Let $\tau : Z \rightarrow \mathbb{Z}^+$ be piecewise constant (constant on each Z_j) with $\mu_Z(\tau > n) = O(e^{-cn})$ for some $c > 0$. We define the *one-sided exponential Young tower* $\Delta = \{(z, \ell) : z \in Z, 0 \leq \ell < \tau(z)\}$ and tower map

$$f_\Delta : \Delta \rightarrow \Delta, \quad f_\Delta(z, \ell) = \begin{cases} (z, \ell + 1), & 0 \leq \ell < \tau(z) - 1 \\ (Fz, 0), & \ell = \tau(z) - 1. \end{cases}$$

An ergodic f_Δ -invariant probability measure is given by $\mu_\Delta = (\mu_Z \times \text{counting})/\bar{\tau}$ where $\bar{\tau} = \int_Z \tau d\mu_Z$.

An observable $V : \Delta \rightarrow \mathbb{R}$ is called *piecewise constant* if V is constant on each partition element $Z_j \times \{\ell\}$ where Z_j is a partition element for the Gibbs-Markov map $F : Z \rightarrow Z$ and $0 \leq \ell < \tau|_{Z_j}$.

It is assumed that there is a distinguished integrable piecewise constant observable $R_0 : \Delta \rightarrow \mathbb{Z}^+$. We are interested in piecewise constant observables JR_0 where $J : \Delta \rightarrow \mathbb{R}$ is bounded and piecewise constant.

Theorem 6.1 *Let $\{\alpha_i : i \in I\} \subset \mathbb{R}$ be a list of the values attained by J . Let $\sigma_i \geq 0$ be constants, $i \in I$, such that $\sum_{i \in I} \sigma_i^2 \in (0, \infty)$. Assume that there are constants $C > 0$, $\epsilon \in (0, 1)$ such that*

$$\sum_{i \in I} |\mu_\Delta(R_0 = n, J = \alpha_i) - 2\sigma_i^2 n^{-3}| \leq Cn^{-(3+\epsilon)} \quad \text{for all } n \geq 1, \quad (6.1)$$

and

$$\mu_\Delta(R_0 = k, R_0 \circ f_\Delta^n = \ell) \leq Ck^{-(2+\epsilon)} \ell^{-(2+\epsilon)} \quad \text{for all } k, \ell, n \geq 1. \quad (6.2)$$

Then JR_0 satisfies a nonstandard CLT with variance $\sigma_J^2 = \sum_{i \in I} \alpha_i^2 \sigma_i^2 \geq 0$. That is, $(n \log n)^{-1/2} \sum_{j=0}^{n-1} (JR_0) \circ f_\Delta^j \rightarrow_d N(0, \sigma_J^2)$.

In the remainder of this section, we prove Theorem 6.1 following [3]. Let $\sigma_{R_0}^2 = \sum_{i \in I} \sigma_i^2 \in (0, \infty)$.

Proposition 6.2 $\mu_\Delta(R_0 = n) = 2\sigma_{R_0}^2 n^{-3} + O(n^{-(3+\epsilon)})$.

Proof This is immediate from condition (6.1). ■

Proposition 6.3 $\int_{R_0 \leq p} (JR_0)^2 d\mu_\Delta = 2\sigma_J^2 \log p + O(1)$ for $p \geq 1$.

Proof By condition (6.1),

$$\begin{aligned} \int_{R_0 \leq p} (JR_0)^2 d\mu_\Delta &= \sum_{m=1}^p \sum_{i \in I} \alpha_i^2 m^2 \mu_\Delta(R_0 = m, J = \alpha_i) \\ &= \sum_{m=1}^p m^2 \left(\sum_{i \in I} 2\alpha_i^2 \sigma_i^2 m^{-3} + O(m^{-(3+\epsilon)}) \right) = 2\sigma_J^2 \log p + O(1) \end{aligned}$$

as required. \blacksquare

For $1 \leq d \leq e \leq \infty$, define

$$A_{d,e} = \left(JR_0 - \frac{1}{\mu_\Delta(d \leq R_0 < e)} \int_{d \leq R_0 < e} JR_0 d\mu_\Delta \right) 1_{\{d \leq R_0 < e\}}.$$

Lemma 6.4 *There are constants $C > 0$ and $\gamma \in (0, 1)$, independent of n, d, e , such that*

$$\left| \int_{\Delta} A_{d,e} (A_{d',e'} \circ f_{\Delta}^n) d\mu_{\Delta} \right| \leq C\gamma^n$$

for all $n \geq 1$, $1 \leq d \leq e \leq \infty$, $1 \leq d' \leq e' \leq \infty$.

Proof We claim that

$$|A_{d,e} 1_{\{R_0=k\}}| \ll k. \quad (6.3)$$

We begin by verifying this claim.

For the first term in $A_{d,e} 1_{\{R_0=k\}}$, we have $|JR_0 1_{\{R_0=k\}}| \leq |J|_{\infty} k$. The remaining term is bounded by $|J|_{\infty} B$, where

$$B = 1_{\{R_0=k, d \leq R_0 < e\}} \mu_{\Delta}(d \leq R_0 < e)^{-1} \int_{d \leq R_0 < e} R_0 d\mu_{\Delta}.$$

Note that if $B \neq 0$, then $d \leq k < e$, so we can suppose throughout that $d \leq k < e$. Also, $B \in [d, e]$ so for $e \leq 2d$ we obtain $B \leq e \leq 2d \leq 2k$. Hence we can suppose that $e > 2d$.

By Proposition 6.2,

$$\int_{d \leq R_0 < e} R_0 d\mu_{\Delta} \leq \sum_{m \geq d} m \mu_{\Delta}(R_0 = m) \ll d^{-1}.$$

Also, there exists $m_0 \geq 1$ such that $\mu_{\Delta}(R_0 = m) > \sigma_{R_0}^2 m^{-3}$ for all $m \geq m_0$. Hence, for $d \geq m_0$, we have $\mu_{\Delta}(d \leq R_0 < e) \gg d^{-2} - e^{-2} \geq \frac{3}{4}d^{-2}$, and we obtain $B \ll d \leq k$ as required.

On the other hand, if $d \leq m_0 < e$, then $\mu_{\Delta}(d \leq R_0 < e) \geq \mu_{\Delta}(R_0 = m_0) > 0$ and we obtain $B \ll d^{-1} \leq 1 \leq k$. This leaves the case where $e \leq m_0$ for which we have the trivial bound $B \leq m_0 \leq m_0 k$. This completes the proof of estimate (6.3).

Now set $B_1 = A_{d,e}$, $B_2 = A_{d',e'}$. By (6.3),

$$|B_1 1_{\{R_0 \leq N\}}| \ll N. \quad (6.4)$$

Since $\int B_i d\mu_\Delta = 0$,

$$\int_{\Delta} B_1 1_{\{R_0 \leq N\}} (B_2 1_{\{R_0 \leq N\}}) \circ f_{\Delta}^n d\mu_{\Delta} = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int_{\Delta} (B_1 1_{\{R_0 \leq N\}} - \int B_1 1_{\{R_0 \leq N\}} d\mu_{\Delta}) (B_2 1_{\{R_0 \leq N\}} - \int B_2 1_{\{R_0 \leq N\}} d\mu_{\Delta}) \circ f_{\Delta}^n d\mu_{\Delta}, \\ I_2 &= \int_{\Delta} B_1 1_{\{R_0 > N\}} d\mu_{\Delta} \int_{\Delta} B_2 1_{\{R_0 > N\}} d\mu_{\Delta}. \end{aligned}$$

The functions $B_i 1_{\{R_0 \leq N\}}$ are piecewise constant. Hence it follows from (6.4) together with exponential decay of correlations for dynamically Hölder observables that there exists $c > 0$ such that

$$|I_1| \ll e^{-cn} |B_1 1_{\{R_0 \leq N\}}|_{\infty} |B_2 1_{\{R_0 \leq N\}}|_{\infty} \ll e^{-cn} N^2.$$

Also, by (6.4) and Proposition 6.2,

$$\left| \int B_i 1_{\{R_0 > N\}} d\mu_{\Delta} \right| \leq \sum_{k>N} \int |B_i| 1_{\{R_0=k\}} d\mu_{\Delta} \ll \sum_{k>N} k \mu_{\Delta}(R_0 = k) \ll N^{-1}.$$

Hence, $|I_2| \ll N^{-2}$ and

$$\left| \int_{\Delta} B_1 1_{\{R_0 \leq N\}} (B_2 1_{\{R_0 \leq N\}}) \circ f_{\Delta}^n d\mu_{\Delta} \right| \ll e^{-cn} N^2 + N^{-2}. \quad (6.5)$$

Next, by condition (6.2) and estimate (6.4),

$$\begin{aligned} \left| \int_{\Delta} B_1 1_{\{R_0 \leq N\}} (B_2 1_{\{R_0 > N\}}) \circ f_{\Delta}^n d\mu_{\Delta} \right| &\leq \sum_{k=1}^N \sum_{\ell=N+1}^{\infty} \left| \int_{\Delta} B_1 1_{\{R_0=k\}} (B_2 1_{\{R_0=\ell\}}) \circ f_{\Delta}^n d\mu_{\Delta} \right| \\ &\ll \sum_{k=1}^N \sum_{\ell=N+1}^{\infty} k \ell \mu_{\Delta}(R_0 = k, R_0 \circ f_{\Delta}^n = \ell) \\ &\ll \sum_{k=1}^N \sum_{\ell=N+1}^{\infty} k^{-(1+\epsilon)} \ell^{-(1+\epsilon)} \ll N^{-\epsilon}. \end{aligned}$$

Similarly,

$$\left| \int_{\Delta} B_1 1_{\{R_0 > N\}} B_2 \circ f_{\Delta}^n d\mu_{\Delta} \right| \ll \sum_{k=N+1}^{\infty} \sum_{\ell=1}^{\infty} k \ell \mu_{\Delta}(R_0 = k, R_0 \circ f_{\Delta}^n = \ell) \ll N^{-\epsilon}.$$

Combining these with (6.5) and taking $N = [e^{cn/(2+\epsilon)}]$, we obtain that

$$\left| \int_{\Delta} B_1 (B_2 \circ f_{\Delta}^n) d\mu_{\Delta} \right| \ll e^{-cn} N^2 + N^{-\epsilon} \ll e^{-\epsilon cn/(2+\epsilon)},$$

as required. ■

Proof of Theorem 6.1 Following [3, Section 3], we take $p = n^{1/2}(\log n)^{-11}$ and $q = n^{1/2} \log \log n$. Using Proposition 6.2, the nonstandard CLT for JR_0 reduces to the nonstandard CLT for $JR_0 1_{\{R_0 \leq q\}}$. Then, using Lemma 6.4, it is shown in [3, Section 3] how to reduce further from $JR_0 1_{\{R_0 \leq q\}}$ to $\hat{A} = JR_0 1_{\{R_0 \leq q\}} - A_{p,q}$.

Finally, the nonstandard CLT for \hat{A} follows by the argument in [3, Section 4]. The ingredients are:

- An exponential multiple decorrelations bound for dynamically Hölder observables ([3, Lemma 4.1]) which is automatically satisfied for systems modelled by Young towers with exponential tails [10, Theorem 7.4.1];
- A locally Hölder estimate on JR_0 ([3, Lemma 4.2]) which is trivially satisfied here (with $d = 1$) since JR_0 is piecewise constant;
- Proposition 6.3 (which corresponds to [3, Lemma 4.3]). ■

If we already know that (f_Δ, μ_Δ) satisfies the Chernov axioms, then checking condition (6.2) can be done following closely the argument in the proof of [3, Lemma 3.2]:

Lemma 6.5 *Assume that (f_Δ, μ_Δ) satisfies the Chernov axioms and that there are constants $a, b, c \geq 0$ with $a \geq c$ such that:*

- (i) *$\{R_0 = k\}$ is foliated by unstable curves of size $\approx k^{-a}$ and by stable curves of size $O(k^{-b})$.*
- (ii) *(Growth-type lemma) f_Δ expands unstable curves inside $\{R_0 = k\}$ by a factor $\approx k^c$.*

Then $\mu_\Delta(R_0 = k, R_0 \circ f_\Delta^n = \ell) \ll k^{-(b+c)} \ell^{-a}$ for all $k, \ell, n \geq 1$.

Proof As mentioned, we follow [3, Lemma 3.2], noting that our set $\{R_0 = k\}$ is equivalent to their set \mathcal{M}_k . By (ii), f_Δ expands $\{R_0 = k\}$ in the unstable direction by a factor of the order of k^c , hence by (i), $f_\Delta \{R_0 = k\}$ is a set foliated by unstable curves of size $\approx k^{c-a}$. Therefore the restriction of μ_Δ to $f_\Delta \{R_0 = k\}$ can be represented by a standard family \mathcal{G}_k with $Z(\mathcal{G}_k) = O(k^{a-c})$; see [10, Sect. 7.4] for the definitions and basic properties of standard families and Z -functions. In particular, the Z -functions of the images $f_\Delta^n \mathcal{G}_k$ are also standard families and satisfy a Lasota-Yorke type inequality

$$Z(f_\Delta^n \mathcal{G}_k) \leq c_1 \vartheta^n Z(\mathcal{G}_k) + c_2$$

for some constants $c_1, c_2 > 0$ and $\vartheta \in (0, 1)$. Hence $Z(f_\Delta^n \mathcal{G}_k) = O(k^{a-c})$ uniformly in k, n .

Denote the u -size of $\{R_0 = \ell\}$ by $u_\ell \approx \ell^{-a}$. By [10, Sect. 7.4],

$$\mu_\Delta(R_0 = k, R_0 \circ f_\Delta^n = \ell) \ll \mu_\Delta(R_0 = k) \cdot Z(f_\Delta^n \mathcal{G}_k) \cdot u_\ell = O(k^{-(b+c)} \ell^{-a})$$

as required. ■

7 Nonstandard WIP for surfaces with flat cylinder

In this section, we prove Theorems A and B. In particular, we prove a nonstandard WIP for the geodesic flow $g_t : M \rightarrow M$ on a surface S with flat cylinder for all $r \geq 5$.

In Subsection 7.1, we construct a new first return map $g = g_h : \Sigma \rightarrow \Sigma$ where the first return time function $h : \Sigma \rightarrow \mathbb{R}^+$ is bounded above and below and Hölder. To distinguish it from f , we call g the *global Poincaré map*. In Subsection 7.2, we obtain asymptotics for the tail of the discrete first return time function $R : \Sigma_0 \rightarrow \mathbb{Z}^+$ to Σ_0 under g . In Subsection 7.3, we complete the proof of Theorems A and B.

7.1 Global Poincaré map $g = g_h : \Sigma \rightarrow \Sigma$ with bounded h

In Section 5, we defined a uniformly hyperbolic first return map $f = g_\Upsilon : \Sigma_0 \rightarrow \Sigma_0$. The return time $\Upsilon : \Sigma_0 \rightarrow (0, \infty) \cup \{\infty\}$ satisfies $\inf \Upsilon > 0$ but is unbounded.

As in [29], we adjoin a section $\Omega_0 \subset T^1\mathcal{R}$ defined in a neighbourhood of the curve $\Xi([-L, L] \times \{0\})$ given by a disjoint union of two open disks

$$\Omega_0 = (-L - \chi, L + \chi) \times \{0\} \times \{(-\chi, \chi) \cup (\pi - \chi, \pi + \chi)\}.$$

Define $\Sigma = \Sigma_0 \cup \Omega_0$. In this way, we obtain a global Poincaré map

$$g = g_h : \Sigma \rightarrow \Sigma, \quad g(x) = g_{h(x)}(x),$$

which is no longer uniformly hyperbolic but for which the return time $h : \Sigma \rightarrow (0, \infty)$ is bounded above and Hölder with $\inf h > 0$. We note that h coincides with Υ on $\Sigma_0 \setminus \Sigma_{\text{in}}$. As in Section 7.1, we define the normalised flux measure μ_Σ on Σ , giving a g -invariant probability measure.

7.2 Discrete first return time function $R : \Sigma_0 \rightarrow \mathbb{Z}^+$

Write $f = g^R$ where

$$R : \Sigma_0 \rightarrow \mathbb{Z}^+, \quad R(x) = \inf\{n \geq 1 : g^n x \in \Sigma_0\}.$$

Then R is bounded on $\Sigma_0 \setminus \Sigma_{\text{in}}$ but unbounded on Σ_{in} ; indeed $R = \Upsilon + O(1)$ since h is bounded above and below. Moreover,

$$R = R_{\mathcal{C}} + R_{\mathcal{N}} + O(1) = R_{\mathcal{C}} + O(R_{\mathcal{N}})$$

where $R_{\mathcal{C}}$ is the transition time in the cylinder \mathcal{C} and $R_{\mathcal{N}}$ is the time spent in the neck \mathcal{N} .

Recall from Section 5.1 the notion of flux measure A which defines the f -invariant probability measure $\mu_{\Sigma_0} = \frac{1}{A(\Sigma_0)}A$ on Σ_0 . The dynamical definition implies that if we are given two sections, then the holonomy map between sections sends the flux measure on the first section to the flux measure on the second section. Set

$$\sigma_R^2 = \frac{4L^2}{A(\Sigma_0)\pi}.$$

Lemma 7.1 (1) $\mu_{\Sigma_0}(R_{\mathcal{C}} = n) = 2\sigma_R^2 n^{-3} + O(n^{-4})$.

(2) There exists $p > 2$ such that $R - R_C \in L^p$.

Proof (1) Clearly, $R_C = 0$ in the bouncing case. In the crossing case, there are four possibilities: trajectories may traverse the cylinder from left to right or vice versa and may spiral clockwise or counterclockwise. Since geodesics are spiralling around a cylinder of length $2L$ and circumference 2π , we see that $R_C = n$ if and only if $\tan \psi \in \pm I$ where $I = \left(\frac{L}{(n+1)\pi}, \frac{L}{n\pi} \right]$. Recalling that the flux measure on $\{-L, L\} \times \mathbb{S}^1 \times (0, \pi)$ is $\sin \psi d\theta d\psi$,

$$\mu_{\Sigma_0}(R_C = n) = \frac{A(R_C = n)}{A(\Sigma_0)} = \frac{1}{A(\Sigma_0)} \cdot 4 \cdot 2\pi \cdot \int_{\tan \psi \in I} \sin \psi d\psi = \frac{8L^2}{A(\Sigma_0)\pi} \frac{1}{n^3} + O(n^{-4}).$$

(2) Let Υ_N be the time spent by a geodesic in the neck N during an excursion in the surface of revolution \mathcal{R} , (so $\Upsilon_N = 2\Upsilon_1$ in Lemma 3.2). By [29, Lemma 5.4], $\mu_{\Sigma_0}(\Upsilon_N > n) \approx n^{-2r/(r-2)}$. Hence $R_N = \Upsilon_N + O(1) \in L^p$ for all $p < 2r/(r-2)$. ■

Lemma 7.2 $\mu_{\Sigma_0}(R_C = k, R_C \circ f^n = \ell) \ll k^{-3}\ell^{-3}$ for all $k, \ell, n \geq 1$.

Proof By Theorem 5.5, $f : \Sigma_0 \rightarrow \Sigma_0$ satisfies the Chernov axioms. Hence we are able to apply Lemma 6.5. We take $b = 0$. It remains to show that we can take $a = c = 3$ in Lemma 6.5.

For geodesics starting in $\mathcal{C}_n^<$, we have $\cos \psi = 1 - n^{-2} + O(n^{-3})$, so $\tan \psi = \sqrt{2}n^{-1} + O(n^{-2})$. Since geodesics are spiralling around a cylinder of length $2L$ and circumference 2π , it holds that $|R_C - \frac{L}{\pi \tan \psi}| \leq 1$. Hence $R_C|_{\mathcal{C}_n^<} = \frac{L}{\sqrt{2}\pi}n + O(1)$. It follows that $\{R_C = k\}$ is contained in a union of homogeneity bands $\mathcal{C}_n^<$ with $n \approx k$.

By Lemma 3.2(3), $|\zeta'(\psi)| \approx n^3 \approx k^3$ on $\mathcal{C}_n^<$. This verifies that $c = 3$ in Lemma 6.5. By Lemma 3.2(3), unstable curves are aligned parallel to the ψ -axis. Hence, by definition of \mathcal{C}_n , unstable curves have size $\approx n^{-3} \approx k^{-3}$ in \mathcal{C}_n . This verifies that $a = 3$ in Lemma 6.5. ■

By Theorem 5.5, the uniformly hyperbolic first return map $f : \Sigma_0 \rightarrow \Sigma_0$ satisfies the Chernov axioms and hence, by [29, Theorem 3.1], is modelled by a Young tower with exponential tails [42]. The discrete return time R lifts to the tower and is piecewise constant, so is well-defined and piecewise constant on the one-sided Young tower $(\Delta, f_\Delta, \mu_\Delta)$ as in Section 6.

Corollary 7.3 $R : \Delta \rightarrow \mathbb{Z}^+$ satisfies a nonstandard CLT on $(\Delta, f_\Delta, \mu_\Delta)$. That is, $(n \log n)^{-1/2}(\sum_{j=0}^{n-1} R \circ f_\Delta^j - n\bar{R}) \rightarrow_d N(0, \sigma_R^2)$ where $\bar{R} = \int_\Delta R d\mu_\Delta$.

Proof By Lemma 7.1(1), condition (6.1) holds (with $J \equiv 1$, $\epsilon = 1$ and $R_0 = R_C$). Similarly, condition (6.2) holds by Lemma 7.2. It thereby follows from Theorem 6.1 that R_C satisfies a nonstandard CLT with variance σ_R^2 .

The following standard argument shows that this limit theorem for R_C is inherited by R . By Lemma 7.1(2), $R = R_C + H$ where $H \in L^p$ for some $p > 2$. Let $H_\tau = \sum_{\ell=0}^{\tau-1} H \circ f_\Delta^\ell : Z \rightarrow \mathbb{R}$. Since τ has exponential tails, $R_\tau \in L^q$ for $q \in (2, p)$. Also, R_τ is piecewise constant so it is well-known (see for example [32, Corollary 2.5]) that R_τ satisfies a standard CLT on (F, Z, μ_Z) (with normalisation $n^{1/2}$). By [34], H satisfies a standard CLT on $(f_\Delta, \Delta, \mu_\Delta)$. Hence $(n \log n)^{-1/2}(\sum_{j=0}^{n-1} H \circ f_\Delta^j - n\bar{H}) \rightarrow_d 0$ where $\bar{H} = \int_\Delta H d\mu_\Delta$. The result follows. ■

7.3 Proof of (non)standard CLT and WIP for the flow

In this subsection, we complete the proof of Theorems A and B. Let $v : M \rightarrow \mathbb{R}$ be a Hölder observable with $\int_M v d\mu = 0$. Define induced observables

$$v_h : \Sigma \rightarrow \mathbb{R}, \quad V : \Sigma_0 \rightarrow \mathbb{R},$$

$$v_h = \int_0^h v \circ g_t dt, \quad V = \sum_{j=0}^{R-1} v_h \circ g^j.$$

Define

$$\alpha_0 = \frac{1}{2L} \int_{\mathbb{S}^1} \int_{-L}^L v(s, \theta, 0) ds d\theta, \quad \alpha_\pi = \frac{1}{2L} \int_{\mathbb{S}^1} \int_{-L}^L v(s, \theta, \pi) ds d\theta. \quad (7.1)$$

Lemma 7.4 *There exists $p > 2$ such that*

$$V - JR_C \in L^p$$

where J takes the values α_0 and α_π each with probability $\frac{1}{2}$ on $(\Sigma_0, \mu_{\Sigma_0})$.

Proof Since v and h are Hölder and g_t is smooth, there exists $\eta > 0$ such that v_h is C^η . We claim that $V = JR_C + O(R_C^{1-\eta}) + O(R_N)$. By Lemma 7.1(2), $R_N \in L^p$ for some $p > 2$. By Lemma 7.1(1), $R_C \in L^q$ for all $q < 2$ so $R_C^{1-\eta} \in L^p$ for some (possibly smaller) $p > 2$, proving the result.

It remains to prove the claim. Recall that the section $\Omega_0 \subset \mathcal{R}$ is contained in $\{\theta = 0\}$ and is a neighbourhood therein of $\{s \in [-L, L], \psi = 0\} \cup \{s \in [-L, L], \psi = \pi\}$. Accordingly, we write $\Omega_0 \cap \mathcal{C} = \Omega_{0,0} \cup \Omega_{0,\pi}$.

For $\chi > 0$ chosen small enough in the definition of Ω_0 , we can ensure that geodesics crossing \mathcal{R} can intersect at most one of $\Omega_{0,0}$ and $\Omega_{0,\pi}$. For $p \in \Omega_{0,0}$, let p_j , $j = 0, \dots, R_C$, be the iterates of p under g that lie in $\Omega_{0,0}$. Then $p_j = (s_j, 0, \psi_0)$ where

$$\tan \psi_0 = L(\pi R_C)^{-1} + O(R_C^{-2}),$$

and $R_C = [L/(\pi \tan \psi_0)]$. Moreover, $s_j = s_0 + 2\pi j \tan \psi_0$ for each $j = 1, \dots, R_C$.

Since v_h is bounded,

$$V(p) = \sum_{j=1}^{R_C} v_h(p_j) + O(R_N).$$

Since v_h is C^η ,

$$|v_h(p_j) - v_h(s_j, 0, 0)| = |v_h(s_j, 0, \psi_0) - v_h(s_j, 0, 0)| \ll |\psi_0|^\eta \ll R_C^{-\eta}.$$

It follows that

$$V(p) = \sum_{j=1}^{R_C} v_h(s_j, 0, 0) + O(R_C^{1-\eta}) + O(R_N)$$

so it remains to estimate $\sum_{j=1}^{R_C} v_h(s_j, 0, 0)$.

Let \bar{S} and \underline{S} be upper and lower Riemann sums for $\int_{-L}^L v_h(s, 0, 0) ds$ with respect to the subdivision $-L \leq s_0 < \dots < s_{R_C} \leq L$. Since $s_j - s_{j-1} \ll R_C^{-1}$ and v_h is C^η , $\bar{S} - \underline{S} \ll R_C^{-\eta}$. Moreover,

$$\underline{S} \leq \int_{-L}^L v_h(s, 0, 0) ds \leq \bar{S}, \quad \underline{S} \leq \sum_{j=1}^{R_C} v_h(s_j, 0, 0)(s_j - s_{j-1}) \leq \bar{S},$$

so

$$\left| \int_{-L}^L v_h(s, 0, 0) ds - \sum_{j=1}^{R_C} v_h(s_j, 0, 0)(s_j - s_{j-1}) \right| \leq \bar{S} - \underline{S} \ll R_C^{-\eta}.$$

We deduce that

$$\begin{aligned} \sum_{j=1}^{R_C} v_h(s_j, 0, 0) &= (2\pi \tan \psi_0)^{-1} \sum_{j=1}^{R_C} v_h(s_j, 0, 0)(s_j - s_{j-1}) \\ &= (2\pi \tan \psi_0)^{-1} \left[\int_{-L}^L v_h(s, 0, 0) ds + O(R_C^{-\eta}) \right] \\ &= R_C \frac{1}{2L} \int_{-L}^L v_h(s, 0, 0) ds + O(R_C^{1-\eta}). \end{aligned}$$

Moreover,

$$v_h(s, 0, 0) = \int_0^{h(s, 0, 0)} (v \circ g_t)(s, 0, 0) dt = \int_{\mathbb{S}^1} v(s, \theta, 0) d\theta,$$

completing the estimate for geodesics intersecting $\Omega_{0,0}$. By time-reversibility, the calculation for geodesics intersecting $\Omega_{0,\pi}$ is identical and the two cases are equally likely. \blacksquare

As in the introduction, we set $I_v = \alpha_0^2 + \alpha_\pi^2$. Define

$$\sigma_J^2 = \frac{1}{2} \sigma_R^2 I_v, \quad \sigma_v^2 = \bar{h}^{-1} \bar{R}^{-1} \sigma_J^2.$$

where $\bar{R} = \int_{\Sigma_0} R d\mu_{\Sigma_0}$ and $\bar{h} = \int_{\Sigma} h d\mu_{\Sigma}$.

Proof of Theorems A and B Since f is modelled by a Young tower with exponential tails, we can apply [30, Theorem 5.1] (with M , Σ and Σ_0 playing the roles of Γ^h , Γ and Δ respectively).

By Lemma 7.1,

$$\mu_{\Sigma_0}(R_C = n, J = \alpha_0) = \mu_{\Sigma_0}(R_C = n, J = \alpha_\pi) = \frac{1}{2} \mu_{\Sigma_0}(R_C = n) = \sigma_R^2 n^{-3} + O(n^{-4}).$$

Hence, condition (6.1) holds (with $\epsilon = 1$ and $R_0 = R_C$). Similarly, condition (6.2) holds by Lemma 7.2. It thereby follows from Theorem 6.1 that $J R_C$ satisfies a nonstandard CLT with variance $\sigma_J^2 \geq 0$. Also, R satisfies a nonstandard CLT with variance σ_R^2 by Corollary 7.3. By Lemma 7.4, the hypotheses of [30, Theorem 5.1] are satisfied.

Notice that $I_v = 0$ if and only if $J \equiv 0$, equivalently $\sigma_J^2 = 0$. If $I_v \neq 0$, then we are in the situation of [30, Theorem 5.1(a)], completing the proof of Theorem B. This implies the nonstandard CLT in the first statement of Theorem A.

If $I_v = 0$, then we are in the situation of [30, Theorem 5.1(b)], proving the standard CLT in the second statement of Theorem A. \blacksquare

8 Decay of correlations for surfaces with flat cylinder

In this section, we consider decay of correlations for the geodesic flow $g_t : M \rightarrow M$ and the global Poincaré map $g : \Sigma \rightarrow \Sigma$. The general setup is the same as in [29, Section 7]. We recall the basics here.⁷

- The flow g_t and map g are related by the identity $g = g_h$ where $h : \Sigma \rightarrow \mathbb{R}^+$ is a Hölder roof function with $0 < \inf h < \sup h < \infty$.
- The first return maps $g : \Sigma \rightarrow \Sigma$ and $f : \Sigma_0 \rightarrow \Sigma_0$ are related by $f = g^R$ where $R : \Sigma_0 \rightarrow \mathbb{Z}^+$ is a discrete first return time.
- The uniformly hyperbolic first return map $f : \Sigma_0 \rightarrow \Sigma_0$ is modelled by a Young tower with exponential tails [42]: there is a subset $Y \subset \Sigma_0$ with product structure and bounded distortion, a probability measure μ_Y on Y , and a “good” return time $\tau : Y \rightarrow \mathbb{Z}^+$ constant along stable leaves with $\mu_Y(\tau > n) = O(e^{-cn})$ for some $c > 0$, such that $f^\tau : Y \rightarrow Y$ is measure-preserving. Moreover, quotienting along stable leaves yields a full-branch Gibbs-Markov map $F : Z \rightarrow Z$ as in Section 6, and τ is constant on the associated partition elements.
- The global Poincaré map $g : \Sigma \rightarrow \Sigma$ is modelled by a subexponential Young tower over $g^{\varphi_*} = (g^R)^\tau = f^\tau : Y \rightarrow Y$ where the return time $\varphi_* : Y \rightarrow \mathbb{Z}^+$ is given by $\varphi_* = \sum_{\ell=0}^{\tau-1} R \circ f^\ell$.
- We can regard $f^\tau : Y \rightarrow Y$ as a (non-first return) Poincaré map for the flow g_t with roof function $\varphi = h_{\varphi_*} = \sum_{\ell=0}^{\varphi_*-1} h \circ g^\ell$.

By Lemma 7.1(1), we have precise asymptotics for the tails of $R_C : \Sigma_0 \rightarrow \mathbb{Z}^+$. In particular, $\mu_{\Sigma_0}(R_C > n) \sim \sigma_R^2 n^{-2}$. Usually, it is difficult to leverage this into precise asymptotics for $\varphi_* : Y \rightarrow \mathbb{Z}^+$. However, here we can exploit the fact that R satisfies a nonstandard limit law by following ideas in [22, 35].

Proposition 8.1 $\mu_Y(\varphi_* > n) \sim \bar{\tau} \sigma_R^2 n^{-2}$.

Proof By Corollary 7.3, R satisfies a nonstandard CLT with variance $\sigma_R^2 > 0$. Now apply [30, Lemma 4.1]. ■

We can now state and prove our result on upper and lower bounds on decay of correlations for the global Poincaré map g as advertised in Remark 1.3.

Theorem 8.2 *For all Hölder observables $v, w : \Sigma \rightarrow \mathbb{R}$, there is a constant $C > 0$ such that*

$$\left| \int_{\Sigma} v \cdot (w \circ g^n) d\mu_{\Sigma} - \int_{\Sigma} v d\mu_{\Sigma} \int_{\Sigma} w d\mu_{\Sigma} \right| \leq C n^{-1} \quad \text{for all } n \geq 1.$$

Moreover, for all Hölder observables $v, w : \Sigma \rightarrow \mathbb{R}$ supported in Σ_0 ,

$$\int_{\Sigma} v \cdot (w \circ g^n) d\mu_{\Sigma} - \int_{\Sigma} v d\mu_{\Sigma} \int_{\Sigma} w d\mu_{\Sigma} \sim \bar{\tau} \sigma_R^2 n^{-1} \int_{\Sigma} v d\mu_{\Sigma} \int_{\Sigma} w d\mu_{\Sigma} \quad \text{for all } n \geq 1.$$

⁷We caution that for example τ here was called σ in [29].

Proof Recall that the global Poincaré map $g : \Sigma \rightarrow \Sigma$ is modelled by a (subexponential) two-sided Young tower Γ with return time $\varphi_* : Y \rightarrow \mathbb{Z}^+$. By Proposition 8.1, $\mu_Y(\varphi_* > n) \sim \bar{\tau}\sigma_R^2 n^{-2}$. Hence the upper bound follows from [43] (a complete argument can be found in [27, Theorem 2.10] or [33, Appendix B]) and the lower bound from [8, Theorem 7.4]. ■

Lemma 8.3 *There is a constant $C > 0$ such that $|\varphi(y) - \varphi(y')| \leq Cd(y, y')$ for $y, y' \in Y$ with $y' \in W_y^s$, and $|\varphi(y) - \varphi(y')| \leq Cd(f^\tau y, f^\tau y')$ for $y, y' \in Y$ with $y' \in W_y^u$.*

Proof This follows from Lemma 5.3 in the same way that [29, Lemma 7.1] follows from [29, Lemma 5.11]. ■

Proof of Theorem C In the terminology of [2, Section 6], we have shown that g_t is a “Gibbs-Markov flow”. Their condition (H) follows from Lemma 8.3 by [2, Lemma 8.3]. As explained in [29, Section 7], the “absence of approximate eigenfunctions” condition in [2] is automatic due to the fact that the geodesic flow g_t has a contact structure. Since h is bounded above and below, it follows from Proposition 8.1 that $\mu_Y(\varphi > t) \approx t^{-2}$. Hence we can apply [2, Theorem 6.4]. ■

A Chernov axioms

For convenience, we list the Chernov axioms as stated in [29, Section 3]. A more in-depth discussion of the axioms, and where they differ from treatments in [5, 9, 11], is given in [29] and not repeated here.

(A1) Let (Σ_0, d) be a compact C^2 Riemannian surface. We let $\mathcal{S}^+, \mathcal{S}^-$ be closed subsets of Σ_0 , and let $f : \text{int } \Sigma_0 \setminus \mathcal{S}^+ \rightarrow \text{int } \Sigma_0 \setminus \mathcal{S}^-$ be a C^2 diffeomorphism.

For $n \geq 1$, define $\mathcal{S}_n = \mathcal{S}^+ \cup f^{-1}\mathcal{S}^+ \cup \dots \cup f^{-n+1}\mathcal{S}^+$.

(A2) There are two families of cones $\{C_x^u\}, \{C_x^s\}$ in the tangent planes $T_x\Sigma_0$, $x \in \Sigma_0$, called unstable and stable cones, and a constant $\Lambda > 1$ such that:

- $\{C_x^u\}, \{C_x^s\}$ are continuous on Σ_0 .
- The axes of $\{C_x^u\}, \{C_x^s\}$ are one-dimensional.
- $\min_{x \in \Sigma_0} \angle(C_x^u, C_x^s) > 0$.
- $Df(C_x^u) \subset C_{fx}^u$ and $Df(C_x^s) \supset C_{fx}^s$ whenever Df exists.
- $\|Df(v^u)\| \geq \Lambda \|v^u\|$ for all $v^u \in C_x^u$, and $\|Df^{-1}(v^s)\| \geq \Lambda \|v^s\|$ for all $v^s \in C_x^s$.

Given a curve $W \subset \Sigma_0$, let d_W be the Riemannian metric on W induced by d . Similarly, let m_W be Lebesgue measure on W induced by Lebesgue measure m on Σ_0 .

Recall the definition of LUM/LSM in Subsection 5.4. Let W_1, W_2 be sufficiently small and close enough LUMs such that small LSMs intersect each of W_1, W_2 at most once. Let $W'_1 = \{x \in W_1 : W_x^s \cap W_2 \neq \emptyset\}$ and $H : W'_1 \rightarrow W_2$ be the *holonomy map*, i.e. $H(x)$ is the unique intersection between W_x^s and W_2 . Also, let $\Lambda(x) = |\det(Df|_{T_x W_x^u})|$.

(A3) The angles between \mathcal{S}^+ and LUMs, and between \mathcal{S}^- and LSMs, are bounded away from zero.

(A4) The map f preserves an ergodic volume measure μ_{Σ_0} such that a.e. $x \in \Sigma_0$ has an LUM W_x^u and the conditional measure on W_x^u induced by μ_{Σ_0} is absolutely continuous with respect to $m_{W_x^u}$. Furthermore, f^n is ergodic for all $n \geq 1$.

(A5) The leaves $W_x^{u/s}$ are uniformly $C^{1+\text{Lip}}$.

(A6) There is a function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{x \rightarrow 0} \psi(x) = 0$ for which the following holds: if W is an LUM, then for all x, y belonging to the same connected component V of $W \cap \mathcal{S}_{n-1}$,

$$\log \left[\prod_{i=0}^{n-1} \frac{\Lambda(f^i x)}{\Lambda(f^i y)} \right] \leq \psi(d_{f^n V}(f^n x, f^n y)).$$

(A7) There is a constant $C > 0$ with the following property: if W_1, W_2 are two sufficiently small and close enough LUMs, then the holonomy map $H : W'_1 \rightarrow W_2$ is absolutely continuous with respect to m_{W_1}, m_{W_2} and

$$C^{-1} \leq \frac{m_{W_2}(H[W'_1])}{m_{W_1}(W'_1)} \leq C.$$

(A8) Write $\mathcal{S}_1 = \mathcal{S}^+$ as a disjoint union $\mathcal{S}_1 = \mathcal{S}_P^+ \cup \mathcal{S}_S^+$, and let

$$\mathcal{S}_{P,n}^+ = \mathcal{S}_P^+ \cup f^{-1}\mathcal{S}_P^+ \cup \dots \cup f^{-n+1}\mathcal{S}_P^+.$$

We assume:

- For each LUM W , the intersection $W \cap \mathcal{S}_1$ is at most countable and $W \cap \mathcal{S}_1$ has at most one accumulation point x_∞ . Moreover, there are constants $C, d > 0$ such that if x_n is the monotone sequence in $W \cap \mathcal{S}_1$ converging to x_∞ then $d(x_n, x_\infty) \leq Cn^{-d}$ for all $n \geq 1$.
- Let $\Lambda_n = \min\{\Lambda(x) : x \in W_n\}$ where $\{W_n\}$ are the connected components of $W \setminus \mathcal{S}_S^+$; then

$$\theta_0 = \liminf_{\delta \rightarrow 0} \sup_{|W| < \delta} \sum_n \Lambda_n^{-1} < 1,$$

where the supremum is taken over LUMs W with $W \cap \mathcal{S}_P^+ = \emptyset$.

- Let $K_n = \lim_{\delta \rightarrow 0} \sup_{|W| < \delta} K_n(W)$ where the supremum is taken over LUMs W and $K_n(W)$ is the number of connected components of $W \setminus \mathcal{S}_{P,n}^+$; then $K_n < \min\{\theta_0^{-1}, \Lambda\}^n$ for some $n \geq 1$.

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