

# WASSERSTEIN CONVERGENCE RATES IN THE INVARIANCE PRINCIPLE FOR NONUNIFORMLY HYPERBOLIC FLOWS

IAN MELBOURNE AND ZHE WANG

**ABSTRACT.** We obtain  $q$ -Wasserstein convergence rates in the invariance principle for nonuniformly hyperbolic flows, where  $q \geq 1$  depends on the degree of nonuniformity. Utilizing a martingale-coboundary decomposition for nonuniformly expanding semiflows, we extend techniques from the discrete-time setting to the continuous-time case. Our results apply to uniformly hyperbolic (Axiom A) flows, nonuniformly hyperbolic flows that can be modelled by suspensions over Young towers with exponential tails (such as dispersing billiard flows and the classical Lorenz attractor), and intermittent solenoidal flows.

## 1. INTRODUCTION

The statistical properties of uniformly expanding/hyperbolic maps and flows are by now well understood, including the central limit theorem (CLT) for Hölder observables [12, 39, 41] and the almost sure invariance principle [17]. The latter result implies both the CLT and its functional form, commonly referred to as the weak invariance principle (WIP). Subsequently, there has been great interest in such statistical properties for nonuniformly expanding/hyperbolic maps and flows (see for example [10, 19, 20, 23, 31, 33, 35]). In addition, sharp Berry-Esseen estimates (convergence rates in the CLT) for nonuniformly expanding maps were obtained in [21].

In recent years, interest has turned to quantifying convergence rates in the WIP for dynamical systems, using metrics such as the Lévy-Prokhorov and Wasserstein distances. Antoniou and Melbourne [4] established convergence rates in the Lévy-Prokhorov distance for nonuniformly hyperbolic maps. More recently, Liu and Wang obtained  $q$ -Wasserstein convergence rates for deterministic hyperbolic maps [28] and for sequential dynamical systems [29]. Dedecker, Merlevède and Rio [16] provided an improved rate in 2-Wasserstein distance. Regarding the multidimensional WIP (i.e.  $\mathbb{R}^d$ -valued observables with  $d > 1$ ), Paviato [36] derived convergence rates in the Lévy-Prokhorov distance and the 1-Wasserstein distance for nonuniformly expanding/hyperbolic maps and flows. Subsequently, Fleming-Vázquez [18] obtained improved convergence rates in the  $q$ -Wasserstein distance for nonuniformly hyperbolic maps.

However, in the continuous-time literature, the study of convergence rates in the CLT and the WIP remains limited. To the best of our knowledge, only three works have addressed this issue. Pène [37, 38] was the first to provide convergence rates in the CLT for dispersive billiards with finite horizon. More recently, the aforementioned results of Paviato [36] on convergence rates in the WIP cover both discrete and continuous time.

---

*Date:* September 5, 2025.

*2010 Mathematics Subject Classification.* 37C10, 37A50, 60F17, 37D20.

*Key words and phrases.* Nonuniformly hyperbolic flow, suspension flow, invariance principle, rate of convergence, Wasserstein distance.

A standard approach for deriving the CLT or WIP for flows from the corresponding result for maps is via inducing (see for example [19, 33, 39]). Indeed, the error rates in the CLT for flows in [37, 38] are obtained in this way, though the rate for flows is weaker than the rate for maps. In this paper we pursue a different method, following Paviato [36], working with the flow directly and obtaining exactly the same error rate as for maps. The key technique in [36] is a secondary martingale-coboundary decomposition for nonuniformly expanding semiflows extending the work of [26] for nonuniformly expanding maps.

For scalar observables, Paviato [36] obtains the Prokhorov convergence rate  $n^{-\frac{1}{4}}(\log n)^{3/4}$  in the WIP for uniformly hyperbolic maps and flow, and the rate  $n^{-\frac{1}{4}+\delta}$  for nonuniformly hyperbolic flows where  $\delta > 0$  depends on the degree of nonuniformity  $p \in (2, \infty)$  and  $\delta \rightarrow 0$  as  $p \rightarrow \infty$ . The rates for nonuniformly hyperbolic flows in [36] are the same as those in [4] for maps. The paper [36] also considers the 1-Wasserstein distance for vector-valued observables of nonuniformly hyperbolic maps and flows, obtaining a rate  $n^{-\frac{1}{6}+\delta}$  where  $\delta \rightarrow 0$  as  $p \rightarrow 3^-$  (independent of the dimension of the observable). However, the obtained rate does not improve for  $p \geq 3$ .

In this paper, as in [28], we restrict to scalar observables but consider convergence rates for the  $q$ -Wasserstein distance  $1 \leq q \leq \frac{p}{2}$ ,  $p \geq 4$ . We show that the same convergence rates  $n^{-\frac{1}{4}+\delta}$  as those obtained in [28] for nonuniformly hyperbolic maps hold also for nonuniformly hyperbolic flows, with  $\delta \rightarrow 0$  as  $p \rightarrow \infty$ . Our results are new for  $q \geq 2$  and significantly improve the results in [36] for  $q = 1$  (scalar observable case) when  $p \geq 4$ .

Our results are applicable to uniformly hyperbolic (Axiom A) flows and a class of nonuniformly hyperbolic flows modelled by suspensions over Young towers with exponential tails, such as planar periodic Lorentz gases with finite horizon, Lorentz gases with cusps and the classical Lorenz attractor. In such examples, we obtain the convergence rate  $n^{-\frac{1}{4}+\delta}$  in the  $q$ -Wasserstein distance for all  $q \geq 1$ , see Section 5.2. Our results also cover intermittent semiflows (Example 2.5) and intermittent solenoidal flows (Example 5.4).

The remainder of this paper is organized as follows. In Section 2, we introduce nonuniformly expanding semiflows and state the main results. Section 3 summarizes some recent results of [36] on martingale approximation. The proof of the main result is given in Section 4. In Section 5, we extend the convergence rate to nonuniformly hyperbolic flows.

*Notation:* Throughout the paper, we use  $1_A$  to denote the indicator function of measurable set  $A$  and  $\|\cdot\|_{L^p}$  means the  $L^p$ -norm. As usual,  $a_n = O(b_n)$  means that there exists a constant  $C > 0$  such that  $|a_n| \leq C|b_n|$  for all  $n \geq 1$ . For simplicity we write  $C$  to denote constants independent of  $n$ , and  $C$  may change from line to line. We use  $\rightarrow_w$  to denote weak convergence in the sense of probability measures and  $\rightarrow_d$  means convergence in distribution. We use  $\mathbb{P}_X$  to denote the law/distribution of random variable  $X$  and use  $X =_d Y$  to mean  $X, Y$  sharing the same distribution.

We denote by  $C[0, 1]$  the space of all continuous functions on  $[0, 1]$  equipped with the supremum distance  $d_C$ , that is

$$d_C(x, y) := \sup_{t \in [0, 1]} |x(t) - y(t)|, \quad x, y \in C[0, 1].$$

Let  $(M, d)$  be a bounded metric space and let  $\eta \in (0, 1]$ . We denote by  $C^\eta(M)$  the space of  $C^\eta$  functions  $v : M \rightarrow \mathbb{R}$  with finite Hölder norm  $\|v\|_{C^\eta} = |v|_\infty + |v|_{C^\eta}$ , where

$$|v|_\infty = \sup_{x \in M} |v(x)|, \quad |v|_{C^\eta} = \sup_{\substack{x, y \in M \\ x \neq y}} \frac{|v(x) - v(y)|}{d(x, y)^\eta}.$$

## 2. STATEMENT OF THE MAIN RESULTS

**2.1. Nonuniformly expanding maps.** Let  $(X, d)$  be a bounded metric space with Borel probability measure  $\rho$  and let  $T : X \rightarrow X$  be a nonsingular, ergodic transformation. Suppose that  $Y$  is a subset of  $X$  with  $\rho(Y) > 0$ , and  $\{Y_j\}$  is an at most countable measurable partition of  $Y$  with  $\rho(Y_j) > 0$ . Let  $r : Y \rightarrow \mathbb{Z}^+$  be constant on each  $Y_j$  and such that  $T^{r(y)}y \in Y$  for all  $y \in Y$ . We call  $r$  the *return time* and  $F = T^r : Y \rightarrow Y$  the *induced map*.

We suppose that there are constants  $\lambda > 1$ ,  $C \geq 1$ ,  $\eta \in (0, 1]$  such that for each  $j \geq 1$ ,

- (1)  $F|_{Y_j} : Y_j \rightarrow Y$  is a (measure-theoretic) bijection,
- (2)  $d(Fx, Fy) \geq \lambda d(x, y)$  for all  $x, y \in Y_j$ ,
- (3)  $d(T^\ell x, T^\ell y) \leq Cd(Fx, Fy)$  for all  $x, y \in Y_j$ ,  $0 \leq \ell < r(j)$ ,
- (4)  $g_j = \frac{d\rho|_{Y_j}}{d\rho|_{Y_j} \circ F}$  satisfies  $|\log g_j(x) - \log g_j(y)| \leq Cd(Fx, Fy)^\eta$  for all  $x, y \in Y_j$ .

Assume that  $r \in L^p(Y)$  for some  $p \geq 1$ ; then we call  $T : X \rightarrow X$  a *nonuniformly expanding map of order  $p$* .

As a consequence of conditions (1), (2) and (4), the map  $F$  is a (*full-branch*) *Gibbs-Markov map* [1]. It is standard that there exists a unique absolutely continuous  $F$ -invariant probability measure  $\mu_Y$  on  $Y$ .

**2.2. Nonuniformly expanding semiflows.** Let  $\Psi_t : M \rightarrow M$  be a semiflow on a bounded metric space  $(M, d)$ , satisfying  $\Psi_0 = \text{Id}$  and  $\Psi_{t+s} = \Psi_t \circ \Psi_s$  for  $s, t \geq 0$ . We assume that there exists  $C > 0$  such that

$$(2.1) \quad d(\Psi_t x, \Psi_t y) \leq Cd(x, y) \quad \text{for all } t \in [0, 1], \ x, y \in M,$$

and

$$(2.2) \quad d(\Psi_t x, \Psi_s x) \leq C|t - s| \quad \text{for all } s, t \geq 0, \ x \in M.$$

Let  $X \subset M$  be a Borel set and define the first return time  $h : X \rightarrow \mathbb{R}^+$ ,  $h(x) = \inf\{t > 0 : \Psi_t x \in X\}$ . We assume that  $h \in C^\eta(X)$  for some fixed  $\eta \in (0, 1]$ , and that  $\inf h \geq 1$ . Such a function  $h$  is often called a *roof function*. The map  $T = \Psi_h : X \rightarrow X$  is called a *Poincaré map*. If  $T$  is a nonuniformly expanding map of order  $p \geq 1$  as described in Subsection 2.1, then we call the semiflow  $\Psi_t : M \rightarrow M$  a *nonuniformly expanding semiflow of order  $p$* .

Define the induced roof function

$$\varphi : Y \rightarrow [1, \infty), \quad \varphi(y) = \sum_{i=0}^{r(y)-1} h(T^i y).$$

Since  $r \in L^p(Y)$  and  $\varphi \leq |h|_\infty r$ , we have  $\varphi \in L^p(Y)$ . We define the suspension  $Y^\varphi = \{(y, u) \in Y \times \mathbb{R} : 0 \leq u \leq \varphi(y)\} / \sim$ , where  $(y, \varphi(y)) \sim (Fy, 0)$ . The suspension semiflow  $F_t : Y^\varphi \rightarrow Y^\varphi$  is given by  $F_t(y, u) = (y, u + t)$  computed modulo identifications.

Also, define the ergodic  $F_t$ -invariant probability measure  $\mu^\varphi = (\mu_Y \times \text{Lebesgue})/\bar{\varphi}$ , where  $\bar{\varphi} = \int_Y \varphi d\mu_Y$ .

We have a projection  $\pi_M : Y^\varphi \rightarrow M$  given by  $\pi_M(y, u) = \Psi_u y$  and it is a semiconjugacy between  $F_t$  and  $\Psi_t$ , satisfying  $\Psi_t \circ \pi_M = \pi_M \circ F_t$ . Hence  $\mu_M = (\pi_M)_* \mu^\varphi$  is an ergodic  $\Psi_t$ -invariant probability measure on  $M$ .

**2.3. Statement of the main results.** Let  $C_0^\eta(M) = \{v \in C^\eta(M) : \int_M v d\mu_M = 0\}$ . Given  $v \in C_0^\eta(M)$ , we denote  $v_t = \int_0^t v \circ \Psi_s ds$  and define the continuous processes  $W_n \in C[0, 1]$ ,  $n \geq 1$ , as

$$W_n(t) := \frac{1}{\sqrt{n}} \int_0^{nt} v \circ \Psi_s ds, \quad t \in [0, 1].$$

The following result is standard; see [25, 27, 31, 33, 34] for details.

**Proposition 2.1.** *Let  $\Psi_t : M \rightarrow M$  be a nonuniformly expanding semiflow of order  $p > 2$  and  $v \in C_0^\eta(M)$ . Then*

(a) (CLT) *The limit  $\sigma^2 = \lim_{t \rightarrow \infty} t^{-1} \int_M v_t^2 d\mu_M$  exists, and  $t^{-1/2} v_t \rightarrow_d N(0, \sigma^2)$  as  $t \rightarrow \infty$ .*

(b) (WIP)  *$W_n \rightarrow_w W$  in  $C[0, 1]$  as  $n \rightarrow \infty$ , where  $W$  is a Brownian motion with mean zero and variance  $\sigma^2 > 0$ .*

(c) (Moment bounds) *There exists  $C > 0$  such that  $\left\| \sup_{t \in [0, K]} |v_t| \right\|_{L^{2(p-1)}(M)} \leq CK^{1/2} \|v\|_{C^\eta}$  for all  $K > 0$ .*

*Proof.* Define the induced observable  $\tilde{v} : X \rightarrow \mathbb{R}$ ,  $\tilde{v} = \int_0^h v \circ \Psi_u du$ . Since  $v \in C_0^\eta(M)$  and  $h \in C^\eta(X)$ , it follows easily from (2.1) that  $\tilde{v} \in C^\eta(X)$  with  $\int_X \tilde{v} d\mu = 0$ . It is well known, see e.g. [26, Corollary 2.13], that the WIP holds for  $\tilde{v} : X \rightarrow \mathbb{R}$ . By a standard inducing argument (see for example [27, Proposition 5.7 and Theorem 5.8]), the WIP holds for  $v : M \rightarrow \mathbb{R}$ .

Similarly, by e.g. [26, Corollary 2.10],  $\left\| \max_{k \leq n} \left| \sum_{j=0}^{k-1} \tilde{v} \circ T^j \right| \right\|_{L^{2(p-1)}(X)} \leq C \|v\|_{C^\eta} n^{1/2}$ . Since the roof function  $h$  is bounded below, the moment estimate for the semiflow follows easily by a standard argument (see for example, [34, proof of Lemma 4.1] or [25, Section 7.2]).  $\square$

In the present paper, we consider the Wasserstein distance to metrize weak convergence. For  $q \geq 1$ , we denote by  $\mathcal{W}_q(\mu, \nu)$  the Wasserstein distance between the distributions  $\mu$  and  $\nu$  on a Polish space  $(\mathcal{X}, d)$  (see [45, Definition 6.1]):

$$\mathcal{W}_q(\mu, \nu) = \inf \left\{ [\mathbb{E} d(X, Y)^q]^{1/q}; \text{law}(X) = \mu, \text{law}(Y) = \nu \right\}.$$

**Proposition 2.2** ([45, Definition 6.8]). *We have that  $\lim_{n \rightarrow \infty} \mathcal{W}_q(\mu_n, \mu) = 0$  if and only if the following two conditions hold:*

(1)  $\mu_n \rightarrow_w \mu$  as  $n \rightarrow \infty$ ;

(2)  $\lim_{n \rightarrow \infty} \int_{\mathcal{X}} d(x, x_0)^q d\mu_n(x) = \int_{\mathcal{X}} d(x, x_0)^q d\mu(x)$  for some (thus any)  $x_0 \in \mathcal{X}$ .

*In particular, if the metric  $d$  is bounded, then the convergence with respect to  $\mathcal{W}_q$  is equivalent to the weak convergence in condition (1).*

In the following, we use the abbreviation  $\mathcal{W}_q(X, Y)$  to mean  $\mathcal{W}_q(\mathbb{P}_X, \mathbb{P}_Y)$ .

**Theorem 2.3.** *Let  $\Psi_t : M \rightarrow M$  be a nonuniformly expanding semiflow of order  $p > 2$  and  $v \in C_0^\eta(M)$ . Then  $\lim_{n \rightarrow \infty} \mathcal{W}_q(W_n, W) = 0$  for all  $1 \leq q < 2(p-1)$ .*

*Proof.* The proof is essentially the same as that of [28, Theorem 3.3], and is included here for completeness. It follows from Proposition 2.1(c) that  $W_n$  has a finite moment of order  $2(p-1)$ . This together with the fact that  $W_n \rightarrow_w W$  in Proposition 2.1(b) implies that for each  $q < 2(p-1)$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E} \sup_{t \in [0,1]} |W_n(t)|^q = \mathbb{E} \sup_{t \in [0,1]} |W(t)|^q$$

by [15, Theorem 4.5.2]. On the other hand, using the fact that  $W_n : M \rightarrow C[0,1]$  and the definition of pushforward measures, we have

$$\int_{C[0,1]} d_C(x, 0)^q d\mu_M \circ W_n^{-1}(x) = \int_M \sup_{t \in [0,1]} |W_n(t, \omega)|^q d\mu_M(\omega) = \mathbb{E} \sup_{t \in [0,1]} |W_n(t)|^q;$$

similarly,

$$\int_{C[0,1]} d_C(x, 0)^q d\mu_M \circ W^{-1}(x) = \mathbb{E} \sup_{t \in [0,1]} |W(t)|^q.$$

Hence

$$\lim_{n \rightarrow \infty} \int_{C[0,1]} d_C(x, 0)^q d\mu_M \circ W_n^{-1}(x) = \int_{C[0,1]} d_C(x, 0)^q d\mu_M \circ W^{-1}(x).$$

By taking  $\mu_n = \mu_M \circ W_n^{-1}$ ,  $\mu = \mu_M \circ W^{-1}$  and  $x_0 = 0$  in Proposition 2.2 and using the fact that  $W_n \rightarrow_w W$  in  $C[0,1]$ , the result follows.  $\square$

Our main result for semiflows is the following:

**Theorem 2.4.** *Let  $\Psi_t : M \rightarrow M$  be a nonuniformly expanding semiflow of order  $p \geq 4$  and  $v \in C_0^\eta(M)$ . Then there exists a constant  $C > 0$  such that  $\mathcal{W}_2(W_n, W) \leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}}$  for all  $n \geq 1$ .*

We postpone the proof of Theorem 2.4 to Section 4.

*Example 2.5* (Intermittent semiflows). We consider a class of intermittent semiflow  $\Psi_t : M \rightarrow M$  with an intermittent Poincaré map, namely the LSV map  $T : [0, 1] \rightarrow [0, 1]$  (see [30]),

$$T(x) = \begin{cases} x(1 + 2^\beta x^\beta) & x \in [0, \frac{1}{2}), \\ 2x - 1 & x \in [\frac{1}{2}, 1], \end{cases}$$

and Hölder return time function  $h : [0, 1] \rightarrow [1, \infty)$ . Here  $\beta > 0$  is a parameter and the map  $T$  and semiflow  $\Psi_t$  are nonuniformly expanding of order  $p$  for all  $p < 1/\beta$ .

Paviato [36] derived the 1-Wasserstein convergence rate  $n^{-\frac{1}{6} + \delta}$  for  $\beta \in (0, \frac{1}{3})$  and  $n^{-\frac{1}{2}(1-2\beta) + \delta}$  for  $\beta \in [\frac{1}{3}, \frac{1}{2})$ . By Theorem 2.4, we obtain the  $q$ -Wasserstein convergence rate  $n^{-\frac{1}{4} + \frac{\beta}{4(1-\beta)} + \delta}$  for  $\beta \in (0, \frac{1}{4})$  and  $q < \frac{1}{2\beta}$ , which improves Paviato's result in this range.

### 3. MARTINGALE-COBOUNDARY DECOMPOSITIONS FOR SEMIFLOWS

In this section, we summarize some results for the suspension semiflow  $F_t : Y^\varphi \rightarrow Y^\varphi$  defined in Section 2. In Subsection 3.1, we recall the notion of Gibbs-Markov semiflow. In Subsection 3.2, we recall the martingale-coboundary decomposition for such semiflows from [36], which extended the approach of [26] to continuous-time systems.

**3.1. Gibbs-Markov semiflows.** Let  $F_t : Y^\varphi \rightarrow Y^\varphi$  be the suspension semiflow over the induced Gibbs-Markov map  $F : Y \rightarrow Y$  with induced roof function  $\varphi \in L^p(Y)$  and ergodic invariant probability measure  $\mu^\varphi$  as in Subsection 2.2. By [36, Proposition 4.1], there is  $C > 0$  such that

$$|\varphi(y) - \varphi(y')| \leq C(\inf_{Y_j} \varphi) d(Fy, Fy')^\eta \quad \text{for all } y, y' \in Y_j, j \geq 1.$$

Following [8, 36], we call  $F_t : Y^\varphi \rightarrow Y^\varphi$  a *Gibbs-Markov semiflow of order  $p$* .

For  $j \geq 1$ , set  $Y_j^\varphi = \{(y, u) \in Y^\varphi : y \in Y_j\}$ . Given  $w : Y^\varphi \rightarrow \mathbb{R}$  and  $\eta \in (0, 1]$ , we define

$$|w|_\infty = \sup_{(y,u) \in Y^\varphi} |w(y, u)|, \quad |w|_{\mathcal{V}^\eta} = \sup_{j \geq 1} \sup_{\substack{(x,u), (y,u) \in Y_j^\varphi \\ x \neq y}} \frac{|w(x, u) - w(y, u)|}{(\inf_{Y_j} \varphi) d(Fx, Fy)^\eta}.$$

We write  $w \in \mathcal{V}^\eta(Y^\varphi)$  if  $\|w\|_{\mathcal{V}^\eta} = |w|_\infty + |w|_{\mathcal{V}^\eta} < \infty$ . Let  $\mathcal{V}_0^\eta(Y^\varphi) = \{w \in \mathcal{V}^\eta(Y^\varphi) : \int_{Y^\varphi} w d\mu^\varphi = 0\}$ .

**3.2. Martingale-coboundary decompositions for Gibbs-Markov semiflows.** Suppose that  $F_t : Y^\varphi \rightarrow Y^\varphi$  is a Gibbs-Markov semiflow of order  $p \geq 2$ . Let  $U_1 w = w \circ F_1$  be the Koopman operator for the time-one map  $F_1$  and  $L_1$  the transfer operator of  $F_1$ , so  $\int L_1 v w d\mu^\varphi = \int v U_1 w d\mu^\varphi$  for  $v \in L^1(Y^\varphi)$ ,  $w \in L^\infty(Y^\varphi)$ .

**Proposition 3.1.** *Given  $w \in \mathcal{V}_0^\eta(Y^\varphi)$ , define a function  $\psi : Y^\varphi \rightarrow \mathbb{R}$ ,  $\psi = \int_0^1 w \circ F_s ds$ . Then there exist  $m \in L^p(Y^\varphi)$ ,  $\chi \in L^{p-1}(Y^\varphi)$  such that  $\psi = m + \chi \circ F_1 - \chi$  and  $m \in \ker L_1$ .*

*Moreover, there exists a constant  $C > 0$  such that for all  $w \in \mathcal{V}_0^\eta(Y^\varphi)$ ,*

$$\|m\|_{L^p} \leq C\|w\|_{\mathcal{V}^\eta}, \quad \left\| \max_{1 \leq k \leq n} |\chi \circ F_k - \chi| \right\|_{L^p} \leq C\|w\|_{\mathcal{V}^\eta} n^{1/p}.$$

*Proof.* This proposition is a summary of parts of [36, Propositions 4.6, 4.8 and 4.9].  $\square$

For  $1 \leq j \leq n$ , define the  $\sigma$ -algebra  $\mathcal{G}_{n,j} = F_{n-j}^{-1} \mathcal{B}$ , where  $\mathcal{B}$  is the underlying  $\sigma$ -algebra on the space  $Y^\varphi$ .

**Proposition 3.2.** *Fix  $n \geq 1$ . Then  $\{m \circ F_{n-j}, \mathcal{G}_{n,j}; 1 \leq j \leq n\}$  is a sequence of martingale differences. That is  $\mathcal{G}_{n,j-1} \subset \mathcal{G}_{n,j}$ ,  $m \circ F_{n-j}$  is  $\mathcal{G}_{n,j}$ -measurable, and  $\mathbb{E}(m \circ F_{n-j} | \mathcal{G}_{n,j-1}) = 0$  for  $0 \leq j \leq n-1$ .*

*Proof.* This is a standard consequence of the facts that  $F_1^{-1} \mathcal{B} \subset \mathcal{B}$ ,  $U_1 L_1 = \mathbb{E}(\cdot | F_1^{-1} \mathcal{B})$  and  $m \in \ker L_1$  (see for example [26, Proposition 2.9]).  $\square$

Following [36, Equation (4.15)], define  $\check{w} = (U_1 L_1 m^2) - \sigma^2 = \mathbb{E}(m^2 - \sigma^2 | F_1^{-1} \mathcal{B})$ . Since  $\int_{Y^\varphi} U_1 L_1 m^2 d\mu^\varphi = \int_{Y^\varphi} L_1 m^2 d\mu^\varphi = \sigma^2$ , we have  $\int_{Y^\varphi} \check{w} d\mu^\varphi = 0$ .

**Proposition 3.3.** *There exists a constant  $C > 0$  such that for all  $n \geq 1$  and  $w \in \mathcal{V}_0^\eta(Y^\varphi)$ ,*

$$\left\| \max_{1 \leq k \leq n} \left| \sum_{i=0}^{k-1} \check{w} \circ F_i \right| \right\|_{L^{2(p-1)}} \leq C\|w\|_{\mathcal{V}^\eta}^2 n^{1/2}.$$

*Proof.* See [36, Corollary 4.18].  $\square$

## 4. PROOF OF THEOREM 2.4

In this section, we prove Theorem 2.4. Recall that  $\Psi_t : M \rightarrow M$  is a nonuniformly expanding semiflow of order  $p \geq 4$  with ergodic probability measure  $\mu_M$ . Let  $F_t : Y^\varphi \rightarrow Y^\varphi$  be the corresponding Gibbs-Markov semiflow with ergodic probability measure  $\mu^\varphi$ . We have the measure-preserving semiconjugacy  $\pi_M : Y^\varphi \rightarrow M$ .

Let  $v \in C_0^\eta(M)$ . By [36, Proposition 4.3], the lifted observable  $w = v \circ \pi_M$  lies in  $\mathcal{V}_0^{\eta^2}(Y^\varphi)$ .

Define continuous processes  $\widetilde{W}_n \in C[0, 1]$ ,  $n \geq 1$ , on  $(Y^\varphi, \mu^\varphi)$  as

$$\widetilde{W}_n(t) := \frac{1}{\sqrt{n}} \int_0^{nt} w \circ F_s ds, \quad t \in [0, 1].$$

Then  $\widetilde{W}_n = W_n \circ \pi_M =_d W_n$ . So it follows from Proposition 2.1(b) that  $\widetilde{W}_n \rightarrow_w W$ , where  $W$  is a Brownian motion with variance  $\sigma^2 = \int_{Y^\varphi} m^2 d\mu^\varphi$ . Moreover,  $\mathcal{W}_{\frac{p}{2}}(W_n, W) = \mathcal{W}_{\frac{p}{2}}(\widetilde{W}_n, W)$ , so we reduce to proving the following:

**Lemma 4.1.** *Suppose that  $F_t : Y^\varphi \rightarrow Y^\varphi$  is a Gibbs-Markov semiflow of order  $p \geq 4$  and let  $\eta \in (0, 1]$ . Suppose that  $w \in \mathcal{V}_0^\eta(Y^\varphi)$ . Then there exists a constant  $C > 0$  such that  $\mathcal{W}_{\frac{p}{2}}(\widetilde{W}_n, W) \leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}}$  for all  $n \geq 1$ .*

Let  $\{m \circ F_{n-j}, \mathcal{G}_{n,j}; 1 \leq j \leq n\}$  be the sequence of martingale differences in Proposition 3.2. We define

$$\zeta_{n,j} := \frac{1}{\sqrt{n}\sigma} m \circ F_{n-j}, \quad \text{for } 1 \leq j \leq n,$$

and define the conditional variances

$$V_{n,k} := \sum_{j=1}^k \mathbb{E}(\zeta_{n,j}^2 | \mathcal{G}_{n,j-1}), \quad \text{for } 1 \leq k \leq n.$$

Define the stochastic process  $X_n$  with sample paths in  $C[0, 1]$  by

$$(4.1) \quad X_n(t) := \sum_{j=1}^k \zeta_{n,j} + \frac{tV_{n,n} - V_{n,k}}{V_{n,k+1} - V_{n,k}} \zeta_{n,k+1}, \quad \text{for } V_{n,k} \leq tV_{n,n} < V_{n,k+1}.$$

We recall the following standard argument from probability theory.

**Proposition 4.2.** *There is a constant  $C > 0$  such that  $\left\| \max_{1 \leq j \leq n} |\zeta_{n,j}| \right\|_{L^p} \leq Cn^{1/p-1/2}$  for all  $n \geq 1$ .*

*Proof.* For all  $n \geq 1$ , we have  $\max_{0 \leq j \leq n-1} |m \circ F_j|^p \leq \sum_{j=0}^{n-1} |m \circ F_j|^p$  so  $\left\| \max_{0 \leq j \leq n-1} |m \circ F_j| \right\|_p \leq n^{1/p} \|m\|_p$ . The result follows by definition of  $\zeta_{n,j}$  and Proposition 3.1.  $\square$

Next we estimate the random variable  $k = k_n(t)$  defined implicitly in (4.1).

**Proposition 4.3.** *There exists a constant  $C > 0$  such that  $\left\| \sup_{t \in [0, 1]} |k - [nt]| \right\|_{L^{2(p-1)}} \leq Cn^{1/2}$  for all  $n \geq 1$ .*

*Proof.* We follow the argument of [36, Proposition 5.5], which is based on [4, Propositions 4.1 and 4.4]. For  $1 \leq k \leq n$ , we can write

$$\begin{aligned} |V_{n,k} - \frac{k}{n}| &= \left| \frac{1}{n\sigma^2} \sum_{j=1}^k \mathbb{E}(m^2 \circ F_{n-j} | F_{n-(j-1)}^{-1} \mathcal{B}) - \frac{k}{n} \right| \\ &= \frac{1}{n\sigma^2} \left| \sum_{j=1}^k \mathbb{E}(m^2 - \sigma^2 | F_1^{-1} \mathcal{B}) \circ F_{n-j} \right| = \frac{1}{n\sigma^2} \left| \sum_{j=1}^k \check{w} \circ F_{n-j} \right|. \end{aligned}$$

Hence it follows from Proposition 3.3 that

$$\left\| \max_{1 \leq k \leq n} |V_{n,k} - \frac{k}{n}| \right\|_{L^{2(p-1)}} \leq \frac{2}{n\sigma^2} \left\| \max_{1 \leq k \leq n} \left| \sum_{j=0}^{k-1} \check{w} \circ F_j \right| \right\|_{L^{2(p-1)}} \leq Cn^{-1/2}.$$

By [4, Proof of Proposition 4.4],  $|k - [nt]| \leq n \max_{j \leq n+1} |V_{n,j} - \frac{j}{n}| + 2$  and the result follows.  $\square$

**Lemma 4.4.** *Let  $B$  denote standard Brownian motion. Let  $p \geq 4$ . Then for any  $\delta > 0$ , there exists a constant  $C > 0$  such that  $\mathcal{W}_{\frac{p}{2}}(X_n, B) \leq Cn^{-(\frac{1}{4}-\delta)}$  for all  $n \geq 1$ .*

*Proof.* The proof is similar to that in [28, Lemma 4.4] and we follow the same steps omitting some arguments that are identical.

(1) By the Skorokhod embedding theorem (see [22, Theorem A.1]), there exists a probability space (depending on  $n$ ) supporting a standard Brownian motion  $B$ , a sequence of nonnegative random variables  $\tau_1, \dots, \tau_n$  with  $T_i = \sum_{j=1}^i \tau_j$ , and a sequence of  $\sigma$ -fields  $\mathcal{F}_i$  generated by all events up to  $T_i$  for  $1 \leq i \leq n$ , such that for all  $1 \leq i \leq n$ ,

- $\sum_{j=1}^i \zeta_{n,j} = B(T_i)$ ;
- $\mathbb{E}(\tau_i | \mathcal{F}_{i-1}) = \mathbb{E}(|\zeta_{n,i}|^2 | \mathcal{G}_{n,i-1})$  a.s.;
- for any  $p \geq 1$ , there exists a constant  $C_p > 0$  such that

$$\mathbb{E}(\tau_i^p | \mathcal{F}_{i-1}) \leq C_p \mathbb{E}(|\zeta_{n,i}|^{2p} | \mathcal{G}_{n,i-1}) \quad \text{a.s.}$$

On this probability space, we show that for any  $\delta > 0$  there exists a constant  $C > 0$  (independent of  $n$ ) such that

$$\left\| \sup_{t \in [0,1]} |X_n(t) - B(t)| \right\|_{L^{\frac{p}{2}}} \leq Cn^{-(\frac{1}{4}-\delta)}.$$

Then the result follows from the definition of the Wasserstein distance.

For ease of exposition, we write  $\zeta_j$  and  $V_k$  instead of  $\zeta_{n,j}$  and  $V_{n,k}$  respectively. By (4.1),

$$(4.2) \quad X_n(t) = B(T_k) + \left( \frac{tV_n - V_k}{V_{k+1} - V_k} \right) (B(T_{k+1}) - B(T_k)), \quad \text{for } V_k \leq tV_n < V_{k+1}.$$

(2) Following the argument in [28, Lemma 4.4], we have

$$\left\| \max_{1 \leq k \leq n} |T_k - V_k| \right\|_{L^{\frac{p}{2}}} \leq Cn^{\frac{1}{2}} \max_{1 \leq k \leq n} \|\zeta_k\|_{L^p}^2 = Cn^{-\frac{1}{2}} \|m\|_{L^p}^2 \leq Cn^{-\frac{1}{2}}.$$

By the proof of Proposition 4.3,

$$\|V_n - 1\|_{L^{\frac{p}{2}}} \leq Cn^{-\frac{1}{2}}.$$



(3) We now estimate  $|X_n - B|$  on the set  $\{|T_n - 1| > 1\}$ . By Chebyshev's inequality and the estimates in (2),

$$\mu^\varphi(|T_n - 1| > 1) \leq \mathbb{E}|T_n - 1|^{\frac{p}{2}} \leq Cn^{-\frac{p}{4}}.$$

Hence we deduce that

$$\begin{aligned} & \left\| 1_{\{|T_n - 1| > 1\}} \sup_{t \in [0,1]} |X_n(t) - B(t)| \right\|_{L^{\frac{p}{2}}} \\ & \leq (\mu^\varphi(|T_n - 1| > 1))^{1/p} \left( \left\| \sup_{t \in [0,1]} |X_n(t)| \right\|_{L^p} + \left\| \sup_{t \in [0,1]} |B(t)| \right\|_{L^p} \right) \leq Cn^{-\frac{1}{4}}. \end{aligned}$$

(4) By (3), it remains to estimate  $|X_n - B|$  on the set  $\{|T_n - 1| \leq 1\}$ . Now

$$\left\| 1_{\{|T_n - 1| \leq 1\}} \sup_{t \in [0,1]} |X_n(t) - B(t)| \right\|_{L^{\frac{p}{2}}} \leq I_1 + I_2$$

where

$$I_1 =: \left\| \sup_{t \in [0,1]} |X_n(t) - B(T_k)| \right\|_{L^{\frac{p}{2}}}, \quad I_2 =: \left\| 1_{\{|T_n - 1| \leq 1\}} \sup_{t \in [0,1]} |B(T_k) - B(t)| \right\|_{L^{\frac{p}{2}}}.$$

By (4.2) and Proposition 4.2,

$$I_1 \leq \left\| \max_{0 \leq k \leq n-1} |\zeta_{k+1}| \right\|_{L^p} \leq Cn^{-\frac{1}{2} + \frac{1}{p}} \leq Cn^{-\frac{1}{4}}.$$

(5) By (3) and (4), it remains to estimate  $I_2$ . Let  $\gamma \in (0, \frac{1}{2})$ . By Kolmogorov's continuity theorem,

$$(4.3) \quad \left\| \sup_{\substack{s, t \in [0,2] \\ s \neq t}} \frac{|B(s) - B(t)|}{|s - t|^\gamma} \right\|_{L^p} < \infty.$$

Following the same argument as in [28, Lemma 4.4], we also have

$$(4.4) \quad \left\| \sup_{t \in [0,1]} |T_k - t|^\gamma \right\|_{L^p} \leq Cn^{-\frac{\gamma}{2}}.$$

On the set  $\{|T_n - 1| \leq 1\}$ , note that

$$\sup_{t \in [0,1]} |B(T_k) - B(t)| \leq \left( \sup_{\substack{s, t \in [0,2] \\ s \neq t}} \frac{|B(s) - B(t)|}{|s - t|^\gamma} \right) \left( \sup_{t \in [0,1]} |T_k - t|^\gamma \right).$$

Hence by Hölder's inequality and (4.3), (4.4),

$$I_2 \leq \left\| \sup_{\substack{s, t \in [0,2] \\ s \neq t}} \frac{|B(s) - B(t)|}{|s - t|^\gamma} \right\|_{L^p} \left\| \sup_{t \in [0,1]} |T_k - t|^\gamma \right\|_{L^p} \leq Cn^{-\frac{\gamma}{2}}.$$

The result now follows by taking  $\gamma$  sufficiently close to  $\frac{1}{2}$ .  $\square$

**Proposition 4.5.** For  $n \geq 1$  and  $\psi = \int_0^1 w \circ F_s ds$ , define  $Z_n := \max_{0 \leq i, \ell \leq \sqrt{n}} \left| \sum_{j=i\sqrt{n}}^{i\sqrt{n}+\ell-1} \psi \circ F_j \right|$ . Then

(a)  $\left| \sum_{j=a}^{b-1} \psi \circ F_j \right| \leq Z_n((b-a)(n^{\frac{1}{2}} - 1)^{-1} + 3)$  for all  $0 \leq a < b \leq n$ .

(b)  $\|Z_n\|_{L^{2(p-1)}} \leq Cn^{\frac{1}{4} + \frac{1}{4(p-1)}}$  for all  $n \geq 1$ .

*Proof.* The proof of part (a) is identical to that in [4, Proposition 4.6] and we omit it. As for part (b), following [36, Proposition 5.6], we have

$$\begin{aligned} \int_{Y^\varphi} |Z_n|^{2(p-1)} d\mu^\varphi &\leq \sum_{i \leq \sqrt{n}} \int_{Y^\varphi} \max_{0 \leq \ell \leq \sqrt{n}} \left| \sum_{j=i\sqrt{n}}^{i\sqrt{n}+\ell-1} \psi \circ F_j \right|^{2(p-1)} d\mu^\varphi \\ &= \sqrt{n} \int_{Y^\varphi} \max_{0 \leq \ell \leq \sqrt{n}} \left| \sum_{j=0}^{\ell-1} \psi \circ F_j \right|^{2(p-1)} d\mu^\varphi = \sqrt{n} \left\| \max_{0 \leq \ell \leq \sqrt{n}} \left| \int_0^\ell w \circ F_s ds \right| \right\|_{L^{2(p-1)}}^{2(p-1)} \\ &\leq Cn^{\frac{1}{2}} \cdot n^{\frac{2(p-1)}{4}} \end{aligned}$$

by Proposition 2.1(c). The result follows.  $\square$

Define  $g : C[0, 1] \rightarrow C[0, 1]$  by  $g(u)(t) := u(1) - u(1 - t)$ .

**Lemma 4.6.** *Let  $p > 2$ . Then there exists a constant  $C > 0$  such that  $\mathcal{W}_{p-1}(g \circ \widetilde{W}_n, \sigma X_n) \leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}}$  for all  $n \geq 1$ .*

*Proof.* The proof follows that in [4, Lemma 4.7] with obvious notional changes. Write

$$\begin{aligned} g(\widetilde{W}_n(t)) - \sigma X_n(t) &= \frac{1}{\sqrt{n}} \left( \int_{n-[nt]}^n w \circ F_s ds - \sum_{j=1}^k m \circ F_{n-j} \right) + E_n(t) \\ &= \frac{1}{\sqrt{n}} \left( \sum_{j=n-[nt]}^{n-k-1} \psi \circ F_j + \chi \circ F_{n-k} - \chi \circ F_n \right) + E_n(t) \end{aligned}$$

where  $|E_n(t)| \leq \frac{1}{\sqrt{n}} \|w\|_\infty + \max_{0 \leq j \leq n} |\zeta_{n,j}|$ , so  $\left\| \sup_{t \in [0,1]} |E_n(t)| \right\|_{L^p} \leq Cn^{-\frac{1}{2} + \frac{1}{p}}$  by Proposition 4.2.

By Proposition 3.1,

$$\left\| \sup_{t \in [0,1]} |\chi \circ F_{n-k} - \chi \circ F_n| \right\|_p \leq 2 \left\| \max_{0 \leq j \leq n} |\chi \circ F_j - \chi| \right\|_p \leq n^{1/p}.$$

Finally,

$$\left\| \sup_{t \in [0,1]} \left| \sum_{j=n-[nt]}^{n-k-1} \psi \circ F_j \right| \right\|_p \leq \|Z_n\|_{p/2} \left( (n^{-1/2} - 1) \left\| \sup_{t \in [0,1]} |k - [nt]| \right\|_{p/2} + 3 \right) \leq Cn^{\frac{1}{4} + \frac{1}{4(p-1)}}$$

by Propositions 4.3 and 4.5. Hence  $\left\| \sup_{t \in [0,1]} |g(\widetilde{W}_n(t)) - \sigma X_n(t)| \right\|_{L^{p-1}} \leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}}$  and the result follows.  $\square$

*Proof of Lemma 4.1.* Note that  $g \circ g = Id$  and  $g$  is Lipschitz with  $\text{Lip } g \leq 2$ . Also,  $g(W) =_d W =_d \sigma B$ . By the Lipschitz mapping theorem (see [28, Proposition 2.4]),

$$\mathcal{W}_{\frac{p}{2}}(\widetilde{W}_n, W) = \mathcal{W}_{\frac{p}{2}}(g(g \circ \widetilde{W}_n), g(g \circ W)) \leq 2\mathcal{W}_{\frac{p}{2}}(g \circ \widetilde{W}_n, g \circ W) = 2\mathcal{W}_{\frac{p}{2}}(g \circ \widetilde{W}_n, \sigma B).$$

By Lemmas 4.4 and 4.6,

$$\begin{aligned} \mathcal{W}_{\frac{p}{2}}(g \circ \widetilde{W}_n, \sigma B) &\leq \mathcal{W}_{\frac{p}{2}}(g \circ \widetilde{W}_n, \sigma X_n) + \mathcal{W}_{\frac{p}{2}}(\sigma X_n, \sigma B) \\ &\leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}} + Cn^{-\frac{1}{4} + \delta} \leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}}, \end{aligned}$$

where the last inequality holds because  $\delta > 0$  can be taken arbitrarily small.  $\square$

## 5. NONUNIFORMLY HYPERBOLIC FLOWS

In this section, we show how the main results in Section 2 extend from nonuniformly expanding semiflows to nonuniformly hyperbolic flows. The techniques for quotienting out stable directions are more-or-less standard following [12, 40, 43] but remain somewhat tricky for flows. For Axiom A flows, this was done in [32]. Then [7] covered the case of singular hyperbolic flows, including the classical Lorenz attractor. In the more general situation considered here, we follow arguments in [36, Section 6]. We note that these techniques are restricted to the case where there is exponential contraction along stable manifolds, even though we allow nonuniform expansion.<sup>1</sup>

In Subsection 5.1, we introduce the setup. In Subsection 5.2, we state our main results for flows and give several examples. In Subsection 5.3, we provide a sketch of the main result for flows.

**5.1. The setup.** Let  $(M, d)$  be a bounded metric space and let  $\Psi_t : M \rightarrow M$  be a flow, satisfying  $\Psi_0 = \text{Id}$  and  $\Psi_{t+s} = \Psi_t \circ \Psi_s$  for  $s, t \in \mathbb{R}$ . As in Section 2, we assume continuous dependence on initial conditions (2.1) and Lipschitz continuity in time (2.2). We suppose that there is a Borel subset  $X \subset M$  with first return time  $h : X \rightarrow \mathbb{R}^+$  satisfying  $h \in C^n(X)$  and  $\inf h \geq 1$ .

- We suppose that there is a “uniformly hyperbolic” subset  $Y \subset X$ , an at most countable measurable partition  $\{Y_j\}$  of  $Y$ , and an integrable return time function  $r : Y \rightarrow \mathbb{Z}^+$  constant on each  $Y_j$  such that  $T^{r(y)}y \in Y$ . Define  $F : Y \rightarrow Y$  as  $Fy = T^{r(y)}y$ . We suppose that  $\mu_Y$  is an ergodic  $F$ -invariant Borel probability measure on  $Y$ . We define a separation time  $s(y, y')$  on  $Y$  as the least integer  $n \geq 0$  such that  $F^n y$  and  $F^n y'$  belong to different partition elements.

- We suppose that there is a measurable partition  $\mathcal{Q}^s$  of  $Y$  consisting of “stable leaves” refining  $\{Y_j\}$ .<sup>2</sup> Let  $Q^s(y)$  be the stable leaf containing  $y \in Y$ . We assume that  $FQ^s(y) \subset Q^s(Fy)$ . Define the quotient space  $\bar{Y} = Y/\mathcal{Q}^s$  with projection  $\bar{\pi} : Y \rightarrow \bar{Y}$  and partition  $\{\bar{Y}_j\}$  where  $\bar{Y}_j = \bar{\pi}Y_j$ . We also have the quotient map  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  with ergodic invariant probability measure  $\bar{\mu}_Y = \bar{\pi}_*\mu_Y$ . We suppose that  $\bar{F}$  is a Gibbs-Markov map as in Section 2.1.

- We require that there is a measurable subset  $\tilde{Y} \subset Y$  such that for every  $y \in Y$ , there is a unique  $\tilde{y} \in \tilde{Y} \cap Q^s(y)$ . Let  $\pi : Y \rightarrow \tilde{Y}$  denote the associated projection. Let  $\beta_n(y) = N$  be the unique integer such that

$$\sum_{j=0}^{N-1} r(F^j y) \leq n < \sum_{j=0}^N r(F^j y).$$

This counts the number of “good” returns of the map  $T$  to  $Y$  by time  $n$ . We suppose that there exist  $C > 0$  and  $\gamma \in (0, 1)$  such that

$$(5.1) \quad d(T^n y, T^n y') \leq C(\gamma^n d(y, y') + \gamma^{s(y, y') - \beta_n(y)}) \quad \text{for all } y, y' \in Y, n \geq 0,$$

and

$$(5.2) \quad d(T^n y, T^n y') \leq C\gamma^{s(y, y') - \beta_n(y)} \quad \text{for all } y, y' \in \tilde{Y}, n \geq 0.$$

<sup>1</sup>The condition of exponential contraction along stable manifolds is relaxed somewhat in [8] in the study of rates of decay of correlations for flows.

<sup>2</sup>More standard notation is  $\mathcal{W}^s$  but we already used  $W$  for processes and  $\mathcal{W}$  for Wasserstein distance.

*Remark 5.1.* In particular, we have contraction of  $T$  along stable leaves:

$$d(T^n y, T^n y') \leq C \gamma^n d(y, y') \quad \text{for all } y, y' \in Y \text{ with } y' \in Q^s(y), n \geq 0.$$

Let  $\varphi : Y \rightarrow [1, \infty)$  be defined as  $\varphi(y) = \sum_{i=0}^{r(y)-1} h(T^i y)$ . Since  $r \in L^1(Y)$  and  $\varphi \leq |h|_\infty r$ , we have  $\varphi \in L^1(Y)$ . Define the suspension  $Y^\varphi = \{(y, u) \in Y \times \mathbb{R} : 0 \leq u \leq \varphi(y)\} / \sim$ , where  $(y, \varphi(y)) \sim (Fy, 0)$ . The suspension flow  $F_t : Y^\varphi \rightarrow Y^\varphi$  is given by  $F_t(y, u) = (y, u + t)$  computed modulo identifications. The projection  $\pi_M : Y^\varphi \rightarrow M$ ,  $\pi_M(y, u) = \Psi_u y$  is a semiconjugacy between  $F_t$  and  $\Psi_t$ . We have an ergodic  $F_t$ -invariant probability measure  $\mu^\varphi = (\mu_Y \times \text{Lebesgue}) / \bar{\varphi}$ , where  $\bar{\varphi} = \int_Y \varphi d\mu_Y$ . Then  $\mu_M = (\pi_M)_* \mu^\varphi$  is an ergodic  $\Psi_t$ -invariant probability measure on  $M$ .

A flow  $(\Psi_t, M, \mu_M)$  satisfying these assumptions is called a *nonuniformly hyperbolic flow of order  $p$*  if  $r$  (and hence  $\varphi$ ) is  $L^p$ .

**5.2. Main results for flows.** Let  $v \in C_0^\eta(M)$  and define continuous processes  $W_n \in C[0, 1]$  as

$$W_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} v \circ \Psi_s ds, \quad t \in [0, 1].$$

Then Proposition 2.1 still holds for nonuniformly hyperbolic flows; see [26, 27, 31, 34] for example. The statements of our main results are completely analogous to those of Theorems 2.3 and 2.4.

**Theorem 5.2.** *Let  $\Psi_t : M \rightarrow M$  be a nonuniformly hyperbolic flow of order  $p > 2$  and  $v \in C_0^\eta(M)$ . Then  $\mathcal{W}_q(W_n, W) \rightarrow 0$  in  $C[0, 1]$  for all  $1 \leq q < 2(p - 1)$ .*

*Proof.* Since  $W_n$  has a finite moment of order  $2(p - 1)$ , together with Proposition 2.2 and the fact that  $W_n \rightarrow_w W$  as  $n \rightarrow \infty$ , we can obtain the conclusion just as in the proof of Theorem 2.3.  $\square$

**Theorem 5.3.** *Let  $\Psi_t : M \rightarrow M$  be a nonuniformly hyperbolic flow of order  $p \geq 4$  and  $v \in C_0^\eta(M)$ . Then there exists a constant  $C > 0$  such that  $\mathcal{W}_{\frac{p}{2}}(W_n, W) \leq Cn^{-\frac{1}{4} + \frac{1}{4(p-1)}}$  for all  $n \geq 1$ .*

*Example 5.4* (Axiom A flows). By [11], Axiom A flows [13, 44] are nonuniformly hyperbolic of order  $p$  for all  $1 \leq p < \infty$  (indeed  $r$  and hence  $\varphi$  are bounded), so Theorem 5.3 applies to  $\mathcal{W}_q$  for all  $q \geq 1$  and we obtain the rate  $\mathcal{W}_q(W_n, W) \leq Cn^{-\frac{1}{4} + \delta}$  for  $\delta > 0$  arbitrarily small. This includes Anosov flows [3] (such as geodesic flows on negatively curved manifolds) and solenoids.

*Example 5.5* (Planar periodic Lorentz gases). The 2-dimensional periodic Lorentz gas is a model of electron gases in metals studied by Sinai [42]. The Lorentz flow is a dispersing billiard flow on  $M = (\mathbb{T}^2 \setminus \Omega) \times S^1$ ,  $\Omega = \bigcup_{i=1}^k \Omega_i$ , where the obstacles  $\Omega_i$  are disjoint convex regions with  $C^3$  boundaries of nonvanishing curvature. The Poincaré map (or collision map)  $T$  is a dispersing billiard defined on  $X = \partial\Omega \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Under the finite horizon condition, which means that the roof function (or collision time)  $h$  is bounded, Young [46] demonstrated that  $T$  has exponential decay of correlations. In particular, it follows from [46] that  $\Psi_t$  is uniformly hyperbolic of order  $p$  for all  $1 \leq p < \infty$ . By Theorem 5.3, we obtain  $\mathcal{W}_q(W_n, W) \leq Cn^{-\frac{1}{4} + \delta}$  for all  $q \geq 1$  and all  $\delta > 0$ .

*Example 5.6* (Dispersing billiards with cusps). Next, we consider dispersing billiard flows with cusps, where the boundary curves are all dispersing but the interior angles at corner points are zero. Chernov and Markarian [14] proved that the billiard map has slow decay of correlations with rate  $1/n$ . However the collision time is not bounded below. By considering an alternative cross section  $X'$  bounded away from the cusps, Bálint and Melbourne [9] proved that the corresponding Lorentz flow is superpolynomially mixing. As a byproduct of proving this, they showed that the flow is nonuniformly hyperbolic of order  $p$  for all  $1 \leq p < \infty$ . Hence, by Theorem 5.3, we obtain  $\mathcal{W}_q(W_n, W) \leq Cn^{-\frac{1}{4}+\delta}$  for all  $q \geq 1$  and all  $\delta > 0$ .

*Example 5.7* (The Lorenz attractor). Statistical limit laws and exponential decay of correlations were proved for the classical Lorenz attractor in [5, 7, 10, 24]. In particular, it follows from [24] that the Lorenz attractor is nonuniformly hyperbolic of order  $p$  for all  $1 \leq p < \infty$ . Hence, by Theorem 5.3, we obtain  $\mathcal{W}_q(W_n, W) \leq Cn^{-\frac{1}{4}+\delta}$  for all  $q \geq 1$  and all  $\delta > 0$ . The same result holds for singular hyperbolic attractors by [6].

*Example 5.8* (Intermittent solenoidal flows). Let  $T_0 : [0, 1] \rightarrow [0, 1]$  be the intermittent map considered in Example 2.5. We consider a diffeomorphism  $T : X \rightarrow X$  introduced in [2], obtained by replacing the expanding map in the classical solenoid map by  $T_0$ , with exponential contraction along stable leaves. Hence we can construct an intermittent flow  $\Psi_t : M \rightarrow M$  with  $T : X \rightarrow X$  as a Poincaré map and Hölder return time function  $h : X \rightarrow [0, \infty)$ . This yields a uniformly expanding flow of order  $p$  for all  $p < 1/\beta$  where  $\beta$  is the parameter in Example 2.5. Hence by Theorem 5.3, we obtain  $\mathcal{W}_q(W_n, W) \leq Cn^{-\frac{1}{4}+\frac{\beta}{4(1-\beta)}+\delta}$  for all  $\beta \in (0, \frac{1}{4})$  and  $q < \frac{1}{2\beta}$ .

**5.3. Proof of Theorem 5.3.** Following the arguments in [36, Section 6], the proof consists of two ingredients:

- (1) Reduction to a roof function that is constant along stable leaves.
- (2) Reduction to an observable that is constant along stable leaves.

We sketch the constructions, referring to [36] for further details.

**Step 1: Reduction to a constant roof function along stable leaves.** Let  $\gamma_1 = \gamma^{n/2}$ ,  $\gamma_2 = \gamma_1^\eta$ . Define

$$\chi_Y : Y \rightarrow \mathbb{R}, \quad \chi_Y(y) = \sum_{n=0}^{\infty} \{ (h \circ T^n)(\pi y) - (h \circ T^n)(y) \}.$$

By (5.1) and (5.2),  $\chi_Y \in L^\infty$  and

$$|\chi_Y(y) - \chi_Y(y')| \leq C(d(y, y')^\eta + \gamma_1^{s(y, y')}) \quad \text{for all } y, y' \in Y,$$

(see [8, Lemma 8.4] or [36, Proposition 6.7]).

Now define

$$\tilde{\varphi} = \varphi + \chi_Y \circ F - \chi_Y.$$

Then  $\tilde{\varphi} : Y \rightarrow \mathbb{R}$  lies in  $L^p$  and, by construction,  $\tilde{\varphi}$  is constant along stable leaves. Replacing  $F$  and  $\varphi$  by  $F^k$  and  $\sum_{j=0}^{k-1} \varphi \circ F^j$  for a fixed sufficiently large  $k$ , we can suppose without loss of generality that  $\inf \tilde{\varphi} \geq 1$  so that  $\tilde{\varphi}$  is a roof function. By [8, Proposition 6.1] or [36, Proposition 6.8],

$$|\tilde{\varphi}(y) - \tilde{\varphi}(y')| \leq C(\inf_{Y_j} \tilde{\varphi}) \gamma_1^{s(y, y')} \quad \text{for all } y, y' \in Y_j, j \geq 1.$$

Let  $Y^{\tilde{\varphi}}$  be the suspension over the map  $F : Y \rightarrow Y$  with roof function  $\tilde{\varphi}$  and let  $\tilde{F}_t : Y^{\tilde{\varphi}} \rightarrow Y^{\tilde{\varphi}}$  be the associated suspension flow with ergodic invariant probability measure  $\mu_{\tilde{\varphi}}$ . We can define a measure-preserving semiconjugacy

$$g : Y^{\tilde{\varphi}} \rightarrow Y^{\varphi}, \quad g(y, u) = (y, u + \chi_Y(y)),$$

between  $\tilde{F}_t$  and  $F_t$ . Then  $\tilde{\pi}_M = \pi_M \circ g : Y^{\tilde{\varphi}} \rightarrow M$  is a measure-preserving semiconjugacy between  $\tilde{F}_t$  and  $\Psi_t$ .

By [36, Proposition 6.10], there exists  $\gamma_3 \in (0, \gamma_2)$  such that

$$(5.3) \quad d(\tilde{\pi}_M \circ \tilde{F}_t(y, 0), \tilde{\pi}_M \circ \tilde{F}_t(y', 0)) \leq C\gamma_3^t$$

for all  $y, y' \in Y$  with  $y' \in Q^s(y)$  and all  $t \geq 0$ .

Finally, let  $v \in C^\eta(M)$ . By [36, Proposition 6.11], the lifted observable  $v \circ \tilde{\pi}_M : Y^{\tilde{\varphi}} \rightarrow \mathbb{R}$  satisfies

$$|v \circ \tilde{\pi}_M(y, u) - v \circ \tilde{\pi}_M(y', s)| \leq C((\inf_{Y_j} \tilde{\varphi})\gamma_2^{s(y, y')} + |u - s|^\eta)$$

for all  $(y, u), (y', s) \in Y^{\tilde{\varphi}}$  such that  $y, y' \in \tilde{Y}_j$ .

**Step 2: Reduction to a constant observable along stable leaves.** By Step 1, we can assume without loss that the flow  $\Psi_t$  is modelled by a suspension flow  $F_t$  defined on a suspension  $Y^\varphi$  with roof function  $\varphi$  constant along stable leaves and satisfying

$$|\varphi(y) - \varphi(y')| \leq C(\inf_{Y_j} \varphi)\gamma_1^{s(y, y')} \quad \text{for all } y, y' \in Y_j, j \geq 1.$$

Moreover, we may assume the exponential contraction (5.3) along stable leaves and we may suppose that the lifted observable  $v \circ \pi_M : Y^\varphi \rightarrow \mathbb{R}$  satisfies

$$|v \circ \pi_M(y, u) - v \circ \pi_M(y', s)| \leq C((\inf_{Y_j} \varphi)\gamma_2^{s(y, y')} + |u - s|^\eta)$$

for all  $(y, u), (y', s) \in Y^\varphi$  such that  $y, y' \in \tilde{Y}_j$ .

Let  $\bar{Y}^{\bar{\varphi}}$  be the suspension over the quotient map  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  with roof function  $\bar{\varphi}$  and let  $\bar{F}_t : \bar{Y}^{\bar{\varphi}} \rightarrow \bar{Y}^{\bar{\varphi}}$  be the associated suspension flow. Since  $\bar{F}$  is a Gibbs-Markov map and the roof function  $\bar{\varphi}$  satisfies

$$|\bar{\varphi}(y) - \bar{\varphi}(y')| \leq C(\inf_{\bar{Y}_j} \bar{\varphi})\gamma_1^{s(y, y')} \quad \text{for all } y, y' \in \bar{Y}_j, j \geq 1,$$

the quotient semiflow  $\bar{F}_t : \bar{Y}^{\bar{\varphi}} \rightarrow \bar{Y}^{\bar{\varphi}}$  is a Gibbs-Markov semiflow w.r.t. the metric  $d(y, y') = \gamma_1^{s(y, y')}$ ,

Given  $v \in C^\eta(M)$ , define

$$\chi : Y^\varphi \rightarrow \mathbb{R}, \quad \chi = \sum_{n=0}^{\infty} \{v \circ \pi_M \circ F_n - v \circ \pi_M \circ F_n \circ \pi\},$$

where  $\pi(y, u) = (\pi y, u)$ . It follows from (5.3) that  $\chi \in L^\infty(Y^\varphi)$ .

Define

$$\hat{v} : Y^\varphi \rightarrow \mathbb{R}, \quad \hat{v} = v \circ \pi_M - \chi + \chi \circ F_1.$$

By construction,  $\hat{v}$  is constant along stable leaves and hence projects to an observable  $\bar{v} : \bar{Y}^{\bar{\varphi}} \rightarrow \mathbb{R}$ .

By [36, Proposition 6.17], there exists  $\gamma_4 \in (0, \gamma_3)$  such that

$$|\hat{v}(y, u) - \hat{v}(y', u)| \leq C(\inf_{Y_j} \varphi)\gamma_4^{s(y, y')}$$

for all  $(y, u), (y', u) \in Y^\varphi$  with  $y, y' \in Y_j$ ,  $j \geq 1$ . Hence  $\bar{v} \in \mathcal{V}^1(\bar{Y}^\varphi)$  with the metric  $d(y, y') = \gamma_4^{s(y, y')}$ .

*Proof of Theorem 5.3.* First we claim that to get the rates for  $W_n$  it suffices to prove them for the sequence

$$\bar{W}_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} \bar{v} \circ \bar{F}_s \, ds$$

defined on the probability space  $(\bar{Y}^\varphi, \bar{\mu}^\varphi)$ .

Consider the sequences

$$W'_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} v \circ \pi_M \circ F_s \, ds, \quad W''_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} \hat{v} \circ F_s \, ds.$$

Note that  $W'_n = W_n \circ \pi_M$  and  $W''_n = \bar{W}_n \circ \bar{\pi}$ . Since  $\pi_M$  and  $\bar{\pi}$  are measure-preserving,  $W_n =_d W'_n$  and  $\bar{W}_n =_d W''_n$ . For all  $t \in \mathbb{R}$ ,

$$\int_0^t (\chi - \chi \circ F_1) \circ F_s \, ds = \int_0^t \chi \circ F_s \, ds - \int_1^{t+1} \chi \circ F_s \, ds = \int_0^1 \chi \circ F_s \, ds - \int_t^{t+1} \chi \circ F_s \, ds.$$

It follows that for all  $q \geq 1$ ,

$$\begin{aligned} \mathcal{W}_q(W_n, \bar{W}_n) &= \mathcal{W}_q(W'_n, W''_n) \leq \left| \sup_{t \in [0, 1]} |W'_n(t) - W''_n(t)| \right|_\infty \\ &= n^{-1/2} \left| \sup_{t \in [0, 1]} \left| \int_0^{nt} (\chi - \chi \circ F_1) \circ F_s \, ds \right| \right|_\infty \leq 2n^{-1/2} |\chi|_\infty. \end{aligned}$$

proving the claim.

Since  $\bar{F}_t$  is a Gibbs-Markov semiflow and  $\bar{v} \in \mathcal{V}_0^1(\bar{Y}^\varphi)$ , the rate for  $\mathcal{W}_{\frac{p}{2}}(\bar{W}_n, W)$  follows from Lemma 4.1.  $\square$

#### ACKNOWLEDGEMENTS

This work is supported by National Key R&D Program of China (No. 2023YFA1009200) and NSFC (Grant 12501239). Zhe Wang is grateful to Prof. Zhenxin Liu for helpful discussions and suggestions.

#### REFERENCES

- [1] J. Aaronson and M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.* **1** (2001) 193–237.
- [2] J. F. Alves and V. Pinheiro. Slow rates of mixing for dynamical systems with hyperbolic structures. *J. Stat. Phys.* **131** (2008) 505–534.
- [3] D. V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.* **90** (1967) 209.
- [4] M. Antoniou and I. Melbourne. Rate of convergence in the weak invariance principle for deterministic systems. *Comm. Math. Phys.* **369** (2019) 1147–1165.
- [5] V. Araújo and I. Melbourne. Exponential decay of correlations for nonuniformly hyperbolic flows with a  $C^{1+\alpha}$  stable foliation, including the classical Lorenz attractor. *Ann. Henri Poincaré* **17** (2016) 2975–3004.
- [6] V. Araújo and I. Melbourne. Mixing properties and statistical limit theorems for singular hyperbolic flows without a smooth stable foliation. *Adv. Math.* **349** (2019) 212–245.
- [7] V. Araújo, I. Melbourne and P. Varandas. Rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps. *Comm. Math. Phys.* **340** (2015) 901–938.

- [8] P. Bálint, O. Butterley and I. Melbourne. Polynomial decay of correlations for flows, including Lorentz gas examples. *Comm. Math. Phys.* **368** (2019) 55–111.
- [9] P. Bálint and I. Melbourne. Decay of correlations and invariance principles for dispersing billiards with cusps, and related planar billiard flows. *J. Stat. Phys.* **133** (2008) 435–447.
- [10] P. Bálint and I. Melbourne. Statistical properties for flows with unbounded roof function, including the Lorenz attractor. *J. Stat. Phys.* **172** (2018) 1101–1126.
- [11] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973) 429–460.
- [12] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics. Vol. 470 Springer-Verlag, Berlin-New York, 1975. i+108 pp.
- [13] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975) 181–202.
- [14] N. Chernov and R. Markarian. Dispersing billiards with cusps: slow decay of correlations. *Comm. Math. Phys.* **270** (2007) 727–758.
- [15] K. L. Chung. *A Course in Probability Theory*. Third edition. Academic Press, Inc., San Diego, CA, 2001. xviii+419 pp.
- [16] J. Dedecker, F. Merlevède and E. Rio. Deviation inequalities for dependent sequences with applications to strong approximations. *Stochastic Process. Appl.* **174** (2024) Paper No. 104377, 16 pp.
- [17] M. Denker and W. Philipp. Approximation by Brownian motion for Gibbs measures and flows under a function. *Ergodic Theory Dynam. Systems* **4** (1984) 541–552.
- [18] N. Fleming-Vázquez. Rates of convergence in the multivariate weak invariance principle for nonuniformly hyperbolic maps. Preprint, 2025. arXiv:2503.16358.
- [19] G. A. Gottwald and I. Melbourne. Central limit theorems and suppression of anomalous diffusion for systems with symmetry. *Nonlinearity* **29** (2016) 2941–2960.
- [20] S. Gouëzel. Central limit theorem and stable laws for intermittent maps. *Probab. Theory Related Fields* **128** (2004) 82–122.
- [21] S. Gouëzel. Berry-Esseen theorem and local limit theorem for non uniformly expanding maps. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 997–1024.
- [22] P. Hall and C. C. Heyde. *Martingale Limit Theory and Its Application*. Probability and Mathematical Statistics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. xii+308 pp.
- [23] F. Hofbauer and G. Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.* **180** (1982) 119–140.
- [24] M. Holland and I. Melbourne. Central limit theorems and invariance principles for Lorenz attractors. *J. London Math. Soc.* **76** (2007) 345–364.
- [25] D. Kelly and I. Melbourne. Smooth approximation of stochastic differential equations. *Ann. Probab.* **44** (2016) 479–520.
- [26] A. Korepanov, Z. Kosloff and I. Melbourne. Martingale-coboundary decomposition for families of dynamical systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35** (2018) 859–885.
- [27] A. Korepanov, Z. Kosloff and I. Melbourne. Deterministic homogenization under optimal moment assumptions for fast-slow systems. Part 1. *Ann. Inst. H. Poincaré Probab. Stat.* **58** (2022) 1305–1327.
- [28] Z. Liu and Z. Wang. Wasserstein convergence rates in the invariance principle for deterministic dynamical systems. *Ergodic Theory Dynam. Systems* **44** (2024) 1172–1191.
- [29] Z. Liu and Z. Wang. Wasserstein convergence rates in the invariance principle for sequential dynamical systems. *Nonlinearity* **37** (2024) Paper No. 125019, 27 pp.
- [30] C. Liverani, B. Saussol and S. Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** (1999) 671–685.
- [31] I. Melbourne and M. Nicol. Almost sure invariance principle for nonuniformly hyperbolic systems. *Comm. Math. Phys.* **260** (2005) 131–146.
- [32] I. Melbourne and A. Török. Central limit theorems and invariance principles for time-one maps of hyperbolic flows. *Comm. Math. Phys.* **229** (2002) 57–71.
- [33] I. Melbourne and A. Török. Statistical limit theorems for suspension flows. *Israel J. Math.* **144** (2004) 191–209.
- [34] I. Melbourne and A. Török. Convergence of moments for Axiom A and non-uniformly hyperbolic flows. *Ergodic Theory Dynam. Systems* **32** (2012) 1091–1100.



- [35] I. Melbourne and P. Varandas. A note on statistical properties for nonuniformly hyperbolic systems with slow contraction and expansion. *Stoch. Dyn.* **16** (2016) 1660012, 13 pages.
- [36] N. Paviato. Rates for maps and flows in a deterministic multidimensional weak invariance principle. Preprint, 2024. arXiv:2406.06123.
- [37] F. Pène. Rates of convergence in the CLT for two-dimensional dispersive billiards. *Comm. Math. Phys.* **225** (2002) 91–119.
- [38] F. Pène. A Berry Esseen result for the billiard transformation. 2007. hal-01101281.
- [39] M. Ratner. The central limit theorem for geodesic flows on  $n$ -dimensional manifolds of negative curvature. *Israel J. Math.* **16** (1973) 181–197.
- [40] D. Ruelle. *Thermodynamic Formalism*. Encyclopedia of Math. and its Applications **5**, Addison Wesley, Massachusetts, 1978.
- [41] Y. G. Sinai. The central limit theorem for geodesic flows on manifolds of constant negative curvature. *Soviet Math. Dokl.* **1** (1960) 983–987.
- [42] Y. G. Sinai. Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. *Uspehi Mat. Nauk* **25** (1970) 141–192.
- [43] J. G. Sinai. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk* **27** (1972) 21–64.
- [44] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967) 747–817.
- [45] C. Villani. *Optimal Transport. Old and New*. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338. Springer-Verlag, Berlin, 2009. xxii+973 pp.
- [46] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.*(2) **147** (1998) 585–650.

I. MELBOURNE: MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK  
*Email address:* i.melbourne@warwick.ac.uk

Z. WANG: SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, P. R. CHINA  
*Email address:* zwangmath@hotmail.com