Global-local mixing for intermittent maps with multiple neutral fixed points

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Abstract

We prove full global-local mixing for a large class of intermittent maps including maps with multiple neutral fixed points, nonMarkovian intermittent maps, and multidimensional nonMarkovian intermittent maps.

1 Introduction

There has been a significant surge of interest recently in the ergodic theory of dynamical system preserving an infinite σ-finite measure [1]. A remark in [1, p. 75] suggests that there is no reasonable notion of mixing for infinite measure systems. Nevertheless, there are at least two notions of mixing that have proved fruitful: *Krickeberg mixing* [14] and more recently *global-local mixing* introduced by Lenci [15].

Let (X, μ) be an infinite, σ-finite measure space. We suppose that $f : X \to X$ is a conservative, exact, measure-preserving transformation. The usual mixing property $\lim_{n\to\infty} \mu(A \cap f^{-n}B) = \mu(A)\mu(B)$ for all (finite measure) measurable sets A, B fails. Moreover, by [12] there always exists a measurable set W with $0 < \mu(W) < \infty$ such that $\mu(W \cap f^{-n}W) = 0$ infinitely often. The system is Krickeberg mixing if there exist constants $a_n \to \infty$ such that $\lim_{n\to\infty} a_n \int_X v\, w \circ f^n \, d\mu = \int_X v \, d\mu \int_X w \, d\mu$ for a sufficiently large class of observables v and w. The more recent notion of global-local mixing [15] asks that $\lim_{n\to\infty} \int_X v\, g \circ f^n \, d\mu = \bar{g} \int_X v \, d\mu$ for all $v \in L^1$ and a sufficiently large class of "global" observables g.

Roughly speaking, an observable $g \in L^\infty$ is *global* if there exists $\bar{g} \in \mathbb{R}$ such that

$$\lim_{m\to\infty} \frac{\int_{Z_m} g \, d\mu}{\mu(Z_m)} = \bar{g}$$

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for suitable nested sequences of measurable subsets $Z_m \subset X$ with $\mu(Z_m) < \infty$ and $\mu(X \setminus Z_m) \to 0$. The precise notion of global observable depends on the context and is made precise for our purposes in Definition 2.2 below.

Prototypical examples are the intermittent maps introduced in [22]. Krickeberg mixing was studied under restrictive conditions in [29] and a full theory was developed by [11, 19, 25, 26]. Likewise, global-local mixing for intermittent maps was studied under restrictive conditions in [3, 4, 5] and a full theory is developed in the current paper. Results on global-local mixing for other classes of dynamical systems can be found in [2, 9, 10].

Bonanno & Lenci [4] proved global-local mixing for a large class of intermittent maps $f : X \to X$ with $X = [0, 1]$, including generalised classical Pomeau-Manneville maps of the form

$$fx = x + bx^{\alpha+1} \mod 1$$

where $\alpha \in (0, 1]$ and $b \in \mathbb{Z}^+$, as well as generalised Liverani-Saussol-Vaienti (LSV) maps [17]. The latter have finitely or infinitely many full branches where the first branch is of the form $x + bx^{\alpha+1}$ with $b > 0$ and the remaining branches are linear with positive slope. They also studied perturbations of these two classes of maps. Full results were obtained for $\alpha \in (0, 1)$ and a weaker result was obtained for $\alpha = 1$.

The aim of the current paper is to remove various restrictions in [4] thereby developing a complete theory of global-local mixing for intermittent maps:

- The examples mentioned so far have a single neutral fixed point at 0. We allow multiple neutral fixed points with the same order of neutrality $\alpha \in (0, 1]$.
- We remove the assumption that f is full-branch or even Markov.
- Full global-local mixing is proved also in the case $\alpha = 1$.
- There is a technical assumption (A5) in [4] which is hard to verify in general but is unnecessary for our approach.

It is worth noting that the initial results of [29] on Krickeberg mixing for intermittent maps also required a technical assumption that was hard to verify and not always true. The approach in [19] eliminated the need for such an assumption, and a key step in the current paper is to apply the result of [19].

Remark 1.1 For certain classes of dynamical systems, global-local mixing seems to be a more flexible property than Krickeberg mixing. Certainly, there are numerous examples in [9] which are shown to be global-local mixing and where proving Krickeberg mixing is way beyond reach. In contrast, a full theory of global-local mixing for intermittent maps seems to be more delicate than Krickeberg mixing: Krickeberg mixing is just one step in our approach (see condition (2.3) below) and further hypotheses and arguments are required to achieve global-local mixing.

The remainder of this paper is organised as follows. In Section 2, we state our main theorem, Theorem 2.6, in a suitable abstract setting. In Section 3, we show

how to reduce the proof of Theorem 2.6 to a calculation involving real sequences, see Lemma 3.1. In Section 4, we prove Lemma 3.1. In Section 5, we apply our result to large classes of one-dimensional intermittent maps, including those in [4] as well as maps with multiple neutral fixed points and nonMarkovian intermittent maps. We also treat a multidimensional nonMarkovian intermittent map studied in [21].

Notation We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $a_n \leq Cb_n$ for all $n \geq n_0$. We write $a_n \approx b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$. As usual, $a_n = o(b_n)$ means that $a_n/b_n \to 0$ and $a_n \sim b_n$ means that $a_n/b_n \to 1$.

2 The abstract setup

Fix a subset $Y \subset X$ with $0 < \mu(Y) < \infty$. We define the first hit map

$$\tau : X \to \mathbb{Z}^+, \qquad \tau(x) = \inf\{n \geq 1 : f^n x \in Y\}.$$

Set

$$Y_n = \{y \in Y : \tau(y) = n\}, \qquad X_n = \{x \in X \setminus Y : \tau(x) = n\}.$$

Proposition 2.1 *$\mu(X_n) < \infty$ for all $n \geq 1$, and $\lim_{n\to\infty} \mu(X_n) = 0$.*

Proof Note that $\mu(X_n) \leq \mu(f^{-n}Y) = \mu(Y) < \infty$ by f-invariance of μ. Also,

$$\sum_{j=1}^n \mu(X_j) = \sum_{j=1}^n \mu(f^{-1}X_j) = \sum_{j=1}^n \mu(X_{j+1}) + \sum_{j=1}^n \mu(Y_{j+1}).$$

Hence

$$\mu(X_1) - \mu(X_{n+1}) = \sum_{j=2}^{n+1} \mu(Y_j) \to \mu(Y) - \mu(Y_1).$$

Similarly, $\mu(Y) = \mu(f^{-1}Y) = \mu(Y_1) + \mu(X_1)$ and it follows that $\mu(X_n) \to 0$.

∎

We can now define our notion of global observable:

Definition 2.2 An observable $g \in L^\infty$ is a *global observable* if there exists $\bar{g} \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\int_{\bigcup_{i=1}^n X_i} g\, d\mu}{\mu(\bigcup_{i=1}^n X_i)} = \bar{g}.$$

The global observable is *centred* if $\bar{g} = 0$.

This coincides with the rough definition in the Introduction, taking $Z_m = X_m \cup Y$.

Note that $f^j Y_{j+n} \subset X_n$ for all $j, n \geq 1$. In general $f^j|_{Y_{j+n}} : Y_{j+n} \to X_n$ need not be a bijection. We assume that there exist integers $n_0 \geq 1$ and $1 \leq q' \leq q$ (where we assume that $q' < \infty$ but allow $q = \infty$), a surjection $\psi : \{1 \leq p \leq q\} \to \{1 \leq p' \leq q'\}$, and decompositions

$$Y_n = \bigcup_{p=1}^{q} Y_{n,p}, \qquad X_n = \bigcup_{p'=1}^{q'} X_{n,p'}, \quad n \geq n_0,$$

into disjoint sets such that f^j maps each $Y_{j+n,p}$ (for $n \geq n_0$) bijectively onto $X_{n,\psi(p)}$.

Remark 2.3 For intermittent maps with one neutral fixed point and two branches (such as the LSV map [17]), we can take $q = q' = 1$.

The abstract setting can be generalised in various ways. For instance, in Example 5.16 we consider bijections $f^j : Y_{j+n,p} \to X_n$ where the decomposition $Y_{n+j} = \bigcup_{p=1}^{q_j}$ depends on j and n. We prefer to emphasise the current setting since it suffices for all of the one-dimensional examples with less onerous notation.

We assume that there are constants $\alpha \in (0, 1]$ and $\gamma_{p'} > 0$, $1 \leq p' \leq q'$, such that

$$\sum_{\psi(p)=p'} \mu(Y_{n,p}) \sim \gamma_{p'}\, n^{-(\alpha+1)} \quad \text{as } n \to \infty \text{ for all } 1 \leq p' \leq q'. \tag{2.1}$$

In particular,

$$\mu(Y_n) \sim \gamma n^{-(\alpha+1)} \quad \text{as } n \to \infty \tag{2.2}$$

where $\gamma = \sum_{p'=1}^{q'} \gamma_{p'} \in (0, \infty)$.

Remark 2.4 When $q' = 1$, conditions (2.1) and (2.2) coincide.

Proposition 2.5 *$\mu(X_{n,p'}) \sim \alpha^{-1}\gamma_{p'}\, n^{-\alpha}$ as $n \to \infty$ for all $1 \leq p' \leq q'$.*

Proof By f-invariance of μ,

$$\mu(X_{n-1,p'}) = \mu(f^{-1}X_{n-1,p'}) = \mu(X_{n,p'}) + \sum_{\psi(p)=p'} \mu(Y_{n,p}).$$

By (2.1),

$$\mu(X_{n-1,p'}) - \mu(X_{n,p'}) \sim \gamma_{p'}\, n^{-(\alpha+1)}.$$

By Proposition 2.1, $\mu(X_{k,p'}) \to 0$ as $k \to \infty$. Hence

$$\mu(X_{n,p'}) = \lim_{k\to\infty} \big(\mu(X_{n,p'}) - \mu(X_{k,p'})\big) = \sum_{i=n}^{\infty} \big(\mu(X_{i,p'}) - \mu(X_{i+1,p'})\big) \sim \alpha^{-1}\gamma_{p'} n^{-\alpha}$$

as required. ∎

Our main assumption is a Krickeberg mixing condition on f. Let $a_n = n^{1-\alpha}$ for $0 < \alpha < 1$ and $a_n = \log n$ for $\alpha = 1$. Let $L : L^1(X, \mu) \to L^1(X, \mu)$ denote the transfer operator of f (so $\int_X Lv\, w\, d\mu = \int_X v\, w \circ f\, d\mu$ for $v \in L^1(X, \mu)$, $w \in L^\infty$). We require that there is a constant $K \geq 0$ such that

$$\lim_{n\to\infty} a_n \big|1_Y(L^n 1_Y - K a_n^{-1})\big|_\infty = 0. \tag{2.3}$$

It turns out that our assumptions so far suffice for global observables that are constant on each $X_{n,p'}$. We have to treat separately the observables in L^∞ that are mean zero when restricted to each $X_{n,p'}$. Our final assumption, which ensures that such observables are negligible, is:

$$\lim_{j\to\infty} j^\alpha \sum_{n=n_0}^{\infty} \sum_{\psi(p)=p'} \int_{Y_{j+n,p}} g \circ f^j\, d\mu = 0 \quad \text{for all } 1 \leq p' \leq q' \tag{2.4}$$

for all $g \in L^\infty$ satisfying $\int_{X_{n,p'}} g\, d\mu = 0$ for all $n \geq n_0$, $1 \leq p' \leq q'$.

We can now state our main theorem.

Theorem 2.6 *Suppose that* (2.1)*,* (2.3) *and* (2.4) *hold. Then f is global-local mixing.*

Remark 2.7 Condition (2.3) has been verified by [11, 19] in many examples and is automatic (given (2.2)) when $F = f^\tau : Y \to Y$ is Gibbs-Markov or when F is AFU (see Section 5.1).

Turning to (2.4) for $g \in L^\infty$ piecewise mean zero, we fix $1 \leq p' \leq q'$ and note that

$$\sum_{\psi(p)=p'} \int_{Y_{j+n,p}} g \circ f^j\, d\mu = \sum_{\psi(p)=p'} \int_{X_{n,p'}} g\, d(f^j|_{Y_{j+n,p}})_*\mu = \int_{X_{n,p'}} g J_{j,n,p'}\, d\mu$$

where

$$J_{j,n,p'} : X_{n,p'} \to (0, \infty), \qquad J_{j,n,p'} = \sum_{\psi(p)=p'} \frac{d(f^j|_{Y_{j+n,p}})_*\mu}{d\mu|_{X_{n,p'}}}.$$

Since g is piecewise mean zero, for any $c \in \mathbb{R}$,

$$\int_{X_{n,p'}} g J_{j,n,p'}\, d\mu = \int_{X_{n,p'}} g(J_{j,n,p'} - c)\, d\mu,$$

so by Proposition 2.5,

$$\Big|\sum_{\psi(p)=p'} \int_{Y_{j+n,p}} g \circ f^j\, d\mu\Big| \ll n^{-\alpha} |g|_\infty \sup_{X_{n,p'}} |J_{j,n,p'} - c|.$$

Suppose that there are constants $c_{j,n,p'} > 0$ such that

$$\sum_{n=n_0}^{\infty} n^{-\alpha} \sup_{X_{n,p'}} |J_{j,n,p'} - c_{j,n,p'}| = o(j^{-\alpha}) \quad \text{as } j \to \infty. \tag{2.5}$$

Then it follows that

$$\sum_{n=n_0}^{\infty} \sum_{\psi(p)=p'} \int_{Y_{j+n,p}} g \circ f^j \, d\mu = o(j^{-\alpha}) \quad \text{as } j \to \infty.$$

Hence to verify (2.4), it suffices to establish (2.5) for each $1 \le p' \le q'$.

3 Proof of Theorem 2.6

In this section, we show how the proof of Theorem 2.6 reduces to a calculation about real sequences contained in Lemma 3.1 below.

3.1 Strategy of the proof

In proving Theorem 2.6, we can suppose without loss of generality that g is centred. Also, by exactness of f, it suffices (see [4, Lemma 3.3(a)]) to prove the result for one $v \in L^1(X,\mu)$ with $\int_X v \, d\mu \ne 0$. Choosing $v = 1_Y$, we reduce to showing that

$$\lim_{n\to\infty} \int_Y g \circ f^n \, d\mu = 0.$$

In Lemma 3.4 below, we use condition (2.3) to reduce to showing that

$$\lim_{n\to\infty} \sum_{j=0}^{n} a_{n-j}^{-1} \sum_{i=n_0}^{\infty} \sum_{\psi(p)=p'} \int_{Y_{j+i,p}} g \circ f^j \, d\mu = 0 \quad \text{for each } 1 \le p' \le q'. \tag{3.1}$$

Next, we define sequences

$$\bar{g}_{n,p'} = \frac{\int_{X_{n,p'}} g \, d\mu}{\mu(X_{n,p'})}, \quad n \ge n_0, \qquad \text{and} \qquad \bar{g}_n = \alpha^{-1} \sum_{p'=1}^{q'} \gamma_{p'} \bar{g}_{n,p'}.$$

The sequences are bounded since $g \in L^\infty$; indeed $|\bar{g}_{n,p}| \le |g|_\infty$ and $|\bar{g}_n| \le \alpha^{-1}\gamma |g|_\infty$. Define

$$\tilde{g} \in L^\infty, \qquad \tilde{g} = g - \sum_{n \ge n_0} \sum_{p'=1}^{q'} 1_{X_{n,p'}} \bar{g}_{n,p'}.$$

In this way, g decomposes into a piecewise constant function $g - \tilde{g}$ with values $\bar{g}_{n,p'}$ on $X_{n,p'}$ and a function $\tilde{g}$ which has mean zero when restricted to each $X_{n,p'}$. In Proposition 3.5, we use assumption (2.4) to show that the contribution of $\tilde{g}$ to (3.1) is zero.

In Section 4, we prove by direct calculation the key estimate:

Lemma 3.1 *Suppose that $\bar{g}_i$ is a bounded real sequence such that*

$$\lim_{n\to\infty} a_n^{-1} \sum_{i=n_0}^{n} i^{-\alpha} \bar{g}_i = 0. \tag{3.2}$$

Then

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} a_{n-j}^{-1} \sum_{i=1}^{\infty} \bar{g}_i (j+i)^{-(\alpha+1)} = 0. \tag{3.3}$$

In Proposition 3.2, we show that the global observable assumption on g (and hence $g - \tilde{g}$) implies that the hypothesis (3.2) of Lemma 3.1 is satisfied. Finally, in Proposition 3.6, we apply Lemma 3.1 to show that the contribution of $g - \tilde{g}$ to (3.1) is zero. This completes the proof of Theorem 2.6.

3.2 Characterisation of global variables

Proposition 3.2 *Let $g \in L^\infty$ be a centred global observable. Then* (3.2) *holds.*

Proof By Proposition 2.5, $\mu(X_n) \sim \alpha^{-1}\gamma n^{-\alpha}$ so there exists $\tilde{\gamma} > 0$ such that $\mu\big(\bigcup_{i=n_0}^n X_i\big) \sim \tilde{\gamma} a_n$. On the other hand, by definition of $\bar{g}_{i,p'}$, $\bar{g}_i$ and Proposition 2.5,

$$\begin{aligned} \int_{\bigcup_{i=n_0}^n X_i} g\, d\mu &= \sum_{i=n_0}^{n} \sum_{p'=1}^{q'} \int_{X_{i,p'}} g\, d\mu = \sum_{i=n_0}^{n} \sum_{p'=1}^{q'} \bar{g}_{i,p'}\, \mu(X_{i,p'}) \\ &= \alpha^{-1} \sum_{i=n_0}^{n} \sum_{p'=1}^{q'} \bar{g}_{i,p'}\, \gamma_{p'}\, i^{-\alpha} + o(a_n) = \sum_{i=n_0}^{n} \bar{g}_i\, i^{-\alpha} + o(a_n). \end{aligned}$$

Hence, by definition of centred global observable,

$$\sum_{i=n_0}^{n} i^{-\alpha} \bar{g}_i = \int_{\bigcup_{i=n_0}^n X_i} g\, d\mu + o(a_n) = o\Big(\mu\Big(\bigcup_{i=n_0}^{n} X_i\Big)\Big) + o(a_n) = o(a_n)$$

as required. ∎

3.3 Reduction of the problem

In this subsection, we carry out the steps described in Subsection 3.1 and reduce the proof of Theorem 2.6 to Lemma 3.1.

Recall that $a_n = n^{1-\alpha}$ for $0 < \alpha < 1$ and $a_n = \log n$ for $\alpha = 1$. Whenever a_n^{-1} is undefined, we set $a_n = 1$.

Proposition 3.3 *The sequence* $\sum_{j=1}^n a_{n-j}^{-1} j^{-\alpha}$ *is bounded. In addition, if* $\lim_{n\to\infty}\delta(n)=0$, *then*

$$\lim_{n\to\infty}\sum_{j=1}^n a_{n-j}^{-1} j^{-\alpha}(\delta(j)+\delta(n-j)) = \lim_{n\to\infty}\sum_{j=1}^n a_{n-j}^{-1} j^{-\alpha}(\delta(j)+\delta(n-j)) = 0.$$

Proof For the first statement, see for example [8, Lemmas A.1 and A.4]. The second statement holds by [8, Lemma A.2] for $0<\alpha<1$ and by [8, Lemma A.4] for $\alpha=1$. ∎

The next result is a consequence of (2.2) and (2.3).

Lemma 3.4 *Let* $g\in L^\infty$. *Then*

$$\lim_{n\to\infty}\Big(\int_Y g\circ f^n\,d\mu - K\sum_{j=0}^n a_{n-j}^{-1}\sum_{i=n_0}^\infty\sum_{p'=1}^{q'}\sum_{\psi(p)=p'}\int_{Y_{j+i,p}} g\circ f^j\,d\mu\Big)=0,$$

where K *is in condition* (2.3).

Proof Write Y as a disjoint union

$$Y=\bigcup_{j=0}^n\big\{y\in Y: f^{n-j}y\in Y,\ \tau(f^{n-j}y)>j\big\}.$$

Then

$$\begin{aligned}\int_Y g\circ f^n\,d\mu &= \sum_{j=0}^n\int_X 1_Y 1_Y\circ f^{n-j}1_{\{\tau>j\}}\circ f^{n-j} g\circ f^n\,d\mu\\ &= \sum_{j=0}^n\int_X 1_Y(L^{n-j}1_Y)\,1_{\{\tau>j\}}\,g\circ f^j\,d\mu.\end{aligned}$$

By (2.3),

$$\int_Y g\circ f^n\,d\mu = K\sum_{j=0}^n a_{n-j}^{-1}\int_Y 1_{\{\tau>j\}} g\circ f^j\,d\mu + O\Big(|g|_\infty\sum_{j=0}^n e_{n-j}\,\mu(y\in Y:\tau(y)>j)\Big)$$

where $e_n=o(a_n^{-1})$ and $\mu(y\in Y:\tau(y)>j)=O(j^{-\alpha})$ by (2.2). By Proposition 3.3, $\lim_{n\to\infty}\sum_{j=0}^n e_{n-j}\,\mu(y\in Y:\tau(y)>j)=0$

By (2.2), $\big|\int_{Y_{j+i}} g\circ f^j\,d\mu\big|\le |g|_\infty\mu(y\in Y:\tau(y)=j+i)=O(j^{-(\alpha+1)})$ for each i. Hence

$$\begin{aligned}\int_Y 1_{\{\tau>j\}} g\circ f^j\,d\mu = \sum_{i=1}^\infty\int_{Y_{j+i}} g\circ f^j\,d\mu &= \sum_{i=n_0}^\infty\int_{Y_{j+i}} g\circ f^j\,d\mu + O(j^{-(\alpha+1)})\\ &= \sum_{i=n_0}^\infty\sum_{p'=1}^{q'}\sum_{\psi(p)=p'}\int_{Y_{j+i,p}} g\circ f^j\,d\mu + O(j^{-(\alpha+1)}).\end{aligned}$$

By Proposition 3.3, $\lim_{n\to\infty}\sum_{j=1}^n a_{n-j}^{-1} j^{-(\alpha+1)} = 0$, so the result follows. ∎

The next result tells us that the contribution from the piecewise mean zero observable $\tilde{g}$ in the decomposition $g = (g - \tilde{g}) + \tilde{g}$ is negligible.

Proposition 3.5 *Let $g \in L^\infty$. Suppose that $\int_{X_{n,p'}} g\,d\mu = 0$ for all $n \geq n_0$, $1 \leq p' \leq q'$. Then $\lim_{n\to\infty}\int_Y g\circ f^n\,d\mu = 0$.*

Proof This is immediate from (2.4), Proposition 3.3 and Lemma 3.4. ∎

Finally, we conclude global-local mixing for the piecewise constant observable $g-\tilde{g}$.

Proposition 3.6 *Let $g \in L^\infty$ be a centred global observable such that $g|_{X_{n,p'}} \equiv \bar{g}_{n,p'}$ for each $n \geq n_0$, $1 \leq p' \leq q'$. Then $\lim_{n\to\infty}\int_Y g\circ f^n\,d\mu = 0$.*

Proof Let $1 \leq p' \leq q'$. Since $f^j Y_{j+i,p} = X_{i,\psi(p)}$.

$$\sum_{\psi(p)=p'}\int_{Y_{j+i,p}} g\circ f^j\,d\mu = \bar{g}_{i,p'}\sum_{\psi(p)=p'}\mu(Y_{j+i,p}).$$

Hence by (2.1),

$$\sum_{i=n_0}^{\infty}\sum_{\psi(p)=p'}\int_{Y_{j+i,p}} g\circ f^j\,d\mu = \sum_{i=n_0}^{\infty}\bar{g}_{i,p'}\gamma_{p'}(j+i)^{-(\alpha+1)} + o(j^{-\alpha}).$$

By Lemma 3.1, $\lim_{n\to\infty}\sum_{j=0}^n a_{n-j}^{-1}\sum_{i=n_0}^{\infty}\bar{g}_i(j+i)^{-(\alpha+1)} = 0$. Hence by Proposition 3.3,

$$\lim_{n\to\infty}\sum_{j=0}^{n} a_{n-j}^{-1}\sum_{i=n_0}^{\infty}\sum_{\psi(p)=p'}\int_{Y_{j+i,p}} g\circ f^j\,d\mu = \lim_{n\to\infty}\sum_{j=1}^{n} a_{n-j}^{-1}\,o(j^{-\alpha}) = 0.$$

The result follows by Lemma 3.4. ∎

4 Proof of Lemma 3.1

In this section, we prove Lemma 3.1, thereby completing the proof of Theorem 2.6. The cases $0<\alpha<1$ and $\alpha = 1$ are covered in Subsections 4.1 and 4.2 respectively.

Recall that $a_n = n^{1-\alpha}$ for $0<\alpha<1$ and $a_n = \log n$ for $\alpha = 1$. Let $b_n = \sum_{i\leq n} i^{-\alpha}\bar{g}_i$. For all $\alpha\in(0,1]$, we have the following resummation argument.

Lemma 4.1 *Let $\alpha\in(0,1]$ and fix $j\geq 1$. Then $\sum_{i\geq 1}\bar{g}_i(j+i)^{-(\alpha+1)} = A_j - B_j$ where*

$$A_j = \sum_{i=1}^{\infty} b_i\, i^{\alpha}\big\{(j+i)^{-(\alpha+1)} - (j+i+1)^{-(\alpha+1)}\big\},$$

$$B_j = \sum_{i=1}^{\infty} b_i\,(j+i+1)^{-(\alpha+1)}\big\{(i+1)^{\alpha} - i^{\alpha}\big\}.$$

Proof Let $d_i = i^\alpha (j+i)^{-(\alpha+1)}$ and $b_0 = 0$. Resummation by parts gives

$$\sum_{i\geq 1} \bar{g}_i (j+i)^{-(\alpha+1)} = \sum_{i\geq 1} i^{-\alpha} \bar{g}_i d_i = \sum_{i\geq 1} (b_i - b_{i-1}) d_i = \sum_{i\geq 1} b_i (d_i - d_{i+1}).$$

But

$$d_i - d_{i+1} = i^\alpha \Big\{ (j+i)^{-(\alpha+1)} - (j+i+1)^{-(\alpha+1)} \Big\} - (j+i+1)^{-(\alpha+1)} \Big\{ (i+1)^\alpha - i^\alpha \Big\},$$

so the result follows. ∎

4.1 The case $\alpha < 1$

In this subsection, we complete the proof of Lemma 3.1 for $0 < \alpha < 1$. By assumption, $b_n = \sum_{i\leq n} i^{-\alpha} \bar{g}_i = o(a_n) = o(n^{1-\alpha})$, so we can write

$$|b_n| = \psi(n) n^{1-\alpha} \quad \text{where} \quad \lim_{n\to\infty} \psi(n) = 0.$$

Note that $(j+i)^{-(\alpha+1)} - (j+i+1)^{-(\alpha+1)} \ll (j+i)^{-(\alpha+2)}$ and $(i+1)^\alpha - i^\alpha \leq i^{\alpha-1}$. It follows that

$$|A_j| \ll \sum_{i=1}^{\infty} \psi(i) i (j+i)^{-(\alpha+2)} \leq Z_j \qquad \text{and} \qquad |B_j| \ll Z_j$$

where

$$Z_j = \sum_{i=1}^{\infty} \psi(i) (j+i)^{-(\alpha+1)}.$$

Hence it suffices to show that

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} (n-j)^{-(1-\alpha)} Z_j = 0.$$

Proposition 4.2 *For any $\delta > 0$, there exists $L(\delta) > 0$ such that*

$$Z_j \leq L(\delta) j^{-(\alpha+1)} + \delta j^{-\alpha} \quad \textit{for all } j \geq 1.$$

Proof Let $M = \max_{i\geq 1} \psi(i)$. Given $\delta > 0$, choose $L_1(\delta) \geq 1$ such that $\psi(i) \leq \delta\alpha$ for all $i > L_1(\delta)$. Then

$$\begin{aligned} Z_j &\leq M \sum_{i=1}^{L_1(\delta)} (j+i)^{-(\alpha+1)} + \delta\alpha \sum_{i=L_1(\delta)+1}^{\infty} (j+i)^{-(\alpha+1)} \\ &\leq M L_1(\delta) j^{-(\alpha+1)} + \delta\alpha \sum_{i=1}^{\infty} (j+i)^{-(\alpha+1)}. \end{aligned}$$

Now use that $\sum_{i=1}^{\infty} (j+i)^{-(\alpha+1)} \leq \int_1^\infty (x+j-1)^{-(\alpha+1)}\, dx = \alpha^{-1} j^{-\alpha}$. ∎

Corollary 4.3 $\lim_{n\to\infty}\sum_{j=1}^{n-1}(n-j)^{-(1-\alpha)}Z_j = 0$.

Proof By Proposition 4.2, $\sum_{j=1}^{n-1}(n-j)^{-(1-\alpha)}Z_j \le L(\delta)U_n + \delta V_n$ where

$$U_n = \sum_{j=1}^{n-1}(n-j)^{-(1-\alpha)}j^{-(\alpha+1)}, \qquad V_n = \sum_{j=1}^{n-1}(n-j)^{-(1-\alpha)}j^{-\alpha}.$$

But

$$\begin{aligned} U_n &= \sum_{1\le j<n/2}(n-j)^{-(1-\alpha)}j^{-(\alpha+1)} + \sum_{n/2\le j<n}(n-j)^{-(1-\alpha)}j^{-(\alpha+1)} \\ &\ll n^{-(1-\alpha)}\sum_{j=1}^{\infty}j^{-(\alpha+1)} + n^{-(\alpha+1)}\sum_{j=1}^{n/2}j^{-(1-\alpha)} \ll n^{-(1-\alpha)}. \end{aligned}$$

Also, V_n is bounded by Proposition 3.3. Hence there is a constant $C>0$ (independent of δ) such that

$$U_n \le Cn^{-(1-\alpha)} \quad \text{and} \quad V_n \le C \quad \text{for all } n\ge 1.$$

We deduce that

$$\sum_{j=1}^{n-1}(n-j)^{-(1-\alpha)}Z_j \le C(L(\delta)n^{-(1-\alpha)}+\delta) \quad \text{for all } n\ge 1$$

and hence $\limsup_{n\to\infty}\sum_{j=1}^{n-1}(n-j)^{-(1-\alpha)}Z_j \le C\delta$. The result follows since δ is arbitrary. ∎

4.2 The case $\alpha = 1$

When $\alpha=1$, we must show that

$$S_n = \sum_{j=1}^{n}\log(n-j)^{-1}\sum_{i=1}^{\infty}\bar{g}_i(j+i)^{-2} \to 0 \quad \text{as } n\to\infty$$

whenever $\bar{g}_i$ is a bounded sequence satisfying $b_n = \sum_{i=1}^{n} i^{-1}\bar{g}_i = o(\log n)$. As before, we write

$$|b_n| = \psi(n)\log n \quad \text{where} \quad \lim_{n\to\infty}\psi(n) = 0.$$

We make use of the standard facts that

$$\sum_{i=n}^{\infty}\frac{\log i}{i^2} \approx \frac{\log n}{n}, \qquad \sum_{j=2}^{n}\frac{1}{\log j} \approx \frac{n}{\log n}.$$

The range $j<n/2$, $i\le n$ has to be treated differently from $\alpha<1$. Accordingly, define

$$S_n^1 = \sum_{1\le j<n/2}\log(n-j)^{-1}\sum_{i=1}^{n}\bar{g}_i(j+i)^{-2}.$$

Lemma 4.4 $\lim_{n\to\infty} S_n^1 = 0$.

Proof Define $\widetilde{S}_n^1 = \sum_{1\le j<n/2} (\log n)^{-1} \sum_{i=1}^n \bar{g}_i (j+i)^{-2}$. For n sufficiently large and $1 \le j < n/2$, we have $\log(\frac{n}{n-j}) \le \log 2$ and $\frac{1}{2}\log n \le \log(n-j) \le \log n$, so $|(\log(n-j))^{-1} - (\log n)^{-1}| \le 2\log 2\,(\log n)^{-2}$. Using also that $\bar{g}_i$ is bounded,

$$|S_n^1 - \widetilde{S}_n^1| \le 2\log 2\,(\log n)^{-2} \sup_i |\bar{g}_i| \sum_{j=1}^n \sum_{i=1}^\infty (j+i)^{-2} \ll (\log n)^{-1}.$$

Also,

$$\begin{aligned}(\log n)|\widetilde{S}_n^1| &= \Big|\sum_{i=1}^n \bar{g}_i \sum_{1\le j<n/2} (j+i)^{-2}\Big| = \Big|\sum_{i=1}^n \bar{g}_i \Big\{\int_1^{n/2} (x+i)^{-2}\,dx + O(i^{-2})\Big\}\Big| \\ &= \Big|\sum_{i=1}^n \bar{g}_i (i^{-1} - (n/2+i)^{-1})\Big| + O(1) \\ &\le \Big\{\Big|\sum_{i=1}^n i^{-1}\bar{g}_i\Big| + 2\sup|\bar{g}_i| \sum_{i=1}^n n^{-1}\Big\} + O(1) = |b_n| + O(1).\end{aligned}$$

The result follows since $b_n = o(\log n)$. ∎

Next, we consider the range $j < n/2$, $i > n$, defining

$$S_n^2 = \sum_{1\le j<n/2} \log(n-j)^{-1} \sum_{i=n+1}^\infty \bar{g}_i (j+i)^{-2}.$$

Lemma 4.5 $\lim_{n\to\infty} S_n^2 = 0$.

Proof Define $\widetilde{S}_n^2 = \sum_{1\le j<n/2} (\log n)^{-1} \sum_{i=n+1}^\infty \bar{g}_i (j+i)^{-2}$. Starting as for S_n^1, we obtain $|S_n^2 - \widetilde{S}_n^2| = O((\log n)^{-1})$ and

$$|\widetilde{S}_n^2| = (\log n)^{-1} \Big|\sum_{i=n+1}^\infty \bar{g}_i (i^{-1} - ([n/2]+i)^{-1})\Big| + O((\log n)^{-1}).$$

Hence, writing $d_i = ([n/2]+i)^{-1}$ and suppressing terms that are $O((\log n)^{-1})$,

$$\begin{aligned}|S_n^2| &\ll n(\log n)^{-1} \Big|\sum_{i=n+1}^\infty i^{-1}\bar{g}_i ([n/2]+i)^{-1}\Big| \\ &= n(\log n)^{-1} \Big|\sum_{i=n+1}^\infty (b_i - b_{i-1}) d_i\Big| = n(\log n)^{-1} \Big|\sum_{i=n+1}^\infty b_i (d_i - d_{i+1}) \;-\; b_n d_{n+1}\Big| \\ &\ll n(\log n)^{-1} \Big\{\sum_{i=n+1}^\infty (\log i)\psi(i) i^{-2} \;+\; \psi(n)(\log n) n^{-1}\Big\} \ll \sup_{j\ge n} \psi(j).\end{aligned}$$

The result follows since $\psi(n) = o(1)$. ∎

To cover the remaining range $n/2 \le j < n$, $i \ge 1$, define

$$S_n^3 = \sum_{n/2 \le j < n} (\log(n-j))^{-1} \sum_{i=1}^{\infty} \bar{g}_i (j+i)^{-2}.$$

Lemma 4.6 $\lim_{n\to\infty} S_n^3 = 0$.

Proof For this term, we proceed almost exactly as in the case $\alpha < 1$ but with the simplification that $j \approx n$. Applying Lemma 4.1 with $\alpha = 1$ yields

$$\Big| \sum_{i \ge 1} \bar{g}_i (j+i)^{-2} \Big| \ll Z_j, \qquad Z_j = \sum_{i \ge 1} \psi(i)(\log i)(j+i)^{-2}.$$

Hence

$$|S_n^3| \ll \sum_{n/2 \le j < n} (\log(n-j))^{-1} Z_j \ll \sum_{n/2 \le j < n} (\log(n-j))^{-1} Z_n \approx n(\log n)^{-1} Z_n,$$

and it remains to show that $\lim_{n\to\infty} n(\log n)^{-1} Z_n = 0$.

Let $M = \max_{i \ge 1} \psi(i)$. Given $\delta > 0$, choose $L(\delta) \ge 1$ such that $\psi(i) \le \delta$ for all $i > L(\delta)$. Then

$$Z_n \le M \sum_{i=1}^{L(\delta)} (\log i)\,(n+i)^{-2} + \delta \sum_{i=1}^{\infty} (\log i)\,(n+i)^{-2}.$$

But

$$\sum_{i=1}^{L(\delta)} (\log i)\,(n+i)^{-2} \le \log L(\delta) \sum_{i=1}^{L(\delta)} n^{-2} = L(\delta)(\log L(\delta)) n^{-2}$$

and

$$\sum_{i=1}^{\infty} (\log i)\,(n+i)^{-2} \le \sum_{i=1}^{\infty} (\log(n+i))\,(n+i)^{-2} = \sum_{i=n+1}^{\infty} (\log i)\, i^{-2} \approx (\log n)\, n^{-1}.$$

It follows that

$$Z_n \le C(L(\delta)(\log L(\delta)) n^{-2} + \delta(\log n)\, n^{-1}) \quad \text{for all } n \ge 1$$

where C is independent of δ. Hence $\limsup_{n\to\infty} n(\log n)^{-1} Z_n \le C\delta$. The result follows since $\delta > 0$ is arbitrary. ∎

5 Examples

In this section, we verify the hypotheses of Theorem 2.6 for a variety of examples of intermittent maps, thereby proving full global-local mixing. The one-dimensional examples, with the exception of Example 5.13, are AFN maps [31, 32] and the necessary background is contained in Subsection 5.1. In Subsection 5.2, we consider nonMarkovian maps with one neutral fixed point. In Subsection 5.3, we consider intermittent maps with several neutral fixed points. In Subsection 5.4, we consider a higher-dimensional nonMarkovian intermittent map introduced in [21].

5.1 Background on AFN maps

Zweimüller [31, 32] studied a class of nonMarkovian interval maps $f : X \to X$ ($X \subset \mathbb{R}$ a closed bounded interval) with indifferent fixed points. It is assumed that there is a measurable partition β of X into open intervals such that f is C^2 and strictly monotone on each $I \in \beta$. The map is said to be AFU if:

(A) *Adler's condition:* $f''/(f')^2$ is bounded on $\bigcup_{I\in\beta} I$;

(F) *Finite images:* $\{fI : I \in \beta\}$ is finite;

(U) *Uniform expansion:* There exists $\rho > 1$ such that $|f'| \ge \rho$ on $\bigcup_{I\in\beta} I$.

The map is AFN if condition (U) is relaxed to condition (N) where $|f'| \ge 1$ and there is a finite number of neutral fixed points $\xi_1, \dots, \xi_d$ satisfying certain properties such that $|f'| > 1$ on $X \setminus \{\xi_1, \dots, \xi_k\}$. In our examples, there exist $\alpha \in (0, 1]$ and $b_1, \dots, b_d > 0$ such that

$$|f(x) - x| \sim b_k|x - \xi_k|^{1+1/\alpha}, \qquad |f'(x) - 1| \sim b_k(1 + 1/\alpha)|x - \xi_k|^{1/\alpha}, \tag{5.1}$$

as $x \to \xi_k$, $k = 1, \dots, d$. Condition (N) in [31, 32] is satisfied for such maps.

Topologically mixing AFN maps are conservative and exact with respect to Lebesgue measure and admit a unique (up to scaling) equivalent σ-finite invariant measure μ. When there is at least one neutral fixed point, $\mu(X) = \infty$. Moreover $\mu(X \setminus \bigcup_{k=1}^{d} B_\epsilon(\xi_1)) < \infty$ for all $\epsilon > 0$.

We induce to an appropriate set Y with $0 < \mu(Y) < \infty$ bounded away from the neutral fixed points and normalise so that $\mu(Y) = 1$. By general arguments [31, 32], the induced map $F = f^\tau : Y \to Y$ is an AFU map; the properties (A) and (F) are inherited from f and clearly $|F'| \ge |f'|$ which is uniformly larger than 1 on Y. Standard arguments, repeated in the examples below, show that $\mu(Y_n) \sim \gamma n^{-(\alpha+1)}$ for some $\gamma > 0$. By [19, Proposition 11.10], condition (2.3) is satisfied for $\alpha \in (\frac{1}{2}, 1]$. If in addition F is topologically mixing and hence mixing for the invariant probability measure $\mu|_Y$, condition (2.3) is satisfied also for $\alpha \in (0, \frac{1}{2}]$ by [11].

Since F is an AFU map, the density $h = d\mu/d\,\mathrm{Leb}$ has the property that $h|_Y$ lies in $\mathrm{BV}(Y)$ and is bounded away from zero [23]. This implies that the one-sided limits $h(y^\pm) \in (0, \infty)$ exist for all $y \in Y$.

Remark 5.1 In the special case where the branches $f|_I$ are full for each $I \in \beta$ (so $\overline{fI} = X$ for $I \in \beta$), these maps were studied by Thaler [27, 28].

Suppose that $F : Y \to Y$ is an AFU map with full branches. (In particular, F is Gibbs-Markov.) Then the density h has the property that $h|_Y$ is Lipschitz (see for example [13, Proposition 2.5]). By uniqueness of the density, it suffices that F is Gibbs-Markov and topologically mixing since then F^k has full branches for some $k \geq 1$.

We will make use of the following basic distortion estimate. Let $M = \sup |f''|/(f')^2$. By Adler's condition, $M < \infty$.

Proposition 5.2 *$|f'(x)^{-1} - f'(y)^{-1}| \leq Me^{2M}|x - y|$ for all $x, y \in I$ and all $I \in \beta$.*

Proof First, note that $|f'(x)/f'(y)| \leq e^M$ for all $x, y \in I$ (see [1, Section 4.3] for details). Hence for $x, y \in I$ with $x < y$,

$$\frac{f'(x) - f'(y)}{f'(x)f'(y)} = \frac{f''(z)}{f'(z)^2} \frac{f'(z)}{f'(x)} \frac{f'(z)}{f'(y)}(y - x) \leq Me^{2M}(y - x)$$

for some $z \in (x, y)$ by the mean value theorem. ∎

Finally, we mention the following convenient formula for $J_{j,n,p'}$ from Remark 2.7 expressed in terms of $h|_Y$. (This result does not use the AFN structure.)

Proposition 5.3 *For $x \in X_{n,p'}$,*

$$J_{j,n,p'}(x) = \frac{\sum_{\psi(p)=p'} h(y_{j,n,p}) \big|(f^j)'(y_{j,n,p})\big|^{-1}}{\sum_{\psi(p)=p'} \sum_{\ell=1}^{\infty} h(y_{\ell,n,p}) \big|(f^\ell)'(y_{\ell,n,p})\big|^{-1}}$$

where $y_{\ell,n,p} \in Y_{n+\ell,p}$ with $f^\ell y_{\ell,n,p} = x$.

Proof It follows from the usual formula $\mu = \sum_{\ell=0}^{\infty} f_*^\ell(\mu|_{\{y \in Y : \tau(y) > \ell\}})$ for constructing μ on X from $\mu|_Y$ that

$$J_{j,n,p'} = d \sum_{\psi(p)=p'} (f^j|_{Y_{j+n,p}})_* \mu \Big/ d \sum_{\ell=1}^{\infty} \sum_{\psi(p)=p'} (f^\ell|_{Y_{\ell+n,p}})_* \mu$$

for $j, n, p' \geq 1$. The result is a direct consequence of this. ∎

For AFN maps, it is natural to define $g \in L^\infty$ to be a global observable if there exists $\bar{g} \in \mathbb{R}$ such that

$$\lim_{\epsilon \to 0^+} \frac{\int_{X \setminus \bigcup_{k=1}^d B_\epsilon(\xi_k)} g \, d\mu}{\mathrm{Leb}(X \setminus \bigcup_{k=1}^d B_\epsilon(\xi_k))} = \bar{g}.$$

For our examples, it is easily seen that this is equivalent to our abstract Definition 2.2.

5.2 Maps with one neutral fixed point

We begin with examples where there is one neutral fixed point at 0 (so $f'(0) = 1$) and $f' > 1$ on $(0, 1]$. Special cases are the Farey map, generalised Pomeau-Manneville maps and generalised LSV maps considered in [3, 4]. We do not assume full branches, nor even that the maps are Markov. For the moment, we restrict to finitely many branches; this restriction is relaxed in Example 5.14.

Example 5.4 Fix $\alpha \in (0, 1]$, $\kappa > 1/\alpha$, $b > 0$, $\eta_1 \in (0, 1)$. We consider piecewise smooth maps $f : [0, 1] \to [0, 1]$ given by $f(x) = \begin{cases} f_0(x), & x < \eta_1 \\ f_1(x), & x > \eta_1 \end{cases}$ with two orientation preserving branches satisfying the following properties: The second branch f_1 is uniformly expanding and maps $[\eta_1, 1]$ onto $[0, 1]$. The first branch f_0 is uniformly expanding on $[\epsilon, \eta_1]$ for all $\epsilon > 0$ with a neutral fixed point at 0:

$$f_0(x) = x + bx^{1+1/\alpha} + O(x^{1+\kappa}), \qquad f_0'(x) = 1 + b(1 + 1/\alpha)x^{1/\alpha} + O(x^\kappa).$$

We assume that $f_0\eta_1 = \eta \in (f_1^{-1}\eta_1, 1]$.

Remark 5.5 This includes the LSV map as the special case $f_0(x) = x(1+2^{1/\alpha}x^{1/\alpha})$, $f_1(x) = 2x - 1$. However, in general f is not full-branch. Even when f is full-branch, these maps are only covered in [4] under an extra monotonicity condition (A5).

This is a simple example of a topologically mixing AFN map with invariant set $X = [0, \eta]$. Note that the first branch $f_0 : [0, \eta_1] \to X$ is full and the second uniformly expanding branch $f_1 : [\eta_1, \eta] \to X$ need not be full. As discussed in Subsection 5.1, f is conservative and exact, there is a unique (up to scaling) σ-finite invariant measure μ equivalent to Lebesgue, and $\mu(X) = \infty$.

We take $Y = [\eta_1, \eta]$. The induced AFU map $F = f^\tau : Y \to Y$ has at most one short branch (on $Y_1 = [f_1^{-1}\eta_1, \eta]$) and the remaining branches are full. It is easily seen that F is topologically mixing and hence mixing.

There exists $n_0 \geq 1$ (depending only on the size of the short branch) such that f^j maps Y_{j+n} bijectively onto X_n for all $j \geq 1$, $n \geq n_0$. In particular, we can take $q = q' = 1$ in the abstract setup of Section 2 and we suppress the subscripts p, p'.

Write $X_n = [x_{n+1}, x_n]$ where $x_n \downarrow 0$. Without loss of generality, we can suppose that $\kappa \in (1/\alpha, 1 + 1/\alpha)$. Let $\kappa' = \alpha\kappa - 1 \in (0, \alpha)$.

Proposition 5.6 *Let $b' = \alpha^\alpha b^{-\alpha}$ and $b'' = \alpha^{\alpha+1}b^{-\alpha}$. Then $x_n = b'n^{-\alpha} + O(n^{-(\alpha+\kappa')})$ and $x_{n-1} - x_n = b''n^{-(\alpha+1)} + O(n^{-(\alpha+\kappa'+1)})$.*

Proof We recall the standard argument (see for example [1, Lemma 4.8.6]). Note that

$$x_{n-1}^{-1/\alpha} = (f_0x_n)^{-1/\alpha} = \{x_n(1 + bx_n^{1/\alpha} + O(x_n^\kappa))\}^{-1/\alpha} = x_n^{-1/\alpha}(1 - \alpha^{-1}bx_n^{1/\alpha} + O(x_n^\kappa)).$$

Hence,

$$x_n^{-1/\alpha} - x_{n-1}^{-1/\alpha} = \alpha^{-1} b + O(x_n^{\kappa - 1/\alpha}). \tag{5.2}$$

Summing (5.2), we obtain the rough estimate $x_n^{-1/\alpha} \approx n$ and substituting this into (5.2) yields the more precise expression

$$x_n^{-1/\alpha} - x_{n-1}^{-1/\alpha} = \alpha^{-1} b + O(n^{-\kappa'}).$$

Summing this gives $x_n^{-1/\alpha} = \alpha^{-1} bn(1 + O(n^{-\kappa'}))$ yielding the estimate for x_n.

Finally,

$$\begin{aligned} x_{n-1} - x_n &= f_0 x_n - x_n = b x_n^{1+1/\alpha} + O(x_n^{1+\kappa}) \\ &= b(b')^{1+1/\alpha} n^{-(\alpha+1)}(1 + O(n^{-\kappa'}) + O(n^{-\alpha(1+\kappa)})) = b'' n^{-(\alpha+1)} + O(n^{-(\alpha+\kappa'+1)}), \end{aligned}$$

completing the proof. ∎

By Proposition 5.6, $\mathrm{Leb}(Y_n) \sim f_1'(\eta_1)^{-1} \,\mathrm{Leb}(X_n) \sim f_1'(\eta_1)^{-1} b''\, n^{-(\alpha+1)}$ as $n \to \infty$. Since $h \in \mathrm{BV}(Y)$,

$$\mu(Y_n) \sim h(\eta_1^+) \,\mathrm{Leb}(Y_n) \sim h(\eta_1^+) f_1'(\eta_1)^{-1} b'' n^{-(\alpha+1)} \quad \text{as } n \to \infty$$

verifying condition (2.1).

It remains to verify condition (2.4). We achieve this via (2.5).

Lemma 5.7 *The diffeomorphism $f^j|_{Y_{j+n}} : Y_{j+n} \to X_n$ satisfies*

$$\{(f^j)'\}^{-1} = \{f_1'(\eta_1)\}^{-1} \Big(\frac{n}{j+n}\Big)^{\alpha+1} \big(1 + O(n^{-\kappa'})\big),$$

where the implied constant is independent of j and n uniformly on Y.

Proof Let $y \in Y_{j+n}$. By Proposition 5.6, $f^k y \in X_{j+n-k}$ satisfies

$$f^k y = b'(j+n-k)^{-\alpha} + O((j+n-k)^{-(\alpha+\kappa')}).$$

Note that

$$\log f_0'(x) = \log(1 + b(1+1/\alpha)x^{1/\alpha} + O(x^\kappa)) = b(1+1/\alpha)x^{1/\alpha} + O(x^\kappa).$$

Since $b(b')^{1/\alpha} = \alpha$, we obtain

$$\begin{aligned} \log f_0'(f^k y) &= (\alpha+1)(j+n-k)^{-1}\big\{1 + O((j+n-k)^{-\kappa'})\big\} + O((j+n-k)^{-\alpha\kappa}) \\ &= (\alpha+1)(j+n-k)^{-1} + O((j+n-k)^{-(1+\kappa')}), \end{aligned}$$

for $1 \le k \le j$. Hence

$$\begin{aligned}\log(f^j)'(y) &= \log f_1'(y) + \sum_{k=1}^{j-1} \log f_0'(f^k y) \\ &= \log f_1'(y) + (\alpha+1) \sum_{k=n+1}^{j+n-1} (k^{-1} + O(k^{-(1+\kappa')})) \\ &= \log f_1'(y) + (\alpha+1)\log\frac{j+n}{n} + O(n^{-\kappa'}).\end{aligned}$$

It follows that $(f^j)'(y) = f_1'(y)\big(\frac{j+n}{n}\big)^{\alpha+1}\big(1+O(n^{-\kappa'})\big)$ and hence

$$\{(f^j)'(y)\}^{-1} = \{f_1'(y)\}^{-1}\Big(\frac{n}{j+n}\Big)^{\alpha+1}\big(1+O(n^{-\kappa'})\big).$$

Finally, by Proposition 5.2 and the estimate $\mathrm{Leb}(Y_n) \ll n^{-(\alpha+1)}$,

$$|\{f'(y)\}^{-1} - \{f'(\eta_1)\}^{-1}| \ll |y-\eta_1| \ll \sum_{k=n+j}^{\infty} k^{-(\alpha+1)} \ll n^{-\alpha} \le n^{-\kappa'}$$

and the result follows. ∎

Corollary 5.8 *Let $J_{j,n}$ be defined as in Remark 2.7. Then,*

$$\lim_{j\to\infty} j^\alpha \sum_{n=n_0}^{\infty} n^{-\alpha} \sup_{X_n} |J_{j,n} - \alpha n^\alpha (j+n)^{-(\alpha+1)}| = 0.$$

Proof Let $x \in X_n$, $y_{j,n} \in Y_{j+n}$ with $f^j y_{j,n} = x$. Note that $y_{j,n} \downarrow \eta_1$ as $n \to \infty$ uniformly in j and $x \in X_n$. Hence $h(y_{j,n}) \to h(\eta_1^+)$ as $n \to \infty$ uniformly in j and $x \in X_n$ by one-sided continuity of h.

Recall from Proposition 5.3 that

$$J_{j,n}(x) = \frac{h(y_{j,n})\big\{(f^j)'(y_{j,n})\big\}^{-1}}{\sum_{\ell=1}^{\infty} h(y_{\ell,n})\big\{(f^\ell)'(y_{\ell,n})\big\}^{-1}}\;.$$

By Lemma 5.7,

$$h(y_{j,n})\{(f^j)'(y_{j,n})\}^{-1} \sim h(\eta_1^+)\{f_1'(\eta_1)\}^{-1} n^{\alpha+1}(j+n)^{-(\alpha+1)}$$

as $n \to \infty$ uniformly in j and $x \in X_n$, and

$$\sum_{\ell=1}^{\infty} h(y_{\ell,n})\{(f^\ell)'(y_{\ell,n})\}^{-1} \sim h(\eta_1^+)\{f_1'(\eta_1)\}^{-1}\alpha^{-1} n$$

as $n \to \infty$ uniformly in $x \in X_n$. The result follows. ∎

By Corollary 5.8, condition (2.5) holds with $c_{j,n} = \alpha n^\alpha (j+n)^{-(\alpha+1)}$, completing the proof of global-local mixing.

Example 5.9 We generalise the previous example to one with several uniformly expanding branches.

Fix $\alpha \in (0,1]$, $b > 0$ and $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{q+1} = 1$ where $q \ge 1$. We suppose that the branches $f_p = f|_{[\eta_p,\eta_{p+1}]}$ are orientation preserving with $f_p(\eta_p) = 0$ for all $p = 0,\dots,q$. Suppose that f_0 has a neutral fixed point at 0 as in Example 5.4 and is uniformly expanding away from zero, and that the remaining branches $f_1,\dots,f_q$ are uniformly expanding. We require that $f_0\eta_1 = \eta \in (f_q^{-1}\eta_q, 1]$ and set $X = [0,\eta]$.

Again, $f : X \to X$ is conservative and exact with at least one full branch f_0. There is a unique (up to scaling) σ-finite absolutely continuous invariant measure μ, and $\mu(X) = \infty$.

We take $Y = [\eta_1, \eta]$ and note that $F = f^\tau : Y \to Y$ is a mixing AFU map. In this example, $f^j|_{Y_{j+n}}$ is q-to-1. Setting $Y_{n,p} = Y_n \cap [\eta_p, \eta_{p+1}]$ for $1 \le p \le q$, we obtain for $n_0 \ge 1$ sufficiently large that f^j maps $Y_{j+n,p}$ bijectively onto X_n for all $j \ge 1$, $n \ge n_0$, $1 \le p \le q$. We take $q' = 1$ and suppress subscripts p'.

Write $X_n = [x_{n+1}, x_n]$ where $x_n \downarrow 0$. The statement and proof of Proposition 5.6 goes through as before. In particular, $\mathrm{Leb}(Y_{n,p}) \sim f_p'(\eta_p)^{-1}\,\mathrm{Leb}(X_n) \sim f_p'(\eta_p)^{-1} b''\, n^{-(\alpha+1)}$ (with $b'' = \alpha^{\alpha+1}b^{-\alpha}$) and $\mu(Y_{n,p}) \sim h(\eta_p^+) f_1'(\eta_p)^{-1} b'' n^{-(\alpha+1)}$, verifying condition (2.1).

Following the proof of Lemma 5.7, we obtain that the diffeomorphism $f^j|_{Y_{j+n,p}} : Y_{j+n,p} \to X_n$ satisfies

$$\{(f^j)'\}^{-1} = \{f_p'(\eta_p)\}^{-1}\Big(\frac{n}{j+n}\Big)^{\alpha+1}\big(1 + O(n^{-\kappa'})\big).$$

Substituting into Proposition 5.3,

$$\sum_{p=1}^{q} h(y_{j,n,p})\{(f^j)'(y_{j,n,p})\}^{-1} \sim \sum_{p=1}^{q} h(\eta_p^+)\{f_p'(\eta_p)\}^{-1} n^{\alpha+1}(j+n)^{-(\alpha+1)}$$

as $n \to \infty$ uniformly in j and $x \in X_n$, and

$$\sum_{p=1}^{q}\sum_{\ell=1}^{\infty} h(y_{\ell,n,p})\{(f^\ell)'(y_{\ell,n,p})\}^{-1} \sim \sum_{p=1}^{q} h(\eta_p^+)\{f_p'(\eta_p)\}^{-1}\alpha^{-1} n$$

as $n \to \infty$ uniformly in $x \in X_n$. Hence we again obtain that $J_{j,n} \sim \alpha n^\alpha (j+n)^{-(\alpha+1)}$ as $n \to \infty$ uniformly in j on X_n, verifying condition (2.5).

Example 5.10 A set of examples parallel to those in Example 5.9 is given by AFN maps $f : [0,1] \to [0,1]$ with $q+1$ branches, $q \ge 1$, and a neutral fixed point at 0 as before, but now we assume that the first q branches are full and the last branch is possibly not full. The analysis is the same as in Example 5.9 with the simplification that $X = [0,1]$, so we omit the details.

This includes the case of Pomeau-Manneville maps $fx = x + bx^{1+1/\alpha} \bmod 1$ for all $b \in [1,\infty)$. Note that the number of branches is given by $\lceil b \rceil$. Previously global-local mixing was studied by [4] for the full-branch case with b an integer, but we do not make this restriction.

Example 5.11 In the previous examples, we assumed for convenience that all branches were orientation preserving. We can easily handle a mixture of branches that preserve and reverse orientation. If a uniform branch $f_p = f|_{[\eta_p,\eta_{p+1}]}$ in Example 5.9 is orientation reversing with $f_p(\eta_{p+1}) = 0$, then the arguments go through with $f_p'(\eta_p)$ and $h(\eta_p^+)$ replaced by $f_p'(\eta_{p+1})$ and $h(\eta_{p+1}^-)$.

In particular, we obtain full global mixing for the Farey map $f : X \to X$. Here, $X = [0, 1]$ and there are two full branches of opposite orientations given by

$$f_0(x) = \frac{x}{1-x}, \quad x \in [0, \tfrac{1}{2}], \qquad \text{and} \qquad f_1(x) = \frac{1-x}{x}, \quad x \in (\tfrac{1}{2}, 1].$$

This is an AFN map with a single neutral fixed point at $x = 0$ and with $\alpha = 1$.

5.3 Maps with several neutral fixed points

In this subsection we consider full-branch AFN maps with $d \ge 2$ equally sticky neutral fixed points $\xi_1, \dots, \xi_d$ satisfying (5.1). In Example 5.12, we consider an example with d branches, each containing a neutral fixed point. In Example 5.14, we consider a generalisation where there can be finitely or infinitely many uniformly expanding branches in addition to the d branches with neutral fixed points.

Example 5.12 Let $X = [0, 1]$. Let $d \ge 2$ and fix $0 = \eta_0 < \eta_1 < \cdots < \eta_d = 1$. We suppose that the branches $f_k = f|_{[\eta_{k-1},\eta_k]}$ are orientation preserving and full for $k = 1, \dots d$ with $f' \ge 1$. Suppose that the d fixed points $\xi_1, \dots, \xi_d$ satisfy (5.1) with $\alpha \in (0, 1]$. As usual, we assume Adler's condition and that f is uniformly expanding away from the fixed points.

We construct the inducing domain Y following [24, Section 7] (see also [8, Section 4.1]). The construction depends on whether $d = 2$ or $d \ge 3$. When $d = 2$, we take $Y = [\eta, f\eta]$ where $\{\eta, f\eta\}$ is the unique orbit of period 2. When $d \ge 3$, we take $Y = \bigcup_{k=1}^{d}(I_k \setminus f^{-1}I_k)$ where $I_k = (\eta_{k-1}, \eta_k)$. From now on, we focus on the case $d \ge 3$.

The induced map $F = f^\tau : Y \to Y$ is topologically mixing (see for example [8, Proposition 4.2]). Moreover F^3 has full branches so it follows from Remark 5.1 that the density h is Lipschitz.

We take $q = d(d-1)$, $q' = d$ and $n_0 = 1$. Set

$$X_{n,p'} = \{x \in X_n \cap I_{p'} : f^n x \in I_{p'}\}, \quad 1 \le p' \le q',\ n \ge 1. \tag{5.3}$$

For each $1 \le p' \le q'$, we note that $Y_n \cap f^{-1}I_{p'}$ is the disjoint union of $d-1$ intervals $Y_{n,p',k} \subset I_k$ where $1 \le k \le d$, $k \ne p'$. In this way, we obtain decompositions

$Y_n = \bigcup Y_{n,p',k}$ and $X_n = \bigcup X_{n,p'}$ such that f^j maps $Y_{j+n,p',k}$ bijectively onto $X_{n,p'}$ for all $j, n \geq 1$, $1 \leq p' \leq q'$, $1 \leq k \leq d$, $k \neq p'$.[1]

We can use Proposition 5.6 as before to show that $\mathrm{Leb}(X_{n,p'}) \sim b''_{p'} n^{-(\alpha+1)}$ where $b''_{p'} = \alpha^{\alpha+1} b_{p'}^{-\alpha}$. Let $\tilde{\xi}_{p',k}$ be the accumulation point of the intervals $Y_{n,p',k}$ and notice that $f\tilde{\xi}_{p',k} = \xi_{p'}$. Thus, $\mathrm{Leb}(Y_{n,p',k}) \sim f'(\tilde{\xi}_{p',k})^{-1} b''_{p'} n^{-(\alpha+1)}$ and so $\mu(Y_{n,p',k}) \sim h(\tilde{\xi}_{p',k}) f'(\tilde{\xi}_{p',k})^{-1} b''_{p'} n^{-(\alpha+1)}$ verifying condition (2.1).

Following the proof of Lemma 5.7, we find for each p', k that the diffeomorphism $f^j : Y_{j+n,p',k} \to X_{n,p'}$ satisfies

$$\{(f^j)'\}^{-1} = \{f'_p(\tilde{\xi}_{p',k})\}^{-1} \Big(\frac{n}{j+n}\Big)^{\alpha+1} \big(1 + O(n^{-\kappa'})\big).$$

Hence, proceeding in the same way as Corollary 5.8, we obtain for each p' that $J_{j,n,p'} \sim \alpha n^{\alpha} (j+n)^{-(\alpha+1)}$ as $n \to \infty$ uniformly in j on $X_{n,p'}$, verifying (2.5).

Example 5.13 We may also treat the interval maps studied in [7, 6]. These are interval maps consisting of two full orientation preserving branches, each with a neutral fixed point. Unlike in Example 5.12, these maps are not AFN as each branch may have either a critical or singular point at the discontinuity. The set of maps we can treat is slightly more general than those introduced in [7]. A precise definition of this class of maps is given in [8, Section 4.2] with the slight modification that we impose the stronger condition (5.1) for the expansion near the fixed points. The definition of the set Y, and the verification of (2.1) and (2.3) can be found in [8, Section 4.2]. With (2.1) established, one can verify (2.5) in the same way as in Example 5.12 with $d = q = q' = 2$.

Example 5.14 Fix $D > d \geq 1$, where we allow the possibility that $D = \infty$. Set $X = [0,1]$ with a partition into D open sub-intervals $I_1, I_2, \ldots$. Assume that the branches $f_k = f|_{I_k}$ are orientation preserving and full for each $k \geq 1$ and that $f' \geq 1$. For $k = 1, \ldots, d$ we suppose that the fixed points $\xi_k \in I_k$ satisfy (5.1) for some $\alpha \in (0,1]$. As before we assume Adler's condition and that $f'x > 1$ for all $x \notin \{\xi_1, \ldots, \xi_d\}$.

Set $Y = \big(\bigcup_{i=1}^{d} (I_k \setminus f^{-1} I_k)\big) \cup \big(\bigcup_{i=d+1}^{D} I_k\big)$. (Unlike in Example 5.12, it is no longer necessary to distinguish low values of d.)

We take $q = d(D-1)$, $q' = d$, $n_0 = 1$. Define $X_{n,p'}$ as in (5.3). For each $1 \leq p' \leq q'$, note that $Y_n \cap f^{-1} I_{p'}$ is the disjoint union of $D-1$ intervals $Y_{n,p',k} \subset I_k$ where $1 \leq k \leq D$, $k \neq p'$. In this way, we obtain decompositions $Y_n = \bigcup Y_{n,p',k}$ and $X_n = \bigcup X_{n,p'}$ such that f^j maps $Y_{j+n,p',k}$ bijectively onto $X_{n,p'}$ for all $j, n \geq 1$, $1 \leq p' \leq q'$, $1 \leq k \leq D$, $k \neq p'$.

For $n \geq 1$, $1 \leq p' \leq q'$, $k \neq p'$, we have $FY_{n,p',k} = f^n Y_{n,p',k} = I_{p'} \setminus f^{-1} I_{p'}$, $F^2 Y_{n,p',k} = Y \setminus I_{p'}$ and $F^3 Y_{n,p',k} = Y$. For the remaining partition elements $Y_{1,p',k}$

[1] We can relabel so that $\{Y_{n,p',k}\}$ is identified with $\{Y_{n,p}\}$ as in the abstract setup, but it seems more convenient to keep the current equivalent labelling.

with $p' > q'$, $k \geq 1$, we have $FY_{1,p',k} = I_{p'}$ and $F^2 Y_{1,p',k} = Y$. Hence F is topologically mixing. By Remark 5.1, the density h is Lipschitz on Y.

To calculate $\mu(Y_{n,p',k})$ we proceed as in Example 5.12 but this time we have to take care about the error terms which appear in the asymptotics to deal with the case $D = \infty$.

Write $|E| = \mathrm{Leb}(E)$ for $E \subset X$ measurable.

Corollary 5.15 $\sup_{I_k}(f')^{-1} \ll |I_k|$.

Proof By the mean value theorem, there exists $y_k \in I_k$ such that $1 = |X| = |fI_k| = f'(y_k)|I_k|$. The result follows from Proposition 5.2. ∎

We emphasise that throughout this example, all the constants implied by the use of big O or $\ll$ are independent of n and k and the appropriate domain contained in X.

Fix $p' \in \{1, \ldots, d\}$ and let $p = (p', k)$ where $1 \leq k \leq D$ and $k \neq p'$. We can again use Proposition 5.6 to show that $|X_{n,p'}| \sim b''_{p'} n^{-(\alpha+1)}$ as $n \to \infty$. As in Example 5.12, let $\tilde{\xi}_p \in I_k$ be the accumulation point of the intervals $Y_{n,p}$. By the mean value theorem, there exists $y_{n,p} \in Y_{n,p}$ such that $|X_{n-1,p'}| = |fY_{n,p}| = f'(y_{n,p})|Y_{n,p}|$. By Corollary 5.15,

$$|Y_{n,p}| \ll |I_k||X_{n-1,p'}| \ll |I_k| n^{-(\alpha+1)}.$$

Hence, by Proposition 5.2,

$$|\{f'(y_{n,p})\}^{-1} - \{f'(\tilde{\xi}_p)\}^{-1}| \ll |y_{n,p} - \tilde{\xi}_p| \ll \sum_{i=n}^{\infty} |Y_{i,p}| \ll |I_k| n^{-\alpha}, \tag{5.4}$$

and so

$$|Y_{n,p}| = \{f'(y_{n,p})\}^{-1}|X_{n-1,p'}| = \{f'(\tilde{\xi}_p)\}^{-1}|X_{n-1,p'}| + O(|I_k| n^{-(\alpha+1)}). \tag{5.5}$$

Since h is Lipschitz on Y,

$$\big|\mu(Y_{n,p}) - h(\tilde{\xi}_p)|Y_{n,p}|\big| \leq \int_{Y_{n,p}} |h - h(\tilde{\xi}_p)| \, d\,\mathrm{Leb} \ll |Y_{n,p}| \sum_{i=n}^{\infty} |Y_{i,p}| \ll |I_k| n^{-(2\alpha+1)}.$$

Hence

$$\big|\mu(Y_{n,p}) - h(\tilde{\xi}_p)\{f'(\tilde{\xi}_p)\}^{-1}|X_{n-1,p'}|\big| \ll |I_k| n^{-(2\alpha+1)}.$$

It follows that $\mu(Y_{n,p}) \sim \gamma_p n^{-(\alpha+1)}$ as $n \to \infty$ uniformly in p where $\gamma_p = h(\tilde{\xi}_p)\{f'(\tilde{\xi}_p)\}^{-1} b''_{p'}$. Also $h(\tilde{\xi}_p)\{f'(\tilde{\xi}_p)\}^{-1} \ll |I_k|$ which is summable, so we have shown that for each $p' \in \{1, \ldots, q'\}$,

$$\sum_{k \neq p'} \mu(Y_{n,p',k}) = \omega_{p'}|X_{n-1,p'}| + O(n^{-(2\alpha+1)}) \sim \omega_{p'} b''_{p'} n^{-(\alpha+1)} \quad \text{as } n \to \infty,$$

where $\omega_{p'} = \sum_{k\neq p'} h(\tilde{\xi}_p)\{f'(\tilde{\xi}_p)\}^{-1} \in (0,\infty)$. Hence we have verified condition (2.1).

We now move on to verifying (2.5). Again, we fix $p' \in \{1,\dots,q'\}$ and consider the $D-1$ indices $p = (p',k)$ where $k \neq p'$. Let $x \in X_{n,p'}$ and let $y_{j,n,p} = y_{j,n,p}(x) \in Y_{j+n,p} \in I_k$ be such that $f^j y_{j,n,p} = x$. We first follow the proof of Lemma 5.7. Note that $f^k y_{j,n,p} = f_{p'}^{-(j-\ell)} x$ for $k = 1,\dots,j$ and hence is independent of p. Hence, when applying Proposition 5.6, we are justified in writing $\log f'(f^k y_{j,n,p}) = (\alpha+1)(j+n-k)^{-1}(1+O((j+n-k)^{-(1+\kappa')})$ for $k = 1,\dots,j$. This in turn yields

$$\log(f^j)'(y_{j,n,p}) = \log f'(y_{j,n,p}) + (\alpha+1)\log\frac{j+n}{n} + O(n^{-\kappa'}).$$

By Proposition 5.2 and (5.5),

$$|\{f'(y_{j,n,p})\}^{-1} - \{f'(\tilde{\xi}_p)\}^{-1}| \ll |y_{j,n,p} - \tilde{\xi}_p| \ll \{f'(\tilde{\xi}_p)\}^{-1} \sum_{i=n+j}^{\infty} i^{-(\alpha+1)} \ll \{f'(\tilde{\xi}_p)\}^{-1} n^{-\alpha}.$$

Thus, we obtain

$$\{(f^j)'(y_{j,n,p})\}^{-1} = \{f'\tilde{(}\xi_p)\}^{-1}\Big(\frac{n}{j+n}\Big)^{\alpha+1}\big(1+O(n^{-\kappa'})\big).$$

Since h is Lipschitz, it follows from (5.4) that $|h(\tilde{\xi}_p) - h(y_{j,n,p})| = O(n^{-\alpha})$, so

$$h(y_{j,n,p})\{(f^j)'(y_{j,n,p})\}^{-1} = n^{\alpha+1}(j+n)^{-(\alpha+1)} h(\tilde{\xi}_p)\{f'(\tilde{\xi}_p)\}^{-1}(1+O(n^{-\kappa'})).$$

Hence

$$\sum_{k\neq p'} h(y_{j,n,p})\{(f^j)'(y_{j,n,p})\}^{-1} = \omega_{p'} n^{\alpha+1}(j+n)^{-(\alpha+1)}(1+O(n^{-\kappa'})),$$

and it follows that

$$\sum_{\ell=1}^{\infty}\sum_{k\neq p'} h(y_{\ell,n,p})\{(f^\ell)'(y_{\ell,n,p})\}^{-1} = \omega_{p'}\alpha^{-1} n(1+O(n^{-\kappa'})).$$

Hence $J_{j,n,p'} = \alpha n^{\alpha}(j+n)^{-(\alpha+1)}(1+O(n^{-\kappa'}))$, verifying condition (2.5).

5.4 A multidimensional example

Example 5.16 We consider a family of multidimensional intermittent maps introduced in [21]. They are nonMarkovian and highly nonconformal (exponentially expanding in one direction and intermittent in the other direction). We stick to the main case in [21] of two-dimensional maps $f : [0,1]\times\mathbb{T} \to [0,1]\times\mathbb{T}$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$), but this generalises to higher-dimensional examples with a general uniformly expanding second branch as discussed in [21, Remark 1.7].

The maps f in [21] take the form $f(x,\theta) = f_0(x,\theta)$ for $0 \le x \le \frac{3}{4}$ and $f(x,\theta) = f_1(x,\theta)$ for $\frac{3}{4} < x \le 1$ where

$$f_0(x,\theta) = (x(1 + x^{1/\alpha} u(x,\theta)),\, 4\theta \bmod 1),$$
$$f_1(x,\theta) = (4x - 3,\, 4\theta \bmod 1).$$

Here, $u : [0, \frac{3}{4}] \times \mathbb{T} \to (0,\infty)$ is a C^2 function satisfying $u(0,\theta) \equiv c_0 > 0$ and $f_{0,1}(\frac{3}{4},\theta) > \frac{15}{16}$ for all $\theta \in \mathbb{T}$. In addition, it is assumed that $|x\partial_x u|_\infty$ and $|\partial_\theta u|_\infty$ are sufficiently small. As usual, $\alpha \in (0,1]$. This example has one neutral invariant circle $\Gamma = \{0\} \times \mathbb{T}$ with uniformly expanding dynamics on this circle.

As in the previous examples, we assume strengthened expansions near the neutral circle, namely that $u(x,\theta) = c_0 + O(x^{\kappa - 1/\alpha})$ uniformly in θ, where $\kappa > 1/\alpha$.

Let $X = \bigcup_{i=0}^{3} f([0,\frac{3}{4}] \times [\frac{i}{4}, \frac{i+1}{4}])$. By [21, Proposition 2.2], f restricts to a topologically exact map $f : X \to X$. Moreover, by [21, Lemma 3.4], there is a unique (up to scaling) absolutely continuous f-invariant σ-finite measure μ on X, and $\mu(X) = \infty$.

For such maps, we prove global-local mixing, taking $g \in L^\infty(X)$ to be a global observable if there exists $\bar{g} \in \mathbb{R}$ such that

$$\lim_{\epsilon \to 0^+} \frac{\int_{[0,\epsilon]\times\mathbb{T}} g\, d\mu}{\mathrm{Leb}([0,\epsilon] \times \mathbb{T})} = \bar{g}.$$

We induce on the set $Y = ([\frac{3}{4}, 1] \times \mathbb{T}) \cap X$ noting that $\mu(Y) \in (0,\infty)$. By [21, Lemma 3.2], the induced map $F = f^\tau : Y \to Y$ is mixing. By [21, Proposition 4.11], there exists $\gamma > 0$ such that $\mu(Y_n) \sim \gamma n^{-(\alpha+1)}$ verifying condition (2.1). Condition (2.3) follows directly from [21, Theorem 1.5(a)].

Proposition 5.17 *f is conservative and exact.*

Proof The first hit time $\tau : X \to \mathbb{Z}^+$ for Y is shown directly in [21] to be finite a.e., and conservativity follows immediately by [18] (see [1, Theorem 1.1.7]).

By [21, Lemma 3.1], $f : X \to X$ is modelled by an aperiodic (that is, topologically mixing) Young tower $(\Delta, \mu_\Delta, f_\Delta)$. Recall that $\Delta = \{(z,\ell) : z \in Z,\, 0 \le \ell \le R(z) - 1\}$ where $R : Z \to \mathbb{Z}^+$ is a return time (not necessarily the first return under f) to a subset $Z \subset Y$ with $0 < \mu(Z) < \infty$. Moreover, by the proof of [21, Lemma 3.1], $\mu_\Delta(\{(z,0) : R(z) = 1\}) > 0$. Under these conditions, we give a simple argument for exactness of f_Δ, and hence f, following [16, 30]. (We note that [30, Lemma 5] already establishes exactness of f_Δ in the finite measure case.)

By definition of a (one-sided) Young tower, the return map $f_\Delta^R : Z \to Z$ is a full-branch Gibbs-Markov map. Identifying Z with the base of the tower, it follows from our assumption on R that $Z \subset f_\Delta Z$. Using bounded distortion of f_Δ^R it follows that there exists $\epsilon > 0$ such that $\mu_\Delta(E \cap f_\Delta E) > 0$ for any $E \subset Z$ with $\mu_\Delta(E) > (1-\epsilon)\mu_\Delta(Z)$.

Let $A \subset \Delta$ with $\mu_\Delta(A) > 0$. By the proof of [30, Lemma 5], there exists $n \geq 1$ such that $\mu_\Delta(Z \cap f_\Delta^n A) > (1-\epsilon)\mu_\Delta(Z)$. Hence

$$\mu_\Delta(f_\Delta^n A \cap f_\Delta^{n+1} A) \geq \mu_\Delta(Z \cap f_\Delta^n A \cap f_\Delta(Z \cap f_\Delta^n A)) > 0.$$

Exactness follows by [20, Lemma 2.1]. ∎

The abstract setup from Section 2 has to be generalised slightly: we can take $q' = 1$ but the value of q now depends on j since the second coordinate of f^j is 4^j-to-one. Specifically, for each $j \geq 1$, $n \geq n_0$, we divide Y_{j+n} into subsets

$$Y_{j,n,p} = \{(y,\theta) \in Y_{j+n} : (p-1)4^{-j} \leq \theta \leq p4^{-j}\}, \quad 1 \leq p \leq q_j = 4^j,$$

so that f^j restricts to a bijection from $Y_{j,n,p}$ onto X_n. Sums of the form $\sum_{\psi(p)=p'}$ are replaced by $\sum_{p=1}^{4^j}$. In particular, (2.4) becomes

$$\lim_{j\to\infty} j^\alpha \sum_{n=n_0}^{\infty} \sum_{p=1}^{4^j} \int_{Y_{j,n,p}} g \circ f^j \, d\mu = 0 \tag{5.6}$$

for all $g \in L^\infty$ satisfying $\int_{X_n} g \, d\mu = 0$ for all $n \geq n_0$. The counterpart of (2.5) is that there are constants $c_{j,n} > 0$ such that

$$\sum_{n=n_0}^{\infty} n^{-\alpha} \sup_{X_n} |J_{j,n} - c_{j,n}| = o(j^{-\alpha}) \quad \text{as } j \to \infty \tag{5.7}$$

where

$$J_{j,n}(x) = \frac{\sum_{p=1}^{4^j} h(y_{j,n,p})(\det Df^j(y_{j,n,p}))^{-1}}{\sum_{\ell=1}^{\infty}\sum_{p=1}^{4^\ell} h(y_{\ell,n,p})(\det Df^\ell(y_{\ell,n,p}))^{-1}}$$

for $x \in X_n$ and $y_{\ell,n,p} \in Y_{\ell,n,p}$ with $f^j y_{\ell,n,p} = x$.

It remains to verify (5.7). By [21, Lemma 2.9], incorporating the strengthened expansion for $u(x,\theta)$, we find that the diffeomorphism $f^j : Y_{j,n,p} \to X_n$ satisfies

$$\{\det Df^j\}^{-1} = 4^{-(j+1)}\Big(\frac{n}{j+n}\Big)^{\alpha+1}\big(1 + O(n^{-\kappa'})\big)$$

where the implied constant is independent of j, p and n uniformly on Y. (There is a factor of 4^j arising from the second coordinate of f^j and a factor of 4 from the first iterate using the second branch $x \mapsto 4x - 3$.)

By the construction in [21], $\lim_{n\to\infty} \operatorname{dist}(y_{j,n,p}, \{x = \frac{3}{4}\}) = 0$ uniformly in j and p. Moreover the one-sided limit $h(\frac{3}{4}^+, \theta)$ exists for a.e. $\theta \in \mathbb{T}$ by [21, Proposition 4.11] and

$$\lim_{n\to\infty} \sum_{p=1}^{4^j} h(y_{j,n,p})4^{-j} = \int_{\mathbb{T}} h(\tfrac{3}{4}^+, \theta)\, d\theta$$

uniformly in j. Once again we find that $J_{j,n} \sim \alpha n^\alpha (j+n)^{-(\alpha+1)}$ as $n \to \infty$ uniformly in j on X_n, verifying condition (2.5).

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