A Very Short Introduction to Simplicial Homology

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Contents

1 Simplicial Complex

A *simplicial complex* is a geometric structure that can be cobbled up as and when one wants using nothing but finitely many elementary units called *simplex (pl. simplices).*

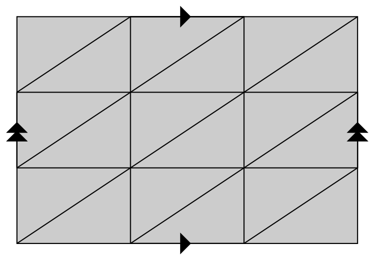

Figure 1: Triangulation of a torus

Definition 1.1. A *k-simplex* is the convex hull of k points in general position. The *k-simplex* is a *k-dimensional* object in $\mathbb{R}^n$.

Let $v_0, \cdots, v_k$ be k points in $\mathbb{R}^n$, the complex hull of $\{v_0, \cdots, v_k\}$ is the smallest convex polygon which contains all of them, i.e. all the points x,

$$x = \lambda_0 v_0 + \cdots + \lambda_k v_k,$$

such that $\sum \lambda_k = 1$. The set of k points spans a subspace of $\mathbb{R}^n$. If each subset of $k-1$ points span a subspace smaller than this then we say the k points are in *general position.*

Definition 1.2. A *simplicial complex* K is a collection of simplices of varying dimensions in $\mathbb{R}^n$ such that the following rules hold.

1. If K contains a simplex it contains all faces of this simplex.
2. The intersection of two simplices of K is either empty or a common face.
3. (In case K is infinite), K is locally finite, i.e. every point of $\mathbb{R}^n$ has a neighborhood that intersects only finitely many simplices of K.

1.1 Triangulable Spaces

The simplicial complex is a polyhedron in $\mathbb{R}^n$. In fact every polyhedron in $\mathbb{R}^n$ can be triangulated using a *simplicial complex*. Similarly if we can construct a homeomorphism $h : |K| \to X$ where X is a topological surface then we can triangulate it. It is a well known theorem that every compact surface admits a triangulation.

1.2 Barycentric Subdivision

We cut up the simplices of a simplicial complex K to obtain another simplicial complex which is called the barycentric subdivision of K. m consecutive barycentric divisions gives us the m^{th} barycentric subdivision denoted by $|K^m|$.

Definition 1.3. A *cone* of a simplex L is denoted by CL. The cone is obtained by appending a point v, called the apex of the cone, to the vertices $\{v_0, \cdots, v_k\}$

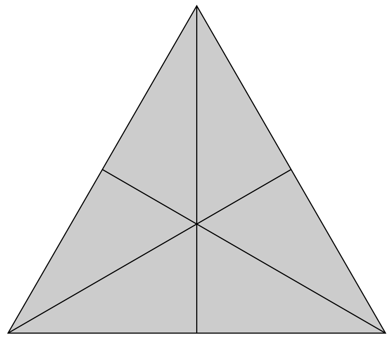

Figure 2: K^1

so that the new sets of vertices $\{v, v_0, \cdots, v_k\}$ are in general position. The cone CL is then the $(k+1)$-simplex got by taking the convex hull of this set. The process is akin to extruding the *k-simplex* to a $(k+1)$-simplex.

To construct the barycentric subdivision we move starting from the lowest dimensional simplices and working our way upwards. The barycenter of a k-simplex A is the point,

$$\hat{A} = \frac{1}{k+1}(v_0 + \cdots + v_k).$$

For each simplex A in our simplicial complex K construct the cone CA using the barycenter of A, i.e. $\hat{A}$, as the vertex. The simplicial complex K^1 obtained from such a maneuver is called the *first barycentric subdivision.* Similarly by repeated barycentric subdivisions we obtain the m^{th}-barycentric subdivision, K^m.

1.3 Orientation

Let A be a k-simplex with vertices $\{v_0, \cdots, v_k\}$. The *orientation* of A is defined to be any permutation of the ordered set of its vertices. Each face of the simplex is given an orientation induced from the parent by deleting one vertex.

Proposition 1.4. *Two orientation of the simplex are equivalent is they differ by an even number of transpositions and different if they differ by an odd number of transpositions. This splits the set of orientations into two.*

Proof. This fact can be proven by induction on the dimension of the simplex. □

Definition 1.5. A *simplicial complex* is orientable if the common faces between each pair of simplices are compatible, i.e. the pair of simplices induce different orientations on the common face (See figure 3).

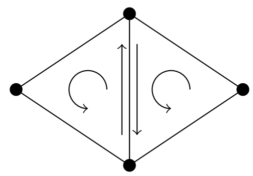

Figure 3: Compatible Simplices

2 Simplicial Homology

The homology groups of a simplicial complex K is denoted by $H_k(K)$. On the simplicial complex K we define a collection of groups called the *chain groups*, i.e. $C_k(K)$ and show that they are elements of a *chain complex,* i.e. a collection of groups and group homeomorphisms,

$$\cdots \xrightarrow{\partial} C_k(K) \xrightarrow{\partial} C_{k-1}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(K) \xrightarrow{\partial} 0,$$

such that $\operatorname{Im} \partial \subseteq \operatorname{Ker} \partial'$ where ∂ and ∂' are adjacent homomorphisms.

Definition 2.1. $C_k(K)$ is the free abelian group generated by the set of k-simplices of K such that $\sigma + \tau = 0$ whenever σ and τ are the same simplex but with opposite orientation.

Now that we have a definition of a *chain group* lets construct the homomorphism for the *chain complex.* We have already defined the boundary mapping in 1.4.

Definition 2.2. The boundary of a k-simplex is the map,

$$\partial(v_0, \cdots, v_k) = \sum_{i=0}^{k} (-1)^i (v_0, \cdots, \hat{v}_i, \cdots, v_k).$$

The term $(v_0, \cdots, \hat{v}_i, \cdots, v_k)$ is simply the $(k-1)$-simplex obtained by deleting the $\hat{v}_i$ vertex, i.e. a face of this simplex. Note that the boundary map, $\partial(-\sigma) = \sigma$. The boundary homomorphism is now defined for each of the generators of $C_k(K)$. Hence it is defined for $C_k(K)$.

Every element in the kernel of $\partial : C_k(K) \to C_{k-1}(K)$ is called a k-cycle. k-cycles are elements of $C_k(K)$ having no boundary.

For any arbitrary element of $C_k(K)$ we may visualize it as a geometric object composed of a collection k-simplices. The boundary homomorphism, as the name implies, gives us the boundary of this figure consisting of all the $(k-1)$-simplicial faces of these k-simplices.

Proposition 2.3. *The collection of chain groups and chain maps together form a chain complex.*

Proof. Consider the following,

$$C_{k+1}(K) \xrightarrow{\partial_{k+1}} C_k(K) \xrightarrow{\partial_k} C_{k-1}(K).$$

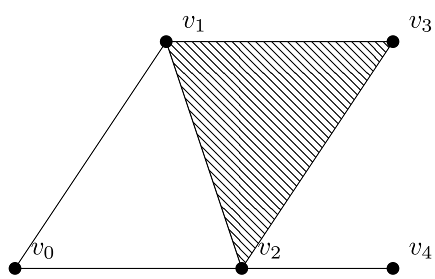

Figure 4: The Simplicial Complex K

Let $\sigma \in C_{k+1}$. We see that $\partial_{k+1}\sigma \in \operatorname{Im} \partial_{k+1}$ is a k-cycle in $C_k(K)$. So $\sigma \in \operatorname{Ker} \partial_k$ and we have $\operatorname{Im} \partial_{k+1} \subseteq \operatorname{Ker} \partial_k$. $\square$

2.1 Homology Groups

Definition 2.4. We have,

$$\cdots \to C_{k+1}(K) \xrightarrow{\partial_{k+1}} C_k(K) \xrightarrow{\partial_k} C_{k-1}(K) \to \cdots .$$

The k^{th}-homology group, $H_k(K)$ is defined as the quotient group,

$$H_k(K) = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}}.$$

Intuitively the k^{th}-homology group is the group formed by k-cycles in $C_k(K)$ but with the group of those k-cycles that are boundaries of elements in $C_{k+1}(K)$ factored out. This makes precise the notion that homology groups intuitively gives us a feel of the *holes* in a surface. To clarify this remark let us consider the triangulation of a surface whose simplicial complex consists of $\{0, 1, 2\}$-simplices only.

In figure 4 the shaded portion is a 2-simplex. So in the homology group $H_2(K)$ the 2-cycles (v_0, v_1, v_2, v_3) and (v_0, v_1, v_3, v_2) are equivalent and hence we have only one generator for $H_2(K)$ so $H_2(K) \cong \mathbb{Z}$.

3 Homotopy

Definition 3.1. Two maps $f, g : X \to Y$ are said to be homotopic to each other when there exists a map $F : X \times I \to Y$ such that given $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Homotopy is denoted by $f \simeq g$.

Proposition 3.2. *The notion of homotopy is an equivalence relation.*

(a) $f \simeq f$.

(b) $f \simeq g \implies g \simeq f$.

(c) $f \simeq g$ *and* $g \simeq h \implies f \simeq h$.

Proof. (a) is trivial.

(b) We have a map $F(x,t); F(x,0) = f(x)$ and $F(x,1) = g(x)$. Replace t by $1-t$.

(c) For the two maps $F(x,t)$ and $G(x,t)$ we define $f \simeq h$ by the map,

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases} .$$

This map is continuous by 3.3.

□

Lemma 3.3. *(Gluing Lemma) Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two maps,* X, Y *closed in* $X \cup Y$*. If* f *and* g *are the same in* $X \cap Y$ *then* $f \cup g : X \cup Y \to Z$ *is a continuous function.*

3.1 Homotopy Types

Definition 3.4. Two spaces X and Y are of the same homotopy type if there exists two maps $f : X \to Y$ and $g : Y \to X$ such that their compositions $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. Two homotopically equivalent spaces are denoted by $X \simeq Y$.

Proposition 3.5. *The relation* $X \simeq Y$ *is an equivalence relation on the class of topological spaces.*

(a) $X \simeq X$.

(b) $X \simeq Y \implies Y \simeq X$.

(c) $X \simeq Y$ *and* $Y \simeq Z \implies X \simeq Z$.

4 Simplicial Maps

Definition 4.1. Let L, M be two simplicial complexes. A function $s : |L| \to |M|$ is called a simplicial map if it takes simplices of L onto simplices of M linearly and if for each simplex A in L, $s(A)$ is a simplex in M.

Let A be a simplex in L. If $x \in A$ then $x = \lambda_0 v_0 + \cdots + \lambda_k v_k$ where $\{v_0, \cdots, v_k\}$ are the vertices of A. Then we have,

$$s(x) = \lambda_0 s(v_0) + \cdots + \lambda_k s(v_k).$$

Proposition 4.2. *A simplicial map is a continuous function.*

Proof. Linear maps are continuous. The gluing lemma 3.3 makes sure that the the simplicial map is continuous too. □

Proposition 4.3. *If s is a simplicial approximation to f then $s \simeq f$.*

Proof. We can think of $|L|$ as being a subset of $\mathbb{R}^n$. Then $F : |L| \times I \to \mathbb{R}^n$ is a straight line homotopy given by $f(x,t) = (1-t)s(x) + tf(x)$. Since $s(x)$ and $f(x)$ lie in the same simplex, $(1-t)s(x) + f(x)$ must lie in the convex hull, i.e. and hence in $|L|$. $\square$

Definition 4.4. A simplicial map $s : |L| \to |M|$ is a simplicial approximation to the map $f : |L| \to |M|$ if for $x \in |L|$, $s(x)$ lies inside the same unique simplex in M as $f(x)$.

4.1 Simplicial Approximation Theorem

Definition 4.5. An open star at vertex v is an open subset of $|K|$ written as $star(v, K)$ consisting of the interiors of those simplices that has v as one of its vertices.

Lemma 4.6. *The vertices $\{v_0, \cdots, v_k\}$ of a simplicial complex K belong to a simplex of K if and only if $\bigcap_0^k star(v_i, K)$ is non empty.*

Proof. The proof is left to the reader. $\square$

Theorem 4.7. *Let $f : |K| \to |L|$ be a continuous map. There exists a simplicial approximation $s : |L^m| \to |M|$ for f when m is sufficiently large.*

Proof. The family $\{star(v_i, L)\}_{i \in I}$ of open sets is an open covering of $|L|$. So $\{f^{-1}(star(v_i, L))\}$ is an open covering of the compact metric space $|K|$. By the *Lebesgue's lemma* we can choose a δ and a number m such that the family $\{star(u_i, K^m)\}$ is of a sufficiently small diameter to fit inside this open covering, i.e.

$$star(u, K^m) \subseteq f^{-1}(star(v, L)) \tag{1}$$

for some vertex v of L. We can choose an association s such that $s(u) = v$. We can extend this to a simplicial map using 4.6. It is now easy to verify that the map s is a simplicial approximation to f. If $x \in K^m$ and $\{u_0, \cdots, u_k\}$ are the vertices of the simplex in which it lies, we have $x \in \bigcap_0^k star(u_i, K^m)$ and so by (1) $s(x)$ lies in the simplex formed by $\{s(u_0), \cdots, s(u_k)\}$. $\square$

5 Invariant Homologies

In this section we show that the homology groups of two homotopically equivalent spaces are isomorphic. But first we show that they are invariant under barycentric subdivision.

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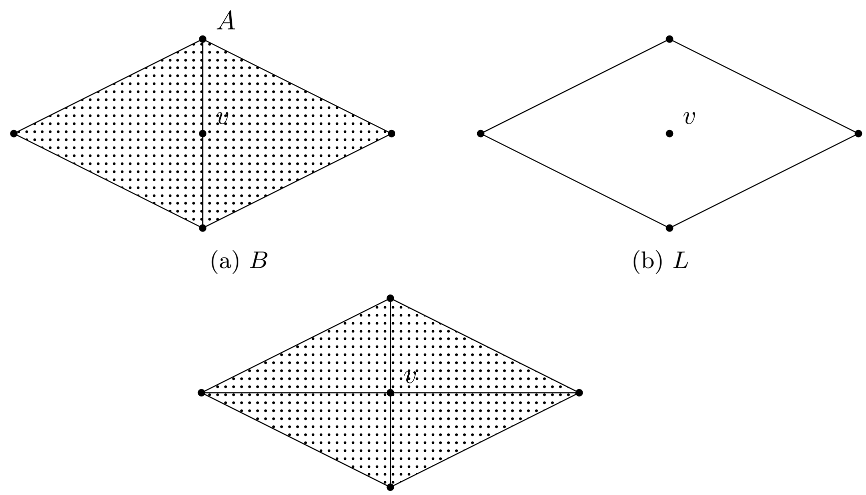

(a) B (b) L

(c) Stellar subdivision of A

Figure 5: Stellar subdivision

Definition 5.1. Let K be a simplicial complex and A any simplex of K. If A is a face of B then let L be the subcomplex of the boundary of B of those simplices that do not have A as a face. The *stellar subdivision* of A is the *cone* obtained with L as the base and v as the vertex. See figure 5.

Proposition 5.2. *The first barycentric subdivision of the a simplicial complex K is obtained from K by finitely many stellar subdivisions.*

Proof. Take the stellar subdivisions of the simplices of K one by one in descending order of their dimension. We will show that the subdivision got by this method is equivalent to the first barycentric subdivision.

(a) Notice that the subdivision obtained by the previous method has the same number of barycenters as the barycentric subdivision.

(b) Secondly, the $(k-1)$-simplices that divide the k-simplices into smaller k-simplices only depends on the positions of the barycenters as one of their vertices and an existing subcomplex of K (by the definition of a *cone*) which is the same in both the methods.

Hence the result follows. The stellar subdivision method is essentially an inverse process to the barycentric subdivision where one starts from the simplex of the lowest dimension and moves upward. □

Corollary 5.3. *The m^{th} order barycentric subdivision K^m is obtained by finitely many stellar subdivisions from the complex K.*

Theorem 5.4. *Barycentric subdivision does not change the homology groups of a complex.*

Proof. Consider two simplicial complexes K and K' that differ by a single stellar subdivision of a simplex A. Let σ be a k-simplex of K and $\chi : C(K) \to C(K')$ a map defined as follows. If σ has A as a face it will be subdivided into smaller k-simplices. $\chi(\sigma)$ is defined to be the sum of these smaller k-simplices taken with their orientation otherwise we set $\chi(\sigma) = \sigma$ We will show that χ is a chain map.

Notice that χ is a homomorphism. Any k-simplex has an inside portion and the boundary outside. Note that the stellar subdivision of A where A is a face of a simplex σ creates subdivisions entirely inside the simplex σ and the face A. Hence the maps $\partial\chi_k$ and $\chi_{k-1}\partial$ are intuitively the same since on the adjacent faces they cancel since the orientations are opposite.

χ is called the *subdivision chain map.* Since χ induces a homomorphism $\chi_* : H(K) \to H'(K)$ we have to show that it is an isomorphism. Define a simplicial map $\theta : K \to K'$ that sends v, the barycentric subdivision of A to v_0 a vertex of A and preserves all other vertices. This map in effect undoes the stellar subdivision. Call the induced chain map as $\theta : C(K') \to C(K)$ itself. Now $\theta\chi : C_k(K) \to C_k(K)$ is the identity homomorphism so that $H_k(K) \xrightarrow{\chi_*} H_k(K') \xrightarrow{\theta_*} H_k(K)$ is the identity. We shall show that θ_* is an inverse of χ_*.

Let z be a k-cycle of K' and consider $z - \chi\theta(z)$. If L denotes the set of all simplices that has v as a vertex along with their faces then L is a cone with v as the apex. $z - \chi\theta(z)$ is a k-cycle in L because outside of L, χ and θ are the identity and $\partial(z - \chi\theta(z)) = \partial(z) - \chi\theta(\partial(z)) = 0$. Since L is a cone by 5.13 the homology group is $H_k(L) = 0, k > 0$. So $z - \chi\theta(z)$ is the boundary of a $(k+1)$-cycle, leading us to the conclusion that both z and $\chi\theta(z)$ lies in the same homology class in $H_k(K')$. If $k = 0$ so that $H_0(L) \cong \mathbb{Z}$ then $\chi\theta$ is trivially the identity. So $H_k(K) \xrightarrow{\theta_*} H_k(K) \xrightarrow{\chi_*} H_k(K')$ is the identity so that χ_* is an isomorphism. □

* * *

Definition 5.5. Two simplicial maps $s, t : |K| \to |L|$ are called close if for each simplex $\sigma \in K$ both $s(\sigma)$ and $t(\sigma)$ are faces of the same simplex $\sigma' \in L$ called the carrier.

Lemma 5.6. *If $s, t : |K^m| \to |L|$ both simplicially approximates $f : |K^m| \to |L|$, show that s and t are close simplicial maps.*

Proof. Let $\{v_0, \cdots, v_k\}$ be the vertices of a simplex σ of K^m. The property of simplicial approximation can be written as,

$$f(star(u, L)) \subseteq star(s(u, L)),$$

if s simplicially approximates f. Now the simplices

$$\{s(v_0), \cdots, s(v_k)\},$$

$$\{t(v_0), \cdots, t(v_k)\}$$

are either entirely different or share some vertices. In either cases since for each $i, f(star(v_i)) \subseteq star(s(v_i)) \cap star(t(v_i))$ is not empty we have by 4.6 a simplex which connects $s(v_i)$ and $t(v_i)$. We have now shown that there exists at least a 1-simplicial skeleton that joins $s(\sigma)$ and $t(\sigma)$. We will now show that there is a higher dimensional simplex. We have,

$$\bigcap_{i}^{k} f(star(v_i)) \subseteq \bigcap_{i}^{k} \{star(s(v_i)) \cap star(t(v_i))\}.$$

But $x \in \bigcup_i^k star(v_i)$ since there form the simplex σ hence we see that the left hand side of the equation above is not empty and by 4.6 their exists a simplex $\{s(v_0), \cdots, s(v_k), t(v_0), \cdots, t(v_k)\}$ so that $s(\sigma)$ and $t(\sigma)$ are two of its faces. □

Lemma 5.7. *If $s, t : |K| \to |L|$ are close then $s_* = t_* : H_k(K) \to H_k(L)$ for all k.*

Proof. We inductively define a sequence of homomorphisms, $d_k : C_k(K) \to C_{k+1}(L)$ called a *chain homotopy* between s and t. If $\sigma = v \in C_0(K)$ then we set $d_0(\sigma) = 0$ if $s(\sigma) = t(\sigma)$ and $d_0 = (s(v), t(v))$ otherwise. Also, $\partial d_0 = t - s : C_0(K) \to C_0(L)$ trivially. For $0 \leq i \leq k-1$ we define $d_i : C_i(K) \to C_{i+1}(L)$ so that,

$$d_{i-1}\partial + \partial d_i = t - s : C_i(K) \to C_i(L)$$

such that $d_i(\sigma)$ is always a chain in the carrier of σ. Now we have a chain homotopy $d_k : C_k(K) \to C_{k+1}(L)$, such that

$$d_{k-1}\partial + \partial\sigma d_k = t - s : C_k(K) \to C_k(L).$$

We have to show that s and t induces the same homomorphisms of homology groups. We only have to work with k-cycles of K, so that $d_{k-1}\partial$ disappears. We see that $(t-s)(\sigma)$ must be a boundary, which is zero in homology and so they induce the same homomorphisms. □

Lemma 5.8. *If $f \simeq g : |K| \to |L|$ then we have a sequence of simplicial maps $s_1, \ldots, s_n : |K^m| \to |L|$ such that s_1 simplicially approximates f, s_n simplicially approximates g and each pair s_i, s_{i+1} are close.*

Proof. (a) Let $f, g : |K| \to |L|$ be maps and write $d(f,g) < \delta$ if for each $x \in |K|$ the distance between $f(x)$ and $g(x)$ is less than δ. If δ is a Lebesgue number for the open covering of $|L|$ by the open stars of its vertices, and if $d(f,g) < \delta/3$, then we will show that the sets

$$f^{-1}(star(v, L)) \cap g^{-1}(star(v, L)),$$

v a vertex of L, form an open covering of $|K|$.

Suppose $x \in |K|$ lies in $f^{-1}(star(v, L))$ but not in $g^{-1}(star(v, L))$. But since $f(x)$ lies in $star(v, L)$ so does $g(x)$, since $d(f,g) < \delta/3$, so that x lies in the set $g^{-1}(star(v, L))$ which is a contradiction. This proves our first assertion.

(b) Since $|K|$ is a compact metric space so we can find a δ and an m such that $\mu(K^m) < \delta$ by the Lebesgue lemma. So $star(u, K^m) \subseteq f^{-1}(star(v, L) \cap g^{-1}(star(v, L)$for some vertex v of L. So that we have a map $s : |K^m| \to |L|$ which simplicially approximates both $f : |K^m| \to |L|$ and $g : |K^m| \to |L|$.

(c) Finally let $f, g : |K| \to |L|$ be homotopic maps with a specific homotopy $f_t(x) = F(x, t)$ between them. Since $f_t(x)$ is continuous, given $\delta > 0$, we can find a series of maps,

$$d(f_{r/n}, f_{(r+1)/n}) < \delta, 0 \leq r < n.$$

Using (b) we can find a common simplicial approximation $s_i : |K^m| \to |L|$ between each pair of such maps. Now since both s_i and s_{i+1} are the simplicial approximations to $f_{(r+1)/n)}$ by 5.6 we see that they are *close* simplicial maps.

□

Theorem 5.9. *Let $f : |K| \to |L|$ be a map. Then f induces a homomorphism $f_* : H_k(K) \to H_k(L)$.*

Proof. Let $s : |K^m| \to |L|$ and $t : |K^n| \to |L|, m \leq n$ be simplicial approximations to $f : |K| \to |L|$. We have two subdivision chain maps and their induced homomorphisms $\chi_{1*} : C_k(K) \to C_k(K^m)$ and $\chi_{2*} : C_k(K^m) \to C_k(K^n$. We also have the standard simplicial map $\theta_* : C_k(K^n) \to C_k(K^m)$ which is the inverse of χ_{2*}.

Now the standard simplicial map $\theta : |K^n| \to |K^m|$ takes each $x \in |K^n|$ to the simplex in $|K^m|$ which is contained in a barycentric subdivision of its carrier in $|K^n|$, i.e. it does not change the carrier when s becomes $s\theta(x)$. So $s\theta : |K^n| \to |L|$ simplicially approximates $f : |K| \to |L|$ along with t. By 5.6, $s\theta$ and t are close simplicial maps and so by 5.7 we have $s_*\theta_* = t*$. Now θ_* and χ_{2*} are inverse to each other so,

$$t_*\chi_{2*}\chi_{1*} = s_*\theta_*\chi_{2*}\chi_{1*} = s_*\chi_{1*} : H_k(K) \to H_k(L).$$

Therefore we define $f_* = s_*\chi_{1*} = t_*\chi_{1*}\chi_{2*}$. □

Theorem 5.10. *If we have two maps $|K| \xrightarrow{f} |L| \xrightarrow{g} |M|$ then $(g \circ g)_* = g_* \circ f_* : H_k(K) \to H_k(M)$ for all k.*

Proof. Choose simplicial approximations $t : |L^n| \to |M|$ for g and $s : |K^m| \to |L^n|$ for f. Let $\chi_1 : C(K) \to C(K^m)$ and $\chi_2 : C(L) \to C(L^n)$ be subdivision chain maps and let $\theta : |L^n| \to |L|$ be a standard simplicial map. We have the following diagram,

$$
\begin{tikzcd}
H_k(K^m) \arrow[r, "s_*"] & H_k(L^n) \arrow[dr, "t_*"] \arrow[d, bend left, "\theta_*"] & \\
H_k(K) \arrow[u, "\chi_{1*}"] \arrow[r, "f_*"'] & H_k(L) \arrow[u, "\chi_{2*}"] \arrow[r, "g_*"'] & H_k(M)
\end{tikzcd}
$$

Now θs and ts simplicially approximates $f : |K^m| \to |L|$ and $gf : |K^m| \to |M|$ respectively so that,

$$g_* \circ f_* = t_* \chi_{2*} \theta_* s_* \chi_{1*} = t_* s_* \chi_{1*} = (ts)_* \chi_{1*} = (g \circ f)_*.$$

□

Theorem 5.11. *If $f, g : |K| \to |L|$ are two maps such that $f \simeq g$, then $f_* = g_* : H_k(K) \to H_k(L)$. Where f_*, g_* are group homomorphisms induced by f, g respectively.*

Proof. From 5.7 we know that simplicial maps induce homomorphisms of homology groups. We apply 5.8 and get a sequence of close simplicial maps $s_1, \cdots, s_n : |K^m| \to |L|$ between f and g. Let $\chi : C(K) \to C(K^m)$ be the subdivision chain map. Then we have, $f_* = s_{1*}\chi_* = \cdots = s_{n*}\chi_* = g_*$. □

* * *

Theorem 5.12. *Let X, Y be topological spaces that admit the triangulations $h : |K| \to X, t : |L| \to Y$ and $X \simeq Y$. Then for each k, their homology groups $H_k(K)$ and $H_k(L)$ are isomorphic.*

Proof. Since $X \simeq Y$ we have $|K| \simeq |L|$. Let $f : |K| \to |K|$ and $g : |L| \to |K|$ be continuous maps. f, g induces a composite homeomorphism,

$$H_k(K) \xrightarrow{f_*} H_k(L) \xrightarrow{g_*} H_k(K),$$

$$H_k(L) \xrightarrow{g_*} H_k(K) \xrightarrow{f_*} H_k(L).$$

Since both these homomorphisms are equal to the identity homomorphism by 5.11 we have an isomorphism $f_* : H_k(K) \to H_k(L)$. □

5.1 The Sphere Alone

Proposition 5.13. *Let L be a simplicial complex and CL the cone of L with vertex v. Then,*

1. $H_0(CL) \cong \mathbb{Z}$,
2. $H_k(CL) = 0, k > 0$.

Proof. In a cone the 0-simplices, i.e. the vertices, are always connected by 1-simplices hence $H_0(CL) \cong \mathbb{Z}$. Assume $k > 0$ and define a homomorphism $d : C_k(K) \to C_{k+1}(K)$ as follows. If $\sigma = (v_0, \cdots, v_k)$ is an oriented k-simplex of K which lies in L, define $d(\sigma) = (v, v_0, \cdots, v_k)$; otherwise set $d(\sigma) = 0$. So d gives a homomorphism from $C_k(K) to C_{k+1}(K)$ since $d(\sigma)$ depends only on the orientation of σ and $d(\sigma) + d(-\sigma) = 0$. Therefore d gives a homomorphism from $C_k(K)$ to $C_{k+1}(K)$. Now if $\sigma \in L$ then

$$\partial d(\sigma) = \partial(v, v_0, \cdots, v_k) = (v_0, \cdots, v_k) + \sum_{i=0}^{k} (-1)^{i+1} (v, v_0, \cdots, v_k) = \sigma - d\partial(\sigma).$$

So if z is a k-cycle of K, we have $\partial d(z) = z - d\partial(z) = z$. This means that every k-cycle is a bounding cycle, and therefore $H_k(K) = 0$ for $q > 0$. □

Proposition 5.14. *The homology groups for the sphere S^n are: $H_0(S^n) \cong \mathbb{Z}, H_k(S^n) = 0, 1 \leq k \leq n-1$ and $H_n(S^n) \cong \mathbb{Z}$.*

Proof. (a) Let Δ^{n+1} be an $(n+1)$-simplex and Σ^n those simplices that lie on the boundary of Δ^{n+1}. We have a triangulation $h : |\Sigma^n| \to S^n$. Therefore the sphere S^n has the same homology groups as $|\Sigma^n|$.

(b) We have, $H_k(\Sigma^n) \cong H_k(\Delta^{n+1})$ for $0 \leq k \leq n-1$ since Σ^n shares all the simplices of dimension less than n with Δ^{n+1}. But Δ^{n+1} is a cone and hence by 5.13 and (a) $H_0(S^n) \cong \mathbb{Z}, H_k(S^n) = 0, 1 \leq k \leq n-1$.

□

5.2 The Sphere and the Euclidean Space

Proposition 5.15. *If $m \neq n$ then S^m and S^n are not homotopically equivalent.*

Proof. $H_m(S^m)$ is isomorphic to $H_m(S^n)$ only if m = n. □

Theorem 5.16. *Two euclidean spaces, $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic if and only if $m = n$.*

Proof. Assume that $h : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism and $m \neq n$. Then $\mathbb{R}^m \cong \mathbb{R}^n$. But we have that $S^{m-1} \cong \mathbb{R}^m - \{0\}$ and $S^{n-1} \cong \mathbb{R}^n - \{0\}$. But by 5.15 we have $m = n$, a contradiction.

□

5.3 Brouwer's Fixed Point Theorem

Theorem 5.17. *Any map from B^n to itself must leave at least one point fixed.*

Proof. (a) For $n = 1$ we use the unit interval $I = [0, 1]$. Assume the result to be false and define a new function $g : I \to \{0, 1\}$ by $g(x) = 0$ if $f(x) > x$ and $g(x) = 1$ if $f(x) < x$. The continuity of f implies the continuity of g and g is onto since $g(0) = 0$ and $g(1) = 1$. But I is a connected space and hence we have a contradiction.

(b) For $n = 2$ we take the unit disc D in the plane and assume we have a map $f : D \to D$ which has no fixed points. Define a function g for each $x \in D$ by sending x to the intersection of the line passing from x to $f(x)$ and the boundary circle C. The continuity of f implies that g is a continuous function. Further $g(x) = x$ for all points on C.

Take the point $p = (1, 0)$ as the base point for both C and D and denote the inclusion of C in D by $i : C \to D$. The spaces and maps $C \xrightarrow{i} D \xrightarrow{g} C$ gives rise to groups and homomorphisms

$$\pi_1(C, p) \xrightarrow{i_*} \pi_1(D, p) \xrightarrow{g_*} \pi_1(C, p).$$

Now $g \circ i(x) = x$ for all x in C so $g_* \circ i_*$ is the identity homomorphism and g_* is onto. But $\pi_(D, p)$ is the trivial group and $\pi_1(C, p) \cong \mathbb{Z}$, so we have a contradiction, thus establishing Brouwer's theorem for $n = 2$.

(c) For $n > 2$ we assume the contrary and turn to the $(n-1)$-homology groups of the n-ball and its boundary the $n-1$-sphere. We define the same straight line continuous map g taking points x in a straight line through $f(x)$ to the boundary. But we know that the $(n-1)^{th}$ homology group of an n-ball is $H_{n-1}(|\Delta^n|) = 0$ but for the $n-1$-sphere it's $H_{n-1}(|\Sigma^{n-1}|) \cong \mathbb{Z}$.

The map $g : B^n \to S^{n-1}$ and the inclusion map $i : S^{n-1} \to B^n$ can be replaced by the maps between triangulations $g : |\Delta^n| \to |\Sigma^{n-1}|$ and $i : |\Sigma^{n-1}| \to |\Delta^n|$ which induces homomorphisms of homology groups,

$$H_{n-1}(|\Sigma^{n-1}|) \xrightarrow{i_*} H_{n-1}(|\Delta^n|) \xrightarrow{g_*} H_{n-1}(|\Sigma^{n-1}|).$$

But $g \circ i(x) = x$ is the identity map for all x in $|\Sigma^{n-1}|$. Then $g_* \circ i_*$ is the identity homomorphism and g_* is onto. Therefore we have a contradiction. □

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The proofs and definition contained in this article are standard. I have referenced most of them from [1]. Sometimes I have made modification to reflect the way I understood them and how I wanted to present them. Any mistakes made in this article are mine alone.

References

[1] Armstrong, Basic Topology*, Springer, (1983)*

[2] *Klaus Jänich,* Topology*, Springer, (1984)*

[3] *Allen Hatcher,* Algebraic Topology*, (2001)*