DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

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Differential Equations and Linear Algebra

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The website with solutions to problems in this textbook is **math.mit.edu/dela**
That site links to video lectures on this book by Gilbert Strang and Cleve Moler.
Linear Algebra and Differential Equations are on MIT's OpenCourseWare site **ocw.mit.edu**.
This provides video lectures of the full courses 18.03 and 18.06.
Course material is on the teaching website: **web.mit.edu/18.06**
Highlights of Calculus (17 lectures and text) are on **ocw.mit.edu**

The front cover shows the Lorenz attractor, drawn for this book by Gonçalo Morais. This is the first example of chaos, found by Edward Lorenz.
The cover was designed by Lois Sellers and Gail Corbett.

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Preface

Differential equations and linear algebra are the two crucial courses in undergraduate mathematics. This new textbook develops those subjects separately and together. Separate is normal—these ideas are truly important. This book presents the basic course on differential equations, in full :

Chapter 1	First order equations
Chapter 2	Second order equations
Chapter 3	Graphical and numerical methods
Chapter 4	Matrices and linear systems
Chapter 6	Eigenvalues and eigenvectors

I will write below about the highlights and the support for readers. Here I focus on the option to include more linear algebra. Many colleges and universities want to move in this direction, by connecting two essential subjects.

More than ever, the central place of linear algebra is recognized. Limiting a student to the mechanics of matrix operations is over. Without planning it or foreseeing it, my lifework has been the presentation of linear algebra in books and video lectures :

Introduction to Linear Algebra (Wellesley–Cambridge Press)
MIT OpenCourseWare (**ocw.mit.edu**, Mathematics 18.06 in 2000 and 2014).

Linear algebra courses keep growing because the need keeps growing. At the same time, a rethinking of the MIT differential equations course 18.03 led to a new syllabus. And independently, it led to this book.

The underlying reason is that time is short and precious. The curriculum for many students is just about full. Still these two topics cannot be missed—and linear differential equations go in parallel with linear matrix equations. The prerequisite is calculus, for a single variable only—the key functions in these pages are inputs $f(t)$ and outputs $y(t)$. For all linear equations, continuous and discrete, the complete solution has two parts :

$$\begin{array}{ll} \textbf{One particular solution } \boldsymbol{y_p} & \boldsymbol{Ay_p = b} \\ \textbf{All null solutions } \boldsymbol{y_n} & \boldsymbol{Ay_n = 0} \end{array}$$

Those right hand sides add to $\boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}$. The crucial point is that the left hand sides add to $\boldsymbol{A}(\boldsymbol{y_p} + \boldsymbol{y_n})$. When the inputs add, and the equation is linear, the outputs add. The equality $\boldsymbol{A}(\boldsymbol{y_p} + \boldsymbol{y_n}) = \boldsymbol{b} + \boldsymbol{0}$ tells us all solutions to $\boldsymbol{Ay} = \boldsymbol{b}$:

The complete solution to a linear equation is $\boldsymbol{y} = (\textbf{one } \boldsymbol{y_p}) + (\textbf{all } \boldsymbol{y_n})$.

The same steps give the complete solution to $dy/dt = f(t)$, for the same reason. We know the answer from calculus—it is the form of the answer that is important here :

$$
\begin{array}{lll}
\dfrac{dy_p}{dt} = f(t) & \text{is solved by} & y_p(t) = \int_0^t f(x)\,dx \\
\dfrac{dy_n}{dt} = 0 & \text{is solved by} & y_n(t) = C \text{ (any constant)} \\
\dfrac{dy}{dt} = f(t) & \text{is completely solved by} & y(t) = y_p(t) + C
\end{array}
$$

For every differential equation $dy/dt = Ay + f(t)$, our job is to find y_p and y_n : one particular solution and all homogeneous solutions. My deeper purpose is to build confidence, so the solution can be understood and used.

Differential Equations

The whole point of learning calculus is to understand movement. An economy grows, currents flow, the moon rises, messages travel, your hand moves. The action is fast or slow depending on forces from inside and outside: competition, pressure, voltage, desire. Calculus explains the meaning of dy/dt, but to stop without putting it into an equation (a differential equation) is to miss the whole purpose.

That equation may describe growth (often exponential growth e^{at}). It may describe oscillation and rotation (with sines and cosines). Very frequently the motion approaches an equilibrium, where forces balance. That balance point is found by linear algebra, when the rate of change dy/dt is zero.

The need is to explain what mathematics can do. I believe in looking partly outside mathematics, to include what scientists and engineers and economists actually remember and constantly use. My conclusion is that first place goes to linear equations. The essence of calculus is to linearize around a present position, to find the direction and the speed of movement.

Section 1.1 begins with the equations $dy/dt = y$ and $dy/dt = y^2$. It is simply wonderful that solving those two equations leads us here :

$$
\begin{array}{lll}
\dfrac{dy}{dt} = y & y = 1 + t + \dfrac{1}{2}t^2 + \dfrac{1}{6}t^3 + \cdots & y = e^t \\
\dfrac{dy}{dt} = y^2 & y = 1 + t + t^2 + t^3 + \cdots & y = 1/(1-t)
\end{array}
$$

To meet the two most important series in mathematics, right at the start, that is pure pleasure. No better practice is possible as the course begins.

Important Choices of $f(t)$

Let me emphasize that a textbook must do more than solve random problems. We could invent functions $f(t)$ forever, but that is not right. Much better to understand a small number of highly important functions:

$f(t) =$	**sines and cosines**	(oscillating and rotating)
$f(t) =$	**exponentials**	(growing and decaying)
$f(t) =$	**1 for $t > 0$**	(a switch is turned on)
$f(t) =$	**impulse**	(a sudden shock)

The solution $y(t)$ is the response to those inputs—frequency response, exponential response, step response, impulse response. These particular functions and particular solutions are the best—the easiest to find and by far the most useful. All other solutions are built from these.

I know that an impulse (a delta function that acts in an instant) is new to most students. This idea deserves to be here! You will see how neatly it works. The response is like the inverse of a matrix—it gives a formula for *all* solutions. The book will be supplemented by video lectures on many topics like this, because a visual explanation can be so effective.

Support for Readers

Readers should know all the support that comes with this book:

math.mit.edu/dela is the key website. The time has passed for printing solutions to odd-numbered problems in the back of the book. The website can provide more detailed solutions and serious help. This includes additional worked problems, and codes for numerical experiments, and much more. Please make use of everything and contribute.

ocw.mit.edu has complete sets of video lectures on both subjects (OpenCourseWare is also on YouTube). Many students know about the linear algebra lectures for 18.06 and 18.06 SC. I am so happy they are helpful. For differential equations, the 18.03 SC videos and notes and exams are extremely useful.

The new videos will be about special topics—possibly even the Tumbling Box.

Linear Algebra

I must add more about linear algebra. My writing life has been an effort to present this subject clearly. Not abstractly, not with a minimum of words, but in a way that is helpful to the reader. It is such good fortune that the central ideas in matrix algebra (a basis for a vector space, factorization of matrices, the properties of symmetric and orthogonal matrices), are exactly the ideas that make this subject so useful. Chapter 5 emphasizes those ideas and Chapter 7 explains the applications of $A^{\mathrm{T}}A$.

Matrices are essential, not just optional. We are constantly acquiring and organizing and presenting data—the format we use most is a matrix. The goal is to see the relation between input and output. Often this relation is linear. In that case we can understand it.

The idea of a vector space is so central. Take *all* combinations of two vectors or two functions. I am always encouraging students to visualize that space—examples are really the best. When you see all solutions to $v_1 + v_2 + v_3 = 0$ and $d^2y/dt^2 + y = 0$, you have the idea of a vector space. This opens up the big questions of linear independence and basis and dimension—by example.

If $f(t)$ comes in continuous time, our model is a differential equation. If the input comes in discrete time steps, we use linear algebra. The model predicts the output $y(t)$ this is created by the input $f(t)$. But some inputs are simply more important than others—they are easier to understand and much more likely to appear. Those are the right equations to present in this course.

Notes to Faculty (and All Readers)

One reason for publishing with Wellesley-Cambridge Press can be mentioned here. I work hard to keep book costs reasonable for students. This was just as important for *Introduction to Linear Algebra*. A comparison on Amazon shows that textbook prices from big publishers are more than double. Wellesley-Cambridge books are distributed by SIAM inside North America and Cambridge University Press outside, and from Wellesley, with the same motive. Certainly quality comes first.

I hope you will see what this book offers. The first chapters are a normal textbook on differential equations, for a new generation. The complete book is a year's course on differential equations and linear algebra, including Fourier and Laplace transforms—plus PDE's (Laplace equation, heat equation, wave equation) and the FFT and the SVD.

This is extremely useful mathematics! I cannot hope that you will read every word. But why should the reader be asked to look elsewhere, when the applications can come so naturally here?

A special note goes to engineering faculty who look for support from mathematics. I have the good fortune to teach hundreds of engineering students every year. My work with finite elements and signal processing and computational science helped me to know what students need—and to speak their language. I see texts that mention the impulse response (for example) in one paragraph or not at all. But this is the fundamental solution from which all particular solutions come. In the book it is computed in the time domain, starting with e^{at}, and again with Laplace transforms. The website goes further.

I know from experience that every first edition needs help. I hope you will tell me what should be explained more clearly. You are holding a book with a valuable goal—to become a textbook for a world of students and readers in a new generation and a new time, with limits and pressing demands on that time. The book won't be perfect. I will be so grateful if you contribute, in any way, to making it better.

Acknowledgments

So many friends have helped this book. In first place is Ashley C. Fernandes, my early morning contact for 700 days. He leads the team at Valutone that prepared the LaTeX files. They gently allowed me to rewrite and rewrite, as the truly essential ideas of differential equations became clear. Working with friends is the happiest way to live.

The book began in discussions about the MIT course 18.03. Haynes Miller and David Jerison and Jerry Orloff wanted *change*—this is the lifeblood of a course. Think more about what we are doing ! Their starting point (I see it repeated all over the world) was to add more linear algebra. Matrix operations were already in 18.03, and computations of eigenvalues—they wanted bases and nullspaces and ideas.

I learned so much from their lectures. There is a wonderful moment when a class gets the point. Then the subject lives. The reader can feel this too, but only if the author does. I guess that is my philosophy of education.

Solutions to the Problem Sets were a gift from Bassel Khoury and Matt Ko. The example of a Tumbling Box came from Alar Toomre, it is the highlight of Section 3.3 (this was a famous experiment in his class, throwing a book in the air). Daniel Drucker watched over the text of Chapters 1-3, the best mathematics editor I know. My writing tries to be personal and direct—Dan tries to make it right.

The cover of this book was an amazing experience. Gonçalo Morais visited MIT from Portugal, and we talked. After he went home, he sent this very unusual picture of a strange attractor—a solution to the Lorenz equation. It became a way to honor that great and humble man, Ed Lorenz, who discovered chaos. Gail Corbett and Lois Sellers are the artists who created the cover—what they have done is beyond my thanks, it means everything.

At the last minute (every book has a crisis at the last minute) Shev MacNamara saved the day. Figures were missing. Big spaces were empty. The S-curve in Section 1.7, the direction fields in Section 3.1, the Euler and Runge-Kutta experiments, those and more came from Shev. He also encourages me to do an online course with new video lectures. I will think more about a MOOC when readers respond.

Thank you all, including every reader.

Gilbert Strang

Outline of Chapter 1 : First Order Equations

1.3	Solve	$dy/dt = ay$	Construct the exponential e^{at}
1.4	Solve	$dy/dt = ay + q(t)$	Four special $q(t)$ and all $q(t)$
1.5	Solve	$dy/dt = ay + e^{st}$	Growth and oscillation : $s = a + i\omega$
1.6	Solve	$dy/dt = a(t)y + q(t)$	Integrating factor $=$ 1/growth factor
1.7	Solve	$dy/dt = ay - by^2$	The equation for $z = 1/y$ is linear
1.8	Solve	$dy/dt = g(t)/f(y)$	Separate $\int f(y)\,dy$ from $\int g(t)\,dt$

The key formula in 1.4 gives the solution $y(t) = e^{at}y(0) + \int_0^t e^{a(t-s)}q(s)ds$.

The website with solutions and codes and extra examples and videos is **math.mit.edu/dela**

Please contact **diffeqla@gmail.com** with questions and book orders and ideas.

Chapter 1

First Order Equations

1.1 Four Examples : Linear versus Nonlinear

A first order differential equation connects a function $y(t)$ to its derivative dy/dt. That rate of change in y is decided by y itself (and possibly also by the time t).

Here are four examples. Example **1** is the most important differential equation of all.

1) $\dfrac{dy}{dt} = y$	2) $\dfrac{dy}{dt} = -y$	3) $\dfrac{dy}{dt} = 2ty$	4) $\dfrac{dy}{dt} = y^2$

Those examples illustrate three **linear** differential equations (**1**, **2**, and **3**) and a **nonlinear** differential equation. The unknown function $y(t)$ is squared in Example **4**. The derivative y or $-y$ or $2ty$ is proportional to the function y in Examples **1**, **2**, **3**. The graph of dy/dt versus y becomes a parabola in Example **4**, because of y^2.

It is true that t multiplies y in Example **3**. That equation is still linear in y and dy/dt. It has a *variable coefficient* $2t$, changing with time. Examples **1** and **2** have *constant coefficient* (the coefficients of y are 1 and -1).

Solutions to the Four Examples

We can write down a solution to each example. This will be one solution but it is not the *complete* solution, because each equation has a family of solutions. Eventually there will be a constant C in the complete solution. This number C is decided by the starting value of y at $t = 0$, exactly as in ordinary integration. The integral of $f(t)$ solves the simplest differential equation of all, with $y(0) = C$:

5) $\dfrac{dy}{dt} = f(t)$ The complete solution is $y(t) = \displaystyle\int_0^t f(s)\, ds + C$.

For now we just write one solution to Examples **1** — **4**. They all start at $y(0) = 1$.

1 $\quad \dfrac{dy}{dt} = y \quad$ is solved by $\quad y(t) = \boldsymbol{e^t}$

2 $\quad \dfrac{dy}{dt} = -y \quad$ is solved by $\quad y(t) = \boldsymbol{e^{-t}}$

3 $\quad \dfrac{dy}{dt} = 2ty \quad$ is solved by $\quad y(t) = \boldsymbol{e^{t^2}}$

4 $\quad \dfrac{dy}{dt} = y^2 \quad$ is solved by $\quad y(t) = \dfrac{\mathbf{1}}{\mathbf{1} - \boldsymbol{t}}.$

Notice: The three linear equations are solved by exponential functions (***powers of e***). The nonlinear equation **4** is solved by a different type of function; here it is $1/(1-t)$. Its derivative is $dy/dt = 1/(1-t)^2$, which agrees with y^2.

Our special interest now is in linear equations with *constant coefficients*, like **1** and **2**. In fact $dy/dt = y$ is the most important property of the great function $y = e^t$. Calculus had to create e^t, because a function from algebra (like $y = t^n$) cannot equal its derivative (the derivative of t^n is nt^{n-1}). But a combination of all the powers t^n can do it. That good combination is e^t in Section 1.3.

The final example extends **1** and **2**, to allow **any constant coefficient** $\boldsymbol{a}$:

6) $\qquad \boldsymbol{\dfrac{dy}{dt} = ay} \quad$ is solved by $\quad \boldsymbol{y = e^{at}} \quad$ (and also $\quad \boldsymbol{y = Ce^{at}}$).

If the constant growth rate a is positive, the solution increases. If a is negative, as in $dy/dt = -y$ with $a = -1$, the slope is negative and the solution e^{-t} decays toward zero. Figure 1.1 shows three exponentials, with dy/dt equal to y and $2y$ and $-y$.

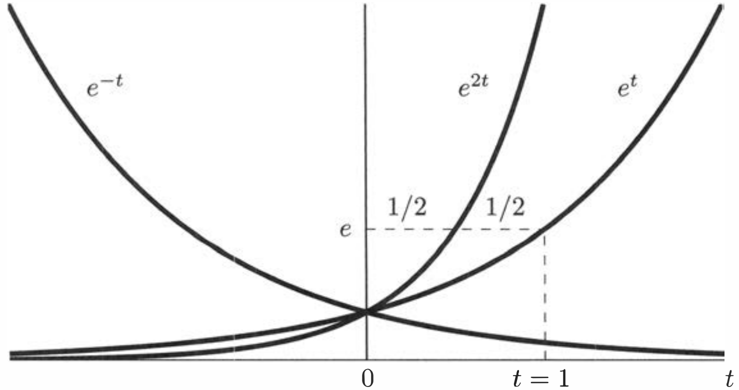


Figure 1.1: Growth, faster growth, and decay. The solutions are e^t and e^{2t} and e^{-t}.

When a is larger than 1, the solution grows faster than e^t. That is natural. The neat thing is that we still follow the exponential curve—but e^{at} climbs that curve faster. You could see the same result by *rescaling the time axis.* In Figure 1.1, the steepest curve (for $a = 2$) is the same as the first curve—but the time axis is compressed by 2.

Calculus sees this factor of 2 from the chain rule for e^{2t}. It sees the factor $2t$ from the chain rule for e^{t^2}. This exponent is t^2, the factor $2t$ is its derivative:

$$\frac{d}{dt}\left(e^u\right) = e^u \frac{du}{dt} \qquad \frac{d}{dt}\left(e^{2t}\right) = \left(e^{2t}\right) \text{ times } 2 \qquad \frac{d}{dt}\left(e^{t^2}\right) = \left(e^{t^2}\right) \text{ times } 2t$$

Problem Set 1.1: Complex Numbers

1 Draw the graph of $y = e^t$ by hand, for $-1 \le t \le 1$. What is its slope dy/dt at $t = 0$? Add the straight line graph of $y = et$. Where do those two graphs cross ?

2 Draw the graph of $y_1 = e^{2t}$ on top of $y_2 = 2e^t$. Which function is larger at $t = 0$? Which function is larger at $t = 1$?

3 What is the slope of $y = e^{-t}$ at $t = 0$? Find the slope dy/dt at $t = 1$.

4 What "logarithm" do we use for the number t (the exponent) when $e^t = 4$?

5 State the chain rule for the derivative dy/dt if $y(t) = f(u(t))$ (chain of f and u).

6 The *second* derivative of e^t is again e^t. So $y = e^t$ solves $d^2y/dt^2 = y$. A second order differential equation should have another solution, different from $y = Ce^t$. What is that second solution ?

7 Show that the nonlinear example $dy/dt = y^2$ is solved by $y = C/(1 - Ct)$ for every constant C. The choice $C = 1$ gave $y = 1/(1 - t)$, starting from $y(0) = 1$.

8 Why will the solution to $dy/dt = y^2$ grow faster than the solution to $dy/dt = y$ (if we start them both from $y = 1$ at $t = 0$) ? The first solution blows up at $t = 1$. The second solution e^t grows exponentially fast but it never blows up.

9 Find a solution to $dy/dt = -y^2$ starting from $y(0) = 1$. Integrate dy/y^2 and $-dt$. (Or work with $z = 1/y$. Then $\boldsymbol{dz/dt} = (dz/dy)\,(dy/dt) = (-1/y^2)(-y^2) = \mathbf{1}$. From $dz/dt = 1$ you will know $z(t)$ and $y = 1/z$.)

10 Which of these differential equations are linear (in y) ?

(a) $y' + \sin y = t$ (b) $y' = t^2(y - t)$ (c) $y' + e^t y = t^{10}$.

11 The product rule gives what derivative for $e^t e^{-t}$? This function is constant. At $t = 0$ this constant is 1. Then $e^t e^{-t} = 1$ for all t.

12 $dy/dt = y + 1$ is not solved by $y = e^t + t$. Substitute that y to show it fails. We can't just add the solutions to $y' = y$ and $y' = 1$. What number c makes $y = e^t + c$ into a correct solution ?

1.2 The Calculus You Need

The prerequisite for differential equations is calculus. This may mean a year or more of ideas and homework problems and rules for computing derivatives and integrals. Some of those topics are essential, but others (as we all acknowledge) are not really of first importance. These pages have a positive purpose, to bring together essential facts of calculus. This section is to read and refer to—it doesn't end with a Problem Set.

I hope this outline may have value also at the end of a single-variable calculus course. Textbooks could include a summary of the crucial ideas, but usually they don't. Certainly the reader will not agree with every choice made here, and the best outcome would be a more perfect list. This one is a lot shorter than I expected.

At the end, a useful formula in differential equations is confirmed by the product rule, the derivative of e^x, and the Fundamental Theorem of Calculus.

1. Derivatives of key functions: $\boldsymbol{x^n}$ $\quad$ $\sin x$ $\quad$ $\cos x$ $\quad$ $\boldsymbol{e^x}$ $\quad$ $\ln x$

The derivatives of $x, x^2, x^3, \ldots$ come from first principles, as limits of $\Delta y/\Delta x$. The derivatives of $\sin x$ and $\cos x$ focus on the limit of $(\sin \Delta x)/\Delta x$. Then comes the great function e^x. It solves the differential equation $dy/dx = y$ starting from $y(0) = 1$. **This is the single most important fact needed from calculus: the knowledge of $\boldsymbol{e^x}$.**

2. Rules for derivatives: Sum rule $\quad$ Product rule $\quad$ Quotient rule $\quad$ Chain rule

When we add, subtract, multiply, and divide the five original functions, these rules give the derivatives. The sum rule is the quiet one, applied all the time to *linear* differential equations. This equation is linear (*a crucial property*):

$$\frac{dy}{dt} = ay + f(t) \text{ and } \frac{dz}{dt} = az + g(t) \text{ add to } \frac{d}{dt}(\boldsymbol{y} + \boldsymbol{z}) = a(\boldsymbol{y} + \boldsymbol{z}) + (\boldsymbol{f} + \boldsymbol{g}).$$

With $a = 0$ that is a straightforward sum rule for the derivative of $y + z$. We can always add equations as shown, because $a(t)y$ is linear in y. This confirms *superposition* of the separate solutions y and z. Linear equations add and their solutions add.

The chain rule is the most prolific, in computing the derivatives of very remarkable functions. The chain $\boldsymbol{y = e^x}$ and $\boldsymbol{x = \sin t}$ produces $\boldsymbol{y = e^{\sin t}}$ (the composite of two functions). The chain rule gives dy/dt by multiplying the derivatives dy/dx and dx/dt:

$$\textbf{Chain rule} \qquad \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = e^x \cos t = y \cos t.$$

Then $e^{\sin t}$ solves that differential equation $\dfrac{dy}{dt} = ay$ with varying growth rate $a = \cos t$.

3. The Fundamental Theorem of Calculus

The derivative of the integral of $f(x)$ is $f(x)$. The integral from 0 to x of the derivative df/dx is $f(x) - f(0)$. One operation inverts the other, when $f(0) = 0$. This is not so easy to prove, because both the derivative and the integral involve a limit step $\Delta x \to 0$.

One way to go forward starts with numbers $y_0, y_1, \ldots, y_n$. Their differences are like derivatives. Adding up those differences is like integrating the derivative:

$$\textbf{Sum of differences}\quad (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_n - y_0. \tag{1}$$

Only y_n and $-y_0$ are left because all other numbers $y_1, y_2, \ldots$ come twice and cancel. To make that equation look like calculus, multiply every term by $\Delta x / \Delta x = 1$:

$$\left[\frac{y_1 - y_0}{\Delta x} + \frac{y_2 - y_1}{\Delta x} + \cdots + \frac{y_n - y_{n-1}}{\Delta x}\right] \Delta x = y_n - y_0. \tag{2}$$

Again, this is true for all numbers $y_0, y_1, \ldots, y_n$. Those can be heights of the graph of a function $y(x)$. The points $x_0, \ldots, x_n$ can be equally spaced between $x = a$ and $x = b$. Then each ratio $\Delta y / \Delta x$ is a *slope* between two points of the graph:

$$\frac{\Delta y}{\Delta x} = \frac{y_k - y_{k-1}}{x_k - x_{k-1}} = \frac{\text{distance up}}{\text{distance across}} = \text{slope}. \tag{3}$$

This slope is exactly correct if the graph is a straight line between the points x_{k-1} and x_k. If the graph is a curve, the approximate slope $\Delta y / \Delta x$ becomes exact as $\Delta x \to 0$.

The delicate part is the requirement $n \Delta x = b - a$, to space the points evenly from $x_0 = a$ to $x_n = b$. Then n will increase as Δx decreases. Equation (2) remains correct at every step, with $y_0 = y(a)$ at the first point and $y_n = y(b)$ at the last point. As $\Delta x \to 0$ and $n \to \infty$, the slopes $\Delta y / \Delta x$ approach the derivative dy/dx. At the same time the sum approaches the integral of dy/dx. Equation (2) turns into equation (4):

Fundamental Theorem of Calculus	$\displaystyle\int_a^b \frac{dy}{dx}\, dx = y(b) - y(a)$	$\displaystyle\frac{d}{dx}\int_a^x f(s)\, ds = f(x)$	(4)

The limits of $\Delta y / \Delta x$ in (3) and the sum in (2) produce dy/dx and its integral. Of course this presentation of the Fundamental Theorem needs more careful attention. But equation (1) holds a key idea: *a sum of differences*. This leads to *an integral of derivatives.*

4. The meaning of symbols and the operations of algebra

Mathematics is a language. The way to learn this language is to use it. So textbooks have thousands of exercises, to practice reading and writing symbols like $y(x)$ and $y(x + \Delta x)$. Here is a typical line of symbols:

$$\textbf{Derivative of } y \qquad \frac{dy}{dt}(t) = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}. \tag{5}$$

I am not very sure that this is clear. One function is y, the other function is its derivative y'.

Could the symbol y' be better than dy/dt? Both are standard in this book. In calculus we know $y(t)$, in differential equations we don't. The whole point of the differential equation is to connect y and y'. From that connection we have to discover what they are.

A first example is $y' = y$. That equation forces the unknown function y to grow exponentially: $y(t) = Ce^t$. At the end of this section I want to propose a more complicated equation and its solution. But I could never find a more important example than e^t.

5. Three ways to use $dy/dx \approx \Delta y/\Delta x$

On the graph of a function $y(x)$, the exact slope is dy/dx and the approximate slope (between nearby points) is $\Delta y/\Delta x$. If we know *any two* of the numbers dy/dx and Δy and Δx, then we have a good approximation to the third number. All three approximations are important, because dy/dx is such a central idea in calculus.

(A) When we know Δx and dy/dx, we have $\Delta y \approx (\Delta x)(dy/dx)$.

This is linear approximation. From a starting point x_0, we move a distance Δx. That produces a change Δy. The graph of $y(x)$ can go up or down, and the best information we have is the slope dy/dx at x_0. (That number gives no way to account for *bending* of the graph, which appears in the next derivative d^2y/dx^2.)

Linear approximation is equivalent to following the tangent line —not the curve:

$$\boldsymbol{\Delta y \approx \Delta x \frac{dy}{dx}} \qquad y(x_0 + \Delta x) \approx y(x_0) + \Delta x \frac{dy}{dx}(x_0) \tag{6}$$

(B) Δy and dy/dx lead to $\Delta x \approx (\Delta y)/(dy/dx)$. This is Newton's Method.

Newton's Method is a way to solve $y(x) = 0$, starting at a point x_0. We want $y(x)$ to drop from $y(x_0)$ to zero at the new point x_1. *The desired change in y is $\Delta y = 0 - y(x_0)$.* What we don't know is Δx, which locates x_1. The exact slope dy/dx will be close to $\Delta y/\Delta x$, and that tells us a good Δx:

$$\textbf{Newton's Method} \qquad \boldsymbol{\Delta x \approx \frac{\Delta y}{dy/dx}} \qquad x_1 - x_0 = \frac{-y(x_0)}{dy/dx(x_0)} \tag{7}$$

Guess x_0, improve to x_1. This is an excellent way to solve nonlinear equations $y(x) = 0$.

(C) Dividing Δy by Δx gives the approximation $dy/dx \approx \Delta y/\Delta x$.

That is the point of equation (5), but something important often escapes our attention. *Are x and $x + \Delta x$ the best two places to compute y?* Writing $\Delta y = y(x + \Delta x) - y(x)$ doesn't seem to offer other choices. If we notice that Δx can be negative, this allows $x + \Delta x$ to be on the left side of x (leading to a backward difference) . The best choice is not forward or backward but *centered around x: a half step each way.*

$$\textbf{Centered difference} \qquad \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \boldsymbol{\frac{y(x + \frac{1}{2}\Delta x) - y(x - \frac{1}{2}\Delta x)}{\Delta x}} \tag{8}$$

Why is centering better? When $y = Cx + D$ has a straight line graph, all ratios $\Delta y/\Delta x$ give the correct slope C. But the parabola $y = x^2$ has the simplest possible bending, and **only this centered difference gives the correct slope $\mathbf{2x}$** (varying with x).

Exact slope for parabolas by centering
$$\frac{\Delta y}{\Delta x} = \frac{(x+\frac{1}{2}\Delta x)^2 - (x-\frac{1}{2}\Delta x)^2}{\Delta x} = \frac{x\,\Delta x - (-x\,\Delta x)}{\Delta x} = \mathbf{2x}$$

The key step in scientific computing is improving first order accuracy (forward differences) to second order accuracy (centered differences). For integrals, rectangle rules improve to trapezoidal rules. This is a big step to good algorithms.

6. Taylor series: Predicting $y(x)$ from all the derivatives at $x = x_0$

From the height y_0 and the slope y_0' at x_0, we can predict the height $y(x)$ at nearby points. But the tangent line in equation (6) assumes that $y(x)$ has constant slope. That first order prediction becomes a second order prediction (*much more accurate*) when we use the second derivative y_0'' at x_0.

Tangent parabola using y_0''
$$y(x_0 + \Delta x) \approx y_0 + (\Delta x) y_0' + \tfrac{1}{2}(\Delta x)^2 y_0''. \tag{9}$$

Adding this $(\Delta x)^2$ term moves us from constant slope to constant bending. For the parabola $y = x^2$, equation (9) is exact: $(x_0 + \Delta x)^2 = (x_0^2) + (\Delta x)(2x_0) + \frac{1}{2}(\Delta x)^2(2)$.

Taylor added more terms—infinitely many. His formula gets *all derivatives correct* at x_0. The pattern is set by $\frac{1}{2}(\Delta x)^2 y_0''$. The n^{th} derivative $y^{(n)}(x)$ contributes a new term $\frac{1}{n!}(\Delta x)^n y_0^{(n)}$. The complete Taylor series includes all derivatives at the point $x = x_0$:

Taylor series
Stop at y' for tangent line
Stop at y'' for parabola
$$\begin{aligned} y(x_0 + \Delta x) &= y_0 + (\Delta x)\, y_0' + \cdots + \frac{1}{n!}(\Delta x)^n\, y_0^{(n)} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!}\, y^{(n)}(x_0) \end{aligned} \tag{10}$$

Those equal signs are not always right. There is no way we can stop $y(x)$ from making a sudden change after x moves away from x_0. Taylor's prediction of $y(x_0 + \Delta x)$ is exactly correct for e^x and $\sin x$ and $\cos x$—good functions like those are "analytic" at all x.

Let me include here the two most important examples in all of mathematics. They are solutions to $dy/dx = y$ and $dy/dx = y^2$ — the most basic linear and nonlinear equations.

Exponential series with $\boldsymbol{y^{(n)}(0) = 1}$
$$y = \boldsymbol{e^x} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \tag{11}$$

Geometric series with $\boldsymbol{y^{(n)}(0) = n!}$
$$y = \frac{\mathbf{1}}{\mathbf{1 - x}} = 1 + x + x^2 + x^3 + \cdots \tag{12}$$

The center point is $x_0 = 0$. The series (11) gives e^x for every x. The series (12) gives $1/(1-x)$ when x is between -1 and 1. Its derivative $1 + 2x + 3x^2 + \cdots$ is $1/(1-x)^2$.

For $x = 2$ that geometric series will certainly not produce $1/(1-2) = -1$. Notice that $1 + x + x^2 + \cdots$ becomes infinite at $x = 1$, exactly where $1/(1-x)$ becomes $1/0$.

The key point for e^x is that its n^{th} derivative is 1 at $x = 0$. The n^{th} derivative of $1/(1-x)$ is $n!$ at $x = 0$. This pattern starts with y, y', y'', y''' equal to $1, 1, 2, 6$ at $x = 0$:

$$y = (1-x)^{-1} \qquad y' = (1-x)^{-2} \qquad y'' = 2(1-x)^{-3} \qquad y''' = 6(1-x)^{-4}.$$

Taylor's formula combines the contributions of all derivatives at $x = 0$, to produce $y(x)$.

7. Application: An important differential equation

The linear differential equation $y' = ay + q(t)$ is a perfect multipurpose model. It includes the growth rate a and the external source term $q(t)$. We want the particular solution that starts from $y(0) = 0$. Creating that solution uses the most essential idea behind integration. Verifying that the solution is correct uses the basic rules for derivatives. Many students in my graduate class had forgotten the derivative of the integral.

Here is the solution $y(t)$ followed by its interpretation, with $a = 1$ for simplicity:

$$\frac{dy}{dt} = y + q(t) \quad \textbf{is solved by} \quad y(t) = \int_0^t e^{t-s} q(s)\, ds. \tag{13}$$

Key idea: At each time s between 0 and t, the input is a source of strength $q(s)$. That input grows or decays over the remaining time $t - s$. **The input $q(s)$ is multiplied by e^{t-s} to give an output at time t.** Then the total output $y(t)$ is the *integral* of $e^{t-s}q(s)$.

We will reach $y(t)$ in other ways. Section 1.4 uses an "integrating factor." Section 1.6 explains "variation of parameters." The key is to see where the formula comes from. *Inputs lead to outputs, the equation is linear, and the principle of superposition applies.* The total output is the sum (in this case, the integral) of all those outputs.

We will confirm formula (13) by computing dy/dt. First, e^{t-s} equals e^t times e^{-s}. Then e^t comes outside the integral of $e^{-s}q(s)$. Use the product rule on those two factors:

$$\textbf{Producing } y + q \qquad \frac{dy}{dt} = \left(\frac{d\,e^t}{dt}\right) \int_0^t e^{-s} q(s)\, ds + \left(e^t\right) \frac{d}{dt} \int_0^t e^{-s} q(s)\, ds. \tag{14}$$

The first term on the right side is exactly $y(t)$. How to recognize that last term as $q(t)$?

We don't need to know the function $q(t)$. What we do know (and need) is the *Fundamental Theorem of Calculus*. **The derivative of the integral of $e^{-t}q(t)$ is $e^{-t}q(t)$.** Then multiplying by e^t gives the hoped-for result $q(t)$, because $e^t e^{-t} = 1$. The linear differential equation $y' = y + q$ with $y(0) = 0$ is solved by the integral of $e^{t-s}q(s)$.

1.3 The Exponentials e^t and e^{at}

Here is the key message from this section: **The solutions to $dy/dt = ay$ are $y(t) = Ce^{at}$. That free constant C matches the starting value $y(0)$. Then $y(t) = y(0)e^{at}$.**

I realize that you already know the function $y = e^t$. It is the star of precalculus and calculus. Now it becomes the key to linear differential equations. Here I focus on the two most important properties of this function e^t:

1. *The slope dy/dt equals the function y.* As y grows, its graph gets steeper:

$$\frac{d}{dt}e^t = e^t. \tag{1}$$

2. $y(t) = e^t$ follows the *addition rule* for exponents:

$$e^t \text{ times } e^T \text{ equals } e^{t+T}. \tag{2}$$

How is this exponential function constructed? Only calculus can do it, because somewhere we must have a "limit step." Functions from ordinary algebra can get close to e^t, but they can't reach it. If we choose those functions to come closer and closer, then their limit is e^t.

This is like using fractions to approach the extraordinary number π. The fractions can start with 3/1 and 31/10 and 314/100. The neat fraction 22/7 is close to π. But "taking the limit" can't be avoided, because π itself is not a fraction.

Similarly e is not a fraction. On this book's home page **math.mit.edu/dela** is an article called *Introducing* e^x. It describes four popular ways to construct this function. The one chosen now is my favorite, because it is the most direct way.

$$\textbf{Construct}\quad \mathbf{y = e^t}\quad \textbf{so that}\quad \frac{\mathbf{dy}}{\mathbf{dt}} = \mathbf{y}\quad (\text{starting from } y = 1 \text{ at } t = 0)$$

To show how this construction works, here are ordinary polynomials y and dy/dt:

1. $y = 1 + t + \frac{1}{2}t^2$ The derivative is $dy/dt = 0 + 1 + t$

2. $y = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3$ The derivative is $dy/dt = 0 + 1 + t + \frac{1}{2}t^2$

You see that dy/dt does not fully agree with y. It always falls one term short of y. We could get $t^3/6$ into the derivative by including $t^4/24$ in y. But now dy/dt will be missing $t^4/24$.

You can see that dy/dt won't catch up to y. *The way out is to have infinitely many terms: Don't stop.* Then you get $dy/dt = y$.

The limit step reaches an infinite series, adding new terms and never stopping. Every term has the form t^n divided by $n\,!$ (*n factorial*). Its derivative is the previous term:

$$\textbf{The derivative of} \quad \frac{t^n}{(n)\dots(1)} = \frac{t^n}{n\,!} \quad \text{is} \quad \frac{t^{n-1}}{(n-1)\dots(1)} = \frac{t^{n-1}}{(n-1)\,!} \tag{3}$$

So if $t^n/n!$ is missing in dy/dt, we will capture it by including $t^{n+1}/(n+1)!$ in y.

Of course dy/dt never completely catches up to y—*until we allow an infinite series. There is a term $t^n/n\,!$ for every n.* The term for $n = 0$ is $t^0/0\,! = 1$.

Construction of e^t

$$y = e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{4}$$

Taking the derivative of every term produces all the same terms. So $dy/dt = y$. Notice: If you change every t to at, **the derivative of $y = e^{at}$ becomes a times e^{at}:**

$$\frac{d}{dt}\left(1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{6} + \cdots\right) = a\left(1 + at + \frac{a^2t^2}{2} + \cdots\right) = ae^{at} \tag{5}$$

This construction of e^t brings up two questions, to be discussed in the Chapter 1 Notes. Does the infinite series add to a finite number (a different number for each choice of t)? Can we add the derivatives of each $t^n/n!$ and safely get the derivative of the sum e^t ? Fortunately both answers are *yes*. The terms get very small, very fast, as n increases. The limiting step is $n \to \infty$, producing the exact e^t.

When $t = 1$, we can watch the terms get small. We *must* do this, because $t = 1$ leads to the all-important number e^1 which is e:

The series for e at $t = 1$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots \approx 2.718$$

The first three terms add to 2.5. The first five terms almost reach 2.71. *We never reach* 2.72. With enough terms you can barely pass 2.71828. It is certain that the total sum e is not a fraction. It never appears in algebra, but it is the key number for calculus.

The Series for e^t is a Taylor Series

The infinite series (4) for e^t is the same as the Taylor series. Section 1.2 went from the tangent line $1 + t$ to the tangent parabola $1 + t + \frac{1}{2}t^2$. The next term will be $\frac{1}{6}t^3$, because that matches the third derivative $y''' = 1$ at $t = 0$. *All derivatives are equal to* 1 *at* $t = 0$, when we start from the basic equation $y' = y$. That equation gives $y'' = y' = y$ and the next derivative gives $y''' = y'' = y' = y$.

Conclusion: $t^n/n\,!$ has the correct n^{th} derivative (which is 1) at the point $t = 0$. All these terms go into the Taylor series. The result is exactly the exponential series (4).

Multiplying Powers by Adding Exponents

We write 3^2 for 3 times 3. We write e^2 for e times e. The question is, does $e = 2.718\ldots$ times $e = 2.718\ldots$ give the same answer as setting $t = 2$ in the infinite series to get e^2?

The answer is again *yes*. I could say "fortunately yes" but that might suggest a lucky accident. The amazing fact is that Property 1 ($y' = y$ is now confirmed) leads automatically to Property 2. The exponential starts from $y(0) = e^0 = 1$ at time $t = 0$.

Property 2. $\boldsymbol{e^t}$ **times** $\boldsymbol{e^T}$ **equals** $\boldsymbol{e^{t+T}}$ so $(e^1)\,(e^1) = e^2$

This is a differential equations course, so the proofs will use Property 1: $dy/dt = y$.

First Proof. We can solve $y' = (a+b)y$ two ways, starting from $y(0) = 1$. We know that $y(t) = \boldsymbol{e^{(a+b)t}}$. Another solution is $y(t) = \boldsymbol{e^{at}e^{bt}}$, as the product rule shows:

$$\frac{d}{dt}\left(e^{at}e^{bt}\right) = \left(ae^{at}\right)e^{bt} + e^{at}\left(be^{bt}\right) = (a+b)e^{at}e^{bt}. \tag{6}$$

This solution $e^{at}e^{bt}$ also starts at $e^0e^0 = 1$. It must be the same as the first solution $e^{(a+b)t}$. The equation $y' = (a+b)y$ only has one solution. At $t = 1$ this says that $e^{a+b} = e^a e^b$. QED.

Second Proof. Starting with $y = 1$ at $t = 0$, the solution out to time t is e^t. The solution to time $t + T$ is e^{t+T}. The question is, do we also get that answer in two steps?

Starting from $y = 1$ at $t = 0$, we go to e^t. Then start from e^t at time t and continue an additional time T. This would give e^T starting from $y = 1$, but here the starting value is e^t. So $C = e^t$ multiplies e^T. At time $t + T$ we have perfect agreement:

$$e^t \text{ times } e^T \quad (\text{which is } C \text{ times } e^T) \text{ agrees with one big step } e^{t+T}.$$

Negative Exponents

Remember the example $dy/dt = -y$ with solution $y = e^{-t}$. That exponent $-t$ is negative. The solution decays toward zero. The exponent rule $e^te^T = e^{t+T}$ still holds for negative exponents. In particular e^t *times* e^{-t} *is* $e^{t-t} = e^0 = 1$:

Negative exponents $$\frac{1}{e^t} = e^{-t} \text{ and } \frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \cdots$$

This number $1/e$ is about .36. The series always succeeds! The graph of $y = e^{-t}$ shows that e^{-t} *stays positive*. It is very small for $t > 32$. Your computer might use 32 bit arithmetic and ignore numbers that are this small.

Why does $\boldsymbol{e^t}$ grow so fast? The slope is y itself. So the slope increases when the function increases. That steep slope makes y increase faster—and then the slope too.

Interest Rates and Difference Equations

There is another approach to e^t and e^{at}, which is not based on an infinite series. (At least, not at the start.) It connects to interest on bank accounts. For e^t the rate is $a = 1 = 100\%$. For e^{at} the differential equation is $dy/dt = ay$ and the interest rate is a.

The different approach is to construct e^t and e^{at} as the limit of compound interest.

$$\boxed{e^t = \lim_{N \to \infty} \left(1 + \frac{t}{N}\right)^N \qquad e^{at} = \lim_{N \to \infty} \left(1 + \frac{at}{N}\right)^N .} \tag{7}$$

The beauty of these formulas is that a bank does exactly what a computational scientist does. They both start with the differential equation $dy/dt = ay$ and the initial condition $y = 1$ at $t = 0$. Banks and scientists don't have computers that give exact solutions, when $y(t)$ changes continuously with time. Both take finite time steps Δt instead of infinitesimal steps dt. **They reach time t in N steps of size $\Delta t = t/N$.** Their approximations are $Y_1, Y_2, \ldots, Y_N$ with $Y_0 = 1$. Compound interest produces a **difference equation**:

$$\frac{dy}{dt} = ay \quad \text{becomes} \quad \frac{Y_{n+1} - Y_n}{\Delta t} = a\,Y_n \quad \text{and} \quad \boldsymbol{Y_{n+1} = (1 + a\,\Delta t)Y_n}. \tag{8}$$

Each step multiplies the bank balance by $1 + a\Delta t$. The new balance is the old balance Y_n plus $a\,\Delta t Y_n$ (the interest on Y_n in the time interval Δt). This is ordinary compound interest that all banks offer, not continuous compounding as in dy/dt. The time step can be $\Delta t = 1$ year or 1 month. The balance at $t = 2$ years $= 24$ months is Y_2 or Y_{24}:

$$Y_2 = (1 + a)^2\, Y_0 \qquad Y_{24} = \left(1 + \frac{a}{12}\right)^{24} Y_0 \approx e^{2a} Y_0. \tag{9}$$

If the rate is $a = 3$ per cent per year $= .03$ per year, continuous compounding for 2 years would produce the exponential factor $e^{.06} \approx 1.06184$. Monthly compounding produces $(1.0025)^{24} \approx 1.06176$. We only lose a little, when the differential equation $y' = ay$ is approximated by the difference equation in (8).

The computational scientist is usually not willing to accept this loss of accuracy in Y. Equation (8) with a forward difference $Y_{n+1} - Y_n$ is called **Euler's method**. Its accuracy is not high and not hard to improve. It is the natural choice for a bank, because a backward difference costs them even more than continuous compounding:

$$\textbf{Backward difference} \quad \frac{Y_n - Y_{n-1}}{\Delta t} = aY_n \quad \text{or} \quad Y_n = \frac{1}{1 - a\Delta t} Y_{n-1}. \tag{10}$$

Y_n connects backward to the earlier Y_{n-1}. Now each step divides by $1 - a\Delta t$. After N steps of size $\Delta t = t/N$, we are again close to e^{at}. But with backward differences and $a > 0$, we overshoot the differential equation and the bank pays a little too much:

$$(1 + a\Delta t)^N \ \text{ is below } \ e^{at} \qquad \frac{1}{(1 - a\Delta t)^N} \ \textit{ is above } \ e^{at}.$$

Complex Exponents

This isn't the time and place to study complex numbers in detail. It will be the pages about oscillations and $e^{i\omega t}$ that cannot go forward without the imaginary number i. Here we are solving $dy/dt = ay$, and all I want to do is to **choose $\boldsymbol{a = i}$.**

I can think of two ways to solve the complex equation $\boldsymbol{dy/dt = iy}$. The fast way uses derivatives of sine and cosine, which we know well:

$$\begin{array}{lrcl} \textbf{Proposed solution} & y & = & \boldsymbol{\cos t \;+\; i \sin t} \\ \textbf{Compare } \boldsymbol{dy/dt} & dy/dt & = & -\sin t \;+\; i\,\cos t \\ \textbf{with the right side } \boldsymbol{iy} & iy & = & i\,\cos t \;+\; i^2\,\sin t \end{array} \qquad (11)$$

To check $dy/dt = iy$, compare the last two lines. **Use the rule $\mathbf{i^2 = -1}$.** (We had to imagine this number, because no real number has $x^2 = -1$.) Then $-\sin t$ is the same as $i^2 \sin t$. So $\boldsymbol{y = \cos t + i \sin t}$ solves the equation $dy/dt = iy$. This solution starts at $y = 1$ when $t = 0$, because $\cos 0 = 1$ and $\sin 0 = 0$.

The slower approach to $dy/dt = iy$ uses the infinite series. Since $a = i$, the solution e^{at} becomes e^{it}. Formally, the series for $y = e^{it}$ certainly solves $dy/dt = iy$:

$$\textbf{Complex exponential} \qquad \boldsymbol{y = e^{it}} = 1 + (it) + \frac{1}{2}(it)^2 + \frac{1}{6}(it)^3 + \cdots \qquad (12)$$

The derivative of each term is i times the previous term. Since the series never stops, the derivative dy/dt perfectly matches iy. And we are still starting at $y = 1$ when we substitute $t = 0$. **This infinite series $\boldsymbol{e^{it}}$ equals the first solution $\boldsymbol{\cos t + i \sin t}$.**

Now use the rule $i^2 = -1$. For $(it)^2$ I will write $-t^2$. And $(it)^3$ equals $-it^3$. The fourth power of i is $i^4 = i^2 i^2 = (-1)^2 = 1$. That sequence $\boldsymbol{i, -1, -i, 1}$ repeats forever.

$$i = i^5 \qquad i^2 = i^6 = -1 \qquad i^3 = i^7 = -i \qquad i^4 = i^8 = 1$$

The infinite series (12) includes those four numbers multiplying powers of t:

$$e^{it} = 1 + \left[it - 1\frac{t^2}{2!} - i\frac{t^3}{3!} + 1\frac{t^4}{4!} \right] + \left[i\frac{t^5}{5!} - 1\frac{t^6}{6!} - i\frac{t^7}{7!} + 1\frac{t^8}{8!} \right] + \cdots$$

This may be the first time a textbook has ever written out nine terms. You can see the full repeat of i, -1, $-i$, 1. That last coefficient divides by $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ which is 40320.

The main point is that the solution $y = \cos t + i\ \sin t$ in equation (11) must be the same as this series solution e^{it}. They both solve $dy/dt = iy$. They both start at $y = 1$ when $t = 0$. The equality between them is one of the greatest formulas in mathematics.

$$\boxed{\textbf{Euler's Formula is} \qquad \boldsymbol{e^{it} = \cos t + i\ \sin t}.} \qquad (13)$$

Then $e^{i\pi} = \cos \pi + i\ \sin \pi = \mathbf{-1}$. And $e^{i2\pi} = 1 + i2\pi + \frac{1}{2}(i2\pi)^2 + \cdots$ must add to $\mathbf{1}$!

I cannot resist comparing $\cos t + i \sin t$ with the series for e^{it}. The **real part** of that series must be $\cos t$. The **imaginary part** (which multiplies i) must be $\sin t$. The even powers $1,\ t^2,\ t^4,\ \ldots$ give cosines. The odd powers $t,\ t^3,\ t^5,\ \ldots$ are multiplied by i:

$$\textbf{Cosine is even} \qquad \cos t = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{t^6}{6!} + \cdots \tag{14}$$

$$\textbf{Sine is odd} \qquad \sin t = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{t^7}{7!} + \cdots \tag{15}$$

These two pieces of the series for e^{it} are famous functions on their own, and now we see their Taylor series . They are beautifully connected by Euler's Formula.

The derivative of the sine series is the cosine series:

$$\frac{\mathbf{d}}{\mathbf{dt}}\,\mathbf{\sin t = \cos t} \qquad \frac{d}{dt}\left(t - \frac{1}{6}t^3 + \cdots\right) = \mathbf{1 - \frac{1}{2}t^2 + \cdots} = \textbf{cosine}$$

The derivative of the cosine series is minus the sine series:

$$\frac{\mathbf{d}}{\mathbf{dt}}\,\mathbf{\cos t = -\sin t} \qquad \frac{d}{dt}\left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \cdots\right) = \mathbf{-t + \frac{1}{6}t^3 \cdots} = -\ \textbf{sine}$$

All this important information came from allowing the exponent in e^{it} to be imaginary. And e^{it} times e^{-it} is exactly $\cos^2 t + \sin^2 t = 1$.

Matrix Exponents

One more thing, which you can safely ignore for now. The exponent in e^{at} could become a **square matrix**. Instead of solving $dy/dt = ay$ by e^{at}, we can solve the matrix equation $d\boldsymbol{y}/dt = A\boldsymbol{y}$ by the matrix e^{At}. Start with the identity matrix I instead of the number 1.

$$\boldsymbol{e^{At}}\ \textbf{is a matrix} \qquad \boxed{\boldsymbol{e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots}} \tag{16}$$

The series has the usual form, with the matrix A instead of the number a. Here I stop, because matrices come in Chapter 4: *Systems of Equations*. When the matrix A is three by three, the equation $dy/dt = Ay$ represents three ordinary differential equations. Still first order linear, still constant coefficients, solved by e^{At} in Section 6.4.

There is one big difference for matrices: $\boldsymbol{e^{At}e^{Bt} = e^{(A+B)t}}$ **is not true**. For numbers a and b this equation is correct. For matrices A and B something goes wrong in equation (6). When you look closely, you see that b moved in front of e^{at}. But $e^{At}B = Be^{At}$ is false for matrices.

■ REVIEW OF THE KEY IDEAS ■

1. In the series for e^t, each term $t^n/n!$ is the derivative of the next term.
2. Then the derivative of e^t is e^t, and the exponent rule holds: $e^t\,e^T = e^{t+T}$.
3. Another approach to $dy/dt = y$ is by finite differences $(Y_{n+1} - Y_n)/\Delta t = Y_n$. $Y_{n+1} = Y_n + \Delta t Y_n$ is the same as compound interest. Then Y_n is close to $e^{n\Delta t}Y_0$.
4. $y = e^{at}$ solves $y' = ay$, and $a = i$ leads to $e^{it} = \cos t + i \sin t$ (Euler's Formula).
5. $\cos t = 1 - t^2/2 + \cdots$ and $\sin t = t - t^3/6 + \cdots$ are the even and odd parts of e^{it}.

Problem Set 1.3

1 Set $t = 2$ in the infinite series for e^2. The sum must be e times e, close to 7.39. How many terms in the series to reach a sum of 7 ? How many terms to pass 7.3 ?

2 Starting from $y(0) = 1$, find the solution to $dy/dt = y$ at time $t = 1$. Starting from that $y(1)$, solve $dy/dt = -y$ to time $t = 2$. Draw a rough graph of $y(t)$ from $t = 0$ to $t = 2$. What does this say about e^{-1} times e ?

3 Start with $y(0) = \$5000$. If this grows by $dy/dt = .02y$ until $t = 5$ and then jumps to $a = .04$ per year until $t = 10$, what is the account balance at $t = 10$?

4 Change Problem 3 to start with \$5000 growing at $dy/dt = .04y$ for the first five years. Then drop to $a = .02$ until $t = 10$. What is now the balance at $t = 10$?

Problems 5–8 are about $y = e^{at}$ and its infinite series.

5 Replace t by at in the exponential series to find e^{at}:

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \cdots$$

Take the derivative of every term (keep five terms). Factor out a to show that *the derivative of* e^{at} *equals* ae^{at}. At what time T does e^{at} reach 2 ?

6 Start from $y' = ay$. Take the derivative of that equation. Take the n^{th} derivative. Construct the Taylor series that matches all these derivatives at $t = 0$, starting from $1 + at + \frac{1}{2}(at)^2$. Confirm that this series for $y(t)$ is the series for e^{at} in Problem 5.

7 At what times t do these events happen ?

(a) $e^{at} = e$ (b) $e^{at} = e^2$ (c) $e^{a(t+2)} = e^{at}e^{2a}$.

8 If you multiply the series for e^{at} in Problem 5 by itself you should get the series for e^{2at}. Multiply the first 3 terms by the same 3 terms to see the first 3 terms in e^{2at}.

9 (recommended) Find $y(t)$ if $dy/dt = ay$ and $\boldsymbol{y(T) = 1}$ (instead of $y(0) = 1$).

10 (a) If $dy/dt = (\ln 2)y$, explain why $y(1) = 2y(0)$.

(b) If $dy/dt = -(\ln 2)y$, how is $y(1)$ related to $y(0)$?

11 In a one-year investment of $y(0) = \$100$, suppose the interest rate jumps from 6% to 10% after six months. Does the equivalent rate for a whole year equal 8%, or more than 8%, or less than 8% ?

12 If you invest $y(0) = \$100$ at 4% interest compounded continuously, then $dy/dt = .04y$. Why do you have more than \$104 at the end of the year ?

13 What linear differential equation $dy/dt = a(t)y$ is satisfied by $y(t) = e^{\cos t}$?

14 If the interest rate is $a = 0.1$ per year in $y' = ay$, how many years does it take for your investment to be multiplied by e ? How many years to be multiplied by e^2 ?

15 Write the first four terms in the series for $y = e^{t^2}$. Check that $dy/dt = 2ty$.

16 Find the derivative of $Y(t) = (1 + \frac{t}{n})^n$. If n is large, this dY/dt is close to Y !

17 Suppose the exponent in $y = e^{u(t)}$ is $u(t) =$ integral of $a(t)$. What equation $dy/dt =$ _____ y does this solve ? If $u(0) = 0$ what is the starting value $y(0)$?

Challenge Problems

18 $e^{d/dx} = 1 + d/dx + \frac{1}{2}(d/dx)^2 + \cdots$ is a sum of higher and higher derivatives. Applying this series to $f(x)$ at $x = 0$ would give $f + f' + \frac{1}{2}f'' + \cdots$ at $x = 0$. The Taylor series says : This is equal to $f(x)$ at $x =$ _____ .

19 (Computer or calculator, 2.xx is close enough) Find the time t when $e^t = 10$. The initial $y(0)$ has increased by an order of magnitude—a factor of 10. The exact statement of the answer is $t =$ _____ . At what time t does e^t reach 100 ?

20 The most important curve in probability is the bell-shaped graph of $e^{-t^2/2}$. With a calculator or computer find this function at $t = -2, -1, 0, 1, 2$. Sketch the graph of $e^{-t^2/2}$ from $t = -\infty$ to $t = \infty$. *It never goes below zero.*

21 Explain why $y_1 = e^{(a+b+c)t}$ is the same as $y_2 = e^{at}e^{bt}e^{ct}$. They both start at $y(0) = 1$. They both solve what differential equation ?

22 For $y' = y$ with $a = 1$, Euler's first step chooses $Y_1 = (1 + \Delta t)Y_0$. Backward Euler chooses $Y_1 = Y_0/(1 - \Delta t)$. Explain why $1 + \Delta t$ is smaller than the exact $e^{\Delta t}$ and $1/(1 - \Delta t)$ is larger than $e^{\Delta t}$. (Compare the series for $1/(1 - x)$ with e^x.)

Note Section 3.5 presents an accurate Runge-Kutta method that captures three more terms of $e^{a\Delta t}$ than Euler. For $dy/dt = ay$ here is the step to Y_{n+1} :

$$\textbf{Runge-Kutta for } \boldsymbol{y' = ay} \qquad Y_{n+1} = \left(1 + a\Delta t + \frac{a^2\Delta t^2}{2} + \frac{a^3\Delta t^3}{6} + \frac{a^4\Delta t^4}{24}\right)Y_n.$$

1.4 Four Particular Solutions

The equation $dy/dt = ay$ is solved by $y(t) = e^{at}y(0)$. All the input is in that starting value $y(0)$. The solution grows exponentially when $a > 0$ and it decays when $a < 0$. *This section allows new inputs $q(t)$ after the starting time $t = 0$.* That input q is a "source" when we add to $y(t)$, and a "sink" when we subtract. If $y(t)$ is the balance in a bank account at time t, then $q(t)$ is the rate of new deposits and withdrawals.

The basic first order linear differential equation (1) is fundamental to this course. We must and will solve this equation. Please pay attention to this section. In every way, this Section 1.4 is important.

$$\boldsymbol{\frac{dy}{dt} = ay + q(t)} \qquad \textbf{starting from } \boldsymbol{y(0)} \textbf{ at } \boldsymbol{t = 0}. \tag{1}$$

Important I will separate the solution $y(t)$ into two parts. One part comes from the starting value $y(0)$. The other part comes from the source term $q(t)$. This separation is a crucial step for all linear equations, and I take this chance to give names to the two parts. The part $y_n = Ce^{at}$ is what we already know. The part y_p from the source $q(t)$ is new.

1 Homogeneous solution or null solution $y_n(t)$ with no source: $q = 0$

This part $y_n(t) = Ce^{at}$ solves the equation $dy/dt = ay$. The source term q is zero (null). We are really solving $y' - ay = 0$, an equation with zero on the right hand side. That equation is **homogeneous**—we can multiply a solution by any constant to get another solution $cy(t)$. This book will choose the simpler word *null* and the subscript n, because this connects differential equations to linear algebra.

2 Particular solution $y_p(t)$ with source $q(t)$

This part $y_p(t)$ comes from the source term $q(t)$. The previous section had no source and therefore no reason to mention $y_p(t)$. Now our whole task is to find a particular solution $y_p(t)$, because the null solutions $y_n(t) = Ce^{at}$ are already set.

3 The complete solution is $y(t) = y_n(t) + y_p(t)$

For linear equations—and only for linear equations—adding the two parts gives the complete solution $y = y_n + y_p$. This is also called the "general solution."

Null	y_n'	$-$	ay_n	$=$	$\mathbf{0}$	y_n can start from $y(0)$
Particular	y_p'	$-$	ay_p	$=$	$\boldsymbol{q(t)}$	y_p can start from $y = 0$
$\boldsymbol{y = y_n + y_p}$	$\boldsymbol{y'}$	$-$	$\boldsymbol{ay}$	$=$	$\boldsymbol{q(t)}$	y must start from $y(0)$

A nonlinear equation could include a quadratic term y^2. In that case adding ${y_n}^2$ to ${y_p}^2$ would not give $(y_n + y_p)^2$. The null equation $y' - y^2 = 0$ would not be homogeneous, and we can't multiply y by a constant C. This will happen for the "logistic equation" in Section 1.7. You will see that $y(0)$ enters the solution $y(t)$ in a more complicated way.

The back cover of this book shows one particular solution y_p combining with all null solutions y_n. This important picture is repeated for matrix equations and linear algebra.

Particular Solutions and the Complete Solution

We can draw the complete solution to $u+v=6$. These points (u,v) fill a straight line. We can also draw all the null solutions to $u+v=0$. They fill a parallel straight line, going through the center point $(0,0)$. Figure 1.2 shows how the null solutions combine with one particular solution $(3,3)$ to give the line of complete solutions.

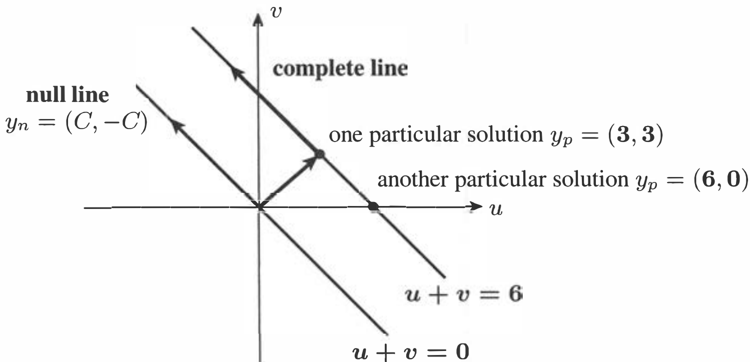


Figure 1.2: By adding all the null solutions to one particular solution, you get every solution (the complete line). You can start from *any* particular y_p that solves $u+v=6$.

Starting from $y_p = (3,3)$, the complete solution has $u = 3+C$ and $v = 3-C$. This includes a **null** solution $C+(-C)=0$, plus the **particular** solution $3+3=6$.

Null	u_n	$+$	v_n	$=$	$\mathbf{0}$		C	$+$	$(-C)$	$=$	0
Particular	u_p	$+$	v_p	$=$	$\mathbf{6}$		3	$+$	3	$=$	6
Complete	u	$+$	v	$=$	$\mathbf{6}$		$\mathbf{(3+C)}$	$+$	$\mathbf{(3-C)}$	$=$	$\mathbf{6}$

The null solution $(C,-C)$ allows any constant C (like $y(0)$). The particular solution could have any numbers u_p and v_p that add to 6. We made a special choice $u_p = 3$ and $v_p = 3$. In the equation $y' - ay = q$ we will often make the special choice $y_p(0) = 0$.

There are many particular solutions! You could say that we chose a *very particular* solution. In the differential equation we chose to start from $y_p(0) = 0$. For the equation $u+v=6$ we chose $u=3$ and $v=3$. *We could equally well choose* $u=6$ *and* $v=0$. This particular solution is different, but we get the same complete solution line:

$$y_{\text{complete}} = (6+c, 0-c) \text{ is the same solution line as } y_{\text{complete}} = (3+C, 3-C).$$

If c is 5, then C is 8. From all c's and all C's, you get the same line.

I want to repeat this pattern of *null solution plus particular solution* by showing how it looks for an ordinary matrix equation $Av = \boldsymbol{b}$ (Chapter 4 explains matrices):

Null solution $A\boldsymbol{v}_n = \mathbf{0}$ **Particular solution** $A\boldsymbol{v}_p = \boldsymbol{b}$ **Complete solution** $\boldsymbol{v} = \boldsymbol{v}_n + \boldsymbol{v}_p$

Always the key is *linearity*: $A\boldsymbol{v}$ *equals* $A\boldsymbol{v}_n + A\boldsymbol{v}_p$. Therefore $A\boldsymbol{v} = \mathbf{0} + \boldsymbol{b} = \boldsymbol{b}$.

Often the only solution to $A\boldsymbol{v}_n = \mathbf{0}$ is $\boldsymbol{v}_n = \mathbf{0}$. Then a particular solution $\boldsymbol{v}_p$ is also the complete solution. This will happen when A is an "invertible matrix."

Inputs $q(t)$ and Responses $y(t)$

For any input source $q(t)$, equation (4) will solve $dy/dt = ay + q(t)$. But when mathematics is applied to science and engineering and our society, problems don't involve "any $q(t)$." *Certain functions $q(t)$ are the most important.* Those functions are constantly met in applied mathematics. Here is a short list of special inputs:

1. **Constant source** $\quad q(t) = \boldsymbol{q}$
2. **Step function at T** $\quad q(t) = \boldsymbol{H(t-T)}$
3. **Delta function at T** $\quad q(t) = \boldsymbol{\delta(t-T)}$
4. **Exponential** $\quad q(t) = \boldsymbol{e^{ct}}$

This section will solve $dy/dt = ay + q(t)$ for the four functions on that short list. The next section adds one more source $q(t)$. It is a combination of sine and cosine. Or $q(t)$ can be a complex exponential (which has one term and is usually easier):

5. **Sinusoid** $\quad q(t) = \boldsymbol{A\cos\omega t + B\sin\omega t}$ or $\boldsymbol{R\,e^{i\omega t}}$

Solving Linear Equations by an Integrating Factor

The equation $y' = ay + q$ is so important that I will solve it in different ways. The first way uses an integrating factor $M(t)$. Put both y terms on the left. Keep $q(t)$ on the right.

Problem $\quad$ **Solve $\boldsymbol{y' - ay = q(t)}$ starting from any $\boldsymbol{y(0)}$**

Method $\quad$ ***Multiply both sides by the integrating factor $\boldsymbol{M(t) = e^{-at}}$.***

We chose that factor e^{-at} so that M times $y' - ay$ is exactly the derivative of My:

Perfect derivative $\qquad e^{-at}(y' - ay)$ agrees with $\dfrac{d}{dt}(e^{-at}y) = \dfrac{d}{dt}(My)$. $\qquad$ (2)

When both sides of $y' - ay = q$ are multiplied by $M = e^{-at}$, our equation is immediately ready to be integrated. The right side is Mq, the left side is the derivative of My.

$$\text{The integral of } \frac{d}{dt}(My) = Mq \quad \text{is} \quad M(t)\,y(t) - M(0)\,y(0) = \int_0^t M(s)\,q(s)\,ds \qquad (3)$$

At $t = 0$ we know that $M(0) = e^0 = 1$. Multiply both sides of equation (3) by e^{at} (which is $1/M$) to see $y(t) = y_n + y_p$. This solution comes many times in the book! To give meaning to formula (4), I will apply it to the most important inputs $q(t)$.

The key formula
Solution to $\boldsymbol{y' = ay + q(t)}$

$$\boldsymbol{y(t) = e^{at}\,y(0) + e^{at}\int_0^t e^{-as}\,q(s)\,ds.} \qquad (4)$$

Constant Source $q(t) = q$

When $q(t)$ is a constant, the integration for the particular solution in equation (4) is easy.

$$\int_0^t e^{-as}\, q\, ds = \left[\frac{qe^{-as}}{-a}\right]_{s=0}^{s=t} = \frac{q}{a}(1 - e^{-at}).$$

Multiply by e^{at} to find $y_p(t)$. An important solution to an important equation.

Solution for constant source q	$\boldsymbol{y(t) = e^{at}\, y(0) + \frac{q}{a}(e^{at} - 1)}$	(5)

Example 1 has a positive growth rate $a > 0$. The solution will increase when $q > 0$. Example 2 will have a *negative* rate $a < 0$. In that case $y(t)$ approaches a *steady state.*

Example 1 Solve $dy/dt - 5y = 3$ starting from $y(0) = 2$. Here $a = 5$ and $q = 3$. This fits perfectly with $y' - ay = q$. Equation (5) gives the solution $y(t)$:

Solution $y(t) = y_n + y_p = \mathbf{2e^{5t} + \frac{3}{5}(e^{5t} - 1)}$. Set $t = 0$ to check that $y(0) = 2$.

Looking at that solution, I have to admit that $y' - 5y = 3$ is not so obvious. This becomes much clearer when the two parts (null + particular) are separated:

$$\begin{aligned} y_n(t) &= 2e^{5t} \text{ certainly has } y_n' - 5y_n = 0 \text{ with } y_n(0) = 2 \\ y_p(t) &= \tfrac{3}{5}(e^{5t} - 1) \text{ has } y_p' = 3e^{5t}. \text{ This agrees with } 5y_p + 3. \end{aligned}$$

Example 2 Solve $dy/dt = 3 - 6y$ starting from $y(0) = 2$.

Formula (5) still gives the answer, but this $y(t)$ is decreasing because $\boldsymbol{a = -6}$ **is negative**:

$$y(t) = 2e^{-6t} + \frac{3}{-6}(e^{-6t} - 1) = \mathbf{\frac{3}{2}e^{-6t} + \frac{1}{2}}.$$

When $t = 0$, that solution starts at $y(0) = 2$. The solution decreases because of e^{-6t}. **As $\boldsymbol{t \to \infty}$ the solution approaches $\boldsymbol{y_\infty = \frac{1}{2}}$.** This value $-q/a$ at $t = \infty$ is a *steady state.*

$$\boxed{\text{At } y = -\frac{q}{a} = \frac{1}{2} \text{ the equation } \frac{dy}{dt} = 3 - 6y \text{ becomes } \boldsymbol{\frac{dy}{dt} = 0}. \text{ Nothing moves.}}$$

Please notice that the steady state is $y_\infty = \frac{1}{2}$ for every initial value $y(0)$. That is because the null solution $y_n = y(0)e^{-6t}$ approaches zero. It is the particular solution that balances the source term $q = 3$ with the decay term $ay = -6y$ to approach $\boldsymbol{y_\infty = -q/a = 3/6}$.

Question If $y(0) = \frac{1}{2}$, what is $y(t)$? *Answer* $\boldsymbol{y(t) = \frac{1}{2}}$ **at all times**. $6y$ balances 3.

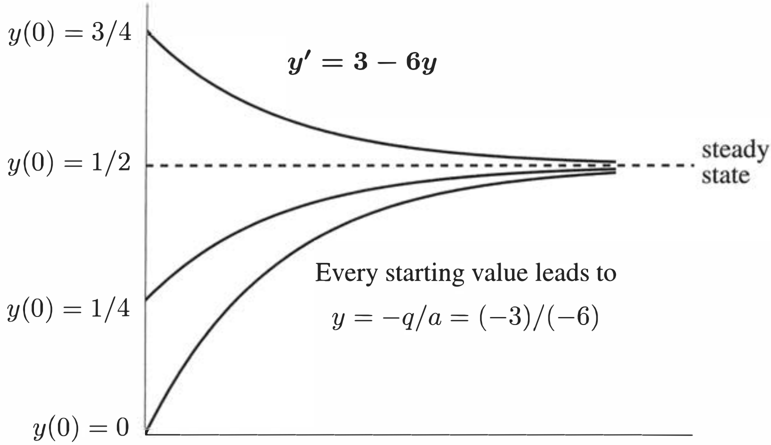


Figure 1.3: When a is negative, e^{at} approaches zero and $y(t)$ approaches $y_\infty = -q/a$.

Here is an important way to rewrite that basic equation $y' = ay + q$ when $a < 0$. The right hand side is the same as $a(y + \frac{q}{a})$. But $\boldsymbol{y + \frac{q}{a}}$ is exactly the distance $\boldsymbol{y - y_\infty}$. **Rewrite $\boldsymbol{y' = ay + q}$ as an easy equation $\boldsymbol{Y' = aY}$ by introducing $\boldsymbol{Y = y - y_\infty}$.**

New unknown $Y = y - y_\infty$ **New equation** $\boldsymbol{Y' = aY}$ **New start** $Y(0) = y(0) - y_\infty$

The solution to $Y' = aY$ is certainly $Y(t) = Y(0)e^{at}$. This approaches $Y_\infty = 0$ when $a < 0$. The original $y = Y + y_\infty$ still approaches y_∞ which is $-q/a$: see Figure 1.3.

$$(y - y_\infty)' = a(y - y_\infty) \quad \textbf{has solution} \quad \boxed{y(t) - y_\infty = e^{at}(y(0) - y_\infty)} \qquad (6)$$

Section 1.6 will present physical examples with $a < 0$: Newton's Law of Cooling, the level of messenger RNA, the decaying concentration of a drug in the bloodstream.

Step Function

The unit step function or "Heaviside step function" $\boldsymbol{H(t)}$ jumps from 0 to 1 at $\boldsymbol{t = 0}$. Figure 1.4 shows its graph. The effect of $H(t)$ is like turning on a switch.

The second graph shows a *shifted step function* $H(t - T)$ which jumps from 0 to 1 at time T. This is the moment when $t - T = 0$, so H jumps at that moment T.

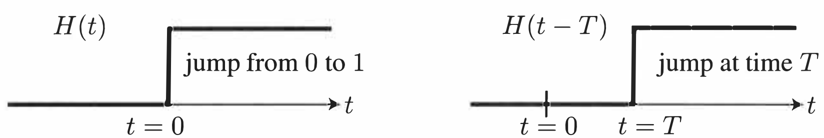


Figure 1.4: **The unit step function is $\boldsymbol{H(t)}$.** Its shift $H(t - T)$ jumps to 1 at $t = T$.

When the step comes at $t = 0$, the solution to $y' - ay = H(t)$ is the step response. That step response is easy to find because this equation is simply $y' - ay = 1$. The starting value is $y(0) = 0$. Put $q = 1$ into formula (5):

$$\textbf{Step response} \qquad \boxed{y(t) = \frac{1}{a}(e^{at} - 1)} \qquad (7)$$

The interesting case is $a < 0$. The solution starts at $y(0) = 0$. It grows to $y(\infty) = -1/a$. The system rises to that steady state after the switch is turned on. The graph of $y(t)$ is the bottom curve in Figure 1.3, except that y_∞ is $1/6$ because the step function has $q = 1$.

The step response is the output $y(t)$ when the step function is the input. We are depositing at a constant rate $q = 1$. But when $a < 0$, we are losing ay in real value because of inflation. Then growth stops at $y = -1/a$, where the deposits just balance the loss.

Now turn on the switch at time T instead of time 0. The step function $H(t - T)$ is piecewise constant with two pieces: zero and one. If I multiply by any constant q, the source $q\,H(t - T)$ jumps from 0 to strength q at time T.

The left side of our differential equation is still $y' - ay$, no change. The integrating factor $M = e^{-at}$ still makes that into a perfect derivative: $M(y' - ay)$ *equals* $(My)'$. The only change is on the right side, where the constant source doesn't start acting until the jump time T. At that time, the step function source $H(t - T)$ is turned on:

$$(e^{-at}y)' = e^{-at}\,H(t-T) \ \text{ now gives } \ e^{-at}y(t) - e^{0t}\,y(0) = \int_T^t e^{-as}\,ds. \qquad (8)$$

The only change for $t \geq T$ is to **start that integral at the turn-on time T**:

$$\int_T^t e^{-as}\,ds = \left[\frac{e^{-as}}{-a}\right]_{s=T}^{s=t} = \frac{1}{a}(e^{-aT} - e^{-at}). \qquad (9)$$

Multiply by e^{at} to get the particular solution $y_p(t)$ beyond time T, and add $y_n = e^{at}y(0)$.

$$\boxed{\textbf{Solution with unit step} \quad y(t) = e^{at}y(0) + \frac{1}{a}\left(e^{a(t-T)} - 1\right) \ \text{ for } \ t \geq T.} \qquad (10)$$

As always, $y(0)$ grows or decays with e^{at} in the null solution y_n. The step response is the particular solution, as soon as the input begins. But nothing enters until time T.

Example 3 Suppose the input turns on at time $t = 0$ and turns off at $t = T$. Find $y(t)$.

Solution The input is $H(t) - H(t-T)$. The output is $y(t) = \frac{1}{a}\left(e^{at} - e^{a(t-T)}\right), t \geq T$.

Delta Function

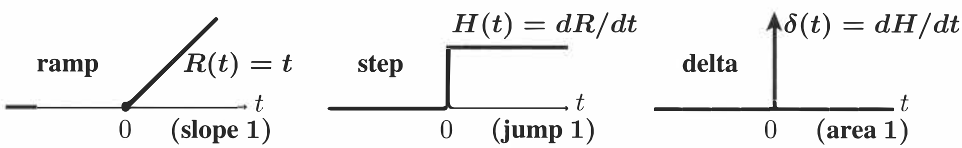
Now we meet a remarkable function $\delta(t)$. This "delta function" is everywhere zero, except at the instant $t = 0$. In that one moment it gives a unit input. Instead of a continuing source spread out over time, $\delta(t)$ is a **point source** completely concentrated at $t = 0$.

For a point source shifted to $\delta(t - T)$, **everything enters exactly at time T**. There is no source before that time or after that time. The delta function is zero except at one point. This "**impulse** " is by no means an ordinary function.

Here is one way to think about $\delta(t)$. **The delta function is the derivative of the unit step function $H(t)$.** But H is constant and dH/dt is zero except at $t = 0$. Take the integral of $\delta(t) = dH/dt$ from any negative number N to any positive number P.

$$\textbf{Integral of } \boldsymbol{\delta(t)} \textbf{ is 1} \qquad \int_N^P \boldsymbol{\delta(t)}\, dt = \int_N^P \frac{\boldsymbol{dH}}{\boldsymbol{dt}}\, dt = H(P) - H(N) = \mathbf{1 - 0}. \qquad (11)$$

"The area under the graph of $\delta(t)$ is 1. All that area is above the single point $t = 0$." Those words are in quotes because area at a point is impossible for ordinary functions. $\delta(t)$ may seem new and strange (it is useful!). Look at $dR/dt = H$ and $dH/dt = \delta$.



Slope of the ramp jumps to 1. Slope of the step function is the delta function.

The value of $\delta(0)$ is infinite. But that one word does not give full information. **The real way to understand delta functions is by their integrals.**

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\, dt = \mathbf{1}} \quad \boxed{\int_{-\infty}^{\infty} \delta(t)\, F(t)\, dt = \boldsymbol{F(0)}} \quad \boxed{\int_{-\infty}^{\infty} \delta(t-T)\, F(t)\, dt = \boldsymbol{F(T)}} \qquad (12)$$

Please visualize a tall thin box function—equal to $1/h$ between $t = 0$ and $t = h$. Now imagine h going to zero. The width h becomes zero and the height $1/h$ becomes infinite. *The area stays at* 1. All integrals of $\delta(t)F(t)$ are concentrated at $t = 0$: the "spike".

Here is a quick way to solve $y' - ay = \delta(t)$, and then we will do it more slowly. We know that the derivative of a step function $H(t)$ is the delta function $\delta(t)$. So the derivative of the step response must be the impulse response:

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\textbf{(step)} = \textbf{delta} \qquad \frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\begin{matrix}\textbf{step}\\ \textbf{response}\end{matrix}\right) = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{e^{at}} - \mathbf{1}}{\boldsymbol{a}}\right) = \boldsymbol{e^{at}} = \begin{matrix}\textbf{impulse}\\ \textbf{response}\end{matrix} \qquad (13)$$

The Impulse Response Solves $y' - ay = \delta(t)$

Start your bank account with one deposit. Start your heart with a sudden shock. Hit a golf ball. Fire a bullet. Many motions start with an "impulse" and then the source term is a delta function $\delta(t)$.

The impulse response $y(t)$ jumps immediately to $y(0) = 1$. You can see that by integrating every term in $dy/dt - ay = \delta(t)$. Integrating $\delta(t)$ from $t = -h$ to h gives $\mathbf{1}$. Integrating dy/dt gives $y(h) - y(-h)$, which is $\boldsymbol{y(h)}$. The integral of ay becomes zero as $h \to 0$. That limit step when $h \to 0$ leaves $\boldsymbol{y(0) = 1}$.

After the jump to $y(0) = 1$, the impulse $\delta(t)$ is immediately zero. So we just have the ordinary null solution to $y' = ay$ starting from $y(0) = 1$:

$$\textbf{Impulse response} \qquad \boldsymbol{y' - ay = \delta(t)} \qquad \boldsymbol{y(t) = e^{at}} \tag{14}$$

Notice the different responses to an impulse and a step function. The impulse deposits everything at $t = 0$. The step function goes on depositing forever. If $a < 0$ and inflation reduces our wealth, the impulse response dies out to $y_\infty = 0$. The step response increases from 0 to $y_\infty = -1/a$, where the deposits balance the loss from inflation.

I want to emphasize: **$\boldsymbol{e^{at}}$ is the growth or decay factor $\boldsymbol{G(t)}$ for all inputs**. When the input is $y(0)$, the output at time t is $e^{at}y(0)$. When the input is $q(s)$ at time s, the output later at t is $e^{a(t-s)}q(s)$. The growth is only over the remaining time $t - s$. **Our main formula (4) is adding up all the outputs that come from all the inputs.**

Delayed Delta Function

The source $q(t) = \delta(t - T)$ turns on at time T. Then immediately it turns off. In that one instant of time, *the value of y jumps by* 1. "We deposited \$1 at that moment." The integral of $dy/dt = \delta(t - T)$ is 1. This is the change in y, before T to after T.

Coming up to time T, the solution is $y(t) = e^{at}y(0)$. At time T we add 1. After time T, that input has the shorter period $t - T$ in which to grow. Multiply 1 by $e^{a(t-T)}$:

$$\boxed{\textbf{Solution for} \quad \boldsymbol{q = \delta(t-T)} \quad \boldsymbol{y(t) = y_n(t) + y_p(t) = e^{at}\,y(0) + e^{a(t-T)}}.} \tag{15}$$

The solution y jumps by $e^{a\,(T-T)} = e^0 = 1$, when that second term appears at $t = T$.

Example 4 Solve the equation $\boldsymbol{y' - 5y = 3\delta(t-4)}$ starting from $y(0) = 2$.
The null solution to $y' - 5y = 0$ starting at $y(0) = 2$ is $\boldsymbol{y_n(t) = 2e^{5t}}$. This we know. The particular solution is $y_p(t) = 0$ up to $t = 4$. At that moment y jumps by 3, from 3δ. Its growth factor is $e^{5(t-4)}$. Then $\boldsymbol{y_p(t) = 3e^{5(t-4)}}$ after $t = 4$.

$$\textbf{Complete solution with jump of 3} \qquad \boldsymbol{y_n + y_p = 2e^{5t} + 3e^{5(t-4)}H(t-4)} \tag{16}$$

The step function $H(t-4)$ combines $y_p = 0$ before the jump and y_p after the jump into one formula. At $t = 4$ the solution jumps by 3. Then this 3 grows to $3e^{5(t-4)}$.

***Remark* 1** This solution makes me realize that the initial value $y(0)$ is like having a delta function at time $t = 0$. *The solution "jumps" to* $y(0)$. I don't know if you agree with that.

***Remark* 2** $q(t) = -\delta(t - T)$ would be negative (a sink instead of a source). A bank account could be earning interest at the rate a, and suddenly you withdraw 1 at time T. The balance $y(T)$ had reached $e^{aT}y(0)$, and it drops by 1. From time T onwards, the growth factor $e^{a(t-T)}$ multiplies the new balance, and $y(t) = e^{at}y(0) - e^{a(t-T)}$.

***Remark* 3** (a little mysterious) We could think of an ordinary continuous input $q(t)$ as a lot of delta functions—a delta function of strength $q(T)$ *at every time* T. Instead of "a lot" I need to say "an integral". Every continuous function $q(t)$ is an integral of delta functions $q(T)\,\delta(t - T)$ at all T. The integral picks out $q(t)$ at the spike point.

$$\textbf{Any } \boldsymbol{q(t)} = \textbf{combination of delta functions} = \int q(T)\,\delta(t-T)\,dT. \qquad (17)$$

Example 5 ($q = 1$) **The integral of all impulses for $\boldsymbol{T \geq 0}$ is the step function $\boldsymbol{H(t)}$.**

Then the integral of all impulse responses is the step response. The integral of e^{at} from 0 to t is $(e^{at} - 1)/a$. *Derivative of step response = impulse response* as in (13).

Exponential Input e^{ct}

The source $q(t) = e^{ct}$ starts at time zero and continues forever. The particular solution $y_p(t)$ is easy to find, because **$\boldsymbol{y_p}$ is a multiple $\boldsymbol{Ye^{ct}}$ of this same exponential $\boldsymbol{e^{ct}}$.** That is the beauty of exponentials . These are the most important functions and the best to work with. They allow growth or decay or oscillation from $c > 0$ and $c < 0$ and $c = i\omega$.

$$\textbf{Substitute } \boldsymbol{y_p = Ye^{ct}} \textbf{ into } \boldsymbol{y' - ay = e^{ct}} \qquad \boldsymbol{cYe^{ct} - aYe^{ct} = e^{ct}}$$

When we cancel e^{ct} this leaves a simple formula for the number Y in Ye^{ct} :

$$\boldsymbol{cY - aY = 1} \quad \textbf{gives} \quad \boxed{\boldsymbol{Y = \frac{1}{c-a}}} \quad \textbf{and} \quad \boxed{\boldsymbol{y_p(t) = \frac{e^{ct}}{c-a}}} \qquad (18)$$

Example 6 Solve $\boldsymbol{y' - 5y = 3e^{4t}}$ starting from $y(0) = 2$. Now $Y = \dfrac{3}{c-a} = \dfrac{3}{4-5}$.

The null solution still involves e^{5t}. The particular solution is Y times e^{4t} !

$$\boldsymbol{y_p(t) = Ye^{4t}} \qquad y_p' - 5y_p = (4Y - 5Y)e^{4t} = 3e^{4t}. \quad \textbf{Then } \boldsymbol{Y = -3}.$$

This particular solution $-3e^{4t}$ starts at -3. Since $y(0) = 2$, the other part starts at $+5$.

$$\textbf{Complete solution} \qquad \boldsymbol{y(t) = 5e^{5t} - 3e^{4t}}.$$

The null solution grows at rate $a = 5$. One particular solution grows at rate $c = 4$. The equation $y' - ay = e^{ct}$ is solved for $c \neq a$ but two final comments are needed.

1. This particular solution $y(t) = e^{ct}/(c-a)$ is not the "very particular" solution that starts from $y_p(0) = 0$. It is still perfectly good, except it starts at $1/(c-a)$. So the complete solution starting at $y(0)$ has to include the usual $y(0)e^{at}$ and **also a term to cancel $1/(c-a)$ at time zero**:

$$y' - ay = e^{ct} \qquad \boxed{y_{\text{complete}} = y(0)\, e^{at} - \frac{e^{at}}{c-a} + \frac{e^{ct}}{c-a}} \tag{19}$$

   There you see a null solution y_n (two terms) and our particular y_p (the last term). Or the last two terms together are the very particular solution $(e^{ct} - e^{at})/(c-a)$.

2. For $c = a$ we are in serious trouble. The formulas fail because we can't divide by $c - a = 0$. This problem $y' - ay = e^{at}$ is a type of **resonance**, when the exponent c in the source happens to equal the exponent a in the natural growth from $y' = ay$. The integral in our main formula (4) becomes $\int e^{-as}\, e^{as}\, ds = \int 1\, ds = t$.

$$\textbf{Resonance} \quad \boxed{c = a} \qquad y' - ay = e^{at} \qquad \boxed{y = y(0)e^{at} + te^{at}} \tag{20}$$

   That extra growth factor t is because y_n resonates with y_p. They both have e^{at}.

■ REVIEW OF THE KEY IDEAS ■

1. **Complete** solution to a linear equation $=$ **null** solution(s) $+$ **particular** solution.

2. The integrating factor e^{-at} multiplies $y' - ay = q(t)$ to give $(e^{-at}\, y)' = e^{-at}\, q(t)$.

   Integrate and multiply by e^{at} : $y(t) = y_n + y_p = e^{at}y(0) + e^{at} \int e^{-as} q(s)\, ds$.

3. For $y' - ay = q =$ **constant**, the particular solution with $y_p(0) = 0$ is $q(e^{at} - 1)/a$.

4. $q(t) = H(t)$: the response to a unit step function is $y_p = (e^{at} - 1)/a$.

5. $q(t) = \delta(t)$: the impulse response to a unit delta function is $y_p = e^{at}$.

6. $q(t) = e^{ct}$ gives $y_p = (e^{ct} - e^{at})/(c-a)$. In case $c = a$, change to $y_p = te^{at}$.

Problem Set 1.4

1 All solutions to $dy/dt = -y + 2$ approach the steady state where dy/dt is zero and $y = y_\infty =$ ___. That constant $y = y_\infty$ is a particular solution y_p.

Which $y_n = Ce^{-t}$ combines with this steady state y_p to start from $y(0) = 4$? This question chose $y_p + y_n$ to be $y_\infty +$ *transient* (decaying to zero).

2 For the same equation $dy/dt = -y + 2$, choose the null solution y_n that starts from $y(0) = 4$. Find the particular solution y_p that starts from $y(0) = 0$.

This splitting chooses the two parts $e^{at}y(0)$ + integral of $e^{a(t-s)}q$ in equation (4).

3 The equation $dy/dt = -2y + 8$ has two natural splittings $\boldsymbol{y_S + y_T = y_N + y_P}$:

1. Steady ($\boldsymbol{y_S = y_\infty}$) + Transient ($\boldsymbol{y_T \to 0}$). What are those parts if $y(0) = 6$?

2. ($y_N' = -2y_N$ from $\boldsymbol{y_N(0) = 6}$) + ($y_P' = -2y_P + 8$ starting from $\boldsymbol{y_P(0) = 0}$).

4 All null solutions to $u - 2v = 0$ have the form $(u, v) = (c,$ ____$)$.

One particular solution to $u - 2v = 3$ has the form $(u, v) = (7,$ ____$)$.

Every solution to $u - 2v = 3$ has the form $(7,$ ____$) + c(1,$ ____$)$.

But also every solution has the form $(3,$ ____$) + C(1,$ ____$)$ for $C = c + 4$.

5 The equation $dy/dt = 5$ with $y(0) = 2$ is solved by $y =$ ____. A natural splitting $y_n(t) =$ __ and $y_p(t) =$ __ comes from $y_n = e^{at}y(0)$ and $y_p = \int e^{a(t-s)}5\,ds$.

This small example has $\boldsymbol{a = 0}$ (so ay is absent) and $\boldsymbol{c = 0}$ (the source is $q = 5e^{0t}$). When $a = c$ we have "resonance." A factor t will appear in the solution y.

Starting with Problem 6, choose the very particular $\boldsymbol{y_p}$ that starts from $\boldsymbol{y_p(0) = 0}$.

6 For these equations starting at $y(0) = 1$, find $y_n(t)$ and $y_p(t)$ and $y(t) = y_n + y_p$.

(a) $y' - 9y = 90$ (b) $y' + 9y = 90$

7 Find a linear differential equation that produces $y_n(t) = e^{2t}$ and $y_p(t) = 5(e^{8t} - 1)$.

8 Find a resonant equation $(a = c)$ that produces $y_n(t) = e^{2t}$ and $y_p(t) = 3te^{2t}$.

9 $y' = 3y + e^{3t}$ has $y_n = e^{3t}y(0)$. Find the resonant y_p with $y_p(0) = 0$.

Problems 10–13 are about $\boldsymbol{y' - ay =}$ constant source $\boldsymbol{q}$.

10 Solve these linear equations in the form $y = y_n + y_p$ with $y_n = y(0)e^{at}$.

(a) $y' - 4y = -8$ (b) $y' + 4y = 8$ Which one has a steady state?

11 Find a formula for $y(t)$ with $y(0) = 1$ and draw its graph. What is y_∞?

(a) $y' + 2y = 6$ (b) $y' + 2y = -6$

12 Write the equations in Problem 11 as $Y' = -2Y$ with $Y = y - y_\infty$. What is $Y(0)$?

13 If a drip feeds $q = 0.3$ grams per minute into your arm, and your body eliminates the drug at the rate $6y$ grams per minute, what is the steady state concentration y_∞ ? Then *in* $=$ *out* and y_∞ is constant. Write a differential equation for $Y = y - y_\infty$.

Problems 14–18 are about $y' - ay =$ step function $H(t-T)$:

14 Why is y_∞ the same for $y' + y = H(t-2)$ and $y' + y = H(t-10)$?

15 Draw the ramp function that solves $y' = H(t-T)$ with $y(0) = 2$.

16 Find $y_n(t)$ and $y_p(t)$ as in equation (10), with step function inputs starting at $T = 4$.

(a) $y' - 5y = 3H(t-4)$ (b) $y' + y = 7H(t-4)$ (*What is* y_∞ ?)

17 Suppose the step function turns on at $T = 4$ and off at $T = 6$. Then $q(t) = H(t-4) - H(t-6)$. Starting from $y(0) = 0$, solve $y' + 2y = q(t)$. What is y_∞ ?

18 Suppose $y' = H(t-1) + H(t-2) + H(t-3)$, starting at $y(0) = 0$. Find $y(t)$.

Problems 19–25 are about delta functions and solutions to $y' - ay = q\,\delta(t-T)$.

19 For all $t > 0$ find these integrals $a(t), b(t), c(t)$ of point sources and graph $b(t)$:

$$\text{(a)} \int_0^t \delta(T-2)\,dT \quad \text{(b)} \int_0^t (\delta(T-2) - \delta(T-3))\,dT \quad \text{(c)} \int_0^t \delta(T-2)\delta(T-3)dT$$

20 Why are these answers reasonable ? (They are all correct.)

$$\text{(a)} \int_{-\infty}^{\infty} e^t \delta(t)dt = 1 \quad \text{(b)} \int_{-\infty}^{\infty} (\delta(t))^2 dt = \infty \quad \text{(c)} \int_{-\infty}^{\infty} e^T \delta(t-T)dT = e^t$$

21 *The solution to* $y' = 2y + \delta(t-3)$ *jumps up by* 1 *at* $t = 3$. Before and after $t = 3$, the delta function is zero and y grows like e^{2t}. Draw the graph of $y(t)$ when (a) $y(0) = 0$ and (b) $y(0) = 1$. Write formulas for $y(t)$ before and after $t = 3$.

22 Solve these differential equations starting at $y(0) = 2$:

(a) $y' - y = \delta(t-2)$ (b) $y' + y = \delta(t-2)$. (*What is* y_∞ ?)

23 Solve $dy/dt = H(t-1) + \delta(t-1)$ starting from $y(0) = 0$: jump and ramp.

24 (My small favorite) What is the steady state y_∞ for $y' = -y + \delta(t-1) + H(t-3)$?

25 Which q and $y(0)$ in $y' - 3y = q(t)$ produce the step solution $y(t) = H(t-1)$?

Problems 26–31 are about exponential sources $q(t) = Qe^{ct}$ and resonance.

26 Solve these equations $y' - ay = Qe^{ct}$ as in (19), starting from y(0) = 2 :

(a) $y' - y = 8e^{3t}$ (b) $y' + y = 8e^{-3t}$ (*What is* y_∞ ?)

27 When $c = 2.01$ is very close to $a = 2$, solve $y' - 2y = e^{ct}$ starting from $y(0) = 1$. By hand or by computer, draw the graph of $y(t)$: near resonance.

28 When $c = 2$ is exactly equal to $a = 2$, solve $y' - 2y = e^{2t}$ starting from $y(0) = 1$. This is resonance as in equation (20). By hand or computer, draw the graph of $y(t)$.

29 Solve $y' + 4y = 8e^{-4t} + 20$ starting from $y(0) = 0$. What is y_∞ ?

30 The solution to $y' - ay = e^{ct}$ didn't come from the main formula (4), but it could. Integrate $e^{-as}e^{cs}$ in (4) to reach the very particular solution $(e^{ct} - e^{at})/(c - a)$.

31 *The easiest possible equation* $y' = 1$ *has resonance*! The solution $y = t$ shows the factor t. What number is the growth rate a and also the exponent c in the source ?

32 Suppose you know two solutions y_1 and y_2 to the equation $y' - a(t)y = q(t)$.

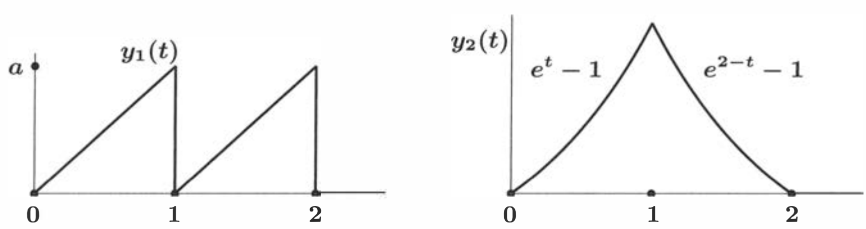
(a) Find a null solution to $y' - a(t)y = 0$.

(b) Find all null solutions y_n. Find all particular solutions y_p.

33 Turn back to the first page of this Section 1.4. Without looking, can you write down a solution to $y' - ay = q(t)$ for all four source functions $\boldsymbol{q}, \boldsymbol{H(t)}, \boldsymbol{\delta(t)}, \boldsymbol{e^{ct}}$?

34 Three of those sources in Problem 33 are actually the same, if you choose the right values for q and c and $y(0)$. What are those values ?

35 What differential equations $y' = ay + q(t)$ would be solved by $y_1(t)$ and $y_2(t)$? Jumps, ramps, corners—maybe harder than expected (math.mit.edu/dela/Pset1.4).



1.5 Real and Complex Sinusoids

Section 1.4 ended with the equation $y' - ay = e^{ct}$. A particular solution was easy to produce, because we kept e^{ct}. We simply chose the correct multiplier $Y = 1/(c-a)$ in $y_p(t) = Ye^{ct}$. **This section changes the real number c to an imaginary number $i\omega$. The multiplier is now $Y = 1/(i\omega - a)$** . The solution formula $Ye^{i\omega t}$ will stay exactly the same, but we need complex numbers (with real part and imaginary part). The payoff is that we can solve all real problems $y' - ay = A\cos\omega t + B\sin\omega t$ at once.

Many scientific and engineering applications are driven by sources $q(t)$ that oscillate like $\cos\omega t$ and $\sin\omega t$ (**sinusoids**). Pistons go up and down to drive a car, voltages go up and down to drive current (alternating current). The input frequency is ω, and the output frequency is also ω. The problem is to find the *amplitude* and the *phase* in the output (the response to the input). The real solution will be $\boldsymbol{y = M\cos\omega t + N\sin\omega t}$.

This $y(t)$ will be a particular solution (steady solution). It is not the transient solution $y_n(t)$ that decays to zero. *We solve $y' - ay = q(t)$ when the source $q(t)$ is a sinusoid.* For this section and the next, applications come from biology and chemistry and medicine and more. The number a is often a rate constant. It tells the speed of a chemical reaction.

Note that RLC circuits (resistor-inductor-capacitor) produce equations with second derivatives. Those will go into Chapter 2, but RC and RL circuits (first order equations) belong here. Our plan for this section is straightforward: *Real then complex.*

1 (Real) Solve $\boldsymbol{dy/dt - ay = q(t) = A\cos\omega t + B\sin\omega t}$.

This leads to two equations for the two coefficients M, N in $y = M\cos\omega t + N\sin\omega t$.

2 (Complex) Solve $\boldsymbol{dy/dt - ay = q(t) = Re^{i\omega t}}$.

This leads to one easy equation for the coefficient in $y = Ye^{i\omega t}$. But that number Y is complex, so we still have two real numbers to find (real and imaginary parts of Y).

3 (A key idea) **Write the complex number $\boldsymbol{1/(i\omega - a)}$ in its polar form $\boldsymbol{Ge^{-i\alpha}}$.**

The positive number G is the **gain** . The angle α is the **phase lag** . Those have important meanings and they are perfect to graph separately. In many problems (most problems) G and α are more useful than the real and imaginary parts of $1/(i\omega - a)$.

So we need to explain and review complex numbers. They are worth knowing and not difficult. The next page will solve the real problem **1** and the complex problem **2**. We can't simplify the real problem by using cosines alone, because the term dy/dt in the equation would unavoidably involve $\sin\omega t$.

The *Review of the Key Ideas* at the end organizes the important steps.

Real Sinusoids

We want a particular real solution $y(t)$ when the source $q(t)$ oscillates with frequency ω.

$$\boxed{\textbf{First order linear equation} \qquad \frac{dy}{dt} - ay = A\cos\omega t + B\sin\omega t.} \tag{1}$$

The solution will have the same form $y = M\cos\omega t + N\sin\omega t$ as the source term. By matching the $\cos\omega t$ terms and separately the $\sin\omega t$ terms, you get two equations for M and N. Just subtract $ay = aM\cos\omega t + aN\sin\omega t$ from $dy/dt = -\omega M\sin\omega t + \omega N\cos\omega t$.

$$\frac{dy}{dt} - ay = q \qquad \begin{array}{ll} \textbf{cos}\,\boldsymbol{\omega t}\ \textbf{terms} & -a\ M + \omega\ N = A \\ \textbf{sin}\,\boldsymbol{\omega t}\ \textbf{terms} & -\omega\ M - a\ N = B \end{array} \tag{2}$$

Those two equations tell us M and N in the real solution $y(t) = M\cos\omega t + N\sin\omega t$. I will write down the solution to equation (2), and then describe two ways to find it.

$$\begin{array}{ll} \textbf{Source} & q = A\cos\omega t + B\sin\omega t \\ \textbf{Solution} & y = M\cos\omega t + N\sin\omega t \end{array} \qquad M = -\frac{aA + \omega B}{\omega^2 + a^2} \qquad N = \frac{\omega A - aB}{\omega^2 + a^2} \tag{3}$$

I would find N by eliminating M in equation (2). If you multiply the first equation by ω and the second equation by a, then *subtraction removes* M. The right side is $\omega A - aB$, the left side is $(\omega^2 + a^2)N$. Then N is correct in equation (3). Similarly we find M.

For two equations it is also practical to find M and N from the 2 by 2 **inverse matrix** :

$$\begin{bmatrix} -a & \omega \\ -\omega & -a \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} M \\ N \end{bmatrix} = \frac{1}{\omega^2 + a^2}\begin{bmatrix} -a & -\omega \\ \omega & -a \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}.$$

The matrix on the left times its inverse on the right gives the identity matrix I in Chapter 4. That denominator $\omega^2 + a^2$ of the inverse matrix appears in M and N, in the solution (3).

Complex Sinusoid $e^{i\omega t}$

Now we come to the very important input $\boldsymbol{q(t) = R\,e^{i\omega t}}$. That input is oscillating with frequency ω radians per second. *The output $y(t)$ will oscillate with the same frequency ω.* This is true because a is constant in the differential equation. When $y(t) = Y e^{i\omega t}$ includes the same factor $e^{i\omega t}$, that factor cancels from every term in the equation:

$$\begin{aligned} q(t) &= Re^{i\omega t} \\ y(t) &= Ye^{i\omega t} \end{aligned} \qquad y' - ay = q \ \text{ becomes } \ i\omega Y e^{i\omega t} - aYe^{i\omega t} = Re^{i\omega t}. \tag{4}$$

When we divide by $e^{i\omega t}$, this leaves an easy algebra problem for the complex number Y :

$$\boxed{\textbf{Response } \boldsymbol{Y(\omega)} \qquad i\omega Y - aY = R \quad \text{gives} \quad \boldsymbol{Y = \frac{R}{i\omega - a}} \ \text{ and } \ \boldsymbol{y = Ye^{i\omega t}}.} \tag{5}$$

The simplicity of the solution $y = Ye^{i\omega t}$ comes from one key fact: The derivative of $e^{i\omega t}$ is a multiple of $e^{i\omega t}$ (the multiplying factor is $i\omega$). This was not true for $\cos\omega t$. Its derivative brings in $\sin\omega t$. So we had to solve two real equations for M and N, while (5) is one complex equation for Y.

Complex Numbers : Rectangular and Polar

The complex number $z = x + iy$ has real part x and imaginary part y. The basic ideas are explained here; more details are in Section 2.2. We plot all z in the **complex plane** (the real-imaginary plane). Figure 1.5 shows the particular number $z = 4 + 3i$ with $x = \text{Re } z = 4$ and $y = \text{Im } z = 3$. No problem with the rectangular form $4 + 3i$, except that multiplying and dividing are not at all convenient in $x - y$ coordinates.

The first figure also shows the **polar form** of the same number z. The magnitude (or modulus) is r. The phase is the angle θ. From x and y we can find r and θ.

The magnitude is $r = \sqrt{x^2 + y^2} = \sqrt{25} = 5$. *The angle* θ *has tangent* $y/x = 3/4$.

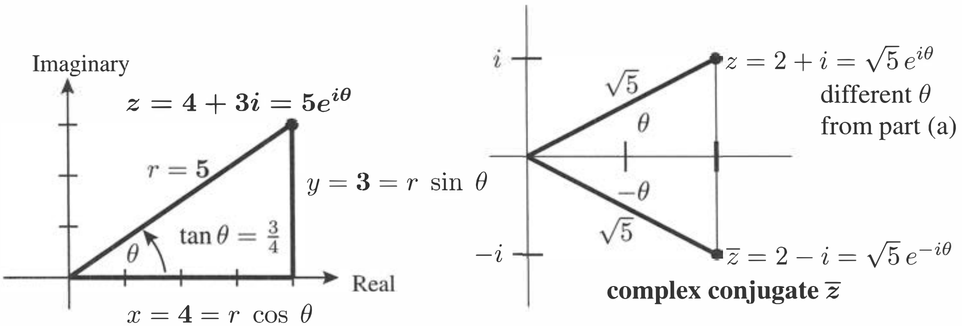


Figure 1.5: (a) $z = 4 + 3i$ is a point in the complex plane. Its polar form is $z = 5e^{i\theta}$.

The polar form is perfect for multiplication and division of complex numbers. To multiply $re^{i\theta}$ times $Re^{i\alpha}$, add the angles and multiply r times R. To divide, subtract the angles and divide r by R.

$$\textbf{Multiply} \quad (re^{i\theta})(Re^{i\alpha}) = \boldsymbol{rR\, e^{i(\theta+\alpha)}} \qquad \textbf{Divide} \quad \frac{re^{i\theta}}{R\,e^{i\alpha}} = \frac{\boldsymbol{r}}{\boldsymbol{R}}\, \boldsymbol{e^{i(\theta-\alpha)}} \tag{6}$$

The polar form is also perfect for squaring a complex number $re^{i\theta}$ and for $1/re^{i\theta}$:

$$\textbf{Square} \quad z^2 = (re^{i\theta})(re^{i\theta}) = \boldsymbol{r^2 e^{2i\theta}} \qquad \textbf{Invert} \quad \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{\boldsymbol{1}}{\boldsymbol{r}}\, \boldsymbol{e^{-i\theta}} \tag{7}$$

Let me compare that polar form of $1/z$ with $1/(x + iy)$. Multiply by $(x - iy)/(x - iy) = 1$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{\boldsymbol{x - iy}}{\boldsymbol{x^2 + y^2}} \qquad\qquad \frac{1}{4+3i} = \frac{4-3i}{4^2+3^2} = \frac{1}{5}e^{-i\theta}$$

This number $x - iy$ appears often. It is the **complex conjugate** $\overline{z}$ of the number $z = x + iy$.

Notice that $x + iy$ times $x - iy$ is $x^2 + y^2$. In other words $\boldsymbol{z}$ **times** $\overline{\boldsymbol{z}}$ **is** $|\boldsymbol{z}|^2 = \boldsymbol{r^2}$.

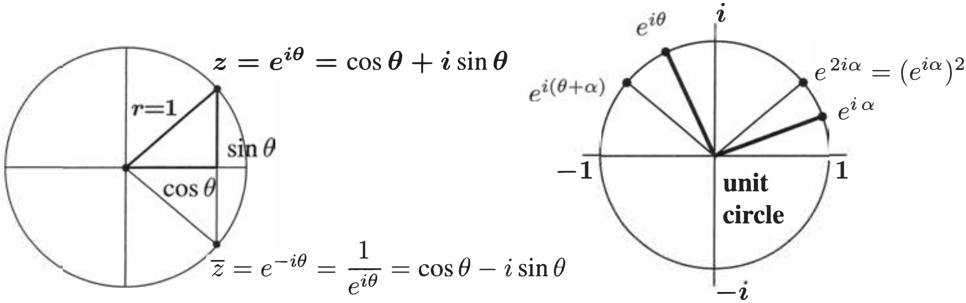


Figure 1.6: Points $e^{i\theta}$ on the unit circle have $r = 1$. When $e^{i\theta}$ multiplies $e^{i\alpha}$, angles add.

The Unit Circle

Figure 1.6 shows the **unit circle** , where every radial distance is $r = 1$. Then we just add the angles to multiply, or double the angles to square, or subtract the angles to divide:

$$\textbf{On the circle} \qquad (e^{i\theta})(e^{i\alpha}) = e^{i(\theta+\alpha)} \qquad (e^{i\theta})(e^{-i\theta}) = 1 \qquad \frac{1}{e^{i\theta}} = e^{-i\theta}$$

$e^{-i\theta}$ is the complex conjugate of $e^{i\theta}$, the mirror image across the axis in Figure 1.6.

Example 1 Describe the paths of the numbers e^{st} and $e^{i\omega t}$ and $e^{(s+i\omega)t}$ in the complex plane (real s and real ω). The time t goes from 0 to ∞. Those paths start at 1.

Solution If $s > 0$, the number e^{st} goes from 1 out the real axis to infinity. If $s < 0$, then e^{st} goes from 1 in to zero. All real.

The path of $e^{i\omega t}$ goes around the unit circle with constant speed. At time $T = 2\pi/\omega$ (and also $2T$, $3T$, ...) it comes back to $e^{2\pi i} = 1$. The path goes clockwise if $\omega < 0$.

The path of $e^{(s+i\omega)t}$ **spirals outward** to infinity if $s > 0$. It spirals inward to zero if $s < 0$. At time $T = 2\pi/\omega$ it is a real number e^{sT}, because the factor $e^{i\omega T} = e^{2\pi i}$ is 1.

The Gain G and the Phase Lag α

The complex number $1/(i\omega - a)$ multiplies the input $q(t) = Re^{i\omega t}$ to give the output $y(t) = Ye^{i\omega t}$. **What is the magnitude of $1/(i\omega - a)$ and what is its angle?** We need its polar form $\boldsymbol{1/(i\omega - a) = Ge^{-i\alpha}}$. Start with $i\omega - a = re^{i\alpha}$ and then invert:

$$i\omega - a = re^{i\alpha} \qquad r = \sqrt{\omega^2 + a^2} \quad \text{and} \quad \tan\alpha = \frac{\text{imaginary part}}{\text{real part}} = -\frac{\omega}{a}.$$

We want $1/(re^{i\alpha})$. This will be $Ge^{-i\alpha}$. The gain is $\boldsymbol{G = 1/r = 1/\sqrt{\omega^2 + \alpha^2}}$:

$$\textbf{Gain } G \quad \textbf{Phase angle } \alpha \qquad \frac{1}{i\omega - a} = \frac{1}{r}\,e^{-i\alpha} = \frac{1}{\sqrt{\omega^2 + a^2}}\,e^{-i\alpha} = Ge^{-i\alpha}. \qquad (8)$$

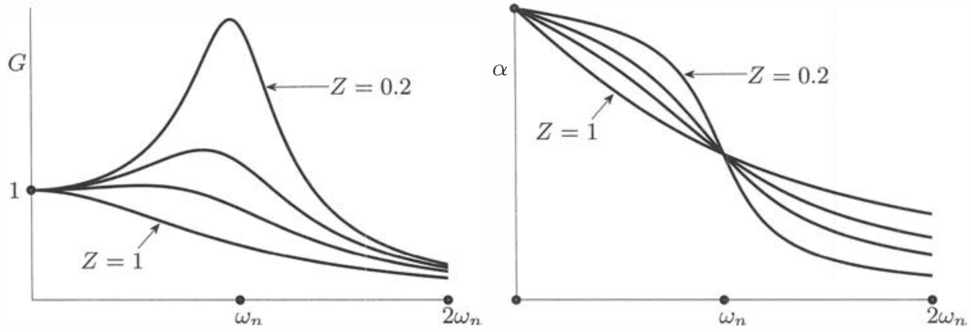


Figure 1.7: Dimensionless gain G and phase angle ϕ as functions of frequency ω.

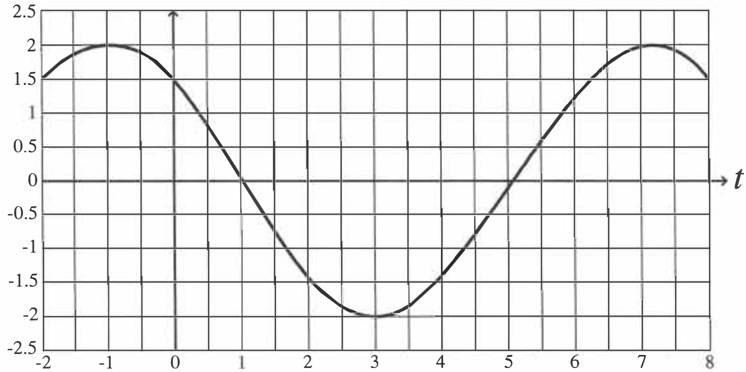
The gain $G(\omega)$ and the angle $\alpha(\omega)$ are often graphed. The graphs below are variations of "*Bode plots*." The amplitude response $G(\omega)$ is especially important, and you are very likely to see that gain G by itself—often including an extra factor $|a|$.

Note One common variation is to include the rate constant a in the forcing term $q(t) = a\,R\,e^{i\omega t}$. We still think of $R\,e^{i\omega t}$ as the input, then a gives q the right physical units. That factor a will appear in the output. So the gain $G = |\text{output}|\,/\,|\text{input}|$ will be increased by that factor $|a|$. *Then* $G = |a|/\sqrt{\omega^2 + a^2}$ is 1 *at the frequency* $\omega = 0$.

Sinusoids $R\,\cos(\omega t - \phi)$

The next page will show that any combination of $\cos\omega t$ and $\sin\omega t$ is a *shifted cosine*. It has frequency ω and amplitude R and phase lag ϕ. If you know ω and R and ϕ, it is no problem to graph $y(t) = \boldsymbol{R}\ \mathbf{cos}(\boldsymbol{\omega t} - \boldsymbol{\phi})$. To go the other way, and *read off those three numbers from the graph*, is much more interesting.

This mystery sinusoid came from lecture notes for MIT's course 18.03. The website **mathlets.org** has interactive experiments. The question here is : **Find $\boldsymbol{\omega}$, $\boldsymbol{R}$, and $\boldsymbol{\phi}$.**



The Sinusoidal Identity

We want to choose the magnitude R and the angle ϕ so that $A\cos\omega t + B\sin\omega t$ is the *real part* of $Re^{i(\omega t-\phi)}$. We can and will solve $y' - ay = Re^{i(\omega t-\phi)}$ quickly. When we take the real part of all terms in this differential equation, the correct input $q(t) = R\cos(\omega t - \phi)$ will appear on the right side and the correct output $y(t)$ will appear on the left side. The real equation will be solved in one step.

So we want this identity for the "sinusoidal" input $q(t)$:

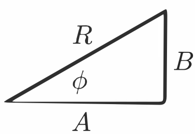
Sinusoidal identity $\quad \boldsymbol{A\cos\omega t + B\sin\omega t = R\cos(\omega t - \phi)}$ (9)

The right side has the same period $2\pi/\omega$ as the left side—and only one term.

To find R and ϕ, expand $R\cos(\omega t - \phi)$ into $R\cos\omega t\cos\phi + R\sin\omega t\sin\phi$. Then match cosines to find A and match sines to find B:

$$A = R\cos\phi \quad\text{and}\quad B = R\sin\phi \qquad \boxed{\boldsymbol{A^2 + B^2 = R^2} \quad\text{and}\quad \boldsymbol{\tan\phi = \frac{B}{A}}.} \tag{10}$$

So we know $R = \sqrt{A^2+B^2}$ and $\phi = \tan^{-1}(B/A)$ in the sinusoidal identity. The beauty of R and ϕ is that they match sinusoids to the polar form of complex numbers.



$A + iB$	$= Re^{i\phi}$	polar form of $A + iB$
R	$= \sqrt{A^2+B^2}$	produces R and ϕ in the
$\tan\phi$	$= B/A$	sinusoidal identity (9)

For practice with this important formula, Problem 1 will develop a slightly different proof.

Example 2 Write $q(t) = \cos 3t + \sin 3t$ as $R\cos(3t - \phi)$: the real part of $Re^{i(3t-\phi)}$.

Solution $A = 1$ and $B = 1$ so that $R = \sqrt{2}$. The angle $\phi = \frac{\pi}{4}$ has $\tan\phi = B/A = 1$. Then $\cos 3t + \sin 3t = \boldsymbol{\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right)}$.

Example 3 Write the real part of $e^{i5t}/(\sqrt{3} + i)$ in the form $A\cos 5t + B\sin 5t$.

Solution $\sqrt{3} + i$ is $2e^{i\pi/6}$ (why ?) Then $e^{i5t}/(\sqrt{3}+i)$ is $\frac{1}{2}e^{i(5t-\pi/6)}$. Its real part is

$$\frac{1}{2}\cos\left(5t - \frac{\pi}{6}\right) = \frac{1}{2}\left(\cos 5t\cos\frac{\pi}{6} + \sin 5t\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4}\cos 5t + \frac{1}{4}\sin 5t.$$

Real Solution y from Complex Solution y_c

The sinusoidal identity solves $y' - ay = A\cos\omega t + B\sin\omega t$ in three steps :

1. This equation is the real part of the complex equation $y_c{}' - ay_c = Re^{i(\omega t-\phi)}$.
2. The complex solution is $y_c = Re^{i(\omega t-\phi)}/(i\omega - a) = \boldsymbol{RG\,e^{i(\omega t-\phi-\alpha)}}$.
3. The real part of that complex solution y_c is the desired real solution $y(t)$.

Those three steps are **1** (real to complex) **2** (solve complex) **3** (complex to real). This will succeed. The second step expresses $1/(i\omega - \alpha)$ as $Ge^{-i\alpha}$ to keep the polar form. The third step produces $y = M\cos\omega t + N\sin\omega t$ directly as $\boldsymbol{y = RG\cos(\omega t - \phi - \alpha)}$.

Example 4 Take those three steps real-complex-real to solve $\boldsymbol{y' - y = \cos t - \sin t}$.

We have to find R, ϕ, G, and α from the numbers $a = 1$, $\omega = 1$, $A = 1$, and $B = -1$. Notice that $\boldsymbol{RG = 1}$.

$$R = \sqrt{A^2 + B^2} = \boldsymbol{\sqrt{2}} \qquad \tan\phi = \frac{B}{A} = -1 \text{ and } \boldsymbol{\phi = -\frac{\pi}{4}} \qquad G = \frac{1}{\sqrt{\omega^2 + a^2}} = \boldsymbol{\frac{1}{\sqrt{2}}}$$

The angle for $i\omega - a = i - 1$ is $\boldsymbol{\alpha = \frac{3\pi}{4}}$. Its tangent is $-\frac{\omega}{a} = -1$.

1. The sinusoidal identity is $\cos t - \sin t = \sqrt{2}\cos(t - \phi) = \boldsymbol{\sqrt{2}\cos(t + \pi/4)}$.

2. $y_{\textbf{complex}} = \dfrac{\boldsymbol{\sqrt{2}\, e^{i(t+\pi/4)}}}{\boldsymbol{i - 1}}$. Here $\dfrac{1}{i\omega - a} = \dfrac{1}{i - 1} = Ge^{-i\alpha} = \dfrac{1}{\sqrt{2}}e^{-3\pi i/4}$.

3. $y_{\textbf{complex}} = \boldsymbol{RG\, e^{i(\omega t - \alpha - \phi)} = e^{i(t - \pi/2)}}$. Then $y_{\textbf{real}} = \boldsymbol{\cos\left(t - \frac{\pi}{2}\right)} = \boldsymbol{\sin t}$.

That example was chosen so that $G = 1/\sqrt{2}$ cancelled $R = \sqrt{2}$. If we keep all the symbols R, ϕ, G, α then the solution $y_{\text{real}} = \boldsymbol{RG\ \cos(\omega t - \phi - \alpha)}$ from Step **3** must agree with the solution $y = M\cos\omega t + N\sin\omega t$ at the start of this section.

The key point in many applications is not necessarily the numbers in the formula for $y(t)$. Very often the goal is to see from the formula how $y(t)$ depends on parameters like a and ω in the differential equation. The gain $G = |\text{output}|/|\text{input}|$ is a convenient and very important guide.

The truth is that the complex solution is better. The sinusoidal identity shows how every combination $A\cos\omega t + B\sin\omega t$ is the real part $R\cos(\omega t - \phi)$ of a complex exponential $Re^{i(\omega t - \phi)}$. So we can convert real to complex and complex back to real.

In between, solve the complex form by using the ***frequency response*** $\boldsymbol{1/(i\omega - a)}$.

Conclusion When the input $q(t)$ is $Re^{i\omega t}$, the output $y(t)$ multiplies by $1/(i\omega - a)$. This multiplying factor is a complex number, and it changes with the frequency ω. We absolutely need to understand that number Y and graph its magnitude G and its phase.

■ REVIEW OF THE KEY IDEAS ■

1. (*Real*) $y' - ay = A\cos\omega t + B\sin\omega t$ leads to $y_{\textbf{real}} = M\cos\omega t + N\sin\omega t$.
2. (*Sinusoidal identity*) $A\cos\omega t + B\sin\omega t$ equals $R\cos(\omega t - \phi)$ with $R^2 = A^2 + B^2$.
3. (*Complex*) $y' - ay = Re^{i(\omega t - \phi)}$ leads to $y_{\textbf{complex}} = Re^{i(\omega t - \phi)}/(i\omega - a)$.
4. (*Complex gain*) $1/(i\omega - a) = \boldsymbol{Ge^{-i\alpha}}$ with $G = 1/\sqrt{\omega^2 + a^2}$ and $\tan\alpha = -\omega/a$.
5. (*Real part of the complex solution*) $y_{\textbf{real}} = \text{Re}(y_{\textbf{complex}}) = \boldsymbol{RG\cos(\omega t - \alpha - \phi)}$.

Problem Set 1.5

Problems 1-6 are about the sinusoidal identity (9). It is stated again in Problem 1.

1 These steps lead again to the sinusoidal identity. This approach doesn't start with the usual formula $\cos(\omega t - \phi) = \cos \omega t \ \cos \phi + \sin \omega t \ \sin \phi$ from trigonometry. The identity says :

$$\textbf{If } \boldsymbol{A + iB = Re^{i\phi}} \textbf{ then } \boldsymbol{A \cos \omega t + B \sin \omega t = R \cos(\omega t - \phi)}.$$

Here are the four steps to find that real part of $Re^{i(\omega t - \phi)}$. Explain $A - iB$ in Step 3.

$$\boldsymbol{R \cos(\omega t - \phi)} = \text{Re}\left[R\, e^{i(\omega t - \phi)}\right] = \text{Re}\left[e^{i\omega t}(R\, e^{-i\phi})\right] = \ (\textit{what is } R\, e^{-i\phi}\,?)$$

$$= \text{Re}\left[(\cos \omega t + i \sin \omega t)\ (A - iB)\right] = \boldsymbol{A \cos \omega t + B \sin \omega t}.$$

2 To express $\sin 5t + \cos 5t$ as $R \cos(\omega t - \phi)$, what are R and ϕ?

3 To express $6 \cos 2t + 8 \sin 2t$ as $R \cos(2t - \phi)$, what are R and $\tan \phi$ and ϕ?

4 Integrate $\cos \omega t$ to find $(\sin \omega t)/\omega$ in this complex way.

(i) $dy_{\text{real}}/dt = \cos \omega t$ is the real part of $dy_{\text{complex}}/dt = e^{i\omega t}$.

(ii) Take the real part of the complex solution.

5 The sinusoidal identity for $A = 0$ and $B = -1$ says that $-\sin \omega t = R \cos(\omega t - \phi)$. Find R and ϕ.

6 Why is the sinusoidal identity useless for the source $q(t) = \cos t + \sin 2t$?

7 Write $2+3i$ as $re^{i\phi}$, so that $\frac{1}{2+3i} = \frac{1}{r}e^{-i\phi}$. Then write $y = e^{i\omega t}/(2+3i)$ in polar form. Then find the real and imaginary parts of y. And also find those real and imaginary parts directly from $(2-3i)e^{i\omega t}/(2-3i)(2+3i)$.

8 Write these functions $A \cos \omega t + B \sin \omega t$ in the form $R \cos(\omega t - \phi)$: Right triangle with sides A, B, R and angle ϕ.

1) $\cos 3t - \sin 3t$ 2) $\sqrt{3} \cos \pi t - \sin \pi t$ 3) $3 \cos(t - \phi) + 4 \sin(t - \phi)$

Problems 9-15 solve real equations using the real formula (3) for M and N.

9 Solve $dy/dt = 2y + 3 \cos t + 4 \sin t$ after recognizing a and ω. Null solutions Ce^{2t}.

10 Find a particular solution to $dy/dt = -y - \cos 2t$.

11 What equation $y' - ay = A \cos \omega t + B \sin \omega t$ is solved by $y = 3 \cos 2t + 4 \sin 2t$?

12 The particular solution to $y' = y + \cos t$ in Section 1.4 is $y_p = e^t \int e^{-s} \cos s \, ds$. Look this up or integrate by parts, from $s = 0$ to t. Compare this y_p to formula (3).

13 Find a solution $y = M \cos \omega t + N \sin \omega t$ to $y' - 4y = \cos 3t + \sin 3t$.

14 Find the solution to $y' - ay = A \cos \omega t + B \sin \omega t$ **starting from $y(0) = 0$**.

15 If $a = 0$ show that M and N in equation (3) still solve $y' = A \cos \omega t + B \sin \omega t$.

Problems 16-20 solve the complex equation $y' - ay = Re^{i(\omega t - \phi)}$.

16 Write down complex solutions $y_p = Ye^{i\omega t}$ to these three equations:

(a) $y' - 3y = 5e^{2it}$ (b) $y' = Re^{i(\omega t - \phi)}$ (c) $y' = 2y - e^{it}$

17 Find complex solutions $z_p = Ze^{i\omega t}$ to these complex equations:

(a) $z' + 4z = e^{8it}$ (b) $z' + 4iz = e^{8it}$ (c) $z' + 4iz = e^{8t}$

18 Start with the real equation $y' - ay = R \cos(\omega t - \phi)$. Change to the complex equation $z' - az = Re^{i(\omega t - \phi)}$. Solve for $z(t)$. Then take its real part $y_p = \text{Re } z$.

19 What is the initial value $y_p(0)$ of the particular solution y_p from Problem 18? If the desired initial value is $y(0)$, how much of the null solution $y_n = Ce^{at}$ would you add to y_p?

20 Find the real solution to $y' - 2y = \cos \omega t$ starting from $y(0) = 0$, in three steps: Solve the complex equation $z' - 2z = e^{i\omega t}$, take $y_p = \text{Re } z$, and add the null solution $y_n = Ce^{2t}$ with the right C.

Problems 21-27 solve real equations by making them complex. First a note on α.

Example 4 was $y' - y = \cos t - \sin t$, with growth rate $a = 1$ and frequency $\omega = 1$. The magnitude of $i\omega - a$ is $\sqrt{2}$ and the polar angle has $\tan \alpha = -\omega/a = -1$. Notice: *Both $\alpha = 3\pi/4$ and $\alpha = -\pi/4$ have that tangent*! How to choose the correct angle α?

The complex number $i\omega - a = i - 1$ is in the *second quadrant*. Its angle is $\alpha = 3\pi/4$. **We had to look at the actual number and not just the tangent of its angle**.

21 Find r and α to write each $i\omega - a$ as $re^{i\alpha}$. Then write $1/re^{i\alpha}$ as $Ge^{-i\alpha}$.

(a) $\sqrt{3}\,i + 1$ (b) $\sqrt{3}\,i - 1$ (c) $i - \sqrt{3}$

22 Use G and α from Problem 21 to solve (a)-(b)-(c). Then take the real part of each equation and the real part of each solution.

(a) $y' + y = e^{i\sqrt{3}t}$ (b) $y' - y = e^{i\sqrt{3}t}$ (c) $y' - \sqrt{3}y = e^{it}$

23 Solve $y' - y = \cos \omega t + \sin \omega t$ in three steps: real to complex, solve complex, take real part. This is an important example.

(1) Find R and ϕ in the sinusoidal identity to write $\cos \omega t + \sin \omega t$ as the real part of $Re^{i(\omega t - \phi)}$.

(2) Solve $y' - y = e^{i\omega t}$ by $y = Ge^{-i\alpha}e^{i\omega t}$. Multiply by $Re^{-i\phi}$ to solve $z' - z = Re^{i(\omega t - \phi)}$.

(3) Take the real part $y(t) = \text{Re}\, z(t)$. Check that $y' - y = \cos \omega t + \sin \omega t$.

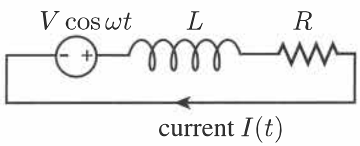
24 Solve $y' - \sqrt{3}y = \cos t + \sin t$ by the same three steps with $a = \sqrt{3}$ and $\omega = 1$.

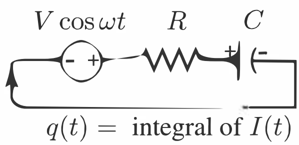
25 (**Challenge**) Solve $y' - ay = A \cos \omega t + B \sin \omega t$ in two ways. First, find R and ϕ on the right and G and α on the left. Show that the final real solution $RG \cos(\omega t - \phi - \alpha)$ agrees with $M \cos \omega t + N \sin \omega t$ in equation (2).

26 We don't have resonance for $y' - ay = Re^{i\omega t}$ when a and $\omega \neq 0$ are real. *Why not*? (Resonance appears when $y_n = Ce^{at}$ and $y_p = Ye^{ct}$ share the exponent $a = c$.)

27 If you took the imaginary part $y = \text{Im}\, z$ of the complex solution to $z' - az = Re^{i(\omega t - \phi)}$, what equation would $y(t)$ solve? Answer first with $\phi = 0$.

Problems 28-31 solve first order circuit equations: not RLC but RL and RC.





28 Solve $\boldsymbol{L\, dI/dt + RI(t) = V \cos \omega t}$ for the current $I(t) = I_n + I_p$ in the RL loop.

29 With $L = 0$ and $\omega = 0$, that equation is Ohm's Law $V = IR$ for direct current. The **complex impedance** $Z = R + i\omega L$ replaces R when $L \neq 0$ and $I(t) = Ie^{i\omega t}$.

$$L\, dI/dt + RI(t) = (\boldsymbol{i\omega L + R})\boldsymbol{I}e^{i\omega t} = \boldsymbol{V}e^{i\omega t} \quad \text{gives} \quad \boldsymbol{Z\, I = V}.$$

What is the magnitude $|Z| = |R + i\omega L|$? What is the phase angle in $Z = |Z|e^{i\theta}$? Is the current $|I|$ larger or smaller because of L?

30 Solve $\boldsymbol{R\dfrac{dq}{dt} + \dfrac{1}{C}q(t) = V \cos \omega t}$ for the charge $q(t) = q_n + q_p$ in the RC loop.

31 Why is the complex impedance now $Z = R + \frac{1}{i\omega C}$? Find its magnitude $|Z|$. **Note that mathematics prefers $\boldsymbol{i = \sqrt{-1}}$, we are not conceding yet to $\boldsymbol{j = \sqrt{-1}}$!**

1.6 Models of Growth and Decay

This is an important section. It combines formulas with their applications. The formulas solve the key linear equation $y' - a(t)y = q(t)$—we are very close to the solution. Now a can vary with t. The final step is to see the *purpose* of those formulas.

The point of this subject and this course is to understand change. **Calculus is about change**. A differential equation is a model of change. It connects dy/dt to the current value of y and to inputs/outputs that produce change. We see this as a math equation and solve it by a formula. If we stop there, we miss the whole reason for differential equations.

I will select five models of growth or decay, and five equations to describe them. Often the hardest part is to get the right equation. (Definitely harder than the right solution formula.) This section presents both steps of applied mathematics:

1. From the **model** to the **equation** 2. From the **equation** to the **solution**.

Our plan is to take the second step (the easier step) first: *Solve the equation*. Find the output $y(t)$ from inputs $a(t)$ and $q(t)$ and $y(0)$. Then come the models.

Here is the differential equation for $y(t)$. We want a formula to solve it—and we want to understand where that formula comes from. The solution $y(t)$ must use the three inputs $a(t)$ and $q(t)$ and $y(0)$, because they define the problem. **Sometimes $a(t)$ changes with time**. This possibility was not allowed in Sections 1.4 and 1.5.

Differential equation $$\boxed{\frac{dy}{dt} = a(t)y + q(t)} \quad \text{starting from } y(0) \text{ at } t = 0. \qquad (1)$$

Up to now, our models had limited options for those inputs (and a was constant):

Growth rate $a(t)$ The classic exponential $y(t) = e^t$ had $\mathbf{a = 1}$

Source term $q(t)$ Sections 1.4 and 1.5 had five particular inputs like $\mathbf{e^{ct}}$ and $\mathbf{e^{i\omega t}}$

Initial value $y(0)$ The starting value for $y(t) = e^t$ was $\mathbf{y(0) = 1}$

The "initial value" $y(0)$ is like a deposit to open a bank account. The source or sink $q(t)$ comes from **saving or spending** as time goes on. The solution $y(t)$ is the balance in the account at time t. I will reveal the final formula now, so you know where we are going.

$$\boxed{\begin{array}{ll}\textbf{Growth factor } \boldsymbol{G(s,t)} & \\ \textbf{from time } \boldsymbol{s} \textbf{ to time } \boldsymbol{t} & \boldsymbol{y(t) = G(0,t)\,y(0) + \int_0^t G(s,t)\,q(s)\,ds.}\end{array}} \qquad (2)$$

Formula (2) has two parts. The first part $y_n = G(0,t)y(0)$ has $q = 0$: no source. The second part y_p introduces the source $q(t)$, which adds fresh growth G times q (or subtracts when $q(t)$ is negative). *Go forward 2 pages to see the factor $G(s,t)$.*

$$y = (\text{ Null solution with } q = 0) + (\text{Particular solution from the input } q).$$

Particular Solution from $q(t)$

On this page a is constant. The particular solution $y_p(t)$ is so important that we will reach it in three ways. Of course those three approaches will be closely related— but they are different enough and valuable enough to be presented separately:

1. Integrating factor 2. Variation of parameters 3. Combine all outputs.

1. The *integrating factor* $M(t) = e^{-at}$ was seen in Section 1.4. It solves $M' = -aM$. For constant growth rate a, multiplying the equation $y' - ay = q(t)$ by $M = e^{-at}$ turns the left side into an exact derivative of My:

$$\frac{d}{dt}(e^{-at}y) = e^{-at}(y' - ay) = e^{-at}q(t). \tag{3}$$

Then we integrate the left and right hand sides to find $y = y_p(t)$ with $y_p(0) = 0$:

$$e^{-at}\,y(t) = \int_0^t e^{-as}\,q(s)\,ds \quad \text{and} \quad y(t) = \int_0^t \boldsymbol{e^{a(t-s)}\,q(s)\,ds}. \tag{4}$$

2. *Variation of parameters* starts with the solutions $y_n = Ce^{at}$ to the null equation $y' - ay = 0$. **The new idea is to let C vary with time in the particular solution.** Substitute $y = C(t)e^{at}$ into the equation $y' - ay = q(t)$ to find $C'e^{at} = q(t)$:

$$(Ce^{at})' - a\,Ce^{at} = C'e^{at} + a\,Ce^{at} - a\,Ce^{at} = \boldsymbol{C'e^{at} = q(t)}. \tag{5}$$

Then $C' = e^{-at}q(t)$. Integrate to find C and the solution formula we want:

$$C(t) = \int_0^t e^{-as}\,q(s)\,ds \qquad y(t) = C(t)e^{at} = \int_0^t \boldsymbol{e^{a(t-s)}\,q(s)\,ds}. \tag{6}$$

The integrating factor M changes the equation. Varying $C(t)$ changes the solution. $C(t)$ will stay important for *systems* of n equations; integrating factors lose out.

3. Each input $q(s)$ grows to $e^{a(t-s)}\,q(s)$ in the time between s and t. Then the solution $y(t)$ comes from these inputs $q(t)$ and growth factor $G = e^{a(t-s)}$. Add up (integrate) all those outputs:

$$\textbf{Growing time for } \boldsymbol{q(s)} \textbf{ is } \boldsymbol{t-s} \qquad \textbf{Output } \boldsymbol{y(t) = \int_0^t e^{a(t-s)}\,q(s)\,ds}. \tag{7}$$

To me, this third approach captures the meaning of the formulas (4) = (6) = (7). I like to think of each input $q(s)$ growing by the factor $G(s,t) = e^{a(t-s)}$ in the time $t - s$.

Changing Growth Rate $a(t)$

The next step is to let $a(t)$ change in time. For example $a(t)$ could be $1+\cos t$, varying between 2 and 0. Certainly interest rates do change. The growth rate a of your bank balance often slows down or speeds up. **Then the growth factor $G(0,t)$ is not just e^{at}.**

The null solution to $y_n' = a(t)y_n$ shows this clearly—the growth from time 0 to time t:

Integrate a from 0 to t
Take the exponential
$$\boxed{G(0,t) = e^{\int_0^t a(s)\,ds}} \qquad y_n(t) = G(0,t)\;y(0). \tag{8}$$

The key point is that $dG/dt = a(t)\,G$. First, the derivative of the integral of $a(t)$ is $a(t)$—by the Fundamental Theorem of Calculus. Second, the chain rule produces the derivative of G, when that integral goes into the exponent. Here is dG/dt:

$$\frac{d}{dt}\left(e^{\text{integral of } a}\right) = \left(e^{\text{integral of } a}\right)\frac{d}{dt}(\text{integral of } a) \qquad \frac{dG}{dt} = (G)(a(t)) \tag{9}$$

When a is constant, that integral is just at. This leads to the usual growth $G = e^{at}$. When a varies, the exponent is messier than at but the idea is the same: $\boldsymbol{dG/dt = aG}$.

Our example is $a(t) = 1+\cos t$. The integral of $a(t)$ is $t+\sin t$. This is the exponent:

Growth factor $G(0,t) = e^{t+\sin t}$ **Null solution** $y_n(t) = e^{t+\sin t}y(0)$

Now we tackle the particular solution that comes from the inputs $q(t)$ when they grow. Again this $y_p(t)$ can come from an *integrating factor* or *variation of parameters* or an *integral of all outputs from all inputs.*

1. The integrating factor is $M(t) = 1/G(t) = e^{-\int_0^t a(s)\,ds}$. This has $M' = -a(t)M$.

Then the derivative of My is exactly Mq, when we use $M' = -aM$.

Product rule
Chain rule
$$\frac{d}{dt}(My) = My' + M'y = M(y' - a(t)y) = Mq(t). \tag{10}$$

Integrate both sides of $(My)' = Mq$ starting from $y_p(0) = 0$. Then divide by M:

$$M(t)y_p(t) = \int_0^t M(s)\,q(s)\,ds \qquad y_p(t) = e^{\int_0^t a(s)\,ds}\int_0^t e^{-\int_0^s a(s)\,ds}\,q(s)\,ds \tag{11}$$

When you multiply those exponentials, the exponents combine. The integral from 0 to t, minus the integral from 0 to s, equals the integral from s to t. Each $q(s)$ enters at s. **The exponential of the integral of a from s to t is the growth factor $G(s,t)$:**

$$\textbf{Growth factor } G(s,t) = e^{\int_s^t a(T)\,dT} \qquad \textbf{Solution } y_p(t) = \int_0^t G(s,t)\,q(s)\,ds \tag{12}$$

2. **Variation of parameters** . I will save this method to use in Chapter 2 for second order equations (with y''). Then all three methods get an equal chance—variation of parameters can solve equations that go beyond $y' = a(t)y + q(t)$.

3. **Integral of outputs** (my own choice). The input $q(s)$ enters at time s. It grows or decays until time t. The growth factor multiplying q over that time is $G(s,t)$. Since $a(t)$ changes, the growth factor needs the integral of a. **The inputs are $q(s)$, the outputs are $G(s,t)\,q(s)$, and the total output $y_p(t)$ agrees with (12):**

$$\boxed{\boldsymbol{G(s,t) = e^{\int_s^t a(T)\,dT}} \qquad \boldsymbol{y_p(t) = \int_0^t G(s,t)\,q(s)\,ds}} \tag{13}$$

When q is a delta function at time s (an impulse), the response is $\boldsymbol{y_p = G(s,t)}$ at time t.

Example 1 The growth rate $a(t) = 2t$ puts the economy into serious inflation. The integral of $a(t)$ is $\int_s^t 2T\,dT = t^2 - s^2$. Then G is the growth from s to t:

$$\boldsymbol{G(s,t) = e^{t^2-s^2}} \qquad y' = 2t\,y + q(t) \text{ has } y_p(t) = \int_0^t e^{t^2-s^2}\,q(s)\,ds.$$

Example 2 Here is an interesting case for investors. **Suppose the interest rate a goes to zero**. What happens to the solution formula? The first term y_n becomes $y(0)$. This deposit doesn't grow or disappear, it stays fixed. The growth factor is $G = 1$ and we just add up all the inputs (they didn't grow):

$$\boldsymbol{a = 0} \qquad y' = q(t) \text{ has the particular solution } y_p(t) = \int_0^t q(s)\,ds.$$

The problem comes when we start with the formula to solve $y' = ay + q$ (*constant* q):

$$y(t) = e^{at}y(0) + \int_0^t e^{a(t-s)}q\,ds = e^{at}y(0) + q\frac{e^{at}-1}{a}.$$

That looks bad at $a = 0$ because of dividing by a. But the factor $e^{at} - 1$ is also zero. *This is a case for l'Hôpital's Rule. Wonderful!* We can make sense of $0/0$:

$$\lim_{a\to 0} \frac{e^{at}-1}{a} = \frac{\text{Derivative with respect to } a}{\text{Derivative with respect to } a} = \frac{t}{1} = \boldsymbol{t}.$$

The particular solution from $y' = q$ reduces to q times t. That is the total savings during the time from 0 to t. With $a = 0$ it doesn't grow. Like putting money under a mattress, $a = 0$ means no risk and no gain. Then $dy/dt = q$ has $y(t) = y(0) + qt$.

Now the solution formula can be applied to real problems.

Models of Growth and Decay

The whole point of a differential equation is to give a mathematical model of a practical problem. It is my duty to show you examples. This section will offer growth equations $(a > 0)$, decay equations $(a < 0)$, and the balance equation that controls the temperature of the Earth. That balance equation is not linear.

Please understand that a linear equation is only an approximation to reality. The approximation can be very good over an important range of values. Newton's Law $F = ma$ is linear and we live by it every day. But Einstein showed that the mass m is not a constant, it increases with the velocity. We don't notice this until we are near the speed of light.

Similarly the stretch in a spring is proportional to the force—for a while. A really large force will stretch the spring way out of shape. That takes us to nonlinear elasticity. Eventually the spring breaks.

The same for analysis of a car crash. Linear at very slow speed, nonlinear at normal speeds, total wreck at high speeds. A crash is a very difficult problem in computational mechanics. So is the effect of dropping a cell phone. This has been studied in great detail.

Back to linear equations, starting with constant a and $y(0)$ and q.

Model 1 $\quad$ $\boldsymbol{y(t)}$ **= money in a savings account**

This is the example we already started. We have a formula for the answer, now we use it. That formula is based on a *continuous* savings rate $q(t)$ (deposits every instant, not every month). It also has *continuous* interest ay (computed every instant, not every month or every year). Continuous compounding does not bring instant riches. Just a little more income, by computing interest day and night.

Suppose we get 3% interest. This number is $a = .03$, but what are the "units" of a? The rate is 3% **per year**. There is a time dimension. If we change to months, the same rate is now $a = \frac{3}{12}\% = .0025$ **per month**.

Units of $\boldsymbol{a}$ are $\dfrac{1}{\text{time}}$ $\quad$ To change from years to months, divide a by 12.

You can see this in the equation $dy/dt = ay$. Both sides have y. So a on the right agrees dimensionally with $1/t$ on the left. Frequency is also $1/$time; $i\omega - a$ is good!

The savings rate q has the same dimension as ay. The dimension of q is **money / time**. We see that in the words too: $q = 100$ **dollars per month**.

Question: Does $\boldsymbol{y(t)}$ grow or decay? $\quad$ This depends on $y(0)$ and a and q.

So far a and q have been positive; we were saving. If we spend money constantly, then q changes to *negative*. Interest is still entering because a is positive. Does q win or does a win? Do we spend all our deposit and drop to $y = 0$, or does the interest $ay(t)$ allow us to keep up the spending level q forever?

Answer: If we start with $\boldsymbol{ay(0) + q > 0}$, then $\boldsymbol{y(t)}$ will grow even if $\boldsymbol{q < 0}$.

The reason is in the differential equation $dy/dt = ay(t) + q$. If the right side is positive at time $t = 0$, then y starts growing. So the right side stays positive, and y keeps growing.

Common sense gives the same answer: If $ay + q > 0$, the interest ay coming in stays ahead of the spending going out.

A question for you. Suppose $a < 0$ but $q > 0$. Your investment is going down at rate a. You are adding new investments at rate q. Overall, does your account go up or down?

You won't actually hit zero, because e^{at} stays positive forever, even if $a < 0$. You approach the steady state $y_\infty = -q/a$. In reality, the end of prosperity has come.

Now I will compare continuous compounding (expressed by a differential equation) with ordinary compounding (a difference equation). The difference equation starts with the same $Y_0 = y(0)$. This changes to Y_1 and then Y_2 and Y_3, taking a finite step each year. When the time step Δt is one year, the interest rate is A **per year** and the saving rate is Q **dollars per year**:

$$\frac{dy}{dt} = ay + q \qquad \text{changes to} \qquad \frac{Y_{n+1} - Y_n}{\Delta t} = AY_n + Q \tag{14}$$

We don't need calculus for difference equations. The derivative enters when the time step Δt approaches zero. The model looks simpler if I multiply equation (14) by Δt:

$$\textbf{One step, n to n + 1} \qquad Y_{n+1} = (1 + A\,\Delta t)Y_n + Q\,\Delta t \tag{15}$$

At the end of year n, the bank adds interest $A\Delta t Y_n$ to the balance Y_n you already have. You also put in new savings (or you spend if $Q < 0$). The new year starts with Y_{n+1}.

In case $A\,\Delta t = at/N$ and $Q = 0$, we are back to $Y_{n+1} = (1 + at/N)Y_n$:

$$\textbf{\textit{N} steps from 0 to \textit{N}} \qquad Y_N = \left(1 + \frac{at}{N}\right)^N Y_0 \to e^{at}y(0) \quad \text{as } N \to \infty.$$

Model 2 Radioactive Decay

The next models will deal with decay. The growth rate a is *negative*. The solution y is decreasing. Decay is an expected and natural result when $a < 0$. In fact the differential equation is called **stable** when all solutions approach zero. In many applications this is highly desired.

Exponential growth with $a > 0$ may be good for bank accounts, but not for a drug in our bloodstream. Here are examples where any starting amount $y(0)$ decays exponentially:

A radioactive isotope like Carbon 14
Newton's Law of Cooling
The concentration of a drug in our bloodstream

I will emphasize the **half-life**—the time for half of the Carbon 14 to decay, or half of the drug to disappear. This is decided by the decay rate $a < 0$ in the equation $y' = ay$.

The half-life H is the opposite of the **doubling time** D, when $a > 0$ and $e^{aD} = 2$.

Half-life and Doubling Time

How long does it take for $y(t)$ to be reduced to half of $y(0)$? The equation $y' = ay$ has the solution $e^{at}y(0)$, and we know that $a < 0$.

$$\textbf{Half-life } \boldsymbol{H} \qquad e^{aH} = \frac{1}{2} \qquad aH = \ln\frac{1}{2} = -\ln 2 \qquad \boldsymbol{H = \frac{-\ln 2}{a}}$$

That answer H is positive because $a < 0$. For Carbon 14 the half-life H is 5730 years.

It has just taken 150 hours on a Cray XT5 supercomputer to find 8 eigenvalues of a matrix of size 1 billion—to explain that long half-life. Other carbon isotopes have $H = 20$ minutes. Going in reverse, H tells us the decay rate:

$$\textbf{Decay rate } \boldsymbol{a} \qquad a = \frac{-\ln 2}{5730} \approx 1.216 \times 10^{-4} \text{ per year.}$$

The "quarter-life" would be $2H$, twice as long as the half-life. The time to divide by e is

$$\textbf{Relaxation time } \boldsymbol{\tau} \qquad e^{a\tau} = e^{-1} \approx 0.368 \qquad a\tau = -1 \qquad \tau = \boldsymbol{\frac{-1}{a}}$$

Question. Suppose we find a sample where 60 % of the Carbon 14 remains. *How old is the sample*? If the carbon came from a tree, its decay started at the moment when the tree died.

Answer. The age T is the time when $e^{aT} = 0.6$. At that time

$$aT = \ln(0.6) \qquad T = \frac{-0.51}{a} = 4200 \text{ years.}$$

The doubling time D uses the same ideas but now the growth rate is $\boldsymbol{a > 0}$:

$$\textbf{Doubling time} \qquad e^{aD} = 2 \qquad aD = \ln 2 \qquad \boldsymbol{D = \frac{\ln 2}{a}}$$

At 5% interest ($a = .05$/year) the doubling time is less than 14 years. Not 20 years.

Model 3 Newton's Law of Cooling

When you put water in a freezer, it cools down. So does a cup of hot coffee on a table. The rate of cooling is proportional to the temperature difference.

$$\textbf{Newton's Law} \qquad \boldsymbol{\frac{dT}{dt} = k(T_\infty - T)} \qquad \boldsymbol{T_\infty =} \textbf{ surrounding temperature}$$

This is a linear constant coefficient equation. The solution approaches T_∞. Include that constant on the left side, to make the equation and the solution clear:

$$\frac{d(T - T_\infty)}{dt} = k(T_\infty - T) \qquad \boldsymbol{T - T_\infty = e^{-kt}(T - T_0)}$$

Question. Suppose the starting temperature difference $T_0 - T_\infty$ is $80°$. After 90 minutes the difference $T_1 - T_\infty$ has dropped to $20°$. At what time will the difference be $10°$? When will the temperature reach T_∞ ?

Answer. The starting difference $80°$ is divided by 4 in 90 minutes. To divide again by 2 takes 45 minutes from $20°$ to $10°$. There you see a fundamental rule for exponentials:

$$\text{If } e^{90k} = 1/4 \text{ then } e^{45k} = \sqrt{1/4} = 1/2. \text{ It is not necessary to know } k.$$

The temperature never reaches T_∞ exactly. The exponential e^{-kt} never reaches 0 exactly.

Model 4 Drug Elimination

The concentration $C(t)$ of a drug in the bloodstream drops at a rate proportional to $C(t)$ itself. Then $dC/dt = -kC$. The elimination constant $k > 0$ is carefully measured, and $C(t) = e^{-kt}C(0)$.

Suppose you want to maintain at least G grams in your body. If you are taking the drug every 8 hours, what dose should you take?

$$t = 8 \text{ hours} \qquad k = \text{ decay rate per hour} \qquad \text{Take } e^{8k}G \text{ grams.}$$

Model 5 Population growth

Certainly the world population is increasing. Its growth rate a is the birth rate minus the death rate. A reasonable estimate for a right now is 1.3% a year, or $a = .013/\text{year}$ (the dimension of a is $1/\text{time}$). A first model assumes this growth rate to be constant, continuing forever: Now we ask for the *doubling time*, a number that is independent of the starting value $y(0)$:

$$\textbf{Doubling time } \boldsymbol{D} \qquad e^{aD} = 2 \quad \text{or} \quad D = \frac{\ln 2}{.013} \text{ years} = 53 \text{ years.}$$

$$\textbf{World population} \qquad \frac{dy}{dt} = .013\,y \quad \text{and} \quad y(t) = e^{.013t}y(0).$$

The "forever" part is unrealistic. After 1000 years, it produces $e^{13}y(0)$. That number e^{13} is enormous. If we start today (so that $t = 0$ is the year we are living in) then eventually we will have about one atom each. Ridiculous. But it is quite possible that the pure growth equation $y' = ay$ does describe the real population for a short time.

Eventually the equation has to be corrected. We need a **nonlinear term** like $-by^2$, to model the effect of competition (y against y). As y gets large, y^2 gets much larger. Then $-by^2$ subtracts from dy/dt and eventually competition stops growth.

This is the famous **"logistic equation"** $\boldsymbol{dy/dt = ay - by^2}$. It is solved in Section 1.7. Here I want to end with a problem of scientific importance—the changing temperature of the Earth. The equations are nonlinear. The data is incomplete. There is no solution formula. This is the reality of science.

Energy Balance Equations

The Earth gets practically all its energy from the Sun. A lot of that energy goes back out into space. This is radiation in and radiation out. The energy that doesn't go back is responsible for changing the Earth's temperature T.

This energy balance is crucial to our lives. It won't permit life on Mercury (too hot), and certainly not on Pluto (too cold). We are extremely fortunate to live on Earth. The form of the temperature equation is completely typical of balance equations in applied mathematics:

Energy in minus energy out **This raises the temperature T**	$\boldsymbol{C\dfrac{dT}{dt} = E_{\text{in}} - E_{\text{out}}}$	(16)

There is a coefficient C in every equation like this. Let me show you another balance equation, to emphasize how the problem can change but the form stays the same.

Flow into a bathtub minus flow out **This raises the water height H**	$\boldsymbol{A\,\dfrac{dH}{dt} = F_{\text{in}} - F_{\text{out}}}$	(17)

The tap controls the incoming flow F_{in}. The drain controls the outgoing flow F_{out}. The volume of water changes according to $dV/dt = F_{\text{in}} - F_{\text{out}}$. That volume change $\boldsymbol{dV/dt}$ is a height change $\boldsymbol{dH/dt}$ multiplied by $\boldsymbol{A} =$ area of the water surface. Check units:

$$H = \textbf{meters} \quad A = (\textbf{meters})^2 \quad V = (\textbf{meters})^3 \quad t = \textbf{seconds} \quad F = (\textbf{meters})^3/\textbf{second}$$

I include this bathtub example because it makes the balance clear:

1. Flow rate in minus flow rate out equals fill rate dV/dt.

2. Volume change dV/dt splits into $(A)\,(dH/dt) =$ area times height change.

In a curved bathtub, the water area A changes with the height H. Then equation (17) is nonlinear. Every scientist looks immediately at the balance equation: Can it be linear? Can its coefficients be constant? The true answer is no, the practical answer is often yes. (Numerical methods are slowed by nonlinearity. Analytical methods are usually destroyed.)

Energy Balance for the Earth

The energy balance equation $CT' = E_{\text{in}} - E_{\text{out}}$ is the start. Temperature is in Kelvin (degrees Celsius are also used). The *heat capacity* C is the energy needed to raise the temperature by 1 degree (just as the area A was the volume of water that raises the height of water by 1 meter). That heat capacity C truly changes between ice and ocean and land. Exactly as predicted, the starting simplification is $C =$ *constant*.

On the right side of the equation, the energy E_{in} is coming from the Sun. A serious fraction α of the arriving energy bounces back and is never absorbed. This fraction α is the **albedo**. It can vary from .80 for snow to .08 for ocean. On a global scale, we have to simplify the albedo formula to a constant, and then improve it:

$$\text{Constant } \alpha = .30 \text{ for all } T \qquad \text{Piecewise linear } \alpha = \begin{cases} .60 & \text{if } \ T \le 255\,K \\ .20 & \text{if } \ T \ge 290\,K \end{cases}$$

The main point is that $\boldsymbol{E_{\text{in}} = (1-\alpha)Q}$, where Q measures energy flow from the Sun to a unit area of the Earth. Now we turn to $\boldsymbol{E_{\text{out}}}$.

Radiation of energy is theoretically proportional to T^4 (the Stefan-Boltzmann law). There is an ideal constant σ from quantum theory, but the Earth is not ideal. The "greenhouse effect" of particles in the atmosphere reduces σ by an emission factor close to $\epsilon = .62$. For a unit area, the radiation E_{out} is $\epsilon\,\sigma\,T^4$ and the radiation E_{in} is $(1-\alpha)Q$:

$$\textbf{Energy balance } \boldsymbol{E_{\text{in}} = E_{\text{out}}} \qquad \boldsymbol{(1-\alpha)Q = \epsilon\,\sigma\,T^4} \qquad \boldsymbol{T = \left(\frac{(1-\alpha)Q}{\epsilon\sigma}\right)^{1/4}}$$

You understand that these are not fixed laws like Einstein's $e = mc^2$. Satellites measure the actual radiation, sensors measure the actual temperature. That nonlinear T^4 formula is often replaced by a linear $A + BT$. This gives the most basic model of a steady state.

Multiple Steady States

I will take one more step with that model—we are on the edge of real science. You know that the albedo α (the bounceback of solar energy) depends on the temperature T. The coefficients A and B and ϵ also depend on T. The temperature balance equation $C\,dT/dt = E_{\text{in}} - E_{\text{out}}$ and the steady equilibrium equation $E_{\text{in}} = E_{\text{out}}$ are **not linear**. From a nonlinear model, what can we learn?

Point 1 $E_{\text{in}}(T) = E_{\text{out}}(T)$ can easily have **more than one solution** $\boldsymbol{T}$.

Point 2 Those steady states when $dT/dt = 0$ can be **stable or unstable**.

Point 3 You can see T_1 and T_3 (**stable**) and T_2 (**unstable**) in this graph of E_{in} and E_{out}.

Why is T_2 unstable? If T is just above T_2, then $E_{\text{in}} > E_{\text{out}}$. Therefore $dT/dt > 0$ and the temperature climbs further away from T_2. If T is just below T_2, then $E_{\text{in}} < E_{\text{out}}$. Therefore $dT/dt < 0$ and T falls further below T_2.

The next section 1.7 shows how to decide stability or instability for any equation $dT/dt = f(T)$ or $dy/dt = f(y)$. Just as here, each steady state has $f(T) = 0$. **Stable steady states also have $\boldsymbol{df/dT < 0}$ or $\boldsymbol{df/dy < 0}$.** Simple and important.

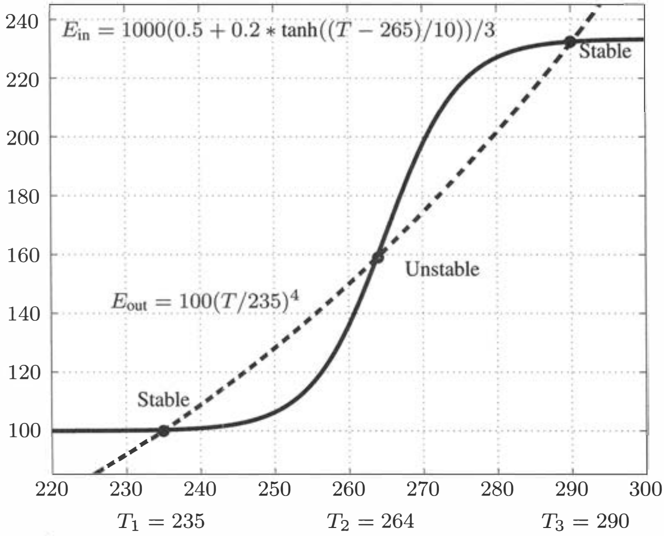


Figure 1.8: The analysis and the graph are from *Mathematics and Climate* by Hans Kaper and Hans Engler (SIAM, 2013). $E_{\text{in}} - E_{\text{out}}$ has slope < 0 at two stable steady states.

Problem Set 1.6

1 Solve the equation $dy/dt = y + 1$ up to time t, starting from $y(0) = 4$.

2 You have \$1000 to invest at rate $a = 1 = 100\,\%$. Compare after one year the result of depositing $y(0) = 1000$ immediately with $q = 0$, or choosing $y(0) = 0$ and $q = 1000/\text{year}$ to deposit continually during the year. In both cases $dy/dt = y + q$.

3 If $dy/dt = y - 1$, when does your original deposit $y(0) = \frac{1}{2}$ drop to zero?

4 Solve $\dfrac{dy}{dt} = y + t^2$ from $y(0) = 1$ with increasing source term t^2.

5 Solve $\dfrac{dy}{dt} = y + e^t$ (resonance $a = c\,!$) from $y(0) = 1$ with exponential source e^t.

6 Solve $\dfrac{dy}{dt} = y - t^2$ from an initial deposit $y(0) = 1$. The spending $q(t) = -t^2$ is growing. When (if ever) does $y(t)$ drop to zero ?

7 Solve $\dfrac{dy}{dt} = y - e^t$ from an initial deposit $y(0) = 1$. This spending term $-e^t$ grows at the same e^t rate as the initial deposit. When (if ever) does y drop to zero ?

8 Solve $\dfrac{dy}{dt} = y - e^{2t}$ from $y(0) = 1$. At what time T is $y(T) = 0$?

9 Which solution (y or Y) is eventually larger if $y(0) = 0$ and $Y(0) = 0$?

$$\frac{dy}{dt} = y + 2t \quad \text{or} \quad \frac{dY}{dt} = 2Y + t.$$

10 Compare the linear equation $y' = y$ to the separable equation $y' = y^2$ starting from $y(0) = 1$. Which solution $y(t)$ must grow faster ? It grows so fast that it blows up to $y(T) = \infty$ at what time T ?

11 $Y' = 2Y$ has a larger growth factor (because $a = 2$) than $y' = y + q(t)$. What source $q(t)$ would be needed to keep $y(t) = Y(t)$ for all time ?

12 Starting from $y(0) = Y(0) = 1$, does $y(t)$ or $Y(t)$ eventually become larger ?

$$\frac{dy}{dt} = 2y + e^t \qquad \frac{dY}{dt} = Y + e^{2t}.$$

Questions 13-18 are about the growth factor $G(s,t)$ from time s to time t.

13 What is the factor $G(s,s)$ in zero time ? Find $G(s,\infty)$ if $a = -1$ and if $a = 1$.

14 Explain the important statement after equation (13): *The growth factor $G(s,t)$ is the solution to $y' = a(t)y + \delta(t-s)$.* The source $\delta(t-s)$ deposits $1 at time s.

15 Now explain this meaning of $G(s,t)$ when *t is less than s*. We go backwards in time. *For $t < s$, $G(s,t)$ is the value at time t that will grow to equal 1 at time s.*

When $t = 0$, $G(s,0)$ is the "present value" of a promise to pay $1 at time s. If the interest rate is $a = 0.1 = 10\,\%$ per year, what is the present value $G(s,0)$ of a million dollar inheritance promised in $s = 10$ years ?

16 (a) What is the growth factor $G(s,t)$ for the equation $y' = (\sin t)y + Q \sin t$?

(b) What is the null solution $y_n = G(0,t)$ to $y' = (\sin t)y$ when $y(0) = 1$?

(c) What is the particular solution $y_p = \int_0^t G(s,t)\, Q \sin s\, ds$?

17 (a) What is the growth factor $G(s,t)$ for the equation $y' = y/(t+1) + 10$?

(b) What is the null solution $y_n = G(0,t)$ to $y' = y/(t+1)$ with $y(0) = 1$?

(c) What is the particular solution $y_p = 10 \int_0^t G(s,t)\, ds$?

18 Why is $G(t,s) = 1/G(s,t)$? Why is $G(s,t) = G(s,S)G(S,t)$?

Problems 19–22 are about the "units" or "dimensions" in differential equations.

19 (recommended) If $dy/dt = ay + qe^{i\omega t}$, with t in seconds and y in meters, what are the units for a and q and ω ?

20 The logistic equation $dy/dt = ay - by^2$ often measures the time t in years (and y counts people). What are the units of a and b ?

21 Newton's Law is $m\,d^2y/dt^2 + ky = F$. If the mass m is in grams, y is in meters, and t is in seconds, what are the units of the stiffness k and the force F ?

22 Why is our favorite example $y' = y + 1$ very unsatisfactory dimensionally ? Solve it anyway starting from $y(0) = -1$ and from $y(0) = 0$.

23 The difference equation $Y_{n+1} = cY_n + Q_n$ produces $Y_1 = cY_0 + Q_0$. Show that the next step produces $Y_2 = c^2Y_0 + cQ_0 + Q_1$. After N steps, the solution formula for Y_N is like the solution formula for $y' = ay + q(t)$. Exponentials of a change to powers of c, the null solution $e^{at}y(0)$ becomes $c^N Y_0$. The particular solution

$$Y_N = c^{N-1}Q_0 + \cdots + Q_{N-1} \quad \text{is like} \quad y(t) = \int_0^t e^{a(t-s)}q(s)ds.$$

24 Suppose a fungus doubles in size every day, and it weighs a pound after 10 days. If another fungus was twice as large at the start, would it weigh a pound in 5 days ?

1.7 The Logistic Equation

This section presents one particular nonlinear differential equation—the *logistic equation*. It is a model of growth *slowed down by competition*. In later chapters, one group y_1 will compete against another group y_2. Here the competition is inside one group. The growth comes from ay as usual. The competition (y against y) comes from $-by^2$.

Logistic equation / nonlinear $$\boldsymbol{\frac{dy}{dt} = ay - by^2} \tag{1}$$

We will discuss the meaning of this equation, and its solution $y(t)$.

One key idea comes right away: the **steady state**. Any time we have $dy/dt = f(y)$, it is important to know when $f(y)$ is zero. Growth stops at that point because dy/dt is zero. If the number Y solves $f(Y) = 0$, the constant function $y(t) = Y$ solves the equation $dy/dt = f(y)$: both sides are zero. For the special starting value $y(0) = Y$, the solution would stay at Y. It is a steady solution, not changing with time.

The logistic equation has two steady states with $\boldsymbol{f(Y) = 0}$:

$$\frac{dy}{dt} = ay - by^2 = 0 \text{ when } aY = bY^2. \text{ Then } \boldsymbol{Y = 0} \text{ or } \boldsymbol{Y = a/b}. \tag{2}$$

That point a/b is where competition balances growth. It is the top of the "S-curve" in Figure 1.9, where the curve goes flat. It is the end of growth. The solution $y(t)$ cannot get past the value a/b. At the start of the S-curve, the other steady state $Y = 0$ is **unstable**. The curve goes *away* from $Y = 0$ and *toward* $Y = a/b$.

In some applications, this number a/b is the **carrying capacity** (K) of the system. If $a/b = K$ then $b = a/K$. So the logistic equation can be written in terms of a and K:

$$\boldsymbol{\frac{dy}{dt}} = ay - by^2 = ay - \frac{a}{K}y^2 = \boldsymbol{ay\left(1 - \frac{y}{K}\right)}. \tag{3}$$

Mathematically, we have done nothing interesting. But the number K may be easier to work with than b. We might have an estimate like $K = 12$ billion people for the maximum population that the world can deal with. Rewriting the equation doesn't change the solution, but it can help our understanding.

Solution of the Logistic Equation

What is $y(t)$? The logistic equation is nonlinear because of y^2, and most nonlinear equations have no solution formula. ($y = Ce^{at}$ is extremely unlikely.) But the particular equation $dy/dt = ay - by^2$ can be solved, and I want to present two ways to do it:

1 (by magic) The equation for $z = 1/y$ happens to be linear: $\boldsymbol{dz/dt = -az + b}$. We can solve that equation and then we know y.

2 (by partial fractions) This systematic approach takes longer. In principle, partial fractions can be used any time dy/dt is a ratio of polynomials in y.

You will appreciate method **1** (only two steps A and B) after you see method **2**.

(A) If $z = \dfrac{1}{y}$, the chain rule gives $\dfrac{dz}{dt} = \dfrac{-1}{y^2}\dfrac{dy}{dt}$. Substitute $ay - by^2$ for $\dfrac{dy}{dt}$:

$$\frac{\boldsymbol{dz}}{\boldsymbol{dt}} = \frac{1}{y^2}\left(-\,ay + by^2\right) = -\frac{a}{y} + b = -\,\boldsymbol{a\,z + b}. \tag{4}$$

(B) This is the linear equation $z' + az = b$ that was solved in the previous sections. Change a to $-\,a$ in the solution formula. Change y and q to z and b:

$$\textbf{Solution} \qquad z(t) = e^{-at}z(0) - \frac{b}{a}\left(e^{-at} - 1\right) = \frac{de^{-at} + b}{a} \tag{5}$$

The number d collects all the constants $a, y(0), b$ in one place:

$$\frac{d}{a} = z(0) - \frac{b}{a} \quad \text{and} \quad z(0) = \frac{1}{y(0)} \quad \text{produce} \quad \boldsymbol{d = \frac{a}{y(0)} - b}. \tag{6}$$

Now turn equation (5) upside down to find $y = 1/z$:

$$\textbf{Solution to the logistic equation} \qquad \boxed{\boldsymbol{y(t) = \frac{a}{de^{-at} + b}}} \tag{7}$$

This is a beautiful solution. Look at its value for large positive t and large negative t:

Approaching $t = +\infty$	$e^{-at} \to 0$	and	$\boldsymbol{y(t) \to \dfrac{a}{b}}$
Approaching $t = -\infty$	$e^{-at} \to \infty$	and	$\boldsymbol{y(t) \to 0}$

Far back in time, the population was near $Y = 0$. Far forward in time, the population will approach $Y = a/b$. Those are the two steady states, the points where $ay - by^2$ is zero and the curve becomes flat. Then dy/dt is zero and y never changes.

In between, the population $y(t)$ is following an **S-curve**, climbing toward a/b. It is symmetric around the halfway point $y = a/2b$. The world is near that point right now.

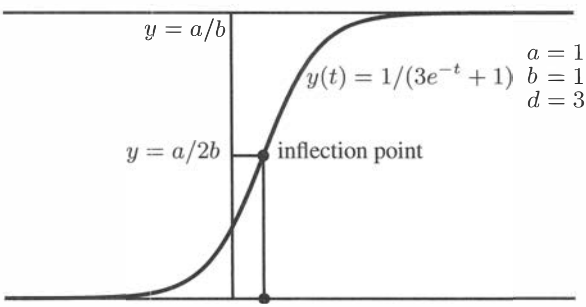


Figure 1.9: The S-curve solves the logistic equation. The inflection point is halfway.

Simplest Example of the S - curve

The best example has $a = b = 1$. The top of the S-curve is $Y = a/b = 1$. The bottom is $Y = 0$. The halfway time is $t = 0$, where $y(0) = \frac{1}{2}$. Then the logistic equation and its solution are as simple as possible:

$$\frac{dy}{dt} = y - y^2 \textbf{ has the solution } \boxed{y(t) = \frac{1}{1+e^{-t}}} \textbf{ starting from } y(0) = \frac{1}{2}. \tag{8}$$

That solution $1/(1+e^{-t})$ approaches 1 when $t \to \infty$. It approaches 0 when $t \to -\infty$. Let me review the "$z = 1/y$ method" to solve the logistic equation $y' = y - y^2$.

$$\frac{dz}{dt} = \frac{-1}{y^2}\frac{dy}{dt} = \frac{-y+y^2}{y^2} = -z+1.$$

Then $z(t) = 1 + Ce^{-t}$. Take $C = 1$ to match $y(0) = \frac{1}{2}$ and $z(0) = 2$. Now $y = \dfrac{1}{1+e^{-t}}$.

World Population and the Carrying Capacity K

What are the numbers a and b for human population ? Ecologists estimate the natural growth rate at $a = .029$ per year. This is not the actual rate, because of b. About 1930, the world population was near $y = 3$ billion. The ay term predicts a one-year increase of (.029) (3 billion) = 87 million. The actual growth was more like $dy/dt = 60$ million/year. In this simple model, that difference of 27 million/year was caused by by^2:

$$27 \text{ million/year} = b\,(3\,\text{billion})^2 \quad \text{leads to} \quad b = 3 \text{ times } 10^{-12}/\text{year}.$$

When we know b, we know the steady state $y(\infty) = K = a/b$. At that point the loss by^2 from competition balances the gain ay from growth:

$$\textbf{Estimated capacity} \quad K = \frac{a}{b} = \frac{.029}{3}10^{12} \approx 9.7 \text{ billion people.}$$

This number is low, and y is growing faster. The estimates I see now are closer to

$$y(\infty) > 10 \text{ billion} \quad \text{and} \quad y(2014) \approx 7.2 \text{ billion.}$$

Our world is beyond the halfway point $y = a/2b$ on the curve. That looks like an inflection point (by symmetry of the graph), and the test $d^2y/dt^2 = 0$ confirms that it is.

The inflection point with $y'' = 0$ is halfway up the curve in Figure 1.9

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(ay - by^2) = (a - 2by)\frac{dy}{dt} = 0 \text{ when } y = \frac{a}{2b} \tag{9}$$

After this halfway point, the S-curve bends downward. The population y is still increasing, but its growth rate dy/dt is decreasing. (Notice the difference.) The inflection point separates "bending up" from "bending down" and the *rate* of growth is a maximum at that point. You will understand that this simple model must be and has been improved.

Partial Fractions

The logistic equation is nonlinear but it is ***separable***. We can separate y from t as follows:

$$\frac{dy}{dt} = ay - by^2 = a\left(y - \frac{b}{a}y^2\right) \quad \text{leads to} \quad \boldsymbol{\frac{dy}{y - \frac{b}{a}y^2} = a\ dt}. \tag{10}$$

In this separated form, the problem is reduced to two ordinary integrations (y-integration on the left side, t-integration on the right side). The integral of $a\,dt$ on the right side is certainly $at + C$. The left side can be looked up in a table of integrals or produced by software like *Mathematica* or discovered by ourselves.

I will explain the idea of **partial fractions** that produces this integral. You may know it as a "Technique of Integration" from first-year calculus (it is really just algebra). The plan is to split the fraction in two pieces so the integration becomes easy:

Partial fractions $$\frac{1}{y - \frac{b}{a}y^2} \quad \text{separates into} \quad \frac{A}{y} + \frac{B}{1 - \frac{b}{a}y} \tag{11}$$

I factored $y - \frac{b}{a}y^2$ into y times $1 - \frac{b}{a}y$. I put those two denominators on the right side. *We need to know A and B.* To compare with the left side, combine those two fractions:

Common denominator $$\frac{A}{y} + \frac{B}{1 - \frac{b}{a}Y} = \frac{A\left(1 - \frac{b}{a}y\right) + By}{y\left(1 - \frac{b}{a}y\right)}. \tag{12}$$

The correct A and B must produce 1 in the numerator, to match the 1 in equation (11):

$$A\left(1 - \frac{b}{a}y\right) + By = 1 \quad \text{when} \quad \boldsymbol{A = 1} \quad \text{and} \quad \boldsymbol{B = \frac{b}{a}}. \tag{13}$$

This completes the algebra of partial fractions, by finding A and B in equation (11):

Two fractions $$\frac{1}{y - \frac{b}{a}y^2} = \frac{1}{y(1 - \frac{b}{a}y)} = \boldsymbol{\frac{1}{y} + \frac{b/a}{1 - \frac{b}{a}y}}. \tag{14}$$

Integrate the Partial Fractions

With $A = 1$ and $B = b/a$, integrate the two partial fractions separately:

$$\int \frac{1\,dy}{y} + \int \frac{(b/a)dy}{1 - (b/a)y} = \ln y - \ln\left(1 - \frac{b}{a}y\right). \tag{15}$$

This is the calculus part (the integration) in solving the logistic equation. After the integration, use algebra to write the answer $y(t)$ in a good form.

Actually that good form of $y(t)$ was already found by our first method. The magic of $z = 1/y$ produced a linear equation $dz/dt = -az + b$. Then returning to $y = 1/z$ put the crucial factor e^{-at} into the denominator of (7), and we repeat that solution here:

$$\textbf{Solution in (7)} \qquad \boldsymbol{y(t) = \frac{a}{de^{-at} + b}} \quad \text{with} \quad \boldsymbol{d = \frac{a}{y(0)} - b.} \tag{16}$$

This same answer must come from the integral (15) that used partial fractions. The integral has the form $\ln y - \ln x$, which is the same as $\ln(y/x)$ (and x is $1 - (b/a)y$).

$$\int \frac{dy}{y - \frac{b}{a}y^2} = \int a\,dt \quad \text{gives} \quad \ln \frac{y}{1 - \frac{b}{a}y} = at + C = at + \ln \frac{y(0)}{1 - \frac{b}{a}y(0)}. \tag{17}$$

I chose the integration constant C to make (17) correct at $t = 0$. Now take exponentials of both sides:

$$\frac{y}{1 - \frac{b}{a}y} = e^{at} \frac{y(0)}{1 - \frac{b}{a}y(0)}. \tag{18}$$

The final algebra part is to solve this equation for y. Let me move that into Problem 3. Then we recover the good formula (16) that came so much faster from $y = 1/z$.

Looking ahead, partial fractions will appear again in Section 2.7. They simplify the Laplace transform so you can recognize the inverse transform. That section gives a formula **PF2** for the numbers A and B in the fractions—it is previewed here in Problem 14.

Again, we solved $dy/dt = f(y)$ by separating $\int dy/f(y)$ from $\int dt$.

Autonomous Equations $dy/dt = f(y)$

The logistic equation is autonomous. This means that f depends only on y, and not on t: $\boldsymbol{dy/dt = f(y)}$. A linear example is $y' = y$. The big advantage of an autonomous equation is that the solution curve can stay the same, when the starting value $y(0)$ is changed. "We just climb onto the curve at height $y(0)$ and keep going."

You saw how Figure 1.9 had the same S-curve for every $y(0)$ between 0 and a/b. The equation $dy/dt = y$ has the same exponential curve $y = e^t$ for every $y(0) > 0$. Just mark the $t = 0$ point wherever the height is $y(0)$.

This means that time t is not essential in the graphs. **The graph of $f(y)$ against y is the key**. For the logistic equation, the parabola $f(y) = ay - by^2$ tells you everything (except the time for each y). $y(t)$ increases when this parabola $f(y)$ is above the axis (because $dy/dt > 0$ when $f > 0$). So I only drew one S-curve.

There is also a decreasing curve starting from $y(0) > a/b$. **It approaches the steady state $Y = a/b$ from above**. Another curve starts below $Y = 0$ and drops to $-\infty$. The up-going S-curve is sandwiched between two downgoing curves, because in Figure 1.10 the positive piece of $ay - by^2$ is sandwiched between two negative pieces.

Stability of Steady States

The steady states of $dy/dt = f(y)$ are solutions of $f(Y) = 0$. The differential equation becomes $0 = 0$ when $y(t) = Y$ is constant (steady). Here is the stability question:

Starting close to Y, does $y(t)$ approach Y (stable) or does it leave Y (unstable)?

We had a formula for the S-curve. So we could answer this stability question. One Y is stable (that is $Y = a/b$ at the end). The steady state $Y = 0$ is unstable. It is important (and not hard) to be able to decide stability *without a formula for* $y(t)$.

Everything depends on the derivative df/dy at the steady value $y = Y$. That slope of $f(y)$ will be called c. Here is the test for stability, followed by a reason and examples.

Stable if $c < 0$ The steady state Y is stable if $df/dy < 0$ at $y = Y$.

Reason: Near the steady state, $f(y)$ is close to $c(y - Y)$. Then $y' = f(y)$ is close to $(y - Y)' = c(y - Y)$. Then **$y - Y$ is like e^{ct}**, and **$y \to Y$ when $c < 0$ and $e^{ct} \to 0$.**

Let me explain in detail for any autonomous equation $dy/dt = f(y)$. Suppose that $Y = 0$ is a steady state. This means that $f(0) = 0$. Calculus gives the linear approximation $f(y) \approx cy$, where c is the slope of the tangent line. That number is $c = df/dy$ at $Y = 0$. **If c is negative then $y(t)$ will move toward $Y = 0$ (stability)**:

For small $y(0) > 0$	$dy/dt = f(y) \approx cy < 0$	$y(t)$ decreases toward 0
For small $y(0) < 0$	$dy/dt = f(y) \approx cy > 0$	$y(t)$ increases toward 0

For any other steady state Y, calculus gives the linear approximation $f(y) \approx c(y - Y)$. Now that number is $c = df/dy$, the slope of the tangent line **at $y = Y$**.

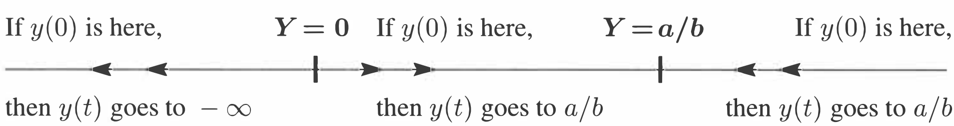
For $y(0)$ just above Y	$dy/dt = f(y) \approx c(y - Y) < 0$	**$y(t)$ decreases toward Y**
For $y(0)$ just below Y	$dy/dt = f(y) \approx c(y - Y) > 0$	**$y(t)$ increases toward Y**

Example 1 (logistic) The derivative of $ay - by^2$ is $\boldsymbol{df/dy = a - 2by}$.

At the steady state $Y = 0$, df/dy is $a > 0$: **$Y = 0$ is unstable.**

At $Y = a/b$, this derivative is $a - 2b(a/b) = -a$. **$Y = a/b$ is stable.**

For $dy/dt = ay - by^2$ this **stability line** shows which way $y(t)$ moves from any $y(0)$.



The steady states have to alternate between stable and unstable, because df/dy will alternate between negative and positive. I am excluding the undecided cases when $f(Y) = 0$ and also $df/dy(Y) = 0$. This is a borderline case for critical harvesting.

The Harvesting Equation

Suppose the logistic equation also includes a constant harvesting rate $-h$. This will reduce the growth rate dy/dt. Let me start with the logistic equation $dy/dt = 4y - y^2$, where the S-curve rises from $Y = 0$ to the other steady state $Y = a/b = 4/1$. If the new harvesting term is $-h = -3$, the steady states change from 0 and 4 to 1 and 3:

$$\boldsymbol{\frac{dy}{dt} = 4y - y^2 - 3} \quad \text{has new steady states} \quad Y = 1 \quad \text{and} \quad Y = 3. \tag{19}$$

I found 1 and 3 by factoring $4Y - Y^2 - 3$ into $-(Y-1)(Y-3)$. Those populations $Y = 1$ and $Y = 3$ are the points where the equation is $dy/dt = 0$. Then $y = Y$ stays steady.

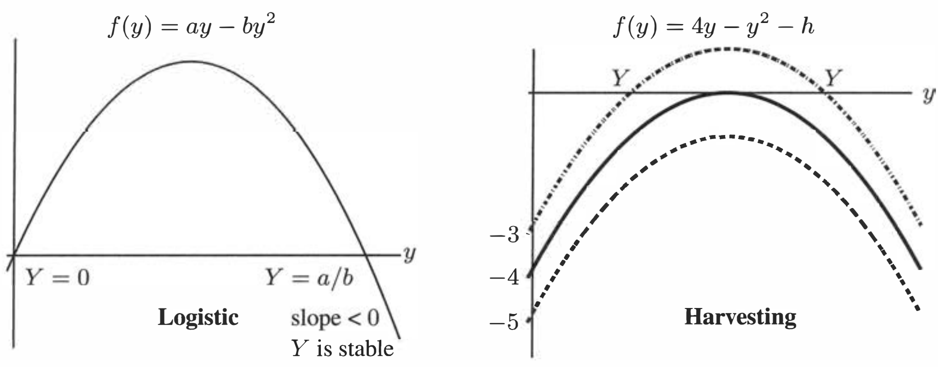


Figure 1.10: Harvesting lowers the parabola $f(y) = ay - by^2 - h$. Steady Y's disappear.

This figure shows the stability or instability of the steady states. $Y = 0$ in the logistic graph and $Y = 1$ in the harvesting graph are **unstable**. At those points $f(y)$ climbs from negative to positive. Above Y, the graph shows $dy/dt = f(y)$ as positive. So $y(t)$ will increase, and it moves away from Y.

$Y = a/b$ in the logistic graph and $Y = 3$ in the harvesting graph are **stable**. Beyond those points $f(y)$ is negative. This is dy/dt. So $y(t)$ decreases back toward Y. The graphs are a little tricky to read, because they don't show $y(t)$. **They show the phase plane with $\boldsymbol{y' = f(y)}$ against $\boldsymbol{y}$**: Velocity versus position, not position versus time!

Looking again at the figure, $h = 4$ gives critical harvesting: **One double stationary point $\boldsymbol{Y = 2}$.** That curve shows $dy/dt = f(y)$ as always negative, so $y(t)$ will decrease. If $y(0)$ is greater than 2, then $y(t)$ must come back toward $Y = 2$. But this is *one-sided stability*, because if $y(0)$ is smaller then 2, then $y(t)$ will decrease and go far away from 2.

The lowest curve has $h = 5$ and **no steady states**. At all points $dy/dt = f(y)$ is negative. All solutions $y(t)$ are decreasing. If we can find a formula for $y(t)$, we can watch this happen: $\boldsymbol{y(t) \to -\infty}$. The logistic and harvesting equations are terrific nonlinear examples, because we can actually find $y(t)$.

Solving the Harvesting Equation

We have three types of harvesting equations, with 2 or 1 or 0 steady states:

$h < 4$ $\quad y' = 4y - y^2 - h$ will reduce to a logistic equation: **underharvesting**

$h = 4$ $\quad y' = -(y-2)^2$ has a double steady state: **critical harvesting**

$h > 4$ $\quad y'$ stays below zero and $y(t)$ approaches $-\infty$: **overharvesting**.

All these equations are autonomous, so they separate into $dy/f(y) = dt$. *Integrate* $1/f(y)$.

Small $h = 3$ $\quad$ Factor $f(y)$ into $-(y-1)(y-3)$ $\quad$ Then $\boldsymbol{Y = 1}$ **and** $\boldsymbol{Y = 3}$

Let me shift those steady states down to $V = 0$ and $V = 2$, by shifting $y(t)$ to $\boldsymbol{v(t) = y(t) - 1}$. The equation for $v(t)$ is logistic, and its S-curve climbs from 0 to 2:

$$(1+v)' = -(v)(v-2) \text{ is } \boldsymbol{v' = 2v - v^2} \tag{20}$$

When you add back the 1 to get $y = 1 + v$, its S-curve climbs from 1 to 3.

Critical $h = 4$ $\quad$ Factor $f(y) = 4y - y^2 - 4 = -(y-2)^2$ $\quad$ Then $\boldsymbol{Y = 2}$ **and 2**

The equation is $y' = -(y-2)^2$. Shifting to $v(t) = y(t) - 2$ gives $\boldsymbol{dv/dt = -v^2}$. Page 1 of this book had the equation $dy/dt = +y^2$ (with time going the other way). The solution looks so innocent:

$$v(t) = \frac{v(0)}{1 + tv(0)} \quad \begin{array}{l} \text{goes gently to } v = 0 \text{ as } t \to \infty \text{ provided } v(0) > 0 \\ \text{goes suddenly to } v = -\infty \text{ when } 1 + tv(0) = 0 \end{array}$$

This shows (one-sided) stability if $y(0) > 2$ and $v(0) > 0$.

When harvesting is more than critical, the population dies out from every $y(0)$.

Overharvesting $h = 5$ Write $y' = 4y - y^2 - 5 = -1 - (y-2)^2$. Always $\boldsymbol{y' < 0}$.

Now $v = y - 2$ simplifies the equation to $\boldsymbol{v' = -1 - v^2}$. Integrate $dv/(1+v^2) = -dt$ to get $\tan^{-1} v = -t + C$. If $v(0) = 0$ then $C = 0$. Now go back to $y = v + 2$:

$$\frac{dv}{dt} = -1 - v^2 \text{ with } v(0) = 0 \text{ gives } v(t) = \tan(-t). \text{ Then } \boldsymbol{y(t) = 2 - \tan t}. \tag{21}$$

When the tangent reaches 2, the population $y = 0$ is all gone. If the solution continues to $t = \pi/2$, then $\tan t$ is infinite. The model loses meaning and $y(\pi/2) = -\infty$.

Overall, I hope you see how a simple stability test tells so much about $y' = f(y)$:

1 Find all solutions to $f(y) = 0$	**2** If $df/dy < 0$ at $y = Y$, that state is stable.

■ REVIEW OF THE KEY IDEAS ■

1. The logistic equation $dy/dt = ay - by^2$ has steady states at $Y = 0$ and $Y = a/b$.

2. The S-curve $y(t) = a/(de^{-at} + b)$ approaches the carrying capacity $y(\infty) = a/b$.

3. The equation for $z = \dfrac{1}{y}$ is linear ! Or we can separate into $dy/\left(y - \dfrac{b}{a}y^2\right) = a\,dt$.

4. The stability test $df/dy = a - 2by < 0$ is passed at $Y = a/b$ and failed at $Y = 0$.

5. This stability test applies to all equations $y' = f(y)$ including $y' = ay - by^2 - h$.

Problem Set 1.7

1 If $y(0) = a/2b$, the halfway point on the S-curve is at $t = 0$. Show that $d = b$ and $y(t) = \dfrac{a}{d\,e^{-at} + b} = \dfrac{a}{b}\dfrac{1}{e^{-at} + 1}$. Sketch the curve from $y_{-\infty} = 0$ to $y_\infty = \dfrac{a}{b}$.

2 If the carrying capacity of the Earth is $K = a/b = 14$ billion people, what will be the population at the inflection point ? What is dy/dt at that point ? The actual population was 7.14 billion on January 1, 2014.

3 Equation (18) must give the same formula for the solution $y(t)$ as equation (16). If the right side of (18) is called R, we can solve that equation for y :

$$y = R\left(1 - \frac{b}{a}y\right) \quad \rightarrow \quad \left(1 + R\frac{b}{a}\right)y = R \quad \rightarrow \quad y = \frac{R}{(1 + R\frac{b}{a})}.$$

Simplify that answer by algebra to recover equation (16) for $y(t)$.

4 Change the logistic equation to $y' = y + y^2$. Now the nonlinear term is positive, and *cooperation of y with y* promotes growth. Use $z = 1/y$ to find and solve a linear equation for z, starting from $z(0) = y(0) = 1$. Show that $y(T) = \infty$ when $e^{-T} = 1/2$. Cooperation looks bad, the population will explode at $t = T$.

5 The US population grew from 313, 873, 685 in 2012 to 316, 128, 839 in 2014. If it were following a logistic S-curve, what equations would give you a, b, d in the formula (4) ? Is the logistic equation reasonable and how to account for immigration ?

6 The **Bernoulli equation** $y' = ay - by^n$ has competition term by^n. Introduce $z = y^{1-n}$ which matches the logistic case when $n = 2$. Follow equation (4) to show that $z' = (n-1)(-az + b)$. Write $z(t)$ as in (5)-(6). Then you have $y(t)$.

Problems 7–13 develop better pictures of the logistic and harvesting equations.

7 $y' = y - y^2$ is solved by $y(t) = 1/(de^{-t}+1)$. This is an S-curve when $y(0) = 1/2$ and $d = 1$. But show that $y(t)$ is very different if $y(0) > 1$ or if $y(0) < 0$.

If $y(0) = 2$ then $d = \frac{1}{2} - 1 = -\frac{1}{2}$. Show that $y(t) \to 1$ from above.
If $y(0) = -1$ then $d = \frac{1}{-1} - 1 = -2$. At what time T is $y(T) = -\infty$?

8 (recommended) Show those 3 solutions to $y' = y - y^2$ in one graph! They start from $y(0) = 1/2$ and 2 and -1. The S-curve climbs from $\frac{1}{2}$ to 1. Above that, $y(t)$ descends from 2 to 1. Below the S-curve, $y(t)$ drops from -1 to $-\infty$.

Can you see 3 regions in the picture? **Dropin curves above $y = 1$ and S-curves sandwiched between 0 and 1 and dropoff curves below $y = 0$.**

9 Graph $f(y) = y - y^2$ to see the unstable steady state $Y = 0$ and the stable $Y = 1$. Then graph $f(y) = y - y^2 - 2/9$ with harvesting $h = 2/9$. What are the steady states Y_1 and Y_2? The 3 regions in Problem 8 now have Z-curves above $y = 2/3$, S-curve sandwiched between $1/3$ and $2/3$, dropoff curves below $y = 1/3$.

10 What equation produces an S-curve climbing to $y_\infty = K$ from $y_{-\infty} = L$?

11 $y' = y - y^2 - \frac{1}{4} = -(y - \frac{1}{2})^2$ shows *critical harvesting* with a double steady state at $y = Y = \frac{1}{2}$. The layer of S-curves shrinks to that single line. Sketch a dropin curve that starts above $y(0) = \frac{1}{2}$ and a dropoff curve that starts below $y(0) = \frac{1}{2}$.

12 Solve the equation $y' = -(y - \frac{1}{2})^2$ by substituting $v = y - \frac{1}{2}$ and solving $v' = -v^2$.

13 With overharvesting, every curve $y(t)$ drops to $-\infty$. There are no steady states. Solve $Y - Y^2 - h = 0$ (quadratic formula) to find only complex roots if $4h > 1$.

The solutions for $h = \frac{5}{4}$ are $y(t) = \frac{1}{2} - \tan(t + C)$. Sketch that dropoff if $C = 0$. Animal populations don't normally collapse like this from overharvesting.

14 With **two partial fractions**, this is my preferred way to find $A = \dfrac{1}{r-s}$, $B = \dfrac{1}{s-r}$

$$\textbf{PF2} \qquad \frac{1}{(y-r)(y-s)} = \frac{1}{(y-r)(r-s)} + \frac{1}{(y-s)(s-r)}$$

Check that equation: The common denominator on the right is $(y-r)(y-s)(r-s)$. The numerator should cancel the $r - s$ when you combine the two fractions.

$$\text{Separate } \frac{1}{y^2-1} \text{ and } \frac{1}{y^2-y} \text{ into two fractions } \frac{A}{y-r} + \frac{B}{y-s}.$$

Note When y approaches r, the left side of **PF2** has a blowup factor $1/(y-r)$. The other factor $1/(y-s)$ correctly approaches $A = 1/(r-s)$. So the right side of **PF2** needs the same blowup at $y = r$. The first term $A/(y-r)$ fits the bill.

15 The **threshold equation** is the logistic equation backward in time:

$$-\frac{dy}{dt} = ay - by^2 \quad \text{is the same as} \quad \frac{dy}{dt} = -ay + by^2.$$

Now $Y = 0$ is the stable steady state. $Y = a/b$ is the unstable state (why?). If $y(0)$ is below the threshold a/b then $y(t) \to 0$ and the species will die out.

Graph $y(t)$ with $y(0) < a/b$ (reverse S-curve). Then graph $y(t)$ with $y(0) > a/b$.

16 (Cubic nonlinearity) The equation $y' = y(1-y)(2-y)$ has **three steady states**: $Y = 0, 1, 2$. By computing the derivative df/dy at $y = 0, 1, 2$, decide whether each of these states is stable or unstable.

Draw the *stability line* for this equation, to show $y(t)$ leaving the unstable Y's. Sketch a graph that shows $y(t)$ starting from $y(0) = \frac{1}{2}$ and $\frac{3}{2}$ and $\frac{5}{2}$.

17 (a) Find the steady states of the **Gompertz equation** $dy/dt = y(1 - \ln y)$.

(b) Show that $z = \ln y$ satisfies the linear equation $dz/dt = 1 - z$.

(c) The solution $z(t) = 1 + e^{-t}(z(0) - 1)$ gives what formula for $y(t)$ from $y(0)$?

18 Decide stability or instability for the steady states of

(a) $dy/dt = 2(1-y)(1-e^y)$ (b) $dy/dt = (1-y^2)(4-y^2)$

19 Stefan's Law of Radiation is $dy/dt = K(M^4 - y^4)$. It is unusual to see fourth powers. Find all real steady states and their stability. Starting from $y(0) = M/2$, sketch a graph of $y(t)$.

20 $dy/dt = ay - y^3$ has how many steady states Y for $a < 0$ and then $a > 0$? Graph those values $Y(a)$ to see a *pitchfork bifurcation*—new steady states suddenly appear as a passes zero. The graph of $Y(a)$ looks like a pitchfork.

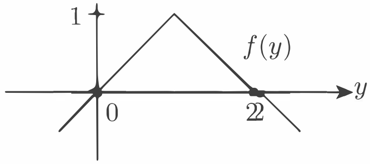
21 (Recommended) The equation $dy/dt = \sin y$ has **infinitely many steady states**. What are they and which ones are stable? Draw the stability line to show whether $y(t)$ increases or decreases when $y(0)$ is between two of the steady states.

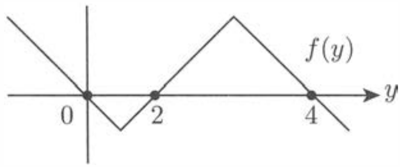
22 Change Problem 21 to $dy/dt = (\sin y)^2$. The steady states are the same, but now the derivative of $f(y) = (\sin y)^2$ is zero at all those states (because $\sin y$ is zero). What will the solution actually do if $y(0)$ is between two steady states?

23 (*Research project*) Find actual data on the US population in the years 1950, 1980, and 2010. What values of a, b, d in the solution formula (7) will fit these values? Is the formula accurate at 2000, and what population does it predict for 2020 and 2100?

You could reset $t = 0$ to the year 1950 and rescale time so that $t = 3$ is 1980.

24 If $dy/dt = f(y)$, what is the limit $y(\infty)$ starting from each point $y(0)$?





25 (a) Draw a function $f(y)$ so that $y(t)$ approaches $y(\infty) = 3$ from every $y(0)$.

(b) Draw $f(y)$ so that $y(\infty) = 4$ if $y(0) > 0$ and $y(\infty) = -2$ if $y(0) < 0$.

26 Which exponents n in $dy/dt = y^n$ produce blowup $y(T) = \infty$ in a finite time? You could separate the equation into $dy/y^n = dt$ and integrate from $y(0) = 1$.

27 Find the steady states of $dy/dt = y^2 - y^4$ and decide whether they are stable, unstable, or one-sided stable. Draw a stability line to show the final value $y(\infty)$ from each initial value $y(0)$.

28 For an autonomous equation $y' = f(y)$, why is it impossible for $y(t)$ to be increasing at one time t_1 and decreasing at another time t_2 ?

The website math.mit.edu/dela has more graph questions for autonomous $y' = f(y)$.

Notes on feedback The S-curve represents a good response from an elevator. The transient response in the middle of the S is the fast movement between floors. The elevator slows down as it approaches steady state (the floor it is going to). There is a *feedback loop* to tell the elevator how far it is from its destination, and control its speed.

An **open-loop** system has no feedback. A simple toaster will keep going and burn your toast. The end time is entirely controlled by the input setting. A **closed-loop** system feeds back the difference between the state $y(t)$ and the desired steady state y_∞. A toaster oven can avoid burning by feeding back the temperature.

The logistic equation is nonlinear because of its feedback term $-by^2$. This is so common in other examples of movement and growth. Our brain controls arm movement and brings it to a stop. Your car has thousands of computer chips and controllers that measure position and speed, to slow down and stop before disaster.

I admit that I don't use cruise control because the car might keep cruising—I am not too sure it will stop. But it does have a feedback loop to keep the car below a set speed.

1.8 Separable Equations and Exact Equations

This section presents two special types of first order nonlinear differential equations. They are a bridge between $y' = ay$ and the very general form $y' = f(t, y)$. These pages explain how to solve the two types in between, by ordinary integration. *Separable* equations are the simplest. For *exact* equations, see formulas (12) and (15).

Separable	Exact
$\dfrac{\mathbf{dy}}{\mathbf{dt}} = \dfrac{\mathbf{g(t)}}{\mathbf{f(y)}}$	$\dfrac{\mathbf{dy}}{\mathbf{dt}} = \dfrac{\mathbf{g(y,t)}}{\mathbf{f(y,t)}}$ when $\dfrac{\partial f}{\partial t} = -\dfrac{\partial g}{\partial y}$

1. Separable Equations $f(y)dy = g(t)dt$

With $f(y)$ on one side and $g(t)$ on the other side, you see the meaning of *separable*. The ordinary way to write this equation would be

$$\frac{dy}{dt} = \frac{g(t)}{f(y)} \quad \text{starting from } y(0) \text{ at time } t = 0. \tag{1}$$

When dy/dt has this separable form, we combine $f(y)$ with dy and $g(t)$ with dt. Those functions f and g need to be integrated. **The integrals $F(y)$ and $G(t)$ start at** $y = y(0)$ **and** $t = 0$:

$$\boldsymbol{F(y)} = \int_{y(0)}^{y} f(u)\,du \qquad \boldsymbol{G(t)} = \int_{x=0}^{t} g(x)\,dx \tag{2}$$

The dummy variables u and x were chosen because y and t are needed in the upper limits of integration. Every author faces this question, to select variables. To show that the letters u and x don't matter, I could change them to Y and T.

After integrating f and g, we have *implicitly* solved the differential equation:

$$\textbf{Solution} \qquad \frac{dy}{dt} = \frac{g(t)}{f(y)} \quad \text{integrates to} \quad \boldsymbol{F(y) = G(t)}. \tag{3}$$

To get an *explicit* solution $y = \ldots$ we have to solve this equation $F(y) = G(t)$ to find y.

Example 1 $\dfrac{dy}{dt} = \dfrac{t}{y}$ is $\boldsymbol{y\,dy = t\,dt}$. Integrate to find $\dfrac{1}{2}\left(y(t)^2 - y(0)^2\right) = \dfrac{1}{2}t^2$.
Solve this implicit equation to find $y(t)$ explicitly:

Solution $\boldsymbol{y(t) = \sqrt{y(0)^2 + t^2}}$. Then $\dfrac{dy}{dt} = \dfrac{t}{\sqrt{y(0)^2 + t^2}} = \dfrac{t}{y}$.

Example 2 $\boldsymbol{dy/dt = 2ty}$ has $\boldsymbol{g(t) = 2t}$ divided by $\boldsymbol{f(y) = 1/y}$.

Solution Separate $1/y$ from $2t$ and integrate to get $F = \ln y - \ln y(0)$ and $G = t^2$:

$$\frac{dy}{y} = 2t\,dt \quad \text{leads to} \quad \int_{y(0)}^{y} \frac{du}{u} = \ln y - \ln y(0) \quad \text{and} \quad \int_0^t 2x\,dx = t^2$$

In this example, $F(y) = G(t)$ produces $\ln y = \ln y(0) + t^2$. Take exponentials of both sides to find the solution y:

$$\mathbf{y = e^{\ln y(0)} e^{t^2} = y(0)\, e^{t^2}}. \tag{4}$$

I always check the derivative dy/dt and the starting value $y(0)$:

$$\frac{d}{dt}\left(y(0)\, e^{t^2}\right) = 2t\left(y(0)\, e^{t^2}\right) = 2ty \qquad y(0)\, e^{t^2} = y(0) \text{ at } t = 0. \tag{5}$$

Example 3 Our favorite equation $\frac{dy}{dt} = ay + q$ is separable when a and q are constant. Move $y + \frac{q}{a}$ to the left side below dy. Keep $a\,dt$ on the right side. Then integrate both sides, and you have solved this equation once more !

$$\frac{dy}{y + \frac{q}{a}} = a\,dt \quad \text{gives} \quad \ln(y + \frac{q}{a}) = at + C. \tag{6}$$

Take exponentials to find y, and set $t = 0$ to find C:

$$\textbf{Exponential growth} \qquad y(t) + \frac{q}{a} = e^{at} e^{C} \quad \text{and} \quad y(0) + \frac{q}{a} = e^{C}. \tag{7}$$

Substitute for e^C in the left equation, to get the answer we know:

$$\boxed{y(t) + \frac{q}{a} = e^{at}\left(y(0) + \frac{q}{a}\right) \quad \text{and then} \quad \boldsymbol{y(t) = e^{at} y(0) + \frac{q}{a}(e^{at} - 1)}.} \tag{8}$$

This answer was the key to Section 1.4. Here the formulas came faster (the first one in that box looks attractive). But I like the old way: *Follow each input as it grows.*

Example 4 **(Logistic equation)**

$$\frac{dy}{dt} = ay - by^2 \qquad \int_{y(0)}^{y} \frac{du}{au - bu^2} = \int_{t(0)}^{t} dx \tag{9}$$

The right side is certainly $G(t) = t - t(0)$. I am including $t(0)$ to show how the system allows any starting value for t as well as y. We don't know a perfect starting time for the Earth's population, so we pick a year like $t(0) = 2000$ and work from there. The key point is that two integrals $F(y)$ and $G(t)$ give the answer.

Section 1.7 computed those integrals and solved the logistic equation.

2. Exact Equations $f(y,t)dy = g(y,t)dt$

A separable equation has $dy/dt = g(t)/f(y)$. We wrote this as $f(y)dy = g(t)dt$. We integrated the two sides separately to get $F(y) = G(t)$. This solved the equation.

Exact equations are not required to be separable. The functions f and g can depend on both variables t and y. The equation does not split into a pure y-integration and a pure t-integration. We now have $\mathbf{f(y,t)\,dy = g(y,t)\,dt}$. But it sometimes succeeds to integrate the left side $f(y,t)$ with respect to y, as if t were a constant which it is not.

Step 1 **Integrate f with respect to y** $$\int f(y,t)\,dy = \boldsymbol{F(y,t) + C(t)}. \tag{10}$$

Normally, any constant C can be added to an integral. The answer stays correct, because the derivative of C is zero. Here, *any function of t can be added to the integral*, because the y derivative of any $C(t)$ is zero. Now $F(y,t) + C(t)$ has more flexibility.

Step 2 **(if possible) Choose** $C(t)$ so that $$\boldsymbol{\frac{\partial}{\partial t}(F(y,t) + C(t)) = -g(y,t)}. \tag{11}$$

If that choice of $C(t)$ is possible, our original equation involving g and f is solved:

Step 3 $$\frac{dy}{dt} = \frac{g(y,t)}{f(y,t)} \quad \text{is solved by} \quad \boldsymbol{F(y,t) + C(t) = \textbf{any constant}}. \tag{12}$$

Before I show when and why this works, here is an example of success.

Example 5 The equation $\dfrac{dy}{dt} = \boldsymbol{\dfrac{2yt - 1}{y^2 - t^2}}$ has $g = 2yt - 1$ and $f = y^2 - t^2$.

Step 1 Integrate $f\,dy = (y^2 - t^2)dy$ to find $F(y,t) = \dfrac{1}{3}y^3 - yt^2$. Then $\dfrac{\partial F}{\partial t} = -2ty$.

Step 2 Solve equation (11) for $C(t)$. For our particular f and g, this is possible :

$$-2ty + \frac{dC}{dt} = -(2yt - 1) \;\text{ gives }\; \frac{dC}{dt} = 1 \text{ and } C(t) = t.$$

Step 3 The original $\dfrac{dy}{dt} = \dfrac{g}{f}$ is solved by $F(y,t) + C(t) = \text{constant}$:

Solution from $F + C$
Constant is set by $y(0)$
$$\boldsymbol{\frac{1}{3}y^3 - yt^2 + t = \frac{1}{3}y(0)^3}.$$

To check this answer, take its time derivative implicitly (which means : just do it).

Implicit differentiation $$y^2\frac{dy}{dt} - t^2\frac{dy}{dt} - 2yt + 1 = 0.$$

This is our equation $dy/dt = \boldsymbol{(2yt - 1)/(y^2 - t^2)}$ as we hoped. Now to explain why.

The Exactness Condition

When is Step 2 possible ? Sometimes there is $C(t)$ to solve equation (11), but usually not. To find the condition for exactness, take the y-derivative of both sides in Step 2:

$$\frac{\partial}{\partial y}\frac{\partial}{\partial t}\left(F(y,t)+C(t)\right)=-\frac{\partial}{\partial y}\left(g(y,t)\right). \tag{13}$$

The order of $\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial t}$ can always be reversed. Certainly $\dfrac{\partial}{\partial y}C(t)=0$ and $\dfrac{\partial}{\partial y}F=f$.

The left side of (13) is $\dfrac{\partial}{\partial y}\dfrac{\partial}{\partial t}F(y,t)=\dfrac{\partial}{\partial t}\dfrac{\partial}{\partial y}F(y,t)$ which is $\dfrac{\partial}{\partial t}f(y,t)$. (14)

Comparing (14) with (13), Step 2 is only possible when our original differential equation $dy/dt=g/f$ is exact:

$$\boxed{\textbf{Exactness condition} \qquad \frac{\partial}{\partial t}f(y,t)=-\frac{\partial}{\partial y}g(y,t).} \tag{15}$$

When the equation is exact, Step 2 will produce $C(t)$. The final question is about Step 3. Why is $F(y,t)+C(t)=$ constant for the original differential equation $dy/dt=g/f$? To see this, take the time derivative of $F(y,t)+C(t)$ using the (implicit) chain rule:

$$\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial t}+\frac{\partial C}{\partial t}=0. \tag{16}$$

Step 1 produced $\dfrac{\partial F}{\partial y}=f$. Step 2 produced $\dfrac{\partial F}{\partial t}+\dfrac{\partial C}{\partial t}=-g$. We have success:

Equation (16) *is* $f\dfrac{dy}{dt}-g=0$. *This is our original problem* $\dfrac{dy}{dt}=\dfrac{g}{f}$.

Example 5 was exact because $g=2yt-1$ and $f=y^2-t^2$ agree on $\boldsymbol{\dfrac{\partial f}{\partial t}=-\dfrac{\partial g}{\partial y}=-2t}$.

Example 6 Steps $1,2,3$ must be possible because this non-separable equation is exact :

$$\frac{dy}{dt}=\frac{t-y}{t+y}=\frac{g(y,t)}{f(y,t)} \quad \text{has} \quad \frac{\partial f}{\partial t}=-\frac{\partial g}{\partial y}=1. \tag{17}$$

Step 1 Integrate $\int f\,dy=\int(t+y)dy$ to find $F=ty+\frac{1}{2}y^2$.

Step 2 Write out $\dfrac{\partial}{\partial t}(F+C)=-g=y-t$ to find $C(t)=-\frac{1}{2}t^2$

Step 3 The example is solved by $F+C=ty+\dfrac{1}{2}y^2-\dfrac{1}{2}t^2=\text{constant}=\frac{1}{2}\,y(0)^2$.

To check that solution, find the total time derivative of $F+C$ by the chain rule :

$$t\frac{dy}{dt}+y+y\frac{dy}{dt}-t=0. \text{ This is } \frac{dy}{dt}=\frac{t-y}{t+y} \text{ as desired.}$$

Final Note : Separable is Exact

Notice that **a separable equation $dy/dt = g(t)/f(y)$ is always exact**:

$$\textbf{(15) is satisfied} \qquad \frac{\partial}{\partial t} f(y) = -\frac{\partial}{\partial y} g(t) \text{ becomes } \mathbf{0 = 0}.$$

No problem with integrating $\int f(y)dy$ and $\int g(t)dt$ to find $F(y)$ and $G(t) = -C(t)$.

■ REVIEW OF THE KEY IDEAS ■

1. A **separable** equation $\dfrac{dy}{dt} = \dfrac{g(t)}{f(y)}$ is solved by $\int f(y)dy = \int g(t)dt +$ any constant.
2. That solution gives y implicitly. Solve to find y explicitly as a function of t.
3. An **exact** equation $\dfrac{dy}{dt} = \dfrac{g(y,t)}{f(y,t)}$ has $\dfrac{\partial g}{\partial y} = -\dfrac{\partial f}{\partial t}$. Then $F(y,t) + C(t) =$ constant.
4. The solution has $F(y,t) = \int f(y,t)dy$ for each t, and $C(t) = -\displaystyle\int \left(\frac{\partial F}{\partial t} + g\right) dt$.
5. The exactness condition in **3** removes y from that integral for $C(t)$ in **4**.

Problem Set 1.8

1 Finally we can solve the example $dy/dt = y^2$ in Section 1.1 of this book.

Start from $y(0) = 1$. Then $\displaystyle\int_1^y \frac{dy}{y^2} = \int_0^t dt$. Notice the limits on y and t. Find $y(t)$.

2 Start the same equation $dy/dt = y^2$ from any value $y(0)$. At what time t does the solution blow up? For which starting values $y(0)$ does it never blow up?

3 Solve $dy/dt = a(t)y$ as a separable equation starting from $y(0) = 1$, by choosing $f(y) = 1/y$. This equation gave the growth factor $G(0,t)$ in Section 1.6.

4 Solve these separable equations starting from $y(0) = 0$:

(a) $\dfrac{dy}{dt} = ty$ (b) $\dfrac{dy}{dt} = t^m y^n$

5 Solve $\dfrac{dy}{dt} = a(t)y^2 = \dfrac{a(t)}{1/y^2}$ as a separable equation starting from $y(0) = 1$.

6 The equation $\dfrac{dy}{dt} = y + t$ is not separable or exact. But it is linear and $y =$ _____ .

7 The equation $\dfrac{dy}{dt} = \dfrac{y}{t}$ has the solution $y = At$ for every constant A. Find this solution by separating $f = 1/y$ from $g = 1/t$. Then integrate $dy/y = dt/t$. Where does the constant A come from ?

8 For which number A is $\dfrac{dy}{dt} = \dfrac{ct - ay}{At + by}$ an exact equation ? For this A, solve the equation by finding a suitable function $F(y, t) + C(t)$.

9 Find a function $y(t)$ different from $y = t$ that has $dy/dt = y^2/t^2$.

10 These equations are separable after factoring the right hand sides :

$$\text{Solve} \quad \frac{dy}{dt} = e^{y+t} \quad \text{and} \quad \frac{dy}{dt} = yt + y + t + 1.$$

11 These equations are linear and separable : Solve $\dfrac{dy}{dt} = (y + 4)\cos t$ and $\dfrac{dy}{dt} = ye^t$.

12 Solve these three separable equations starting from $y(0) = 1$:

$$\text{(a)} \quad \frac{dy}{dt} = -4ty \qquad \text{(b)} \quad \frac{dy}{dt} = ty^3 \qquad \text{(c)} \quad (1+t)\frac{dy}{dt} = 4y$$

Test the exactness condition $\partial g/\partial y = -\partial f/\partial t$ and solve Problems 13-14.

13 (a) $\dfrac{dy}{dt} = \dfrac{-3t^2 - 2y^2}{4ty + 6y^2}$ (b) $\dfrac{dy}{dt} = -\dfrac{1 + ye^{ty}}{2y + te^{ty}}$

14 (a) $\dfrac{dy}{dt} = \dfrac{4t - y}{t - 6y}$ (b) $\dfrac{dy}{dt} = -\dfrac{3t^2 + 2y^2}{4ty + 6y^2}$

15 Show that $\dfrac{dy}{dt} = -\dfrac{y^2}{2ty}$ is exact but the same equation $\dfrac{dy}{dt} = -\dfrac{y}{2t}$ is not exact. Solve both equations. (This problem suggests that many equations become exact when multiplied by an integrating factor.)

16 Exactness is really the condition to solve two equations with the same function $H(t, y)$:

$$\frac{\partial H}{\partial y} = f(t, y) \ \text{ and } \ \frac{\partial H}{\partial t} = -g(t, y) \ \text{ can be solved if } \ \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial y}.$$

Take the t derivative of $\partial H/\partial y$ and the y derivative of $\partial H/\partial t$ to show that exactness is *necessary*. It is also *sufficient* to guarantee that a solution H will exist.

17 The linear equation $\dfrac{dy}{dt} = aty + q$ is not exact or separable. Multiply by the integrating factor $e^{-\int at\,dt}$ and solve the equation starting from $y(0)$.

Second order equations $F(t, y, y', y'') = 0$ involve the second derivative y''. This reduces to a first order equation for y' (not y) in two important cases:

I. When y is missing in F, set $y' = v$ and $y'' = v'$. Then $\boldsymbol{F(t, v, v') = 0}$.
II. When t is missing in F, set $y'' = \dfrac{dv}{\cancel{dt}} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$. Then $\boldsymbol{F\left(y, v, v\dfrac{dv}{\cancel{dy}}\right) = 0}$.

See the website for **reduction of order** when one solution $y(t)$ is known.

18 (y is missing) Solve these differential equations for $v = y'$ with $v(0) = 1$. Then solve for y with $y(0) = 0$.

(a) $y'' + y' = 0$ (b) $2ty'' - y' = 0$.

19 Both y and t are missing in $\boldsymbol{y'' = (y')^2}$. Set $v = y'$ and go two ways:

I. (y missing) Solve $\dfrac{dv}{dt} = v^2$ for $v(t)$ and then $\dfrac{dy}{dt} = v(t)$ with $y(0) = 0$, $y'(0) = 1$.

II. (t missing) Solve $v\dfrac{dv}{\cancel{dy}} = v^2$ for $v(y)$ and then $\dfrac{dy}{\cancel{dt}} = v(y)$ with $y(0) = 0$, $y'(0) = 1$.

20 An **autonomous equation $\boldsymbol{y' = f(y)}$** has no terms that contain t (t is missing).

Explain why every autonomous equation is separable. A non-autonomous equation could be separable or not. For a linear equation we usually say LTI (**linear time-invariant**) when it is autonomous: coefficients are constant, not varying with t.

21 $my'' + ky = 0$ is a highly important LTI equation. Two solutions are $\cos\omega t$ and $\sin\omega t$ when $\omega^2 = k/m$. Solve differently by reducing to a first order equation for $y' = dy/dt = v$ with $y'' = v\, dv/dy$ as above:

$$mv\frac{dv}{dy} + ky = 0 \;\text{ integrates to }\; \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \text{ constant } E.$$

For a mass on a spring, kinetic energy $\frac{1}{2}mv^2$ plus potential energy $\frac{1}{2}ky^2$ is a constant energy E. What is E when $y = \cos\omega t$? What integral solves the separable $m(y')^2 = 2E - ky^2$? I would not solve the linear oscillation equation this way.

22 $my'' + k\sin y = 0$ is the *nonlinear* oscillation equation: not so simple. Reduce to a first order equation as in Problem 21:

$$mv\frac{dv}{\cancel{dy}} + k\sin y = 0 \;\text{ integrates to }\; \frac{1}{2}mv^2 - k\cos y = \text{ constant } E.$$

With $v = dy/dt$ what impossible integral is needed for this first order separable equation? Actually that integral gives the period of a nonlinear pendulum—this integral is extremely important and well studied even if impossible.

■ CHAPTER 1 NOTES ■

The great function of calculus is e^t. How best to define this exponential function? Section 1.3 constructed $y = e^t$ from its infinite series $1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots$. Euler would approve! Taking the derivative of each term brings back e^t. This property $dy/dt = y$ is the most important tool we have—it is the foundation of our subject.

I like this approach to e^t for at least two reasons:

1. It is based on the derivatives of t and t^2 and t^n : well known.

2. The Chapter 3 Notes solve nonlinear equations in exactly the same way.

The limiting step required here is to add up an infinite series. We don't expect a simple answer like $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. But the numbers $1/n!$ in e^t are (*much smaller*) than these numbers $1/2^n$.

This is really the key point, to see that the terms $t^n/n!$ approach zero quickly.

The infinite series $1 + t + t^2/2 + \cdots + t^n/n! + \cdots$ converges for every t.

Proof. Each term $t^n/n!$ multiplies the previous term $t^{n-1}/(n-1)!$ by t/n. At some point $n = N$, that number t/N goes below $\frac{1}{2}$. From this point on, we know that

$$\frac{t^N}{N!} + \frac{t^{N+1}}{(N+1)!} + \frac{t^{N+2}}{(N+2)!} + \cdots \quad \text{is less than} \quad \frac{t^N}{N!}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

The right side is $t^N/N!$ times 2. The left side is smaller. The first N terms that come before $t^N/N!$ have no effect on convergence of the series (they just enter the final sum). *So the series for e^t always converges.*

If t is negative, use its absolute value $|t|$ and the proof still succeeds. The series for the derivative of e^t is the same as the series for e^t. So we know: This series is absolutely convergent. We can safely say that $y' = y$.

Four approaches to e^t Looking back at my own teaching and writing, I really missed the importance of this big step in calculus. Just another function? *Not at all.* Textbooks offer four main ways to construct $y = e^t$:

1. Add all the terms $t^n/n!$. The derivative of each term is the previous $t^{n-1}/(n-1)!$

2. Take the nth power of $(1 + t/n)$ as in compound interest. Let n approach infinity.

3. The slope of b^t is C times b^t. Choose e as the value of b that makes $C = 1$.

4. Integrate $1/y$ to construct $t = \ln y$. Invert this function to find $y = e^t$.

I believe that **3** and **4** are too tricky. Explicit constructions are the winners. You want to say, "*Here is the function.*" In method **2** you are working with $(1 + t/n)^n$: not too bad. In **1** you see step by step and term by term that $dy/dt = y$.

Chapter 2

Second Order Equations

2.1 Second Derivatives in Science and Engineering

Second order equations involve the second derivative d^2y/dt^2. Often this is shortened to y'', and then the first derivative is y'. In physical problems, y' can represent velocity v and the second derivative $y'' = a$ is **acceleration**: the rate dy'/dt that velocity is changing.

The most important equation in dynamics is Newton's Second Law $\boldsymbol{F = ma}$. Compare a second order equation to a first order equation, and allow them to be nonlinear:

$$\textbf{First order} \quad y' = f(t, y) \qquad \textbf{Second order} \quad y'' = F(t, y, y') \tag{1}$$

The second order equation needs **two initial conditions**, normally $y(0)$ and $y'(0)$—the initial velocity as well as the initial position. Then the equation tells us $y''(0)$ and the movement begins.

When you press the gas pedal, that produces acceleration. The brake pedal also brings acceleration but it is *negative* (the velocity decreases). The steering wheel produces acceleration too ! Steering changes the direction of velocity, not the speed.

Right now we stay with straight line motion and one–dimensional problems :

$$\frac{d^2y}{dt^2} > 0 \quad \text{(speeding up)} \qquad\qquad \frac{d^2y}{dt^2} < 0 \quad \text{(slowing down).}$$

The graph of $y(t)$ bends upwards for $y'' > 0$ (the right word is *convex*). Then the velocity y' (slope of the graph) is increasing. The graph bends downwards for $y'' < 0$ (*concave*). Figure 2.1 shows the graph of $\boldsymbol{y = \sin t}$, when the acceleration is $a = d^2y/dt^2 = -\sin t$. The important equation $\boldsymbol{y'' = -y}$ leads to $\sin t$ and $\cos t$.

Notice how the velocity dy/dt (slope of the graph) changes sign in between zeros of y.

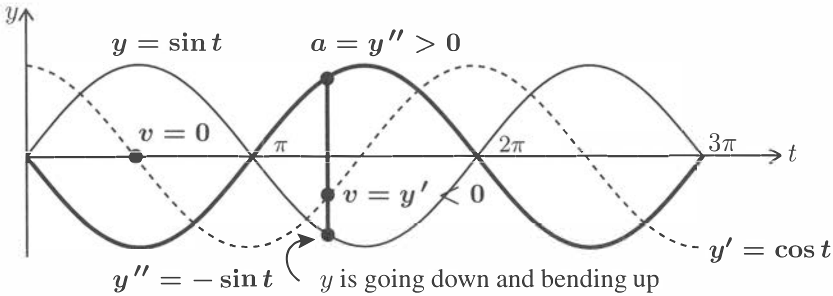


Figure 2.1: $y'' > 0$ means that velocity y' (or slope) increases. The curve bends upward.

The best examples of $F = ma$ come when the force F is $-ky$, a constant k times the "position" or "displacement" $y(t)$. This produces the oscillation equation.

Fundamental equation of mechanics $$m\frac{d^2y}{dt^2} + ky = 0 \qquad (2)$$

Think of a mass hanging at the bottom of a spring (Figure 2.2). The top of the spring is fixed, and the spring will stretch. Now stretch it a little more (move the mass downward by $y(0)$) and let go. The spring pulls back on the mass. Hooke's Law says that the force is $F = -ky$, proportional to the stretching distance y. Hooke's constant is k.

The mass will oscillate up and down. The oscillation goes on forever, because equation (2) does not include any friction (damping term $b\,dy/dt$). The oscillation is a perfect cosine, with $y = \cos\omega t$ and $\omega = \sqrt{k/m}$, because the second derivative has to produce k/m to match $y'' = -(k/m)y$.

Oscillation at frequency $\omega = \sqrt{\dfrac{k}{m}}$ $$y = y(0)\cos\left(\sqrt{\frac{k}{m}}\,t\right). \qquad (3)$$

At time $t = 0$, this shows the extra stretching $y(0)$. The derivative of $\cos\omega t$ has a factor $\omega = \sqrt{k/m}$. The second derivative y'' has the required $\omega^2 = k/m$, so $my'' = -ky$.

The movement of one spring and one mass is especially simple. There is only one frequency ω. When we connect N masses by a line of springs there will be N frequencies—then Chapter 6 has to study the eigenvalues of N by N matrices.

$$m\frac{d^2y}{dt^2} = -ky$$

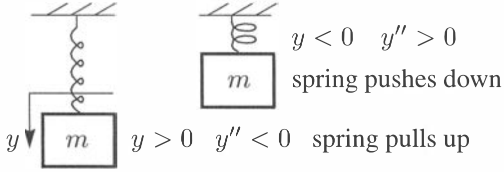


Figure 2.2: Larger k = stiffer spring = *faster* ω. Larger m = heavier mass = *slower* ω.

Initial Velocity $y'(0)$

Second order equations have *two* initial conditions. The motion starts in an initial position $y(0)$, and its initial velocity is $y'(0)$. We need both $y(0)$ and $y'(0)$ to determine the two constants c_1 and c_2 in the complete solution to $my'' + ky = 0$:

$$\textbf{"Simple harmonic motion"} \qquad y = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right). \tag{4}$$

Up to now the motion has started from rest ($y'(0) = 0$, no initial velocity). Then c_1 is $y(0)$ and c_2 is zero: only cosines. As soon as we allow an initial velocity, the sine solution $y = c_2 \sin \omega t$ must be included. But its coefficient c_2 is not just $y'(0)$.

$$\text{At} \quad t = 0, \quad \frac{dy}{dt} = c_2\, \omega \cos \omega t \quad \text{matches} \quad y'(0) \quad \text{when} \quad \boldsymbol{c_2 = \frac{y'(0)}{\omega}}. \tag{5}$$

The original solution $y = y(0) \cos \omega t$ matched $y(0)$, with zero velocity at $t = 0$. The new solution $y = (y'(0)/\omega) \sin \omega t$ has the right initial velocity and it starts from zero. When we combine those two solutions, $y(t)$ matches both conditions $y(0)$ and $y'(0)$:

$$\textbf{Unforced oscillation} \qquad \boldsymbol{y(t) = y(0) \cos \omega t + \frac{y'(0)}{\omega} \sin \omega t \ \text{ with } \ \omega = \sqrt{\frac{k}{m}}}. \tag{6}$$

With a trigonometric identity, I can combine those two terms (cosine and sine) into one.

Cosine with Phase Shift

We want to rewrite the solution (6) as $\boldsymbol{y(t) = R \cos(\omega t - \alpha)}$. The amplitude of $y(t)$ will be the positive number R. The phase shift or lag in this solution will be the angle α. By using the right identity for the cosine of $\omega t - \alpha$, we match both $\cos \omega t$ and $\sin \omega t$:

$$R \cos(\omega t - \alpha) = R \cos \omega t \, \cos \alpha + R \sin \omega t \, \sin \alpha. \tag{7}$$

This combination of $\cos \omega t$ and $\sin \omega t$ agrees with the solution (6) if

$$R \cos \alpha = y(0) \quad \text{and} \quad R \sin \alpha = \frac{y'(0)}{\omega}. \tag{8}$$

Squaring those equations and adding will produce R^2:

$$\textbf{Amplitude } \boldsymbol{R} \qquad R^2 = R^2(\cos^2 \alpha + \sin^2 \alpha) = (y(0))^2 + \left(\frac{y'(0)}{\omega}\right)^2. \tag{9}$$

The ratio of the equations (8) will produce the tangent of α:

$$\textbf{Phase lag } \boldsymbol{\alpha} \qquad \tan \alpha = \frac{R \sin \alpha}{R \cos \alpha} = \frac{y'(0)}{\omega\, y(0)}. \tag{10}$$

Problem 14 will discuss the angle α we should choose, since different angles can have the same tangent. The tangent is the same if α is increased by π or any multiple of π.

The pure cosine solution that started from $y'(0) = 0$ has *no phase shift*: $\alpha = 0$. Then the new form $y(t) = R \cos(\omega t - \alpha)$ is the same as the old form $y(0) \cos \omega t$.

Frequency ω or f

If the time t is measured in *seconds*, the frequency ω is in *radians per second*. Then ωt is in radians—it is an angle and $\cos \omega t$ is its cosine. But not everyone thinks naturally about radians. Complete cycles are easier to visualize. So frequency is also measured in *cycles per second*. A typical frequency in your home is $f = 60$ cycles per second. One cycle per second is usually shortened to $\boldsymbol{f = 1}$ **Hertz**. A complete cycle is 2π radians, so $f = 60$ *Hertz is the same frequency as* $\omega = 120\pi$ *radians per second*.

The **period** is the time T for one complete cycle. Thus $T = 1/f$. This is the only page where f is a frequency—on all other pages $f(t)$ is the driving function.

Frequency $\quad \boldsymbol{\omega = 2\pi f} \qquad$ **Period** $\quad \boldsymbol{T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}}$.

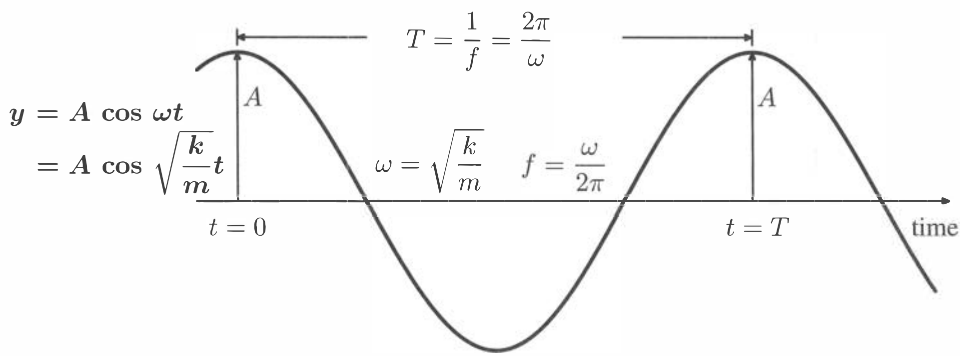


Figure 2.3: Simple harmonic motion $y = A \cos \omega t$: amplitude A and frequency ω.

Harmonic Motion and Circular Motion

Harmonic motion is up and down (or side to side). **When a point is in circular motion, its projections on the x and y axes are in harmonic motion**. Those motions are closely related, which is why a piston going up and down can produce circular motion of a flywheel. The harmonic motion "speeds up in the middle and slows down at the ends" while the point moves with constant speed around the circle.

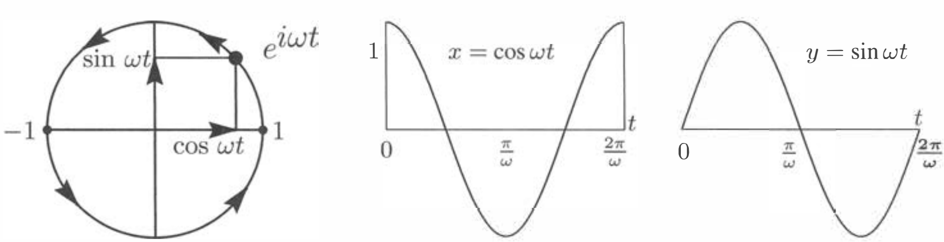


Figure 2.4: Steady motion around a circle produces cosine and sine motion along the axes.

Response Functions

I want to introduce some important words. The **response** is the output $y(t)$. Up to now the only inputs were the initial values $y(0)$ and $y'(0)$. In this case $y(t)$ would be the *initial value response* (but I have never seen those words). When we only see a few cycles of the motion, initial values make a big difference. In the long run, what counts is the response to a *forcing function* like $f = \cos \omega t$.

Now ω is the **driving frequency** on the right hand side, where the **natural frequency** $\omega_n = \sqrt{k/m}$ is decided by the left hand side: ω comes from y_p, ω_n comes from y_n. When the motion is driven by $\cos \omega t$, a particular solution is $y_p = Y \cos \omega t$:

Forced motion $y_p(t)$ at frequency ω $\quad my'' + ky = \cos \omega t \quad \boldsymbol{y_p(t) = \dfrac{1}{k - m\omega^2} \cos \omega t.}$ (11)

To find $y_p(t)$, I put $Y \cos \omega t$ into $my'' + ky$ and the result was $(k - m\omega^2) Y \cos \omega t$. This matches the driving function $\cos \omega t$ when $Y = 1/(k - m\omega^2)$.

The initial conditions are nowhere in equation (11). Those conditions contribute the null solution y_n, which oscillates at the natural frequency $\omega_n = \sqrt{k/m}$. Then $k = m\omega_n^2$.

If I replace k by $m\omega_n^2$ in the response $y_p(t)$, I see $\omega_n^2 - \omega^2$ in the denominator:

$$\boxed{\textbf{Response to } \cos \omega t \qquad \boldsymbol{y_p(t) = \frac{1}{m\left(\omega_n^2 - \omega^2\right)} \cos \omega t.}} \qquad (12)$$

Our equation $my'' + ky = \cos \omega t$ has no damping term. That will come in Section 2.3. It will produce a phase shift α. Damping will also reduce the amplitude $|Y(\omega)|$. The amplitude is all we are seeing here in $Y(\omega) \cos \omega t$:

Frequency response $\quad \boldsymbol{Y(\omega) = \dfrac{1}{k - m\omega^2} = \dfrac{1}{m\left(\omega_n^2 - \omega^2\right)}.}$ (13)

The mass and spring, or the inductance and capacitance, decide the natural frequency ω_n. The response to a driving term $\cos \omega t$ (or $e^{i\omega t}$) is multiplication by the frequency response $Y(\omega)$. *The formula changes when $\omega = \omega_n$—we will study resonance!*

With damping in Section 2.3, the frequency response $Y(\omega)$ will be a complex number. We can't escape complex arithmetic and we don't want to. The magnitude $|Y(\omega)|$ will give the **magnitude response** (or amplitude response). The angle θ in the complex plane will decide the **phase response** (then $\alpha = -\theta$ because we measure the phase lag).

The response is $Y(\omega)e^{i\omega t}$ to $f(t) = e^{i\omega t}$ and the response is $g(t)$ to $f(t) = \delta(t)$. These show the frequency response Y from equation (13) and the impulse response g from equation (15). *$Ye^{i\omega t}$ and $g(t)$ are the two key solutions to $my'' + ky = f(t)$.*

Impulse Response = Fundamental Solution

The most important solution to a linear differential equation will be called $g(t)$. In mathematics g is the *fundamental solution.* In engineering g is the *impulse response.* It is a particular solution when the right side $f(t) = \delta(t)$ is an impulse (a delta function).

The same $g(t)$ solves $mg'' + kg = 0$ when the initial velocity is $g'(0) = 1/m$.

Fundamental solution	$mg'' + kg = \delta(t)$	**with zero initial conditions**	(14)
Null solution also	$g(t) = \dfrac{\sin \omega_n t}{m\omega_n}$	**has** $g(0) = 0$ **and** $g'(0) = \dfrac{1}{m}$.	(15)

To find that null solution, I just put its initial values 0 and $1/m$ into equation (6). The cosine term disappeared because $g(0) = 0$.

I will show that those two problems give the same answer. Then this whole chapter will show why $g(t)$ is so important. For first order equations $y' = ay + q$ in Chapter 1, the fundamental solution (impulse response, growth factor) was $g(t) = e^{at}$. The first two names were not used, but you saw how e^{at} dominated that whole chapter.

I will first explain the response $g(t)$ in physical language. *We strike the mass and it starts to move.* All our force is acting at one instant of time: *an impulse.* A finite force within one moment is impossible for an ordinary function, only possible for a delta function. Remember that the integral of $\delta(t)$ jumps to 1 when we pass the point $t = 0$.

If we integrate $mg'' = \delta(t)$, nothing happens before $t = 0$. In that instant, the integral jumps to 1. The integral of the left side mg'' is mg'. Then $mg' = 1$ instantly at $t = 0$. This gives $g'(0) = 1/m$. You see that computing with an impulse $\delta(t)$ needs some faith.

The point of $g(t)$ is that it solves the equation for any forcing function $f(t)$:

$$\boxed{my'' + ky = f(t) \text{ has the particular solution } y(t) = \int_0^t g(t-s) f(s)\, ds.} \tag{16}$$

That was the key formula of Chapter 1, when $g(t-s)$ was $e^{a(t-s)}$ and the equation was first order. Section 2.3 will find $g(t)$ when the differential equation includes damping. The coefficients in the equation will stay constant, to allow a neat formula for $g(t)$.

You may feel uncertain about working with delta functions—a means to an end. We will verify this final solution $y(t)$ in three different ways:

1 Substitute $y(t)$ from (16) directly into the differential equation (Problem 21)

2 Solve for $y(t)$ by variation of parameters (Section 2.6)

3 Solve again by using the Laplace transform $Y(s)$ (Section 2.7).

■ REVIEW OF THE KEY IDEAS ■

1. $my'' + ky = 0$: A mass on a spring oscillates at the natural frequency $\omega_n = \sqrt{k/m}$.

2. $my'' + ky = \cos\omega t$: This driving force produces $y_p = (\cos\omega t)/m\left(\omega_n^2 - \omega^2\right)$.

3. There is resonance when $\omega_n = \omega$. The solution $y_p = t\sin\omega t$ includes a new factor t.

4. $mg''+kg = \delta(t)$ gives $\boldsymbol{g(t) = (\sin\omega_n t)/m\omega_n}$ = null solution with $g'(0) = 1/m$.

5. Fundamental solution g: Every driving function f gives $y(t) = \int_0^t g(t-s)f(s)\,ds$.

6. Frequency: ω radians per second or f cycles per second (f Hertz). Period $T = 1/f$.

Problem Set 2.1

1 Find a cosine and a sine that solve $d^2y/dt^2 = -9y$. This is a second order equation so we expect *two constants* C and D (from integrating twice):

Simple harmonic motion $\qquad y(t) = C\cos\omega t + D\sin\omega t.\quad$ What is ω?

If the system starts from rest (this means $dy/dt = 0$ at $t = 0$), which constant C or D will be zero?

2 In Problem 1, which C and D will give the starting values $y(0) = 0$ and $y'(0) = 1$?

3 Draw Figure 2.3 to show simple harmonic motion $y = A\cos(\omega t - \alpha)$ with phases $\alpha = \pi/3$ and $\alpha = -\pi/2$.

4 Suppose the circle in Figure 2.4 has radius 3 and circular frequency $f = 60$ Hertz. If the moving point starts at the angle $-45°$, find its x-coordinate $A\cos(\omega t - \alpha)$. The phase lag is $\alpha = 45°$. When does the point first hit the x axis?

5 If you drive at 60 miles per hour on a circular track with radius $R = 3$ miles, what is the time T for one complete circuit? Your circular frequency is $f =$ _____ and your angular frequency is $\omega =$ _____ (with what units?). The period is T.

6 The total energy E in the oscillating spring-mass system is

$$E = \textbf{kinetic}\text{ energy in mass} + \textbf{potential}\text{ energy in spring} = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2.$$

Compute E when $y = C\cos\omega t + D\sin\omega t$. The energy is constant!

7 Another way to show that the total energy E is constant:

Multiply $\boldsymbol{my'' + ky = 0}$ by $\boldsymbol{y'}$. Then integrate $my'y''$ and kyy'.

8 A **forced oscillation** has another term in the equation and $A\cos\omega t$ in the solution:

$$\frac{d^2y}{dt^2} + 4y = F\cos\omega t \quad \text{has} \quad y = C\cos 2t + D\sin 2t + A\cos\omega t.$$

(a) Substitute y into the equation to see how C and D disappear (they give y_n). Find the forced amplitude A in the particular solution $y_p = A\cos\omega t$.

(b) In case $\omega = 2$ (forcing frequency = natural frequency), what answer does your formula give for A? The solution formula for y breaks down in this case.

9 Following Problem 8, write down the complete solution $y_n + y_p$ to the equation

$$m\frac{d^2y}{dt^2} + ky = F\cos\omega t \;\text{ with }\; \omega \neq \omega_n = \sqrt{k/m} \;\text{ (no resonance)}.$$

The answer y has free constants C and D to match $y(0)$ and $y'(0)$ (A *is fixed* by F).

10 Suppose Newton's Law $F = ma$ has the force F in the *same* direction as a:

$$my'' = +ky \qquad \text{including} \qquad y'' = 4y.$$

Find two possible choices of s in the exponential solutions $y = e^{st}$. The solution is not sinusoidal and s is real and the oscillations are gone. Now y is unstable.

11 Here is a *fourth order* equation: $d^4y/dt^4 = 16y$. Find *four* values of s that give exponential solutions $y = e^{st}$. You could expect four initial conditions on y: $y(0)$ is given along with what three other conditions?

12 To find a particular solution to $y'' + 9y = e^{ct}$, I would look for a multiple $y_p(t) = Ye^{ct}$ of the forcing function. What is that number Y? When does your formula give $Y = \infty$? (Resonance needs a new formula for Y.)

13 In a particular solution $y = Ae^{i\omega t}$ to $y'' + 9y = e^{i\omega t}$, what is the amplitude A? The formula blows up when the forcing frequency ω = what natural frequency?

14 Equation (10) says that the tangent of the phase angle is $\tan\alpha = y'(0)/\omega y(0)$. First, check that $\tan\alpha$ is dimensionless when y is in meters and time is in seconds. Next, if that ratio is $\tan\alpha = 1$, should you choose $\alpha = \pi/4$ or $\alpha = 5\pi/4$? Answer:

$$\textbf{Separately you want } R\cos\alpha = y(0) \textbf{ and } R\sin\alpha = y'(0)/\omega.$$

If those right hand sides are positive, choose the angle α between 0 and $\pi/2$.
If those right hand sides are negative, add π and choose $\alpha = 5\pi/4$.

Question: If $y(0) > 0$ and $y'(0) < 0$, does α fall between $\pi/2$ and π or between $3\pi/2$ and 2π? If you plot the vector from $(0,0)$ to $(y(0), y'(0)/\omega)$, its angle is α.

15 Find a point on the sine curve in Figure 2.1 where $y > 0$ but $v = y' < 0$ and also $a = y'' < 0$. The curve is sloping down and bending down.

Find a point where $y < 0$ but $y' > 0$ and $y'' > 0$. The point is below the x-axis but the curve is sloping _____ and bending _____.

16 (a) Solve $y'' + 100y = 0$ starting from $y(0) = 1$ and $y'(0) = 10$. (**This is $\boldsymbol{y_n}$.**)

(b) Solve $y'' + 100y = \cos\omega t$ with $y(0) = 0$ and $y'(0) = 0$. (**This can be $\boldsymbol{y_p}$.**)

17 Find a particular solution $y_p = R\cos(\omega t - \alpha)$ to $y'' + 100y = \cos\omega t - \sin\omega t$.

18 Simple harmonic motion also comes from a linear pendulum (like a grandfather clock). At time t, the height is $A\cos\omega t$. What is the frequency ω if the pendulum comes back to the start after 1 second? The period does not depend on the amplitude (a large clock or a small metronome or the movement in a watch can all have $T = 1$).

19 If the phase lag is α, what is the time lag in graphing $\cos(\omega t - \alpha)$?

20 What is the response $y(t)$ to a delayed impulse if $my'' + ky = \delta(t - T)$?

21 (Good challenge) Show that $y = \int_0^t g(t-s)f(s)\,ds$ has $my'' + ky = f(t)$.

1 Why is $y' = \int_0^t g'(t-s)f(s)\,ds + g(0)f(t)$? Notice the two t's in y.

2 Using $g(0) = 0$, explain why $y'' = \int_0^t g''(t-s)f(s)\,ds + g'(0)f(t)$.

3 Now use $g'(0) = 1/m$ and $mg'' + kg = 0$ to confirm $my'' + ky = f(t)$.

22 With $f = 1$ (direct current has $\omega = 0$) verify that $my'' + ky = 1$ for this y:

$$\textbf{Step response}\quad y(t) = \int_0^t \frac{\sin\omega_n(t-s)}{m\omega_n}\,1\,ds = y_p + y_n \text{ equals } \frac{\mathbf{1}}{\boldsymbol{k}} - \frac{\mathbf{1}}{\boldsymbol{k}}\,\mathbf{cos}\,\boldsymbol{\omega_n t}.$$

23 (Recommended) For the equation $d^2y/dt^2 = 0$ find the null solution. Then for $d^2g/dt^2 = \delta(t)$ find the fundamental solution (start the null solution with $g(0) = 0$ and $g'(0) = 1$). For $y'' = f(t)$ find the particular solution using formula (16).

24 For the equation $d^2y/dt^2 = e^{i\omega t}$ find a particular solution $y = Y(\omega)e^{i\omega t}$. Then $Y(\omega)$ is the frequency response. Note the "resonance" when $\omega = 0$ with the null solution $y_n = 1$.

25 Find a particular solution $Ye^{i\omega t}$ to $my'' - ky = e^{i\omega t}$. The equation has $-ky$ instead of ky. What is the frequency response $Y(\omega)$? For which ω is Y infinite?

2.2 Key Facts About Complex Numbers

The solutions to differential equations involve *real* numbers $\boldsymbol{a}$ and *imaginary* numbers $\boldsymbol{i\omega}$. They combine into *complex* numbers $\boldsymbol{s = a + i\omega}$ (real plus imaginary). Here are three equations and their solutions:

$\dfrac{dy}{dt} = ay$	$\dfrac{d^2y}{dt^2} + \omega^2 y = 0$	$\dfrac{d^2y}{dt^2} - 2a\dfrac{dy}{dt} + (\omega^2 + a^2)y = 0$
$y = Ce^{at}$	$y = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$	$y = c_1 e^{(a+i\omega)t} + c_2 e^{(a-i\omega)t}$

Chapter 1 solved $y' = ay$. Section 2.1 solved $y'' + \omega^2 y = 0$. Section 2.3 will solve the last equation $Ay'' + By' + Cy = 0$. The balance between real and imaginary (between a and $i\omega$) will come down to a competition between B^2 and $4AC$.

This course cannot go forward without complex numbers. You see their rectangular form in $s = a + i\omega$ (real part and imaginary part). What you must also see is their **polar form**. It is e^{st}, more than s by itself, that demands to be seen in polar form:

$$e^{st} = e^{(a+i\omega)t} = e^{at}\, e^{i\omega t}$$

$$e^{at} \textbf{ gives growth or decay} \qquad e^{i\omega t} \textbf{ gives oscillation and rotation}$$

The real part a is the rate of growth. The imaginary part ω is the frequency of oscillation. The addition $a + i\omega$ turns into the multiplication $e^{at}e^{i\omega t}$ because of the rule for exponentials. We will surely see exponentials everywhere, because they solve all constant coefficient equations: *The solution to* $y' = sy$ *is* $y = Ce^{st}$. With a forcing function $e^{i\omega t}$, a particular solution to $y' - sy = e^{i\omega t}$ is $y_p = e^{i\omega t}/(i\omega - s)$: a complex function.

Euler's formula $e^{i\omega t} = \cos\omega t + i\sin\omega t$ brings back two real functions (cosine and sine). Real equations have real solutions. When the forcing function on the right side is $f = A\cos\omega t + B\sin\omega t$, a good particular solution is $y_p = M\cos\omega t + N\sin\omega t$.

In this real world, the amplitudes $\sqrt{A^2+B^2}$ and $\sqrt{M^2+N^2}$ are all-important. The amplitude is what we see (in light) and hear (in sound) and feel (in vibration).

The null solutions y_n and the particular solution y_p need complex numbers. The form of y_n is Ce^{st}. The form of y_p is $Ye^{i\omega t}$. The complex gain is Y. Notice that the ω in $s = a + i\omega$ is the *natural frequency* in the null solution y_n. The ω in the right hand side $e^{i\omega t}$ is the *driving frequency* in the particular solution y_p.

If $\omega_{natural} = \omega_{driving}$, we will see "resonance" and we will need new formulas.

Here is the plan for this section.

1 Multiply complex numbers s_1 and s_2 (review).

2 Use the polar form $s = re^{i\theta}$ to find the powers $s^n = r^n e^{in\theta}$ (review).

3 Look especially at the equation $s^n = 1$. It has n roots, all on the **unit circle**.

4 Find the **exponential** $\boldsymbol{e^{st}}$ and watch it move in the complex plane.

Complex Numbers : Rectangular and Polar

A complex number $a + i\omega$ has a real part a and an imaginary part ω. Two complex numbers are easy to add: real part $a_1 + a_2$, imaginary part $\omega_1 + \omega_2$. It is multiplication that looks messy in equation (1). The good way is in equation (5).

Multiplication $$(a_1 + i\omega_1)\,(a_2 + i\omega_2) = (a_1a_2 - \omega_1\omega_2) + i(a_1\omega_2 + a_2\omega_1). \quad (1)$$

Just multiply each part a_1 and $i\omega_1$ by each part a_2 and $i\omega_2$.

Important case s times $\overline{s}$ $$(a + i\omega)\,(a - i\omega) = a^2 + \omega^2 \text{ : Real number.} \quad (2)$$

$\overline{s} = a - i\omega$ is the **complex conjugate** of $s = a + i\omega$. Equation (2) says that $\boldsymbol{s\overline{s} = |s|^2}$.

$|s| = \boldsymbol{\sqrt{a^2+\omega^2}}$ is the *absolute value* or *magnitude* or *modulus* of $s = a + i\omega$.

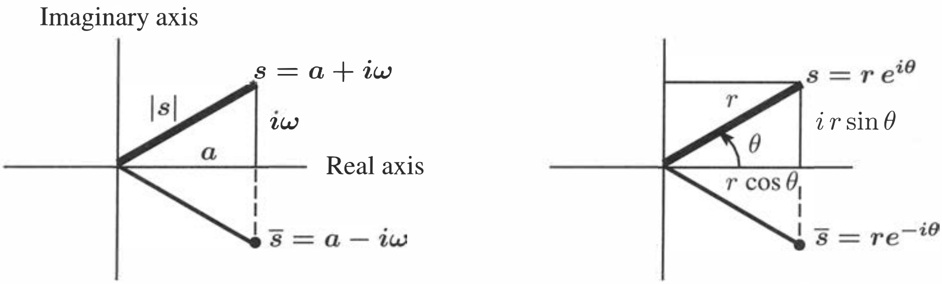


Figure 2.5: (i) The rectangular form $s = a + i\omega$. (ii) The polar form $s = re^{i\theta}$ with absolute value $r = |s| = \sqrt{a^2+\omega^2}$. The complex conjugate of s is $\overline{s} = a - i\omega = re^{-i\theta}$.

The polar form of s uses that distance $r = |s|$ to the center point $(0, 0)$. The real numbers a and ω (rectangular) are connected to r and θ (polar) by

$$\boxed{\boldsymbol{a = r\cos\theta} \qquad \boldsymbol{\omega = r\sin\theta} \qquad s = a + i\omega = r\,(\cos\theta + i\,\sin\,\theta) = \boldsymbol{re^{i\theta}}.} \quad (3)$$

At that moment you see Euler's Formula $e^{i\theta} = \cos\theta + i\,\sin\,\theta$. I could regard this as the complex *definition* of the exponential. Or I can separate the infinite series for $e^{i\theta}$ into its real part (the series for $\cos\theta$) and imaginary part (the series for $\sin\theta$).

Euler's Formula is used all the time, to express $e^{i\theta}$ in terms of $\cos\theta$ and $\sin\,\theta$. It is useful to go the other way, and express the cosine and sine in terms of $e^{i\theta}$ and $e^{-i\theta}$:

Cosines from exponentials $$\cos\theta = \frac{\boldsymbol{e^{i\theta} + e^{-i\theta}}}{\boldsymbol{2}} \qquad \sin\,\theta = \frac{\boldsymbol{e^{i\theta} - e^{-i\theta}}}{\boldsymbol{2i}} \quad (4)$$

The sine comes from subtraction. Cancel $\cos\theta$ to get $2i\,\sin\,\theta$. We need to divide by $2i$.

The Polar Form of s^n and $1/s$

The polar form is perfect for multiplication and for powers s^n. We just multiply absolute values of s_1 and s_2, and *add* their angles. Multiply $r_1 r_2$ and add $\theta_1 + \theta_2$.

Multiplication $s_1 s_2$	$\left(r_1\, e^{i\theta_1}\right)\left(r_2\, e^{i\theta_2}\right) = \boldsymbol{r_1 r_2}\, e^{i(\boldsymbol{\theta_1 + \theta_2})}$	(5)
Powers of $s = re^{i\theta}$	$s^n = \left(r\, e^{i\theta}\right)^n = \boldsymbol{r^n}\, e^{\boldsymbol{in\theta}}$	(6)

If $n = 2$, we are multiplying $re^{i\theta}$ times $re^{i\theta}$ to get $r^2 e^{i2\theta}$. (θ is added to θ.) If $n = -1$, we are dividing. The rectangular form of $1/(a + i\omega)$ matches the polar form of $1/(re^{i\theta})$:

$$\frac{\mathbf{1}}{\boldsymbol{a + i\omega}} = \frac{1}{a + i\omega}\,\frac{a - i\omega}{a - i\omega} = \frac{\boldsymbol{a - i\omega}}{\boldsymbol{a^2 + \omega^2}} \qquad \frac{\mathbf{1}}{\boldsymbol{re^{i\theta}}} = \frac{1}{r}\,\frac{1}{e^{i\theta}} = \frac{\mathbf{1}}{\boldsymbol{r}}\boldsymbol{e^{-i\theta}}. \tag{7}$$

That magnitude is $r = |a + i\omega| = \sqrt{a^2 + \omega^2}$. Equation (7) says that $1/s$ equals $\overline{s}/|s|^2$. In solving $y' - ay = e^{i\omega t}$, what we meet is $y = e^{i\omega t}/(i\omega - a)$:

$$\textbf{Gain } G \textbf{ and Phase } \boldsymbol{\alpha} \qquad i\omega - a = re^{i\alpha} \qquad \frac{1}{i\omega - a} = \frac{1}{r}\, e^{-i\alpha} = Ge^{-i\alpha} \tag{8}$$

I prefer this polar form. When $s = re^{i\theta}$, the absolute value of $1/s$ is $1/r$. The angle is $-\theta$.

Examples The polar form of $1 + i$ is $\sqrt{2}e^{i\pi/4}$: absolute value $r = \sqrt{1+1} = \boldsymbol{\sqrt{2}}$.
The polar form of its conjugate $1 - i$ is $\boldsymbol{\sqrt{2}}e^{-\pi i/4}$.
The polar form of its reciprocal $1/(1 + i)$ is $(\mathbf{1}/\boldsymbol{\sqrt{2}})e^{-\pi i/4}$.

Notice that *we can add 2π to the angle θ.* That brings us around a circle and back to the same point. Then $e^{i\theta} = e^{i(\theta + 2\pi)}$ and $e^{-i\pi/4} = e^{7\pi i/4}$.

The Unit Circle

The polar form brings out the importance of the unit circle in the complex plane. That circle contains all complex numbers with absolute value $r = |s| = 1$. The numbers on the unit circle are exactly $\boldsymbol{s = e^{i\theta} = \cos\theta + i\,\sin\theta}$.

Since $r = 1$, every r^n is also 1. All powers like s^2 and s^{-1} stay on the unit circle. The angles in Figure 2.6 become 2θ and $-\theta$. The n^{th} power s^n has angle $n\theta$.

Here is a nice application of complex numbers to trigonometry. The "double angle" formulas for $\cos 2\theta$ and $\sin 2\theta$ are not so easy to remember. The "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ are even harder. But all these formulas come from one simple fact:

$$\boldsymbol{(e^{i\theta})^n = e^{in\theta}} \qquad (\cos\theta + i\,\sin\theta)^{\boldsymbol{n}} = \cos n\theta + i\,\sin n\theta. \tag{9}$$

If you take $n = 2$, you are squaring $e^{i\theta} = \cos\theta + i\,\sin\theta$ to get $e^{i2\theta}$:

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\,\cos\theta\,\sin\theta = \cos 2\theta + i\,\sin 2\theta. \tag{10}$$

The real part $\cos^2\theta - \sin^2\theta$ is $\boldsymbol{\cos 2\theta}$. The imaginary part $2\,\sin\theta\cos\theta$ is $\boldsymbol{\sin 2\theta}$. For triple angles, multiply again by $\cos\theta + i\,\sin\theta$ (in Problem 4).

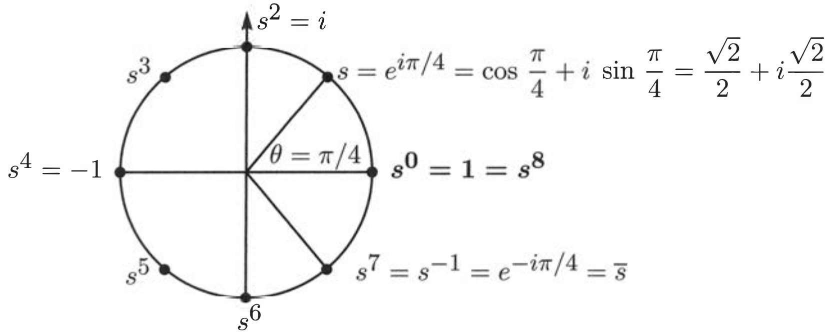


Figure 2.6: The number $s = e^{i\theta}$ has $s^2 = e^{i2\theta}$ and $s^{-1} = e^{-i\theta}$, all on the circle with $r = 1$. Here $\theta = 45°$ which is $\pi/4$ radians. So $2\theta = 90°$ and $\boldsymbol{s^2 = i}$. **Then** $\boldsymbol{s^8 = 1}$.

The Equation $s^n = 1$

There are two numbers with $s^2 = 1$ (they are $s = 1$ and -1). There are four numbers with $s^4 = 1$ (they are 1 and -1 and i and $-i$). Those four numbers are *equally spaced around the unit circle*. This is the pattern for every equation $s^n = 1$: n numbers equally spaced around the unit circle, starting with $s = 1$. The Fundamental Theorem of Algebra says that n^{th} degree equations have n (possibly complex) solutions. The equation $s^n = 1$ is no exception, and all its roots are on the unit circle.

$$\boldsymbol{n} \textbf{ roots of } \boldsymbol{s^n = 1} \qquad s = e^{2\pi i/n}, s = e^{4\pi i/n}, \ldots, s = e^{2n\pi i/n} = e^{2\pi i} = 1.$$

These are the powers $s, s^2, \ldots, s^n$ of the special complex number $s = e^{2\pi i/n}$. This number $s = e^{2\pi i/8}$ is the first of the 8 solutions to $s^8 = 1$, going around the circle in Figure 2.6.

Here is a remarkable fact about the solutions to $s^n = 1$. **Those $\boldsymbol{n}$ numbers add to zero.** In Figure 2.6, you can see that $s^5 = -s$ and $s^6 = -s^2$ and $s^7 = -s^3$ and $s^8 = -s^4$. The roots pair off. Each pair adds to zero. So the 8 roots add to zero.

For $n = 3$ or 5 or 7, this pairing off will not work. The three solutions to $s^3 = 1$ are at $120°$ angles. (s and s^2 are $e^{2\pi i/3}$ and $e^{4\pi i/3}$, at angles $120°$ and $240°$. Then comes $360°$.) To show that those three numbers add to zero, I will factor $s^3 - 1 = 0$:

$$0 = s^3 - 1 = (s-1)(s^2 + s + 1) \quad \text{leads to} \quad \boldsymbol{s^2 + s + 1 = 0}. \tag{11}$$

The n numbers on the unit circle go into the Fourier matrix. They are the key to the overwhelming success of the Fast Fourier Transform in Section 8.2.

The Exponentials $e^{i\omega t}$ and e^{ist}

We use complex numbers to solve differential equations. For $dy/dt = ay$ the solution $y = Ce^{at}$ is real. But second order equations can bring oscillations $e^{i\omega t}$ together with growth/decay from e^{at}. Now y has sines and cosines, or complex exponentials.

$$\boxed{y = c_1\, e^{(a+i\omega)t} + c_2\, e^{(a-i\omega)t} \quad \text{or} \quad y = C_1\, e^{at} \cos \omega t + C_2\, e^{at} \sin \omega t.} \tag{12}$$

Our goal is to follow those pieces of the complete solution to $Ay'' + By' + Cy = 0$. *Where does the point $e^{(a+i\omega)t}$ travel in the complex plane ?* The next section connects a and ω to the numbers A, B, C and solves the differential equation.

The best way to track the path of $e^{(a+i\omega)t}$ is to separate a from $i\omega$. The path of $e^{i\omega t}$ is a circle. The factor e^{at} turns the circle into a spiral.

$$\textbf{Rule for exponentials} \qquad e^{(a+i\omega)t} = e^{at}e^{i\omega t}. \tag{13}$$

This is the polar form ! The factor e^{at} is the absolute value r. The angle ωt is the phase angle θ. As the time t increases, we follow those two parts :

Absolute value	e^{at} grows with t if $a > 0$ $\quad e^{at}$ decays if $a < 0$
Phase angle	$e^{i\omega t}$ goes around the unit circle when t increases by $2\pi/\omega$

The real part a decides stability. This is just like Chapter 1. We will see that damping produces $a < 0$ which is stability. In that case $B > 0$ in $y'' + By' + Cy = 0$.

This section is about the $i\omega$ part of the exponent s. That produces the $e^{i\omega t}$ part of the solution $y = e^{st}$. The pure oscillations in Section 2.1 came from $my'' + ky = 0$ with no damping. They had only this $e^{i\omega t}$ part (along with $e^{-i\omega t}$, which travels in the opposite direction around the unit circle). The frequency is $\omega = \sqrt{k/m}$.

Watch $e^{i\omega t}$ as it goes around the circle. If you follow its horizontal motion (its shadow on the x axis) you will see $\cos \omega t$. If you follow its height on the y axis, you will see $\sin \omega t$. The circle is complete when $\omega t = 2\pi$. So the period is $T = 2\pi/\omega$.

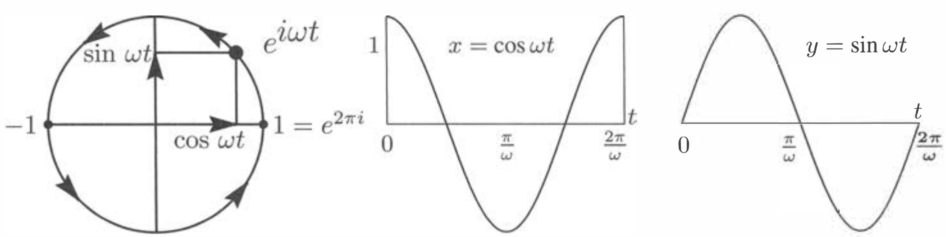


Figure 2.7: $y'' + \omega^2 y = 0$: One complex solution $e^{i\omega t}$ produces two real solutions.

When we multiply $e^{i\omega t}$ by e^{at}, **their product e^{st} gives a spiral.** The spiral goes in to the center if a is negative. The spiral goes outward $a > 0$. You are seeing the benefit of complex numbers, to merge oscillation and decay into one function. The real functions are $e^{at} \cos \omega t$ and $e^{at} \sin \omega t$. The complex function is $e^{at}\, e^{i\omega t} = e^{st}$.

Question What will be the time T and the crossing point X, when the spiral completes one loop and returns to the positive x–axis ?

Answer The time T will be $2\pi/\omega$, to complete each loop of the spiral. The crossing point on the x–axis will be $X = e^{aT}$. At time $2T$, the crossing will be at X^2.

Problem Set 2.2

1 Mark the numbers $s_1 = 2 + i$ and $s_2 = 1 - 2i$ as points in the complex plane. (The plane has a real axis and an imaginary axis.) Then mark the sum $s_1 + s_2$ and the difference $s_1 - s_2$.

2 Multiply $s_1 = 2 + i$ times $s_2 = 1 - 2i$. Check absolute values: $|s_1||s_2| = |s_1 s_2|$.

3 Find the real and imaginary parts of $1/(2+i)$. Multiply by $(2-i)/(2-i)$:

$$\frac{1}{2+i}\,\frac{2-i}{2-i} = \frac{2-i}{|2+i|^2} = ?$$

4 *Triple angles* Multiply equation (10) by another $e^{i\theta} = \cos\theta + i\sin\theta$ to find formulas for $\cos 3\theta$ and $\sin 3\theta$.

5 *Addition formulas* Multiply $e^{i\theta} = \cos\theta + i\sin\theta$ times $e^{i\phi} = \cos\phi + i\sin\phi$ to get $e^{i(\theta+\phi)}$. Its real part is $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. What is its imaginary part $\sin(\theta+\phi)$?

6 Find the real part and the imaginary part of each cube root of 1. Show directly that the three roots add to zero, as equation (11) predicts.

7 The three cube roots of 1 are z and z^2 and 1, when $z = e^{2\pi i/3}$. What are the three cube roots of 8 and the three cube roots of i ? (The angle for i is 90° or $\pi/2$, so the angle for one of its cube roots will be _____. The roots are spaced by 120°.)

8
(a) The number i is equal to $e^{\pi i/2}$. Then its i^{th} power $\boldsymbol{i^i}$ comes out equal to a real number, using the fact that $(e^s)^t = e^{st}$. What is that real number $\boldsymbol{i^i}$?

(b) $e^{i\pi/2}$ is also equal to $e^{5\pi i/2}$. Increasing the angle by 2π does not change $e^{i\theta}$ — it comes around a full circle and back to i. Then i^i has another real value $(e^{5\pi i/2})^i = e^{-5\pi/2}$. What are all the possible values of i^i ?

9 The numbers $s = 3 + i$ and $\overline{s} = 3 - i$ are complex conjugates. Find their sum $s + \overline{s} = -B$ and their product $(s)(\overline{s}) = C$. Then show that $s^2 + Bs + C = 0$ and also $\overline{s}^2 + B\overline{s} + C = 0$. Those numbers s and $\overline{s}$ are the two roots of the quadratic equation $x^2 + Bx + C = 0$.

10 The numbers $s = a + i\omega$ and $\overline{s} = a - i\omega$ are complex conjugates. Find their sum $s + \overline{s} = -B$ and their product $(s)(\overline{s}) = C$. Then show that $s^2 + Bs + C = 0$. The two solutions of $x^2 + Bx + C = 0$ are s and $\overline{s}$.

11
(a) Find the numbers $(1+i)^4$ and $(1+i)^8$.

(b) Find the polar form $re^{i\theta}$ of $(1 + i\sqrt{3})/(\sqrt{3} + i)$.

12 The number $z = e^{2\pi i/n}$ solves $z^n = 1$. The number $Z = e^{2\pi i/2n}$ solves $Z^{2n} = 1$. How is z related to Z? (This plays a big part in the Fast Fourier Transform.)

13 (a) If you know $e^{i\theta}$ and $e^{-i\theta}$, how can you find $\sin\theta$?

(b) Find all angles θ with $e^{i\theta} = -1$, and all angles ϕ with $e^{i\phi} = i$.

14 Locate all these points on one complex plane:

(a) $2 + i$ (b) $(2 + i)^2$ (c) $\dfrac{1}{2 + i}$ (d) $|2 + i|$

15 Find the absolute values $r = |z|$ of these four numbers. If θ is the angle for $6 + 8i$, what are the angles for these four numbers?

(a) $6 - 8i$ (b) $(6 - 8i)^2$ (c) $\dfrac{1}{6 - 8i}$ (d) $8i + 6$

16 What are the real and imaginary parts of $e^{a + i\pi}$ and $e^{a + i\omega}$?

17 (a) If $|s| = 2$ and $|z| = 3$, what are the absolute values of sz and s/z?

(b) Find upper and lower bounds in $L \le |s + z| \le U$. When does $|s + z| = U$?

18 (a) Where is the product $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta)$ in the complex plane?

(b) Find the absolute value $|S|$ and the polar angle ϕ for $S = \boldsymbol{\sin\theta + i\cos\theta}$.

This is my favorite problem, because S combines $\cos\theta$ and $\sin\theta$ in a new way. To find ϕ, you could plot S or add angles in the multiplication of part (a).

19 Draw the spirals $e^{(1 - i)t}$ and $e^{(2 - 2i)t}$. Do those follow the same curves? Do they go clockwise or anticlockwise? When the first one reaches the negative x-axis, what is the time T? What point has the second one reached at that time?

20 The solution to $d^2y/dt^2 = -y$ is $y = \cos t$ if the initial conditions are $y(0) =$ _____ and $y'(0) =$ _____. The solution is $y = \sin t$ when $y(0) =$ _____ and $y'(0) =$ _____. Write each of those solutions in the form $c_1 e^{it} + c_2 e^{-it}$, to see that real solutions can come from complex c_1 and c_2.

21 Suppose $y(t) = e^{-t} e^{it}$ solves $y'' + By' + Cy = 0$. What are B and C? If this equation is solved by $y = e^{3it}$, what are B and C?

22 From the multiplication $e^{iA}\, e^{-iB} = e^{i(A - B)}$, find the "subtraction formulas" for $\cos(A - B)$ and $\sin(A - B)$.

23 (a) If r and R are the absolute values of s and S, show that rR is the absolute value of sS. (Hint: Polar form!)

(b) If $\overline{s}$ and $\overline{S}$ are the complex conjugates of s and S, show that $\overline{s}\,\overline{S}$ is the complex conjugate of sS. (Polar form!)

24 Suppose a complex number s solves a real equation $s^3 + As^2 + Bs + C = 0$ (with A, B, C real). Why does the complex conjugate $\overline{s}$ also solve this equation? "*Complex solutions to real equations come in conjugate pairs s and $\overline{s}$.*"

25 (a) If two complex numbers add to $s + S = 6$ and multiply to $sS = 10$, what are s and S? (They are complex conjugates.)

(b) If two numbers add to $s + S = 6$ and multiply to $sS = -16$, what are s and S? (Now they are real.)

26 If two numbers s and S add to $s + S = -B$ and multiply to $sS = C$, show that s and S solve the quadratic equation $s^2 + Bs + C = 0$.

27 Find three solutions to $s^3 = -8i$ and plot the three points in the complex plane. What is the sum of the three solutions?

28 (a) For which complex numbers $s = a + i\omega$ does e^{st} approach 0 as $t \to \infty$? Those numbers s fill which "half–plane" in the complex plane?

(b) For which complex numbers $s = a + i\omega$ does s^n approach 0 as $n \to \infty$? Those numbers s fill which part of the complex plane? Not a half-plane!

2.3 Constant Coefficients A, B, C

Section 2.1 presented the important equation $my'' + ky = 0$. That is a special case of this second order constant coefficient equation. We still need two initial conditions:

$$\boxed{A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0} \quad \text{starting from } y(0) \text{ and } y'(0). \tag{1}$$

The coefficients A, B, C can be *any constants*. For pure oscillation, A was the mass m and C was the spring constant k, both positive. **$B > 0$ introduces damping.** In this section the numbers A, B, C can be positive or negative or zero, so we may have exponential growth or decay or (damped) oscillation. With zero on the right hand side of equation (1), this section is finding null solutions y_n: *unforced motion.*

Our first job is to solve equation (1). When the coefficients are constant, we always look for exponentials e^{st}. That number s can be positive (y will grow) or negative (y decays) or pure imaginary (y oscillates). If s is a complex number $a + i\omega$, then its real part a controls growth or decay. The imaginary part ω controls oscillation.

We will see the solutions clearly, because A, B, C are constant. The right choice of $y(0)$ and $y'(0)$ will produce the growth factor $g(t)$ that multiplies all inputs to give y_p.

The key step is to find the rate s in $y = e^{st}$. A second order equation normally has two possible rates s_1 and s_2. To find those numbers, substitute $y = e^{st}$ into equation (1):

$$As^2e^{st} + B\,se^{st} + Ce^{st} = 0. \tag{2}$$

The factor e^{st} can be divided out because it is never zero. This leaves an all-important equation to determine s:

Characteristic equation $\quad As^2 + Bs + C = 0. \quad$ (3)

This is an ordinary quadratic equation for s. Every quadratic has two roots s_1 and s_2. They could be real, they could be complex, they could be equal. The two roots come from the quadratic formula:

Two values for s

$$s_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \qquad s_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{4}$$

Those roots add up to $s_1 + s_2 = -B/A$. The roots multiply to give $s_1 s_2 = C/A$. The question of real roots or complex roots is highly important, and it has a direct answer:

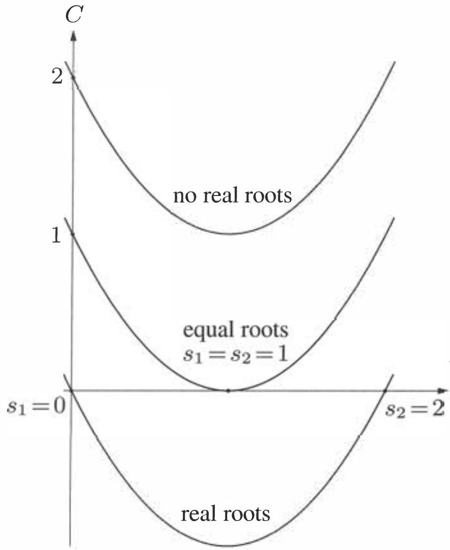
Real roots $B^2 > 4AC$	Equal roots $B^2 = 4AC$	Complex roots $B^2 < 4AC$

When $B^2 - 4AC$ is positive, its square root is real. Then we have real roots $s_1 > s_2$. When $B^2 - 4AC = 0$, its square root is zero and $s_1 = s_2$ (borderline case: equal roots). When $B^2 - 4AC$ is negative, its square root is *imaginary*. The quadratic formula (4) produces two complex numbers $a + i\omega$ and $a - i\omega$ with the same real part $a = -B/2A$.

Let me look at all three cases, starting with examples.

Two Real Roots, One Double Root, No Real Roots

A picture will show you how $B^2 - 4AC$ decides real vs. complex. The three parabolas in Figure 2.8 have $C = 0$ and $C = 1$ and $C = 2$. **By increasing C we lift the parabolas.** The critical value is $C = 1$, when the middle parabola barely touches $y = 0$ at $s = 1$. $C = 1$ gives a double root and in this case $B^2 = 4AC = 4$.



$$y = s^2 - 2s + \mathbf{2}\,; \qquad \boldsymbol{s = 1 \pm i}$$

$$y = s^2 - 2s + \mathbf{1} = (s-1)^2 \qquad \boldsymbol{s = 1, 1}$$

$$y = s^2 - 2s + \mathbf{0} = s(s-2) \qquad \boldsymbol{s = 0, 2}$$

Figure 2.8: Lowest curve: Two roots for $C = 0$. Middle curve: Double root for $C = 1$. Highest curve misses the axis: No real roots for $C = 2 \longrightarrow$ *complex roots* $a + i\omega$.

All three parabolas have $A = 1$ and $B = -2$ and $B^2 = 4$. The test that compares B^2 to $4AC$ is comparing 4 to $4C$. This shows again that $C = 1$ is at the critical borderline $B^2 = 4AC$. Any value $C > 1$ will lift the parabola above the $y = 0$ axis. The roots of $s^2 - 2s + C = 0$ will be complex, and $y'' - 2y' + Cy = 0$ will give damped oscillation.

For $C = 2$ that equation becomes $(s-1)^2 = -1$. Then $s - 1 = i$ or $s - 1 = -i$. The two complex roots are $s = 1 + i$ and $s = 1 - i$. The quadratic formula (4) agrees.

Real Roots $s_1 > s_2$

Example 1 $y'' + 3y' + 2y = 0$ with $\boldsymbol{y = e^{st}}$ Substitute $A, B, C = 1, 3, 2$ to find s.

$$As^2 + Bs + C = \mathbf{s^2 + 3s + 2} = 0 \text{ factors into } (\boldsymbol{s+1})(\boldsymbol{s+2}) = 0. \tag{5}$$

The roots are both negative: $\boldsymbol{s_1 = -1}$ and $\boldsymbol{s_2 = -2}$. Those numbers come from the quadratic formula (4) and they come faster from the factors in (5): The first factor $s + 1$ is zero when $s_1 = -1$, and $s + 2 = 0$ when $s_2 = -2$. **Damping $\longrightarrow$ negative $\boldsymbol{s}$ $\longrightarrow$ stability.**

The complete solution to our linear differential equation is any combination of the two pure exponential solutions. These are null solutions (homogeneous solutions).

Null solutions $\quad \boldsymbol{y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} = c_1 e^{-t} + c_2 e^{-2t}}$ (6)

The numbers c_1 and c_2 are chosen to make $y(0)$ and $y'(0)$ correct when $t = 0$:

$$\textbf{Set } \boldsymbol{t = 0} \qquad y(0) = c_1 + c_2 \qquad \text{and} \qquad y'(0) = -c_1 - 2c_2. \tag{7}$$

Those two equations safely determine $c_1 = 2y(0) + y'(0)$ and $c_2 = -y(0) - y'(0)$:

$$\textbf{Final solution} \quad \boldsymbol{y(t) = c_1 e^{-t} + c_2 e^{-2t}} = y(0)(2e^{-t} - e^{-2t}) + y'(0)(e^{-t} - e^{-2t}).$$

Example 2 Solve $y'' - 3y' + 2y = 0$. The coefficient B has changed from 3 to -3.

Solution Substitute $y = e^{st}$ as before. *Negative damping gives positive s.*

$$s^2 - 3s + 2 = 0 \qquad (s-1)(s-2) = 0 \qquad s_1 = 2 \text{ and } s_2 = 1.$$

The complete solution is now $\boldsymbol{y(t) = c_1 e^{2t} + c_2 e^{t}}$. Exponential growth = instability.

Equal Roots $s_1 = s_2$

The roots of $As^2 + Bs + C$ will be equal when $B^2 = 4AC$. When you factor the quadratic, you see $(s - s_1)^2$ times A. The factor $s - s_1$ appears twice: $\boldsymbol{s = s_1}$ **is now a double root.**

Our e^{st} method has a problem when it finds one double root $s = s_1$. After $y = e^{s_1 t}$, what is a *second solution* to our second order equation ?

We will show that $\boldsymbol{y = te^{s_1 t}}$ **is also a solution** when $s_2 = s_1$.

Example 3 Solve $y'' - 2y' + y = 0$. Those coefficients $1, -2, 1$ have $B^2 = 4AC$.

Solution Substitute $y = e^{st}$ as usual. The root $s = 1$ is repeated: *two equal roots.*

$$s^2 - 2s + 1 = 0 \qquad (s-1)^2 = 0 \qquad \boldsymbol{s_1 = 1 = s_2}$$

With that root, $y = e^t$ solves the equation: easy to check. A second solution is needed ! We now confirm that $\boldsymbol{y = te^{st} = te^{t}}$ is also a solution of $y'' - 2y' + y = 0$:

$$y' = (te^t)' = te^t + e^t \qquad y'' - 2y' + y = (te^t + 2e^t) - 2(te^t + e^t) + (te^t) = 0$$

A double root of $\boldsymbol{As^2 + Bs + C = 0}$ must be $\boldsymbol{s_1 = -B/2A}$.
Then $\boldsymbol{y_1 = e^{s_1 t}}$ and also $\boldsymbol{y_2 = te^{s_1 t}}$ solve $\boldsymbol{Ay'' + By' + Cy = 0}$.

Proof With simple roots, the lowest parabola in Figure 2.8 cuts across $Y = 0$. The middle parabola $Y = (s-1)^2$ is *tangent* to the $Y = 0$ axis at the double root $1, 1$. "*The graph touches twice at the same point* $s = s_1$." The root is $s_1 = s_2 = -B/2A$.

$$\begin{array}{ll}\textbf{Height zero} \\ \textbf{Slope zero}\end{array} \qquad Y = As_1^2 + Bs_1 + C = 0 \quad \text{and also} \quad \frac{dY}{ds} = 2As_1 + B = 0. \tag{8}$$

To confirm that $Ay'' + By' + Cy$ is zero for $y = te^{s_1 t}$, look at y and y' and y'' :

$$\begin{aligned} \boldsymbol{y'} &= s_1 te^{s_1 t} + e^{s_1 t} = \boldsymbol{s_1 y + e^{s_1 t}} \\ \boldsymbol{y''} &= s_1 y' + s_1 e^{s_1 t} = s_1(s_1 y + e^{s_1 t}) + s_1 e^{s_1 t} = \boldsymbol{s_1^2 y + 2s_1 e^{s_1 t}} \end{aligned}$$

Substituting y'' and y' and y into $Ay'' + By' + Cy$, we get $0 + 0$ from equation (8):

$$A(s_1^2 y + 2s_1 e^{s_1 t}) + B(s_1 y + e^{s_1 t}) + Cy = (As_1^2 + Bs_1 + C)y + (2As_1 + B)e^{s_1 t} = 0 + 0.$$

The quadratic formula agrees with $s_1 = -B/2A = s_2$, because $B^2 - 4AC = 0$. The square root disappears, leaving $-B/2A$ for both solutions. Here is the simplest example of a double root $s_1 = s_2$ and a factor t in the second solution.

Example 4 **Solve $\boldsymbol{y'' = 0}$.** The coefficients $1, 0, 0$ have $B^2 = 4AC$.

Solution Substitute $y = e^{st}$ to find $s^2 e^{st} = 0$ and $\boldsymbol{s^2 = 0}$. *The double root is* $s = 0$. The usual solution $y = e^{st} = e^{0t} = 1$ does have $y'' = 0$. We need a second solution.

The rule $y = te^{st}$ still applies when $s = 0$. That second solution is $\boldsymbol{y = te^{0t} = t}$. We know this already: $y = 1$ *and* $y = t$ *solve* $y'' = 0$.

Higher Order Equations

Problem 18 will extend these ideas to n^{th} order equations (still constant coefficients!). Substitute $y = e^{st}$ to get an n^{th} degree polynomial in s. *Now there are n roots.* If those roots $s_1, s_2, \ldots, s_n$ are all different, they give n independent solutions $y = e^{st}$. But if a root s_1 is repeated two or three or m times, we need m different solutions for $s = s_1$:

$$\textbf{Multiplicity } \boldsymbol{m} \qquad \text{The } m \text{ solutions are } \; y = \boldsymbol{e^{s_1 t}}, \; y = \boldsymbol{t\, e^{s_1 t}}, \ldots, \; y = \boldsymbol{t^{m-1}\, e^{s_1 t}}. \tag{9}$$

A simple example would be the equation $y'''' = 0$. Substituting $y = e^{st}$ leads to $s^4 = 0$. This equation has four zero roots (multiplicity $m = 4$). The four solutions predicted by equation (9) are $y = 1, t, t^2, t^3$. No surprise that those all satisfy the equation $y'''' = 0$: their fourth derivatives are zero.

Here is a fourth order equation that produces two real roots and two complex roots:

$$\boldsymbol{y'''' - y = 0} \qquad y = e^{st} \text{ leads to } \boldsymbol{s^4 - 1 = 0} \tag{10}$$

The four roots are $s_1 = \mathbf{1}$ and $s_2 = \mathbf{-1}$ and $s_3 = \boldsymbol{i}$ and $s_4 = \boldsymbol{-i}$. Then the complete solution to $y'''' = y$ is $y = c_1 e^t + c_2 e^{-t} + c_3 e^{it} + c_4 e^{-it}$.

Complex Roots $s_1 = a + i\omega$ and $s_2 = a - i\omega$

The formula for the roots of a quadratic includes the square root of $B^2 - 4AC$. When that number is negative, the square root is *imaginary*. The example $y'' + y = 0$ has A, B, C equal to $1, 0, 1$, so $B^2 - 4AC = -4$. The quadratic is $As^2 + Bs + C = \boldsymbol{s^2 + 1}$.

The solutions to $s^2 + 1 = 0$ are $s = i$ and $s = -i$. The solutions to $s^2 + 4 = 0$ are $s = 2i$ and $s = -2i$. The oscillations from $y'' + 4y = 0$ can be written in two ways:

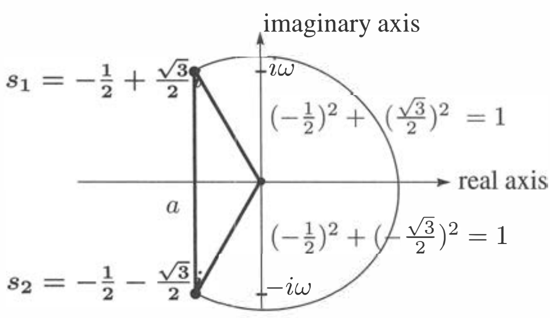
$$\boldsymbol{B = 0}\textbf{: No damping} \qquad \boxed{y = c_1 e^{2it} + c_2 e^{-2it} = C_1 \cos 2t + C_2 \sin 2t.} \qquad (11)$$

The real part of s is zero when $B = 0$: pure oscillation.

Now bring in damping: $\boldsymbol{y'' + y' + y = 0}$. For the solutions to $s^2 + s + 1 = 0$, go to the quadratic formula: A, B, C are $1, 1, 1$ and $B^2 - 4AC$ is -3:

$$\boldsymbol{s^2 + s + 1 = 0} \qquad s_1 = \frac{-1 + \sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \qquad s_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\, i.$$

The two complex roots s_1 and s_2 have the same real part $a = -1/2$. Their imaginary parts ω and $-\omega$ have opposite signs (as in $\sqrt{3}/2$ and $-\sqrt{3}/2$). Those are the plus and minus signs on the square root of $B^2 - 4AC$. Assuming that A, B, C are real numbers, the two roots of $As^2 + Bs + C = 0$ are *complex conjugates*. If I place s_1 and s_2 onto the complex plane, they are symmetric mirror images across the real axis.



The roots are $\boldsymbol{a + i\omega}$ and $\boldsymbol{a - i\omega}$. Their product is $\boldsymbol{a^2 + \omega^2 = C/A = 1}$.

The **conjugate** of $s = a + i\omega$ is $\overline{s} = a - i\omega$. The magnitude is $|s| = \sqrt{a^2 + \omega^2}$.

In the example with $a = -1/2$ and $\omega = \sqrt{3}/2$, the magnitude is exactly $|s| = 1$. This is because $(-1/2)^2 + (\sqrt{3}/2)^2 = 1$. The circle in the picture has radius 1. The unit circle is extremely important to recognize. The complex numbers on that circle have the form $s = \cos\theta + i \sin\theta$, because $(\text{cosine})^2 + (\text{sine})^2 = 1$. The angle θ is measured from the positive real axis. In the figure this angle is $120°$ or $\pi/3$.

The points on the unit circle are given by Euler's Formula $\boldsymbol{e^{i\theta} = \cos\theta + i \sin\theta}$.

We can switch between the complex form for $y(t)$ and its equivalent real form.

$$\textbf{Complex } y(t) = e^{at}(c_1 e^{i\omega t} + c_2 e^{-i\omega t}) \qquad \textbf{Real } y(t) = e^{at}(C_1 \cos\omega t + C_2 \sin\omega t)$$

Euler's formula for $e^{i\omega t}$ and $e^{-i\omega t}$ shows that $C_1 = c_1 + c_2$ and $C_2 = ic_1 - ic_2$.

With those key facts about complex numbers $a + i\omega$, we come back to the example $s^2 + s + 1 = 0$ and the differential equation it comes from:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0 \qquad \boxed{y_1 = e^{s_1 t} = e^{(a+i\omega)t}} \qquad \boxed{y_2 = e^{s_2 t} = e^{(a-i\omega)t}}$$

This number $e^{(a+i\omega)t}$ is *not* on the unit circle. The real part $a = -1/2$ is responsible. When $a = 0$, $e^{i\omega t}$ goes around the circle. When $a < 0$, $e^{(a+i\omega)t}$ spirals to zero: **damped**.
The magnitude of $e^{i\omega t}$ is 1, but e^{at} grows large or small depending on the sign of a:

Growth	$a > 0$	Magnitude $\lvert e^{(a+i\omega)t}\rvert = \boldsymbol{e^{at}}$	$\rightarrow \infty$
Decay	$a < 0$	Magnitude $\lvert e^{(a+i\omega)t}\rvert = \boldsymbol{e^{at}}$	$\rightarrow \mathbf{0}$

That real part is always $\boldsymbol{a = -B/2A}$. Every equation $Ay'' + By' + Cy = 0$ will have damping and decay if A and B are positive. Here is an example with $B = -1$:

Negative damping $\rightarrow$ growth $\qquad \boxed{y'' - y' + y = 0} \qquad \boxed{s^2 - s + 1 = 0}$.

That changes a to $+\frac{1}{2}$. The roots $a \pm i\omega$ are now coming from $s^2 - s + 1 = 0$:

$$s_1 = a + i\omega = +\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{has magnitude} \quad |s_1| = \sqrt{a^2 + \omega^2} = 1.$$

This point s_1 is on the unit circle, because $|s_1| = 1$. Its real part a is $+\frac{1}{2}$, so s_1 is on the right side (not left side) of the imaginary axis. The angle in $s_1 = e^{i\theta}$ changes to $\theta = 60°$. Now s_1 and s_2 are on the *right half* of the unit circle (the unstable half: e^{st} grows).

"Anti-damping" $B = -1$ $\qquad$ **Growth rate** $a = \dfrac{1}{2}$ $\qquad$ **Magnitude** $|e^{st}| = e^{at} = e^{t/2}$

In most physical problems we expect positive damping $B > 0$ and negative growth rate $a < 0$. Then the differential equation is stable and its null solutions die out as $t \to \infty$.

Overdamping versus Underdamping

This section emphasizes the difference between $B^2 > 4AC$ and $B^2 < 4AC$. That is the difference between real roots and complex roots. This is a difference you can see—with your own eyes and not just with formulas. For damping coefficients $B = 1, 2, 3$ the solutions to $y'' + By' + y = 0$ will approach zero in different ways (Figure 2.9).

At this time I want to vary the damping B instead of the stiffness C.

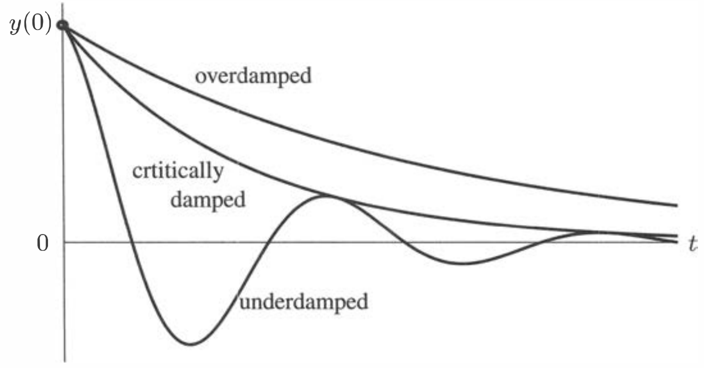


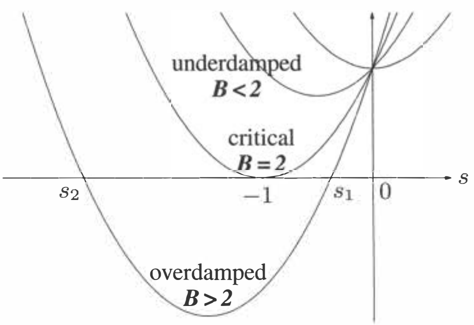
Figure 2.9: $y(t)$ goes directly to zero (overdamped) or it oscillates (underdamped).

The four damping possibilities match the four possibilities for roots of $As^2 + Bs + C = 0$. This table brings the whole section together:

Overdamping	$B^2 > 4AC$	**Real roots**	$e^{s_1 t}$ **and** $e^{s_2 t}$
Critical damping	$B^2 = 4AC$	**Double root**	$e^{s_1 t}$ **and** $te^{s_1 t}$
Underdamping	$B^2 < 4AC$	**Complex roots**	$e^{at}\cos\omega t,\ e^{at}\sin\omega t$
No damping	$B = 0$	**Imaginary roots**	$\cos\omega t$ **and** $\sin\omega t$

Figure 2.9 shows how *the graph crosses zero and comes back*, for underdamping. This is like a child's swing that is settling to zero (so the child can get off the swing). When $B = 0$ we have $a = 0$ and imaginary roots $\pm i\omega$ and pure spring–mass oscillation.

Figure 2.10 shows four parabolas all with $A = C = 1$. The damping coefficients are $B = 0, 1, 2, 3$. When $B = 3$ the damping is strong and $s^2 - 3s + 1 = 0$ has *real roots*. When $B = 2$ the damping is critical and $s^2 - 2s + 1 = 0$ has a *double root* $s = 1, 1$. When $B = 1$ the damping is weak and the roots are *complex*. The solutions $y = e^{at}\cos\omega t$ and $y = e^{at}\sin\omega t$ oscillate as the e^{at} term goes to zero. When $B = 0$ there is no decay.



$$y = s^2 + 0\,s + 1 \qquad s = i, -i$$
$$y = s^2 + 1\,s + 1 \qquad s = (-1 \pm \sqrt{3}i)/2$$
$$y = s^2 + 2\,s + 1 \qquad s = -1, -1$$
$$y = s^2 + 3\,s + 1 \qquad s = (-3 \pm \sqrt{5})/2$$

Figure 2.10: As B increases, the lowest point on the parabola moves left and down.

Fundamental Solution = Growth Factor = Impulse Response

One special choice of initial conditions is all-important: $\boldsymbol{g(0) = 0}$ **and** $\boldsymbol{g'(0) = 1/A}$. The letter g instead of y picks out this fundamental solution. This is a null solution with the jump start $g'(0)$. It is also a particular solution to $Ag'' + Bg' + Cg = \delta(t)$. This fundamental solution from the delta function will lead us to *all* solutions.

Review: The roots of $As^2 + Bs + C = 0$ are s_1 and s_2. They give two solutions $e^{s_1 t}$ and $e^{s_2 t}$ to the null equation, if $s_1 \neq s_2$. We want the combination $g = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ that matches $g(0) = 0$ and $g'(0) = 1/A$. Choose the right c_1 and c_2:

$$\begin{array}{lll} g(0) = c_1 + c_2 = 0 & \textit{Multiply by } s_2 & s_2 c_1 + s_2 c_2 = 0 \\ \boldsymbol{g'(0) = s_1 c_1 + s_2 c_2 = 1/A} & \textit{Then subtract} & \boldsymbol{(s_1 - s_2) c_1 = 1/A} \end{array}$$

$$\textbf{The fundamental solution } \boxed{g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{A(s_1 - s_2)}} \textbf{ has } c_1 = \frac{1}{A(s_1 - s_2)} = -c_2 \quad (12)$$

No damping For the oscillation equation $my'' + ky = 0$, the roots of $ms^2 + k = 0$ are imaginary: $s_1 = i\sqrt{k/m} = i\omega$ and $s_2 = -i\sqrt{k/m} = -i\omega$. Then the fundamental solution has a simple form with $A = m$:

$$g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{m(s_1 - s_2)} = \frac{e^{i\omega t} - e^{-i\omega t}}{m(2i\omega)} = \frac{2i \sin \omega t}{2i\, m\omega} = \frac{\sin \omega t}{A\omega}. \quad (13)$$

This is exactly the impulse response from Section 2.1. Clearly $g(0) = 0$ and $g'(0) = 1/A$.

Underdamping Now $s_1 = a + i\omega$ and $s_2 = a - i\omega$. There is decay from $a = -B/2A$ and oscillation from ω. Soon we will write p for $B/2A$ and ω_d for ω.

$$g(t) = \frac{e^{(a+i\omega)t} - e^{(a-i\omega)t}}{A(2i\omega)} = e^{at} \frac{\sin \omega t}{A\omega} = e^{-pt} \frac{\sin \omega_d t}{A\omega_d}. \quad (14)$$

Critical damping Now $B^2 = 4AC$ and the roots are equal: $\boldsymbol{s_1 = s_2 = -B/2A}$. The second solution to the differential equation (after $e^{s_1 t}$) is $g(t) = te^{s_1 t}$. Dividing by A, this is exactly the solution that has $g(0) = 0$ and $g'(0) = 1/A$.

$$g(t) = \frac{te^{s_1 t}}{A} = \frac{t\, e^{-Bt/2A}}{A}. \quad (15)$$

Overdamping When $B^2 > 4AC$, the roots s_1 and s_2 are real. Formula (12) is best.

The real purpose of $g(t)$ is to solve $Ay'' + By' + Cy = f(t)$ with any right side $f(t)$. This impulse response g is the fundamental solution that gives all other solutions:

$$\textbf{Solution for any } \boldsymbol{f(t)} \qquad \boxed{y_p(t) = \int_0^t g(t-s) f(s)\, ds} \quad (16)$$

The **step response** *to* $f(t) = 1$ *is* $y_p =$ *integral of* $g(t)$. This comes in Section 2.5.

Delta Function and Impulse Response

In this section $g(t)$ is a **null solution** with initial velocity $g'(0) = 1/A$. The same $g(t)$ is a **particular solution** in the next section, with initial velocity zero but driven by an impulse $f(t) = \delta(t)$. Only a delta function could make this possible: $g(t)$ is y_n for one problem and y_p for another problem.

The informal explanation is to integrate all terms in $Ag'' + Bg' + Cg = \delta(t)$. On the right side the integral is 1. The integration is over a *very short interval* 0 to Δ. On the left side the integral of Ag'' is $Ag'(\Delta)$, plus terms of order Δ going to 0. To match 1 on the right side, the impulse response $g(t)$ starts immediately with $g' = 1/A$.

Example 5 **The best example is** $\boldsymbol{g''(t) = \delta(t)}$ **with ramp function** $\boldsymbol{g(t) = t}$.

The derivative of the ramp is a step function. You see the sudden jump to $g' = 1$. The ramp $g(t) = t$ agrees with formula (15) in this case with $A = 1$ and $B = C = 0$. The null equation $g'' = 0$ starting from $g(0) = 0$ and $g'(0) = 1$ is solved by $g(t) = t$. Everything is zero for $t < 0$. Then we see the ramp $g(t)$ and the step $g'(t)$ and $g'' = \delta(t)$. This is the limiting case of equation (12) when B and C and s_1 and s_2 approach zero.

A personal note Thank you for accepting the slightly illegal input $\delta(t)$ and its response $g(t)$. I could have left those out of the book. But I couldn't have lived with myself. They are truly the key to theory and applications.

Shift Invariance from Constant Coefficients

For a constant coefficient equation, the growth from time s to time t is exactly equal to the growth from 0 to $t - s$. The problem is **shift invariant**. We can start the time interval anywhere we want. For all intervals of the same length, we will see the same growth factor $g(t - s)$. This is the growth of input

$$\textbf{Inputs } \boldsymbol{f(s)} \textbf{ at times } \boldsymbol{s} \qquad \textbf{Total output } \boldsymbol{y(t) = \int_0^t g(t-s)\, f(s)\, ds}. \tag{17}$$

This is exactly like the main formula $y(t) = \int e^{a(t-s)} q(s)\, ds$ in Chapter 1. There the growth factor was $g(t) = e^{at}$. The equation $dy/dt - ay = q(t)$ had *constant* a.

Shift invariance is lost if any of the coefficients A, B, C change with time. The growth factor becomes $g(s, t)$, depending on the specific start s and end t (*not just on the elapsed time* $t - s$). In this harder case the solution is $y(t) = \int g(s, t)\, f(s)\, ds$.

For a first order equation, Section 1.6 found $g(s, t)$. But second order equations with time-varying coefficients are usually impossible to solve with familiar functions. We often have no formula for $g(s, t)$—the response at time t to an impulse at time s. *Shift invariance* (constant coefficients) is the key to successful solution formulas.

Better Formulas for s_1 and s_2

The solutions to $As^2 + Bs + C = 0$ are s_1 and s_2. The formula for those two roots involves $B^2 - 4AC$. We have seen that $B^2 > 4AC$ is very different from $B^2 < 4AC$. Overdamping leads to real roots, underdamping leads to complex roots and oscillations. The formulas are so important that the whole world of science and engineering has tried to make them simpler.

Here is the natural way to start. Assign letters to the ratios $B/2A$ and C/A. We know C/A as ω_n^2. This is k/m in mechanics. It gives the "natural frequency" with no damping. *For the ratio $B/2A$ I will use the letter p.* The main point is to simplify s_1 and s_2 :

$$s_1, s_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \boxed{-p \pm \sqrt{p^2 - \omega_n^2}} \tag{18}$$

A big improvement! Two symbols instead of three, which makes sense because we can divide $As^2 + Bs + C = 0$ by A. By introducing $p = B/2A$ we remove the 2 and the 4 in equation (18).

The comparison of B^2 to $4AC$ is now the comparison of p^2 to ω_n^2. When $p^2 > \omega_n^2$, the roots are real (overdamping). When $p^2 - \omega_n^2$ is negative, s_1 and s_2 will be complex. **We have oscillation at a damped frequency ω_d, lower than the natural frequency ω_n :**

$$\boxed{\omega_d^2 = \omega_n^2 - p^2} \qquad s_1 \text{ and } s_2 = -p \pm i\sqrt{\omega_n^2 - p^2} = \boxed{-p \pm i\omega_d} \tag{19}$$

The Damping Ratio Z

The presentation could stop there. We see that the ratio of p to ω_n is highly important. This fact suggests one final step, that we take now: $Z = p/\omega_n$ **is the damping ratio** Z. In engineering this ratio is called zeta (the Greek letter is ζ). To make it easier to write, allow me to use Z (capital zeta in Greek = capital Z in Roman.) *Then we can replace p by $Z\omega_n$.* Now the formula $s = -p \pm i\omega_d$ uses ω_n and Z :

$$\textbf{Damping ratio } Z = \frac{p}{\omega_n} \qquad \boxed{s = -Z\omega_n \pm i\omega_d = -Z\omega_n \pm i\omega_n\sqrt{1 - Z^2}} \tag{20}$$

The damped ω_d^2 is $\omega_n^2 - p^2 = \omega_n^2(1 - Z^2)$. Its square root ω_d is the damped frequency. The null solutions are $y_n(t) = e^{-Z\omega_n t}(c_1 \cos \omega_d t + c_2 \sin \omega_d t)$.

Underdamping is $Z < 1$, critical damping is $Z = 1$, and overdamping is $Z > 1$. The key points become clear because *this ratio Z is dimensionless* :

$$\textbf{Damping ratio} \quad Z = \frac{p}{\omega_n} = \frac{B/2A}{\sqrt{C/A}} = \frac{B}{\sqrt{4AC}} = \frac{b}{\sqrt{4mk}}. \tag{21}$$

If time is measured in minutes instead of seconds, the numbers A, B, C are changed by 60^2 and 60 and 1. **The ratio of B to $\sqrt{4AC}$ is not changed**: a factor of 60 for both. This confirms that $B^2 - 4AC$ is a suitable quantity to appear in the quadratic formula, because B^2 and $4AC$ have the same units.

One last point is a good approximation when Z is small. The square root of $1 - Z^2$ is close to $1 - \frac{1}{2}Z^2$. This comes from calculus (linear approximation using the tangent line). The good way to confirm it is to square both sides. Then $Z^4/4$ is very small.

$$\boldsymbol{\sqrt{1 - Z^2} \approx 1 - \frac{1}{2}Z^2} \quad \text{becomes} \quad 1 - Z^2 \approx 1 - Z^2 + \frac{1}{4}Z^4. \tag{22}$$

The good measure of damping is the **ratio** $Z = B/\sqrt{4AC}$. This key dimensionless number decides everything:

$Z > 1$	$B^2 > 4AC$ and real roots: *Overdamping and no oscillation.*
$Z < 1$	$B^2 < 4AC$ and complex roots: *Underdamping and slow oscillation.*
$Z = 1$	$B^2 = 4AC$ and a double root $-B/2A$: *critical damping.*

Here is a curious fact. For very large B, the roots are approximately $s_1 = -1/B$ and $s_2 = -B$. That root s_2 gives fast decay. But the actual decay of $y(t)$ is controlled by s_1, which approaches zero! So increasing B actually slows down this dominant decay mode.

Note that many authors refer to s_1 and s_2 as **poles**. They are poles of the transfer function $Y(s) = 1/(As^2 + Bs + C)$, where Y becomes $1/0$. We will come back to transfer functions! Some authors emphasize **time constants** rather than exponents. The exponential e^{-pt} has time constant $\tau = 1/p$. In that time τ, e^{-pt} decays by a factor e.

■ REVIEW OF THE KEY IDEAS ■

1. The equation $Ay'' + By' + Cy = 0$ is solved by $y = e^{st}$ when $As^2 + Bs + C = 0$.

2. The roots s_1, s_2 are *real* if $B^2 > 4AC$, *equal* if $B^2 = 4AC$, *complex* if $B^2 < 4AC$.

3. Negative real roots give stability and overdamping: $y(t) = \boldsymbol{c_1 e^{s_1 t} + c_2 e^{s_2 t}} \to 0$.

4. Equal roots $s = -B/2A$ when $B^2 = 4AC$. Change the second solution to $\boldsymbol{y_2 = t\, e^{st}}$.

5. Complex roots $a \pm i\omega$ give underdamped oscillations: $e^{at}(C_1 \cos\omega t + C_2 \sin\omega t)$.

6. The initial values $g(0) = 0$ and $g'(0) = 1/A$ give $g(t) = \boldsymbol{\left(e^{s_1 t} - e^{s_2 t}\right)/A(s_1 - s_2)}$. The same $g(t)$ solves $Ag'' + Bg' + Cg = \delta(t)$. This is the fundamental solution.

7. s_1 and s_2 become $\boldsymbol{-p \pm i\omega_d}$ with $p = B/2A$ and $\omega_d^2 = \omega_n^2 - p^2$. With damping ratio $Z = B/\sqrt{4AC} < 1$, those complex s_1 and s_2 are $\boldsymbol{-Z\omega_n \pm i\omega_n\sqrt{1 - Z^2}}$.

Problem Set 2.3

1 Substitute $y = e^{st}$ and solve the characteristic equation for s:

(a) $2y'' + 8y' + 6y = 0$ (b) $y'''' - 2y'' + y = 0$.

2 Substitute $y = e^{st}$ and solve the characteristic equation for $s = a + i\omega$:

(a) $y'' + 2y' + 5y = 0$ (b) $y'''' + 2y'' + y = 0$

3 Which second order equation is solved by $y = c_1e^{-2t} + c_2e^{-4t}$? Or $y = te^{5t}$?

4 Which second order equation has solutions $y = c_1e^{-2t}\cos 3t + c_2e^{-2t}\sin 3t$?

5 Which numbers B give (under) (critical) (over) damping in $4y'' + By' + 16y = 0$?

6 If you want oscillation from $my'' + by' + ky = 0$, then b must stay below _____ .

Problems 7–16 are about the equation $As^2 + Bs + C = 0$ and the roots s_1, s_2.

7 The roots s_1 and s_2 satisfy $s_1 + s_2 = -2p = -B/2A$ and $s_1s_2 = \omega_n^2 = C/A$. Show this two ways:

(a) Start from $As^2 + Bs + C = A(s - s_1)(s - s_2)$. Multiply to see s_1s_2 and $s_1 + s_2$.

(b) Start from $s_1 = -p + i\omega_d$, $s_2 = -p - i\omega_d$

8 Find s and y at the bottom point of the graph of $y = As^2 + Bs + C$. At that minimum point $s = s_{\text{min}}$ and $y = y_{\text{min}}$, the slope is $dy/ds = 0$.

9 The parabolas in Figure 2.10 show how the graph of $y = As^2 + Bs + C$ is raised by increasing B. Using Problem 8, show that the bottom point of the graph moves left (change in s_{min}) and down (change in y_{min}) when B is increased by ΔB.

10 (recommended) Draw a picture to show the paths of s_1 and s_2 when $s^2 + Bs + 1 = 0$ and the damping increases from $B = 0$ to $B = \infty$. At $B = 0$, the roots are on the _____ axis. As B increases, the roots travel on a circle (why?). At $B = 2$, the roots meet on the real axis. For $B > 2$ the roots separate to approach 0 and $-\infty$. *Why is their product s_1s_2 always equal to 1?*

11 (this too if possible) Draw the paths of s_1 and s_2 when $s^2 + 2s + k = 0$ and the stiffness increases from $k = 0$ to $k = \infty$. When $k = 0$, the roots are _____ . At $k = 1$, the roots meet at $s =$ _____. For $k \to \infty$ the two roots travel up/down on a _____ in the complex plane. *Why is their sum $s_1 + s_2$ always equal to -2?*

12 If a polynomial $P(s)$ has a double root at $s = s_1$, then $(s - s_1)$ is a double factor and $P(s) = (s - s_1)^2Q(s)$. Certainly $P = 0$ at $s = s_1$. Show that also $dP/ds = 0$ at $s = s_1$. Use the product rule to find dP/ds.

13 Show that $y'' = 2ay' - (a^2 + \omega^2)y$ leads to $s = a \pm i\omega$. Solve $y'' - 2y' + 10y = 0$.

14 The undamped *natural frequency* is $\omega_n = \sqrt{k/m}$. The two roots of $ms^2 + k = 0$ are $s = \pm\, i\omega_n$ (pure imaginary). With $p = b/2m$, the roots of $ms^2 + bs + k = 0$ are $\boldsymbol{s_1, s_2 = -p \pm \sqrt{p^2 - \omega_n^2}}$. The coefficient $p = b/2m$ has the units of $1/\text{time}$.

Solve $s^2 + 0.1s + 1 = 0$ and $s^2 + 10s + 1 = 0$ with numbers correct to two decimals.

15 With large overdamping $\boldsymbol{p >> \omega_n}$, the square root $\sqrt{p^2 - \omega_n^2}$ is close to $p - \omega_n^2/2p$. Show that the roots of $ms^2 + bs + k$ are $s_1 \approx \boldsymbol{-\omega_n^2/2p} = (\text{small})$ and $s_2 \approx -2p = \boldsymbol{-b/m}$ (large).

16 With small underdamping $\boldsymbol{p << \omega_n}$, the square root of $p^2 - \omega_n^2$ is approximately $i\omega_n - ip^2/2\omega_n$. Square that to come close to $p^2 - \omega_n^2$. Then the frequency for small underdamping is reduced to $\omega_d \approx \omega_n - p^2/2\omega_n$.

17 Here is an 8th order equation with eight choices for solutions $y = e^{st}$:

$$\frac{d^8y}{dt^8} = y \quad \text{becomes} \quad s^8 e^{st} = e^{st} \quad \text{and} \quad \boldsymbol{s^8 = 1} : \quad \text{Eight roots in Figure 2.6.}$$

Find two solutions e^{st} that don't oscillate (s is real). Find two solutions that only oscillate (s is imaginary). Find two that spiral in to zero and two that spiral out.

18 $A_n \dfrac{d^ny}{dt^n} + \cdots + A_1 \dfrac{dy}{dt} + A_0 y = 0$ leads to $\boldsymbol{A_n s^n + \cdots + A_1 s + A_0 = 0}$.

The n roots $s_1, \ldots, s_n$ produce n solutions $y(t) = e^{st}$ (if those roots are distinct). Write down n equations for the constants c_1 to c_n in $y = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$ by matching the n initial conditions for $y(0)$, $y'(0)$, ..., $D^{n-1}y(0)$.

19 **Find two solutions to $\boldsymbol{d^{2015}y/dt^{2015} = dy/dt}$.** Describe all solutions to $s^{2015} = s$.

20 The solution to $y'' = 1$ starting from $y(0) = y'(0) = 0$ is $y(t) = t^2/2$. The fundamental solution to $g'' = \delta(t)$ is $g(t) = t$ by Example 5. Does the integral $\int g(t-s)f(s)ds = \int (t-s)ds$ from 0 to t give the correct solution $y = t^2/2$?

21 The solution to $y'' + y = 1$ starting from $y(0) = y'(0) = 0$ is $y = 1 - \cos t$. The solution to $g'' + g = \delta(t)$ is $\boldsymbol{g(t) = \sin t}$ by equation (13) with $\omega = 1$ and $A = 1$. Show that $1 - \cos t$ agrees with the integral $\int g(t-s)f(s)ds = \int \sin(t-s)ds$.

22 The step function $H(t) = 1$ for $t \geq 0$ is the integral of the delta function. **So the step response $\boldsymbol{r(t)}$ is the integral of the impulse response.** This fact must also come from our basic solution formula :

$$Ar'' + Br' + Cr = 1 \ \text{ with } \ r(0) = r'(0) = 0 \quad \text{has} \quad \boldsymbol{r(t) = \int_0^t g(t-s)\, 1\, ds}$$

Change $t - s$ to τ and change ds to $-d\tau$ to confirm that $r(t) = \int\limits_0^t g(\tau)d\tau$.

Section 2.5 will find two good formulas for the step response $r(t)$.

2.4 Forced Oscillations and Exponential Response

The equation $Ay''+By'+Cy=0$ has no forcing term. Its right side is zero. This equation is *homogeneous*. The null solution $y_n(t) = c_1e^{s_1t} + c_2e^{s_2t}$ is controlled by the initial conditions $y(0)$ and $y'(0)$. If those are zero, the system never moves.

The equation $Ay'' + By' + Cy = f(t)$ is **forced** or **driven** by that new term $f(t)$. Previously $y = 0$ was a possible solution. Now we can expect a particular solution y_p.

This section is about driving forces $f = e^{st}$ and $e^{i\omega t}$ and $\cos\omega t$ and $\sin\omega t$. For $f = e^{st}$, the next example will show you how to find y_p.

Exponential Driving Force

In this example, one particular solution $y_p(t) = Ye^{st}$ is a multiple of the input e^{4t}. All we have to do is find that number Y, by substituting into the differential equation.

Example 1 Solve $\boldsymbol{y'' + 5y' + 6y = e^{4t}}$. One particular solution will be $\boldsymbol{y_p = Ye^{4t}}$. When Ye^{4t} is substituted into the equation, all terms contain e^{4t} :

$$y'' + 5y' + 6y = 16Ye^{4t} + 20Ye^{4t} + 6Ye^{4t} = e^{4t}. \tag{1}$$

The left side is $42\,Ye^{4t}$. This matches the right side e^{4t} when $Y = 1/42$:

$$\textbf{Particular } y_p \qquad \boxed{42\,Ye^{4t} = e^{4t} \text{ gives } 42\,Y = 1 \qquad y_p(t) = e^{4t}/42} \tag{2}$$

The complete solution has the form $y = y_p + y_n$. There are two arbitrary constants c_1 and c_2 in the solution $y_n(t)$ to the homogeneous equation (the null equation with forcing term = zero). Look for the two exponents s_1 and s_2 that solve the quadratic equation $As^2 + Bs + C = 0$. We know how to find the null solution y_n.

Substitute $\boldsymbol{y = e^{st}}$ **into** $\boldsymbol{y'' + 5y' + 6y = 0}$. **Cancel** $\boldsymbol{e^{st}}$ **to find** $\boldsymbol{s^2 + 5s + 6 = 0}$.

That quadratic factors into $(s + 2)(s + 3)$. This is zero for $\boldsymbol{s = -2}$ and $\boldsymbol{s = -3}$. Those roots of the "characteristic equation" are the exponents in the null solution $y_n(t)$. This is the homogeneous solution = complementary solution = **transient solution**, which decays to zero at $t = \infty$ when there is damping.

$$\textbf{Null solution} \qquad y_n(t) = c_1e^{-2t} + c_2e^{-3t}.$$

The final step is to choose c_1 and c_2 so that $y = y_p + y_n = \frac{1}{42}e^{4t} + y_n$ satisfies the initial conditions. This will complete Example 1, by getting it right at $t = 0$.

$$\begin{aligned} \textbf{Initial position} \quad y(0) &= \frac{1}{42} + c_1 + c_2 \\ \textbf{Initial velocity} \quad y'(0) &= \frac{4}{42} - 2\,c_1 - 3\,c_2 \end{aligned}$$

Those two equations tell us the correct values c_1 and c_2, when $y(0)$ and $y'(0)$ are given.

Exponential Response Formula

We can turn that example into a formula for Y that almost always succeeds. Put $y = Ye^{st}$ into the equation. Each derivative multiplies y by s. So $Ay'' + By' + Cy$ will multiply $y = Ye^{st}$ by the number $As^2 + Bs + C$. Divide by that number to see Y:

$$\boxed{\boldsymbol{Ay'' + By' + Cy = e^{st}}} \quad \textbf{is solved by} \quad \boxed{\boldsymbol{y = Ye^{st} = \frac{1}{As^2 + Bs + C}\, e^{st}}} \tag{3}$$

That fraction Y is called the **transfer function**. It 'transfers' the exponential input e^{st} into the exponential output $y_p = Ye^{st}$. The formula allows s to be an imaginary $i\omega$ or any complex number $s = a + i\omega$. Use the exponent s that is in the driving force f:

$$\boxed{Ay'' + By' + Cy = \boldsymbol{e^{i\omega t}} \quad \text{leads to} \quad y_p(t) = \frac{\mathbf{1}}{\boldsymbol{A(i\omega)^2 + B(i\omega) + C}}\, \boldsymbol{e^{i\omega t}}.} \tag{4}$$

Example 2 $\boldsymbol{y'' + y' = e^{it}}$ **has** $\boldsymbol{s = i\omega = i}$. Substitute $y = Ye^{it}$ and solve for Y:

$$i^2\, Ye^{it} + i\, Ye^{it} = e^{it} \qquad (i^2 + i)Y = 1 \qquad \boldsymbol{y_p(t) = \frac{1}{-1+i}\, e^{it}}. \tag{5}$$

Example 3 (important) **Solve** $\boldsymbol{y'' + y' = \cos t}$. The cosine is the real part of e^{it}. Warning: The solution will *not* have the form $y = Y\cos t$. The derivative $-Y\sin t$ would appear in the differential equation, with no other term to cancel it. The correct solution involves *both* $\cos t$ *and* $\sin t$. Damping from y' delays the cosine.

Here $y_p(t)$ in Example 3 is the real part of $y_p(t)$ in Example 2. Please use this idea:

The real part of the input $\boldsymbol{e^{i\omega t}}$ produces the real part of the output $\boldsymbol{Ye^{i\omega t}}$.

Step 1 Write $Y = \dfrac{1}{-1+i} = \dfrac{1}{-1+i}\left(\dfrac{-1-i}{-1-i}\right) = \dfrac{-1-i}{2}$.

Step 2 The real part of $Ye^{it} = \dfrac{-1-i}{2}(\cos t + i\sin t)$ is $\boldsymbol{y_p = \frac{1}{2}(-\cos t + \sin t)}$.

The exponential response formulas are (3) and (4). The only time they fail is when the denominator in the fraction is zero. The formula would then contain $\mathbf{1/0}$. That happens when the exponent s in the driving term equals one of the exponents s_1 and s_2 in the null solution $y_n = c_1e^{s_1t} + c_2e^{s_2t}$. This is called **resonance**: $\boldsymbol{s = s_1}$ **or** $\boldsymbol{s = s_2}$.

You see that we cannot allow y_p to be included among the null solutions y_n. If the right side is $f \neq 0$ for y_p, it cannot also be $f = 0$ as required by y_n. We will see that the correct form for a resonant solution y_p includes an extra factor $\boldsymbol{t}$ in $Y\boldsymbol{t}e^{st}$.

A special effort goes into the oscillating case $s = i\omega$. Null solutions $y_n = e^{st}$ depend only on A, B, C. That part comes from the roots of $As^2 + Bs + C = 0$. The new part is the forced oscillation $\boldsymbol{y_p(t)}$, a particular solution that is driven by $\cos\omega t$. It will be $\boldsymbol{y_p(t) = G\cos(\omega t - \alpha)}$ **with a phase shift $\boldsymbol{\alpha}$ and a gain $\boldsymbol{G}$ in the amplitude**.

Equations of Order N and Order 2

I would like to outline the work ahead, because this section is important. It started with a specific example $y''+5y'+6y = e^{4t}$. Those numbers $1, 5, 6, 4$ changed to letters A, B, C, s. We solved the second order equation $Ay''+By'+Cy = e^{st}$. The solution Ye^{st} introduced the transfer function $Y = 1/(As^2 + Bs + C)$.

Now we have two ways to go, both essential. One is to see the same formula $y = Ye^{st}$ for **every constant coefficient equation**. Y comes from the "exponential response formula" because Ye^{st} is the response to the exponential $f(t) = e^{st}$. One formula covers almost all equations (but resonance is special and Y has to change).

The other crucial step is to focus on **second order equations driven by $f = e^{i\omega t}$**. Yes, this is covered by the formula. But if we are serious, we won't stop with $Y(i\omega)$. We truly need the rectangular and polar forms of that complex number:

$$\boldsymbol{Y(i\omega) = \frac{1}{A(i\omega)^2 + B(i\omega) + C} = M - iN = Ge^{-i\alpha}}. \tag{6}$$

M, N, G, α will be in equations (23) to (27). The solution driven by $f = \cos\omega t$ becomes $y = M \cos \omega t + N \sin \omega t$. Damped motion $(B > 0)$ can be compared with undamped. And the big applications in Section 2.5 need the better notation using Z:

$$\textbf{Natural frequency} \quad \omega_n^2 = \frac{C}{A} \qquad \textbf{Damping ratio} \quad Z = \frac{B}{\sqrt{4AC}} \qquad \textbf{Damped frequency} \quad \omega_d^2 = \omega_n^2(1 - Z^2) \tag{7}$$

The damping ratio Z and those frequencies ω_n and ω_d give meaning to the solution $y(t)$.

Complete Solution $y_p + y_n$

Let me summarize the case of **undamped forced oscillation** (driving force $F\cos\omega t$). If $B = 0$, the complete solution to $Ay'' + Cy = F\cos\omega t$ is one particular solution y_p plus any null solution y_n at the natural frequency $\omega_n = \sqrt{C/A}$. Notice the two ω's:

$$\begin{array}{l}\textbf{Particular solution } (\omega) \\ \textbf{Unforced solution } (\omega_n)\end{array} \qquad \boxed{y = \frac{F}{C - A\omega^2}\cos\omega t + c_1 e^{i\omega_n t} + c_2 e^{-i\omega_n t}} \tag{8}$$

To repeat: Any time we have a linear equation $\boldsymbol{Ly = f}$, the complete solution has the form $y = y_p + y_n$. The particular solution solves $Ly_p = f$. The null solution solves $Ly_n = 0$. Linearity of L guarantees that $y = y_p + y_n$ solves $Ly = f$:

$$\textbf{Complete solution } \boldsymbol{y = y_p + y_n} \qquad \text{If } Ly_p = f \text{ and } Ly_n = 0 \text{ then } Ly = f. \tag{9}$$

This book emphasizes linear equations. You will see $y_p + y_n$ again, always with the rule of linearity $Ly = Ly_p + Ly_n$. This applies to linear differential equations and matrix equations. In differential equations, L is called a *linear operator*.

Linear operator $Ly = Ay'' + By' + Cy$ or $Ly = A_N \dfrac{d^N y}{dt^N} + \cdots + A_1 \dfrac{dy}{dt} + A_0 y$

For an operator L, the inputs y and the outputs Ly are functions.

Every solution to $Ly = f$ *has the form* $y_p + y_n$. Suppose we start with one particular solution y_p. If y is any other solution, then $L(y - y_p) = 0$:

$$\boldsymbol{y_n = y - y_p} \textbf{ is a null solution} \qquad \boldsymbol{Ly_n} = Ly - Ly_p = f - f = \mathbf{0}. \qquad (10)$$

Example 4 Suppose the linear equation is just $\boldsymbol{Ly} = x_1 - x_2 = 1$: one equation in two unknowns x_1 and x_2. The solutions are vectors $\boldsymbol{y} = (x_1, x_2)$. The right side $f = 1$ is not zero. The bold line in Figure 2.11 is the graph of all solutions.

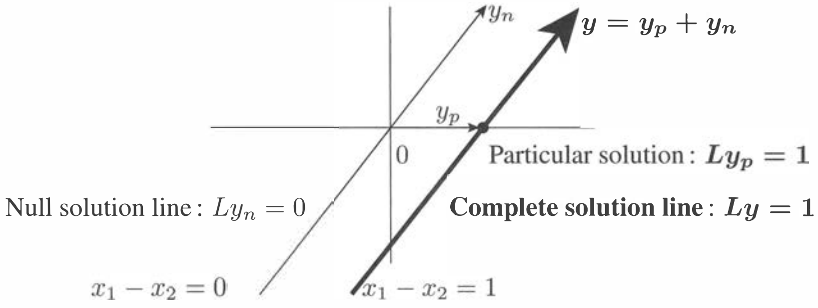


Figure 2.11: Complete solution = *one* particular solution + *all* null solutions.

Every point on that bold line is a particular solution to $x_1 - x_2 = 1$. We marked only one y_p. **Null solutions lie on a parallel line $\boldsymbol{x_1 - x_2 = 0}$ through the center $\boldsymbol{(0, 0)}$.**

Example 5 Second order equations $Ay'' + By' + Cy = e^{st}$ or $e^{i\omega t}$ have complete solutions $y = y_p + y_n$. The particular solution $y_p = Ye^{st}$ is a multiple of e^{st}. The null solutions are $y_n = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. If $s_2 = s_1$, replace $e^{s_2 t}$ by $te^{s_1 t}$.

Example 6 The complete solution to the impressive equation $5y = 10$ is $y = 2$. This is our only choice for the particular solution, $y_p = 2$. The null solutions solve $5y_n = 0$, and the only possibility is $y_n = 0$. *The one and only solution is* $y = y_p + y_n = 2 + 0$.

That seems boring, when $y_n = 0$ is the only null solution. But this is what we want (and usually get) for matrix equations. If A is an invertible matrix, the only solution to $Ay = b$ is $y = y_p = A^{-1}b$. Then the only null solution to $Ay_n = 0$ is $y_n = 0$.

Higher Order Equations

Up to this moment, third derivatives have not been seen. They don't arise often in physical problems. But exponential solutions Ye^{st} and $Ye^{i\omega t}$ still appear. The one essential requirement is that the equation must have *constant coefficients*.

Equation of order N	$$A_N \frac{d^N y}{dt^N} + \cdots + A_1 \frac{dy}{dt} + A_0 y = f(t)$$	(11)

When $f = 0$, the best solutions of the null equation are still exponentials $y_n = e^{st}$. Substitute e^{st} into the equation to find N possible exponents $\boldsymbol{s_1, s_2, \ldots, s_N}$.

$$\boldsymbol{f = 0 \text{ and } y_n = e^{st}} \qquad \left(\boldsymbol{A_N s^N + \cdots + A_1 s + A_0}\right) e^{st} = 0. \tag{12}$$

The exponents s in y_n are the N roots of that polynomial. So we (usually) have N independent solutions $e^{s_1 t}, \ldots, e^{s_N t}$. All their combinations are still solutions. If the polynomial in (12) happens to have a double root at s, our two solutions are e^{st} and te^{st}.

Example 7 **Solve the third order equation** $\boldsymbol{y''' + 2y'' + y' = e^{3t}}$.

Solution To find the null solutions y_n, substitute $y_n = e^{st}$ with right hand side zero:

$$\boldsymbol{s^3 + 2s^2 + s = 0} \qquad s(s^2 + 2s + 1) = 0 \qquad \boldsymbol{s(s+1)^2 = 0}.$$

The exponents are $s = 0, -1, -1$. The null solutions are $c_1 e^{0t}$ and $c_2 e^{-t}$ and $c_3 t e^{-t}$ (the extra t comes from the double root). A particular solution y_p is Ye^{3t} (since 3 is not one of the exponents 0 and -1 in y_n). Substitute Ye^{3t} to find $\boldsymbol{Y = 1/48}$:

$$27Ye^{3t} + 18Ye^{3t} + 3Ye^{3t} = e^{3t} \text{ and } 48Y = 1 \text{ and } \boldsymbol{y_p = e^{3t}/48}.$$

The transfer function is $\boldsymbol{Y(s) = 1/(s^3 + 2s^2 + s)}$. For $\boldsymbol{e^{3t}}$ put $\boldsymbol{s = 3}$. Then $\boldsymbol{Y = 1/48}$.

Here is the plan for this section on constant coefficient equations with forced oscillations.

1 Find the **exponential response** $\boldsymbol{y(t) = Y(s)e^{st}}$ to the driving function $f(t) = e^{st}$.

2 Adjust that formula when $Y(s) = \infty$ because of **resonance**.

3 Solve the **real equation** $Ay'' + By' + Cy = \cos \omega t$ to see the effect of damping.

This is the key example for applications: y is the real part of $Y(s)e^{st}$ when $s = i\omega$. The solution in equation (23) is $\boldsymbol{y(t) = M \cos \omega t + N \sin \omega t = G \cos(\omega t - \alpha)}$.

Exponential Response Function = Transfer Function

This book concentrates on first and second order equations. When the coefficients are constant and the right side is an exponential, we have solved three important problems:

First order	$y' - ay = \boldsymbol{e^{ct}}$	$y_p = e^{ct}/(\boldsymbol{c - a})$
Oscillation	$my'' + ky = \boldsymbol{e^{i\omega t}}$	$y_p = e^{i\omega t}/(\boldsymbol{k - m\omega^2})$
Second order	$Ay'' + By' + Cy = \boldsymbol{e^{st}}$	$y_p = \boldsymbol{e^{st}}/(\boldsymbol{As^2 + Bs + C})$

It is natural (natural to a mathematician) to try to solve all constant coefficient equations of all orders by one formula. We can almost do it, but resonance gets in the way.

Let me write D for each derivative d/dt. Then D^2 is d^2/dt^2. All our equations involve powers of D, and equations of order N involve D^N. Here $N = 2$.

$$\textbf{Polynomial } \boldsymbol{P(D)} \qquad Ay'' + By' + Cy = (\boldsymbol{AD^2 + BD + C})\, y = \boldsymbol{P(D)}\, y. \tag{13}$$

The null solutions and the particular solution all come from this polynomial $P(D)$.

$$\textbf{Find } \boldsymbol{N} \textbf{ null solutions } \boldsymbol{y_n = e^{st}} \qquad As^2 + Bs + C = 0 \text{ is exactly } \boldsymbol{P(s) = 0} \tag{14}$$

$$\textbf{Find a particular } \boldsymbol{y_p = Ye^{ct}} \qquad P(D)y = e^{ct} \text{ gives the number } \boldsymbol{Y = 1/P(c)} \tag{15}$$

The value Y of the transfer function gives the exponential response $\quad \boldsymbol{y_p = e^{ct}/P(c)}$.

Please understand: In the null solutions, s has N specific values $s_1, \ldots, s_N$. Those are the roots of the Nth degree characteristic equation $P(s) = 0$. In the particular solution $e^{ct}/P(c)$, the specific value $s = c$ is the exponent in the right hand side $f = e^{ct}$.

The exponents c and s are completely allowed to be imaginary or complex.

$$\boldsymbol{P(D)y = e^{ct}} \qquad \boxed{y = y_p + y_n = \frac{e^{ct}}{P(c)} + c_1 e^{s_1 t} + \cdots + c_N e^{s_N t}} \tag{16}$$

That fraction $Y = 1/P(c)$ "transfers" the input $f = e^{ct}$ into the output $y = Ye^{ct}$. You often see it as $1/P(s)$ with the variable s. It is sometimes called the **system function**.

There is only one exception to this simple and beautiful exponential response formula. The forcing exponent c might be one of the exponents $s_1, \ldots, s_N$ in the null solution. **In this case $P(c)$ is zero**. We cannot divide by $P(c)$ when it is zero.

$$\textbf{Exception} \qquad \text{If } P(c) = 0 \text{ then } y = e^{ct}/P(c) \text{ cannot solve } P(D)y = e^{ct}.$$

$P(c) = 0$ is the exceptional case of **resonance**. The formula $e^{ct}/P(c)$ has to change.

Resonance

We may be pushing a swing at its natural frequency. Then $c = i\omega_n = i\sqrt{k/m}$. The polynomial $P(D)$ from $my'' + ky$ is $mD^2 + k$, and we have $P(c) = 0$ at this natural frequency. Here is the exponential response formula adjusted for resonance.

$$\textbf{Resonant response} \qquad \boxed{\textbf{If } \boldsymbol{P(c) = 0} \textbf{ then } \boldsymbol{y_p = \frac{t}{P'(c)}\, e^{ct}}} \tag{17}$$

That extra factor t enters the solution when $P(c) = 0$. We replace $1/P(c)$ by $\boldsymbol{t/P'(c)}$. This succeeds unless there is "double resonance" and $P'(c)$ is also zero. Then the formula moves on to the second derivative of P, and $y_p(t) = t^2 e^{ct}/P''(c)$.

The odds against double resonance are pretty high. The point is that the equation $\boldsymbol{P(D)y = e^{ct}}$ has a neat solution in terms of the polynomial P: usually $\boldsymbol{y = e^{ct}/P(c)}$.

I can explain that resonant solution $y = te^{ct}/P'(c)$ when $P(c) = 0$ and $P'(c) \neq 0$. We have seen this happen in Section 1.5 for the first order equation $y' - ay = e^{ct}$. That equation has $P(D) = D - a$ and $P(c) = c - a$ and resonance when $c = a$:

$$y' - ay = e^{ct} \quad \text{has the very particular solution} \quad \boldsymbol{y_{vp}} = \frac{\boldsymbol{e^{ct} - e^{at}}}{\boldsymbol{c - a}}$$

$$\text{As } c \text{ approaches } a, \ y_{vp} \text{ approaches } \frac{\text{derivative of top}}{\text{derivative of bottom}} = \frac{\boldsymbol{te^{at}}}{\boldsymbol{1}}$$

That is l'Hôpital's Rule! The only unusual thing is that we have c in place of x, and c-derivatives in place of x-derivatives. The very particular solution is the one starting from $y_{vp} = 0$ at $t = 0$. The resonant solution te^{at} fits our formula $te^{ct}/P'(c)$ because $c = a$ and $P(c) = c - a$ and $P'(c) = 1$.

When the equation has order N, the polynomial P has degree N. Suppose the exponent c is *close* to a—which is one of the exponents $s_1, \ldots, s_N$ in the null solution. Then $P(a) = 0$ and e^{at} is a null solution and $e^{ct}/P(c)$ is one particular solution:

$$\text{A very particular solution to } P(D)y = e^{ct} \text{ is } \boldsymbol{y_{vp}} = \frac{\boldsymbol{e^{ct} - e^{at}}}{\boldsymbol{P(c) - P(a)}}. \qquad (18)$$

To emphasize: c close to a is fine. But $c = a$ is *not* fine. Formula (16) changes at $c = a$:

$$\textbf{Resonance} \qquad \textbf{If } \boldsymbol{c = a} \textbf{ then l'Hôpital's limit in (16) is} \quad \boxed{y_{vp} = \frac{\boldsymbol{te^{at}}}{\boldsymbol{P'(a)}}.} \qquad (19)$$

Take the c-derivatives of $e^{ct} - e^{at}$ and $P(c) - P(a)$ at $c = a$, to get $te^{at}$and $P'(a)$.

Summary The transfer function is $Y(s) = 1/P(s)$. It has "poles" at the N roots of $P(s) = 0$. Those are the exponents in the null solutions $y_n(t)$. The particular solution $y_p = Ye^{ct}$ has the same exponent c as the driving term $f = e^{ct}$. The transfer function $Y(c) = 1/P(c)$ decides the amplitude of $y_p(t)$. If c is a pole of Y, we have resonance.

Example 8 The 4th degree equation $\boldsymbol{D^4y = d^4y/dt^4 = 1}$ has *4-way resonance.*

What are the null solutions to $y'''' = 0$? By trying $y = e^{st}$ we get $s^4 = 0$. This has *all four roots at* $s = 0$. Then one null solution is $y = e^{0t}$, which is $y = 1$. The other null solutions have factors t, t^2, t^3 because of the four-way zero. Altogether:

The null solutions to $\boldsymbol{y'''' = 0}$ **have the form** $\boldsymbol{y_n(t) = c_1 + c_2t + c_3t^2 + c_4t^3}$.

Now find a particular solution to $y'''' = e^{ct}$. For most exponents c we get $\boldsymbol{y_p = e^{ct}/c^4}$. This is exactly $e^{ct}/P(c)$. *But* $c = 0$ *gives* **quadruple resonance**: $c^4 = 0$ has a 4-way root. A quadruple l'Hôpital rule gives the fourth derivative P'''' and the very particular solution to $y'''' = 1$ that you knew before taking this course and seeing this book:

$$\boldsymbol{y'''' = 1 = e^{0t}} \ \textbf{ has } \ \boldsymbol{c = a = 0} \ \textbf{ and } \ \boldsymbol{P = s^4} \qquad y_p(t) = \frac{\boldsymbol{t^4e^{0t}}}{\boldsymbol{P''''(0)}} = \frac{\boldsymbol{t^4}}{\boldsymbol{24}}.$$

Real Second Order Equations with Damping

Now we focus on the key equation: **second order**. The left side is $Ay'' + By' + Cy$. The transfer function is $Y(s) = 1/(As^2 + Bs + C)$. When the right side is $f(t) = e^{i\omega t}$, the exponent is $s = i\omega$. When A, B, C are nonzero, we won't have resonance:

$$\textbf{No resonance} \quad A(i\omega)^2 + B(i\omega) + C = (C - A\omega^2) + i(B\omega) \neq 0.$$

We know that the response to $f(t) = e^{i\omega t}$ is $y_p(t) = Y(i\omega)e^{i\omega t}$. This is a perfect example, except that those functions are not real.

In applications to real life (and this equation has many), we want $f(t) = \cos \omega t$. We *must* solve this problem. You will say, just solve for $e^{i\omega t}$ and $e^{-i\omega t}$, and take half of each solution. Even faster than that, **solve for $e^{i\omega t}$ and take the real part of $y_p(t)$**. Or you could stay entirely real and look for a solution $y(t) = M \cos \omega t + N \sin \omega t$.

All those ideas will succeed. They all give the same answer (in different forms). The best form has to bring out the most important number in the answer $y(t)$. That number is the **amplitude G of the forced oscillation**. So first place goes to the **polar form $y(t) = G\cos(\omega t - \alpha)$**, because this shows the gain G.

The null solutions decay because the solutions s_1 and s_2 to $As^2 + Bs + C = 0$ have negative real parts $-B/2A$. The particular solution $G\cos(\omega t - \alpha)$ does not decay, because it is driven by a forcing function $f = \cos \omega t$ that never stops.

The next pages will find G and α. This is algebra put to good use. We are working with letters A, B, C that represent physical quantities. In Section 2.5 they will be mass-damping-stiffness or inductance-resistance-inverse capacitance. Those are not the only possible examples! Biology and chemistry and management and the economics of a whole country also see damped oscillations. I hope you will find those models.

Damped Oscillations in Rectangular Form

I will start with the rectangular form $y(t) = M \cos \omega t + N \sin \omega t$. It is not as useful as the polar form, but it is easier to compute. Substitute this $y(t)$ into the differential equation $Ay'' + By' + Cy = \cos \omega t$. Match the cosine terms and the sine terms:

Cosines on both sides	$-A\omega^2 M + B\omega N + CM \; = 1$	(20)
Sines on the left side	$-A\omega^2 N - B\omega M + CN \; = 0$	(21)

To solve for M, multiply equation (20) by $C - A\omega^2$. Then multiply equation (21) by $B\omega$ and subtract from (20). The coefficient of N will be zero. So N is eliminated and we have an equation for M alone. M is multiplied by the important number D:

$C - A\omega^2$ times (20)
minus $B\omega$ times (21)

$$[(\boldsymbol{C - A\omega^2})^2 + (\boldsymbol{B\omega})^2]M = \boldsymbol{D}M = C - A\omega^2. \quad (22)$$

We divide by D to find $M = (C - A\omega^2)/D$. Then equation (21) tells us $N = B\omega/D$. And equation (27) will tell us that $\boldsymbol{M^2 + N^2 = 1/D}$.

$$\textbf{Real solution } y_p \textbf{ is } M\cos\omega t + N\sin\omega t \qquad M = \frac{C - A\omega^2}{D} \qquad N = \frac{B\omega M}{C - A\omega^2} = \frac{B\omega}{D} \tag{23}$$

Let me say right away: **The complex number $Y(i\omega)$ is just $M - iN$.** This calculation will connect real to complex and rectangular to polar. When I multiply and divide by $Y(-i\omega)$, you will see that the denominator of $Y(i\omega)$ is $D = (C - A\omega^2)^2 + (B\omega)^2$:

$$\frac{1}{(C - A\omega^2) + iB\omega} \times \frac{(C - A\omega^2) - iB\omega}{(C - A\omega^2) - iB\omega} = \frac{(C - A\omega^2) - iB\omega}{D} = M - iN. \tag{24}$$

$Y = M - iN$ is exactly what we want and need. The input $f = \cos\omega t$ is the real part of $e^{i\omega t}$, so the output y is the real part of $Ye^{i\omega t}$. That real part is the rectangular form $y = M\cos\omega t + N\sin\omega t$:

$$\mathrm{Re}\,(Ye^{i\omega t}) = \mathrm{Re}\,[(M - iN)(\cos\omega t + i\sin\omega t)] = M\cos\omega t + N\sin\omega t \tag{25}$$

Damped Oscillations in Polar Form

The solution we want is the real part of $Y(i\omega)e^{i\omega t}$. Equation (25) computed that solution in its rectangular form. To compute $y(t)$ in polar form, the first step (almost the only step) is to put $Y(i\omega)$ in polar form. This number is the complex gain:

$$\textbf{Complex gain} \quad Y(i\omega) = M - iN = Ge^{i\alpha} \text{ with } G = \frac{1}{\sqrt{D}} \text{ and } \tan\alpha = \frac{N}{M}. \tag{26}$$

That amplitude G is simply called the "gain". It is the most important quantity in all these pages of calculations. The input $\cos\omega t$ had amplitude 1, the output $y(t)$ has amplitude G. Of course that output is not $y = G\cos\omega t$! Damping produces a phase lag α. At the same time damping reduces the amplitude of the output.

The undamped amplitude $|Y| = 1/|C - A\omega^2|$ is reduced to $G = 1/\sqrt{D}$:

$$G = \sqrt{M^2 + N^2} = \left(\frac{(C - A\omega^2)^2}{D^2} + \frac{(B\omega)^2}{D^2}\right)^{1/2} = \left(\frac{D}{D^2}\right)^{1/2} = \frac{1}{\sqrt{D}}. \tag{27}$$

I will collect all these beautiful (?) important (!) formulas after one example.

Example 9 Solve $y'' + y' + 2y = \cos t$ in rectangular form and also in polar form.

Solution The equation has $A = 1, B = 1, C = 2$, and $\omega = 1$. We are finding a particular solution. Let me use the formulas directly and then comment briefly. The numbers give $C - A\omega^2 = 1$ and $B\omega = 1$, so $D = 1^2 + 1^2 = 2$.

Therefore the solution has $G = \sqrt{1/2}$ and $M = N = \frac{1}{2}$ and $\tan\alpha = 1$ and $\alpha = \pi/4$:

Rectangular $\quad y(t) = M \cos \omega t + N \sin \omega t = \frac{1}{2}(\cos t + \sin t)$

Polar $\quad y(t) = \text{Re}\left(Ge^{-i\alpha}e^{i\omega t}\right) = G\cos(\omega t - \alpha) = \frac{1}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right).$

For this example we verify directly that polar = rectangular:

$$G\cos\left(t - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos t \,\cos\frac{\pi}{4} + \sin t\,\sin\frac{\pi}{4}\right) = \frac{1}{2}(\cos t + \sin t).$$

The rectangular form has simpler numbers. But the polar form has the most important number $G = 1/\sqrt{2}$. That gain G is less than the undamped gain $|Y|$ by a factor $\cos\alpha$.

$$\text{Undamped} \quad |Y| = \frac{1}{|C - A\omega^2|} = 1 \qquad \text{Damped} \quad G = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{2}} = \cos\alpha.$$

Undamped versus Damped

The undamped equation $Ay'' + Cy = \cos\omega t$ has $B = 0$ and $Y = 1/(C - A\omega^2)$. Compare that amplitude of $y(t) = Y\cos\omega t$ from Section 2.1 with the harder problem we just solved. The comparison lets you see how the damping contributes $Bs = Bi\omega$ in the transfer function that multiplies the input $e^{i\omega t}$. Damping causes a phase lag α. Damping also reduces the amplitude to $G = Y\cos\alpha$. Here are the key formulas:

	Undamped	**Damped**
Equation	$Ay'' + Cy = \cos\omega t$	$Ay'' + By' + Cy = \cos\omega t$
Solution	$y = Y\cos\omega t$	$y = G\cos(\omega t - \alpha)$
Magnitude	$\lvert Y\rvert = \dfrac{1}{\lvert C - A\omega^2\rvert}$	$G = \dfrac{1}{\sqrt{D}} = Y\cos\alpha$
Phase lag	zero	$\tan\alpha = \dfrac{N}{M} = \dfrac{B\omega}{C - A\omega^2}$

When the driving function is $F\cos\omega t$, the solutions include that extra factor F. When the driving function is $\sin\omega t$, that is the same as $\cos\left(\omega t - \frac{\pi}{2}\right)$. So the solutions have $\phi = \pi/2$ as an additional phase lag: $y = G\cos(\omega t - \alpha - \pi/2) = G\sin(\omega t - \alpha)$.

When the driving function is $A\cos\omega t + B\sin\omega t$, that equals $R\cos(\omega t - \phi)$. This is the sinusoidal identity from Section 1.5. Then the solution is $\boldsymbol{RG\cos(\omega t - \alpha - \phi)}$. This is the particular solution y_p that oscillates with the same frequency ω as the input.

Let me show why the gain is reduced to $G = Y\cos\alpha$ from its undamped value $|Y| = 1/|C - A\omega^2|$. We know from (27) that $G = \sqrt{M^2 + N^2} = 1/\sqrt{D}$. And we know from (23) that $YM = 1/D$:

$$\textbf{Damped gain} \qquad Y\cos\alpha = \frac{YM}{\sqrt{M^2+N^2}} = \frac{1/D}{1/\sqrt{D}} = G. \tag{28}$$

Better Notation

A good plan is to divide $my'' + by' + ky = kF(t)$ by the mass m, for several reasons:

$$y'' + \frac{b}{m}y' + \frac{k}{m}y = \frac{k}{m}F(t). \tag{29}$$

First, the coefficient of y'' becomes 1. Second, replacing k/m by ω_n^2 gives it meaning. Third, the input F has the same units as the output y. So now the gain $G = |y|/|F|$ is dimensionless. This happened because the original $f(t)$ with unsuitable units was replaced by $kF(t)$—which is now divided by m.

Most valuable of all is a new way to write the damping term b/m, which is B/A. The key point is that $\boldsymbol{b^2}$ **and** $\boldsymbol{mk}$ **have the same dimensions**. From the equation, my'' and by' and ky have the same dimensions. Then so do $(by')^2$ and $(my'')(ky)$. And also $(y')^2$ and $(y'')(y)$—they both contain $1/(\text{time})^2$. *This leaves* b^2 *and* mk.

This quantity $Z = b/\sqrt{4mk}$ is highly useful. Overdamping is $Z > 1$. Underdamping is $Z < 1$. The coefficient b/m in equation (29) has a better form $2Z\omega_n$ in (30).

$$\boxed{\frac{b}{m} = \frac{2b}{\sqrt{4mk}}\sqrt{\frac{k}{m}} = 2Z\omega_n} \qquad \boxed{y'' + 2Z\omega_n y' + \omega_n^2 y = \omega_n^2 F(t)} \tag{30}$$

***Z* is the damping ratio. The correct symbol is a Greek zeta** (ζ). But a capital zeta $= Z$ is so much easier to read and write. (The MATLAB command is also named zeta.) Watch how this ratio of B to $\sqrt{4AC}$ brings out the important parts of every formula. If $Z < 1$, the natural frequency ω_n is reduced to the **damped frequency** $\boldsymbol{\omega_d = \omega_n\sqrt{1-Z^2}}$.

Roots s_1 and s_2	$s^2 + 2Z\omega_n s + {\omega_n}^2 = 0$ gives $s = -Z\omega_n \pm \omega_n\sqrt{Z^2-1}$	(31)
Underdamping	$Z^2 = \dfrac{b^2}{4mk} < 1$ and $\boldsymbol{s = -Z\omega_n \pm i\omega_d}$	(32)
Null solutions	$y_n(t) = \boldsymbol{e^{-Z\omega_n t}\,(c_1\cos\omega_d t + c_2\sin\omega_d t)}$	(33)

The null solutions are not pure oscillations. They include the exponential $e^{-Z\omega_n t}$. Their frequency changes to ω_d. The graph of $y(t)$ oscillates as it approaches zero, and the peak times when $y = y_{\max}$ are spaced by $2\pi/\omega_d$.

The page after Problem Set 2.4 collects our solution formulas in one place.

■ REVIEW OF THE KEY IDEAS ■

1. A particular solution to $Ay'' + By' + Cy = e^{st}$ is $e^{st}/(As^2 + Bs + C)$.
2. This is a constant coefficient equation $P(D)y = e^{ct}$ with solution $\boldsymbol{y_p = e^{ct}/P(c)}$.
3. Resonance occurs if e^{ct} is a null solution of $P(D)y = 0$. This means that $P(c) = 0$.
4. Resonance leads to an extra t: $\boldsymbol{y_p(t) = te^{ct}/P'(c)}$ when $P(c) = 0$ and $P'(c) \neq 0$.
5. For second order equations with $f = \cos\omega t$ the gain is $G = 1/|P(i\omega)| = 1/\sqrt{D}$.
6. The real solution is $M\cos\omega t + N\sin\omega t = G\cos(\omega t - \alpha)$ with $\tan\alpha = N/M$.
7. With damping ratio $Z = B/\sqrt{4AC}$, the equation is $y'' + 2\omega_n Z y' + \omega_n^2 y = \omega_n^2 F(t)$.
8. If $Z < 1$, the damped frequency is $\omega_d = \omega_n\sqrt{1 - Z^2}$. Then s_1, s_2 are $-Z\omega_n \pm i\omega_d$.

Problem Set 2.4

Problems 1-4 use the exponential response $\boldsymbol{y_p = e^{ct}/P(c)}$ to solve $\boldsymbol{P(D)y = e^{ct}}$.

1 Solve these constant coefficient equations with exponential driving force:

(a) $y_p'' + 3y_p' + 5y_p = e^t$ (b) $2y_p'' + 4y_p = e^{it}$ (c) $y'''' = e^t$

2 These equations $P(D)y = e^{ct}$ use the symbol D for d/dt. Solve for $y_p(t)$:

(a) $(D^2 + 1)y_p(t) = 10e^{-3t}$ (b) $(D^2 + 2D + 1)y_p(t) = e^{i\omega t}$

(c) $(D^4 + D^2 + 1)y_p(t) = e^{i\omega t}$

3 How could $y_p = e^{ct}/P(c)$ solve $y'' + y = e^t e^{it}$ and then $y'' + y = e^t \cos t$?

4 (a) What are the roots s_1 to s_3 and the null solutions to $y_n''' - y_n = 0$?

(b) Find particular solutions to $y_p''' - y_p = e^{it}$ and to $y_p''' - y_p = e^t - e^{i\omega t}$.

Problems 5-6 involve repeated roots $\boldsymbol{s}$ in $\boldsymbol{y_n}$ and resonance $\boldsymbol{P(c) = 0}$ in $\boldsymbol{y_p}$.

5 Which value of C gives resonance in $y'' + Cy = e^{i\omega t}$? Why do we never get resonance in $y'' + 5y' + Cy = e^{i\omega t}$?

6 Suppose the third order equation $P(D)y_n = 0$ has solutions $y = c_1e^t + c_2e^{2t} + c_3e^{3t}$. What are the null solutions to the sixth order equation $P(D)P(D)y_n = 0$?

7 Complete this table with equations for roots s_1 and s_2 and solutions y_n and y_p:

Undamped free oscillation	$my'' + ky = 0$	$y_n =$ ____
Undamped forced oscillation	$my'' + ky = e^{i\omega t}$	$y_p =$ ____
Damped free motion	$my'' + by' + ky = 0$	$y_n =$ ____
Damped forced motion	$my'' + by' + ky = e^{ct}$	$y_p =$ ____

8 Complete the same table when the coefficients are 1 and $2Z\omega_n$ and ω_n^2 with $Z < 1$.

Undamped and free	$y'' + \omega_n^2 y = 0$	$y_n =$ ____
Undamped and forced	$y'' + \omega_n^2 y = e^{i\omega t}$	$y_p =$ ____
Underdamped and free	$y'' + 2Z\omega_n y' + \omega_n^2 y = 0$	$y_n =$ ____
Underdamped and forced	$y'' + 2Z\omega_n y' + \omega_n^2 y = e^{ct}$	$y_p =$ ____

9 What equations $y'' + By' + Cy = f$ have these solutions ?

(a) $y = c_1 \cos 2t + c_2 \sin 2t + \cos 3t$

(b) $y = c_1 e^{-t} \cos 4t + c_2 e^{-t} \sin 4t + \cos 5t$

(c) $y = c_1 e^{-t} + c_2 t e^{-t} + e^{i\omega t}$

10 If $y_p = te^{-6t} \cos 7t$ solves a second order equation $Ay'' + By' + Cy = f$, what does that tell you about A, B, C, and f ?

11 (a) Find the steady oscillation $y_p(t)$ that solves $y'' + 4y' + 3y = 5 \cos \omega t$.

(b) Find the amplitude A of $y_p(t)$ and its phase lag α.

(c) Which frequency ω gives maximum amplitude (maximum gain) ?

12 Solve $y'' + y = \sin \omega t$ starting from $y(0) = 0$ and $y'(0) = 0$. Find the limit of $y(t)$ as ω approaches 1, and the problem approaches resonance.

13 Does critical damping and a double root $s = 1$ in $y'' + 2y' + y = e^{ct}$ produce an extra factor t in the null solution y_n or in the particular y_p (proportional to e^{ct}) ? What is y_n with constants c_1, c_2 ? What is $y_p = Ye^{ct}$?

14 If $c = i\omega$ in Problem 13, the solution y_p to $y'' + 2y' + y = e^{i\omega t}$ is ____. That fraction Y is the transfer function at $i\omega$. What are the magnitude and phase in $Y = Ge^{-i\alpha}$?

By rescaling both t and y, we can reach $A = C = 1$. Then $\omega_n = 1$ and $B = 2Z$. The model problem is $y'' + 2Zy' + y = f(t)$.

15 What are the roots of $s^2 + 2Zs + 1 = 0$? Find two roots for $Z = 0, \frac{1}{2}, 1, 2$ and identify each type of damping. The natural frequency is now $\omega_n = 1$.

16 Find two solutions to $y'' + 2Zy' + y = 0$ for every Z except $Z = 1$ and -1. Which solution $g(t)$ starts from $g(0) = 0$ and $g'(0) = 1$? What is different about $Z = 1$?

17 The equation $my'' + ky = \cos \omega_n t$ is exactly at resonance. The driving frequency on the right side equals the natural frequency $\omega_n = \sqrt{k/m}$ on the left side. Substitute $y = Rt \sin(\sqrt{k/m}\, t)$ to find R. This resonant solution grows in time because of the factor t.

18 Comparing the equations $Ay''+By'+Cy = f(t)$ and $4Az''+Bz'+(C/4)z = f(t)$, what is the difference in their solutions ?

19 Find the fundamental solution to the equation $g'' - 3g' + 2g = \delta(t)$.

20 (Challenge problem) Find the solution to $y'' + By' + y = \cos t$ that starts from $y(0) = 0$ and $y'(0) = 0$. Then let the damping constant B approach zero, to reach the resonant equation $y'' + y = \cos\, t$ in Problem 17, with $m = k = 1$.

Show that your solution $y(t)$ is approaching the resonant solution $\frac{1}{2}t \sin\, t$.

21 Suppose you know three solutions $y_1,\ y_2,\ y_3$ to $y'' + B(t)y' + C(t)y = f(t)$. How could you find $B(t)$ and $C(t)$ and $f(t)$?

Solution Page   Linear Constant Coefficient Equations

First order $\dfrac{dy}{dt} = ay + f(t)$ **Second order** $A\dfrac{d^2y}{dt^2} + B\dfrac{dy}{dt} + Cy = f(t)$

Nth order $A_N\dfrac{d^Ny}{dt^N} + \cdots + A_1\dfrac{dy}{dt} + A_0y = (A_N D^N + \cdots + A_0)y = P(D)y = f(t)$

Null solutions y_n have $f(t) = 0$ Substitute $y = e^{st}$ to find the N exponents s

First order	$\dfrac{d}{dt}(e^{st}) = ae^{st}$	$s = a$ and $y_n = ce^{at}$
Second order	$As^2 + Bs + C = 0$	$y_n = c_1e^{s_1t} + c_2e^{s_2t}$
Nth order	$P(s) = 0$	$y_n = c_1e^{s_1t} + \cdots + c_Ne^{s_Nt}$

Exponential response to $f(t) = e^{ct}$ Step response for $c = 0$ Look for $y = Ye^{ct}$

First order	$\dfrac{d}{dt}\left(Ye^{ct}\right) - aYe^{ct} = e^{ct}$	$y_p = \dfrac{e^{ct}}{c-a}$ has $Y = \dfrac{1}{c-a}$
Second order	$Y(Ac^2 + Bc + C)e^{ct} = e^{ct}$	$y_p = \dfrac{e^{ct}}{Ac^2 + Bc + C} = Ye^{ct}$
Nth order	$YP(c)e^{ct} = e^{ct}$	$y_p = \dfrac{e^{ct}}{P(c)}$ or $\dfrac{te^{ct}}{P'(c)}$ when $P(c) = 0$

Fundamental solution $g(t)$ = Impulse response when $f(t) = \delta(t)$

First order	$g(t) = e^{at}$	starting from $g(0) = 1$
Second order	$g(t) = \dfrac{e^{s_1t} - e^{s_2t}}{A(s_1 - s_2)}$	starting from $g(0) = 0$ and $g'(0) = 1/A$
Undamped	$g(t) = \dfrac{\sin \omega_n t}{A\omega_n}$	underdamped $g(t) = e^{-Z\omega_n t}\dfrac{\sin \omega_d t}{A\omega_d}$
Nth order	$g(t) = y_n(t)$	$g(0) = g'(0) = \ldots = 0, g^{(N-1)}(0) = 1/A_N$

Very particular solution for each driving function $f(t)$: zero initial conditions on y_{vp}

Multiply input at every time s
by the growth factor over $t - s$

$$y(t) = \int_0^t g(t-s)\, f(s)\, ds$$

Undetermined coefficients	Direct solution for special $f(t)$ in Section 2.6
Variation of parameters	$y_p(t)$ comes from $y_n(t)$ in Section 2.6
Solution by Laplace transform	Transfer function = transform of $g(t)$ in Section 2.7
Solution by convolution	$y(t) = g(t) * f(t)$ in Section 8.6

2.5 Electrical Networks and Mechanical Systems

Section 2.4 solved the equation $Ay'' + By' + Cy = \cos \omega t$. Now we want to understand the meaning of A, B, C in real applications. This is the fundamental equation of engineering for a one-unknown system, when the forcing function is a sinusoid. It is a perfect opportunity to use the **transfer function**. This connects the input to the response.

For mechanical engineers the unknown y gives the position of one mass—oscillating or rotating or vibrating. For electrical engineers the unknown y is the voltage $V(t)$ or the current $I(t)$ in a one-loop RLC circuit. Those letters R, L, C represent a resistor, an inductor, and a capacitor. For a chemical engineer or a scientist or an economist the equation is a model of ..... I have to stop or this presentation will go out of control.

The great differential equations of applied mathematics are *first order or second order*. The equations we understand best are *linear with constant coefficients*.

In later chapters the single unknown becomes a vector. Its coefficients become square matrices in $\boldsymbol{dy/dt = Ay}$ and $\boldsymbol{d^2y/dt^2 = -Sy}$. We have a system of n equations for voltages at nodes or currents along edges or positions of n masses. Linear algebra will organize the equations and their solutions. *Matrix differential equations give us the right language to express applied mathematics.*

Our goals are to find and solve the equations for $y(t)$ in real applications. These are **balance equations**: balance of forces and balance of currents. **Flow in equals flow out**.

Spring-Mass-Dashpot Equation and Loop Equation

In mechanics, y and y' and y'' are the position, the velocity, and the acceleration. The numbers A, B, C represent the *mass* m, the *damping* b, and the *stiffness* k:

Newton's Law $\boldsymbol{F = ma}$ $\qquad \boldsymbol{my'' + by' + ky =}$ **applied force.** (1)

The picture in Figure 2.12 shows the mass m attached to a spring and also a dashpot. Those two are responsible for the forces $-ky$ and $-by'$. The stretched spring pulls back on the mass. By Hooke's Law that force is $-ky$. The damping force comes from a dashpot (old-fashioned word, key idea). You could visualize the mass moving in a heavy liquid like oil. The friction force is $-by'$, proportional to velocity and in the opposite direction.

For an electrical network, it was Kirchhoff and not Newton who provided the balance equations. **Kirchhoff's Voltage Law says that the sum of voltage drops around any closed loop is zero**. The current is $I(t)$ and we start with one loop:

Voltage law KVL : $\qquad \boldsymbol{L\dfrac{dI}{dt} + RI + \dfrac{1}{C}\int I\,dt =}$ **applied voltage.** (2)

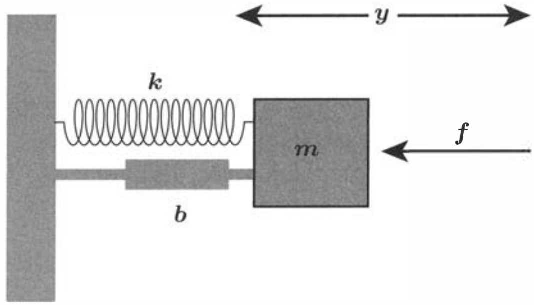


Figure 2.12: Three forces enter $F = my''$: spring force ky, friction by', driving force f.

The numbers L, R, C are the inductance, the resistance, and the capacitance. (Unfortunately we divide by the capacitance C. In the end the equation has constant coefficients and regardless of the letters we solve it.) To produce a second order differential equation for $I(t)$, and to remove the integration in equation (2), take the derivative of every term:

Loop equation for the current $I(t)$ $$LI'' + RI' + \frac{1}{C}I = F\cos\omega t. \tag{3}$$

That force $F\cos\omega t$ comes from a battery or a generator, when we close the switch. We will be looking for a **particular solution $I_p(t)$**. That solution is produced by the applied force. We are *not* looking at initial conditions and $y_n(t)$. Those null solutions y_n are transient, with $f = 0$. They die out exponentially fast.

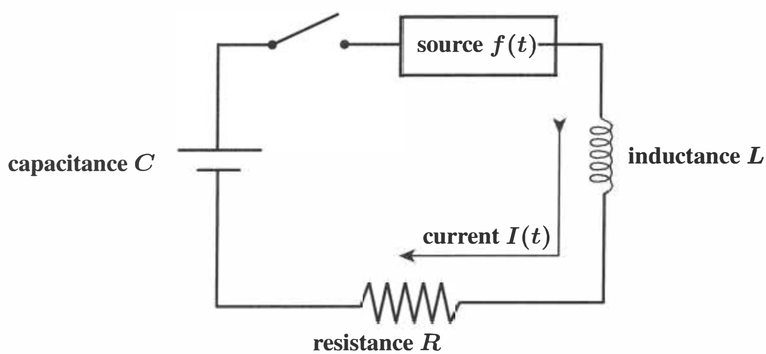


Figure 2.13: A one-loop RLC circuit with a source and a switch.

The Mechanical-Electrical Analogy

Both applications produce second order equations $Ay'' + By' + Cy = f(t)$. This means we can solve both problems at once—not only mathematically but also physically. We can predict the behavior of a mechanical system by testing an electrical analog, when simple circuit elements are more convenient to work with. The basic idea is to match the three numbers m, b, k with the numbers L, R, and $1/C$.

Mechanical System		**Electrical System**
Mass m	$\longleftrightarrow$	Inductance L
Damping constant $\boldsymbol{b}$	$\longleftrightarrow$	Resistance R
Spring constant k	$\longleftrightarrow$	Reciprocal capacitance $1/C$
Natural frequency $\omega_n^2 = k/m$	$\longleftrightarrow$	Natural frequency $\omega_n^2 = 1/LC$

Before solving for the loop current $I(t)$, let me outline three solution methods—our past method, our present method, and our future method.

$\cos\omega t$ to $e^{i\omega t}$ to $Y(\omega)$

Past method Section 2.4 solved $Ay'' + By' + Cy = F\cos\omega t$. The equation was real and the solution was real. That solution had a sine-cosine form and also an amplitude-phase form:

$$y(t) = M\cos\omega t + N\sin\omega t = G\cos(\omega t - \alpha). \tag{4}$$

The connections between inputs F and outputs M, N came by substituting $y(t)$ into the differential equation and matching terms. Then $G^2 = M^2 + N^2$ and $M = G\cos\alpha$.

Present method Instead of working with $\cos\omega t$ and $\sin\omega t$, it is much cleaner to work with a **complex input** $\boldsymbol{V e^{i\omega t}}$. Then the output (the current) is *a multiple of* $Ve^{i\omega t}$. That multiple Y is a complex number. It tells us amplitudes and also phase shifts.

This is the right way to see the response of a one-loop RLC circuit. When the input frequency is ω, the output frequency is also ω.

Equation $$L\frac{dI}{dt} + RI + \frac{1}{C}\int I\,dt = \text{applied voltage} = Ve^{i\omega t} \tag{5}$$

Solution $$I(t) = \frac{Ve^{i\omega t}}{i\omega L + R + 1/i\omega C} = \frac{\text{input}}{\text{impedance}} \tag{6}$$

We will study that complex impedance in detail.

Future method Once we see the advantages of a complex $e^{i\omega t}$, we won't go back. What we are really doing is *to change a differential equation for y in the* **time domain** *into an algebraic equation for Y in the* **frequency domain**:

$$\textbf{Set } \boldsymbol{y = Ye^{i\omega t}} \qquad Ay'' + By' + Cy = e^{i\omega t} \text{ becomes } (i^2\omega^2 A + i\omega B + C)Y = 1.$$

Derivatives of $y(t)$ become multiplications by $i\omega$. We are talking here about the most important and useful simplification in applied mathematics. It requires constant coefficients A, B, C. This allows us to factor out $e^{i\omega t}$.

The **transfer function** $\boldsymbol{Y(s)}$ takes two more steps from derivatives to algebra. First, it changes $e^{i\omega t}$ to e^{st}. That exponent s can be pure imaginary ($s = i\omega$). It can also be any complex number ($s = a + i\omega$). We recover the freedom of Chapter 1, to allow growth or decay from $a > 0$ or $a < 0$. We are interested in *all* s and not just the special s_1 and s_2 that came from solving $As^2 + Bs + C = 0$.

The exponentials $e^{s_1 t}$ and $e^{s_2 t}$ went into the transient solution $y_n(t)$. Now we are working with the long-time solution $y_p(t)$ coming from an applied force Fe^{st}.

The second contribution of the transfer function is to give a name to the all-important multiplier in the system. It multiplies the input to give the output.

$$\textbf{The transfer function is } Y(s) = \frac{1}{As^2 + Bs + C}. \quad \textbf{The output is } \boldsymbol{Y(s)} \textbf{ times } \boldsymbol{e^{st}}.$$

Derivatives and integrals become multiplications and divisions (by s). One more name is needed. $Y(s)$ is the **Laplace transform** of the impulse response $g(t)$.

Input $f = \boldsymbol{\delta(t)}$	Output $y = g(t) =$ **impulse response**	Transform $\boldsymbol{Y(s)}$
Input $f =$ **step**	Output $y = r(t) =$ **step response**	Transform $\boldsymbol{Y(s)/s}$

The step function is the integral of the impulse $\delta(t)$. The step response is the integral of the impulse response $g(t)$. For their Laplace transforms, integration becomes division by s. *Calculus in the time domain becomes algebra in the frequency domain.*

The rules for the transforms of dy/dt and $\int y(t)\,dt$, and also a table of inverse Laplace transforms to recover $y(t)$ from $Y(s)$, will come in Section 2.7.

Complex Impedance

The present method uses $Ve^{i\omega t}$ for the alternating current input. The output divides that input by the impedance Z. This is like Ohm's Law $I = E/R$, but the resistance R changes to the impedance Z for this RLC loop:

$$\textbf{Current} \qquad I(t) = \frac{Ve^{i\omega t}}{i\omega L + R + 1/i\omega C} = \frac{Ve^{i\omega t}}{Z} = \frac{\textbf{input}}{\textbf{impedance}}. \tag{7}$$

The complex impedance Z depends on ω. The real part of Z is the resistance R. The imaginary part of Z is the "reactance" $\omega L - 1/\omega C$. From those rectangular coordinates $\operatorname{Re} Z$ and $\operatorname{Im} Z$, we know the polar form $|Z|e^{i\alpha}$ of this complex number:

$$\textbf{Magnitude} \qquad |Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \tag{8}$$

$$\textbf{Phase angle} \qquad \tan\alpha = \frac{\operatorname{Im} Z}{\operatorname{Re} Z} = \frac{\omega L - 1/\omega C}{R} \tag{9}$$

$$\textbf{Loop current} \qquad I(t) = \frac{Ve^{i\omega t}}{Z} = \frac{V}{|Z|}\, e^{i(\omega t - \alpha)} \tag{10}$$

The phase angle α tells us the time lag of the current behind the voltage.

Remember that R is the damping constant, like the coefficient B in $Ay'' + By' + Cy$. In the language of Section 2.4, we have *forced damped motion*. The damping keeps us away from exact resonance with the natural frequency of free undamped motion—which has $\omega L = 1/\omega C$ and $\omega = 1/\sqrt{LC}$. The magnitude $|Z|$ is smallest and $V/|Z|$ is largest at that natural frequency. We tune a radio to this ω to get a loud clear signal.

Example 1 Suppose the RLC circuit has resistance $R = 10$ ohms and inductance $L = 0.1$ henry and capacitance $C = 10^{-4}$ farad. The units of R and ωL and $1/\omega C$ must agree. Since frequency ω is measured in inverse seconds, all three units can be given in terms of $V =$ volts and $A =$ amps (for current) and seconds:

R	Ohm Ω	$= V/A$	$=$ 1 volt per amp
L	Henry H	$= V \cdot \sec/A$	$=$ 1 volt-second per amp
C	Farad F	$= A \cdot \sec/V$	$=$ 1 amp-second per volt

Example 2 Find the impedance Z, its magnitude $|Z|$, and the phase angle α for an RLC loop when the frequency is $\omega = 60$ cycles/second $= 60$ Hz $= 120\pi$ radians/second.

The impedance of this loop is $\quad Z = R + i\left(\omega L - \dfrac{1}{\omega C}\right) = |Z|e^{-i\alpha}.$

The magnitude of the impedance is $\quad |Z| = \ldots$

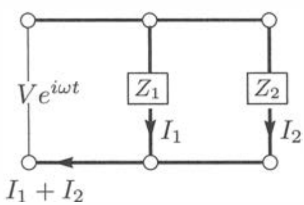
The phase angle producing time delay is $\quad \alpha = \ldots$

Example 3 To tune a radio to a station with frequency ω, what should be the capacitance C (which you adjust)? Suppose R and L are fixed and known.

Solution The goal of tuning is to achieve $\omega L = 1/\omega C$. Then the imaginary part of Z is zero: *inductance cancels capacitance*. Tuning achieves $Z = R$, that real part R is fixed.

$$\omega L = \frac{1}{\omega C} \qquad \omega^2 = \frac{1}{LC} \qquad C = \frac{1}{L\omega^2}$$

Example 4 Suppose the network contains **two RLC branches in parallel**. Find the total impedance Z_{12} from the impedances Z_1 and Z_2 of the two separate branches.



$$\frac{1}{Z_{12}} = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_1 + Z_2}{Z_1 Z_2}$$

$$I_{12} = I_1 + I_2 = \frac{Z_1 Z_2}{Z_1 + Z_2} V e^{i\omega t}$$

Loop Equations Versus Node Equations : KVL or KCL

Equation (2) expressed Kirchhoff's Voltage Law. **The sum of voltage drops around a closed loop is zero**. In principle, we could find a set of independent loops in any larger electrical network. Then the Voltage Law will give an equation like (2) around each of the independent loops. Those loop currents determine the currents on all the edges of the network and the voltages at all the nodes.

Most codes to solve problems on large networks *do not use the voltage law*! The preferred approach is **Kirchhoff's Current Law** : **The net current into each node is zero**. The balance equations of KCL say that "current in = current out" at every node.

Let me illustrate nodal analysis using the network in Figure 2.14. The unknowns are the voltages V_1 and V_2. The currents are easy to find once those voltages are known.

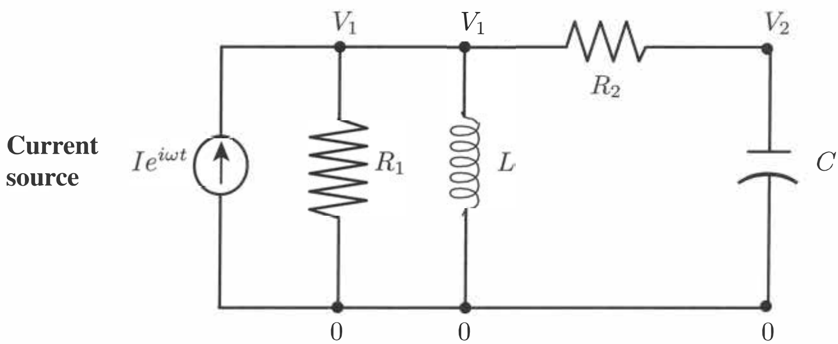


Figure 2.14: Four currents in and out of Node 1. *Node* 2 : Current in, current out.

A problem of this size can be solved symbolically or numerically :

Symbolically Work in the s-domain and find the transfer function. Since R_1 is in parallel with L, and R_2 is in series with C, we can find the currents on all the edges in terms of V_1 and V_2. Here is Kirchhoff's Current Law at those nodes :

$$\frac{V_1}{R_1} + \frac{V_1}{Ls} + \frac{V_1 - V_2}{R_2} = I \qquad \text{and} \qquad \frac{V_2 - V_1}{R_2} + sCV_2 = 0 \qquad (11)$$

Numerically Assign values to R_1, L, R_2, C and ω. Compute V_1 and V_2 from current balance at the nodes. Compute the currents from V_1/R_1 and $V_2/iL\omega$.

For a larger network, the algebra in the s-domain ($i\omega$ domain) becomes humanly impossible. A symbolic package could go further but in the end (and for nonlinear networks) the numerical approach will win. Widely known codes developed from the original SPICE code created at UC Berkeley. The SPICE codes use nodal analysis instead of loop analysis, for realistic networks.

Computational mechanics faced the same choice between nodal analysis and loop analysis. It reached the same conclusion. A complicated structure is broken up into *finite elements*—small pieces in which linear or quadratic approximation is adequate.

The choice is between displacements at nodes or stresses inside the elements, as the primary unknowns. The finite element community has made the same decision as the circuit simulation community: *Work with displacements* (and work with voltages) *at the nodes.*

A network produces a large system of equations—linear equations with simple RLC elements and nonlinear equations for circuit elements like transistors. **The nodes connected by the edges form a graph**. To organize the equations, you need the basic concepts of graph theory in Section 5.5:

An **incidence matrix** $\boldsymbol{A}$ tells which pairs of nodes are connected by which edges.

A **conductivity matrix** C expresses the physical properties along each edge.

Then the overall conductance matrix is $\boldsymbol{K} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}$. The system we solve, for linear problems in circuit simulation and in structural mechanics, has the matrix form $\boldsymbol{Ky} = \boldsymbol{f}$.

Chapter 4 will explain matrices and Section 5.5 will focus on the incidence matrix A of a graph. Those are necessary preparations for Kirchhoff's Current Law at all the nodes. Then Sections 7.4 and 7.5 create the stiffness matrix K (for mechanics) and the graph Laplacian matrix (for networks): basic ideas in applied mathematics.

Step Response

This book has emphasized the two fundamental problems for differential equations. One is the response to a delta function. The other is the response to a step function. For second order equations the impulse response $g(t)$ was computed in Section 2.3. This is our chance to find the step response, and we have to take it.

The two responses are closely related because the two inputs are related. The delta function is the derivative of the step function $H(t)$. The step function is the integral of the delta function. For constant coefficient equations, we can integrate every term. **The integral of the impulse response $g(t)$ is the step response $r(t)$.**

Impulse response $\boldsymbol{g(t)}$	$Ag'' + Bg' + Cg = C\delta(t)$	(12)
Step response $\boldsymbol{r(t)}$	$Ar'' + Br' + Cr = CH(t)$	(13)

We are following the "better notation" convention that includes the coefficient C on the right hand side. Its purpose is to give the output y or g or r the same units as the forcing term. Then the gain $G = |\text{output/input}|$ is dimensionless. For the step function with input $H(t) = 1$, **the steady state of the step response will be $\boldsymbol{r(\infty) = 1}$.**

I see two ways to compute that step response. One is to integrate the impulse response. The other is to solve equation (13) directly. The particular solution is $\boldsymbol{r_p(t)} = 1$. The null solution is a combination of $e^{s_1 t}$ and $e^{s_2 t}$, using the two roots of $As^2 + Bs + C = 0$.

To be safe, it seems reasonable to find $r(t)$ both ways.

Method 1 Integrate the impulse response $g(t) = \dfrac{C}{A}\dfrac{e^{s_1 t} - e^{s_2 t}}{s_1 - s_2}$ (14)

Method 2 Solve $Ar'' + Br' + Cr = C$ with $r(0) = r'(0) = 0$. (15)

Computing the Step Response

Method 2 is the normal way to solve differential equations. Substitute e^{st} to find s_1

Null solutions e^{st} $As^2 + Bs + C = 0$ has roots s_1 and s_2.

The complete solution to $Ar'' + Br' + Cr = C$ is *particular* + *null*:

$$r(t) = 1 + c_1 e^{s_1 t} + c_2 e^{s_2 t}. \tag{16}$$

The step response starts from $r(0) = 0$ and $r'(0) = 0$. A switch is turned on at $t = 0$, and the solution rises to $r(\infty) = 1$. The conditions at $t = 0$ determine c_1 and c_2:

$$r(0) = 1 + c_1 + c_2 = 0 \qquad r'(0) = c_1 s_1 + c_2 s_2 = 0. \tag{17}$$

Those coefficients are $c_1 = s_2/(s_1 - s_2)$ and $c_2 = -s_1/(s_1 - s_2)$. Then we know $r(t)$:

Step response $$r(t) = 1 + \frac{1}{s_1 - s_2}\left(s_2 e^{s_1 t} - s_1 e^{s_2 t}\right). \tag{18}$$

The same answer must come from integrating $g(t)$ in equation (14) from 0 to t. Remember that the roots of any quadratic multiply to give $s_1 s_2 = C/A$.

Step response = integral of $g(t)$ $$r(t) = \frac{s_1 s_2}{s_1 - s_2}\left[\frac{e^{s_1 t} - 1}{s_1} - \frac{e^{s_2 t} - 1}{s_2}\right]. \tag{19}$$

The coefficient of $e^{s_1 t}$ is the same $s_2/(s_1 - s_2)$ as in (18). Similarly for the coefficient of $e^{s_2 t}$. The constant term equals 1, so (18) and (19) are the same:

$$\frac{s_1 s_2}{s_1 - s_2}\left[-\frac{1}{s_1} + \frac{1}{s_2}\right] = \frac{s_1 s_2}{s_1 - s_2}\left[\frac{s_1 - s_2}{s_1 s_2}\right] = 1.$$

Better Notation

Our formula for the step response $r(t)$ can't stop with equation (18). Those roots s_1 and s_2 will depend on the physical parameters A, B, C. In mechanics these numbers are m, b, k. For a one-loop network the numbers are L, R, $1/C$. We need to express $r(t)$ with numbers we know, instead of s_1 and s_2.

Remember that *combinations* of A, B, C are especially useful. The simplest choices are $p = B/2A$ and ω_n^2 :

$$r'' + \frac{B}{A}r' + \frac{C}{A}r = \frac{C}{A} \quad \text{becomes} \quad \boldsymbol{r'' + 2pr' + \omega_n^2 r = \omega_n^2}. \tag{20}$$

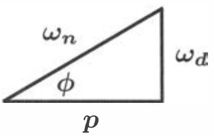
The same exponents s_1 and s_2 are now roots of $s^2 + 2ps + \omega_n^2 = 0$. Suppose $p < \omega_n$:

$$\textbf{Null solutions } \boldsymbol{e^{st}} \qquad \boldsymbol{s_1, s_2} = -p \pm \sqrt{p^2 - \omega_n^2} = \boldsymbol{-p \pm i\omega_d}. \tag{21}$$

Substituting for s_1 and s_2 in equation (18) gives a beautiful expression for $r(t)$:

$$\textbf{Step response} \qquad \boxed{\boldsymbol{r(t) = 1 - \frac{\omega_n}{\omega_d} e^{-pt} \sin(\omega_d t + \phi)}.} \tag{22}$$

That angle ϕ is in the right triangle that connects ω_n to p and ω_d :



$$\boldsymbol{\omega_d^2 + p^2 = \omega_n^2} \qquad \boldsymbol{\sin\phi = \frac{\omega_d}{\omega_n}} \qquad \boldsymbol{\cos\phi = \frac{p}{\omega_n}}$$

Now we check that $r(0) = 0$ and $r'(0) = 0$—then formula (22) must be correct:

$$r(0) = 1 - \frac{\omega_n}{\omega_d} \sin\phi = 0 \qquad r'(0) = \frac{\omega_n}{\omega_d}(p \sin\phi - \omega_d \cos\phi) = 0.$$

That final solution (22) combines $e^{-pt} \sin \omega_d t$ and $e^{-pt} \cos \omega_d t$. This null solution is a combination of $e^{s_1 t}$ and $e^{s_2 t}$ with $s = -p \pm i\omega_d$, as required. The particular solution is $r(\infty) = 1$. We see this steady state appear when the transients decay to zero with e^{-pt}. *The step response rises to* 1.

The number $p = B/2A$ can be replaced by ω_n times the damping ratio, if preferred.

Practical Resonance: Minimum D, Maximum Gain

The gain is $1/\sqrt{D}$. *If D is small then the gain is large.* That is how you tune a radio, by choosing the frequency ω_{res} that minimizes D and maximizes G. Then you can hear the signal. It is not perfect resonance—the gain does not become infinite—but it is resonance in practice.

$$\textbf{Practical resonance} \qquad \text{Minimize} \qquad D = \left(C - A\omega^2\right)^2 + (B\omega)^2$$

$$\textbf{Derivative of } \boldsymbol{D} \textbf{ is zero} \qquad -4A\omega(C - A\omega^2) + 2B^2\omega = 0.$$

When you cancel ω and solve $2B^2 = 4A(C - A\omega^2)$, that gives the frequency ω_{res} with largest gain. When $B = 0$ this is the natural frequency ω_n with infinite gain: $A\omega_n^2 = C$.

For $2Z^2 < 1$ there is practical resonance when $2B^2 = 4A(C - A\omega^2)$ at ω_{res} :

Largest gain $\quad \omega_{res}^2 = \dfrac{C}{A} - \dfrac{B^2}{2A^2} = \dfrac{C}{A}\left(1 - \dfrac{B^2}{2AC}\right) = \omega_n^2(1 - 2Z^2).$

■ REVIEW OF THE KEY IDEAS ■

1. L, R, C in $LI'' + RI' + \frac{1}{C}I = e^{i\omega t}$ are the inductance, resistance, capacitance.

2. For networks, node equations replace that loop equation: KCL instead of KVL.

3. The response to a step function rises from $r(0) = 0$ to a steady value $r(\infty) = 1$.

4. Practical resonance (the maximum gain) is at the frequency $\omega_{res} = \omega_n\sqrt{1 - 2\zeta^2}$.

Important note We computed the step response $r(t)$ in the time domain. Using the Laplace transform in Section 2.7, this computation can be moved to the s-domain. The transform of a unit step is $1/s$. *Derivatives in t become multiplications by s*:

The state equation $Ar'' + Br' + Cr = C$ transforms to $(As^2 + Bs + C)R(s) = \dfrac{C}{s}$.

The problem is to find the inverse Laplace transform $r(t)$ of this function $R(s)$. There are excellent control engineering textbooks that leave this as an exercise in partial fractions. The time domain (state space) solution in this section reached $r(t)$ successfully.

Problem Set 2.5

1 (Resistors in parallel) Two parallel resistors R_1 and R_2 connect a node at voltage V to a node at voltage zero. The currents are V/R_1 and V/R_2. What is the total current I between the nodes? Writing R_{12} for the ratio V/I, what is R_{12} in terms of R_1 and R_2?

2 (Inductor and capacitor in parallel) Those elements connect a node at voltage $Ve^{i\omega t}$ to a node at voltage zero (grounded node). The currents are $(V/i\omega L)e^{i\omega t}$ and $V(i\omega C)e^{i\omega t}$. The total current $Ie^{i\omega t}$ between the nodes is their sum. Writing Z_{12} for the ratio $Ve^{i\omega t}/Ie^{i\omega t}$, what is Z_{12} in terms of $i\omega L$ and $i\omega C$?

3 The impedance of an RLC loop is $Z = i\omega L + R + 1/i\omega C$. This impedance Z is real when $\omega =$ _____. This impedance is pure imaginary when _____. This impedance is zero when _____.

4 What is the impedance Z of an RLC loop when $R = L = C = 1$? Draw a graph that shows the magnitude $|Z|$ as a function of ω.

5 Why does an LC loop with no resistor produce a 90° phase shift between current and voltage? Current goes around the loop from a battery of voltage V in the loop.

6 The mechanical equivalent of zero resistance is zero damping: $my'' + ky = \cos\omega t$. Find c_1 and Y starting from $y(0) = 0$ and $y'(0) = 0$ with $\omega_n^2 = k/m$.

$$y(t) = c_1 \cos\omega_n t + Y \cos\omega t.$$

That answer can be written in two equivalent ways:

$$y = Y(\cos\omega t - \cos\omega_n t) = 2Y \sin\frac{(\omega_n - \omega)t}{2} \sin\frac{(\omega_n + \omega)t}{2}.$$

7 Suppose the driving frequency ω is close to ω_n in Problem 6. A fast oscillation $\sin[(\omega_n + \omega)t/2]$ is multiplying a very slow oscillation $2Y\sin[(\omega_n - \omega)t/2]$. By hand or by computer, draw the graph of $y = (\sin t)(\sin 9t)$ from 0 to 2π.

You should see a fast sine curve inside a slow sine curve. This is a **beat**.

8 What m, b, k, F equation for a mass-dashpot-spring-force corresponds to Kirchhoff's Voltage Law around a loop? What force balance equation on a mass corresponds to Kirchhoff's Current Law?

9 If you only know the natural frequency ω_n and the damping coefficient b for one mass and one spring, why is that *not enough* to find the damped frequency ω_d? If you know all of m, b, k what is ω_d?

10 Varying the number a in a first order equation $y' - ay = 1$ changes the *speed* of the response. Varying B and C in a second order equation $y'' + By' + Cy = 1$ changes the *form* of the response. Explain the difference.

11 Find the step response $r(t) = y_p + y_n$ for this overdamped system:

$$r'' + 2.5r' + r = 1 \;\text{ with }\; r(0) = 0 \;\text{ and }\; r'(0) = 0.$$

12 Find the step response $r(t) = y_p + y_n$ for this critically damped system. The double root $s = -1$ produces what form for the null solution?

$$r'' + 2r' + r = 1 \;\text{ with }\; r(0) = 0 \;\text{ and }\; r'(0) = 0.$$

13 Find the step response $r(t)$ for this underdamped system using equation (22):

$$r'' + r' + r = 1 \;\text{ with }\; r(0) = 0 \;\text{ and }\; r'(0) = 0.$$

14 Find the step response $r(t)$ for this undamped system and compare with (22):

$$r'' + r = 1 \;\text{ with }\; r(0) = 0 \;\text{ and }\; r'(0) = 0.$$

15 For $b^2 < 4mk$ (underdamping), what parameter decides the speed at which the step response $r(t)$ rises to $r(\infty) = 1$? Show that the **peak time** is $T = \pi/\omega_d$ when $r(t)$ reaches its maximum before settling back to $r = 1$. At peak time $r'(T) = 0$.

16 If the voltage source $V(t)$ in an RLC loop is a unit step function, what resistance R will produce an overshoot to $r_{\text{max}} = 1.2$ if $C = 10^{-6}$ Farads and $L = 1$ Henry? (Problem 15 found the peak time T when $r(T) = r_{\text{max}}$).

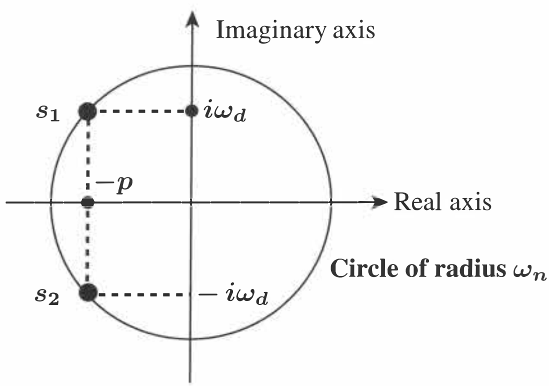
Sketch two graphs of $r(t)$ for $p_1 < p_2$. Sketch two graphs as ω_d increases.

17 What values of m, b, k will give the step response $r(t) = 1 - \sqrt{2}e^{-t}\sin(t + \frac{\pi}{4})$?

18 What happens to the $p - \omega_d - \omega_n$ right triangle as the damping ratio ω_n/p increases to 1 (critical damping)? At that point the damped frequency ω_d becomes _____. The step response becomes $r(t) =$ _____.

19 **The roots $s_1, s_2 = -p \pm i\omega_d$ are poles of the transfer function $1/(As^2+Bs+C)$**

Show directly that the product of the roots $s_1 = -p + i\omega_d$ and $s_2 = -p - i\omega_d$ is $s_1 s_2 = \omega_n^2$. The sum of the roots is $-2p$. The quadratic equation with those roots is $s^2 + 2ps + \omega_n^2 = 0$.



20 Suppose p is increased while ω_n is held constant. How do the roots s_1 and s_2 move?

21 Suppose the mass m is increased while the coefficients b and k are unchanged. What happens to the roots s_1 and s_2?

22 **Ramp response** How could you find $y(t)$ when $F = t$ is a ramp function?

$$y'' + 2py' + \omega_n^2 y = \omega_n^2 t \quad \text{starting from} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

A particular solution (straight line) is $y_p =$ _____. The null solution still has the form $y_n =$ _____. Find the coefficients c_1 and c_2 in the null solution from the two conditions at $t = 0$.

This ramp response $y(t)$ can also be seen as the integral of _____.

2.6 Solutions to Second Order Equations

Up to now, all forcing terms $f(t)$ for second order equations have been e^{st} or $\cos \omega t$. How can you find a particular solution when $f(t)$ is not a sinusoid or exponential? This section gives one answer for constant coefficients A, B, C and then a general answer $\boldsymbol{VP}$:

$\boldsymbol{UC}$ If $f(t)$ is a polynomial in t, then $y_p(t)$ is also a polynomial in t.

$\boldsymbol{VP}$ Suppose we know the null solutions $y_n = c_1 y_1(t) + c_2 y_2(t)$. Then a particular solution has the form $y_p = c_1(t) y_1(t) + c_2(t) y_2(t)$.

Those methods are called "**undetermined coefficients**" and "**variation of parameters**".

The special method is simple to execute (you will like it). When $f(t)$ is a quadratic, then one solution is also a quadratic: $y_p(t) = at^2 + bt + c$. Those numbers a, b, c are the **undetermined coefficients**. The differential equation will determine them. This succeeds for any constant coefficient differential equation—always limited to special $f(t)$.

That method $\boldsymbol{UC}$ can be pushed further. If $f(t)$ is a polynomial times an exponential, then $y_p(t)$ has the same form. The highest power of t allowed in y_p is the same as in f. Those polynomials normally have the same degree.

Only in the case of resonance must we allow an extra factor t in the solution. This is like the exponential response to $f(t) = e^{ct}$ in Section 2.4. That presented a perfect example of an undetermined coefficient Y in $y_p(t) = Ye^{st}$. The coefficient $Y = 1/(As^2 + Bs + C)$ was determined by the equation. This is $Y = 1/P(s)$ for all equations $P(D)y = e^{st}$. With resonance we move to $y_p = te^{st}/P'(s)$.

Variation of parameters is a more powerful method. It applies to all $f(t)$. It even applies when the equation $A(t)y'' + B(t)y' + C(t)y = f(t)$ has variable coefficients. But it starts with a big assumption: **We have to know the null solutions** $y_1(t)$ **and** $y_2(t)$.

The method will succeed completely when the coefficients A, B, C are constant. This important case gives formula (17). Variation of parameters also succeeded in Chapter 1, for first order equations $y' - a(t)y = q(t)$. In that case we could solve the null equation $y' = a(t)y$. For second order equations with variable coefficients, like Airy's equation $y'' = ty$, the null equation is a difficult obstacle.

I guess we have to realize that not all problems lead to simple formulas.

The Method of Undetermined Coefficients

This direct approach finds a particular solution y_p, when the forcing term $f(t)$ has a special form. I can explain the method of undetermined coefficients by four examples.

Example 1 $\boldsymbol{y'' + y = t^2}$ **has a solution of the form** $\boldsymbol{y = at^2 + bt + c}$.

The reason for this choice of y is that y' and y'' will have a similar form. They will also be combinations of t^2 and t and 1. *All the terms in* $y'' + y = t^2$ *will have this special form.*

Choose the numbers a, b, c to satisfy that equation:

$$\boldsymbol{y'' + y = (at^2 + bt + c)'' + (at^2 + bt + c) = t^2.} \qquad (1)$$

Key idea: **We can separately match the coefficients of t^2 and t and 1 in equation (1)**:

$$(\boldsymbol{t^2}) \quad a = 1 \qquad (\boldsymbol{t}) \quad b = 0 \qquad (\boldsymbol{1}) \quad 2a + c = 0 \qquad (2)$$

Then $c = -2a = -2$ and the answer is $y = at^2 + c = \boldsymbol{t^2 - 2}$. This solves $y'' + y = t^2$.

Example 2 Find the complete solution to $\boldsymbol{y'' + 4y' + 3y = e^{-t} + t}$.

Answer First find the null solution to $y_n'' + 4y_n' + 3y_n = 0$, by substituting $y_n = e^{st}$:

$$(s^2 + 4s + 3)e^{st} = 0 \;\text{ leads to }\; s^2 + 4s + 3 = (s+1)(s+3) = 0.$$

The roots are $s_1 = -1$ and $s_2 = -3$. The null solutions are $y_n = c_1e^{-t} + c_2e^{-3t}$.

Now find one particular solution. With $f = e^{-t} + t$, the usual form with undetermined coefficients would be $y_p = ae^{-t} + bt + c$ (notice c in the polynomial). *But e^{-t} is a null solution.* Therefore the assumed form for y needs an extra factor t multiplying e^{-t}. **Substitute $\boldsymbol{y = ate^{-t} + bt + c}$** into the differential equation, so $y' = ae^{-t} - ate^{-t} + b$:

$$y'' + 4y' + 3y = (-2ae^{-t} + ate^{-t}) + 4(ae^{-t} - ate^{-t} + b) + 3(ate^{-t} + bt + c) = e^{-t} + t.$$

The coefficients of te^{-t} are $a - 4a + 3a = 0$. No problem with this te^{-t} term. We must balance the coefficients of e^{-t} and t and 1:

$$\textbf{Find } \boldsymbol{a, b, c} \qquad \boldsymbol{-2a + 4a = 1} \qquad \boldsymbol{3b = 1} \qquad \boldsymbol{4b + 3c = 0}$$

Then $a = \frac{1}{2}$ and $b = \frac{1}{3}$ and $c = -\frac{4}{9}$ produce the particular $y_p = \boldsymbol{\frac{1}{2}\,te^{-t} + \frac{1}{3}\,t - \frac{4}{9}}$. The null solution is $c_1e^{-t} + c_2e^{-3t}$. The complete solution is always $y = y_p + y_n$.

The method only applies to very special forcing functions, but when it succeeds it is as fast and simple as possible. Let me list special inputs $f(t)$ and the form of a solution $y(t)$ when the differential equation $Ay'' + By + Cy = f(t)$ has **constant coefficients**.

1.	$\boldsymbol{f(t)} =$ **polynomial in** $\boldsymbol{t}$	$\boldsymbol{y(t)} =$ **polynomial in** $\boldsymbol{t}$ **(same degree)**
2.	$\boldsymbol{f(t) = A\cos\omega t + B\sin\omega t}$	$\boldsymbol{y(t) = M\cos\omega t + N\sin\omega t}$
3.	$\boldsymbol{f(t)} =$ **exponential** $\boldsymbol{e^{st}}$	$\boldsymbol{y(t) = Ye^{st}}$
4.	$\boldsymbol{f(t)} =$ **product** $\boldsymbol{t^2\,e^{st}}$	$\boldsymbol{y(t) = (at^2 + bt + c)\,e^{st}}$

$t^2\,e^{st}$ is included in **4** by multiplying possibilities **1** and **3**. The good form for $y(t)$ multiplies the solutions to **1** and **3**. The coefficients M, N, Y, a, b, c are "undetermined" until you substitute $y(t)$ into the differential equation. *That equation determines a, b, c.*

Note to professors It seems to me that a polynomial times e^{t^2} shares the key property. Its derivatives have the same form. But their polynomial degree goes up. Not good.

Example 3 Find a particular solution to $\boldsymbol{y'' + y = t\, e^{st}}$ = **polynomial times** $\boldsymbol{e^{st}}$.
The good form to assume for $y(t)$ is $(at+b)\,e^{st}$. *Please notice that* $b\,e^{st}$ *is included.* Even though f doesn't have e^{st} by itself, that will appear in the derivatives of $t\,e^{st}$. To be sure we capture every derivative, $at+b$ must include that constant b.
I need to find the second derivative of the undetermined $\boldsymbol{y(t) = (at+b)\,e^{st}}$.

$$\boldsymbol{y'} = s(at+b)\,e^{st} + a\,e^{st} \qquad \boldsymbol{y''} = s^2(at+b)\,e^{st} + 2as\,e^{st}.$$

Substitute y and y'' into the equation $y'' + y = t\,e^{st}$ and match terms to find a and b:

$$\begin{array}{lrcl} \text{Coefficient of } t\,e^{st} & as^2 + a & = & 1 \\ \text{Coefficient of } e^{st} & bs^2 + 2as + b & = & 0 \end{array}$$

Those two equations produce $\quad a = \dfrac{1}{1+s^2} \quad$ and $\quad b = \dfrac{-2as}{1+s^2} = \dfrac{-2s}{(1+s^2)^2}.$ (3)

Now $y(t) = (at+b)\,e^{st}$ is a particular solution of $y'' + y = t\,e^{st}$.

Possible difficulty of the method **Suppose** $\boldsymbol{s = i}$ **or** $\boldsymbol{-i}$ **in the forcing term** $\boldsymbol{f = t\,e^{st}}$

Those exponents $s = i$ and $s = -i$ have $1 + s^2 = 0$. Our answer in (3) for a and b is dividing by zero. The result is useless. What went wrong ?

Explanation If $s = i$, the assumed form $y = (at+b)e^{it}$ includes a solution be^{it} of $y'' + y = 0$. We have accidentally included a null solution $y_n = be^{it}$. There is no hope of determining b. That coefficient is truly undetermined and it stays that way.

We are seeing a problem of resonance, when the hoped-for y_p is already a part of y_n. The result in Section 2.4 was that *resonant solutions have and need an extra factor* t. The same is true here. When $s = i$ or $s = -i$, the good form to assume is $y_p = \boldsymbol{t}(at+b)\,e^{st}$.

When you substitute this y_p into $y'' + y = t\,e^{st}$, the coefficients a and b will be properly determined. If $s = i$, could you verify that $a = -1/4$ and $b = i/4$?

Example 4 Let me apply "undetermined coefficients" to an equation you already know :

$$Ay'' + By' + Cy = \cos \omega t. \tag{4}$$

Solution by undetermined coefficients Look for $y(t) = M\cos\omega t + N\sin\omega t$. Those coefficients M and N are also in equation (21) of Section 2.4.

$$M = \frac{C - A\,\omega^2}{D} \qquad N = \frac{B\,\omega}{D} \qquad D = (C - A\omega^2)^2 + B^2\omega^2.$$

Is this perfect ? Not quite. In case the denominator is $D = 0$, the method will fail. That is exactly the case of resonance, when $A\,\omega^2 = C$ and $B = 0$. The coefficients M and N become $0/0$. The equation becomes $A\,(y'' + \omega^2\,y) = \cos\omega t$. The particular y_p cannot be $M\cos\omega t + N\sin\omega t$ ***because*** $\boldsymbol{\cos\omega t}$ ***and*** $\boldsymbol{\sin\omega t}$ ***are null solutions*** $\boldsymbol{y_n}$. They have $y'' + \omega^2\,y = 0$. The same ω is on both sides of the equation.

Resonant solutions In case $D = 0$, the particular solution again has an extra factor t.

Then put $y_p = Mt\cos\omega t + Nt\sin\omega t$ into equation (4) to find $M = 0$ and $N = 1/2$.

Summary of the Method of Undetermined Coefficients

When the forcing term $f(t)$ is a polynomial or a sinusoid or an exponential, look for a particular solution $y_p(t)$ of the same form. Derivatives of polynomials are polynomials, derivatives of sinusoids are sinusoids, derivatives of exponentials are exponentials. Then all terms in $Ay'' + By' + Cy = f$ will share the same form.

When $f(t) =$ sum of exponentials, look for $y(t) =$ sum of exponentials. When f is a polynomial times a sinusoid or an exponential, $y(t)$ has the same form. When a sinusoid or an exponential in f happens to be a null solution (*resonance*), include an extra t in y_p.

Question What form would you assume for $y(t)$ when $f(t) = 4e^t + 5\cos 2t + t$?

Answer Look for $y(t) = Ye^t + M\cos 2t + N\sin 2t + at + b$. The coefficients in the differential equation need to be constants. Then Ay'', By', Cy and f all look like y.

Variation of Parameters

Now we want to allow any forcing function $f(t)$. The equation might even have variable coefficients. If we know the null solutions, the method called "variation of parameters" can find a particular solution.

Suppose the null solution with $f = 0$ is $y_n(t) = c_1y_1(t) + c_2y_2(t)$. We know y_1 and y_2. *For a particular solution when* $f(t) \neq 0$, *allow* c_1 *and* c_2 *to vary with time*:

Variation of parameters $$\boxed{\boldsymbol{y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)}} \quad (5)$$

This idea applies to any second order linear differential equation like

$$\frac{d^2y}{dt^2} + B(t)\frac{dy}{dt} + C(t)y = f(t). \quad (6)$$

Substituting $y_p(t)$ from (5) gives a first equation for $c_1{}'$ and $c_2{}'$. Those are the parameters varying with t. To recognize a convenient second equation for $c_1{}'$ and $c_2{}'$, compute the derivative of y_p by the product rule :

$$y_p{}' = (c_1(t)y_1{}' + c_2(t)y_2{}') + (c_1{}'(t)y_1 + c_2{}'(t)y_2). \quad (7)$$

A good choice is to require that the second sum be zero :

Second equation for $c_1{}', c_2{}'$ $$\boldsymbol{c_1{}'(t)y_1(t) + c_2{}'(t)y_2(t) = 0.} \quad (8)$$

Now the second sum in (7) drops out and we compute $y_p{}''$ (product rule again) :

$$y_p{}'' = (c_1(t)y_1{}'' + c_2(t)y_2{}'') + (c_1{}'(t)y_1{}' + c_2{}'(t)y_2{}'). \quad (9)$$

Put y_p, $y_p{}'$, $y_p{}''$ from (5), (7), (9) into the differential equation to get a wonderful result :

First equation for $c_1{}', c_2{}'$ $$\boldsymbol{c_1{}'(t)y_1{}'(t) + c_2{}'(t)y_2{}'(t) = f(t).} \quad (10)$$

That became simple because the null solutions y_1 and y_2 satisfy $y'' + B(t)y' + C(t)y = 0$.

We now have two equations (8) and (10) for two unknowns $c_1{}'(t)$ and $c_2{}'(t)$. At each time t, the four coefficients P, Q, R, S in the two equations are the numbers $y_1(t)$, $y_2(t)$, $y_1{}'(t)$, $y_2{}'(t)$. Solve those two equations, first using P, Q, R, S:

$$\begin{aligned} Pc_1{}' + Qc_2{}' &= 0 \\ Rc_1{}' + Sc_2{}' &= f \end{aligned} \quad \text{lead to} \quad c_1{}' = \frac{-Qf}{PS-QR} \quad \text{and} \quad c_2{}' = \frac{Pf}{PS-QR}. \tag{11}$$

When you multiply those fractions by P and Q, they cancel. When you multiply the fractions by R and S and add, the result is the second equation $Rc_1{}' + Sc_2{}' = f(t)$.

Linear equations come at the beginning of linear algebra in Chapter 4. Here we have a separate problem for each time t, and the solution (11) becomes (12) when P, Q, R, S are $y_1(t)$, $y_2(t)$, $y_1{}'(t)$, $y_2{}'(t)$. I will write W for $PS - QR$:

$$\boldsymbol{c_1{}'(t) = \frac{-y_2(t)f(t)}{W(t)} \qquad c_2{}'(t) = \frac{y_1(t)f(t)}{W(t)} \qquad W(t) = y_1y_2{}' - y_2y_1{}'} \tag{12}$$

This denominator $W(t)$ is the **Wronskian** of the two null solutions $y_1(t)$ and $y_2(t)$. It was introduced in Section 2.1, and the independence of $y_1(t)$ and $y_2(t)$ guarantees that $W(t) \neq 0$. The divisions by $W(t)$ in (12) are safe. **The varying parameters $\boldsymbol{c_1(t)}$ and $\boldsymbol{c_2(t)}$ are the integrals of $\boldsymbol{c_1{}'(t)}$ and $\boldsymbol{c_2{}'(t)}$ in** (12).

We have found a particular solution $c_1y_1 + c_2y_2$ to the differential equation (6):

If y_1 and y_2 are independent null solutions to $y'' + B(t)y' + C(t)y = 0$, then a particular solution $y_p(t)$ with right side $f(t)$ is $c_1(t)y_1(t) + c_2(t)y_2(t)$:

Variation of Parameters
$$\boldsymbol{y_p(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}\,dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}\,dt.} \tag{13}$$

Example 5 **Variation of parameters: Find a particular solution for $\boldsymbol{y'' + y = t}$.**

The right side $f(t) = t$ is not a sinusoid. No problem to find the independent solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ to the null equation $y'' + y = 0$. The Wronskian is 1:

$$W(t) = y_1y_2{}' - y_2y_1{}' = \cos^2 t + \sin^2 t = 1 \quad \text{(never zero as predicted).}$$

The particular solution $y_p(t) = c_1(t)\ \cos t + c_2(t)\ \sin t$ needs integrals of $c_1{}'$ and $c_2{}'$:

$$c_1(t) = \int \frac{(-\sin t)t\,dt}{1} = \boldsymbol{t\cos t - \sin t} \qquad c_2(t) = \int \frac{(\cos t)t\,dt}{1} = \boldsymbol{t\sin t + \cos t}.$$

Variation of parameters has found a particular solution $c_1y_1 + c_2y_2$, and it simplifies:

$$y_p = \boldsymbol{(t\ \cos\ t - \sin\ t)}\ \cos\ t + \boldsymbol{(t\ \sin\ t + \cos\ t)}\ \sin\ t = \boldsymbol{t}. \tag{14}$$

Apologies! We could have seen by ourselves that $y = t$ solves $y'' + y = t$. And the method of undetermined coefficients would find $y = t$ much faster: no integrations.

Example 6 Solve $y'' + y = \delta(t)$ by variation of parameters. The null solutions $\cos t$ and $\sin t$ still give $W(t) = 1$. The delta function f goes into the integrals for c_1 and c_2:

$$c_1 = \int \frac{(\sin t)\,\delta(t)\,dt}{1} = \sin 0 = \mathbf{0} \qquad c_2 = \int \frac{(\cos t)\,\delta(t)\,dt}{1} = \cos 0 = \mathbf{1}$$

Then $y_p(t) = (1)y_2(t) = \textbf{sin } \boldsymbol{t}$. With $f = \delta(t)$, this is the fundamental solution $g(t)$ (the impulse response). Then $\sin t$ is also the solution to $y''+y=0$ that starts from $y(0)=0$ and $y'(0)=1$. We will find this growth factor again in (17) with $s_1 = -s_2 = i$.

Constant Coefficients and the Solution Formula

The one time we are sure to know the null solutions y_1 and y_2 is when the differential equation has constant coefficients. Substituting $y = e^{st}$ into $Ay'' + By' + Cy = 0$ leads to $As^2 + Bs + C = 0$. The roots are s_1 and s_2. The null solutions are $e^{s_1 t}$ and $e^{s_2 t}$. Notice that we are free to assume that $\boldsymbol{A = 1}$. (If not, divide the equation by A.)

Variation of parameters gives the solution (13). All we need is the Wronskian $W(t)$, and for these null solutions it is beautiful:

$$\boldsymbol{W(t)} = y_1 y_2{}' - y_2 y_1{}' = (e^{s_1 t})(s_2 e^{s_2 t}) - (e^{s_2 t})(s_1 e^{s_1 t}) = \boldsymbol{(s_2 - s_1) e^{s_1 t} e^{s_2 t}}. \quad (15)$$

Immediately we know that $W(t) \neq 0$ unless $s_1 = s_2$. With equal roots we expect to need the special null solution $y_2 = te^{st}$. Even in that case the Wronskian looks terrific:

$$\boldsymbol{W(t)} = (e^{st})(te^{st})' - (te^{st})(e^{st})' = (e^{st})(ste^{st} + e^{st}) - (te^{st})(se^{st}) = \boldsymbol{e^{2st}}. \quad (16)$$

When you substitute y_1 and y_2 and W into (13), that "VP formula" produces $y_p(t)$.

Unequal roots $\boldsymbol{s_1 \neq s_2}$. The first integral has $y_2/W = e^{-s_1 t}/(s_2 - s_1)$. The second integral has $y_1/W = e^{-s_2 t}/(s_2 - s_1)$. Put those into (13):

Particular solution
Constant coefficients

$$y_p(t) = \frac{-e^{s_1 t}}{s_2 - s_1} \int_0^t e^{-s_1 T} f(T)\,dT + \frac{e^{s_2 t}}{s_2 - s_1} \int_0^t e^{-s_2 T} f(T)\,dT$$

To me, a growth factor $g(t-T)$ is multiplying the inputs $f(T)$. *The integrals just sum up the outputs.* Here is the same formula for $y_p(t)$ written so it uses $g(t)$:

$$\textbf{Growth factor} \quad \boldsymbol{g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{s_1 - s_2}} \qquad \textbf{Solution} \quad \boldsymbol{y_p(t) = \int_0^t g(t-T) f(T)\,dT} \quad (17)$$

That might be the nicest formula in the book. Probably I am writing those words because I didn't see this formula coming. Section 2.3 discovered the same response $g(t)$!

Forgive me for that personal note. I will go on to the other case, with $s_1 = s_2$.

Equal roots $s_1 = s_2 = s$ with $W = e^{2st}$. The first integral in (13) still has $y_1 = e^{st}$ and now $y_2/W = te^{-st}$. The second integral has $y_2 = te^{st}$ and $y_1/W = e^{-st}$:

Particular solution y_p
Null solutions e^{st}, te^{st}

$$y_p(t) = -e^{st}\int_0^t Te^{-sT}f(T)dT + te^{st}\int_0^t e^{-sT}f(T)dT.$$

This also has a perfect form when you identify the factor $g(t-T)$ that is multiplying f:

$$\textbf{Growth factor}\quad \boxed{g(t) = te^{st}} \qquad \textbf{Solution}\quad \boxed{y_p(t) = \int_0^t g(t-T)f(T)dT} \tag{18}$$

Formulas that good never happen by accident, $g(t)$ must mean something important:

The growth factor $g(t)$ is the impulse response: **$y_p(t)$ is $g(t)$ when $f(t)$ is $\delta(t)$.**

Let me close Section 2.6 on that high note. Then Section 2.7 will take the Laplace transform of the growth factors $g(t)$ to get the **transfer function $Y(s)$**:

$$\text{The transform of } g(t) = \frac{e^{s_1t} - e^{s_2t}}{s_1 - s_2} \text{ is } \frac{1}{(s-s_1)(s-s_2)} = \frac{1}{s^2 + Bs + C} = Y(s).$$

$$\text{The transform of } g(t) = te^{s_1t} \text{ is } \frac{1}{(s-s_1)^2} = \frac{1}{s^2 + Bs + C} \text{ when } s_1 = s_2.$$

$Y(s)$ comes from B and C. **The solution $y(t)$ comes from $g(t)$ = "Green's function."** The last pages of the book will see the integral of $g(t-T)f(T)$ as a convolution.

■ REVIEW OF THE KEY IDEAS ■

1. **Undetermined coefficients** in y_p apply when $f(t)$ has only e^{st}, $\cos\omega t$, $\sin\omega t$, t^n.
2. Set y_p = exponential/sinusoid/polynomial. Find coefficients $a, b, \ldots$ to match $f(t)$.
3. **Variation of parameters**: c_1 and c_2 vary with t in $y_p = c_1(t)\,y_1(t) + c_2(t)\,y_2(t)$.
4. Two equations for $c_1{}'$ and $c_2{}'$ lead to c_1 and c_2 = integrals of $-y_2\,f/W$ and $y_1\,f/W$.
5. For constant coefficients c_1 and c_2 those are integrals of $e^{-s_1t}f(t)$ and $e^{-s_2t}f(t)$.
6. Then $y_p = \int g(t-s)f(s)ds$ when $g(t)$ = response to the impulse $f = \delta(t)$.

Problem Set 2.6

Find a particular solution by inspection (or the method of undetermined coefficients)

1 (a) $y'' + y = 4$ (b) $y'' + y' = 4$ (c) $y'' = 4$

2 (a) $y'' + y' + y = e^t$ (b) $y'' + y' + y = e^{ct}$

3 (a) $y'' - y = \cos t$ (b) $y'' + y = \cos 2t$ (c) $y'' + y = t + e^t$

4 For these $f(t)$, predict the form of $y(t)$ with undetermined coefficients:

(a) $f(t) = t^3$ (b) $f(t) = \cos 2t$ (c) $f(t) = t \cos t$

5 Predict the form for $y(t)$ when the right hand side is

(a) $f(t) = e^{ct}$ (b) $f(t) = te^{ct}$ (c) $f(t) = e^t \cos t$

6 For $f(t) = e^{ct}$ when is the prediction for $y(t)$ different from Ye^{ct} ?

Use the method of undetermined coefficients to find a solution $y_p(t)$.

7 (a) $y'' + 9y = e^{2t}$ (b) $y'' + 9y = te^{2t}$

8 (a) $y'' + y' = t + 1$ (b) $y'' + y' = t^2 + 1$

9 (a) $y'' + 3y = \cos t$ (b) $y'' + 3y = t \cos t$

10 (a) $y'' + y' + y = t^2$ (b) $y'' + y' + y = t^3$

11 (a) $y'' + y' + y = \cos t$ (b) $y'' + y' + y = t \sin t$

Problems 12–14 involve resonance. Multiply the usual form of y_p by t.

12 (a) $y'' + y = e^{it}$ (b) $y'' + y = \cos t$

13 (a) $y'' - 4y' + 3y = e^t$ (b) $y'' - 4y' + 3y = e^{3t}$

14 (a) $y' - y = e^t$ (b) $y' - y = te^t$ (c) $y' - y = e^t \cos t$

15 For $y'' + 4y = e^t \sin t$ (exponential times sinusoidal) we have two choices:

1 (Real) Substitute $y_p = Me^t \cos t + Ne^t \sin t$: determine M and N
2 (Complex) Solve $z'' + 4z = e^{(1+i)t}$. Then y is the imaginary part of z.

Use both methods to find the same $y(t)$—which do you prefer?

16 (a) Which values of c give resonance for $y'' + 3y' - 4y = te^{ct}$?

(b) What form would you substitute for $y(t)$ if there is no resonance?

(c) What form would you use when c produces resonance?

17 This is the rule for equations $P(D)y = e^{ct}$ with resonance $P(c) = 0$:

If $P(c) = 0$ and $P'(c) \neq 0$, look for a solution $y_p = Cte^{ct}$ $(m = 1)$
If c is a root of multiplicity m, then y_p has the form ______.

18 (a) To solve $d^4y/dt^4 - y = t^3e^{5t}$, what form do you expect for $y(t)$?

(b) If the right side becomes $t^3 \cos 5t$, which 8 coefficients are to be determined?

19 For $y' - ay = f(t)$, the method of undetermined coefficients is looking for all $f(t)$ so that the usual formula $y_p = e^{at}\int e^{-as}f(s)ds$ is easy to integrate. Find these integrals for the "nice functions" $f = e^{ct}$, $f = e^{i\omega t}$, and $f = t$:

$$\int e^{-as}e^{cs}ds \qquad \int e^{-as}e^{i\omega s}ds \qquad \int e^{-as}sds$$

Problems 20–27 develop the method of variation of parameters.

20 Find two solutions y_1, y_2 to $y'' + 3y' + 2y = 0$. Use those in formula (13) to solve

(a) $y'' + 3y' + 2y = e^t$ (b) $y'' + 3y' + 2y = e^{-t}$

21 Find two solutions to $y'' + 4y' = 0$ and use variation of parameters for

(a) $y'' + 4y' = e^{2t}$ (b) $y'' + 4y' = e^{-4t}$

22 Find an equation $y'' + By' + Cy = 0$ that is solved by $y_1 = e^t$ and $y_2 = te^t$. If the right side is $f(t) = 1$, what solution comes from the VP formula (13)?

23 $y'' - 5y' + 6y = 0$ is solved by $y_1 = e^{2t}$ and $y_2 = e^{3t}$, because $s = 2$ and $s = 3$ come from $s^2 - 5s + 6 = 0$. Now solve $y'' - 5y' + 6y = 12$ in two ways:

1. Undetermined coefficients (or inspection) **2.** Variation of parameters using (13)

The answers are different. Are the initial conditions different?

24 What are the initial conditions $y(0)$ and $y'(0)$ for the solution (13) coming from variation of parameters, starting from any y_1 and y_2?

25 The equation $y'' = 0$ is solved by $y_1 = 1$ and $y_2 = t$. Use variation of parameters to solve $y'' = t$ and also $y'' = t^2$.

26 Solve $y_s{}'' + y_s = 1$ for the step response using variation of parameters, starting from the null solutions $y_1 = \cos t$ and $y_2 = \sin t$.

27 Solve $y_s{}'' + 3y_s{}' + 2y_s = 1$ for the step response starting from the null solutions $y_1 = e^{-t}$ and $y_2 = e^{-2t}$.

28 Solve $A\boldsymbol{y}'' + Cy = \cos \omega t$ when $A\omega^2 = C$ (the case of resonance). Example 4 suggests to substitute $y = Mt\cos\omega t + Nt\sin\omega t$. Find M and N.

29 Put $g(t)$ into the great formulas (17)-(18) to see the equations above them.

2.7 Laplace Transforms $Y(s)$ and $F(s)$

If you think about the functions that have dominated this book, the list is not very long. They are the right hand sides of linear differential equations and also the solutions $y(t)$:

1. Exponentials e^{at}
2. Sinusoids $\cos \omega t$ and $\sin \omega t$
3. Polynomials starting with 1 and t and t^2
4. Step functions $H(t-T)$
5. Delta functions $\delta(t-T)$
6. Products of 1 to 5

Why are these functions special? I believe this is an important question.

The answer that strikes me first is something I had not thought about:

The derivatives and integrals of these functions are also on the list (*almost*).

That was true from the very start of Chapter 1. Example 1 on page 1 was $y = e^t$. Its fundamental property is $dy/dt = y$. The derivative leaves it unchanged, which puts it on the list. And the product of two exponentials is another exponential. In fact exponentials could be a short list by themselves.

Cosines and sines were written separately, but those are combinations of $e^{i\omega t}$ and $e^{-i\omega t}$. They just move us to complex numbers. The constant polynomial is $e^{0t} = 1$. Integrals and derivatives of polynomials are polynomials. The product rule for derivatives (and the reverse rule which is integration by parts) keep the list self-contained: no new functions.

There is one flaw but it is easily fixed. The delta function $\delta(t)$ is the derivative of the step function $H(t)$, but we need all derivatives and integrals. Include them on the list! Solving $dy/dt =$ step function gives $\boldsymbol{y(t)} =$ *ramp function*. This is zero for $t \le 0$, and $y(t) = t$ for $t \ge 0$. Its graph has a corner and its slope has a jump. The integral of that linear ramp is a *parabolic ramp*. The next integral leads toward a *cubic spline*. The derivative of a delta function is a very singular object (see Problem 25).

In the end, all these ideal functions can go on the list which is now complete.

The Algebra of Differential Equations

With those special functions, solving a constant coefficient linear differential equation is not so difficult. It reduces to an algebra problem. The null solution y_n is a combination of exponentials (possibly times powers of t). The particular solution y_p has a known form like $Ye^{i\omega t}$—the differential equation will decide the undetermined coefficient Y. For functions $\mathbf{1}$ to $\mathbf{6}$, the integrals using variation of parameters are already on the list.

The Laplace transform gives a systematic way to do the algebra. *Functions of t become functions of s.* Instead of derivatives dy/dt, we have multiplications $sY(s)$. Then differential equations in t become algebra equations in s. Start with these examples:

Left side $\boldsymbol{y(t) \to Y(s)}$	$y'(t) \to \boldsymbol{sY(s)}$ and $y''(t) \to \boldsymbol{s^2 Y(s)}$ when $y(0) = y'(0) = 0$
Right side $\boldsymbol{f(t) \to F(s)}$	$f = e^{at} \to F = \boldsymbol{1/(s-a)}$ and impulse $f = \delta(t) \to F = \boldsymbol{1}$.

Solving a differential equation by using the Laplace transform involves three steps:

1 Transform every term **2** Solve for $Y(s)$ **3** Find $y(t)$ whose transform is $Y(s)$.

You will see how initial values for $y(0)$ and $y'(0)$ go into the s-equation for $Y(s)$. And most important, you will see how the **zeros** of the polynomial $s^2 + Bs + C$ become "**poles**" of $Y(s)$. Those exponents s_1 and s_2 give us the null solution $y_n(t)$. Dividing by that polynomial gives the transfer function $1/(s^2 + Bs + C)$. Now we see all of this as a natural part of the Laplace transform.

Example 1 Start from $y(0) = 0$ and $y'(0) = 0$. With those initial conditions, the transform of y' is sY and the transform of y'' is s^2Y. We can transform a whole equation:

Step 1 $\boldsymbol{y'' - 4y' + 3y = e^{at}}$ transforms to $\boldsymbol{(s^2 - 4s + 3)\, Y(s) = \dfrac{1}{s-a}}$

Step 2 The transform of $y(t)$ is $Y(s) = \dfrac{1}{(s^2 - 4s + 3)(s-a)} = \boldsymbol{\dfrac{1}{(s-3)(s-1)(s-a)}}$

Step 3 The inverse Laplace transform of $Y(s)$ is $\boldsymbol{y(t) = C_1 e^{3t} + C_2 e^{t} + G e^{at}}$.

C_1 and C_2 come from matching the initial conditions $y(0) = 0$ and $y'(0) = 0$. The gain $G = 1/(a^2 - 4a + 3)$ is the transfer function at $s = a$. The inverse transform of $Y(s)$ is computed in equations (12) and (14). Step 2 revealed the poles of $Y(s)$:

$$\frac{1}{\boldsymbol{(s-3)(s-1)(s-a)}} \quad \textbf{has poles at } \boldsymbol{s = 3} \textbf{ and } \boldsymbol{s = 1} \textbf{ and } \boldsymbol{s = a}.$$

Those three numbers are the all-important exponents in $y(t) = C_1 e^{3t} + C_2 e^{t} + G e^{at}$. Now they are seen as the poles **3**, **1**, $\boldsymbol{a}$ where $\boldsymbol{Y(s)}$ becomes infinite.

Example 2 Change from $f = e^{at}$ to $\boldsymbol{f = \delta(t)}$ = **impulse**. Keep $y(0) = y'(0) = 0$.

Step 1 $\boldsymbol{y'' + By' + Cy = \delta(t)}$ transforms to $\boldsymbol{(s^2 + Bs + C)\, Y(s) = 1}$.

Step 2 The transform of $y(t)$ is $Y(s) = \boldsymbol{\dfrac{1}{s^2 + Bs + C}}$ = **transfer function.**

Step 3 **The inverse transform is** $\boldsymbol{y(t) = g(t) = \dfrac{e^{s_1 t} - e^{s_2 t}}{s_1 - s_2}}$ = **impulse response.**

Those roots s_1, s_2 of $s^2 + Bs + C = (s - s_1)(s - s_2)$ give poles in $Y(s)$ and exponentials in $y(t)$. You have to be impressed by how quickly steps 1-2-3 led to this central fact.

When $f = \delta(t)$, the transform of the impulse response g is the transfer function Y.

The Laplace Transform

Our first Table of Transforms will include the most essential functions and no more. A more complete presentation of this transform will be saved for the last sections of the book. We will define $Y(s)$ here, but the shift rule for transforms will be developed there. All step functions $H(t - T)$ are left for Chapter 8, except for one comment below.

Especially we point to the final Section 8.6 on "*convolutions*". These are the inverse transforms of products $Y(s) = F(s)G(s)$. Convolution is exactly what we need when $f(t)$ is not a simple function like e^{at} and $F(s)$ is not a simple function like $1/(s - a)$.

To create the Table of Transforms we start with the integral that defines $F(s)$:

$$\textbf{The Laplace transform of } f(t) \textbf{ is } F(s) = \int_0^\infty f(t)\, e^{-st}\, dt. \tag{1}$$

The first function to transform is certainly $f(t) = e^{at}$. Then $F(s) = 1/(s - a)$ as expected:

$$F(s) = \int_0^\infty e^{at} e^{-st}\, dt = \left[\frac{e^{(a-s)t}}{a - s}\right]_{t=0}^{t=\infty} = 0 - \frac{1}{a - s} = \frac{1}{s - a}. \tag{2}$$

That integral would be infinite if $a \geq s$. It is typical of Laplace transforms to require $s > a$. Then the factor e^{-st} in the integral brings us safely to zero at $t = \infty$. The following rule is natural for all functions $f(t)$, when you look at the integral (1) from $t = 0$ to $t = \infty$:

By definition $f(t) = 0$ for all $t < 0$. Functions don't start until $t = 0$.

Then the step function $H(t)$ and the constant function $f = 1$ have the same transform!

$$\textbf{The transform of } f(t) = 1 \textbf{ is } F(s) = \int_0^\infty 1 e^{-st}\, dt = \frac{1}{s}. \tag{3}$$

This is the transform of e^{at} when the exponent a goes to 0 and $1/(s - a)$ goes to $1/s$.

Transform of the Derivative

Now comes the most important rule—the whole basis for solving differential equations. If the transform of $y(t)$ is $Y(s)$, what is the transform of the derivative dy/dt?

Derivative Rule $\boxed{\text{The transform of } dy/dt \text{ is } sY(s) - y(0).}$

The derivative rule shows how the initial conditions enter the transformed problem—not as separate side conditions, but directly into the equation for $Y(s)$. The proof uses integration by parts. The integral of dy/dt is $y(t)$ and the derivative of e^{-st} is $-se^{-st}$:

$$\int_0^\infty \frac{dy}{dt}\, e^{-st}\, dt \quad = \quad -\int_0^\infty y(t)(-se^{-st})\, dt + \left[y(t)e^{-st}\right]_0^\infty$$

$$\textbf{Transform of } dy/dt \qquad = \quad \boldsymbol{sY(s) - y(0)} \qquad (4)$$

Again s must be large enough—or more exactly, the real part of s must be large enough—to assure that $y(t)e^{-st}$ drops to zero at $t = \infty$.

We can immediately solve the model problem of Chapter 1: A first order linear equation. The solution steps 1, 2, 3 produce $Y(s)$ with poles (blowup values for s) at the two key exponents $s = a$ and $s = c$:

Example 3 Solve $\boldsymbol{\dfrac{dy}{dt} - ay = e^{ct}}$ starting from any $y(0)$.

Step 1 Transform the equation to $sY(s) - y(0) - aY(s) = \dfrac{1}{s-c}$. (5)

Step 2 $(s-a)Y(s) = y(0) + \dfrac{1}{s-c}$ gives $Y(s) = \dfrac{y(0)}{s-a} + \dfrac{1}{(s-a)(s-c)}$. (6)

Step 3 The inverse transform of $\dfrac{y(0)}{s-a}$ is the null solution $\boldsymbol{y_n(t) = y(0)e^{at}}$. (7)

The inverse transform of $\dfrac{1}{(s-a)(s-c)}$ is the very particular solution $\boldsymbol{\dfrac{e^{ct} - e^{at}}{c-a}}$. (8)

I have to say, this is beautiful. The effort we made in Chapter 1 has been reduced to its bare minimum. All that is left is the derivative rule, the transform of exponentials, and "partial fractions." Those partial fractions were the algebra from Step 2 to Step 3: separating $1/(s-a)(s-c)$ with two poles a and c into **two fractions with one pole each**.

$$\textbf{PF2} \qquad \frac{1}{(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{c})} = \frac{1}{(\boldsymbol{s}-\boldsymbol{a})(a-c)} + \frac{1}{(c-a)(\boldsymbol{s}-\boldsymbol{c})} \qquad (9)$$

PF2 was used in Example 2 to find the impulse response. In that case a and c were s_1 and s_2. Partial fractions were also used in Example 1, with $f = e^{at}$ and *three* poles 3, 1 a.

Partial Fractions

Example 1 reached $Y(s) = 1/(s+3)(s+1)(s-a)$. We didn't immediately know its inverse transform $y(t)$. But finding $y(t)$ becomes simple when $Y(s)$ is separated into **three terms with one pole each. Those three pieces are the Partial Fractions in PF3**:

$$\frac{1}{(\boldsymbol{s}-\boldsymbol{3})(\boldsymbol{s}-\boldsymbol{1})(\boldsymbol{s}-\boldsymbol{a})} = \frac{1}{(\boldsymbol{s}-\boldsymbol{3})(3-1)(3-a)} + \frac{1}{(1-3)(\boldsymbol{s}-\boldsymbol{1})(1-a)} + \frac{1}{(a-3)(a-1)(\boldsymbol{s}-\boldsymbol{a})}$$

Usually I would show you where this PF3 formula comes from. In this case I would rather show you that it is correct. Above all, you must see the main point: The three separate terms with one pole each lead immediately to the three parts $\boldsymbol{C_1 e^{3t}}$ and $\boldsymbol{C_2 e^{t}}$ and $\boldsymbol{Y e^{at}}$.

Officially, correctness can be proved by multiplying PF3 by $(s-3)(s-1)(s-a)$.

$$1 = \frac{(s-1)(s-a)}{(3-1)(3-a)} + \frac{(s-3)(s-a)}{(1-3)(1-a)} + \frac{(s-3)(s-1)}{(a-3)(a-1)}. \tag{10}$$

At $s = 3$, the last two terms disappear and we have $1 = 1$ (as desired). At $s = 1$, the second term equals 1. At $s = a$, the third term equals 1. Thus (10) is an equation of the form $1 = As^2 + Bs + C$, and the equation is correct at three values $s = 3, 1, a$. Therefore the equation must be always correct, and PF3 is shown to be true.

Remark The theory of partial fractions usually computes C_1 and C_2 and Y so that

$$\boxed{\frac{\boldsymbol{1}}{(\boldsymbol{s}-\boldsymbol{3})(\boldsymbol{s}-\boldsymbol{1})(\boldsymbol{s}-\boldsymbol{a})} = \frac{\boldsymbol{C_1}}{\boldsymbol{s}-\boldsymbol{3}} + \frac{\boldsymbol{C_2}}{\boldsymbol{s}-\boldsymbol{1}} + \frac{\boldsymbol{Y}}{\boldsymbol{s}-\boldsymbol{a}}.} \tag{11}$$

The idea is to put the right side over a common denominator, which is on the left side. Matching the coefficients of s^2 and s and 1 gives three equations for C_1 and C_2 and Y. My shortcut was to go directly to the answers C_1, C_2, G that you see in PF3:

$$C_1 = \frac{1}{(3-1)(3-a)} \qquad C_2 = \frac{1}{(1-3)(1-a)} \qquad Y = \frac{1}{(a-3)(a-1)}. \tag{12}$$

I think it is easier to remember this pattern than to solve for a new C_1 and C_2 and Y, every time you change the poles 3 and 1 and a. *To repeat, from the three partial fractions in* PF3 *we read off the coefficients C_1, C_2, Y in equation* (12).

Very Particular Solution

Look at what we have in those three parts. The last part Ye^{at} is a particular solution—the one that comes from the transfer function and the exponential response formula. The equation was $y'' - 4y' + 3y = e^{at}$, and the response to e^{at} is

$$y_p(t) = Ye^{at} = \frac{1}{a^2 - 4a + 3}\, e^{at} = \frac{1}{(a-3)(a-1)}\, e^{at}. \tag{13}$$

That is old news. This is not the very particular solution, it doesn't start at $y(0) = 0$ and $y'(0) = 0$. The solution with that particular start is the one from the Laplace transform:

$$\textbf{The very particular solution is all of } y_{vp}(t) = C_1 e^{3t} + C_2 e^{t} + Y e^{at}. \tag{14}$$

Remember, any null solution y_n can be added to one particular y_p. That gives another y_p. The very particular solution y_{vp} starts from rest.

The complete solution adjusts the free constants c_1 and c_2 (note the small c) to match any starting values $y(0)$ and $y'(0)$:

$$y_{\text{complete}} = c_1 e^{3t} + c_2 e^{t} + Y e^{at}. \tag{15}$$

You could solve for c_1 and c_2 as usual, by setting $t = 0$ in y and y'. Then you are working in the time domain. Or you could use $y(0)$ and $y'(0)$ in finding $Y(s)$, when you transform the equation in the first place. Let me show you that way, compared to the usual way.

Including $y(0)$ and $y'(0)$ in the Transform

We know that the transform of y' is $sY(s) - y(0)$. To find the transform of y'', use that first derivative rule twice. This brings in $y'(0)$ along with $y(0)$.

$$\begin{aligned} \textbf{transform of } y'' &= s(\text{transform of } y') - y'(0) \\ &= s(sY(s) - y(0)) - y'(0) \\ &= \boldsymbol{s^2 Y(s) - s y(0) - y'(0).} \end{aligned} \tag{16}$$

Now we can solve the equation $y'' - 4y' + 3y = e^{at}$ entirely by Laplace transform:

Step 1 Transform to $(s^2Y(s) - sy(0) - y'(0)) - 4(sY(s) - y(0)) + 3Y(s) = \dfrac{1}{s-a}$

Step 2 Rewrite as $(s^2 - 4s + 3)Y(s) = (s-4)y(0) + y'(0) + 1/(s-a)$.

$$\text{Solve for } Y(s): \quad Y(s) = \frac{(s-4)y(0) + y'(0)}{s^2 - 4s + 3} + \frac{1}{(s^2 - 4s + 3)(s-a)}. \tag{17}$$

Step 3 Invert both pieces of $Y(s)$ to find $y_n(t) + y_p(t)$.

This looks more painful to me! The last part of $Y(s)$ is fine—that is what we already worked with to find y_p. Its inverse transform is the very particular solution in (14). The first part of $Y(s)$ involves $y(0)$ and $y'(0)$. We have to do partial fractions again: *not good.*

The denominator $s^2 - 4s + 3$ has two factors $(s-3)(s-1)$ and not three factors. But I would prefer to find c_1 and c_2 in the complete solution (15), by setting $t = 0$ and solving these two equations:

$$\begin{aligned} c_1 + c_2 + Y &= y(0) \\ 3\,c_1 + c_2 + aY &= y'(0) \end{aligned} \tag{18}$$

When $y(0)$ and $y'(0)$ are zero, that's when c_1 and c_2 and y equal C_1 and C_2 and y_{vp}.

Transforms at Resonance

The reader will remember that when two exponents come together, and two solutions become one solution like e^{at}, another solution is born. It is like atomic fission or fusion. The new solution has the form $\boldsymbol{te^{at}}$. We want to find its Laplace transform.

Equal exponents can happen in two different ways for $y'' + By' + Cy = f(t)$.

1 (*Null solution*) Two roots s_1 and s_2 of the characteristic polynomial become equal.

2 (*Particular solution*) The exponent in $f = e^{at}$ equals s_1 or s_2 in the null solution.

In a truly extreme case we might have $s_1 = s_2 = a$, three equal exponents. Then the null solution is $c_1 e^{at} + c_2 te^{at}$, and a particular solution is $G\boldsymbol{t^2}e^{at}$.

We are seeing these possibilities in the "time domain" and we can see them in the "frequency domain". **Double roots in the t-domain become double poles in $Y(s)$.**

$$\textbf{The Laplace transform of } te^{at} \textbf{ is } \frac{1}{(s-a)^2} \textbf{ with a double pole.} \tag{19}$$

A nice proof starts with a simple pole in the transform. The transform of e^{at} is $1/(s-a)$. Now take derivatives of both sides *with respect to* a :

$$\int_0^\infty e^{at}e^{-st}\,dt = \frac{1}{s-a} \qquad \int_0^\infty te^{at}e^{-st}\,dt = \frac{d}{da}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$$

If we take another a-derivative, the transform of t^2e^{at} is seen as $2(s-a)^{-3}$ with a triple pole. The simplest example of this extreme case would be the equation $y'' = 2$.

$$y'' = 2 \text{ has exponents } 0 \text{ and } 0 \text{ in } y_n(t) = c_1 + c_2 t \quad \text{and } a = 0 \text{ in } y_p(t) = t^2 e^{0t} = t^2.$$

The initial conditions give $c_1 = y(0)$ and $c_2 = y'(0)$. The solution is easy to check :

$$y = y(0) + ty'(0) + t^2 \quad \textbf{solves} \quad y'' = 2. \tag{20}$$

To find this solution by Laplace transform, start by transforming y'' and 2 :

$$s^2Y(s) - y(0)s - y'(0) = \frac{2}{s} \qquad \text{gives} \quad Y(s) = \frac{y(0)}{s} + \frac{y'(0)}{s^2} + \frac{2}{s^3}. \tag{21}$$

The inverse transforms of $1/s$ and $1/s^2$ are 1 and t. The inverse transform of $2/s^3$ is t^2. So the inverse transform of $Y(s)$ is the correct $y = y(0) + ty'(0) + t^2$ in (20).

Those are really e^{0t} and te^{0t} and t^2e^{0t} : three zero exponents, a truly extreme case.

The inverse of equation (19) tells us the fundamental solution $g(t)$ when the transfer function $1/(s^2 + Bs + C)$ has a double pole and $s^2 + Bs + C = 0$ has $s_1 = s_2$:

If $s^2 + Bs + C = (s - s_1)^2$ then the fundamental solution is $g(t) = te^{s_1 t}$.

The Transforms of $\cos \omega t$ and $\sin \omega t$

In all of this section on Laplace transforms, there is no requirement that a must be real. That exponent can be $i\omega$ or $-i\omega$ or any complex number $a + i\omega$. From the identity $\cos \omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$, and from the linearity of the formula for $F(s) = \int f(t)e^{-st}dt$, we can combine the known transforms of $e^{i\omega t}$ and $e^{-i\omega t}$:

The transform of $f(t) = \cos \omega t$ is $F(s) = \dfrac{1}{2}\left(\dfrac{1}{s-i\omega} + \dfrac{1}{s+i\omega}\right) = \dfrac{s}{s^2+\omega^2}$ (22)

The twin identity $\sin \omega t = \dfrac{1}{2i}(e^{i\omega t} - e^{-i\omega t})$ also comes from Euler's formula.

The transform of $f(t) = \sin \omega t$ is $F(s) = \dfrac{1}{2i}\left(\dfrac{1}{s-i\omega} - \dfrac{1}{s+i\omega}\right) = \dfrac{\omega}{s^2+\omega^2}$. (23)

Those transforms appear in the fundamental example of a mass hanging from a spring:

Step 1 $\quad my'' + ky = \cos \omega t$ transforms to $m(s^2Y(s) - sy(0) - y'(0)) + kY(s) = \dfrac{s}{s^2+\omega^2}$.

The transform $Y(s)$ is multiplied by $ms^2 + k$. **The transfer function is $\boldsymbol{1/(ms^2+k)}$.**

The transfer function multiplies the input to give the output. The input is on the right hand side, the output is the solution. Both of those are now in transform space!

Step 2 $\quad$ Solve for $Y(s) = \dfrac{1}{ms^2+k}\left(sy(0) + y'(0) + \dfrac{s}{s^2+\omega^2}\right)$. (24)

We are ready for Step 3, but it doesn't look so easy. It requires the inverse transform of this $Y(s)$. Our simple mass-spring problem has led us to a *fourth degree* denominator $(ms^2 + k)(s^2 + \omega^2)$. We need partial fractions to separate $Y(s)$ into two pieces with *second degree* denominators. That algebra is not so bad, and it can be left for Problem 26.

The result is that $y(t)$ has a term in $\cos \omega t$ and another term in $\cos \omega_n t$. The driving frequency is ω, the natural frequency $\omega_n = \sqrt{k/m}$ comes from the zeros of $ms^2 + k$. **The frequencies in the solution $\boldsymbol{y(t)}$ are the poles $\boldsymbol{\pm i\omega}$ and $\boldsymbol{\pm i\omega_n}$ in its transform $\boldsymbol{Y(s)}$.**

That bold statement is really the important message from a Laplace transform. We engineer the system or the network by moving those poles. Often we keep them well separated to avoid instability. And we add damping to push the zeros of $ms^2 + bs + k$ (poles of $Y(s)$) off the imaginary axis and into the stable left halfplane where Re $s < 0$.

$f(t)$	$1, t, t^2$	$e^{at}, te^{at}, t^2e^{at}$	$\cos \omega t, \sin \omega t$	y, y', y''
$F(s)$	$\frac{1}{s}, \frac{1}{s^2}, \frac{2}{s^3}$	$\frac{1}{s-a}, \frac{1}{(s-a)^2}, \frac{2}{(s-a)^3}$	$\frac{s}{s^2+\omega^2}, \frac{\omega}{s^2+\omega^2}$	$Y,\ sY - y(0),$ $s^2Y - sy(0) - y'(0)$

Complex Roots $a \pm i\omega$

Finally we come to the most typical case for physical systems. It has damping, and it has oscillation. *The roots of* $s^2 + 2s + 5$ *are complex.* Their real parts are $a = -2/2 = -1$. Their imaginary parts $\pm\sqrt{B^2 - 4AC}/2$ are $\pm i\omega = \pm\sqrt{-16}/2 = \pm 2i$. We are in the underdamped case and the solutions to $y'' + 2y' + 5y = 0$ can be written two ways:

$$y = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \quad \text{or} \quad y = e^{-t}(C_1 \cos 2t + C_2 \sin 2t). \tag{25}$$

What does this problem look like in the s-domain, after a Laplace transform?

$$y'' + 2y' + 5y = 0 \quad \text{transforms to} \quad (s^2 + 2s + 5)\,Y(s) - (s+2)\,y(0) - y'(0) = 0. \tag{26}$$

That quadratic $s^2 + 2s + 5$ will go into the denominator of $Y(s)$, as always. **This part of $Y(s)$ is the transfer function $1/(s^2 + 2s + 5)$.** The numerator is $(s+2)y(0) + y'(0)$ from the initial conditions. The right hand side of our null equation (26) is zero and the transfer function is connecting the inputs $y(0)$ and $y'(0)$ to the solution:

$$\text{The transform of } y(t) \quad \text{is} \quad Y(s) = \frac{(s+2)\,y(0) + y'(0)}{s^2 + 2s + 5}. \tag{27}$$

This is the point where partial fractions can enter, if we choose. We can separate $s^2 + 2s + 5$ into its linear factors $(s - s_1)(s - s_2)$. *I suggest not to do it.* Those roots s_1 and s_2 are complex numbers, and it is easier to stay with one real quadratic.

We are close to the transforms of $\cos \omega t$ and $\sin \omega t$, already in the Table above. The new factor is $e^{at} = e^{-t}$ from the real part, and it gives decay.

$$\boldsymbol{e^{at} \cos \omega t \text{ and } e^{at} \sin \omega t \text{ transform to } \frac{s-a}{(s-a)^2 + \omega^2} \text{ and } \frac{\omega}{(s-a)^2 + \omega^2}}. \tag{28}$$

For (27), the key is to separate $s^2 + 2s + 5$ into $(s+1)^2 + 4$. From this we recognize $a = -1$ and $\omega = 2$ as expected. Then the inverse transform combines $e^{-t} \cos 2t$ and $e^{-t} \sin 2t$. The numerator in (27) is linear, call it $Hs + K$. To fit perfectly with the numerator $s - a$ in (28), we can split any $Hs + K$ into $H(s-a) + (K + Ha)$:

$$\text{The inverse transform of} \quad \frac{Hs + K}{(s-a)^2 + \omega^2} \quad \text{is} \quad He^{at} \cos \omega t + (K + Ha)\, e^{at} \frac{\sin \omega t}{\omega} \tag{29}$$

For higher order equations, and for equations with exponential driving functions $f(t)$, the transform $Y(s)$ involves polynomials of higher degree. In principle, partial fractions can reduce to degree 1 and degree 2. Those produce the real poles and complex poles of $Y(s)$—the real and complex exponentials e^{st} in $y(t)$. I would certainly turn first to the method of undetermined coefficients in Section 2.6.

The best contribution of Laplace transforms is to focus attention on transfer functions like $1/(As^2 + Bs + C)$ and their poles.

■ REVIEW OF THE KEY IDEAS ■

1. The Laplace transform of $f(t)$ is $F(s) = \int_0^\infty f(t)e^{-st}dt$. $\boldsymbol{f = e^{at} \to F = \frac{1}{s-a}}$.
2. $Ay'' + By' + Cy$ transforms to $(As^2 + Bs + C)\,Y(s) - (As + B)y(0) - Ay'(0)$.
3. Step 1 transforms the equation, Step 2 solves for $Y(s)$, Step 3 inverts $Y(s)$ to $y(t)$.
4. The exponents in the solutions $y_n(t)$ and $y_p(t)$ are the poles in $Y(s)$.
5. Partial fractions can simplify $Y(s)$ using **PF**2 and **PF**3, to help invert to $y(t)$.

Problem Set 2.7

1 Take the Laplace transform of each term in these equations and solve for $Y(s)$, with $y(0) = 0$ and $y'(0) = 1$. Find the roots s_1 and s_2 — the poles of $Y(s)$:

Undamped	$y'' + 0y' + 16y = 0$
Underdamped	$y'' + 2y' + 16y = 0$
Critically damped	$y'' + 8y' + 16y = 0$
Overdamped	$y'' + 10y' + 16y = 0$

For the overdamped case use PF2 to write $Y(s) = A/(s - s_1) + B/(s - s_2)$.

2 Invert the four transforms $Y(s)$ in Problem 1 to find $y(t)$.

3 (a) Find the Laplace transform $Y(s)$ from the equation $y' = e^{at}$ with $y(0) = A$.
(b) Use PF2 to break $Y(s)$ into two fractions $C_1/(s - a) + C_2/s$.
(c) Invert $Y(s)$ to find $y(t)$ and check that $y' = e^{at}$ and $y(0) = A$.

4 (a) Find the transform $Y(s)$ when $y'' = e^{at}$ with $y(0) = A$ and $y'(0) = B$.
(b) Split $Y(s)$ into $C_1/(s - a) + C_2/(s - a)^2 + C_3/s$.
(c) Invert $Y(s)$ to find $y(t)$. Check $y'' = e^{at}$ and $y(0) = A$ and $y'(0) = B$.

5 Transform these differential equations to find $Y(s)$:

(a) $y'' - y' = 1$ with $y(0) = 4$ and $y'(0) = 0$
(b) $y'' + y = \cos \omega t$ with $y(0) = y'(0) = 0$ and $\omega \neq 1$
(c) $y'' + y = \cos t$ with $y(0) = y'(0) = 0$. What changed for $\omega = 1$?

6 Find the Laplace transforms F_1, F_2, F_3 of these functions f_1, f_2, f_3:

$$f_1(t) = e^{at} - e^{bt} \qquad f_2(t) = e^{at} + e^{-at} \qquad f_3(t) = t\cos t$$

7 For any real or complex a, the transform of $f = te^{at}$ is ____. By writing $\cos \omega t$ as $(e^{i\omega t} + e^{-i\omega t})/2$, transform $g(t) = t\cos \omega t$ and $h(t) = te^t \cos \omega t$. (*Notice that the transform of h is new.*)

8 Invert the transforms F_1, F_2, F_3 using PF2 and PF3 to discover f_1, f_2, f_3:

$$F_1(s) = \frac{1}{(s-a)(s-b)} \qquad F_2(s) = \frac{s}{(s-a)(s-b)} \qquad F_3(s) = \frac{1}{s^3 - s}$$

9 Step 1 transforms these equations and initial conditions. Step 2 solves for $Y(s)$. Step 3 inverts to find $y(t)$:

(a) $y' - ay = t$ with $y(0) = 0$

(b) $y'' + a^2 y = 1$ with $y(0) = 1$ and $y'(0) = 2$

(c) $y'' + 3y' + 2y = 1$ with $y(0) = 4$ and $y'(0) = 5$.

What particular solution y_p to (c) comes from using "undetermined coefficients"?

Questions 10-16 are about partial fractions.

10 Show that PF2 in equation (9) is correct. Multiply both sides by $(s-a)(s-b)$:

$$(*) \qquad 1 = \_\_\_\_ + \_\_\_\_ .$$

(a) What do those two fractions in $(*)$ equal at the points $s = a$ and $s = b$?

(b) The equation $(*)$ is correct at those two points a and b. It is the equation of a straight ____. So why is it correct for every s?

11 Here is the PF2 formula with numerators. Formula $(*)$ had $K = 1$ and $H = 0$:

$$\textbf{PF2}' \qquad \frac{Hs+K}{(s-a)(s-b)} = \frac{Ha+K}{(s-a)(a-b)} + \frac{Hb+K}{(b-a)(s-b)}$$

To show that PF2$'$ is correct, multiply both sides by $(s-a)(s-b)$. You are left with the equation of a straight ____. Check your equation at $s = a$ and at $s = b$. Now it must be correct for all s, and PF2$'$ is proved.

12 Break these functions into two partial fractions using PF2 and PF2$'$:

(a) $\dfrac{1}{s^2-4}$ (b) $\dfrac{s}{s^2-4}$ (c) $\dfrac{Hs+K}{s^2-5s+6}$

13 Find the integrals of $(a)(b)(c)$ in Problem 12 by integrating each partial fraction. The integrals of $C/(s-a)$ and $D/(s-b)$ are logarithms.

14 Extend PF3 to PF3$'$ in the same way that PF2 extended to PF2$'$:

$$\textbf{PF3}' \qquad \frac{Gs^2+Hs+K}{(s-a)(s-b)(s-c)} = \frac{Ga^2+Ha+K}{(s-a)(a-b)(a-c)} + \frac{?}{?} + \frac{?}{?}.$$

15 The linear polynomial $(s-b)/(a-b)$ equals 1 at $s=a$ and 0 at $s=b$. Write down a quadratic polynomial that equals 1 at $s=a$ and 0 at $s=b$ and $s=c$.

16 What is the number C so that $C(s-b)(s-c)(s-d)$ equals 1 at $s=a$?

Note A complete theory of partial fractions must allow double roots (when $b=a$). The formula can be discovered from l'Hôpital's Rule (in PF3 for example) when b approaches a. Multiple roots lose the beauty of PF3 and PF3′—we are happy to stay with simple roots a, b, c.

Questions 17-21 involve the transform $F(s)=1$ of the delta function $f(t)=\delta(t)$.

17 Find $F(s)$ from its definition $\int_0^\infty f(t)e^{-st}dt$ when $f(t)=\delta(t-T)$, $T\geq 0$.

18 Transform $y''-2y'+y=\delta(t)$. The **impulse response** $y(t)$ transforms into $Y(s)=$ **transfer function**. The double root $s_1=s_2=1$ gives a double pole and a new $y(t)$.

19 Find the inverse transforms $y(t)$ of these transfer functions $Y(s)$:

(a) $\dfrac{s}{s-a}$ (b) $\dfrac{s}{s^2-a^2}$ (c) $\dfrac{s^2}{s^2-a^2}$

20 Solve $y''+y=\delta(t)$ by Laplace transform, with $y(0)=y'(0)=0$. If you found $y(t)=\sin t$ as I did, this involves a serious mystery: *That sine solves $y''+y=0$, and it doesn't have $y'(0)=0$. Where does $\delta(t)$ come from*? In other words, what is the derivative of $y'=\cos t$ if all functions are zero for $t<0$?

If $y=\sin t$, explain why $y''=-\sin t+\delta(t)$. Remember that $y=0$ for $t<0$.

Problem (20) connects to a remarkable fact. The same impulse response $y=g(t)$ solves both of these equations: **An impulse at $t=0$ makes the velocity $y'(0)$ jump by 1**. Both equations start from $y(0)=0$.

$$y''+By'+Cy=\boldsymbol{\delta(t)} \text{ with } y'(0)=\mathbf{0} \qquad y''+By'+Cy=\mathbf{0} \text{ with } y'(0)=\mathbf{1}.$$

21 (Similar mystery) These two problems give the same $Y(s)=s/(s^2+1)$ and the same impulse response $y(t)=g(t)=\cos t$. How can this be?

$$y'=-\sin t \text{ with } y(0)=\mathbf{1} \qquad y'=-\sin t+\boldsymbol{\delta(t)} \text{ with “}y(0)=0\text{”}$$

Problems 22-24 involve the Laplace transform of the integral of $y(t)$.

22 If $f(t)$ transforms to $F(s)$, what is the transform of the integral $h(t)=\int_0^t f(T)dT$? Answer by transforming the equation $dh/dt=f(t)$ with $h(0)=0$.

23 Transform and solve the integro-differential equation $y' + \int_0^t y\,dt = 1,\ y(0) = 0.$

A mystery like Problem 20: $y = \cos t$ seems to solve $y' + \int_0^t y dt = 0, y(0) = 1.$

24 Transform and solve the amazing equation $dy/dt + \int_0^t y\,dt = \delta(t).$

25 The derivative of the delta function is not easy to imagine—it is called a "doublet" because it jumps up to $+\infty$ and back down to $-\infty$. Find the Laplace transform of the doublet $d\delta/dt$ from the rule for the transform of a derivative.

A doublet $\delta'(t)$ is known by its integral: $\int \delta'(t)F(t)dt = -\int \delta(t)F'(t)dt = \boldsymbol{-F'(0)}.$

26 (Challenge) What function $y(t)$ has the transform $Y(s) = 1/(s^2+\omega^2)(s^2+a^2)$? First use partial fractions to find H and K:

$$Y(s) = \frac{H}{s^2+\omega^2} + \frac{K}{s^2+a^2}$$

27 Why is the Laplace transform of a unit step function $H(t)$ the same as the Laplace transform of a constant function $f(t) = 1$?

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Chapter 3

Graphical and Numerical Methods

The world of differential equations is large (very large). This page aims to see what is already done and what remains to do.

Chapters 1 and 2 concentrated on *equations we can solve*. Compared to digging for coal or drilling for oil, this was the equivalent of picking up gold. Solutions were waiting for us. Looking back honestly, we just wrote them down (not so easy in Chapter 2).

Above all I am thinking of e^{at} in Chapter 1 and e^{st} in Chapter 2 and $e^{\lambda t}\boldsymbol{x}$ coming in Chapter 6 (with eigenvalues and eigenvectors). When the equation is linear, and its coefficients are constant, then its solutions are exponentials.

Chapter 1 First order equations (linear or separable or exact or special)

Chapter 2 Second order equations $Ay'' + By' + Cy = f(t)$

Chapter 6 First order systems $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$ with matrices A and vectors $\boldsymbol{y}$.

Chapter 3 will be different. Instead of $f(t)$ we have $f(t, y)$. Most nonlinear problems don't allow a formula for $y(t)$. "A solution exists but it has no formula." This is the hard reality of differential equations $y' = f(t, y)$. The equations are important but they don't have exponential answers. This chapter **pictures** the solution, **computes** the solution, and decides if the solution is **stable**.

Section 3.1 Pictures for nonlinear equations $y' = f(t, y)$: Stability decided by $\partial f/\partial y$.

Section 3.2 Pictures for linear second order equations and 2 by 2 systems : Stable or not.

Section 3.3 Test for stability at critical points by linearizing systems of equations.

Section 3.4 Euler methods (safe but slow) for computing approximations to y.

Section 3.5 Fast and accurate computations, by methods more efficient than Euler. Science and engineering and finance constantly use Runge-Kutta.

After this chapter, the book will move into high dimensions : **the world of linear algebra**. One particle and one resistor and one spring and one of anything : that was only a start. The reality is a network of connections : a brain, a living body, a modern machine, a web of processors. Every network leads to a matrix. *You will learn how to read a matrix*.

In my opinion, linear algebra is pure gold.

3.1 Nonlinear Equations $y' = f(t, y)$

This section aims to get a picture of $y(t)$, not a formula. The pictures will be graphs in the $t - y$ plane (t across and $y(t)$ up). The differential equation is $dy/dt = f(t, y)$ and everything depends on that function f. I can start with a linear equation $y' = 2y$.

> The solutions to $y' = 2y$ are $y(t) = Ce^{2t}$. For every number C this gives a solution curve from $t = -\infty$ to $t = \infty$. Those curves cover every point in the $t - y$ plane. This is the "solution picture" we want for nonlinear equations $y' = f(t, y)$.

That solution $y = Ce^{2t}$ has a graph. The plane is filled with those graphs. Every point t, y has one of those curves going through it (choose the right C). A different equation $y' = \sin ty$ won't have a formula. Its picture starts with just this one fact:

$\boldsymbol{dy/dt = \sin ty}$ **The solution curve through the point $\boldsymbol{t}$, $\boldsymbol{y}$ has the slope $\boldsymbol{\sin ty}$.**

From that *point* picture we have to build a *curve* picture. This section tries to connect small arrows at points into solution curves through those points. The arrow at the point t, y has the right slope $f(t, y)$. Connecting with other arrows is the hard part.

I will separate this section into facts about $y(t)$ and pictures of $y(t)$.

Facts About $y(t)$

The facts will be answers to these questions, and the Chapter 3 Notes add more:

1. Starting from $y(0)$ at $t = 0$, **does $\boldsymbol{dy/dt = f(t, y)}$ have a solution**?
2. **Could there be two or more solutions** that start from the same $y(0)$?

Question **1** is about *existence* of $y(t)$. Is there a solution curve through $t=0$, $y=y(0)$?
Question **2** is about *uniqueness* of $y(t)$. Could two solution curves go through one point?

When $f(t, y)$ is reasonable, we expect exactly one curve through every point t, y: *existence and also uniqueness*. Which functions are reasonable? Here are answers:

1. A solution exists if $f(t, y)$ is a continuous function for t near 0 and y near $y(0)$.
2. There can't be two solutions with the same $y(0)$ when $\partial f/\partial y$ is also continuous.

The word "continuous" has a precise technical meaning. Let me be imprecise and nontechnical. Continuity at a point rules out jumps and infinities in a small neighborhood of that point. The particular function $f = y/t$ is certainly ruled out at points where $t = 0$:

$$\frac{dy}{dt} = \frac{y}{t} \text{ with } y(0) = 0 \text{ has infinitely many solutions } \boldsymbol{y = Ct}.$$

The particular function $f = t/y$ is also ruled out when $y(0) = 0$ (no division by 0):

$$\frac{dy}{dt} = \frac{t}{y} \text{ with } y(0) = 0 \text{ has two solutions } \boldsymbol{y(t) = t} \textbf{ and } \boldsymbol{y(t) = -t}.$$

In those examples, y/t and t/y are starting from $0/0$. Solutions do exist (that fact wasn't guaranteed). Solutions are not unique (no surprise). We ask more from $f(t, y)$.

There is one important point that we emphasize here, because it could easily be missed.

Continuity of f and $\dfrac{\partial f}{\partial y}$ at all points does not guarantee that solutions reach $t = \infty$.

Yes, there will be a solution starting from $y(0)$. That solution will be unique. But $y(t)$ could blow up at some finite time t. The first nonlinear equation in the book (Section 1.1) was an example of early explosion:

$$\textbf{Blow-up at } \boldsymbol{t = 1} \quad \text{The solution to } \frac{dy}{dt} = y^2 \text{ with } y(0) = 1 \text{ is } y(t) = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{t}}.$$

That function $f = y^2$ is certainly continuous. Its derivative $\partial f/\partial y = 2y$ is also continuous. But the derivative $2y$ grows when the solution grows. To be sure there is no explosion at a finite time t, we ask for an upper bound L on the continuous function $\partial f/\partial y$:

$$\textbf{If } \left|\frac{\partial f}{\partial y}\right| \leq \boldsymbol{L} \textbf{ for all } \boldsymbol{t} \textbf{ and } \boldsymbol{y} \textbf{ there is a unique solution through } \boldsymbol{y(0)} \textbf{ reaching all } \boldsymbol{t}.$$

For a linear differential equation $y' = a(t)y + q(t)$, the derivative $\partial f/\partial y$ of the right hand side is just $a(t)$. Then if $|a(t)| \leq L$ and $q(t)$ is continuous for all time, solution curves go from $t = -\infty$ to $t = \infty$. Chapter 1 found a formula for $y(t)$ in this linear case.

I will end with one final nonlinear fact. The condition $|\partial f/\partial y| \leq L$ is pushed to its limit when $\partial f/\partial y = L$ exactly. Then $y' = Ly + q(t)$. A comparison with this linear equation gives information about the nonlinear equation, when $|\partial f/\partial y| \leq L$:

$$\text{If } y' = f(t, y) \text{ and } z' = f(t, z), \text{ then } |y(t) - z(t)| \leq e^{\boldsymbol{Lt}}|y(0) - z(0)|. \tag{1}$$

If $y(t)$ and $z(t)$ start very close, they stay close. This is the opposite of what you see on the cover of this book. The cover shows a famous example of **chaos**: solutions go wild. A slight change in $y(0)$ will send the solution on a completely different (and distant) path. We now know that Pluto's orbit is chaotic: very very unpredictable. The equations allow it, because they don't have $|\partial f/\partial y| \leq L$. Pluto is not a planet.

Pictures of the Solution

Example 1 $\quad dy/dt = 2 - y \quad$ **Solution** $\boldsymbol{y(t) = 2 + Ce^{-t}} \quad \boldsymbol{y(\infty) = 2}$

The perfect picture of $y' = 2 - y$ would show a small arrow at every point t, y. **The arrow would have slope $\boldsymbol{s = 2 - y}$.** Along the all-important "steady state line" $y = 2$, this slope would be *zero*. The arrows are flat $(s = 0)$ along that line: a constant solution.

Above that steady line, the slope $2 - y$ is negative. The vectors have components dt across and $dy = (2 - y)dt$ down. We don't have space for an arrow at every point, but Figure 3.1 gives the idea. MATLAB calls the field of arrows a "quiver".

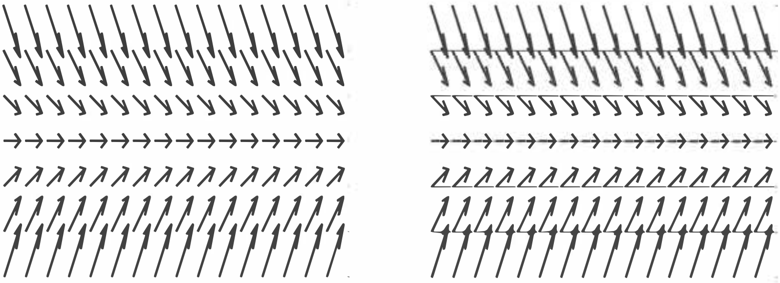


Figure 3.1: (a) Arrows with slopes $f(t, y)$ show the direction of the solution curves $y(t)$. (b) **Along an isocline $f(t, y) = s$, all arrows have the same slope s.** Here $s = 2 - y$.

Notice that all arrows point **toward** the line $y = 2$. That steady state solution is **stable**. The formula $y(t) = 2 + Ce^{-t}$ confirms that the solutions approach $y = 2$.

First key idea: **The solution curves $y(t) = 2 + Ce^{-t}$ are tangent to the arrows.** Tangent means: The curves have the same slope $s = 2 - y$ as the arrows! The curves solve the equation, the equation specifies the slopes, the arrows have correct slopes.

Second key idea: **Put your arrows along isoclines**. An isocline (meaning "same slope") is a curve $f(t, y) =$ constant. This idea makes the arrows much easier to draw. All the isoclines $2 - y = s$ are horizontal lines for this equation $y' = 2 - y$. When the differential equation is $dy/dt = f(t, y)$, **each choice of slope s produces an isocline $f(t, y) = s$.**

In our example, those isoclines $2-y = s$ are flat because $f(t, y) = 2-y$ does not depend on t (autonomous equation). I start the picture by drawing a few isoclines. I always draw the isocline $f(t, y) = 0$ (here $2 - y = 0$ is the steady state line $y = 2$). For this equation, that "nullcline" or "zerocline" with $s = 0$ **is also a solution curve**. The arrows have slope zero when $y = 2$, so they point along the flat line.

How to understand these pictures? **The arrows are pointing along the solution curves**. The curves cross over isoclines. But they don't cross over the zero isocline $y = 2$.

All arrows are pointing toward the line $y = 2$. Those arrows will eventually take us across every other isocline. The pictures say that the solution curves $y(t)$ are asymptotic to that line $y = 2$. For this equation $dy/dt = 2 - y$ we know the solutions $y = 2 + Ce^{-t}$.

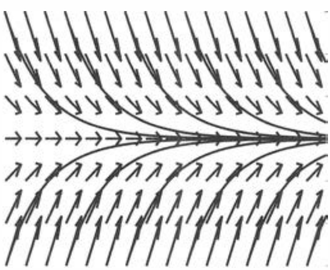


Figure 3.2: Solution curves (tangent to arrows) go through isoclines: $y' = 2 - y$.

Example 2 $\quad \dfrac{dy}{dt} = y - y^2 \qquad$ **Solutions** $y(t) = \dfrac{1}{1 + Ce^{-t}} \qquad y(t) \to 1$ or $-\infty$

The slope of every small arrow is $y - y^2$. In the range $0 < y < 1$, y will be larger than y^2. The arrows have positive slope $y - y^2$ in this range (small slope near $y = 0$, small slope near $y = 1$, all up and to the right). The other two ranges are above $y = 1$ and below $y = 0$. There the slopes $y - y^2$ are negative—arrows go down and right. *The solution curves are steep when y is large*, because $y^2 >> y$.

Figure 3.3 shows the isoclines $f(t, y) = y - y^2 = s =$ constant. Again f does not depend on t ! The equation is autonomous, the isoclines are flat lines. There are **two zeroclines** $\boldsymbol{y = 1}$ **and** $\boldsymbol{y = 0}$ (where $dy/dt = 0$ and y is constant). Those arrows have zero slope and the graph of $y(t)$ runs along each zerocline : a steady state.

The question is about all the other solution curves : What do they do ? We happen to have a formula for $y(t)$, but the point is that *we don't need it*. Figure 3.3 shows the three possibilities for the solution curves to the *logistic equation* $y' = y - y^2$:

1. Curves above $y = 1$ go from $+\infty$ down toward the line $y = 1$ (**dropin curves**)
2. Curves between $y = 0$ and $y = 1$ go up toward that line $y = 1$ (**S-curves**)
3. Curves below $y = 0$ go down (fast) toward $y = -\infty$ (**dropoff curves**).

The solution curves go across all isoclines except the two zeroclines where $y - y^2 = 0$.

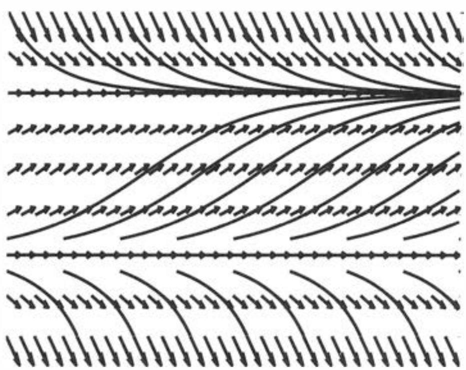


Figure 3.3: The arrows form a "direction field". Isoclines $y - y^2 = s$ attract or repel.

You see the S-curves between 0 and 1. The arrows are flat as they leave $y = 0$, steepest at $y = \frac{1}{2}$, flat again as they approach $y = 1$. The dropoff curves are below $y = 0$. Those arrows get very steep and the curves never reach $t = \infty$: $y = 1/(1 - e^{-t})$ gives $1/0 =$ *minus infinity* when $t = 0$. That dropoff curve never gets out of the third quadrant.

Important Solution curves have a special feature for autonomous equations $y' = f(y)$. Suppose the curve $y(t)$ is shifted right or left to the curve $Y(t) = y(t + C)$. Then $Y(t)$ solves the same equation $Y' = f(Y)$—both sides are just shifted in the same way.

Conclusion: The solution curves for autonomous equations $y' = f(y)$ just shift along *with no change in shape*. You can also see this by integrating $dy/f(y) = dt$ (separable equation). The right side integrates to $t + C$. We get all solutions by allowing all C.

In the logistic example, all S-curves and dropin curves and dropoff curves come from shifting *one* S-curve and *one* dropin curve and *one* dropoff curve.

Solution Curves Don't Meet

Is there a solution curve through every point (t, y)? Could two solution curves meet at that point? Could a solution curve suddenly end at a point? These "picture questions" are already answered by the facts.

At the start of this section, the functions f and $\partial f/\partial y$ were required to be continuous near $t = 0$, $y = y(0)$. Then there is a unique solution to $y' = f(t, y)$ with that start. In the picture this means: **There is exactly one solution curve going through the point.** The curve doesn't stop. By requiring f and $\partial f/\partial y$ to be continuous at and near *all* points, we guarantee one non-stopping solution curve through every point.

Example 3 will fail! The solution curves for $dy/dt = -t/y$ are half-circles and not whole circles. **They start and stop and meet on the line $\boldsymbol{y = 0}$** (where $f = -t/y$ is **not continuous**). Exactly one semicircular curve passes through every point with $y \neq 0$.

Example 3 $\boldsymbol{dy/dt = -t/y}$ is separable. Then $y\,dy = -t\,dt$ leads to $\boldsymbol{y^2 + t^2 = C}$.

Start again with pictures. The isocline $f(t, y) = -t/y = s$ is the line $y = (-1/s)t$. All those isoclines go through $(0, 0)$ which is a very singular point. In this example the direction arrows with slope s are perpendicular to the isoclines with slope $dy/dt = -1/s$.

The isoclines are rays out from $(0, 0)$. The arrow directions are perpendicular to those rays and tangent to the solution curves. **The curves are half-circles $\boldsymbol{y^2 + t^2 = C}$.** (There is another half-circle on the opposite side of the axis. So two solutions start from $y = 0$ at time $-T$ and go forward to $y = 0$ at time T.) The solution curves stop at $y = 0$, where the function $f = -t/y$ loses its continuity and the solution loses its life.

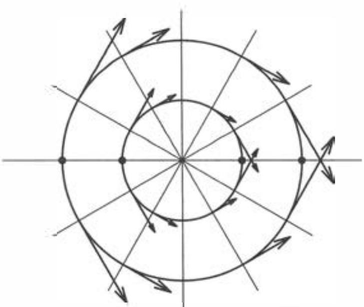


Figure 3.4: For $y' = -t/y$ the isoclines are rays. The solution curves are half-circles.

Example 4 $y' = 1 + t - y$ is linear but not separable. The isoclines trap the solution.

Trapping between isoclines is a neat part of the picture. It is based on the arrows. **All arrows go one way across an isocline, so all solution curves go that way**. Solutions that cross the isocline can't cross back. The zero isocline $f(t, y) = 1 + t - y = 0$ in Figure 3.5 is the line $y = t + 1$. Along that isocline the arrows have slope 0. The solution curves must cross from above to below.

The central isocline $1 + t - y = 1$ in Figure 3.5 is the $45°$ line $y = t$. This solves the differential equation! The arrow directions are exactly along the line: slope $s = 1$. Other solution curves could never touch this one.

The picture shows solution curves in a "lobster trap" between the lines: the curves can't escape. They are trapped between the line $y = t$ and every isocline $1 + t - y = s$ above or below it. The trap gets tighter and tighter as s increases from 0 to 1, and the isocline gets closer to $y = t$. *Conclusion from the picture*: **The solution $\boldsymbol{y(t)}$ must approach $\boldsymbol{t}$.**

This is a linear equation $y' + y = 1 + t$. The null solutions to $y' + y = 0$ are Ce^{-t}. The forcing term $1 + t$ is a polynomial. A particular solution comes by substituting $y_p(t) = at + b$ into the equation and solving for those undetermined coefficients a and b:

$$(at + b)' = 1 + t - (at + b) \qquad \boldsymbol{a = 1 \text{ and } b = 0} \qquad \boldsymbol{y = y_n + y_p = Ce^{-t} + t} \tag{2}$$

The solution curves $y = Ce^{-t} + t$ do approach the line $y = t$ asymptotically as $t \to \infty$.

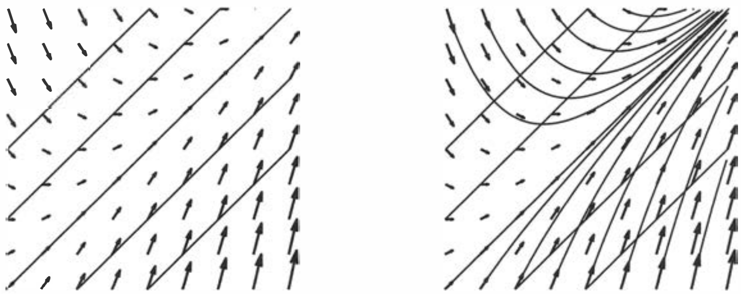


Figure 3.5: The solution curves for $y' = 1 + t - y$ get trapped between the $45°$ isoclines.

■ REVIEW OF THE KEY IDEAS ■

1. The direction field for $y' = f(t, y)$ has an arrow with slope f at each point t, y.
2. Along the isocline $f(t, y) = s$, all arrows have the same slope s.
3. The solution curves $y(t)$ are tangent to the arrows. One way through isoclines!
4. Fact: When f and $\partial f/\partial y$ are continuous, the curves cover the plane and don't meet.
5. The solution curves for autonomous $y' = f(y)$ shift left-right to $Y(t) = y(t - T)$.

Problem Set 3.1

1 (a) Why do two isoclines $f(t,y) = s_1$ and $f(t,y) = s_2$ never meet ?

(b) Along the isocline $f(t,y) = s$, what is the slope of all the arrows ?

(c) Then all solution curves go only one way across an _____ .

2 (a) Are isoclines $f(t,y) = s_1$ and $f(t,y) = s_2$ always parallel ? Always straight ?

(b) An isocline $f(t,y) = s$ is a solution curve when its slope equals _____ .

(c) The zerocline $f(t,y) = 0$ is a solution curve only when y is _____ : slope 0.

3 If $y_1(0) < y_2(0)$, what continuity of $f(t,y)$ assures that $y_1(t) < y_2(t)$ for all t ?

4 The equation $dy/dt = t/y$ is completely safe if $y(0) \neq 0$. Write the equation as $y\,dy = t\,dt$ and find its unique solution starting from $y(0) = -1$. The solution curves are hyperbolas—can you draw two on the same graph ?

5 The equation $dy/dt = y/t$ has many solutions $y = Ct$ in case $y(0) = 0$. It has no solution if $y(0) \neq 0$. When you look at all solution curves $y = Ct$, which points in the t, y plane have no curve passing through ?

6 For $y' = ty$ draw the isoclines $ty = 1$ and $ty = 2$ (those will be hyperbolas). On each isocline draw four arrows (they have slopes 1 and 2). Sketch pieces of solution curves that fit your picture between the isoclines.

7 The solutions to $y' = y$ are $y = Ce^t$. Changing C gives a higher or lower curve. But $y' = y$ is autonomous, its solution curves should be shifting right and left !

Draw $y = 2e^t$ and $y = -2e^t$ to show that they really are *right-left shifts* of $y = e^t$ and $y = -\,e^t$. The shifted solutions to $y' = y$ are e^{t+C} and $-\,e^{t+C}$.

8 For $y' = 1 - y^2$ the flat lines $y =$ constant are isoclines $1 - y^2 = s$. Draw the lines $y = 0$ and $y = 1$ and $y = -1$. On each line draw arrows with slope $1 - y^2$. The picture says that $y =$ _____ and $y =$ _____ are steady state solutions. From the arrows on $y = 0$, guess a shape for the solution curve $y = (e^t - e^{-t})/(e^t + e^{-t})$.

9 The parabola $y = t^2/4$ and the line $y = 0$ are both solution curves for $\boldsymbol{y' = \sqrt{|y|}}$. Those curves meet at the point $t = 0$, $y = 0$. What continuity requirement is failed by $f(y) = \sqrt{|y|}$, to allow more than one solution through that point ?

10 Suppose $y = 0$ up to time T is followed by the curve $y = (t-T)^2/4$. Does this solve $\boldsymbol{y' = \sqrt{|y|}}$? Draw this $y(t)$ going through flat isoclines $\sqrt{|y|} = 1$ and 2.

11 The equation $\boldsymbol{y' = y^2 - t}$ is often a favorite in MIT's course 18.03 : not too easy. Why do solutions $y(t)$ rise to their maximum on $y^2 = t$ and then descend ?

12 Construct $f(t,y)$ with two isoclines so solution curves go *up* through the higher isocline and other solution curves go *down* through the lower isocline. *True or false* : Some solution curve will stay between those isoclines : **A continental divide**.

3.2 Sources, Sinks, Saddles, and Spirals

The pictures in this section show solutions to $Ay'' + By' + Cy = 0$. These are linear equations with constant coefficients A, B, and C. The graphs show solutions y on the horizontal axis and their slopes $y' = dy/dt$ on the vertical axis. These pairs $(y(t), y'(t))$ depend on time, *but time is not in the pictures*. The paths show *where* the solution goes, but they don't show when.

Each specific solution starts at a particular point $(y(0), y'(0))$ given by the initial conditions. The point moves along its path as the time t moves forward from $t = 0$. We know that the solutions to $Ay'' + By' + Cy = 0$ depend on the two solutions to $As^2 + Bs + C = 0$ (an ordinary quadratic equation for s). When we find the roots s_1 and s_2, we have found all possible solutions:

$$y = c_1 e^{s_1 t} + c_2 e^{s_2 t} \qquad y' = c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t} \tag{1}$$

The numbers s_1 and s_2 tell us which picture we are in. Then the numbers c_1 and c_2 tell us which path we are on.

Since s_1 and s_2 determine the picture for each equation, it is essential to see the six possibilities. We write all six here in one place, to compare them. Later they will appear in six different places, one with each figure. The first three have real solutions s_1 and s_2. The last three have complex pairs $\boldsymbol{s = a \pm i\omega}$.

Sources	**Sinks**	**Saddles**	**Spiral out**	**Spiral in**	**Center**
$s_1 > s_2 > 0$	$s_1 < s_2 < 0$	$s_2 < 0 < s_1$	$a = \text{Re } s > 0$	$a = \text{Re } s < 0$	$a = \text{Re } s = 0$

In addition to those six, there will be limiting cases $s = 0$ and $s_1 = s_2$ (as in resonance).

Stability This word is important for differential equations. *Do solutions decay to zero*? The solutions are controlled by $e^{s_1 t}$ and $e^{s_2 t}$ (and in Chapter 6 by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$). We can identify the two pictures (out of six) that are displaying full stability: the sinks. **A center $\boldsymbol{s = \pm i\omega}$ is at the edge of stability** ($e^{i\omega t}$ is neither decaying or growing).

2. Sinks are stable	$s_1 < s_2 < 0$	**Then** $\boldsymbol{y(t) \to 0}$
5. Spiral sinks are stable	$\text{Re } s_1 = \text{Re } s_2 < 0$	**Then** $\boldsymbol{y(t) \to 0}$

Special note. May I mention here that the same six pictures also apply to a system of *two first order equations*. Instead of y and y', the equations have unknowns y_1 and y_2. Instead of the constant coefficients A, B, C, the equations will have a 2 by 2 matrix. Instead of the roots s_1 and s_2, that matrix will have eigenvalues λ_1 and λ_2. **Those eigenvalues are the roots of an equation $\boldsymbol{A\lambda^2 + B\lambda + C = 0}$**, just like s_1 and s_2.

We will see the same six possibilities for the λ's, and the same six pictures. The eigenvalues of the 2 by 2 matrix give the growth rates or decay rates, in place of s_1 and s_2.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ has solutions } \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}.$$

The eigenvalue is λ and the eigenvector is $\boldsymbol{v} = (v_1, v_2)$. The solution is $\boldsymbol{y(t) = ve^{\lambda t}}$.

The First Three Pictures

We are starting with the case of *real roots* s_1 and s_2. In the equation $Ay'' + By' + Cy = 0$, this means that $\boldsymbol{B^2 \geq 4AC}$. Then B is relatively large. The square root in the quadratic formula produces a real number $\sqrt{B^2 - 4AC}$. If A, B, C have the same sign, we have overdamping and **negative roots** and stability. The solutions decay to $(0,0)$: a **sink**.

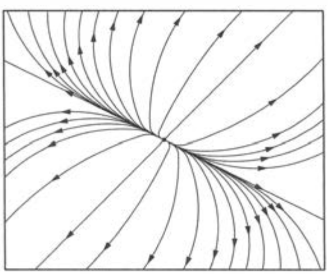
If A and C have opposite sign to B as in $y'' - 3y' + 2y = 0$, we have negative damping and **positive roots** s_1, s_2. The solutions grow (this is instability : a **source** at $(0,0)$).

Suppose A and C have different signs, as in $y'' - 3y' - 2y = 0$. Then s_1 and s_2 also have **different signs** and the picture shows a **saddle**. The moving point $(y(t), y'(t))$ can start in toward $(0,0)$ before it turns out to infinity. The positive s gives $e^{st} \to \infty$. *Second example for a saddle*: $y'' - 4y = 0$ leads to $s^2 - 4 = (s-2)(s+2) = 0$. The roots $s_1 = \mathbf{2}$ and $s_2 = \mathbf{-2}$ have opposite signs. Solutions $c_1 e^{2t} + c_2 e^{-2t}$ grow unless $c_1 = 0$. Only that one line with $c_1 = 0$ has arrows inward.

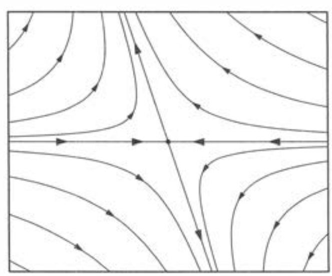
In every case with $B^2 \geq 4AC$, the roots are real. The solutions $y(t)$ have growing exponentials or decaying exponentials. We don't see sines and cosines and oscillation.

The first figure shows growth : $0 < s_2 < s_1$. Since $e^{s_1 t}$ grows faster than $e^{s_2 t}$, the larger number s_1 will dominate. The solution path for (y, y') will approach the straight line of slope s_1. That is because the ratio of $y' = c_1 s_1 e^{s_1 t}$ to $y = c_1 e^{s_1 t}$ is exactly s_1.

If the initial condition is on the "s_1 line" then the solution (y, y') stays on that line: $c_2 = 0$. If the initial condition is exactly on the "s_2 line" then the solution stays on that secondary line : $c_1 = 0$. You can see that if $c_1 \neq 0$, the $c_1 e^{s_1 t}$ part takes over as $t \to \infty$.



Reverse all the arrows in the left figure. Paths go in toward $\mathbf{(0,0)}$



$\boldsymbol{0 < s_2 < s_1}$ **Source : Unstable**

$\boldsymbol{s_1 < s_2 < 0}$ **Sink : Stable**

$\boldsymbol{s_2 < 0 < s_1}$ **Saddle : Unstable**

Figure 3.6: **Real roots $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$.** The paths of the point $(y(t), y'(t))$ lead out when roots are positive and lead in when roots are negative. With $s_2 < 0 < s_1$, the s_2-line leads in but all other paths eventually go out near the s_1-line: *The picture shows a saddle point.*

Example for a source: $y'' - 3y' + 2y = 0$ leads to $s^2 - 3s + 2 = (s-2)(s-1) = 0$. The roots **1** and **2** are positive. The solutions grow and e^{2t} dominates.

Example for a sink: $y'' + 3y' + 2y = 0$ leads to $s^2 + 3s + 2 = (s+2)(s+1) = 0$. The roots $\mathbf{-2}$ and $\mathbf{-1}$ are negative. The solutions decay and e^{-t} dominates.

The Second Three Pictures

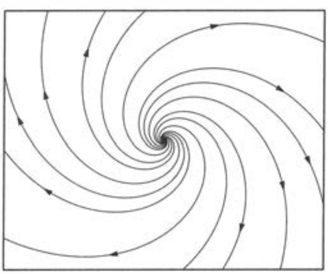
We move to the case of **complex roots** s_1 and s_2. In the equation $Ay'' + By' + Cy = 0$, this means that $B^2 < 4AC$. Then A and C have the same signs and B is relatively small (underdamping). The square root in the quadratic formula (2) is an imaginary number. *The exponents s_1 and s_2 are now a complex pair $a \pm i\omega$:*

Complex roots of
$\boldsymbol{As^2 + Bs + C = 0}$

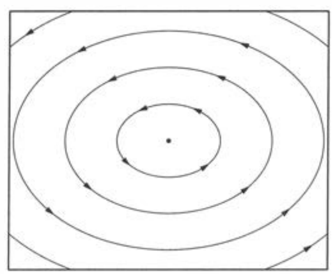
$$s_1, s_2 = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} = a \pm i\omega. \qquad (2)$$

The path of (y, y') **spirals around the center**. Because of e^{at}, the spiral goes out if $a > 0$: **spiral source**. Solutions spiral in if $a < 0$: **spiral sink**. The frequency ω controls how fast the solutions oscillate and how quickly the spirals go around $(0,0)$.

In case $a = -B/2A$ is zero (no damping), we have a **center** at (0, 0). The only terms left in y are $e^{i\omega t}$ and $e^{-i\omega t}$, in other words $\cos\omega t$ and $\sin\omega t$. Those paths are ellipses in the last part of Figure 3.7. The solutions $y(t)$ are **periodic**, because increasing t by $2\pi/\omega$ will not change cos ωt and sin ωt. That circling time $2\pi/\omega$ is the **period**.



Reverse all the arrows in the left figure. Paths go in toward $\mathbf{(0,0)}$.



$\boldsymbol{a = \text{Re}\, s > 0}$
Spiral source : Unstable

$\boldsymbol{a = \text{Re}\, s < 0}$
Spiral sink : Stable

$\boldsymbol{a = \text{Re}\, s = 0}$
Center : Neutrally stable

Figure 3.7: **Complex roots s_1 and s_2.** The paths go once around $(0,0)$ when t increases by $2\pi/\omega$. The paths spiral in when A and B have the same signs and $a = -B/2A$ is negative. They spiral out when a is positive. If $B = 0$ (no damping) and $4AC > 0$, we have a center. The simplest center is $y = \sin t, y' = \cos t$ (circle) from $y'' + y = 0$.

First Order Equations for y_1 and y_2

On the first page of this section, a "Special Note" mentioned another application of the same pictures. Instead of graphing the path of $(y(t), y'(t))$ for one second order equation, we could follow the path of $(y_1(t), y_2(t))$ for **two first order equations**. The two equations look like this:

$$\textbf{First order system } \boldsymbol{y}' = A\boldsymbol{y} \qquad \begin{aligned} dy_1/dt &= ay_1 + by_2 \\ dy_2/dt &= cy_1 + dy_2 \end{aligned} \tag{3}$$

The starting values $y_1(0)$ and $y_2(0)$ are given. The point (y_1, y_2) will move along a path in one of the six figures, depending on the numbers a, b, c, d.

Looking ahead, those four numbers will go into a 2 by 2 matrix A. Equation (3) will become $d\boldsymbol{y}/dt = A\boldsymbol{y}$. The symbol $\boldsymbol{y}$ in boldface stands for the vector $\boldsymbol{y} = (y_1, y_2)$. And most important for the six figures, *the exponents s_1 and s_2 in the solution $\boldsymbol{y}(t)$ will be the* **eigenvalues $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$** of the matrix A.

Companion Matrices

Here is the connection between a second order equation and two first order equations. All equations on this page are linear and all coefficients are constant. I just want you to see the special "*companion matrix*" that appears in the first order equations $\boldsymbol{y}' = A\boldsymbol{y}$.

Notice that $\boldsymbol{y}$ is printed in **boldface type** because it is a **vector**. It has two components y_1 and y_2 (those are in lightface type). The first y_1 is the same as the unknown y in the second order equation. The second component y_2 is the velocity dy/dt:

$$\begin{matrix} y_1 = y \\ y_2 = y' \end{matrix} \qquad \boldsymbol{y'' + 4y' + 3y} = 0 \quad \text{becomes} \quad \boldsymbol{y_2' + 4y_2 + 3y_1} = 0. \tag{4}$$

On the right you see one of the first order equations connecting y_1 and y_2. We need a second equation (two equations for two unknowns). *It is hiding at the far left!* There you see that $\boldsymbol{y_1' = y_2}$. In the original second order problem this is the trivial statement $y' = y'$. In the vector form $\boldsymbol{y}' = A\boldsymbol{y}$ it gives the first equation in our system. The first row of our matrix is **0 1**. When y and y' become y_1 and y_2,

$$y'' + 4y' + 3y = 0 \quad \text{becomes} \quad \begin{aligned} y_1' &= \phantom{-3y_1 - {}} y_2 \\ y_2' &= -3y_1 - 4y_2 \end{aligned} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-3} & \mathbf{-4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{5}$$

That first row **0 1** makes this a 2 by 2 **companion matrix**. It is the companion to the second order equation. The key point is that the first order and second order problems have the same "characteristic equation" because they are the same problem.

The equation $s^2 + 4s + 3 = 0$ gives the exponents $s_1 = \mathbf{-3}$ and $s_2 = \mathbf{-1}$

The equation $\lambda^2 + 4\lambda + 3 = 0$ gives the eigenvalues $\lambda_1 = \mathbf{-3}$ and $\lambda_2 = \mathbf{-1}$

The problems are the same, the exponents -3 and -1 are the same, the figures will be the same. Those figures show a **sink** because -3 and -1 are real and both negative. Solutions approach $(0,0)$. These equations are **stable**.

$$\textbf{The companion matrix for } \boldsymbol{y'' + By' + Cy = 0} \textbf{ is } \boldsymbol{A = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}}.$$

Row 1 of $y' = Ay$ is $y_1' = y_2$. Row 2 is $y_2' = -Cy_1 - By_2$. When you replace y_2 by y_1', this means that $y_1'' + By_1' + Cy_1 = 0$: *correct.*

Stability for 2 by 2 Matrices

I can explain when a 2 by 2 system $\boldsymbol{y}' = A\boldsymbol{y}$ is stable. This requires that all solutions $\boldsymbol{y}(t) = (y_1(t), y_2(t))$ approach zero as $t \to \infty$. When the matrix A is a companion matrix, this 2 by 2 system comes from one second order equation $y'' + By' + Cy = 0$. In that case we know that stability depends on the roots of $s^2 + Bs + C = 0$. **Companion matrices are stable when $B > 0$ and $C > 0$.**

From the quadratic formula, the roots have $s_1 + s_2 = -B$ and $s_1 s_2 = C$.
If s_1 and s_2 are negative, this means that $B > 0$ and $C > 0$.
If $s_1 = a + i\omega$ and $s_2 = a - i\omega$ and $a < 0$, this again means $B > 0$ and $C > 0$

Those complex roots add to $s_1 + s_2 = 2a$. Negative a (stability) means positive B, since $s_1 + s_2 = -B$. Those roots multiply to $s_1 s_2 = a^2 + \omega^2$. This means that C is positive, since $s_1 s_2 = C$.

For companion matrices, stability is decided by $B > 0$ and $C > 0$. **What is the stability test for any 2 by 2 matrix** ? This is the key question, and Chapter 6 will answer it properly. We will find the equation for the eigenvalues of any matrix (Section 6.1). We will test those eigenvalues for stability (Section 6.4). Eigenvalues and eigenvectors are a major topic, the most important link between differential equations and linear algebra. Fortunately, the eigenvalues of 2 by 2 matrices are especially simple.

$$\text{The eigenvalues of the matrix } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ have } \lambda^2 - T\lambda + D = 0.$$

The number T is $a + d$. The number D is $ad - bc$.

Companion matrices have $a = 0$ and $b = 1$ and $c = -C$ and $d = -B$. Then the characteristic equation $\lambda^2 - T\lambda + D = 0$ is exactly $s^2 + Bs + C = 0$.

$$\text{Companion matrices have } \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \quad T = a + d = -B \text{ and } D = ad - bc = C.$$

The stability test $B > 0$ and $C > 0$ is turning into the stability test $T < 0$ and $D > 0$.

This is the test for any 2 by 2 matrix. Stability requires $T < 0$ and $D > 0$. Let me give four examples and then collect together the main facts about stability.

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \text{ is } \textit{unstable} \text{ because } T = 0+3 \text{ is positive}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \text{ is } \textit{unstable} \text{ because } D = -(1)(2) \text{ is negative}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ is } \textit{stable} \text{ because } T = -3 \text{ and } D = +2$$

$$A_4 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \text{ is } \textit{stable} \text{ because } T = -1-1 \text{ is negative and } D = 1+1 \text{ is positive}$$

The eigenvalues always come from $\lambda^2 - T\lambda + D = 0$. For that last matrix A_4, this eigenvalue equation is $\lambda^2 + 2\lambda + 2 = 0$. The eigenvalues are $\lambda_1 = \mathbf{-1+i}$ and $\lambda_2 = \mathbf{-1-i}$. They add to $T = -2$ and they multiply to $D = +2$. **This is a spiral sink and it is stable**.

Stability for 2 by 2 matrices	$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is stable if $\begin{matrix} T = a + d < 0 \\ D = ad - bc > 0 \end{matrix}$

The six pictures for (y, y') become six pictures for (y_1, y_2). The first three pictures have real eigenvalues from $T^2 \geq 4D$. The second three pictures have complex eigenvalues from $T^2 < 4D$. This corresponds perfectly to the tests for $y'' + By' + Cy = 0$ and its companion matrix :

Real eigenvalues	$T^2 \geq 4D$	$B^2 \geq 4C$	Overdamping
Complex eigenvalues	$T^2 < 4D$	$B^2 < 4C$	Underdamping

That gives one picture of eigenvalues λ: *Real or complex*. The second picture is different : *Stable or unstable*. Both of those splittings are decided by T and D (or $-B$ and C).

1.	**Source**	$T > 0,\ D > 0,\ T^2 \geq 4D$	*Ustable*
2.	**Sink**	$T < 0,\ D > 0,\ T^2 \geq 4D$	*Stable*
3.	**Saddle**	$D < 0$ and $T^2 \geq 4D$	*Unstable*
4.	**Spiral source**	$T > 0,\ D > 0,\ T^2 < 4D$	*Unstable*
5.	**Spiral Sink**	$T < 0,\ D > 0,\ T^2 < 4D$	*Stable*
6.	**Center**	$T = 0,\ D > 0,\ T^2 < 4D$	*Neutral*

That neutrally stable center has eigenvalues $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ and undamped oscillation.

Section 3.3 will use this information to decide the stability of *nonlinear* equations.

Eigenvectors of Companion Matrices

Eigenvalues of A come with eigenvectors. If we stay a little longer with a companion matrix, we can see its eigenvectors. Chapter 6 will develop these ideas for any matrix, and we need more linear algebra to understand them properly. But our vectors (y_1, y_2) come from (y, y') in a differential equation, and that connection makes the eigenvectors of a companion matrix especially simple.

The fundamental idea for constant coefficient linear equations is always the same: **Look for exponential solutions.** For a second order equation those solutions are $y = e^{st}$. For a system of two first order equations those solutions are $\boldsymbol{y} = \boldsymbol{v}e^{\lambda t}$. **The vector $\boldsymbol{v} = (v_1, v_2)$ is the eigenvector that goes with the eigenvalue $\boldsymbol{\lambda}$.**

Substitute $\begin{aligned} y_1 &= v_1 e^{\lambda t} \\ y_2 &= v_2 e^{\lambda t} \end{aligned}$ into the equations $\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned}$ and factor out $e^{\lambda t}$.

Because $e^{\lambda t}$ is the same for both y_1 and y_2, it will appear in every term. When all factors $e^{\lambda t}$ are removed, we will see the equations for v_1 and v_2. That vector $\boldsymbol{v} = (v_1, v_2)$ will satisfy the eigenvector equation $A\boldsymbol{v} = \lambda \boldsymbol{v}$. This is the key to Chapter 6.

Here I only look at eigenvectors for companion matrices, because v has a specially nice form. The equations are $y_1' = y_2$ and $y_2' = -Cy_1 - By_2$.

Substitute $\begin{aligned} y_1 &= v_1 e^{\lambda t} \\ y_2 &= v_2 e^{\lambda t} \end{aligned}$ Then $\begin{aligned} \lambda v_1 e^{\lambda t} &= v_2 e^{\lambda t} \\ \lambda v_2 e^{\lambda t} &= -Cv_1 e^{\lambda t} - Bv_2 e^{\lambda t}. \end{aligned}$

Cancel every $e^{\lambda t}$. The first equation becomes $\lambda v_1 = v_2$. This is our answer:

$$\text{Eigenvectors of companion matrices are multiples of the vector } \boldsymbol{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

■ REVIEW OF THE KEY IDEAS ■

1. If $B^2 \neq 4AC \neq 0$, six pictures show the paths of (y, y') for $Ay'' + By' + Cy = 0$.

2. Real solutions to $As^2 + Bs + C = 0$ lead to sources and sinks and saddles at $(0, 0)$.

3. Complex roots $s = a \pm i\omega$ give spirals around $(0, 0)$ (or closed loops if $a = 0$).

4. Roots s become eigenvalues λ for $\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$. Same six pictures.

Problem Set 3.2

1 Draw Figure 3.6 for a sink (the missing middle figure) with $y = c_1e^{-2t} + c_2e^{-t}$. Which term dominates as $t \to \infty$? The paths approach the dominating line as they go in toward zero. **The slopes of the lines are -2 and -1** (the numbers s_1 and s_2).

2 Draw Figure 3.7 for a spiral sink (the missing middle figure) with roots $s = -1 \pm i$. The solutions are $y = C_1e^{-t}\cos t + C_2e^{-t}\sin t$. They approach zero because of the factor e^{-t}. They spiral around the origin because of $\cos t$ and $\sin t$.

3 Which path does the solution take in Figure 3.6 if $y = e^t + e^{t/2}$? Draw the curve $(y(t), y'(t))$ more carefully starting at $t = 0$ where $(y, y') = (2, 1.5)$.

4 Which path does the solution take around the saddle in Figure 3.6 if $y = e^{t/2} + e^{-t}$? Draw the curve more carefully starting at $t = 0$ where $(y, y') = (2, -\frac{1}{2})$.

5 Redraw the first part of Figure 3.6 when the roots are equal: $s_1 = s_2 = 1$ and $y = c_1e^t + c_2te^t$. *There is no s_2-line.* Sketch the path for $y = e^t + te^t$.

6 The solution $y = e^{2t} - 4e^t$ gives a source (Figure 3.6), with $y' = 2e^{2t} - 4e^t$. Starting at $t = 0$ with $(y, y') = (-3, -2)$, where is (y, y') when $e^t = 1.1$ and $e^t = .25$ and $e^t = 2$?

7 The solution $y = e^t(\cos t + \sin t)$ has $y' = 2e^t\cos t$. This spirals out because of e^t. Plot the points (y, y') at $t = 0$ and $t = \pi/2$ and $t = \pi$, and try to connect them with a spiral. Note that $e^{\pi/2} \approx 4.8$ and $e^{\pi} \approx 23$.

8 The roots s_1 and s_2 are $\pm 2i$ when the differential equation is _____. Starting from $y(0) = 1$ and $y'(0) = 0$, draw the path of $(y(t), y'(t))$ around the center. Mark the points when $t = \pi/2, \pi, 3\pi/2, 2\pi$. Does the path go clockwise?

9 The equation $y'' + By' + y = 0$ leads to $s^2 + Bs + 1 = 0$. For $B = -3, -2, -1, 0, 1, 2, 3$ decide which of the six figures is involved. For $B = -2$ and 2, why do we not have a perfect match with the source and sink figures?

10 For $y'' + y' + Cy = 0$ with damping $B = 1$, the characteristic equation will be $s^2 + s + C = 0$. Which C gives the changeover from a *sink* (overdamping) to a spiral *sink* (underdamping)? Which figure has $C < 0$?

Problems 11–18 are about $dy/dt = Ay$ with companion matrices $\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}$.

11 The eigenvalue equation is $\boldsymbol{\lambda^2 + B\lambda + C = 0}$. Which values of B and C give complex eigenvalues? Which values of B and C give $\lambda_1 = \lambda_2$?

12 Find λ_1 and λ_2 if $B = 8$ and $C = 7$. Which eigenvalue is more important as $t \to \infty$? Is this a sink or a saddle?

13 Why do the eigenvalues have $\lambda_1 + \lambda_2 = -B$? Why is $\lambda_1 \lambda_2 = C$?

14 Which second order equations did these matrices come from?

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (saddle)} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ (center)}$$

15 The equation $y'' = 4y$ produces a saddle point at $(0,0)$. Find $s_1 > 0$ and $s_2 < 0$ in the solution $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. If $c_1 c_2 \neq 0$, this solution will be (large) (small) as $t \to \infty$ and also as $t \to -\infty$.

The only way to go toward the saddle $(y, y') = (0,0)$ as $t \to \infty$ is $c_1 = 0$.

16 If $B = 5$ and $C = 6$ the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. The vectors $\boldsymbol{v} = (1,3)$ and $\boldsymbol{v} = (1,2)$ are *eigenvectors* of the matrix A: Multiply $A\boldsymbol{v}$ to get $3\boldsymbol{v}$ and $2\boldsymbol{v}$.

17 In Problem 16, write the two solutions $\boldsymbol{y} = \boldsymbol{v}e^{\lambda t}$ to the equations $\boldsymbol{y}' = A\boldsymbol{y}$. Write the complete solution as a combination of those two solutions.

18 The eigenvectors of a companion matrix have the form $v = (1, \lambda)$. Multiply by A to show that $Av = \lambda v$ gives one trivial equation and the characteristic equation $\lambda^2 + B\lambda + C = 0$.

$$\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \quad \text{is} \quad \begin{aligned} \lambda &= \lambda \\ -C - B\lambda &= \lambda^2 \end{aligned}$$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

19 An equation is stable and all its solutions $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$ go to $y(\infty) = 0$ exactly when

$(s_1 < 0$ **or** $s_2 < 0)$ $\qquad$ $(s_1 < 0$ **and** $s_2 < 0)$ $\qquad$ $(\text{Re } s_1 < 0$ **and** $\text{Re } s_2 < 0)$?

20 If $Ay'' + By' + Cy = D$ is stable, what is $y(\infty)$?

3.3 Linearization and Stability in 2D and 3D

The logistic equation $y' = y - y^2$ has two steady states $Y = 0$ and $Y = 1$. Those are **critical points**, where the function $f(y) = y - y^2$ is zero. Along the lines $Y = 0$ and $Y = 1$ the equation $y' = f(y)$ becomes $0 = 0$. We have those two steady solutions, and their stability or instability is important. Do nearby solutions approach Y or not?

The stability test requires $df/dy < 0$ at Y. This is the slope of the tangent to $f(y)$:

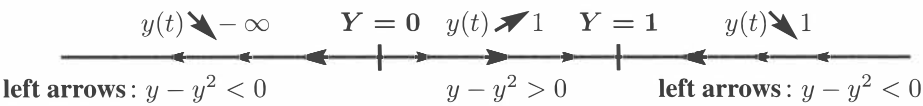
$$\boldsymbol{f(y - Y)} \approx f(Y) + \left(\frac{df}{dy}\right)(y - Y) = \mathbf{0} + \boldsymbol{A(y - Y)}. \tag{1}$$

The linearization of $y' = f(y)$ at the critical point $y = Y$ comes from $f \approx A(y - Y)$. Replace f by this linear part and include the constant Y on the left side too:

Linearized equation near a critical point Y	$\boldsymbol{(y - Y)' = A(y - Y)}.$	(2)

The solution $y - Y = Ce^{At}$ grows if $A > 0$ (instability). The solution decays if $A < 0$. The logistic equation has $f(y) = y - y^2$ with derivative $A = 1 - 2y$. At the steady state $Y = 0$ this shows instability $(A = +1)$. The other critical point $Y = 1$ is stable $(A = -1)$.

The **stability line** or **phase line** in Section 1.7 showed $Y = 1$ as the attractor:



The arrows in Section 3.1 had slopes $f(t, y)$. Stability is decided by the slope df/dy.

Note The most basic example is $y' = y$. The only steady state solution is $Y = 0$. That must be unstable, because $f = y$ has $A = df/dy = 1$. All other solutions $y(t) = Ce^t$ travel far away from $Y = 0$, even when $C = y(0)$ is close to zero.

Opposite case: $y' = 6 - y$ is stable $(A = -1)$. Solutions approach $Y = y_\infty = 6$.

Solution Curves in the yz Plane

Those paragraphs were review for one unknown $y(t)$. Section 3.2 had two unknowns y and z in two linear first order equations (or y and y' in a linear second order equation).

Move now to nonlinear. The equations will be **autonomous**, the same at all times t:

$\dfrac{\boldsymbol{dy}}{\boldsymbol{dt}} = \boldsymbol{f(y, z)}$ **and** $\dfrac{\boldsymbol{dz}}{\boldsymbol{dt}} = \boldsymbol{g(y, z)}$	**starting from $y(0)$ and $z(0)$.**	(3)

A **critical point** Y, Z solves $f(Y, Z) = 0$ and $g(Y, Z) = 0$. It is a steady solution: constant $y = Y$ and constant $z = Z$.

Critical point $\quad \boldsymbol{f(Y,Z)=0}$ and $\boldsymbol{g(Y,Z)=0}$ (4)

For every critical point Y, Z we must decide: **stable or unstable or neutral**?

To graph the solutions, there is a problem with y and z and t. Three variables won't fit into a 2D picture. Our solution curves for autonomous equations will omit t. The curves $y(t)$, $z(t)$ show the paths of solutions in the y, z plane but not the times along those paths.

Those pictures do not show the time t, as the solution moves. Different equations $dy/dt = cf(y,z)$ and $dz/dt = cg(y,z)$ will produce the same picture for all $c \neq 0$. That constant c just rescales the time and the speed along the same path $y(ct), z(ct)$. Time and speed are not shown by the pictures.

Each steady state $y(t) = Y, z(t) = Z$ will be one point in the picture! The stability question is whether paths near that point (those are nearby solutions) go in toward Y, Z or away from Y, Z or around Y, Z: stable or unstable or neutrally stable.

That stability question is answered by the eigenvalues of a 2 by 2 matrix A.

Solutions Near a Critical Point

Here is the key to this section. **Very close to a critical point where $\boldsymbol{f(Y,Z)=0}$ and $\boldsymbol{g(Y,Z)=0}$, solution curves have the same six possibilities that we already know**:

Stable	Sink	**Unstable**	Source
	Spiral sink		Spiral source
Neutral	Center		Saddle point

The pictures for linear equations were in Section 3.2. They came from six possibilities for the roots of $As^2 + Bs + C = 0$, and from six types of 2 by 2 matrices A:

Linear equations
Constant coefficients

$$\begin{aligned} y' &= ay + bz \\ z' &= cy + dz \end{aligned} \qquad \begin{bmatrix} y \\ z \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{5}$$

Those model problems in 2D have the critical point $Y = 0, Z = 0$. That is the point where $f(y,z) = ay + bz = 0$ and $g(y,z) = cy + dz = 0$. There is one critical point $(0,0)$ at the center of each picture in Section 3.2. Now we are saying that **nonlinear equations look like linear equations when you look near each critical point**.

This is the 2D equivalent of one equation $(y - Y)' = A(y - Y)$. That number A was df/dy. Now we have two unknowns y and z, and two functions $f(y,z)$ and $g(y,z)$. **There are four partial derivatives of $\boldsymbol{f}$ and $\boldsymbol{g}$, and they go into the 2 by 2 matrix $\boldsymbol{A}$**:

First derivative matrix
"Jacobian matrix"

$$A = \begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} \tag{6}$$

Linearization of a Nonlinear Equation

For one equation, linearization was based on the tangent line. The beginning of the Taylor series around Y is $f(Y) + (df/dy)(y - Y)$. Critical points have $f(Y) = 0$, removing the constant term. Two variables y and z lead to the same idea, but now it is a tangent plane:

$$\begin{aligned} f(y,z) &\approx f(Y,Z) + \left(\frac{\partial f}{\partial y}\right)(y-Y) + \left(\frac{\partial f}{\partial z}\right)(z-Z) \\ g(y,z) &\approx g(Y,Z) + \left(\frac{\partial g}{\partial y}\right)(y-Y) + \left(\frac{\partial g}{\partial z}\right)(z-Z) \end{aligned} \tag{7}$$

A critical point has $f(Y,Z) = g(Y,Z) = 0$. The four linear terms take over:

$$\boxed{\begin{bmatrix} (y-Y)' \\ (z-Z)' \end{bmatrix} \approx \begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} \begin{bmatrix} y-Y \\ z-Z \end{bmatrix} = A \begin{bmatrix} y-Y \\ z-Z \end{bmatrix}.} \tag{8}$$

There stands the linearized equation. It is centered and linearized around the special point (Y, Z). If we reset by shifting (Y, Z) to $(0, 0)$, equation (8) is one of our model problems:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = A \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \tag{9}$$

Example 1 Linearize $y' = \sin(ay + bz)$ and $z' = \sin(cy + dz)$ at $Y = 0$, $Z = 0$.

Solution Check first: $f = \sin(ay + bz)$ and $g = \sin(cy + dz)$ are zero at $(Y, Z) = (0, 0)$. *This is a critical point.* The first derivatives of f and g at that point go into A.

$$\partial f/\partial y = a\cos(ay + bz) = a\cos 0 = a \text{ when } (y, z) = (0, 0)$$

The other three partial derivatives give b and c and d. They enter the matrix A:

$$\begin{aligned} y' &= \sin(ay+bz) \\ z' &= \sin(cy+dz) \end{aligned} \quad \text{linearizes to} \quad \begin{aligned} y' &= ay + bz \\ z' &= cy + dz \end{aligned} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \tag{10}$$

That example just moved the simple linearization $\sin x \approx x$ into two variables.

Example 2 **(Predator-Prey) Linearize** $\begin{aligned} y' &= y - yz \\ z' &= yz - z \end{aligned}$ **at all critical points.**

Meaning of these predator-prey equations The prey y is like rabbits, the predator z is like foxes. On their own with no foxes, the rabbits grow by nibbling grass: $y' = y$. On their own with no rabbits, the foxes don't eat well and $z' = -z$. Then the multiplication yz accounts for the interactions between y rabbits and z foxes. Those interactions end up in more foxes and fewer rabbits.

This example has simplified coefficients 1 and -1 multiplying y and z and yz. The predator-prey model is a great example and we will develop it further.

Linearize Predator – Prey at Critical Points

Set $f = Y - YZ = 0$ and also $g = YZ - Z = 0$. Solve for all critical points Y, Z.

$$\boldsymbol{Y - YZ = Y(1 - Z) = 0} \qquad \text{and} \qquad \boldsymbol{YZ - Z = (Y - 1)Z = 0}.$$

The critical points Y, Z are $0, 0$ and $1, 1$. Track their stability using the matrix A.

$$\textbf{At } \boldsymbol{Y, Z = 0, 0} \qquad A = \begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} \boldsymbol{1 - Z} & \boldsymbol{-Y} \\ \boldsymbol{Z} & \boldsymbol{Y - 1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{1} & 0 \\ 0 & \boldsymbol{-1} \end{bmatrix}.$$

This is a **saddle point**: **unstable**. Starting near $0, 0$ the rabbit population $y(t)$ will grow. The eigenvalues are 1 (for the rabbits) and -1 (for the foxes) from $\boldsymbol{y' = y}$ and $\boldsymbol{z' = -z}$. An all-fox population would decay (this is the only path in to the saddle point).

$$\textbf{At } \boldsymbol{Y, Z = 1, 1} \qquad A = \begin{bmatrix} 1 - Z & -Y \\ Z & Y - 1 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{-1} \\ \boldsymbol{1} & 0 \end{bmatrix}.$$

This matrix has imaginary eigenvalues $\lambda_1 = \boldsymbol{i}$ and $\lambda_2 = \boldsymbol{-i}$. Their real parts are **zero**. The stability is **neutral**. **The critical point $\boldsymbol{Y = 1, Z = 1}$ is a center**. A solution that starts near that point will go around $1, 1$ and return where it started:

Extra rabbits $\longrightarrow$ Foxes increase $\longrightarrow$ Rabbits decrease $\longrightarrow$ Foxes decrease $\longrightarrow$ **Extra rabbits**

We can see without eigenvalues that the solution to the linearized equations makes a perfect circle around $(1, 1)$. The matrix A has -1 in row 1 and $+1$ in row 2.

$$\begin{aligned} (y-1)' &= -\,(z-1) \\ (z-1)' &= +\,(y-1) \end{aligned} \qquad \text{is solved by} \qquad \begin{aligned} y - 1 &= r\cos t \\ z - 1 &= r\sin t \end{aligned} \tag{11}$$

The actual nonlinear solution $y(t)$, $z(t)$ won't make a perfect circle. Usually we can't find its exact path, but in this case we can. The $y - z$ equation is separable and solvable:

$$\frac{dy}{dz} = \frac{dy/dt}{dz/dt} = \frac{f}{g} = \frac{y(1-z)}{(y-1)z} \quad \text{separates into} \quad \boldsymbol{\frac{y-1}{y}\,dy = \frac{1-z}{z}\,dz}. \tag{12}$$

Integration of 1 and $1/y$ and $1/z$ gives $\boldsymbol{y - \ln y = \ln z - z + C}$. That constant is $C = 2$ when $y = z = 1$ (critical). These solution curves are drawn in Figure 3.8 for $C = 2.1, 2.2, 2.3, 2.4$. They are nearly circular near $C = 2$. That is linearization!

As C increases, y and z move further away from 1 and the circles are lost. But the nonlinear solution is still **periodic**. The rabbit-fox population comes back to its starting point and goes around again. Populations can be close to cyclic.

Equation (12) took time out of the picture. A numerical solution (Euler or Runge-Kutta) puts time back. This famous model came from Lotka and Volterra in 1925.

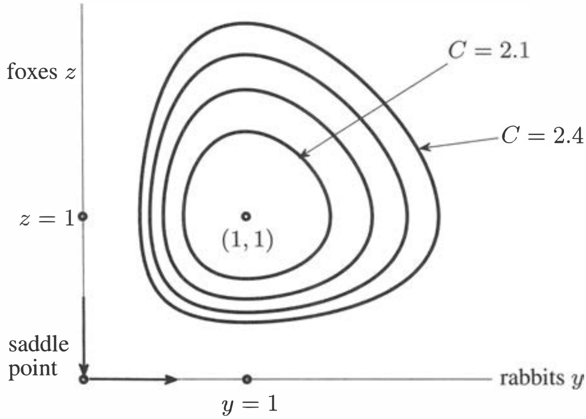


Figure 3.8: Solution paths $y + z - \ln y - \ln z = C$ around the critical point: a *center*.

Predator – Prey – Logistic Equation

When Example 2 has no foxes ($z = 0$), the rabbit equation is $y' = y$. There is no control of rabbits and $y = Ce^t$. When we add a logistic term like $-qy^2$ (rabbits eventually competing with rabbits for available lettuce) this makes the equations more realistic.

We also allow different coefficients p, r, s, t (not all 1 or -1) in the other terms:

Rabbits $\boldsymbol{y' = y\,(p - qy - rz)}$

Foxes $\boldsymbol{z' = z\,(-\,s + wy)}$

First critical point $\boldsymbol{(Y, Z) = (0, 0)}$

Second point $\boldsymbol{(Y, Z) = (p/q, 0)}$

Third $\boldsymbol{s = wY}$ **and** $\boldsymbol{p = qY + rZ}$

At those critical points, y' and z' are zero. The solutions are steady states $y = Y$, $z = Z$.

Near those points we linearize the equation to decide stability. The derivatives of $f(y, z)$ and $g(y, z)$ are in control, because $f = g = 0$ at the critical points:

First derivatives
Jacobian at 0, 0
$$\begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} = \begin{bmatrix} p - 2qy - rz & -ry \\ wz & -s + wy \end{bmatrix} = \begin{bmatrix} \boldsymbol{p} & 0 \\ 0 & \boldsymbol{-s} \end{bmatrix}.$$

$(0, 0)$ is a saddle point: unstable. Small populations have $y' \approx py$ and $z' \approx -sz$. Rabbits increase and foxes decrease. One eigenvalue p is positive, the other eigenvalue $-s$ is negative. Near this $(0, 0)$ point, the competition terms $-qy^2$ and $-ryz$ and wyz are higher order. Those terms disappear in the linearization.

The second critical point has $Y = p/q$ and $Z = 0$. This point is a sink or a saddle:

Linearization around $\boldsymbol{(p/q, 0)}$
$$\begin{bmatrix} y - Y \\ z - Z \end{bmatrix}' = A \begin{bmatrix} y - Y \\ z - Z \end{bmatrix} \quad \textbf{with} \quad A = \begin{bmatrix} -q & -rp/q \\ 0 & -s + wp/q \end{bmatrix}$$

If $s > wp/q$, that last entry is negative. So is $-q$, and we have a sink: two negative eigenvalues.
If $s < wp/q$, that last entry is positive. In this case we have a saddle.

The third critical point (Y, Z) is different. At this point $p = qY + rZ$ and $s = wY$. This leaves only three simple terms in the first derivative matrix above:

Linearization around (Y, Z) $\quad \begin{bmatrix} y - Y \\ z - Z \end{bmatrix}' = A \begin{bmatrix} y - Y \\ z - Z \end{bmatrix}$ with $A = \begin{bmatrix} -qY & -rY \\ wZ & 0 \end{bmatrix}$

The new term $-qy^2$ in the rabbit equation has produced $-qY = -qs/w$ in the matrix A. This is a negative number, it stabilizes the equation. It pulls both of the eigenvalues (previously imaginary) to negative real parts. **Neutral stability changes to full stability.**

2 by 2 matrices are special (with only two eigenvalues λ_1 and λ_2). I can reveal the two facts that produce those two eigenvalues of A: Add the λ's and multiply the λ's.

Sum	$\lambda_1 + \lambda_2$ equals the sum $\boldsymbol{T}$ of diagonal entries	$\boldsymbol{T = -qY}$
Product	$\lambda_1\lambda_2$ equals the determinant $\boldsymbol{D}$ of the matrix	$\boldsymbol{D = rYwZ}$

Our matrix has $\boldsymbol{\lambda_1 + \lambda_2 < 0}$ and $\boldsymbol{\lambda_1\lambda_2 > 0}$. This suggests two negative eigenvalues λ_1 and λ_2 (a sink). It also allows $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ ($a < 0$, a spiral sink). Our conclusion is: *The third critical point Y, Z is stable.*

Final Tests for Stability: Trace and Determinant

We can bring this whole section together. It started with finding the critical points Y, Z and linearizing the differential equations. Now we can give simple tests on the 2 by 2 linearized matrix A. We don't need to compute the eigenvalues before testing them—because the matrix immediately tells us their sum $\lambda_1 + \lambda_2$ and their product $\lambda_1\lambda_2$. *That sum and product* (**the trace and determinant of $\boldsymbol{A}$**) *are all we need.*

Step 1 Find all critical points (steady states) of $y' = f(y, z)$ and $z' = g(y, z)$ by solving $f(Y, Z) = 0$ and $g(Y, Z) = 0$.

Step 2 At each critical point find the matrix A from derivatives of f and g

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial y & \partial g/\partial z \end{bmatrix} \quad \text{at the point } Y, Z$$

Step 3 Decide stability from the **trace $\boldsymbol{T = a + d}$ and determinant $\boldsymbol{D = ad - bc}$**

Unstable	$T > 0$ or $D < 0$ or both
Neutral	$T = 0$ and $D \geq 0$
Stable	$T < 0$ and $D > 0$

If $T^2 \geq 4D > 0$, the stable critical point is a **sink**: real eigenvalues less than zero. If $T^2 < 4D$, the stable critical point is a **spiral sink**: complex eigenvalues with Re $\lambda < 0$. Section 6.4 will explain these rules and draw the stable region $T < 0, D > 0$.

The solution curves $y(t)$, $z(t)$ are paths in the yz plane. Near each critical point Y, Z, the paths are close to one of the six possibilities in Section 3.2. **Source**, **sink**, or **saddle** for real eigenvalues; **Spiral source**, **spiral sink**, or **center** for complex eigenvalues.

A Special 3 by 3 System: A Tumbling Box

You understand that 3 by 3 systems will be more complicated. The pictures don't stay in a plane. There are 9 partial derivatives of f, g, h with respect to x, y, z. The matrix A with those entries is 3 by 3. Its three eigenvalues decide stability (T and D are not enough).

But we live in three dimensions. The most ordinary motions will follow a space curve and not a plane curve. We can imagine the whole of three-dimensional space filled with those curves—that picture is hard to draw. Still there are important special motions that we can understand (and even test for ourselves). Here is a beautiful example.

Throw a closed box up in the air. **Throw a cell phone**. **Throw this book**. Those all have unequal sides $s_1 < s_2 < s_3$. Gravity will bring the book or the box back down, but that is not the interesting part. The key is *how it turns in space*.

There are three special ways to throw the box. It can rotate around the short side s_1. It can rotate around the longest side s_3. The box can try to rotate around its middle side s_2. Those three motions will be critical points. Your throwing experiment will quickly find that **two of the rotations are stable and one is unstable**. In this book on differential equations, we want to understand why. Please put a rubber band around the book.

Since the up and down motion from gravity is not important, we will remove it. Keep the origin $(0,0,0)$ at the center of the box. The box turns around that center point. At every moment in time, a 3D rotation is around an **axis**. If the box tumbles around in the air, that rotation axis is changing with time.

After writing about boxes I thought of another important example. **Throw a football**. If you throw it the right way, spinning around its long axis, it flies smoothly. Any quarterback does that automatically. But if your arm is hit while throwing, the ball wobbles. A football has one long axis and two equal short axes, $s_1 = s_2 < s_3$.

One more: A well-thrown frisbee spins around its short axis (very short). Its long axes go out to the edges of the frisbee, so $s_1 < s_2 = s_3$. A bad throw will make it tumble.

Tumbling indicates an unstable critical point for the equations of motion.

Equations of Motion: Simplest Form

For a box of the right shape, Euler found these three equations. The unknowns x, y, z give the angular momentum around axes $1, 2, 3$ (short, medium, long).

$f(x,y,z)$	$dx/dt = yz$	**Critical points X, Y, Z have $f = g = h = 0$**
$g(x,y,z)$	$dy/dt = -2xz$	**There are 6 critical points on a sphere**
$h(x,y,z)$	$dz/dt = xy$	$(X,Y,Z) = (\pm 1,0,0)\ (0,\pm 1,0)\ (0,0,\pm 1)$

Multiply the three equations by x, y, z and add them together, to see the sphere:

$$x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = xyz - 2xyz + xyz = 0 \qquad \boldsymbol{x^2 + y^2 + z^2 = \textbf{constant}}.$$

The point x, y, z travels on a sphere. There are six critical points X, Y, Z (steady rotations). The question is, which steady states are stable? Try the experiment. Toss up a book.

Linearize at Each Critical Point

When you take 9 partial derivatives of $f = yz$ and $g = -2xz$ and $h = xy$, you get the 3 by 3 Jacobian matrix J. Its first row $\mathbf{0} \quad \boldsymbol{z} \quad \boldsymbol{y}$ contains the partial derivatives of $f = yz$. At each critical point, substitute X, Y, Z into J to see the matrix A in the linearized equations. The six critical points (X, Y, Z) are $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

$$\boldsymbol{J} = \begin{bmatrix} 0 & z & y \\ -2z & 0 & -2x \\ y & x & 0 \end{bmatrix} \qquad \pm\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{-2} \\ 0 & \mathbf{1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} & 0 \\ \mathbf{-2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

That middle matrix A with two ones gives instability around the point $(0, 1, 0)$. Start the linearized equations from the nearby point $(c, 1, c)$.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is} \quad \begin{aligned} \boldsymbol{x'} &= \boldsymbol{z} \\ \boldsymbol{y'} &= \mathbf{0} \\ \boldsymbol{z'} &= \boldsymbol{x} \end{aligned} \quad \text{Then} \quad \begin{aligned} \boldsymbol{x} &= \boldsymbol{ce^t} \\ \boldsymbol{y} &= \mathbf{1} \\ \boldsymbol{z} &= \boldsymbol{ce^t} \end{aligned} \tag{13}$$

Those solutions with e^t are leaving the critical point. You are seeing the eigenvalue $\lambda = 1$. The other eigenvalues are 0 and -1: a **saddle point**. When you try to spin a box around its middle axis, the wobble quickly gets worse. *It is humanly impossible to spin the box perfectly because that axis is unstable.*

The other two axes are neutrally stable. Their matrices A have -2 and $+1$. Their eigenvalues are $\sqrt{2}\,i$ and $-\sqrt{2}\,i$ and 0. Around the short axis $(1, 0, 0)$, the essential part of A is 2 by 2. We see sines and cosines (not e^t and instability):

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ \boldsymbol{z'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{-2} \\ 0 & \mathbf{1} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{-2}\boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix}. \quad \text{Then} \quad \begin{aligned} x &= 1 \\ y &= \sqrt{2}c \cos(\sqrt{2}\,t) \\ z &= c \sin(\sqrt{2}\,t) \end{aligned}$$

The turning axis (x, y, z) travels in an ellipse around $(1, 0, 0)$. This indicates a *center*. Let me go back to the nonlinear equations to see that elliptical cylinder $y^2 + 2z^2 = C$.

Multiply $x' = yz, y' = -2xz, z' = xy$ by $0, y, 2z$. Add to get $\boldsymbol{yy' + 2zz' = 0}$.

The derivative of $y^2 + 2z^2$ is zero. Every path $x(t), y(t), z(t)$ is an ellipse on the sphere.

Alar Toomre's Picture of the Solutions

At this point we know a lot about every solution to $x' = yz$ and $y' = -2xz$ and $z' = xy$.

Stays on a sphere	$x^2 + y^2 + z^2 = C_1$	Multiply the equations by x, y, z.
Stays on an elliptical cylinder	$2x^2 + y^2 = C_2$	Multiply by $2x, y, 0$ and add.
Stays on an elliptical cylinder	$y^2 + 2z^2 = C_3$	Multiply by $0, y, 2z$ and add.
Stays on a hyperbolic cylinder	$x^2 - z^2 = C_4$	Multiply by $x, 0, -z$ and add.

Professor Alar Toomre made the tumbling box famous among MIT students. The year when I went to his 18.03 lecture, he tossed up a book several times (in all three ways). The book turned or tumbled around its short and middle and long axes: *stable*, *unstable*, and *stable*. Actually the stability is only neutral, and wobbles don't grow or disappear.

Maybe you can see those ellipses around two critical points: cylinders intersect a sphere. The website will show one of those cylinders going around $(1, 0, 0)$: a neutrally stable case. It is harder to visualize the hyperbolas $x^2 - z^2 = C_4$ around the unstable point $(0, 1, 0)$.

This figure shows the value of *seeing* a solution—not just its formula. With good fortune a video of this experiment will go onto the book's website **math.mit.edu/dela**.

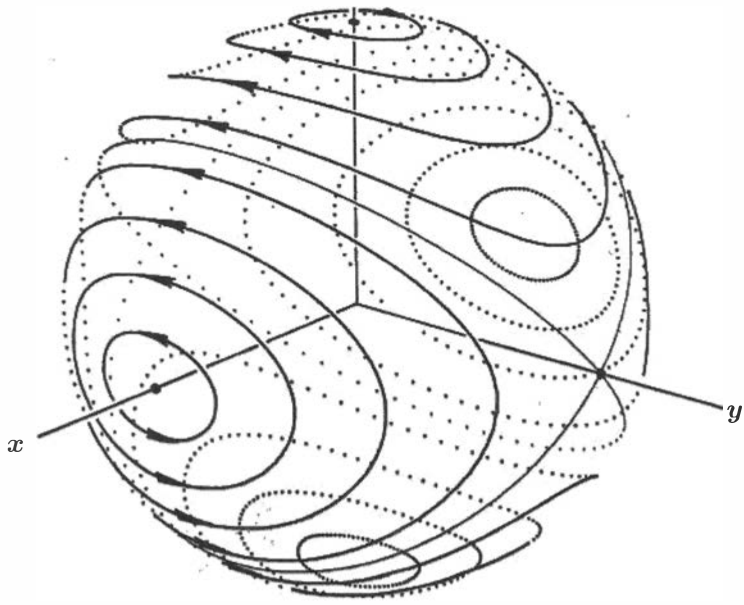


Figure 3.9: Toomre's picture of solution paths $x(t), y(t), z(t)$ from Euler's three equations.

I will end this example with a square box: two equal axes. The symmetry of a football also produces two equal axes. The Earth itself is flatter near the North Pole and South Pole, and symmetric around that short axis. *Fortunately for us this case is neutrally stable.*

The Earth's wobble doesn't go away, at the same time it doesn't get worse. The spin axis passes about five meters from the North Pole.

Flattened sphere	$dx/dt = 0$	**Critical points $(\pm 1, 0, 0)$ at Poles**
Square book	$dy/dt = -xz$	**Critical plane $(0, y, z)$**
Two equal axes	$dz/dt = xy$	**(the plane of the Equator)**

The partial derivatives of $-xz$ and xy are quick to compute at $(X, Y, Z) = (1, 0, 0)$:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{has eigenvalues } \lambda = i \text{ and } \lambda = -i \text{ and } \lambda = 0$$

The path of x, y, z is a circle around the North Pole (for the nonlinear equations too). The Earth wobbles as it spins, but it stays stable. Not like a tumbling box.

Epidemics and the SIR Model

An epidemic can spread until a serious fraction of the population gets sick—or the epidemic can die out early. Unstable or stable: always the important question. Suppose it is a flu epidemic on a closed campus (with no flu shots). The population divides into three groups:

S = Susceptible (may catch the flu)
I = Infected (sick with the flu)
R = Recovered (after having the flu)

The equations for $S(t)$, $I(t)$, $R(t)$ will involve an infection constant β and a recovery constant α. The infection rate is βSI, proportional to the susceptible fraction S times the infected (and infectious) fraction I. The recovery rate is simply αI. This simple model has been improved in many ways—SIR is now a highly developed technique. Epidemiology has major importance, and we want to present this small model:

$$\begin{aligned} dS/dt &= -\beta SI = f(S, I) \\ dI/dt &= \beta SI - \alpha I = g(S, I) \\ dR/dt &= \alpha I \end{aligned}$$

We work with fractions of the total population, so $S + I + R = 1$. Adding the equations confirms that $S + I + R$ is constant (their derivatives add to zero). It is enough to study S and I. We are ignoring births and deaths—our system is closed and the epidemic is fast.

The important critical point is $S = 1, I = 0$. The population is well, but everyone is susceptible. Flu is coming. Is that critical point stable if a few people get sick?

$$\begin{bmatrix} \partial f/\partial S & \partial f/\partial I \\ \partial g/\partial S & \partial g/\partial I \end{bmatrix} = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\beta \\ 0 & \boldsymbol{\beta - \alpha} \end{bmatrix} \quad \text{at } S = 1, I = 0$$

The eigenvalues of that matrix are 0 and $\beta - \alpha$. We certainly need $\beta < \alpha$ for stability. "*Sick must get well faster than well get sick.*" The other eigenvalue $\lambda = 0$ needs a closer analysis, and the model itself requires improvement.

A neutral eigenvalue like $\lambda = 0$ can be pushed either way by nonlinear terms. One way to establish nonlinear stability is to solve the equations—*after removing* t:

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{(\beta S - 1)I}{-\beta SI} = -1 + \frac{1}{\beta S} \quad \text{gives} \quad \boldsymbol{I = -S + \frac{\ln S}{\beta} + C}.$$

The moving point travels along the curve $I + S - (\ln S)/\beta = I(0) + S(0) - (\ln S(0))/\beta$.

An important fact about epidemics is the serious difficulty of estimating α and β. Their ratio $R_0 = \beta/\alpha$ controls the spread of disease: The epidemic dies out if $R_0 < 1$. One comment about estimating β: When the epidemic is over, you could compare $I + S - (\ln S)/\beta$ at $t = 0$ and $t = \infty$. Much more is in the books by Brauer and Castillo-Chavez, especially *Mathematical Models in Population Biology and Epidemiology*.

The Law of Mass Action

When two chemical species react, the law of mass action decides the rate:

$$\boldsymbol{S + E \to SE} \qquad \boldsymbol{\frac{dy}{dt} = kse} \qquad \begin{array}{l} \boldsymbol{s} = \textbf{concentration of } \boldsymbol{S} \\ \boldsymbol{e} = \textbf{concentration of } \boldsymbol{E} \end{array}$$

This is like predator-prey and epidemics (multiply one population times the other, s times e). Then y is the concentration of SE. When E is an enzyme, there is also a reverse reaction $SE \to S + E$ and a forward reaction $SE \to P + E$. For a chemist, the desired product is P. For us, there are three mass action laws with rates k_1, k_{-1}, k_2:

$$\frac{dy}{dt} = k_1 se - k_{-1}y - k_2 y \qquad \frac{ds}{dt} = -k_1 se + k_{-1}y \qquad \frac{de}{dt} = -k_1 se + k_{-1}y + k_2 y = -\frac{dy}{dt}$$

Life depends on enzymes: Very low concentrations $e(0) << s(0)$ and very fast reactions. Without E, blood would take years to clot. Steaks would take decades to digest. This math course might take a century to learn. The enzyme is the **catalyst** (like platinum in a catalytic converter).

After the fast reaction that uses E, the slower reactions bring the enzyme back. Beautifully, separating the two time scales leads to a separable equation for y:

$$\textbf{Michaelis-Menten equation} \qquad \frac{dy}{dt} = -\frac{cy}{y + K} \tag{14}$$

Maini and Baker have shown how matching fast time to slow time leads to (14).

This is just one example of the *nonlinear* differential equations of biology. Mathematics can reveal the main features of the solution. For a detailed picture we turn to accurate numerical methods—and those come in the next section.

Continuous Chaos and Discrete Chaos

This section about stability will now end with extreme instability: **Chaos**. For this we need three differential equations (or two difference equations). Chaotic problems are a recent discovery, but now we know they are everywhere: Chaos is more common than stable equations and even more common than ordinary instability.

This is a deep subject, but you can see its remarkable features from simple experiments. Here are suggestions for one equation, then two, then the big one (Lorenz):

1. **Newton's method** on page 6 finds square roots by solving $f(x) = x^2 - c = 0$. Compute x_1, then x_2, then $x_3, \ldots$ **Then x_n approaches $\pm\sqrt{c}$.**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

   But if $c = -1$, these real x's cannot approach the imaginary square roots $x = \pm i$. The x_n will move around wildly when $x_{n+1} = \frac{1}{2}\left(x_n - x_n^{-1}\right)$. Try 100 steps from $x_0 = \sqrt{3}$ and $x_0 = 2$.

2. The **Hénon map** approaches a "strange attractor" in the xy plane:

   Stretching and folding $\quad x_{n+1} = 1 + y_n - 1.4x_n^2$ and $y_n = 0.3x_n$

   Try four steps, starting from many different x_0, y_0 between -1 and 1.

3. The **Lorenz equations** arise in trying to predict atmospheric convection and weather:

$$x' = a(y - x) \qquad y' = x(b - z) - y \qquad z' = xy - cz$$

   Lorenz himself chose $a = 10$, $b = 28$, $c = 8/3$. The system becomes chaotic. The solutions are extremely sensitive to changes in the starting values. Harvey Mudd College has an ODE Architect Library that includes Lorenz and suggests great experiments. Try it!

■ REVIEW OF THE KEY IDEAS ■

1. The critical points of $y' = f(y, z)$, $z' = g(y, z)$ solve $f(Y, Z) = g(Y, Z) = 0$. *Steady state* $y(t) = Y$, $z(t) = Z$.

2. Near that steady state, $f(y, z) \approx (\partial f/\partial y)(y - Y) + (\partial f/\partial z)(z - Z)$. Similarly $g(y, z)$ is "linearized" at Y, Z. These derivatives of f and g go in a 2×2 matrix A.

3. The equations $(y, z)' = (f, g)$ are stable at Y, Z when the linearized equations $(y - Y, z - Z)' = A(y - Y, z - Z)$ are stable. Then λ_1 and λ_2 have real parts < 0.

4. Stability at Y, Z requires $\dfrac{\partial f}{\partial y} + \dfrac{\partial g}{\partial z} < 0$ and $\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z} > \dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y}$. This means that the eigenvalues have $\lambda_1 + \lambda_2 = a + d < 0$ and $\lambda_1\lambda_2 = ad - bc > 0$.

5. Boxes and books tumble unstably around their middle axes. Footballs are neutral.

6. Epidemics and kinetics are nonlinear when species 1 multiplies species 2 : $y' = kyz$.

Problem Set 3.3

1 If $y' = 2y + 3z + 4y^2 + 5z^2$ and $z' = 6z + 7yz$, how do you know that $Y = 0$, $Z = 0$ is a critical point ? What is the 2 by 2 matrix A for linearization around $(0, 0)$? This steady state is certainly unstable because _____ .

2 In Problem 1, change $2y$ and $6z$ to $-2y$ and $-6z$. What is now the matrix A for linearization around $(0, 0)$? How do you know this steady state is stable ?

3 The system $y' = f(y, z) = 1 - y^2 - z$, $z' = g(y, z) = -5z$ has a critical point at $Y = 1$, $Z = 0$. Find the matrix A of partial derivatives of f and g at that point: stable or unstable ?

4 This linearization is wrong but the zero derivatives are correct. *What is missing* ? $Y = 0$, $Z = 0$ is not a critical point of $y' = \cos\,(ay + bz)$, $z' = \cos\,(cy + dz)$.

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} -a\sin 0 & -b\sin 0 \\ -c\sin 0 & -d\sin 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

5 Find the linearized matrix A at every critical point. Is that point stable ?

(a) $\begin{aligned} y' &= 1 - yz \\ z' &= y - z^3 \end{aligned}$ (b) $\begin{aligned} y' &= -y^3 - z \\ z' &= y + z^3 \end{aligned}$

6 Can you create two equations $y' = f(y, z)$ and $z' = g(y, z)$ with four critical points: $(1, 1)$ and $(1, -1)$ and $(-1, 1)$ and $(-1, -1)$?

I don't think all four points could be stable ? This would be like a surface with four minimum points and no maximum.

7 The second order nonlinear equation for a damped pendulum is $y'' + y' + \sin y = 0$. Write z for the damping term y', so the equation is $z' + z + \sin y = 0$.

Show that $Y = 0$, $Z = 0$ is a stable critical point at the bottom of the pendulum. Show that $Y = \pi$, $Z = 0$ is an unstable critical point at the top of the pendulum.

8 Those pendulum equations $y' = z$ and $z' = -\sin y - z$ have infinitely many critical points ! What are two more and are they stable ?

9 The Liénard equation $y'' + p(y)y' + q(y) = 0$ gives the first order system $y' = z$ and $z' =$ _____. What are the equations for a critical point ? When is it stable ?

10 Are these matrices stable or neutrally stable or unstable (source or saddle) ?

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 9 \\ -1 & -1 \end{bmatrix}$$

11 Suppose a predator x eats a prey y that eats a smaller prey z:

$$\begin{aligned} dx/dt &= -x + xy \\ dy/dt &= -xy + y + yz \\ dz/dt &= -yz + 2z \end{aligned}$$

Find all critical points X, Y, Z
Find A at each critical point
(9 partial derivatives)

12 The damping in $y'' + (y')^3 + y = 0$ depends on the velocity $y' = z$. Then $z' + z^3 + y = 0$ completes the system. Damping makes this nonlinear system stable—is the linearized system stable ?

13 Determine the stability of the critical points $(0,0)$ and $(2,1)$:

(a) $\begin{aligned} y' &= -y + 4z + yz \\ z' &= -y - 2z + 2yz \end{aligned}$ (b) $\begin{aligned} y' &= -y^2 + 4z \\ z' &= y - 2x^4 \end{aligned}$

Problems 14–17 are about Euler's equations for a tumbling box.

14 The correct coefficients involve the moments of inertia I_1, I_2, I_3 around the axes. The unknowns x, y, z give the angular momentum around the three principal axes:

$$\begin{aligned} dx/dt &= ayz && \text{with} && a = (1/I_3 - 1/I_2) \\ dy/dt &= bxz && \text{with} && b = (1/I_1 - 1/I_3) \\ dz/dt &= cxy && \text{with} && c = (1/I_2 - 1/I_1). \end{aligned}$$

Multiply those equations by x, y, z and add. This proves that $x^2 + y^2 + z^2$ is _____.

15 Find the 3 by 3 first derivative matrix from those three right hand sides f, g, h. What is the matrix A in the 6 linearizations at the same 6 critical points ?

16 You almost always catch an unstable tumbling book at a moment when it is flat. That tells us: The point $x(t), y(t), z(t)$ spends most of its time (near) (far from) the critical point $(0,1,0)$. This brings the travel time t into the picture.

17 In reality what happens when you

(a) throw a baseball with no spin (a knuckleball) ?

(b) hit a tennis ball with overspin ?

(c) hit a golf ball left of center ?

(d) shoot a basketball with underspin (a free throw) ?

3.4 The Basic Euler Methods

For most differential equations, solutions are numerical. We solve model equations to understand what to expect in more complicated problems. Then the numbers we need—close to exact but never perfect—come from finite time steps Δt.

This section will show you the key ideas. The approximations will be simple and clear, but not highly accurate. The next section comes closer to the reality of modern codes. The Runge-Kutta method is still frequently used, with refinements that those two creators certainly did not anticipate. The cycle of **predicting at $t + \Delta t$, correcting at $t + \Delta t$**, and **adjusting the stepsize Δt** for the next step is now highly developed.

Local accuracy comes from small steps, but speed comes from larger steps. The right balance depends on the particular equation and the user's need for accuracy. Always there is a requirement of *stability*—because small errors are unavoidable. But after the numerical errors enter the calculation, they must not grow faster than the solution itself.

Euler's First Step $y_1 = y_0 + \Delta t\, f_0$

The equation to solve is $\boldsymbol{dy/dt = f(t, y)}$. The initial value $y(0)$ is given—this will be our starting y_0. A *difference equation* will go forward to y_1. That is our approximation to the exact solution at $t_1 = \Delta t$ (the end of the first time step and the start of the next step). By going forward in steps of size $\Delta t_1, \Delta t_2, \ldots$ we compute values $y_1, y_2, \ldots$ that are close to the exact solution.

We know two facts at $t = 0$. The value of y is y_0 and the slope dy/dt at that point is given by f in the equation. *That slope is called* f_0. It is the right side $f(t, y)$ when $y = y_0$ and $t = 0$. With value y_0 and slope f_0, we know the tangent line $y = y_0 + tf_0$ to the curve $y(t)$. So we can take a step Δt along that tangent line—not too large a step or we will wander too far from the exact curve $y(t)$.

Step Δt along tangent line	$\boldsymbol{y_1 = y_0 + \Delta t\, f_0}$	(1)

Figure 3.10 shows y_1 for the model equation $y' = 2y$. At $y_0 = 1$ the slope is $f_0 = 2$ (since $f(y) = 2y$). **We follow that tangent line as far as $\boldsymbol{y_1 = 1 + 2\Delta t}$.**

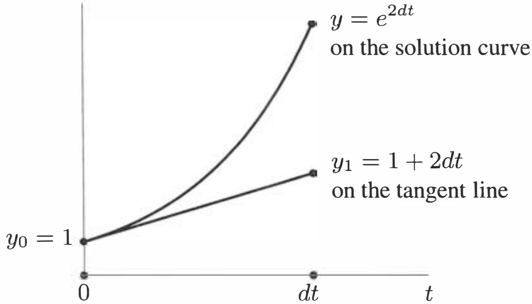


Figure 3.10: The tangent line $y = y_0 + tf_0$ starts at y_0. Euler stops at $\boldsymbol{y_1 = y_0 + \Delta t f_0}$.

Euler's Method $y_{n+1} = y_n + \Delta t f_n$

On the graph, we are following pieces of tangent lines. This is the same as approximating the derivative dy/dt (which changes during a time step) by the forward difference $\Delta y/\Delta t$ (which is held constant during a time step):

$$\frac{dy}{dt} = f(t,y) \qquad \text{becomes} \qquad \frac{\boldsymbol{y_1 - y_0}}{\boldsymbol{\Delta t}} = \boldsymbol{f_0}. \tag{2}$$

There is a new tangent line for the second time step. That step starts at y_1 (which we just computed). *The slope at that point in time is* $f_1 = f(\Delta t, y_1)$. We are using the differential equation $y' = f(t,y)$ to tell us the slopes f_0, f_1, f_2, ... at the start of every time step:

$$\boldsymbol{n^{\text{th}}} \textbf{ time step} \qquad \boxed{\frac{\boldsymbol{\Delta y}}{\boldsymbol{\Delta t}} = \boldsymbol{f(t_n, y_n)}} \quad \textbf{is Euler's method} \quad \boxed{\frac{\boldsymbol{y_{n+1} - y_n}}{\boldsymbol{\Delta t}} = \boldsymbol{f_n}} \tag{3}$$

The model equation $\boldsymbol{dy/dt = 2y}$ has the exact solution $y(t) = e^{2t}$. Euler's method $y_{n+1} = y_n + \Delta t f_n$ will multiply y_n at every step by the number $1 + 2\Delta t$:

$$y_{n+1} = y_n + \Delta t(2y_n) = (1 + 2\Delta t)y_n \qquad \text{leads to} \qquad \boldsymbol{y_n = (1 + 2\Delta t)^n y_0}. \tag{4}$$

We have seen powers of $(1 + \frac{1}{n})$ and $(1 + \frac{a}{n})$ in Section 1.3 from *compound interest.* The current balance was y_n and the interest at rate a was $a\Delta t y_n$. Then the new balance was $y_{n+1} = (1 + a\Delta t)y_n$. This is exactly Euler's method to solve $dy/dt = ay$, and our example has $a = 2$.

$$\textbf{Approximating } \boldsymbol{e^{2t}} \qquad \boldsymbol{y_n = (1 + 2\Delta t)^n \approx (e^{2\Delta t})^n = e^{2n\Delta t}}. \tag{5}$$

The errors $y_n - y$ grow as n increases. But the errors at each step also shrink as $\Delta t \to 0$. If we hold $n\,\Delta t$ fixed at some value T, then we are taking n steps to reach that time T. As n increases and Δt decreases, the steps are smaller—the tangent lines stay closer. Then Euler's y_n approaches the exact $y(T) = e^{2T}$.

Euler's Error

The error E_n is $y(n\,\Delta t) - y_n$. This is the exact solution minus the computed solution y_n at time $n\,\Delta t$. It comes from accumulating small errors at every time step—the tangent lines move away from the true graph of $y(t)$.

First, estimate those small errors at the n separate time steps. *How far is a tangent line from a curve, after a step* Δt? The answer comes from calculus.

$$\begin{array}{l}\textbf{Local error}\\ \textbf{Taylor series}\end{array} \qquad y(t + \Delta t) = y(t) + \Delta t\, y'(t) + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{(\Delta t)^2 y''(t)} + \cdots \tag{6}$$

When we keep two terms and omit the third term, the error is $\leq \frac{1}{2}(\Delta t)^2\, |y''|_{\max}$.

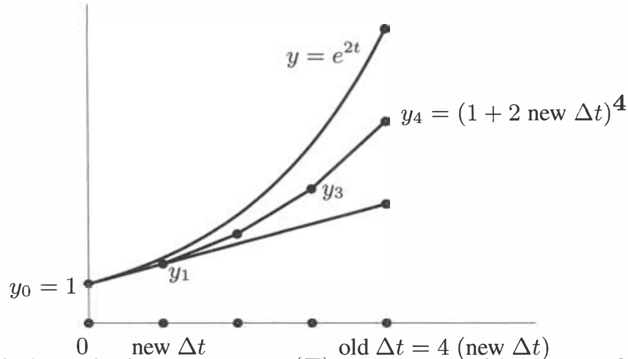


Figure 3.11: Euler's method converges to $y(T)$ as $n \to \infty$, with n steps of size $\Delta t = T/n$.

The Mean Value Theorem would establish that bound of order $(\Delta t)^2$. This is the error in one step—a tangent line moving away from the curve. We will take n steps to reach the time $n\ \Delta t = T$. If all goes well, the 1-step error $C(\Delta t)^2$ grows in n steps to $CT\Delta t$.

$$\textbf{The error at time } \boldsymbol{T} \textbf{ after } \boldsymbol{n} \textbf{ steps is } \ |\boldsymbol{y(T) - y_n}| \le \boldsymbol{Cn(\Delta t)^2 = CT\Delta t}. \qquad (7)$$

Conclusion: Euler's method is *first-order accurate*. The error is proportional to Δt. If we take $2n$ steps of size $\Delta t/2$, and do twice as much work, that will divide the error by 2 (approximately). This is really minimum accuracy.

The Runge-Kutta method has error proportional to $(\Delta t)^4$. Then reducing Δt to $\Delta t/2$ improves the error by a factor near 16. We will be matching many more terms in the Taylor series, where Euler only matched the first derivative. In the example $y' = 2y$, we know that $y(T) = e^{2T}$:

$$\textbf{First-order accuracy} \qquad (1 + 2\ \Delta t)^n = \left(1 + \frac{2T}{n}\right)^n \approx e^{2T} \text{ with error } \frac{C}{n}. \qquad (8)$$

This table shows the slow improvement as n increases, compared to the superfast improvement from keeping more terms in the Taylor series :

$\boldsymbol{n}$	$\boldsymbol{(1 + \frac{1}{n})^n}$ **from Euler**	**Taylor series for** $\boldsymbol{e}$
1	2.0000000	2.0000000
2	2.2500000	2.5000000
3	2.3703704	2.6666667
4	2.4414062	2.7083333
5	2.4883200	2.7166667
6	2.5216264	2.7180556
7	2.5464997	2.7182540
8	2.5657845	2.7182788
9	2.5811748	2.7182815
10	2.5937425	**2.7182818**

Stability

We jumped over an important point when we converted n local errors of size $(\Delta t)^2$ to one global error of size Δt. The local errors occur in each step. The global error at T is the composite of n local errors. We assumed that local errors at early times would not grow much before the final time T.

Think of the local error as a small bank deposit every day. The global error at the end of a year ($T = 365\ \Delta t$) includes 365 small errors. Those small deposits should grow during the year (they earn interest too). The constant C in equation (8) allows for this growth.

What if the equation is $\boldsymbol{dy/dt = -100y}$? This shows decay, not growth. The solution starting at $y(0) = 1$ is $y(T) = \boldsymbol{e^{-100T}}$, very small. But does Euler's method show the same fast decay in the approximate solution, when the equation has $f_n = -100y_n$?

$$y_{n+1} = y_n + \Delta t f_n = (1 - 100\ \Delta t)y_n \qquad \boldsymbol{y_n = (1 - 100\ \Delta t)^n y_0} \tag{9}$$

If $100\Delta t$ is small, then $1 - 100\Delta t$ is less than 1 and its powers decay as they should. But we will have $\boldsymbol{100\Delta t = 3}$ when $\Delta t = 0.03$. That step seems small but *it is not*. The number $1 - 100\Delta t$ will be -2. Equation (9) shows that every step multiplies by -2. The powers of -2 grow exponentially!

$$y_n = 1, -2, 4, -8, \dots \quad y_n = (1 - 100\Delta t)^n y_0 = (-2)^n y_0 \text{ is } \textit{exponentially unstable}.$$

Conclusion: Stability for $y' = -100y$ requires $|1 - 100\Delta t| \le 1$. **We need $\boldsymbol{\Delta t \le 2/100}$.**

In a way this limit on Δt is acceptable. Euler is missing the $\frac{1}{2}(100\Delta t)^2$ term in the Taylor series for e^{-100t}. We would want $100\Delta t < 1$ just for reasonable accuracy. The stability requirement $100\Delta t < 2$ is not a heavy burden. But read further.

Stiff Equations

Imagine an equation with solutions e^{-t} and e^{-100t}. Then e^{-t} will dominate, because it has much slower decay than e^{-100t}. We have decay rates $s = -1$ and $s = -100$:

$$y'' + 101y' + 100y = 0 \quad \text{with} \quad s^2 + 101s + 100 = (s+1)(s+100). \tag{10}$$

This is certainly *overdamped*. The roots $s = -1$ and $s = -100$ are real. Euler's method needs to follow e^{-t} accurately, because that is the important solution. **But stability still requires $\boldsymbol{\Delta t \le 2/100}$.**

The unimportant solution e^{-100t} is getting in the way. It reduces Δt and therefore adds more work (many steps), beyond the ordinary demand of first order accuracy. A problem like equation (10) is called **stiff**: stability can be too expensive for ordinary Euler.

We can see this second order problem as two first order equations. Introduce y' as a second unknown. As in Section 3.1, a "companion matrix" multiplies the vector (y, y'):

$$y'' + 101y' + 100y = 0 \text{ is the same as } \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -101 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}. \tag{11}$$

The eigenvalues of the matrix are the same roots -1 and -100. That is a ***stiff problem*: *slow decay together with fast decay*.**

Euler's method for this matrix equation is just like Euler for $y' = Ay$:

$$\frac{\boldsymbol{y}_{n+1} - \boldsymbol{y}_n}{\Delta t} = A\boldsymbol{y}_n \quad \text{or} \quad \boldsymbol{y}_{n+1} = (\boldsymbol{I} + \boldsymbol{A}\boldsymbol{\Delta t})\boldsymbol{y}_n. \tag{12}$$

Every step multiplies by $I + A\Delta t$. That matrix has eigenvalues $1 - \Delta t$ and $1 - 100\Delta t$. Normally $1 - \Delta t$ is more important and larger. But if $100\Delta t$ is greater than 2, then the second number $1 - 100\Delta t$ is below -1. Its powers will show extreme instability.

The cure for stiff systems is to switch to an **implicit method**.

Backward Euler = Implicit Euler

The idea of implicit methods is to use backward differences. Go back from y_{n+1} and t_{n+1} and f_{n+1}, instead of going forward from y_n and t_n and f_n.

Backward Euler
$$\frac{\boldsymbol{y}^B_{n+1} - \boldsymbol{y}_n}{\boldsymbol{\Delta t}} = \boldsymbol{f}_{n+1} = \boldsymbol{f}(\boldsymbol{t}_{n+1}, \boldsymbol{y}^B_{n+1}). \tag{13}$$

The example $y' = -100y$ will divide by $1 + 100\Delta t$ instead of multiplying by $1 - 100\Delta t$:

$$\frac{y^B_{n+1} - y_n}{\Delta t} = -100\, y^B_{n+1} \quad \text{is} \quad \boldsymbol{(1 + 100\Delta t) y^B_{n+1} = y_n}.$$

That division happens at every time step. After n steps this method remains very stable:

"Implicit Euler" $\quad y^B_n = \left(\dfrac{1}{1 + 100\Delta t}\right)^n y_0$ is decreasing correctly.

For this linear equation, division is no more expensive than multiplication. Implicit is the way to go. But we pay a much higher price for implicit when the problem is nonlinear. Instead of substituting the known y_n to find $f_n = f(n\,\Delta t, y_n)$ in ordinary "explicit" Euler, we now have to solve a nonlinear equation to find the unknown y^B_{n+1}:

Each step must solve for $\boldsymbol{y^B_{n+1}}$ $\quad y^B_{n+1} - \Delta t f(t_{n+1}, y^B_{n+1}) = y_n. \qquad (14)$

If the forcing function f is complicated, even an approximate solution for y^B_{n+1} will be expensive. You see the struggle that is constantly presented: **Implicit methods are more stable but much slower**. For $\boldsymbol{y}' = A\boldsymbol{y}$, the matrix to invert is in $(I - \Delta t\, A)\boldsymbol{y}^B_{n+1} = \boldsymbol{y}_n$.

Difference Equations vs Differential Equations

Compare a^n with e^{at}: powers and exponentials. The powers come from a difference equation $Y_{n+1} = aY_n$. The exponentials come from a differential equation $y' = ay$. Stability means that those solutions *approach zero*. For ordinary numbers (this includes complex numbers) the test on a is easy.

$$\boldsymbol{a^n \to 0 \ \text{ when } \ |a| < 1} \qquad\qquad \boldsymbol{e^{at} \to 0 \ \text{ when } \ \mathrm{Re}\, a < 0}.$$

When we have a matrix A, the same tests are applied to the eigenvalues:

$$\boldsymbol{A^n \to 0 \ \text{ when all } \ |\lambda| < 1} \qquad\qquad \boldsymbol{e^{At} \to 0 \ \text{ when all } \ \mathrm{Re}\, \lambda_i < 0}.$$

■ REVIEW OF THE KEY IDEAS ■

1. Euler's method is $(y_{n+1} - y_n)/\Delta t = f_n$ or $\boldsymbol{y_{n+1} = y_n + \Delta t\, f(n\,\Delta t, y_n)}$.

2. That step to y_{n+1} follows the tangent line at y_n, not the curve $y(t)$. Error $\approx (\Delta t)^2$.

3. After n steps to time $T = n\,\Delta t$, the error is proportional to Δt: *First order accuracy.*

4. Stability requires y_n to grow no faster than the exact $y(t)$: Often a *size limit on* Δt.

5. **Backward Euler** is $y^B_{n+1} - y_n = \Delta t f(y^B_{n+1})$. Harder to find y^B_{n+1} but more stable.

Problem Set 3.4

1 Apply Euler's method $y_{n+1} = y_n + \Delta t f_n$ to find y_1 and y_2 with $\Delta t = \frac{1}{2}$:

(a) $y' = y$ (b) $y' = y^2$ (c) $y' = 2ty$ (all with $y(0) = y_0 = 1$)

2 For the equations in Problem 1, find y_1 and y_2 with the step size reduced to $\Delta t = \frac{1}{4}$. Now the value y_2 is an approximation to the exact $y(t)$ at what time t? Then y_2 in this question corresponds to which y_n in Problem 1?

3 (a) For $dy/dt = y$ starting from $y_0 = 1$, what is Euler's y_n when $\Delta t = 1$?

(b) Is it larger or smaller than the true solution $y = e^t$ at time $t = n$?

(c) What is Euler's y_{2n} when $\Delta t = \frac{1}{2}$? This is closer to the true $y(n) = e^n$.

4 For $dy/dt = -y$ starting from $y_0 = 1$, what is Euler's approximation y_n after n steps of size Δt? Find all the y_n's when $\Delta t = 1$. Find all the y_n's when $\Delta t = 2$. Those time steps are *too large* for this equation.

5 The true solution to $y' = y^2$ starting from $y(0) = 1$ is $y(t) = 1/(1-t)$. This explodes at $t = 1$. Take 3 steps of Euler's method with $\Delta t = \frac{1}{3}$ and take 4 steps with $\Delta t = \frac{1}{4}$. Are you seeing any sign of explosion?

6 The true solution to $dy/dt = -2ty$ with $y(0) = 1$ is the bell-shaped curve $y = e^{-t^2}$. It decays quickly to zero. Show that step $n + 1$ of Euler's method gives $y_{n+1} = (1 - 2n\Delta t^2)y_n$. Do the y_n's decay toward zero? Do they stay there?

7 The equations $y' = -y$ and $z' = -10z$ are uncoupled. If we use Euler's method for both equations with the same Δt between $\frac{2}{10}$ and 2, show that $y_n \to 0$ but $|z_n| \to \infty$. The method is failing on the solution $z = e^{-10t}$ that should decay fastest.

8 What values y_1 and y_2 come from *backward Euler* for $dy/dt = -y$ starting from $y_0 = 1$? Show that $y^B_1 < 1$ and $y^B_2 < 1$ even if Δt is very large. We have *absolute stability*: no limit on the size of Δt.

9 The logistic equation $y' = y - y^2$ has an S-curve solution in Section 1.7 that approaches $y(\infty) = 1$. This is a steady state because $y' = 0$ when $y = 1$.

Write Euler's approximation $y_{n+1} =$ _____ to this logistic equation, with stepsize Δt. Show that this has the same steady state: y_{n+1} equals y_n if $y_n = 1$.

10 The important question in Problem 9 is whether the steady state $y_n = 1$ is stable or unstable. Subtract 1 from both sides of Euler's $y_{n+1} = y_n + \Delta t(y_n - y_n^2)$:

$$y_{n+1} - 1 = y_n + \Delta t(y_n - y_n^2) - 1 = (y_n - 1)(1 - \Delta t y_n).$$

Each step multiplies the distance from 1 by $(1 - \Delta t y_n)$. Near the steady $y_\infty = 1$, $1 - \Delta t\, y_n$ has size $|1 - \Delta t|$. For which Δt is this smaller than 1 to give stability?

11 Apply backward Euler $y_{n+1}^B = y_n + \Delta t f_{n+1}^B = y_n + \Delta t \left[y_{n+1}^B - \left(y_{n+1}^B \right)^2 \right]$ to the logistic equation $y' = f(y) = y - y^2$. What is y_1^B if $y_0 = \frac{1}{2}$ and $\Delta t = \frac{1}{4}$?

You have to solve a quadratic equation to find y_1^B. I am finding two answers for y_1^B. A computer code might choose the answer closer to y_0.

12 For the bell-shaped curve equation $y' = -2ty$, show that backward Euler divides y_n by $1 + 2n(\Delta t)^2$ to find y_{n+1}^B. As $n \to \infty$, what is the main difference from forward Euler in Problem 6?

13 The equation $y' = \sqrt{|y|}$ has *many solutions* starting from $y(0) = 0$. One solution stays at $y(t) = 0$, another solution is $y = t^2/4$. (Then $y' = t/2$ agrees with $\sqrt{y}$.) Other solutions can stay at $y = 0$ up to $t = T$, and then switch to the parabola $y = (t - T)^2/4$. As soon as y leaves the bad point $y = 0$, where $f(y) = y^{1/2}$ has infinite slope, the equation has only one solution.

Backward Euler $y_1 - \Delta t \sqrt{|y_1|} = y_0 = 0$ gives two correct values $y_1^B = 0$ and $y_1^B = (\Delta t)^2$. What are the three possible values of y_2^B?

14 Every finite difference person will think of averaging forward and backward Euler:

$$\textbf{Centered Euler / Trapezoidal} \qquad \boldsymbol{y_{n+1}^C - y_n = \Delta t \left(\frac{1}{2} f_n + \frac{1}{2} f_{n+1}^C \right)}.$$

For $y' = -y$ the key questions are **accuracy** and **stability**. Start with $y(0) = 1$.

$$y_1^C - y_0 = \Delta t \left(-\frac{1}{2} y_0 - \frac{1}{2} y_1^C \right) \quad \text{gives} \quad \boldsymbol{y_1^C = \frac{1 - \Delta t/2}{1 + \Delta t/2} y_0}.$$

Stability Show that $|1 - \Delta t/2| < |1 + \Delta t/2|$ for all Δt. *No stability limit on* Δt.

Accuracy For $y_0 = 1$ compare the exact $y_1 = e^{-\Delta t} = 1 - \Delta t + \frac{1}{2}\Delta t^2 - \cdots$ with $y_1^C = (1 - \frac{1}{2}\Delta t)/(1 - \frac{1}{2}\Delta t) = (1 - \frac{1}{2}\Delta t)(1 - \frac{1}{2}\Delta t + \frac{1}{4}\Delta t^2 - \cdots)$.

An extra power of Δt is correct: *Second order accuracy.* A good method.

The website has codes for Euler and Backward Euler and Centered Euler. Those methods are slow and steady with first order and second order accuracy. The test problems give comparisons with faster methods like Runge-Kutta.

3.5 Higher Accuracy with Runge-Kutta

The section on basic Euler methods contained two messages. First, those methods are simple to understand (they follow a tangent line). Second, those methods are too simple to give good or even adequate accuracy. This section brings major improvements. The fourth order Runge-Kutta method is the basis for **ode45**, the workhorse among all of MATLAB's codes for solving $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$.

Notice that this equation—linear or more likely nonlinear—involves first derivatives $\boldsymbol{y}'$ and no higher derivatives. In case the original equation is $y'' = F(t, y, y')$, introduce $y' = y_2$ as a new equation together with the original $y_2' = F(t, y, y_2)$. The unknowns $y_1 = y$ and $y_2 = y'$ go into a vector $\boldsymbol{y}$. The right hand sides y_2 and F go into a vector $\boldsymbol{f}$.

n equations for n unknown $\boldsymbol{y}'s$ $\qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= F(t, y_1, y_2) \end{aligned} \qquad \begin{aligned} y_1' &= f_1(t, y_1, \ldots, y_n) \\ &\cdots\cdots \\ y_n' &= f_n(t, y_1, \ldots, y_n) \end{aligned}$

In the middle is a system of two equations coming from $y'' = F$. On the right is a system of n equations for the vector $\boldsymbol{y}$ of n unknowns. The n equations $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ start from n initial conditions $y_1(0), \ldots, y_n(0)$, and $\boldsymbol{f}$ is a vector of n right hand sides.

We are ready for more accurate approximations to $y' = f(t, y)$ and $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$.

Improved Euler = Simplified Runge-Kutta

Euler's first order method is $y_{n+1}^E = y_n + \Delta t\, f_n$. Let me describe an improvement to *second order accuracy*, which means an error of size $(\Delta t)^2$. This uses the Runge-Kutta idea: **Substitute Euler's y_{n+1}^E once more into f. Use that output to get a better y_{n+1}^S** :

$$\textbf{Improved Euler Simplified R-K} \qquad \frac{\boldsymbol{y_{n+1}^S - y_n}}{\boldsymbol{\Delta t}} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{f(t_n, y_n)} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{f}\left(\boldsymbol{t_{n+1}, y_{n+1}^E}\right). \tag{1}$$

Let me show you the improvement for $y' = ay$. In this case $f(t, y)$ is ay. You can see y^E as a **prediction** of the next value y_{n+1} and y^S as a **correction**:

$$\boldsymbol{y^E = y_n + a\,\Delta t\, y_n} \quad \text{goes into} \quad \boldsymbol{y^S} = y_n + \frac{1}{2}a\,\Delta t\, y_n + \frac{1}{2}a\,\Delta t(\boldsymbol{y_n + a\,\Delta t\, y_n}). \tag{2}$$

When that last term is multiplied out, we see the correct $(\Delta t)^2$ term included in y_{n+1}^S :

$$\textbf{Linear case } \boldsymbol{y' = ay} \qquad y_{n+1}^S = y_n + a\,\Delta t\, y_n + \frac{1}{2}a^2(\Delta t)^2 y_n. \tag{3}$$

We are following the *tangent parabola* starting at y_n. The parabola stays much closer to the true $y(t)$ curve than the tangent line. This improvement means a $(\Delta t)^3$ error at each step. With stability, those errors produce a $(\Delta t)^2$ overall error after $n = T/\Delta t$ steps.

The exact $y(t + \Delta t)$ is $e^{a\Delta t}y(t)$. Equation (3) has *three* correct terms of $e^{a\Delta t}$. Euler uses the slope $y' = f(t, y)$ only at the *start* of the time step, but *the improvement* y^S *in equation* (1) *averages the slope at the start and the end of the step.*

Simplified Adams Method

Here is another way to achieve second order accuracy. **Save and reuse the computed value y_{n-1} at the previous time $t - \Delta t$.** With the right coefficients $3/2$ and $-1/2$, and essentially no extra work, we can again capture the term $\frac{1}{2}(\Delta t)^2 y''$ that Euler missed.

Adams-Bashforth Multistep method
$$y^{A}_{n+1} = y_n + \frac{3}{2}\Delta t f(t_n, y_n) - \frac{1}{2}\Delta t f(t_{n-1}, y_{n-1}). \qquad (4)$$

All we do is to save each computed value of f_n for one more step. That number becomes the f_{n-1} term in (4). The right hand side of (4) gives the correct y' and y'' terms:

$$y_n + \frac{3}{2}\Delta t y'_n - \frac{1}{2}\Delta t y'_{n-1} \approx y_n + \frac{3}{2}\Delta t y'_n - \frac{1}{2}\Delta t(y'_n - \Delta t y''_n) = \boldsymbol{y_n + \Delta t y'_n + \frac{1}{2}(\Delta t)^2 y''_n}$$

Each extra step back to $y_{n-2}, y_{n-3}, \ldots$ can increase the accuracy by 1. Those multi-step methods compete with Runge-Kutta and eventually they win. But fourth order is still mostly on the R-K side. One reason is that Adams needs a special effort to compute y_{-1} before the first step can begin. Runge-Kutta starts cold.

Runge-Kutta easily changes Δt from one step to the next. On the other hand, its four evaluations of $f(t, y)$ could be expensive. Stiff systems need backward differences.

Fourth Order Runge-Kutta

The famous version of Runge-Kutta uses *four* evaluations of the right side. It starts at time t_n with solution y_n^{RK}. It reaches time $t_{n+1} = t_n + \Delta t$ with approximate solution y_{n+1}^{RK}. On the way, Runge-Kutta stops twice for $\boldsymbol{k_2}$ and $\boldsymbol{k_3}$ at $t_{n+1/2} = t_n + \frac{1}{2}\Delta t$.

At each step from t_n to t_{n+1} compute $\boldsymbol{k_1, k_2, k_3, k_4}$

$$\begin{aligned} \boldsymbol{k_1} &= f(t_n, y_n)/2 \\ \boldsymbol{k_2} &= f(t_{n+1/2}, y_n + \Delta t\, \boldsymbol{k_1})/2 \\ \boldsymbol{k_3} &= f(t_{n+1/2}, y_n + \Delta t\, \boldsymbol{k_2})/2 \\ \boldsymbol{k_4} &= f(t_{n+1}, y_n + 2\Delta t\, \boldsymbol{k_3})/2 \end{aligned}$$

A combination of those four $\boldsymbol{k}$'s gives fourth-order accuracy for y_{n+1}^{RK}:

Runge-Kutta step
$$\boldsymbol{\frac{y_{n+1}^{RK} - y_n}{\Delta t} = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)} \qquad (5)$$

That short line is one of the most important formulas in this book. Among highly accurate methods, Runge-Kutta is especially easy to code and run—probably the easiest there is. Before each step, we decide on Δt. For the model problem $y' = y$ the R-K combination produces five correct terms in the series for $e^{\Delta t}$. You can see evaluations of f inside evaluations of f, starting with $\boldsymbol{k_1} = f_n/2 = y/2$:

$$\boldsymbol{k_2} = \frac{1}{2}\left(y + \frac{\Delta t}{2}y\right) \quad \boldsymbol{k_3} = \frac{1}{2}\left(y + \frac{\Delta t}{2}\left(y + \frac{\Delta t}{2}y\right)\right) \quad \boldsymbol{k_4} = \frac{1}{2}\left(y + \Delta t\left(y + \frac{\Delta t}{2}\left(y + \frac{\Delta t}{2}y\right)\right)\right)$$

Problem 1 will simplify $k_1 + 2k_2 + 2k_3 + k_4$. The new y_{n+1} at the end of the step is $\boldsymbol{y_{n+1} = (1 + \Delta t + \cdots + \frac{1}{4!}(\Delta t)^4)y_n}$. All terms correct for $e^{\Delta t}$ and 4^{th} order accuracy.

The Stability of Runge-Kutta

To determine the limit of stability, apply the method to $y' = -y$. The true solution $y = e^{-t}y(0)$ will decrease. But if Δt is too large, the approximations y_n will *increase* in size. The first example of possible instability was Euler's method:

$$\textbf{Euler instability for } \boldsymbol{\Delta t > 2} \quad y^E_{n+1} = (1-\Delta t)y_n \text{ has } |1-\Delta t| > 1$$

When we apply the same test to Runge-Kutta, instability enters for $\boldsymbol{\Delta t > 2.78}$:

$$\textbf{RK instability for } \boldsymbol{\Delta t \geq 3} \qquad 1 - 3 + \frac{1}{2}9 - \frac{1}{6}27 + \frac{1}{24}81 = \boldsymbol{\frac{11}{8}} > 1.$$

The full infinite series would give the small number e^{-3}. But these five terms give a multiplier $11/8$ that is larger than 1. If we take this over-large step n times, the Runge-Kutta approximation $y_n = (11/8)^n$ will be enormous and completely wrong. The more exact stability limit is $a\,\Delta t < 2.78$ for $y' = ay$.

Example 1 Apply all three methods to $\boldsymbol{dy/dt = y}$. The true solution $\boldsymbol{y = e^t}$ reaches $y = e = 2.71828\ldots$ at time $t = 1$. Try $\Delta t = 0.2$ and 0.1.

$\boldsymbol{\Delta t = 0.2}$	$\boldsymbol{y^E}$	$\boldsymbol{y^S}$	$\boldsymbol{y^{RK}}$	$\boldsymbol{\Delta t = 0.1}$	$\boldsymbol{y^E}$	$\boldsymbol{y^S}$	$\boldsymbol{y^{RK}}$
$t = 0$	1	1	1	$t = 0$	1	1	1
				.1	1.10	1.1050	1.1051708
$t = .2$	1.20	1.220	1.221400	.2	1.21	1.2210	1.2214026
				.3	1.33	1.3492	1.3498585
$t = .4$	1.44	1.488	1.491818	.4	1.46	1.4909	1.4918242
				.5	1.61	1.6474	1.6487206
$t = .6$	1.73	1.816	1.822106	.6	1.77	1.8204	1.8221180
				.7	1.95	2.0116	2.0137516
$t = .8$	2.07	2.215	2.225521	.8	2.14	2.2228	2.2255396
				.9	2.36	2.4562	2.4596014
$t = 1$	2.49	2.703	2.718251	1.0	2.59	2.7141	2.7182797

The error in y^S is divided by 4 (from .015 to .004 at $t = 1$) when Δt is cut in half. This indicates second order accuracy for simplified Runge-Kutta, as the theory predicted. The work is only doubled.

ode 45 and ODEPACK and More

Runge-Kutta is accurate and easy to code. The final value y_{n+1} can be made even better. With *six* evalutions of f (not four) we can also compute a value Y^5_{n+1} that has *fifth* order accuracy. By comparing with y^{RK}_{n+1} we get an estimate of the error, which indicates whether a larger Δt is possible or a smaller Δt is necessary. This is the heart of Matlab's **ode 45** code. A good solver for stiff systems is **ode 15s**.

ODEPACK and SUNDIALS are open collections of Fortran 77 codes from Livermore Laboratory. Those emphasize Adams methods (backward differences for stiff problems).

Mathematica has DSolve for solution formulas and NDSolve for numerical solutions. Wolfram Alpha is remakable for the very wide range of problems it solves. SciPy and SymPy and Scilab are also free and high quality. **See the web** !

■ REVIEW OF THE KEY IDEAS ■

1. Higher order equations like $y'' + y' + y = F(t, y, y')$ reduce to $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$. Most finite difference methods prefer this first order system with $\boldsymbol{y} = (y, y')$.

2. $y_{n+1}^E = y_n + \Delta t f_n$ improves to second order accuracy by also using $f(t_{n+1}, y_{n+1}^E)$.

3. Fourth order Runge-Kutta uses that substitution into $f(t, y)$ four times in each step.

4. The Runge-Kutta error is divided by almost $2^4 = 16$ when Δt is divided by 2.

5. Stability for $y' = ay$ requires $a\Delta t^E > \mathbf{-2}$ and $a\Delta t^S > \mathbf{-2}$ and $a\Delta t^{RK} > \mathbf{-2.78}$. Otherwise disaster for $a < 0$: the approximations Y_n will start to grow.

Problem Set 3.5

Runge-Kutta can only be appreciated by using it. A simple code is on math.mit.edu/dela. Professional codes are ode 45 (in MATLAB) and ODEPACK and many more.

1 For $y' = y$ with $y(0) = 1$, show that simplified Runge-Kutta and full Runge-Kutta give these approximations y_1 to the exact $y(\Delta t) = e^{\Delta t}$:

$$y_1^S = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 \qquad y_1^{RK} = 1 + \Delta t + \frac{1}{2}(\Delta t)^2 + \frac{1}{6}(\Delta t)^3 + \frac{1}{24}(\Delta t)^4$$

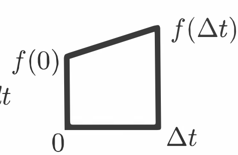
2 With $\Delta t = 0.1$ compute those numbers y_1^S and y_1^{RK} and subtract from the exact $y = e^{\Delta t}$. The errors should be close to $(\Delta t)^3/6$ and $(\Delta t)^5/120$.

3 Those values y_1^S and y_1^{RK} have errors of order $(\Delta t)^3$ and $(\Delta t)^5$. Errors of this size at every time step will produce total errors of size _____ and _____ at time T, from N steps of size $\Delta t = T/n$.

Those estimates of total error are correct provided errors don't grow (*stability*).

4 $dy/dt = f(t)$ with $y(0) = 0$ is solved by integration when f does not involve y. From time $t = 0$ to Δt, simplified Runge-Kutta approximates the integral of $f(t)$:

$$y_1^S = \Delta t\left(\frac{1}{2}f(0) + \frac{1}{2}f(\Delta t)\right) \text{ is close to } y(\Delta t) = \int_0^{\Delta t} f(t)\,dt$$



Suppose the graph of $f(t)$ is a straight line as shown. Then the region is a *trapezoid*. Check that its area is exactly y_1^S. Second order means exact for linear f.

5 Suppose again that f does not involve y, so $dy/dt = f(t)$ with $y(0) = 0$. Then full Runge-Kutta from $t = 0$ to Δt approximates the integral of $f(t)$ by y_1^{RK} :

$$y_1^{RK} = \Delta t \left(c_1 f(0) + c_2 f(\Delta t/2) + c_3 f(\Delta t)\right). \qquad \textbf{Find } \boldsymbol{c_1, c_2, c_3}.$$

This approximation to $\int_0^{\Delta t} f(t)\, dt$ is called Simpson's Rule. It has 4^{th} order accuracy.

6 Reduce these second order equations to first order systems $\boldsymbol{y}' = \boldsymbol{f}(t, y)$ for the vector $\boldsymbol{y} = (y, y')$. Write the two components of $\boldsymbol{y}_1^E$ (Euler) and $\boldsymbol{y}_1^S$.

(a) $y'' + yy' + y^4 = 1$ (b) $my'' + by' + ky = \cos t$

7 When $my'' + by' + ky = \cos t$ in Problem 6 is reduced to a vector equation $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}$ find $\boldsymbol{y}_1^E$ and $\boldsymbol{y}_1^S$ from the initial vector $\boldsymbol{y}_0$.

8 For $y' = -y$ and $y_0 = 1$ the exact solution $y = e^{-t}$ is approximated at time Δt by 2 or 3 or 5 terms:

$$y_1^E = 1-\Delta t \quad y_1^S = 1-\Delta t+\frac{1}{2}(\Delta t)^2 \quad y_1^{RK} = 1-\Delta t+\frac{1}{2}(\Delta t)^2-\frac{1}{6}(\Delta t)^3+\frac{1}{24}(\Delta t)^4$$

(a) With $\Delta t = 1$ compare those three numbers to the exact e^{-1}. What error E ?

(b) With $\Delta t = 1/2$ compare those three numbers to $e^{-1/2}$. Is the error near $E/16$?

9 For $y' = ay$, simplified Runge-Kutta gives $y_{n+1}^S = (1 + a\Delta t + \frac{1}{2}(a\Delta t)^2)y_n$. This multiplier of y_n reaches $1 - 2 + 2 = \mathbf{1}$ when $a\Delta t = -2$: *the stability limit.*

(**Computer experiment**) For $N = 1, 2, \ldots, 10$ discover the stability limit $L = L_N$ when the series for e^{-L} is cut off after $N + 1$ terms:

$$\left|1 - L + \frac{1}{2}L^2 - \frac{1}{6}L^3 + \cdots \pm \frac{1}{N!}L^N\right| = 1.$$

We know $L = 2$ for $N = 1$ and $N = 2$. Runge-Kutta has $L = 2.78$ for $N = 4$.

CHAPTER 3 NOTES

Proof that $y' = f(t,y)$ has a solution **Functions $y_0, y_1, y_2, \ldots$ approach $y(t)$**

Section 3.1 stated a fact: $dy/dt = f(t,y)$ has one solution starting from $y(0)$, when f is a good function: Assume f and df/dy are continuous at all points. Since we have no formula for y (and we don't expect one), how can we know that a solution exists?

One good answer constructs y_1 from $y_0 = y(0)$, then y_2 from y_1, then y_3 from $y_2, \ldots$

$$\textbf{Equation} \quad \boxed{\frac{dy_{n+1}}{dt} = f(t, y_n(t))} \qquad \textbf{Solution} \quad \boxed{y_{n+1} = y_0 + \int_0^t f(s, y_n(s))ds} \tag{6}$$

Let me practice with $y' = y$ and $y(0) = 1$. The solution is e^t. Take three steps to y_3:

$$\begin{array}{llll} y_0' = 0 & y_1' = y_0 & y_2' = y_1 & y_3' = y_2 \\ y(0) = 1 & y_1 = 1 + t & y_2 = 1 + t + \dfrac{t^2}{2} & y_3 = 1 + t + \dfrac{t^2}{2} + \dfrac{t^3}{6} \end{array}$$

The same construction of e^t was in Section 1.3. Now we go much further, to solve nonlinear equations $y' = f(t,y)$. The key idea is to compare $y_{n+1} - y_n$ with the previous $y_n - y_{n-1}$. Subtract equation (6) for y_n from equation (6) for y_{n+1}:

$$y_{n+1}(t) - y_n(t) = \int_0^t [f(s, y_n(s)) - f(s, y_{n-1}(s))]\, ds. \tag{7}$$

When $|\partial f/\partial y| \le L$, the difference $|f(y_n) - f(y_{n-1})|$ is not larger than $L|y_n - y_{n-1}|$.

$$\begin{aligned} |y_2 - y_1| &\le \int_0^t L|y_1 - y_0|\, ds \le Lt|y_1 - y_0|_{\max} \\ |y_3 - y_2| &\le \int_0^t L|y_2 - y_1|\, ds \le \int_0^t L^2 t|y_1 - y_0|_{\max} = \frac{L^2t^2}{2}|y_1 - y_0|_{\max} \end{aligned}$$

We are seeing Lt and $L^2t^2/2$ and next will be $L^3t^3/6$. Those numbers $L^n t^n/n!$ approach zero quickly because of $n!$ If n is large and N is larger, then

$$|y_N - y_n| \le |y_N - y_{N-1}| + |y_{N-1} - y_{N-2}| + \cdots + |y_{n+1} - y_n| \le C\frac{L^n t^n}{n!}$$

This is what we need to know: the differences $y_N(t) - y_n(t)$ approach zero. Cauchy showed that the numbers $y_n(t)$ *must approach a limit* $y(t)$. (Of course y_{n+1} will approach the same limit.) That limiting function $y(t)$ will be our desired solution:

$$\boldsymbol{y_{n+1}}(t) = y_0 + \int_0^t f(s, \boldsymbol{y_n}(s))\, ds \;\rightarrow\; \boldsymbol{y}(t) = y_0 + \int_0^t f(s, \boldsymbol{y(s)})\, ds. \text{ Then } y' = f(t,y).$$

Chapter 4

Linear Equations and Inverse Matrices

4.1 Two Pictures of Linear Equations

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers—we never see x^2 or x times y. Our first linear system is deceptively small, only "2 by 2." But you will see how far it leads:

$$\begin{array}{lrcrcl} \textbf{Two equations} & x & - & 2y & = & 1 \\ \textbf{Two unknowns} & 2x & + & y & = & 7 \end{array} \tag{1}$$

We begin *a row at a time*. The first equation $\boldsymbol{x - 2y = 1}$ produces a straight line in the xy plane. The point $x = 1$, $y = 0$ is on the line because it solves that equation. The point $x = 3$, $y = 1$ is also on the line because $3 - 2 = 1$. For $x = 101$ we find $y = 50$.

The slope of this line in Figure 4.1 is $\frac{1}{2}$, because y increases by 1 when x changes by 2. But slopes are important in calculus and this is linear algebra!

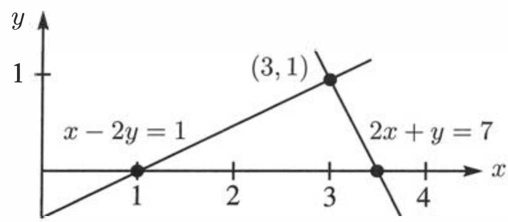


Figure 4.1: *Row picture*: The point $(3, 1)$ where the two lines meet is the solution.

The second line in this "row picture" comes from the second equation $2x + y = 7$. You can't miss the intersection point where the two lines meet. *The point $x = 3, y = 1$ lies on both lines*. It solves both equations at once. This is the solution to our two equations.

ROWS ***The row picture shows two lines meeting at a single point (the solution).***

Turn now to the column picture. I want to recognize the same linear system as a "vector equation." Instead of numbers we need to see *vectors*. If you separate the original system into its columns instead of its rows, you get a vector equation:

Combination equals b
$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix} = \boldsymbol{b}. \tag{2}$$

This has two column vectors on the left side. The problem is *to find the combination of those vectors that equals the vector on the right*. We are multiplying the first column by x and the second column by y, and adding vectors. With the right choices $x = 3$ and $y = 1$ (the same numbers as before), this produces $3(\textbf{\textit{column 1}}) + 1(\textbf{\textit{column 2}}) = \boldsymbol{b}$.

COLUMNS ***The column picture combines the column vectors on the left side of the equations to produce the vector b on the right side.***

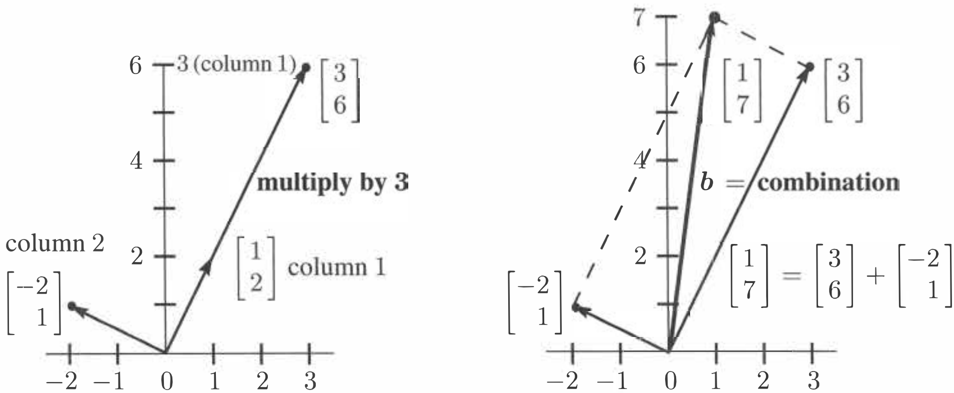


Figure 4.2: *Column picture*: A combination **3** (column 1) + **1** (column 2) gives the vector $\boldsymbol{b}$.

Figure 4.2 is the "column picture" of two equations in two unknowns. The left side shows the two separate columns, and column 1 is multiplied by 3. This multiplication by a *scalar* (a number) is one of the two basic operations in linear algebra:

Scalar multiplication
$$3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}.$$

If the components of a vector $\boldsymbol{v}$ are v_1 and v_2, then $c\boldsymbol{v}$ has components cv_1 and cv_2.

The other basic operation is *vector addition.* We add the first components and the second components separately. $3-2$ and $6+1$ give the vector sum $(1,7)$ as desired:

$$\textbf{Vector addition} \qquad \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

The right side of Figure 4.2 shows this addition. The sum along the diagonal is the vector $\boldsymbol{b} = (1,7)$ on the right side of the linear equations.

To repeat: The left side of the vector equation is a ***linear combination*** of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That combination step is crucially important, because it contains both of the basic operations on vectors: *multiply and add.*

$$\textbf{Linear combination of the 2 columns} \qquad 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Of course the solution $x = 3, y = 1$ is the same as in the row picture. I don't know which picture you prefer! Two intersecting lines are more familiar at first. You may like the row picture better, but only for a day. My own preference is to combine column vectors. It is a lot easier to see a combination of four vectors in four-dimensional space, than to visualize how four "planes" might possibly meet at a point. (*Even one three-dimensional plane in four-dimensional space is hard enough. . .*)

The ***coefficient matrix*** on the left side of equation (1) is the 2 by 2 matrix A:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

This is very typical of linear algebra, to look at a matrix by rows and also by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $A\boldsymbol{v} = \boldsymbol{b}$:

$$\textbf{Matrix multiplies vector} \qquad \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

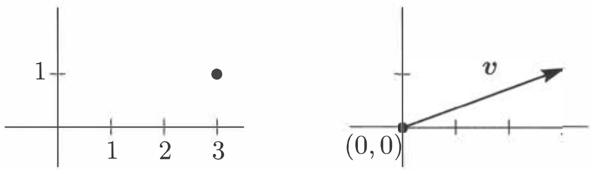
The row picture deals with the two rows of A. The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into the solution vector $\boldsymbol{v}$. Here is matrix-vector multiplication, matrix A times vector $\boldsymbol{v}$. Please look at this multiplication $A\boldsymbol{v}$!

$$\begin{array}{l}\textbf{Dot products with rows} \\ \textbf{Combination of columns}\end{array} \qquad A\boldsymbol{v} = \boldsymbol{b} \;\text{ is }\; \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}. \qquad (3)$$

Linear Combinations of Vectors

Before I go to three dimensions, let me show you the most important operation on vectors. We can see a vector like $\boldsymbol{v} = (3, 1)$ as a pair of numbers, or as a point in the plane, or as an arrow that starts from $(0, 0)$. The arrow ends at the point $(3, 1)$ in Figure 4.3.

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

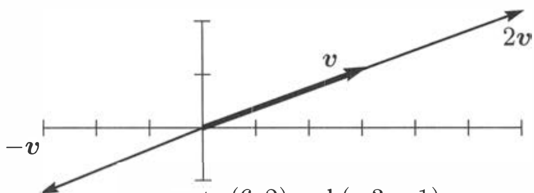


column vector | point $(3, 1)$ | arrow to $(3, 1)$

Figure 4.3: The vector $\boldsymbol{v}$ is given by two numbers or a point or an arrow from $(0, 0)$.

A first step is to multiply that vector by any number c. If $c = 2$ then the vector is doubled to $2\boldsymbol{v}$. If $c = -1$ then it changes direction to $-\boldsymbol{v}$. Always the "scalar" c multiplies each separate component (here 3 and 1) of the vector $\boldsymbol{v}$. The arrow doubles the length to show $2\boldsymbol{v}$ and it reverses direction to show $-\boldsymbol{v}$:

$$2\boldsymbol{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad -\boldsymbol{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$



column vectors | arrows to $(6, 2)$ and $(-3, -1)$

Figure 4.4: Multiply the vector $\boldsymbol{v} = (3, 1)$ by scalars $c = 2$ and -1 to get $c\boldsymbol{v} = (3c, c)$.

If we have another vector $\boldsymbol{w} = (-1, 1)$, we can add it to $\boldsymbol{v}$. Vector addition $\boldsymbol{v} + \boldsymbol{w}$ can use numbers (the normal way) or it can use the arrows (to visualize $\boldsymbol{v} + \boldsymbol{w}$). The arrows in Figure 4.5 go head to tail: **At the end of $\boldsymbol{v}$, place the start of $\boldsymbol{w}$.**

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

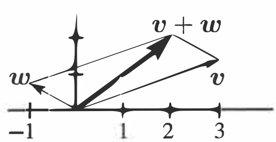


Figure 4.5: The sum of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (-1, 1)$ is $\boldsymbol{v} + \boldsymbol{w} = (2, 2)$. This is also $\boldsymbol{w} + \boldsymbol{v}$.

Allow me to say, adding $\boldsymbol{v} + \boldsymbol{w}$ and multiplying $c\boldsymbol{v}$ will soon be second nature. In themselves they are not impressive. What really counts is when you do both at once.

Multiply cv and also dw, then add to get the **linear combination** $cv + dw$.

$$\textbf{Linear combination } 2v + 3w \qquad 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

This is the basic operation of linear algebra! If you have two 5-dimensional vectors like $\boldsymbol{v} = (1,1,1,1,2)$ and $\boldsymbol{w} = (3,0,0,1,0)$, you can multiply $\boldsymbol{v}$ by 2 and $\boldsymbol{w}$ by 1. You can combine to get $2\boldsymbol{v} + \boldsymbol{w} = (5,2,2,3,4)$. Every combination $c\boldsymbol{v} + d\boldsymbol{w}$ is a vector in the big 5-dimensional space $\mathbf{R}^5$.

I admit that there is no picture to show these vectors in $\mathbf{R}^5$. Somehow I imagine arrows going to $\boldsymbol{v}$ and $\boldsymbol{w}$. If you think of all the vectors $c\boldsymbol{v}$, *they form a line in* $\mathbf{R}^5$. The line goes in both directions from $(0,0,0,0,0)$ because c can be positive or negative or zero.

Similarly there is a line of all vectors $d\boldsymbol{w}$. The hard but all-important part is to imagine all the combinations $c\boldsymbol{v} + d\boldsymbol{w}$. Add all vectors on one line to all vectors on the other line, and what do you get? It is a "2-dimensional plane" inside the big 5-dimensional space. I don't lose sleep trying to visualize that plane. (There is no problem in working with the five numbers.) For linear combinations in high dimensions, algebra wins.

Dot Product of v and w

The other important operation on vectors is a kind of multiplication. This is not ordinary multiplication and we don't write $\boldsymbol{vw}$. The output from $\boldsymbol{v}$ and $\boldsymbol{w}$ will be one number and it is called the **dot product** $\boldsymbol{v} \cdot \boldsymbol{w}$.

DEFINITION The **dot product** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2. \tag{4}$$

The dot product of $\boldsymbol{v} = (3,1)$ and $\boldsymbol{w} = (-1,1)$ is $\boldsymbol{v} \cdot \boldsymbol{w} = (3)(-1) + (1)(1) = -2$.

Example 1 The column vectors $(1,2)$ and $(-2,1)$ have a *zero* dot product:

$$\begin{array}{l}\textbf{Dot product is zero} \\ \textbf{Perpendicular vectors}\end{array} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular*. The angle between them is 90°.

The clearest example of two perpendicular vectors is $\boldsymbol{i} = (1,0)$ along the x axis and $\boldsymbol{j} = (0,1)$ up the y axis. Again the dot product is $\boldsymbol{i} \cdot \boldsymbol{j} = 0 + 0 = 0$. Those vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ form a right angle. They are the columns of the 2 by 2 **identity matrix** I.

The dot product of $\boldsymbol{v} = (3,1)$ and $\boldsymbol{w} = (1,2)$ is 5. Soon $\boldsymbol{v} \cdot \boldsymbol{w}$ will reveal the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ (not 90°). Please check that $\boldsymbol{w} \cdot \boldsymbol{v}$ is also 5.

Multiplying a Matrix A and a Vector v

Linear equations have the form $A\boldsymbol{v} = \boldsymbol{b}$. The right side $\boldsymbol{b}$ is a column vector. On the left side, the coefficient matrix A multiplies the unknown column vector $\boldsymbol{v}$ (we don't use a "dot" for $A\boldsymbol{v}$). The all-important fact is that $A\boldsymbol{v}$ is computed by *dot products in the row picture*, and $A\boldsymbol{v}$ is a **combination of the columns in the column picture**.

I put those words "combination of the columns" in boldface, because this is an essential idea that is sometimes missed. One definition is usually enough in linear algebra, but $A\boldsymbol{v}$ has two definitions—the rows and the columns produce the same output vector $A\boldsymbol{v}$.

The rules stay the same if A has n columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Then $\boldsymbol{v}$ has n components. The vector $A\boldsymbol{v}$ is still a combination of the columns, $A\boldsymbol{v} = v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + \cdots + v_n\boldsymbol{a}_n$. **The numbers in $\boldsymbol{v}$ multiply the columns in $\boldsymbol{A}$.** Let me start with $n = 2$.

By rows $\quad A\boldsymbol{v} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{v} \\ (\text{row } 2) \cdot \boldsymbol{v} \end{bmatrix}$ $\qquad$ **By columns** $\quad A\boldsymbol{v} = v_1(\text{column } 1) + v_2(\text{column } 2)$.

Example 2 In equation (3) I wrote "dot products with rows" and "combination of columns." Now you know what those mean. They are the two ways to look at $A\boldsymbol{v}$:

Dot products with rows
Combination of columns

$$\begin{bmatrix} a\,v_1 + b\,v_2 \\ c\,v_1 + d\,v_2 \end{bmatrix} = v_1 \begin{bmatrix} a \\ c \end{bmatrix} + v_2 \begin{bmatrix} b \\ d \end{bmatrix}. \tag{5}$$

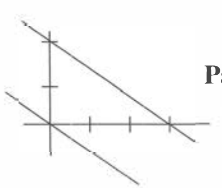
You might naturally ask, *which way to find $A\boldsymbol{v}$* ? My own answer is this : I compute by rows and I visualize (and understand) by columns. Combinations of columns are truly fundamental. But to calculate the answer $A\boldsymbol{v}$, I have to find one component at a time. Those components of $A\boldsymbol{v}$ are the dot products with the rows of A.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 3v_2 \\ 4v_1 + 5v_2 \end{bmatrix} = v_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + v_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Singular Matrices and Parallel Lines

The row picture and column picture can fail—and they will fail together. For a 2 by 2 matrix, the row picture fails when the lines from row 1 and row 2 are parallel. The lines don't meet and $A\boldsymbol{v} = \boldsymbol{b}$ has no solution :

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \qquad \begin{aligned} 2v_1 - 3v_2 &= 6 \\ 4v_1 - 6v_2 &= 0 \end{aligned}$$



Parallel lines
no solution

The row picture shows the problem and so does the algebra : 2 times equation 1 produces $4v_1 - 6v_2 = \mathbf{12}$. But equation 2 requires $4v_1 - 6v_2 = \mathbf{0}$. Notice that this line goes through the center point $(0, 0)$ because the right side is zero.

How does the column picture fail? *Columns* 1 *and* 2 *point in the same direction.* When the rows are "dependent", the columns are also dependent. All combinations of the columns $(2,4)$ and $(3,6)$ lie in the same direction. Since the right side $\boldsymbol{b} = (6,0)$ is not on that line, $\boldsymbol{b}$ is *not* a combination of those two column vectors of A. Figure 4.6 (a) shows that there is *no solution* to the equation.

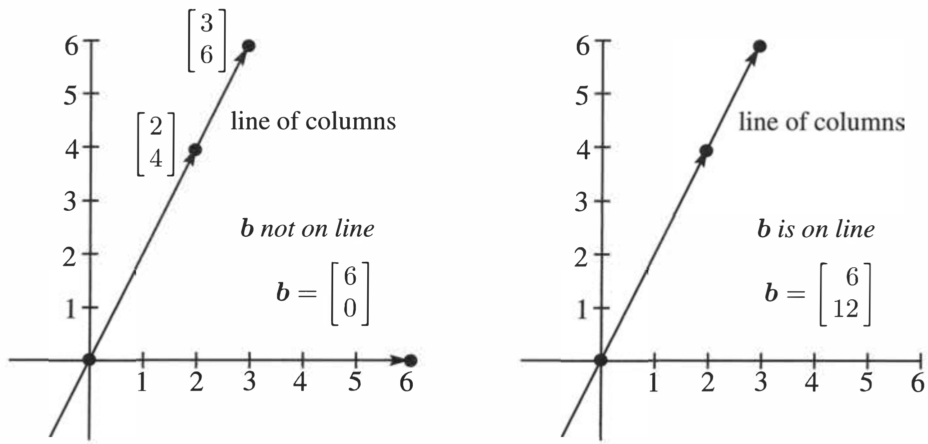
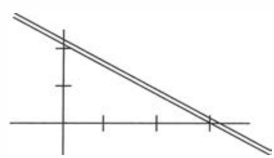


Figure 4.6: Column pictures (a) **No solution** (b) **Infinity of solutions**

Example 3 Same matrix A, now $\boldsymbol{b} = (6,12)$, infinitely many solutions to $A\boldsymbol{v} = \boldsymbol{b}$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \qquad \begin{aligned} 2v_1 - 3v_2 &= \mathbf{6} \\ 4v_1 - 6v_2 &= \mathbf{12} \end{aligned}$$



In the row picture, the two lines are the same. *All points* on that line solve both equations. Two times equation 1 gives equation 2. Those close lines are one line.

In the column picture above, the right side $\boldsymbol{b} = (6,12)$ falls right onto the line of the columns. Later we will say: ***b*** *is in the column space of* A. There are infinitely many ways to produce $(6,12)$ as a combination of the columns. They come from infinitely many ways to produce $\boldsymbol{b} = (0,0)$ (**choose any** $\boldsymbol{c}$). Add one way to produce $\boldsymbol{b} = (6,12) = 3(2,4)$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{3c}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \mathbf{2c}\begin{bmatrix} -3 \\ -6 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \mathbf{3}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \mathbf{0}\begin{bmatrix} -3 \\ -6 \end{bmatrix}. \tag{6}$$

The vector $\boldsymbol{v}_n = (\mathbf{3c}, \mathbf{2c})$ is a **null solution** and $\boldsymbol{v}_p = (\mathbf{3}, \mathbf{0})$ is a **particular solution.** $\boldsymbol{A}\boldsymbol{v}_{\boldsymbol{n}}$ **equals zero and** $\boldsymbol{A}\boldsymbol{v}_{\boldsymbol{p}}$ **equals** $\boldsymbol{b}$. Then $A(\boldsymbol{v}_p + \boldsymbol{v}_n) = \boldsymbol{b}$. Together, $\boldsymbol{v}_p$ and $\boldsymbol{v}_n$ give the **complete solution**, all the ways to produce $\boldsymbol{b} = (6,12)$ from the columns of A:

$$\boxed{\textbf{Complete solution to } A\boldsymbol{v} = \boldsymbol{b} \qquad \boldsymbol{v}_{\text{complete}} = \boldsymbol{v}_p + \boldsymbol{v}_n = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3c \\ 2c \end{bmatrix}.} \tag{7}$$

Equations and Pictures in Three Dimensions

In three dimensions, a linear equation like $x+y+2z=6$ produces a *plane*. The plane would go through $(0,0,0)$ if the right side were 0. In this case the "6" moves us to a parallel plane that misses the center point $(0,0,0)$.

A second linear equation will produce another plane. Normally the two planes meet in a *line*. Then a third plane (from a third equation) normally cuts through that line at a *point*. That point will lie on all three planes, so it solves all three equations.

This is the *row picture*, three planes in three–dimensional space. They meet at the solution. One big problem is that this row picture is hard to draw. Three planes are too many to see clearly how they meet (maybe Picasso could do it).

The *column picture* of $A\boldsymbol{v}=\boldsymbol{b}$ is easier. It starts with three column vectors in three-dimensional space. We want to combine those columns of A to produce the vector $v_1(\text{column }1)+v_2(\text{column }2)+v_3(\text{column }3)=\boldsymbol{b}$. Normally there is one way to do it. That gives the solution (v_1,v_2,v_3) — which is also the meeting point in the row picture.

I want to give an example of success (one solution) and an example of failure (no solution). Both examples are simple, but they really go deeply into linear algebra.

Example 4 Invertible matrix A, one solution $\boldsymbol{v}$ for any right side $\boldsymbol{b}$.

$$A\boldsymbol{v}=\boldsymbol{b} \qquad \text{is} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}. \tag{8}$$

This matrix is **lower triangular**. It has zeros above the main diagonal. Lower triangular systems are quickly solved by forward substitution, top to bottom. The top equation gives v_1, then move down. First $v_1=\mathbf{1}$. Then $-v_1+v_2=3$ gives $v_2=\mathbf{4}$. Then $-v_2+v_3=5$ gives $v_3=\mathbf{9}$.

Figure 4.7 shows the three columns $\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3$. When you combine them with $1,4,9$ you produce $\boldsymbol{b}=(1,3,5)$. In reverse, $\boldsymbol{v}=(1,4,9)$ must be the solution to $A\boldsymbol{v}=\boldsymbol{b}$.

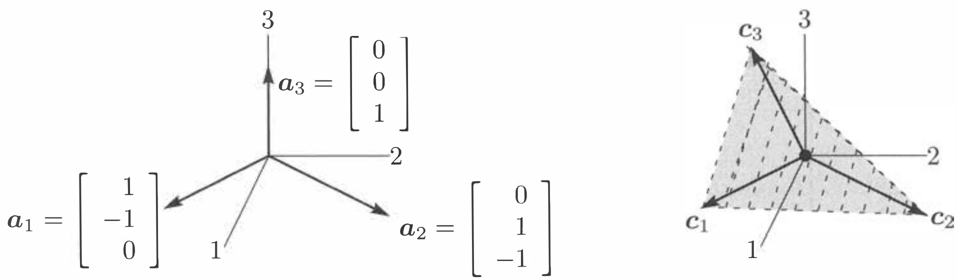


Figure 4.7: **Independent columns $\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3$ not in a plane**. Dependent columns $\boldsymbol{c}_1,\boldsymbol{c}_2,\boldsymbol{c}_3$ are three vectors all in the same plane.

Example 5 **Singular matrix**: no solution to $C\boldsymbol{v} = \boldsymbol{b}$ or infinitely many solutions (depending on $\boldsymbol{b}$).

$$\begin{aligned} \boldsymbol{w_1} - \boldsymbol{w_3} &= \boldsymbol{b_1} \\ -\boldsymbol{w_1} + \boldsymbol{w_2} &= \boldsymbol{b_2} \\ -\boldsymbol{w_2} + \boldsymbol{w_3} &= \boldsymbol{b_3} \end{aligned} \quad \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}. \tag{9}$$

This matrix C is a "circulant." The diagonals are constants, all 1's or all 0's or all -1's. The diagonals circle around so each diagonal has three equal entries. Circulant matrices will be perfect for the Fast Fourier Transform (**FFT**) in Chapter 8.

To see if $C\boldsymbol{w} = \boldsymbol{b}$ has a solution, add those three equations to get $0 = b_1 + b_2 + b_3$.

$$\textbf{Left side} \qquad (w_1 - w_3) + (-w_1 + w_2) + (-w_2 + w_3) = 0. \tag{10}$$

$C\boldsymbol{w} = \boldsymbol{b}$ cannot have a solution unless $0 = b_1 + b_2 + b_3$. The components of $\boldsymbol{b} = (1, 3, 5)$ do not add to zero, so $C\boldsymbol{w} = (1, 3, 5)$ has no solution.

Figure 4.7 shows the problem. **The three columns of C lie in a plane**. **All combinations $C\boldsymbol{w}$ of those columns will lie in that same plane**. If the right side vector $\boldsymbol{b}$ is not in the plane, then $C\boldsymbol{w} = \boldsymbol{b}$ cannot be solved. The vector $\boldsymbol{b} = (1, 3, 5)$ is off the plane, because the equation of the plane requires $b_1 + b_2 + b_3 = 0$.

Of course $C\boldsymbol{w} = (0, 0, 0)$ always has the zero solution $\boldsymbol{w} = (0, 0, 0)$. But when the columns of C are in a plane (as here), there are additional nonzero solutions to $C\boldsymbol{w} = \boldsymbol{0}$. Those three equations are $w_1 = w_3$ and $w_1 = w_2$ and $w_2 = w_3$. The **null solutions** are $\boldsymbol{w}_n = (\boldsymbol{c}, \boldsymbol{c}, \boldsymbol{c})$. When all three components are equal, we have $C\boldsymbol{w}_n = \boldsymbol{0}$.

The vector $\boldsymbol{b} = (1, 2, -3)$ is also in the plane of the columns, because it does have $b_1 + b_2 + b_3 = 0$. In this good case there must be a **particular solution** to $C\boldsymbol{w}_p = \boldsymbol{b}$. There are many particular solutions $\boldsymbol{w}_p$, since any solution can be a particular solution. I will choose the particular $\boldsymbol{w}_p = (1, 3, 0)$ that ends in $w_3 = 0$:

$$C\boldsymbol{w}_p = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

> The **complete solution** is
> $\boldsymbol{w}_{\text{complete}} = \boldsymbol{w}_p + \text{ any } \boldsymbol{w}_n$

Summary These two matrices A and C, with third columns $\boldsymbol{a}_3$ and $\boldsymbol{c}_3$, allow me to mention two key words of linear algebra: *independence and dependence*. This book will develop those ideas much further. I am happy if you see them early in the two examples:

$\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are independent	**A is invertible**	**$A\boldsymbol{v} = \boldsymbol{b}$ has one solution $\boldsymbol{v}$**
$\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are dependent	**C is singular**	**$C\boldsymbol{w} = \boldsymbol{0}$ has many solutions $\boldsymbol{w}_n$**

Eventually we will have n column vectors in n-dimensional space. The matrix will be n by n. The key question is whether $A\boldsymbol{v} = \boldsymbol{0}$ has only the zero solution. Then the columns don't lie in any "hyperplane." When columns are independent, the matrix is invertible.

Problem Set 4.1

Problems 1–8 are about the row and column pictures of $Av = b$.

1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $\boldsymbol{v} = (x, y, z) = (2, 3, 4)$:

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= 3 \\ 0x + 0y + 1z &= 4 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Draw the four vectors in the column picture. Two times column 1 plus three times column 2 plus four times column 3 equals the right side $\boldsymbol{b}$.

2 If the equations in Problem 1 are multiplied by $2, 3, 4$ they become $DV = B$:

$$\begin{aligned} 2x + 0y + 0z &= 4 \\ 0x + 3y + 0z &= 9 \\ 0x + 0y + 4z &= 16 \end{aligned} \quad \text{or} \quad DV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} = \boldsymbol{B}$$

Why is the row picture the same? Is the solution $\boldsymbol{V}$ the same as $\boldsymbol{v}$? What is changed in the column picture—the columns or the right combination to give $\boldsymbol{B}$?

3 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

4 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.

5 The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line $\mathbf{L}$). Find three solutions on $\mathbf{L}$.

6 Move the third plane in Problem 5 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line $\mathbf{L}$, but the third plane doesn't _____ that line.

7 In Problem 5 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is _____. Find two combinations of the columns that give $\boldsymbol{b} = (2, 3, 5)$. This is only possible for $\boldsymbol{b} = (4, 6, c)$ if $c =$ _____.

8 Normally 4 "planes" in 4-dimensional space meet at a ______. Normally 4 vectors in 4-dimensional space can combine to produce $\boldsymbol{b}$. What combination of $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$, $(1,1,1,1)$ produces $\boldsymbol{b}=(3,3,3,2)$?

Problems 9–14 are about multiplying matrices and vectors.

9 Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:

$$\text{(a)}\ \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

10 Compute each $A\boldsymbol{x}$ in Problem 9 as a combination of the columns:

$$\text{9(a) becomes}\quad A\boldsymbol{x} = 2\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

How many separate multiplications for $A\boldsymbol{x}$, when the matrix is "3 by 3"?

11 Find the two components of $A\boldsymbol{x}$ by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

12 Multiply A times $\boldsymbol{x}$ to find three components of $A\boldsymbol{x}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

13 (a) A matrix with m rows and n columns multiplies a vector with ______ components to produce a vector with ______ components.

(b) The planes from the m equations $A\boldsymbol{x}=\boldsymbol{b}$ are in ______-dimensional space. The combination of the columns of A is in ______-dimensional space.

14 Write $2x+3y+z+5t=8$ as a matrix A (how many rows?) multiplying the column vector $\boldsymbol{x}=(x,y,z,t)$ to produce $\boldsymbol{b}$. The solutions $\boldsymbol{x}$ fill a plane or "hyperplane" in 4-dimensional space. *The plane is 3-dimensional with no 4D volume.*

Problems 15–22 ask for matrices that act in special ways on vectors.

15 (a) What is the 2 by 2 identity matrix? I times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} y \\ x \end{bmatrix}$.

16 (a) What 2 by 2 matrix R rotates every vector by 90° ? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} y \\ -x \end{bmatrix}$.

(b) What 2 by 2 matrix R^2 rotates every vector by 180° ?

17 Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

18 What 2 by 2 matrix E subtracts the first component from the second component ? What 3 by 3 matrix does the same ?

$$E\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

19 What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z + x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z - x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1}, the two results are (____) and (____).

20 What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (____) and (____).

21 What 2 by 2 matrix R rotates every vector through 45° ? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R.

22 Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication $A\boldsymbol{v}$. The matrix A has one row. The solutions to $A\boldsymbol{v} = \boldsymbol{0}$ lie on a ____ perpendicular to the vector ____. The columns of A are only in ____ -dimensional space.

23 In MATLAB notation, write the commands that define this matrix A and the column vectors $\boldsymbol{v}$ and $\boldsymbol{b}$. What command would test whether or not $A\boldsymbol{v} = \boldsymbol{b}$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

24 If you multiply the 4 by 4 all-ones matrix A = ones(4) and the column v = ones(4,1), what is A∗v ? (Computer not needed.) If you multiply B = eye(4) + ones(4) times w = zeros(4,1) + 2∗ones(4,1), what is B∗w ?

Questions 25–27 review the row and column pictures in 2, 3, and 4 dimensions.

25 Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$.

26 For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a ____.

27 For four linear equations in two unknowns x and y, the row picture shows four ____. The column picture is in ____ -dimensional space. The equations have no solution unless the vector on the right side is a combination of ____.

Challenge Problems

28 Invent a 3 by 3 **magic matrix** M_3 with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be $8, 3, 4$. What is M_3 times $(1, 1, 1)$? What is M_4 times $(1, 1, 1, 1)$ if a 4 by 4 magic matrix has entries $1, \ldots, 16$?

29 Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are the first two columns of a 3 by 3 matrix A. Which third columns $\boldsymbol{w}$ would make this matrix singular ? Describe a typical column picture of $A\boldsymbol{v} = \boldsymbol{b}$ in that singular case, and a typical row picture (for a random $\boldsymbol{b}$).

30 **Multiplying by A is a "linear transformation".** Those important words mean:

If $\boldsymbol{w}$ is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

It is this "***linearity***" $A\boldsymbol{w} = cA\boldsymbol{u} + dA\boldsymbol{v}$ that gives us the name *linear algebra*.

If $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\boldsymbol{u}$ and $A\boldsymbol{v}$ are the columns of A.

Combine $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. **If $\boldsymbol{w} = \begin{bmatrix} \mathbf{5} \\ \mathbf{7} \end{bmatrix}$ how is $A\boldsymbol{w}$ connected to $A\boldsymbol{u}$ and $A\boldsymbol{v}$** ?

31 A 9 by 9 ***Sudoku matrix*** S has the numbers $1, \ldots, 9$ in every row and column, and in every 3 by 3 block. For the all-ones vector $\boldsymbol{v} = (1, \ldots, 1)$, what is $S\boldsymbol{v}$?

A better question is: **Which row exchanges will produce another Sudoku matrix** ? Also, which exchanges of block rows give another Sudoku matrix ?

Section 4.5 will look at all possible permutations (reorderings) of the rows. I see 6 orders for the first 3 rows, all giving Sudoku matrices. Also 6 permutations of the next 3 rows, and of the last 3 rows. And 6 block permutations of the block rows ?

32 Suppose the second row of A is some number c times the first row :

$$A = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix}.$$

Then if $a \neq 0$, the second column of A is what number d times the first column ? **A square matrix with dependent rows will also have dependent columns**. This is a crucial fact coming soon.

4.2 Solving Linear Equations by Elimination

This section explains a systematic way to solve linear equations—the best way we know. The method is called ***"elimination"***, and you can see it in this 2 by 2 example. Before elimination, x and y appear in both equations. After elimination, the first unknown x has disappeared from the second equation $5y = 5$.

$$\begin{array}{ll} x - 2y = 1 & (\textit{multiply equation } 1 \textit{ by } 2) \\ 2x + y = 7 & (\textit{subtract to eliminate } 2x) \end{array}$$

After elimination

$$\begin{array}{rl} \boldsymbol{x - 2y} & \mathbf{= 1} \\ \boldsymbol{5y} & \mathbf{= 5} \end{array}$$

The new equation $5y = 5$ instantly gives $y = 1$. Substituting $y = 1$ back into the first equation leaves $x - 2 = 1$. Therefore $x = 3$ and the solution $(x, y) = (3, 1)$ is complete.

Elimination produces an ***upper triangular system***—this is the goal. The nonzero coefficients $1, -2, 5$ form a triangle. That system is solved from the bottom upwards, first $y = 1$ and then $x = 3$. This quick process is called ***back substitution***. It is used for upper triangular systems of any size, after elimination produces a triangle.

Important point: The original equations have the same solution $x = 3$ and $y = 1$. Before and after elimination, the lines meet at the same point $(3, 1)$. Every step worked with both sides of correct equations.

The step that eliminated x from equation 2 is the fundamental operation in this chapter. We use it so often that we look at it closely:

To eliminate $\boldsymbol{2x}$: Subtract a multiple of equation $\mathbf{1}$ from equation $\mathbf{2}$.

Two times $x - 2y = 1$ gives $2x - 4y = 2$. When this is subtracted from $2x + y = 7$, the right side becomes $7 - 2 = 5$. The main point is that $2x$ cancels $2x$. **The system becomes triangular.**

Ask yourself how that multiplier $\ell = 2$ was found. The first equation contains $\mathbf{1}x$. ***So the first pivot was* $\mathbf{1}$** (the coefficient of x). The second equation contains $\mathbf{2}x$, **so the multiplier was $\mathbf{2}$**. Then subtraction $2x - 2x$ produced the zero and the triangle.

You will see the multiplier rule if I change the first equation to $3x - 6y = 3$. (Same straight line but the first pivot becomes 3.) The correct multiplier is now $\ell = \frac{2}{3}$. *To find that multiplier, divide the coefficient "$\mathbf{2}$" to be eliminated by the pivot "$\mathbf{3}$"*:

$$\begin{array}{lll} \mathbf{3}x - 6y = 3 & \textbf{Multiply equation 1 by } \tfrac{\mathbf{2}}{\mathbf{3}} & 3x - 6y = 3 \\ \mathbf{2}x + y \;= 7 & \textbf{Subtract from equation 2} & \qquad 5y = 5. \end{array}$$

The final system is triangular and the last equation still gives $y = 1$. Back substitution produces $3x - 6 = 3$ and $3x = 9$ and $x = 3$. We changed the numbers but not the lines or the solution. ***Divide by the pivot to find that multiplier $\boldsymbol{\ell = \frac{2}{3}}$***:

Pivot	$=$	***first nonzero in the row that does the elimination***
Multiplier	$=$	***(entry to eliminate) divided by (pivot)***

The new second equation starts with the second pivot, which is 5. We would use it to eliminate y from the third equation if there were one. *To solve n equations we want n pivots.* ***The pivots are on the diagonal of the triangle after elimination.***

You could have solved those equations for x and y without reading this book. It is an extremely humble problem, but we stay with it a little longer. Even for a 2 by 2 system, elimination might break down. By understanding the possible breakdown (when we can't find a full set of pivots), you will understand the whole process of elimination.

Breakdown of Elimination

Normally, elimination produces the pivots that take us to the solution. But failure is possible. At some point, the method might ask us to *divide by zero*. We can't do it. The process has to stop. There might be a way to adjust and continue—or failure may be unavoidable.

Example 1 fails with ***no solution*** to $0y = 5$. Example 2 fails with ***too many solutions*** to $0y = 0$. Example 3 succeeds by exchanging the equations.

Example 1 ***Permanent failure with no solution***. Elimination makes this clear:

$$\begin{array}{rcr} x - 2y = 1 & \text{Subtract } 2 \text{ times} & x - 2y = 1 \\ 2x - 4y = 7 & \text{eqn. 1 from eqn. 2} & \mathbf{0}y = 5. \end{array}$$

There is *no* solution to $0y = 5$. *This system has no second pivot.* ***(Zero is never allowed as a pivot!)*** If there is no solution, elimination discovers that fact by reaching an impossible equation like $0y = 5$.

The row picture of failure shows parallel lines—which never meet. The column picture shows the two columns $(1, 2)$ and $(-2, -4)$ in the same direction. *All combinations of the columns lie along a line.* But the column from the right side is in a different direction $(1, 7)$. No combination of the columns can produce this right side—therefore no solution.

When we change the right side from $(1, 7)$ to $(1, 2)$, failure shows as a whole line of solution points. Instead of no solution, Example 2 changes to **infinitely many solutions**.

Example 2 ***Failure with infinitely many solutions. Change b*** $= (1, 7)$ ***to*** $(1, 2)$.

$$\begin{array}{rcrl} x - 2y = 1 & \text{Subtract } 2 \text{ times} & x - 2y = 1 & \textbf{Too few pivots} \\ 2x - 4y = 2 & \text{eqn. 1 from eqn. 2} & \mathbf{0}y = \mathbf{0} & \textbf{Too many solutions} \end{array}$$

Every y satisfies $0y = 0$. There is really only one equation $x - 2y = 1$. The unknown y is ***"free"***. After y is freely chosen, x is determined as $x = 1 + 2y$. I prefer to see a *particular solution* $\boldsymbol{v}_p = (1, 0)$ and a line of *null solutions* $\boldsymbol{v}_n = c\,(2, 1)$ in $\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n$.

$$\boxed{\textbf{Complete solution} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \textbf{particular } \boldsymbol{v}_p + \textbf{null } \boldsymbol{v}_n.} \qquad (1)$$

In the row picture, the parallel lines have become the same line. Every point (x, y) on that line satisfies both equations.

In the column picture, $\boldsymbol{b} = (1, 2)$ is now the same as column 1. So we can choose $x = 1$ and $y = 0$. We can also choose $x = 0$ and $y = -\frac{1}{2}$; column 2 times $-\frac{1}{2}$ equals $\boldsymbol{b}$. Every (x, y) that solves the row problem also solves the column problem.

Failure For n equations we do not get n pivots. The rows combine into a zero row.

Success We do get n pivots. **But we may have to exchange the n equations**.

Elimination can go wrong in a third way—but this time it can be fixed. *Suppose the first pivot position contains zero.* We refuse to allow zero as a pivot. When the first equation has no term involving x, we can *exchange* it with an equation below:

Example 3 *Temporary failure (zero in pivot). A row exchange produces two pivots* :

$$\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned} \qquad \begin{matrix}\text{Exchange the}\\ \text{two equations}\end{matrix} \qquad \boxed{\begin{aligned} 3x - 2y &= 5 \\ 2y &= 4. \end{aligned}}$$

The new system is already triangular. This small example is ready for back substitution. The last equation gives $y = 2$, and then the first equation gives $x = 3$. The row picture is normal (two intersecting lines). The column picture is also normal (column vectors not in the same direction). The pivots 3 and 2 are normal—but a ***row exchange*** was required.

Examples 1 and 2 are ***singular***—there is no second pivot. Example 3 is ***nonsingular***—there is a full set of pivots and exactly one solution. Singular equations have no solution or infinitely many solutions. Pivots must be nonzero because we have to divide by them.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems. Three by three is enough to see the pattern. For now the matrices are square—an equal number of rows and columns. Here is a 3 by 3 system, specially constructed so that all steps lead to whole numbers and not fractions:

$$\begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \tag{2}$$

What are the steps? The first pivot is the boldface **2** (upper left). Below that pivot we want to eliminate the 4. *The first multiplier is the ratio* $4/2 = 2$. Multiply the pivot equation by $\ell_{21} = 2$ and subtract. Subtraction removes the $4x$ from the second equation:

Step 1 Subtract 2 times equation 1 from equation 2. This leaves $y + z = 4$.

We also eliminate $-2x$ from equation 3, still using the first pivot. The quick way is to add equation 1 to equation 3. Then $2x$ cancels $-2x$. We do exactly that, but the rule in this book is to *subtract rather than add*. The systematic pattern has multiplier $\ell_{31} = -2/2 = -1$. Subtracting -1 times an equation is the same as adding:

Step 2 Subtract -1 times equation 1 from equation 3. This leaves $y + 5z = 12$.

The two new equations involve only y and z. The second pivot (in boldface) is 1:

$$\textbf{x is eliminated} \qquad \begin{aligned} \mathbf{1}y + 1z &= 4 \\ 1y + 5z &= 12 \end{aligned}$$

We have reached a 2 *by* 2 *system.* The final step eliminates y to make it 1 by 1:

Step 3 Subtract equation 2_{new} from 3_{new}. The multiplier is $1/1 = 1$. Then $4z = 8$.

The original $A\boldsymbol{v} = \boldsymbol{b}$ has been converted into an upper triangular $U\boldsymbol{v} = \boldsymbol{c}$:

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \qquad \begin{matrix} A\boldsymbol{v} = \boldsymbol{b} \\ \text{has become} \\ U\boldsymbol{v} = \boldsymbol{c} \end{matrix} \qquad \begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ \mathbf{1}y + 1z &= 4 \\ \mathbf{4}z &= 8. \end{aligned} \tag{3}$$

The goal is achieved—forward elimination is complete from A to U. ***The pivots are*** $\mathbf{2, 1, 4}$ ***on the diagonal of*** U. The pivots 1 and 4 were hidden in the original system. Elimination brought them out. $U\boldsymbol{v} = \boldsymbol{c}$ is ready for ***back substitution***, which is quick:

$$(4z = 8 \text{ gives } z = \mathbf{2}) \quad (y + z = 4 \text{ gives } y = \mathbf{2}) \quad (\text{equation 1 gives } x = \mathbf{-1})$$

The solution is $(x, y, z) = (-1, 2, 2)$. The row picture has three planes from the three equations. All the planes go through this solution. This picture is not easy to draw (it is totally impossible for larger systems).

The column picture shows a combination $A\boldsymbol{v}$ of column vectors producing the right side $\boldsymbol{b}$. The coefficients in that combination are $-1, 2, 2$ (the solution):

$$A\boldsymbol{v} = (\mathbf{-1})\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + \mathbf{2}\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + \mathbf{2}\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} \text{ equals } \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}. \tag{4}$$

The numbers x, y, z multiply columns 1, 2, 3 in $A\boldsymbol{v} = \boldsymbol{b}$ and also in the triangular $U\boldsymbol{v} = \boldsymbol{c}$.

For a 4 by 4 problem, or an n by n problem, elimination proceeds the same way. Here is the whole idea, column by column from A to U, when elimination succeeds.

Column 1. ***Use the first equation to create zeros below the first pivot.***

Column 2. ***Use the new equation 2 to create zeros below the second pivot.***

Columns 3 to n. ***Keep going to find all*** n ***pivots and the triangular*** U.

$$\text{After column 2 we have} \begin{bmatrix} \boldsymbol{x} & x & x & x \\ 0 & \boldsymbol{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}. \quad \text{We want } U = \begin{bmatrix} \boldsymbol{x} & x & x & x \\ & \boldsymbol{x} & x & x \\ & & \boldsymbol{x} & x \\ & & & \boldsymbol{x} \end{bmatrix}. \tag{5}$$

The result of forward elimination is an upper triangular system. The matrix will be nonsingular (= *invertible*) if and only if there is a full set of n pivots (never zero!).

Here is a final example to show the original $A\boldsymbol{v} = \boldsymbol{b}$, the triangular system $U\boldsymbol{v} = \boldsymbol{c}$, and the solution $\boldsymbol{v} = (x, y, z)$ from back substitution:

$$\begin{array}{lll} x + y + z = 6 & & x+y+z=6 \\ x + 2y + 2z = 9 & \textbf{Forward} & y+z=3 \\ x + 2y + 3z = 10 & \textbf{Forward} & z=1 \end{array} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{array}{l} \textbf{Back} \\ \textbf{Back} \\ \end{array}$$

All multipliers are 1. All pivots are 1. All planes meet at the solution $\boldsymbol{v} = (3, 2, 1)$. The columns of A combine with coefficients $3, 2, 1$ to give $\boldsymbol{b} = (6, 9, 10)$:

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 10 \end{bmatrix}.$$

The numbers $6, 9, 10$ are *dot products*. The first number 6 is the dot product of the first row $(1, 1, 1)$ with $\boldsymbol{v} = (3, 2, 1)$.

Question What coefficient of z in equation 3 would make the system singular?
Answer The third pivot would drop from 1 to 0 if the original $3z$ dropped to $2z$. Then the planes in the row picture have no point in common.

There is no solution to the new $\boldsymbol{Av} = \boldsymbol{b}$. The three columns in the column picture would lie in the same plane, and $\boldsymbol{b} = (6, 9, 10)$ is not in that plane. So $\boldsymbol{b}$ will not be a combination of the columns, if the third column becomes $(1, 2, 2)$. In this example column 3 becomes the same as column 2—useless, we need "independent" columns!

Question What coefficient of y in equation 2 would become 0 in the first elimination step? Would the system become singular or not?

Answer Change equation 2 to $x + y + 2z = 7$ (for example). The coefficient of y is now 1. Subtracting equation 1 leaves $\mathbf{0}y + z = 3$. **Now we can exchange equations 2 and 3**. This system is nonsingular. No problem except equations in the wrong order.

■ REVIEW OF THE KEY IDEAS ■

1. A linear system $A\boldsymbol{v} = \boldsymbol{b}$ becomes upper triangular $(U\boldsymbol{v} = \boldsymbol{c})$ by elimination.

2. We subtract ℓ_{ij} times equation j from equation i, to make the (i, j) entry zero.

3. The multiplier is $\ell_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$. Pivots can not be zero!

4. A zero in the pivot position can be exchanged if there is a nonzero below it.

5. Back substitution solves the upper triangular system (bottom to top).

6. When breakdown is permanent, the system has no solution or infinitely many.

Problem Set 4.2

Problems 1–10 are about elimination on 2 by 2 systems.

1 What multiple ℓ_{21} of equation 1 should be subtracted from equation 2 ?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this step, solve the triangular system by back substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution ?

2 If you find solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ to $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{c}$, what is the solution $\boldsymbol{u}$ to $A\boldsymbol{u} = \boldsymbol{b} + \boldsymbol{c}$? What is the solution $\boldsymbol{U}$ to $A\boldsymbol{U} = 3\boldsymbol{b} + 4\boldsymbol{c}$? (We saw superposition for linear differential equations, it works in the same way for all linear equations.)

3 What multiple of equation 1 should be *subtracted* from equation 2 ?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution ?

4 What multiple ℓ of equation 1 should be subtracted from equation 2 to remove cx ?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot ? The second pivot is missing when $ad = bc$: that is the *singular case*.

5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions ?

Singular system $$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

7 For which a does elimination break down (1) permanently or (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the temporary breakdown by a row exchange.

8 For which three numbers k does elimination break down? Which is fixed by a row exchange? In these three cases, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

9 What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture for $\boldsymbol{b} = (1, 2)$ and $(1, 0)$.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

10 In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ _____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c =$ _____ .

11 (Recommended) A system of linear equations can't have exactly two solutions. If (x, y) and (X, Y) are two solutions to $A\boldsymbol{v} = \boldsymbol{b}$, what is another solution?

Problems 12–20 study elimination on 3 by 3 systems (and possible failure).

12 Reduce this system to upper triangular form by two row operations:

$$\begin{array}{lr} & 2x + 3y + z = 8 \\ \text{Eliminate } x \ \rightarrow & 4x + 7y + 5z = 20 \\ \text{Eliminate } y \ \rightarrow & -2y + 2z = 0. \end{array}$$

Circle the pivots. Solve by back substitution for z, y, x.

13 Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+ z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract _____ times row _____ from row _____ .

14 Which number d forces a row exchange? What is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

15 Which number b leads later to a row exchange? Which b leads to a singular problem that row exchanges cannot fix? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \quad &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

16 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form.

(b) Construct a 3 by 3 system that needs a row exchange for pivot 2, but breaks down for pivot 3.

17 If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{array}{ll} \textbf{Equal} & 2x - y + z = 0 \\ \textbf{rows} & 2x - y + z = 0 \\ & 4x + y + z = 2 \end{array} \qquad \begin{array}{ll} 2x + 2y + z = 0 & \textbf{Equal} \\ 4x + 4y + z = 0 & \textbf{columns} \\ 6x + 6y + z = 2. & \end{array}$$

18 Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\boldsymbol{b} = (1, 10, 100)$ and how many with $\boldsymbol{b} = (0, 0, 0)$?

19 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

20 Three planes can fail to have an intersection point, *even if no planes are parallel.* The system is singular if row 3 is a combination of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

21 Find the pivots and the solution for both systems ($A\boldsymbol{v} = \boldsymbol{b}$ and $S\boldsymbol{w} = \boldsymbol{b}$):

$$\begin{aligned} 2x + y \qquad\qquad &= 0 \\ x + 2y + z \qquad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned} \qquad\qquad \begin{aligned} 2x - y \qquad\qquad &= 0 \\ -x + 2y - z \qquad &= 0 \\ - y + 2z - t &= 0 \\ - z + 2t &= 5. \end{aligned}$$

22 If you extend Problem 21 following the $1, 2, 1$ pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot? S is my favorite matrix.

23 If elimination leads to $x + y = 1$ and $2y = 3$, find three possible original problems.

24 For which two numbers a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

25 For which three numbers a will elimination fail to give three pivots ?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of } a.$$

26 Look for a matrix that has row sums 4 and 8, and column sums 2 and s :

$$\text{Matrix } = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if $s =$ _____. Then find two different matrices that have the correct row and column sums. *Extra credit*: Write down the 4 by 4 system $A\boldsymbol{v} = (4, 8, 2, s)$ with $\boldsymbol{v} = (a, b, c, d)$ and make A triangular by elimination.

27 Elimination in the usual order gives what matrix U and what solution (x, y, z) to this "lower triangular" system ? We are really solving by *forward substitution*:

$$\begin{aligned} 3x \qquad\qquad &= 3 \\ 6x + 2y \qquad &= 8 \\ 9x - 2y + z &= 9. \end{aligned}$$

28 Create a MATLAB command `A(2, : ) = ...` for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix A is already known.

29 If the last corner entry of A is $A(5, 5) = 11$ and the last pivot of A is $U(5, 5) = 4$, what different entry $A(5, 5)$ would have made A singular ?

Challenge Problems

30 Suppose elimination takes A to U without row exchanges. Then row i of U is a combination of which rows of A ? If $A\boldsymbol{v} = \mathbf{0}$, is $U\boldsymbol{v} = \mathbf{0}$? If $A\boldsymbol{v} = \boldsymbol{b}$, is $U\boldsymbol{v} = \boldsymbol{b}$?

31 Start with 100 equations $A\boldsymbol{v} = \mathbf{0}$ for 100 unknowns $\boldsymbol{v} = (v_1, \ldots, v_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".

(a) Elimination takes linear combinations of the rows. So this singular system has the singular property : Some linear combination of the 100 **rows** is _____.

(b) Singular systems $A\boldsymbol{v} = \mathbf{0}$ have infinitely many solutions. This means that some linear combination of the 100 ***columns*** is _____.

(c) Invent a 100 by 100 singular matrix with no zero entries.

(d) For your matrix, describe in words the row picture and the column picture of $A\boldsymbol{v} = \mathbf{0}$. Not necessary to draw 100-dimensional space.

4.3 Matrix Multiplication

We know how to multiply A times a column vector $\boldsymbol{v}$. Now we want to multiply A times a matrix B (matrix-matrix multiplication). The rule is exactly what we would hope for:

> **Multiply A times each column of B to get a column of AB**
>
> The entry in row i, column j of AB is (**row $\boldsymbol{i}$ of $\boldsymbol{A}$**) $\cdot$ (**column $\boldsymbol{j}$ of $\boldsymbol{B}$**)

If B has only one column (call it $\boldsymbol{v}$), this is the same matrix-vector multiplication as before. When B has n columns, so has AB. The rule for matrix sizes makes dot products possible.

Rule The number of columns in A must match the number of rows in B.

Figure 4.8 shows a typical (row i) $\cdot$ (column j) in the matrix multiplication AB.

$$\begin{bmatrix} * & & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ * & & & \\ * & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

A is **4** by **5** $\qquad$ B is **5** by **6** $\qquad$ AB is **4** by **6**

Figure 4.8: Here $i = 2$ and $j = 3$. Then $(AB)_{23}$ is (row 2 of A) $\cdot$ (column 3 of B).

Let me say right away that normally AB is entirely different from BA. Those have different shapes unless A and B are square and the same size. But even the top left corner of BA has nothing to do with the top left corner of AB (and then $\boldsymbol{BA \neq AB}$).

Top left (row 1 of B) $\cdot$ (column 1 of A) $\neq$ (row 1 of A) $\cdot$ (column 1 of B).

Example 1 Here A has two columns and B has two rows. We can multiply AB.

$$A_{\mathbf{2}\times\mathbf{2}}\, B_{\mathbf{2}\times\mathbf{3}} = (AB)_{\mathbf{2}\times\mathbf{3}} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} a & b & \boldsymbol{a+b} \\ c & d & \boldsymbol{c+d} \end{bmatrix}.$$

Column 3 of B is $(1, 1)$. Then column 3 of AB is A times $(1, 1)$.

Example 2 Here B is the 3 by 3 **identity matrix** (very special, always written $B = I$).

$\boldsymbol{B}$ = Identity matrix $\boldsymbol{I}$
$\boldsymbol{AI = A}$ when sizes are right

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The first column of that answer is A times the first column (1, 0, 0) of $B = I$. This just reproduces the first column of A. Each column of A is unchanged in AI.

Now put the identity matrix first, as in IB. Multiplication gives $IB = B$ for every B (including $B = A$). We have here an unusual case, when the order AI gives the same answer as IA. If A is any square matrix and I has the same size, then $\boldsymbol{AI = IA = A}$.

Example 3 Another special matrix is the **inverse** of A. That matrix B is written A^{-1}:

$$\textbf{\textit{A} times } \boldsymbol{A^{-1}} \textbf{ is } \boldsymbol{I} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The dot product of a row of A with a column of A^{-1} is 1 or 0. A^{-1} times A is also I.

To find that matrix A^{-1}, I had to look ahead to Section 4.4—this is a long calculation. We avoid computing A^{-1} wherever possible, and so does any good linear algebra code.

$$\boxed{\textbf{The key fact about matrix multiplication is that} \quad (\boldsymbol{AB})\,\boldsymbol{C} = \boldsymbol{A}\,(\boldsymbol{BC}).} \qquad (1)$$

To multiply three matrices A, B, C you must keep them in order. But you can choose to multiply AB first or BC first. *Parentheses can be moved, and parentheses can be removed.*

Example 4 Suppose A and C are 3 by 1 matrices (those are column vectors). Suppose B is 1 by 3 (a row vector). Compute and compare $(AB)C$ and $A(BC)$.

Solution BC is (1×3) times $(3 \times 1) = 1 \times 1$. One number d from one dot product:

$$\textbf{\textit{A} times \textit{BC}} \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \left(\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) = \begin{bmatrix} a_1 d \\ a_2 d \\ a_3 d \end{bmatrix}. \qquad (2)$$

On the other hand, AB is (3×1) times $(1 \times 3) = 3 \times 3$. This AB is a full-size matrix !

$$\textbf{\textit{AB} times \textit{C}} \qquad \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \qquad (3)$$

If you multiply that first row of AB times C, you will see $a_1 d$. Multiplying the other rows by C gives $a_2 d$ and $a_3 d$. $\boldsymbol{(AB)C}$ **in equation (3) equals** $\boldsymbol{A(BC)}$ **in equation (2).**

The Laws for Matrix Operations

May I put on record six laws that matrices do obey, while emphasizing an equation they don't obey ? The matrices can be square or rectangular, and the laws involving $A + B$ are all simple and all obeyed. Here are three addition laws:

$$\begin{array}{lll} A + B & = B + A & \text{(commutative law)} \\ c\,(A + B) & = c\,A + c\,B & \text{(distributive law)} \\ A + (B + C) & = (A + B) + C & \text{(associative law).} \end{array}$$

Three more laws hold for multiplication, but $AB = BA$ is not one of them:

$AB \neq BA$ (the commutative "law" is *usually broken*)

$A(B + C) = AB + AC$ (distributive law from the left)

$(A + B)C = AC + BC$ (distributive law from the right)

$A(BC) = (AB)C$ (associative law for ABC) (***parentheses not needed***).

When A and B are not square, AB is a different size from BA. These matrices can't be equal—even if both multiplications are allowed. For square matrices, almost any example shows that AB is different from BA:

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}.$$

It is true that $AI = IA$. All square matrices commute with I and also with cI. Only these matrices cI commute with all other matrices.

The law $A(B + C) = AB + AC$ is proved a column at a time. Start with $A(\boldsymbol{b} + \boldsymbol{c}) = A\boldsymbol{b} + A\boldsymbol{c}$ for the first column. That is the key to everything—***linearity***. Say no more.

Powers of Matrices

Look at the special case when $A = B = C =$ square matrix. Then $(A \textit{ times } A^2)$ is equal to $(A^2 \textit{ times } A)$. The product in either order is A^3. The matrix powers A^p follow the same rules as numbers:

$$A^p = AAA \cdots A \ (p \text{ factors}) \qquad (A^p)(A^q) = A^{p+q} \qquad (A^p)^q = A^{pq}.$$

Those are the ordinary laws for exponents. A^3 times A^4 is A^7 (seven factors). A^3 to the fourth power is A^{12} (twelve A's). When p and q are zero or negative these rules still hold, provided A has a "-1 power"—which is the *inverse matrix* A^{-1}. Then $A^0 = I$ is the identity matrix (no factors).

For a number, a^{-1} is $1/a$. For a matrix, the inverse is written A^{-1}. (It is *never* I/A. But backslash $A\backslash I$ is allowed in MATLAB.) Every number has an inverse except $a = 0$. To decide when A has an inverse is a central problem in linear algebra. This section is like a Bill of Rights for matrices, to say when A and B can be multiplied and how.

Elimination Matrices

We now combine two ideas—elimination and matrices. The goal is to express all the steps of elimination in the clearest possible way. You will see how to subtract a multiple ℓ_{ij} times row j from row i—using a matrix E.

The column vector $\boldsymbol{b}$ is multiplied by the elimination matrix E:

$$\textbf{Subtract } 2b_1 \textbf{ from } b_2 \qquad E\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & \mathbf{1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ \boldsymbol{b_2 - 2b_1} \\ b_3 \end{bmatrix}. \tag{4}$$

Whatever we do to one side of $A\boldsymbol{v} = \boldsymbol{b}$, we do to the other side. *Elimination is multiplying both sides by E.* On the left side, we see row operations.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & \mathbf{1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{bmatrix} = \begin{bmatrix} \text{row 1} \\ \textbf{row 2} - \textbf{2 row 1} \\ \text{row 3} \end{bmatrix}. \tag{5}$$

EA will be our matrix after the first elimination step. The multiplier 2 was chosen to produce 0 in the 2, 1 position (row 2, column 1). This matrix E should be named E_{21} because it eliminates the original entry a_{21} to leave zero.

The next step of elimination comes from a matrix E_{31} (producing zero in place of a_{31}). Then E_{32} produces zero in row 3, column 2, using a multiplier ℓ_{32}. Altogether, the three steps from A to the upper triangular U come from three elimination matrices:

Elimination by matrices $\quad A$ becomes $\quad E_{32}E_{31}E_{21}A = U$ (upper triangular).

We do the same operations on the right side. $E_{32}E_{31}E_{21}\boldsymbol{b}$ becomes the new right side vector $\boldsymbol{c}$. Then back substitution solves $U\boldsymbol{v} = \boldsymbol{c}$.

Example 5 Choose the multiplier $\ell_{21} = c/a$ to produce zero in U_{21}, using $E = E_{21}$:

$$EA = \begin{bmatrix} 1 & 0 \\ -\boldsymbol{c/a} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d-(c/a)b \end{bmatrix} = U. \tag{6}$$

Undo this elimination by **adding** c/a times row 1 of U to row 2 of U:

$$E^{-1}U = \begin{bmatrix} 1 & 0 \\ \boldsymbol{c/a} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d-(c/a)b \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$

Thus $U = EA$ and $A = E^{-1}U$. Often we write this as $A = LU$.

Four Ways to Multiply AB

I will end this section by writing down four different ways to compute AB. All four ways give the same answer. In the end we are doing the same calculations, but we are seeing those steps in different orders.

1. (Rows of A) times (columns of B) (*dot products*)
2. A times (columns of B) (matrix-vector multiplications)
3. (Rows of A) times B (vector-matrix multiplications)
4. (Columns of A) times (rows of B) (add up n column-times-row matrices)

Let me look at the 1, 1 entry in the top corner of AB. The usual way is a dot product:

$$(\text{row 1 of } A) \cdot (\text{column 1 of } B) = (AB)_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \tag{7}$$

Orders **2** and **3** give that same dot product in AB. Here is order **4**, *columns times rows*:

$$(\text{column 1 of } A)(\text{row 1 of } B) = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdot \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{8}$$

The next column-times-row matrix is (column 2 of A)(row 2 of B). That starts with $a_{12}b_{21}$ in the top left corner. We get $a_{1j}b_{j1}$ when column j of A multiplies row j of B.

Adding these simple matrices will produce the correct dot product (the sum of $a_{1j}b_{j1}$) in the top left corner—and in every entry of AB.

When A and B are n by n matrices, so is AB. It contains n^2 dot products. So it needs n^3 separate multiplications. For matrices of order $n = 100$ this is a million multiplications. No problem, that may only take one second (on the computer).

When A is an $\boldsymbol{m}$ **by** $\boldsymbol{n}$ matrix and B is $\boldsymbol{n}$ **by** $\boldsymbol{p}$, the product AB is $\boldsymbol{m}$ **by** $\boldsymbol{p}$. It contains mp dot products. So it needs mnp separate multiplications.

Matrices of order $n = 10,000$ need a trillion (10^{12}) multiplications. Codes avoid multiplying full matrices whenever possible. And they watch especially for *sparse matrices*, when many of the entries (almost all) are zero. The codes don't waste time multiplying by zero.

Problem Set 4.3

Problems 1–16 are about the laws of matrix multiplication .

1 A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results ?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B+C).$$

2 What rows or columns or matrices do you multiply to find

(a) the third column of AB ?

(b) the first row of AB ?

(c) the entry in row 3, column 4 of AB ?

(d) the entry in row 1, column 1 of CDE ?

3 Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

4 In Problem 3, multiply A times BC. Then multiply AB times C.

5 Compute A^2 and A^3. Make a prediction for A^5 and A^n :

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

6 Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \underline{\qquad} + B^2$.

7 True or false. Give a specific example when false:

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

(d) $(AB)^2 = A^2B^2$.

8 How is each row of DA and EA related to the rows of A, when

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ?$$

How is each column of AD and AE related to the columns of A?

9 Row 1 of A is added to row 2. This gives EA below. Then column 1 of EA is added to column 2 to produce $(EA)F$. Notice E and F in boldface.

$$EA = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

$$(EA)F = (EA) \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+c & a+c+b+d \end{bmatrix}.$$

Do those steps in the opposite order, first multiply AF and then $E(AF)$. Compare with $(EA)F$. What law is obeyed by matrix multiplication?

10 Row 1 of A is added to row 2 to produce EA. Then F adds row 2 of EA to row 1. Now F is on the left, for row operations. The result is $F(EA)$:

$$F(EA) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix}.$$

Do those steps in the opposite order: first add row 2 to row 1 by FA, then add row 1 of FA to row 2. What law is or is not obeyed by matrix multiplication?

11 (3 by 3 matrices) Choose the only B so that for every matrix A

(a) $BA = 4A$

(b) $BA = 4B$ (tricky)

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged

(d) All rows of BA are the same as row 1 of A.

12 Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I. The only matrices that commute with B and C and all other 2 by 2 matrices are $A =$ multiple of I.

13 Which of the following matrices are guaranteed to equal $(A-B)^2$: $A^2 - B^2$, $(B-A)^2$, $A^2 - 2AB + B^2$, $A(A-B) - B(A-B)$, $A^2 - AB - BA + B^2$?

14 True or false :

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

15 If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector $\boldsymbol{x}$ with n components ?

(b) A multiplies an n by p matrix B ?

(c) A multiplies itself to produce A^2 ? Here $m = n$ and A is square.

16 For $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, compute these answers *and nothing more* :

(a) column 2 of AB (b) row 2 of AB (c) row 2 of A^2

(d) row 2 of A^3.

Problems 17–19 use a_{ij} for the entry in row i, column j of A.

17 Write down the 3 by 3 matrix A whose entries are

(a) $a_{ij} =$ minimum of i and j (b) $a_{ij} = (-1)^{i+j}$ (c) $a_{ij} = i/j$.

18 What words would you use to describe each of these classes of matrices ? Give a 3 by 3 example in each class. Which matrix belongs to all four classes ?

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i < j$ (c) $a_{ij} = a_{ji}$

(d) $a_{ij} = a_{1j}$.

19 The entries of A are a_{ij}. Assuming that zeros don't appear, what is

(a) the first pivot ?

(b) the multiplier ℓ_{31} of row 1 to be subtracted from row 3 ?

(c) the new entry that replaces a_{32} after that subtraction ?

(d) the second pivot ?

Problems 20–24 involve powers of A.

20 Compute A^2, A^3, A^4 and also $A\boldsymbol{v}$, $A^2\boldsymbol{v}$, $A^3\boldsymbol{v}$, $A^4\boldsymbol{v}$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

21 Find all the powers $A^2, A^3, \ldots$ and $AB, (AB)^2, \ldots$ for

$$A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

22 By trial and error find real nonzero 2 by 2 matrices such that

$$A^2 = -I \qquad BC = 0 \qquad DE = -ED \text{ (not allowing } DE = 0\text{)}.$$

23 (a) Find a nonzero matrix A for which $A^2 = 0$.

(b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.

24 By experiment with $n = 2$ and $n = 3$ predict A^n for these matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Problems 25–31 use column-row multiplication and block multiplication.

25 Multiply A times I using columns of A (3 by 3) times rows of I.

26 Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\qquad} = \underline{\qquad}.$$

27 Show that the product of two upper triangular matrices is always upper triangular:

$$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x & & \\ 0 & & \\ 0 & 0 & x \end{bmatrix}.$$

Proof using dot products (*Row-times-column*) (Row 2 of A) $\cdot$ (column 1 of B) $= 0$. Which other dot products give zeros?

Proof using full matrices (*Column-times-row*) Draw x's and 0's in (column 2 of A) times (row 2 of B). Also show (column 3 of A) times (row 3 of B).

28 If A is 2 by 3 with rows 1, 1, 1 and 2, 2, 2, and B is 3 by 4 with columns 1, 1, 1 and 2, 2, 2 and 3, 3, 3 and 4, 4, 4, use each of the four multiplication rules to find AB:

(1) Rows of A times columns of B. **Inner products** (each entry in AB)

(2) Matrix A times columns of B. **Columns of AB**

(3) Rows of A times the matrix B. **Rows of AB**

(4) Columns of A times rows of B. **Outer products** (3 matrices add to AB)

29 Which matrices E_{21} and E_{31} produce zeros in the $(2,1)$ and $(3,1)$ positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA.

30 **Block multiplication** produces zeros below the pivot in one big step:

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & D - \boldsymbol{c}\,\boldsymbol{b}/a \end{bmatrix} \text{ with vectors } \mathbf{0}, \boldsymbol{b}, \boldsymbol{c}.$$

In Problem 29, what are $\boldsymbol{c}$ and D and what is the block $D - \boldsymbol{c}\,\boldsymbol{b}/a$?

31 With $i^2 = -1$, the product of $(A + iB)$ and $(\boldsymbol{x} + i\boldsymbol{y})$ is $A\boldsymbol{x} + iB\boldsymbol{x} + iA\boldsymbol{y} - B\boldsymbol{y}$. Use blocks to separate the real part without i from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

32 (*Very important*) Suppose you solve $A\boldsymbol{v} = \boldsymbol{b}$ for three special right sides $\boldsymbol{b}$:

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are the columns of a matrix X, what is A times X?

33 If the three solutions in Question 32 are $\boldsymbol{v}_1 = (1,1,1)$ and $\boldsymbol{v}_2 = (0,1,1)$ and $\boldsymbol{v}_3 = (0,0,1)$, solve $A\boldsymbol{v} = \boldsymbol{b}$ when $\boldsymbol{b} = (3,5,8)$. Challenge problem: What is A?

34 **Practical question** Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The same answer comes from A times BC, now with $mnq + npq$ separate multiplications. Notice npq for BC.

(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?

(b) With N-component vectors, would you choose $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ or $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$?

(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

35 **Unexpected fact** A friend in England looked at powers of a 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \quad A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \quad A^4 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

He noticed that the ratios $2/3$ and $10/15$ and $54/81$ are all the same. This is true for all powers. It doesn't work for an $n \times n$ matrix, unless A is tridiagonal. One neat proof is to look at the equal $(1,1)$ entries of A^nA and AA^n. Can you use that idea to show that $B/C = 2/3$ in this example?

4.4 Inverse Matrices

Suppose A is a square matrix. We look for an "***inverse matrix***" A^{-1} of the same size, so that *A^{-1} times A equals I.* Whatever A does, A^{-1} undoes. Their product is the identity matrix—which leaves all vectors unchanged, so $A^{-1}A\boldsymbol{v} = \boldsymbol{v}$. *But A^{-1} might not exist.*

What a matrix mostly does is to multiply a vector $\boldsymbol{v}$. Multiplying $A\boldsymbol{v} = \boldsymbol{b}$ by A^{-1} gives $A^{-1}A\boldsymbol{v} = A^{-1}\boldsymbol{b}$. ***This is $\boldsymbol{v} = A^{-1}\boldsymbol{b}$.*** The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. A number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix A^{-1} is called "A inverse."

DEFINITION The matrix A is ***invertible*** if there exists a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \tag{1}$$

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible? We don't mean that we immediately calculate A^{-1}. In most problems we never compute it! Here are six "notes" about A^{-1}.

Note 1 ***A^{-1} exists if and only if elimination produces n pivots*** (row exchanges are allowed). Elimination solves $A\boldsymbol{v} = \boldsymbol{b}$ without explicitly using the matrix A^{-1}.

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \tag{2}$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix.*

Note 3 If A is invertible, the one and only solution to $A\boldsymbol{v} = \boldsymbol{b}$ is $\boldsymbol{v} = A^{-1}\boldsymbol{b}$:

$$\textit{Multiply} \quad A\boldsymbol{v} = \boldsymbol{b} \quad \textit{by} \quad A^{-1}. \quad \textit{Then} \quad \boldsymbol{v} = A^{-1}A\boldsymbol{v} = A^{-1}\boldsymbol{b}.$$

Note 4 (Important) ***Suppose there is a nonzero vector $\boldsymbol{v}$ such that $A\boldsymbol{v} = \boldsymbol{0}$. Then A cannot have an inverse.*** No matrix can bring $\boldsymbol{0}$ back to $\boldsymbol{v}$.

If A is invertible, then $A\boldsymbol{v} = \boldsymbol{0}$ can only have the zero solution $\boldsymbol{v} = A^{-1}\boldsymbol{0} = \boldsymbol{0}$.

Note 5 A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\textbf{2 by 2 Inverse} \atop \textbf{Divide by } \boldsymbol{ad-bc} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad - bc$ is the **determinant** of A. A matrix is invertible if its determinant is not zero. A^{-1} always involves a division by the determinant of A.

Note 6 A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

Example 1 The 2 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $A\boldsymbol{v} = \mathbf{0}$ when $\boldsymbol{v} = (2, -1)$. It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix A into a zero row.

The Inverse of a Product AB

For two nonzero numbers a and b, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For two matrices A and B, the situation is similar. It is hard to say much about the invertibility of $A + B$. But the *product* AB has an inverse, if and only if the two factors A and B are separately invertible (and the same size). The important point is that A^{-1} and B^{-1} come in ***reverse order***:

If A and B are invertible then so is AB. The inverse of a product AB is

$$\boldsymbol{(AB)^{-1} = B^{-1}A^{-1}}. \tag{4}$$

To see why the order is reversed, multiply AB times $B^{-1}A^{-1}$. Inside that is $BB^{-1} = I$:

$$\textbf{Inverse of } AB \qquad (AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I. This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____.

The same reverse order applies to three or more matrices:

$$\textbf{Reverse order} \qquad \boldsymbol{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}. \tag{5}$$

Example 2 ***Inverse of an elimination matrix.*** If E subtracts 5 times row 1 from row 2, then E^{-1} *adds* 5 times row 1 to row 2:

$$\begin{matrix}\textbf{\textit{E} subtracts} \\ \boldsymbol{E^{-1}} \textbf{ adds}\end{matrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiply EE^{-1} to get the identity matrix I. Also multiply $E^{-1}E$ to get I. We are adding and subtracting the same 5 times row 1. Whether we add and then subtract (this is EE^{-1}) or subtract and then add (this is $E^{-1}E$), we are back at the start.

For square matrices, an inverse on one side is automatically an inverse on the other side.

If $AB = I$ then automatically $BA = I$ for square matrices. In that case B is A^{-1}. This is extremely useful to know but we are not ready to prove it.

Example 3 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 2 to find FE. Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the required order $(FE)^{-1} = E^{-1}F^{-1}$ for the inverses.

Right order
Good inverse

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{5} & 1 & 0 \\ 0 & \mathbf{4} & 1 \end{bmatrix}. \tag{6}$$

The result is beautiful and correct. The product FE contains "20" but its inverse doesn't. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the *new* row 2 (changed by row 1) from row 3. ***In this order FE, row 3 feels an effect from row 1.***

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn't change again. ***In this order $E^{-1}F^{-1}$, row 3 feels no effect from row 1.***

$E^{-1}F^{-1}$ is quick. The multipliers 5, 4 fall into place below the diagonal of 1's.

Calculating A^{-1} by Gauss-Jordan Elimination

I hinted that A^{-1} might not be explicitly needed. The equation $A\boldsymbol{v} = \boldsymbol{b}$ is solved by $\boldsymbol{v} = A^{-1}\boldsymbol{b}$. But it is not necessary or efficient to compute A^{-1} and multiply it times $\boldsymbol{b}$. *Elimination goes directly to $\boldsymbol{v}$.* Elimination is also the way to find A^{-1}, as we now show.

The Gauss-Jordan idea is to solve $AA^{-1} = I$. Find each column of A^{-1}.

A multiplies the first column of A^{-1} (call that $\boldsymbol{v}_1$) to give the first column of I (call that $\boldsymbol{e}_1$). This is our equation $A\boldsymbol{v}_1 = \boldsymbol{e}_1 = (1,0,0)$. There will be two more equations. Each of the columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of A^{-1} is multiplied by A to produce a column of I:

3 columns of A^{-1}

$$AA^{-1} = A\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = I. \tag{7}$$

To invert a 3 by 3 matrix A, we have to solve three systems of equations: $A\boldsymbol{v}_1 = \boldsymbol{e}_1$ and $A\boldsymbol{v}_2 = \boldsymbol{e}_2 = (0,1,0)$ and $A\boldsymbol{v}_3 = \boldsymbol{e}_3 = (0,0,1)$. Gauss-Jordan finds A^{-1} this way.

The ***Gauss-Jordan method*** computes A^{-1} by solving ***all n equations together***.

Usually the "augmented matrix" $[A\ \ b]$ has one extra column b. Now we have three right sides (the columns of I). So the augmented matrix is the block matrix $[\,A\ \ I\,]$.

$$[\,A\ \ e_1\ \ e_2\ \ e_3\,] = \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \qquad \textbf{Start Gauss-Jordan on } [\,A\ \ I\,]$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{\frac{3}{2}} & \mathbf{-1} & \mathbf{\frac{1}{2}} & \mathbf{1} & \mathbf{0} \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad (\tfrac{1}{2}\ \boldsymbol{row}\ \mathbf{1} + \boldsymbol{row}\ \mathbf{2})$$

$$\to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{\frac{4}{3}} & \mathbf{\frac{1}{3}} & \mathbf{\frac{2}{3}} & \mathbf{1} \end{bmatrix} \qquad (\tfrac{2}{3}\ \boldsymbol{row}\ \mathbf{2} + \boldsymbol{row}\ \mathbf{3})$$

We are halfway to A^{-1}. The matrix in the first three columns is U (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. *Jordan's idea is to continue with elimination*! He goes all the way to the ***identity matrix***.

Rows are subtracted from rows *above*, to produce ***zeros above the pivots***:

$$\begin{pmatrix}\text{Zero above}\\ \text{third pivot}\end{pmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{\frac{3}{2}} & \mathbf{0} & \mathbf{\frac{3}{4}} & \mathbf{\frac{3}{2}} & \mathbf{\frac{3}{4}} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \qquad (\tfrac{3}{4}\ \boldsymbol{row}\ \mathbf{3} + \boldsymbol{row}\ \mathbf{2})$$

$$\begin{pmatrix}\text{Zero above}\\ \text{second pivot}\end{pmatrix} \to \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{\frac{3}{2}} & \mathbf{1} & \mathbf{\frac{1}{2}} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \qquad (\tfrac{2}{3}\ \boldsymbol{row}\ \mathbf{2} + \boldsymbol{row}\ \mathbf{1})$$

The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1. We have reached I in the first half of the matrix, because A is invertible.

The three columns of A^{-1} are in the second half of $[\,I\ \ A^{-1}\,]$:

$$\begin{matrix}(\text{divide by } 2)\\ (\text{divide by } \frac{3}{2})\\ (\text{divide by } \frac{4}{3})\end{matrix} \quad \left[\begin{array}{ccc|ccc} \mathbf{1} & 0 & 0 & \mathbf{\frac{3}{4}} & \mathbf{\frac{1}{2}} & \mathbf{\frac{1}{4}} \\ 0 & \mathbf{1} & 0 & \mathbf{\frac{1}{2}} & \mathbf{1} & \mathbf{\frac{1}{2}} \\ 0 & 0 & \mathbf{1} & \mathbf{\frac{1}{4}} & \mathbf{\frac{1}{2}} & \mathbf{\frac{3}{4}} \end{array}\right] = [\,I\ \ \boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \boldsymbol{x}_3\,] = [\,\boldsymbol{I}\ \ \boldsymbol{A}^{-1}\,].$$

Starting from the 3 by 6 matrix $[\,A\ \ I\,]$, we ended with $[\,I\ \ A^{-1}\,]$. Here is the whole Gauss-Jordan process on one line for any invertible matrix A:

Gauss-Jordan ***Multiply $[\,\boldsymbol{A}\ \ \boldsymbol{I}\,]$ by $\boldsymbol{A}^{-1}$ to get $[\,\boldsymbol{I}\ \ \boldsymbol{A}^{-1}\,]$.***

The elimination steps create the inverse matrix while changing A to I. For large matrices, we probably don't want A^{-1} at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular A^{-1} because it is an important example. We introduce the words *symmetric*, *tridiagonal*, and *determinant*:

1. A is ***symmetric*** across its main diagonal. So is A^{-1}.

2. A is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a sparse matrix is generally a full matrix.

3. The *product of pivots* is $2(\frac{3}{2})(\frac{4}{3}) = 4$. This number 4 is the ***determinant*** of A.

A^{-1} involves division by the determinant
$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \tag{8}$$

This is why an invertible matrix cannot have a zero determinant.

Example 4 Find A^{-1} by Gauss-Jordan elimination starting from $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$. There are two row operations and then a division to put 1's in the pivots:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & 1 & 0 \\ \mathbf{4} & \mathbf{7} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix} \quad \left(\text{this is } \begin{bmatrix} U & L^{-1} \end{bmatrix}\right)$$

$$\to \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{7} & \mathbf{-3} \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \mathbf{\frac{7}{2}} & -\mathbf{\frac{3}{2}} \\ 0 & 1 & \mathbf{-2} & \mathbf{1} \end{bmatrix} \quad \left(\text{this is } \begin{bmatrix} I & \boldsymbol{A^{-1}} \end{bmatrix}\right).$$

That A^{-1} involves division by the determinant $ad - bc = 2 \cdot 7 - 3 \cdot 4 = 2$. The matrix A must be invertible, or elimination cannot reduce it to I (in the left half of $\begin{bmatrix} I & A^{-1} \end{bmatrix}$).

Gauss-Jordan shows why A^{-1} is expensive. We must solve n equations for its n columns.

To solve $Av = b$ without A^{-1}, we deal with *one* column b to find one column v.

In defense of A^{-1}, we want to say that its cost is not n times the cost of one system. Surprisingly, the cost for n columns is only multiplied by 3. This saving is because the n equations $A\boldsymbol{v}_i = \boldsymbol{e}_i$ all involve the same matrix A. Working with the right sides is relatively cheap, because elimination only has to be done once on A.

The complete A^{-1} needs n^3 elimination steps, where one equation needs $n^3/3$.

Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: ***A^{-1} exists exactly when A has a full set of n pivots.*** (Row exchanges are allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With n pivots, elimination solves all the equations $A\boldsymbol{v}_i = \boldsymbol{e}_i$. The columns $\boldsymbol{v}_i$ go into A^{-1}. Then $AA^{-1} = I$ and A^{-1} is at least a ***right-inverse***.

2. Elimination is really a sequence of multiplications by E's and P's and D^{-1}:

Left-inverse of A
$$(D^{-1} \cdots E \cdots P \cdots E)A = I. \tag{9}$$

D^{-1} divides by the pivots. The matrices E produce zeros below and above the pivots. Permutations P will exchange rows if needed. The product matrix in equation (9) is a ***left-inverse***. With n pivots we have reached $A^{-1}A = I$.

The right-inverse equals the left-inverse. That was Note 2 at the start of in this section. So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will now show that A ***must have n pivots if*** $AC = I$. (Then we deduce that C is also a left-inverse and $CA = I$.) Here is one route to those conclusions:

1. If A doesn't have n pivots, elimination will lead to a *zero row*.

2. Those elimination steps are taken by an invertible M. *So a row of MA is zero.*

3. If $AC = I$ had been possible, then $MAC = M$. The zero row of MA, times C, gives a zero row of M itself.

4. An invertible matrix M can't have a zero row! So A *must* have n pivots if $AC = I$.

That argument took four steps, but the outcome is short and important.

Elimination gives a complete test for invertibility of a square matrix. ***A^{-1} exists (and Gauss-Jordan finds it) exactly when A has n pivots***. The argument above shows more:

$$\boxed{\text{If} \quad AC = I \quad \text{then} \quad CA = I \quad \text{and} \quad C = A^{-1}}$$

Example 5 Here L is lower triangular with 1's on the diagonal. *Then L^{-1} is too.*

A triangular matrix is invertible if and only if no diagonal entries are zero.

Here L has 1's so L^{-1} also has 1's. Use the Gauss-Jordan method to construct L^{-1}. Start by subtracting multiples of pivot rows from rows *below*. Normally this gets us halfway to the inverse, but for L it gets us all the way. L^{-1} appears on the right when I appears on the left. Notice how L^{-1} contains 11, from 3 times 5 minus 4.

Gauss-Jordan on triangular L

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{1} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L} & I \end{bmatrix}$$

$$\begin{array}{c} \\ \to \\ \to \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{(3 times row 1 from row 2)} \\ \text{(4 times row 1 from row 3)} \\ \text{(then 5 times row 2 from row 3)} \end{array}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{-3} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{11} & \mathbf{-5} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} I & \boldsymbol{L^{-1}} \end{bmatrix}.$$

L goes to I by a product of elimination matrices $E_{32}E_{31}E_{21}$. So that product is L^{-1}. The 11 in L^{-1} does not come into L, to spoil 3, 4, 5 in the good order $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$.

■ REVIEW OF THE KEY IDEAS ■

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.

2. A is invertible if and only if it has n pivots (row exchanges allowed).

3. If $A\boldsymbol{v} = \mathbf{0}$ for a nonzero vector $\boldsymbol{v}$, then A has no inverse.

4. The inverse of AB is the reverse product $B^{-1}A^{-1}$. And $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the n columns of A^{-1}. The augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ is row-reduced to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

Problem Set 4.4

1 Find the inverses of A, B, C (directly or from the 2 by 2 formula):

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

2 For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

3 Solve for the first column (x, y) and second column (t, z) of A^{-1}:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4 Show that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ is not invertible by trying to solve $AA^{-1} = I$ for column 1 of A^{-1}:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \left(\begin{array}{l} \textit{For a different } A, \text{ could column 1 of } A^{-1} \\ \text{be possible to find but not column 2?} \end{array}\right)$$

5 Find an upper triangular U (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$.

6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, find two different matrices such that $AB = AC$.

7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $Av = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{v} = \boldsymbol{b}$?

(c) What happens to row 3 in elimination?

8 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{0}$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Why is no third pivot ?

9 Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible and how would you find B^{-1} from A^{-1}?

10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

14 If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

Notice the order. The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} A$ is _____.

15 Prove that a matrix with a column of zeros cannot have an inverse.

16 Multiply $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ times $\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. What is the inverse of each matrix if $ad \neq bc$?

17 (a) What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

18 If B is the inverse of A^2, show that AB is the inverse of A.

19 (Recommended) A is a 4 by 4 matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

20 Find the numbers a and b that give the inverse of 5 * eye(4) − ones(4,4) :

$$[\,5I - \text{ones}\,]^{-1} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 * eye(5) − ones(5,5) ? In MATLAB, $I =$ eye.

21 Sixteen 2 by 2 matrices contain only 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1}.

22 Change I into A^{-1} as you reduce A to I (by row operations):

$$[\,A \;\; I\,] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [\,A \;\; I\,] = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

23 Follow the 3 by 3 text example of Gauss-Jordan but with all plus signs in A. Eliminate above and below the pivots to reduce $[\,A \;\; I\,]$ to $[\,I \;\; A^{-1}\,]$:

$$[\,A \;\; I\,] = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

24 Use Gauss-Jordan elimination on $[\,U \;\; I\,]$ to find the upper triangular U^{-1}:

$$\boldsymbol{UU^{-1} = I} \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

25 Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A \;\; I\,]$ and $[\,B \;\; I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{6} \end{bmatrix}$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1}.

27 Invert these matrices A by the Gauss-Jordan method starting with $[\,A \quad I\,]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

28 Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1's down the main diagonal is invertible.

(c) If A is invertible then A^{-1} and A^2 are invertible.

30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A \quad I\,]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

33 **(Puzzle)** Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

34 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

4.5 Symmetric Matrices and Orthogonal Matrices

This section introduces the **transpose** of a matrix. Start with any m by n matrix A. Then the rows of A become the columns of A^{T} (called "A *transpose*"). The columns of A are the rows of A^{T}. The m by n matrix is flipped across its main diagonal. **Then $\boldsymbol{A^{\mathrm{T}}}$ is $\boldsymbol{n}$ by $\boldsymbol{m}$.**

$$\textbf{Transpose} \qquad \text{If} \quad A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{6} \\ 0 & 0 & 5 \end{bmatrix} \quad \text{then} \quad A^{\mathrm{T}} = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{2} & 0 \\ \mathbf{6} & 5 \end{bmatrix}.$$

The entry in row i, column j of A^{T} comes from row j, column i of A. So $\boldsymbol{(A^{\mathrm{T}})_{ij} = A_{ji}}$.

The transpose of a lower triangular matrix is upper triangular. Two key rules:

Products $\boldsymbol{AB}$	The transpose of AB is $\boldsymbol{(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}}$	(1)
Inverses $\boldsymbol{A^{-1}}$	The transpose of A^{-1} is $\boldsymbol{(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}}$.	(2)

Notice especially how $B^{\mathrm{T}}A^{\mathrm{T}}$ comes in reverse order. For inverses, this reverse order is quick to check: $B^{-1}A^{-1}$ times AB produces $B^{-1}(A^{-1}A)B = I$. For transposes, rules (1) and (2) are tested and explained in the problem set. We want to move to the essential matrices of this section because they are the most important matrices in mathematics:

Symmetric matrices	$\boldsymbol{A^{\mathrm{T}}}$ **equals** $\boldsymbol{A}$. Then A is square and $a_{ij} = a_{ji}$.
Orthogonal matrices	$\boldsymbol{A^{\mathrm{T}}}$ **equals** $\boldsymbol{A^{-1}}$. Then A is square and $A^{\mathrm{T}}A = I$.

Here is a symmetric example S and also an orthogonal example Q:

$$\textbf{Symmetric } \boldsymbol{S} = \begin{bmatrix} 1 & 4 \\ 4 & 6 \end{bmatrix} \qquad \textbf{Orthogonal } \boldsymbol{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Symmetry of S is easy to see: $\mathbf{4} = \mathbf{4}$. For orthogonality I will check that $Q^{\mathrm{T}}Q = I$:

$$\begin{array}{l}\textbf{Columns are orthogonal} \\ \textbf{Columns are unit vectors}\end{array} \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Those words at the left tell you the key facts about the columns $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$:

$$\boldsymbol{Q^{\mathrm{T}}Q = I} \qquad \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 \\ \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_1 & \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

Off the diagonal you see $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_1 = 0$. The columns are orthogonal vectors. On the diagonal $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$. The $\boldsymbol{q}$'s are unit column vectors: **length 1**.

Symmetric matrices will have the special letter S and orthogonal matrices will be Q.

Symmetric Matrices $S = A^{\mathrm{T}}A$

The full glory of symmetric matrices comes with their eigenvalues λ and eigenvectors $\boldsymbol{x}$. Those strange words, half German and half English, are at the heart of Chapter 6. You will see the key equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (this puts $A\boldsymbol{x}$ in the same direction as $\boldsymbol{x}$). Let me write here only two facts that show why symmetric matrices are special:

$\boldsymbol{S\boldsymbol{x} = \lambda \boldsymbol{x}}$ Symmetric matrices have **real eigenvalues $\boldsymbol{\lambda}$ and orthogonal eigenvectors** $\boldsymbol{x}$.

Those facts will be crucial in solving symmetric systems $\boldsymbol{y}' = S\boldsymbol{y}$ and $\boldsymbol{y}'' + S\boldsymbol{y} = \mathbf{0}$.

It is equally important to know where symmetric matrices come from. One part of applied mathematics and engineering mathematics is solving equations. We have solved $A\boldsymbol{v} = \boldsymbol{b}$ and we will soon solve $d\boldsymbol{y}/dt = A\boldsymbol{y}$. Solving is one half of our subject, the other half is discovering the equations in the first place.

Start with a physical or biological or economic problem. Model it by equations. Solving $F = ma$ and $e = mc^2$ may take thought, but we give first place to Newton and Einstein for *discovering* those equations.

To repeat: Where do symmetric matrices come from? In my experience, you start with a matrix A. Often this matrix is rectangular (m by n). Its transpose is also rectangular (A^{T} is n by m). Sooner or later, you are almost sure to see the matrix $A^{\mathrm{T}}A$. At that moment you have a square symmetric n by n matrix:

$$\boldsymbol{S = A^{\mathrm{T}}A} \textbf{ is always symmetric. Its transpose is } \boldsymbol{S^{\mathrm{T}} = (A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}A^{\mathrm{TT}} = S}. \qquad (5)$$

This matrix $A^{\mathrm{T}}A$ is automatically square, because (n by m) times (m by n) is (n by n).

Example 1
$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 11 & \mathbf{12} \\ \mathbf{12} & 16 \end{bmatrix}.$$

The number 12 comes *twice* in $A^{\mathrm{T}}A$. It is (row 1 of A^{T}) $\cdot$ (column 2 of A) and also (row 2 of A^{T}) $\cdot$ (column 1 of A). The numbers 11 and 16 on the diagonal are dot products of a column with itself. So they give the *length squared* of the columns. These diagonal entries of $A^{\mathrm{T}}A$ cannot be negative.

Comment. Since A is 3 by 2, the system $A\boldsymbol{v} = \boldsymbol{b}$ has three equations but only two unknowns v_1 and v_2. Almost surely there will be no solution. But if those numbers b_1, b_2, b_3 came from careful and expensive measurements, we cannot say "no solution" and stop. We want to find the "best solution" or "closest solution" to $A\boldsymbol{v} = \boldsymbol{b}$.

In practice we usually choose the vector $\widehat{\boldsymbol{v}}$ that makes $A\widehat{\boldsymbol{v}}$ as close as possible to $\boldsymbol{b}$. The error vector $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{v}}$ is as short as possible. We are minimizing $||\boldsymbol{e}||^2 = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$, the squared length of the error. The best vector $\widehat{\boldsymbol{v}}$ is the **least squares solution**.

In Section 7.1, minimizing the error is a calculus problem and also a linear algebra problem. Both approaches lead to the equation $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$. The best $\widehat{\boldsymbol{v}}$ involves $A^{\mathrm{T}}A$.

Difference Matrices

I want to show you larger examples of $A^{\mathrm{T}}A$ that are truly important. Start with a *backward difference matrix* A. It can have $n+1$ rows and n columns. Here $n = 3$:

$$\textbf{Difference matrix}\quad \textbf{Differences of } \boldsymbol{v}\textbf{'s} \qquad A = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix} \qquad A\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 - v_1 \\ v_3 - v_2 \\ - v_3 \end{bmatrix} \qquad (6)$$

That vector $A\boldsymbol{v}$ in linear algebra corresponds to the derivative dv/dx in calculus. You see backward differences $\Delta v = \big[v(x) - v(x - \Delta x)\big]/\Delta x$ in calculus. This is before the stepsize Δx approaches zero and $\Delta v/\Delta x$ approaches dv/dx.

More often you see forward differences $\big[v(x + \Delta x) - v(x)\big]/\Delta x$, where the small Δx goes forward from x. Those appear in linear algebra when we transpose the matrix A. But first differences are "anti-symmetric" and A^{T} will be *minus* a forward difference. So the vector $A^{\mathrm{T}}\boldsymbol{w}$ corresponds to the derivative $-dw/dx$:

$$\textbf{3 by 4 matrix}\quad \textbf{Differences of } \boldsymbol{w}\textbf{'s} \qquad A^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{bmatrix} \qquad A^{\mathrm{T}}\boldsymbol{w} = \begin{bmatrix} w_1 - w_2 \\ w_2 - w_3 \\ w_3 - w_4 \end{bmatrix} \qquad (7)$$

Now comes the symmetric matrix $S = A^{\mathrm{T}}A$. It will be 3 by 3. Since A and A^{T} are "first differences" with 1 and -1, $A^{\mathrm{T}}A$ will be a **second difference matrix** with $-1, 2, -1$:

$$\textbf{Second differences} \quad S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad S\boldsymbol{v} = \begin{bmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ - v_2 + 2v_3 \end{bmatrix} \qquad (8)$$

The main diagonal of S has 2's, because each column of A produces $1^2 + (-1)^2 = 2$. The subdiagonal and superdiagonal of S have -1's, because this is the dot product of a column of A with the next column.

Let me admit quietly that S is my favorite matrix. You are seeing the 3 by 3 version, what I really like is n by n. Chapter 7 makes the link with calculus, where the first derivative of the first derivative is the *second derivative*:

$$S\boldsymbol{v} \textbf{ corresponds to } -\frac{\boldsymbol{d^2v}}{\boldsymbol{dx^2}} \qquad \frac{v(x+\Delta x) - 2v(x) + v(x - \Delta x)}{(\Delta x)^2} \approx \frac{d^2v}{dx^2}. \qquad (9)$$

All of Chapter 2 was about second order equations involving y''. Newton's Law $F = ma$ puts second derivatives (the acceleration a) at the heart of physics. When springs oscillate, and when current goes through a network, this matrix $S = A^{\mathrm{T}}A$ will appear.

The truth is that we need to know everything about S—its pivots, its determinant, its inverse, its eigenvalues, its eigenvectors. We will.

The matrix $L = AA^{\mathrm{T}}$ is almost as important. Please recognize that L is also symmetric, but L is different from S. When A has n columns and $n+1$ rows, $S = A^{\mathrm{T}}A$ is n by n. But $L = AA^{\mathrm{T}}$ is square of size $n+1$. We keep $n = 3$ and $n+1 = 4$:

Second differences in L
New boundary conditions

$$L = AA^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} \tag{10}$$

This matrix has no inverse! Can you see a vector $\boldsymbol{w}$ that has $L\boldsymbol{w} = \mathbf{0}$? It is the vector of all ones, $\boldsymbol{w} = (1,1,1,1)$. Each row of L adds to zero and that will produce $L\boldsymbol{w} = \mathbf{0}$.

Permutation Matrices

A quick way to produce orthogonal matrices is to use the columns of the identity matrix. In any order, the columns of I are orthonormal. The new order is called a "permutation" of the original order. So the new matrix is called a **permutation matrix**.

Important: We could put the *rows* of I into the new order. That also produces a permutation matrix. If this row exchange matrix is P, then the column exchange matrix is P^{T}. You can see the transpose in this 3 by 3 example starting from I:

Rows in the order 2, 3, 1 $\quad P = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 \end{bmatrix}$ $\quad$ **Columns** in order 2, 3, 1 $\quad P^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \end{bmatrix}$. $\quad$ (11)

When P multiplies a vector $\boldsymbol{v}$, it puts the components of $\boldsymbol{v}$ in the new order y, z, x. Then P^{T} puts them back in the original order x, y, z:

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \quad \text{and} \quad P^{\mathrm{T}}\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

These are orthogonal matrices, so P^{-1} is the same as P^{T}. Then $P^{\mathrm{T}}P = PP^{\mathrm{T}} = I$.

We can complete the list of all 3 by 3 permutation matrices (including the identity matrix itself, which exchanges nothing: the identity permutation). The other permutations exchange two rows or two columns of I. There are P and P^{T} in (11), and four more.

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \; P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Altogether 6 permutation matrices when $n = 3$. And $n!$ permutation matrices of size n.

The effect of P_{12} is to exchange (*permute*) rows 1 and 2, when we multiply $P_{12}A$ or $P_{12}\boldsymbol{b}$.

$$P_{12}\begin{bmatrix}\text{row 1 of } A\\ \text{row 2 of } A\\ \text{row 3 of } A\end{bmatrix}=\begin{bmatrix}\textbf{row 2 of } \boldsymbol{A}\\ \textbf{row 1 of } \boldsymbol{A}\\ \text{row 3 of } A\end{bmatrix} \qquad P_{12}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix}\boldsymbol{b_2}\\ \boldsymbol{b_1}\\ b_3\end{bmatrix}.$$

This is exactly what we do in elimination, when a zero appears in the first pivot position. If $a_{11}=0$ and $a_{21}\neq 0$, P_{12} exchanges rows to produce a nonzero pivot.

Elimination by matrices *Eliminate by E_{ij}, exchange rows by P_{jk}.*

The elimination matrix E_{ij} subtracts a multiple ℓ_{ij} of row j from a lower row $i > j$. Before that, a permutation matrix P_{jk} may put row k into row j, to produce a better number (a larger number) in the pivot position.

We *must* use P_{jk} to get a nonzero pivot. We *may* use P_{jk} to get a larger pivot. The LAPACK code (open source) chooses the largest available number as the pivot. The jth pivot (in column j) will be the largest number in row j or below. LAPACK is the foundation for the linear algebra part of many important software systems, including MATLAB.

Orthogonal Matrices

When A has orthogonal columns, the symmetric matrix $A^{\mathrm{T}}A$ is *diagonal*. The off-diagonal entries are dot products of different columns of A, so they are all zero.

When the columns of A are *unit vectors* (length 1), all diagonal entries of $A^{\mathrm{T}}A$ are 1. Those entries are (row i of A^{T}) $\cdot$ (column i of A) $=$ length squared $=$ 1. Dot products of columns with themselves are on the main diagonal of $A^{\mathrm{T}}A$.

The best case is **orthonormal columns**. Those are orthogonal unit vectors, both properties at the same time. In this case we write $\boldsymbol{q}$ for the vectors and Q for the matrix:

$$\begin{array}{ll}\textbf{Orthogonal} & \boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j=\mathbf{0}\\ \textbf{Unit vectors} & \boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i=\mathbf{1}\end{array} \qquad Q^{\mathrm{T}}Q=\begin{bmatrix}\boldsymbol{q}_1^{\mathrm{T}}\\ \cdots\\ \boldsymbol{q}_n^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_1\cdots\boldsymbol{q}_n\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}. \tag{12}$$

When Q is square, I call it an **orthogonal matrix**. (The name "orthonormal matrix" might have been better.) I still use the letter Q when the matrix is rectangular, with $m > n$. But a rectangular Q^{T} is only a ***left–inverse*** of Q:

$$(\boldsymbol{m}=\boldsymbol{n})\quad Q^{\mathrm{T}}Q=QQ^{\mathrm{T}}=I \qquad (\boldsymbol{m}>\boldsymbol{n})\quad Q^{\mathrm{T}}Q=I \text{ but } \boldsymbol{QQ^{\mathrm{T}}}\neq \boldsymbol{I}. \tag{13}$$

$Q^{\mathrm{T}}Q = I$ is a very powerful property. When we multiply any vector by Q, its length will not change:

$$\textbf{Same length} \qquad ||\boldsymbol{Qv}|| = ||\boldsymbol{v}|| \textbf{ for every vector } \boldsymbol{v}. \tag{14}$$

The proof comes directly from $||Q\boldsymbol{v}||^2 = (Q\boldsymbol{v})^{\mathrm{T}}(Q\boldsymbol{v}) = \boldsymbol{v}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{v}$. The matrix $Q^{\mathrm{T}}Q$ is the identity. So we are left with $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = ||\boldsymbol{v}||^2$.

The fact that lengths don't change makes orthogonal matrices very safe to compute with. Nothing blows up, nothing becomes too small (no overflow and no underflow). The basic computation in linear algebra is the solution of a linear system, and for (square) orthogonal matrices this is incredibly easy:

$$\boldsymbol{Q^{-1} = Q^{\mathrm{T}}} \qquad \text{The solution of } Q\boldsymbol{v} = \boldsymbol{b} \text{ is } \boldsymbol{v} = Q^{\mathrm{T}}\boldsymbol{b}. \tag{15}$$

To solve the equations, we just transpose the matrix. The greatest example is the **Fourier matrix**, which breaks up a signal $\boldsymbol{b}$ into separate pure frequencies. The vector $\boldsymbol{b}$ in the time domain is transformed to $\boldsymbol{v}$ in the frequency domain. The "energy" can be measured in either domain, because $||\boldsymbol{b}||^2$ is equal to $||\boldsymbol{v}||^2$—as we saw above.

The Fourier matrix F is exceptional because multiplications by F and F^{-1} are extremely fast. They break up into diagonal matrices and permutation matrices. This is the insight behind the Fast Fourier Transform. (*The FFT is in Section 8.2.*)

The equation $Q\boldsymbol{v} = \boldsymbol{b}$ has a clear geometrical meaning when Q is 2 by 2. $Q\boldsymbol{v}$ is expressing that vector $\boldsymbol{b}$ as a combination of the columns of Q. *Those columns $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ give the perpendicular axes in* Figure 4.9. We are finding the component of $\boldsymbol{b}$ in each direction.

Those two components are $\boldsymbol{v}_1 = \boldsymbol{q}_1 \cdot \boldsymbol{b}$ and $\boldsymbol{v}_2 = \boldsymbol{q}_2 \cdot \boldsymbol{b}$. Solving $Q\boldsymbol{v} = \boldsymbol{b}$ by $\boldsymbol{v} = Q^{\mathrm{T}}\boldsymbol{b}$ is just a change from x, y axes to q_1, q_2 axes.

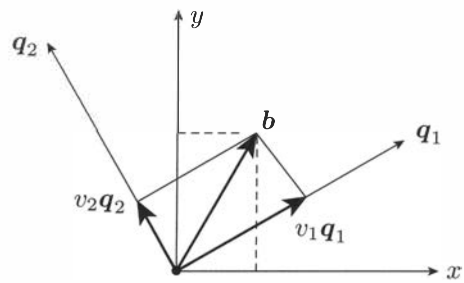


Figure 4.9: Every $\boldsymbol{b} = (x, y)$ splits into $\boldsymbol{b} = v_1\boldsymbol{q}_1 + v_2\boldsymbol{q}_2$. And $||\boldsymbol{b}||^2 = x^2 + y^2 = v_1^2 + v_2^2$.

Both Symmetric and Orthogonal

Symmetric matrices are the best, they are everywhere in applied mathematics. Orthogonal matrices are a strong second, starting with rotation matrices and the Fourier matrix. Most symmetric matrices are not orthogonal and most orthogonal matrices are not symmetric. It is natural to wonder when and if we can have both properties at once.

Exchange and reflection and "Hadamard" matrices are symmetric and orthogonal:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \boldsymbol{H} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}. \tag{16}$$

Notice that the columns of H are unit vectors: $\frac{1}{4}((-1)^2 + 1^2 + 1^2 + 1^2) = 1$. Nobody knows which dimensions allow n orthogonal vectors of 1's and -1's (not odd dimensions !). Wikipedia describes this unsolved problem on its "Hadamard matrix" page.

To find more symmetric orthogonal matrices, and eventually all of them, we can use an important fact about orthogonal matrices:

If Q_1 and Q_2 are orthogonal, so is their product $Q = Q_1Q_2$.

The test is always to check $Q^{\mathrm{T}}Q = I$. Here this is $(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2$. In the middle is $Q_1^{\mathrm{T}}Q_1 = I$. Then the outside has $Q_2^{\mathrm{T}}Q_2 = I$.

Conclusion: We can multiply orthogonal matrices and stay orthogonal.

Problem: We can't always multiply symmetric matrices and stay symmetric.

Here is one approach that succeeds with both properties. Start with any diagonal matrix D of 1's followed by -1's:

Symmetric and orthogonal $\qquad D = \operatorname{diag}(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{-1}, \ldots, \mathbf{-1}).$ (17)

Multiply D on the left side by any orthogonal Q and on the right side by Q^{T}. That "symmetric multiplication" keeps the matrix QDQ^{T} symmetric:

Symmetric and orthogonal $\qquad (QDQ^{\mathrm{T}})^{\mathrm{T}} = Q^{\mathrm{TT}}D^{\mathrm{T}}Q^{\mathrm{T}} = QDQ^{\mathrm{T}}.$ (18)

This product of orthogonal matrices is also orthogonal. When you meet eigenvalues in Chapter 6, you will see that *all* symmetric orthogonal matrices have this form QDQ^{T}. Possibly that small fact is appearing for the first time in a textbook.

Factoring a Matrix

That was for fun, this is more important. "A symmetric matrix S is like a real number r." "An orthogonal matrix Q is like a complex number $e^{i\theta}$ with absolute value 1." Every complex number can be written in polar form $re^{i\theta}$, and what we hope for is true:

Every square matrix A can be written in polar form $A = SQ$.

$A = SQ$ is equivalent to the Singular Value Decomposition (this is explained in Section 7.2). The SVD is the last and most remarkable step in the *Fundamental Theorem of Linear Algebra*. The polar form is in the Chapter 7 Notes.

■ REVIEW OF THE KEY IDEAS ■

1. The transpose has $A^{\mathrm{T}}_{ij} = A_{ji}$. Then $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ and $A\boldsymbol{v}\cdot\boldsymbol{w}$ equals $\boldsymbol{v}\cdot A^{\mathrm{T}}\boldsymbol{w}$.

2. Symmetric matrices have $S^{\mathrm{T}} = S$. Orthogonal matrices have $Q^{\mathrm{T}} = Q^{-1}$.

3. $A^{\mathrm{T}}A$ is always a symmetric matrix. Key examples are second difference matrices.

4. The columns of Q are orthogonal vectors of length 1. Then $||Q\boldsymbol{x}|| = ||\boldsymbol{x}||$ for all $\boldsymbol{x}$.

5. The $n\,!$ permutation matrices P reorder the rows of I (n by n), and $P^{\mathrm{T}} = P^{-1}$.

Problem Set 4.5

Questions 1–9 are about transposes A^{T} and symmetric matrices $S = S^{\mathrm{T}}$.

1 Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

2 (a) Find 2 by 2 symmetric matrices A and B so that AB is not symmetric.

(b) With $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$, show that $AB = BA$ ensures that AB will now be symmetric. The product is symmetric only when A commutes with B.

3 (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order*?

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is _____ triangular.

4 Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matrix).

5 Every square matrix A has a symmetric part and an antisymmetric part:

$$A = \text{symmetric} + \text{antisymmetric} = \left(\frac{A + A^{\mathrm{T}}}{2}\right) + \left(\frac{A - A^{\mathrm{T}}}{2}\right).$$

Transpose the antisymmetric part to get *minus* that part. Split these in two parts:

$$A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

6 The transpose of a block matrix $M = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ is $M^{\mathrm{T}} =$ _____. Test an example to be sure. Under what conditions on A, B, C, D is the block matrix symmetric?

7 True or false:

(a) The block matrix $\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$ is automatically symmetric.

(b) If A and B are symmetric then their product AB is symmetric.

(c) If A is not symmetric then A^{-1} is not symmetric.

(d) When A, B, C are symmetric, the transpose of ABC is CBA.

8 (a) How many entries of S can be chosen independently, if $S = S^{\mathrm{T}}$ is 5 by 5?

(b) How many entries can be chosen if A is *skew-symmetric*? ($A^{\mathrm{T}} = -A$).

9 Transpose the equation $A^{-1}A = I$. The result shows that the inverse of A^{T} is _____. If S is symmetric, **how does this show that S^{-1} is also symmetric**?

Questions 10–14 are about permutation matrices.

10 Why are there $n!$ permutation matrices of size n? They give $n!$ orders of $1, \ldots, n$.

11 If P_1 and P_2 are permutation matrices, so is $P_1 P_2$. This still has the rows of I in some order. Give examples with $P_1 P_2 \neq P_2 P_1$ and $P_3 P_4 = P_4 P_3$.

12 There are 12 "*even*" permutations of $(1, 2, 3, 4)$, with an *even number of exchanges*. Two of them are $(1, 2, 3, 4)$ with no exchanges and $(4, 3, 2, 1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, just order the numbers.

13 If P has 1's on the antidiagonal from $(1, n)$ to $(n, 1)$, describe PAP. Is P even?

14 (a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

(b) Find a 4 by 4 permutation with $P^4 \neq I$.

Questions 15–18 are about first differences A and second differences $A^{\mathrm{T}}A$ and AA^{T}.

15 Write down the 5 by 4 backward difference matrix A.

(a) Compute the symmetric second difference matrices $S = A^{\mathrm{T}}A$ and $L = AA^{\mathrm{T}}$.

(b) Show that S is invertible by finding S^{-1}. Show that L is singular.

16 In Problem 15, find the pivots of S and L (4 by 4 and 5 by 5). The pivots of S in equation (8) are $2, 3/2, 4/3$. The pivots of L in equation (10) are $1, 1, 1, 0$ (fail).

17 (Computer problem) Create the 9 by 10 backward difference matrix A. Multiply to find $S = A^{\mathrm{T}}A$ and $L = AA^{\mathrm{T}}$. If you have linear algebra software, ask for the determinants $\det(S)$ and $\det(L)$.

Challenge : By experiment find $\det(S)$ when $S = A^{\mathrm{T}}A$ is n by n.

18 (Infinite computer problem) Imagine that the second difference matrix S is infinitely large. The diagonals of 2's and -1's go from minus infinity to plus infinity:

$$\textbf{Infinite tridiagonal matrix} \qquad S = \begin{bmatrix} \cdot & \cdot & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & \cdot & \cdot \end{bmatrix}$$

(a) Multiply S times the infinite *all-ones* vector $\boldsymbol{v} = (\ldots, 1, 1, 1, 1, \ldots)$

(b) Multiply S times the infinite *linear* vector $\boldsymbol{w} = (\ldots, 0, 1, 2, 3, \ldots)$

(c) Multiply S times the infinite *squares* vector $\boldsymbol{u} = (\ldots, 0, 1, 4, 9, \ldots)$.

(d) Multiply S times the infinite *cubes* vector $\boldsymbol{c} = (\ldots, 0, 1, 8, 27, \ldots)$.

The answers correspond to second derivatives (with minus sign) of 1 and x^2 and x^3.

Questions 19–28 are about matrices with $Q^{\mathrm{T}}Q = I$. If Q is square, then it is an orthogonal matrix and $Q^{\mathrm{T}} = Q^{-1}$ and $QQ^{\mathrm{T}} = I$.

19 Complete these matrices to be orthogonal matrices:

(a) $Q = \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix}$ (b) $Q = \frac{1}{3}\begin{bmatrix} -1 & & \\ 2 & & \\ 2 & & \end{bmatrix}$ (c) $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ 1 & -1 & & \\ 1 & -1 & & \end{bmatrix}$.

20 (a) Suppose Q is an orthogonal matrix. Why is $Q^{-1} = Q^{\mathrm{T}}$ also an orthogonal matrix ?

(b) From $Q^{\mathrm{T}}Q = I$, the columns of Q are orthogonal unit vectors (orthonormal vectors). Why are the rows of Q (square matrix) also orthonormal vectors ?

21 (a) Which vectors can be the first column of an orthogonal matrix ?

(b) If $Q_1^{\mathrm{T}}Q_1 = I$ and $Q_2^{\mathrm{T}}Q_2 = I$, is it true that $(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = I$? Assume that the matrix shapes allow the multiplication Q_1Q_2.

22 If $\boldsymbol{u}$ is a unit column vector (length 1, $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$), show why $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is

(a) a symmetric matrix: $H = H^{\mathrm{T}}$ (b) an orthogonal matrix: $H^{\mathrm{T}}H = I$.

23 If $\boldsymbol{u} = (\cos\theta, \sin\theta)$, what are the four entries in $H = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$? Show that $H\boldsymbol{u} = -\boldsymbol{u}$ and $H\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} = (-\sin\theta, \cos\theta)$. This H is a **reflection matrix**: the $\boldsymbol{v}$-line is a mirror and the $\boldsymbol{u}$-line is reflected across that mirror.

24 Suppose the matrix Q is orthogonal and also upper triangular. What can Q look like ? Must it be diagonal ?

25 (a) To construct a 3 by 3 orthogonal matrix Q whose first column is in the direction $\boldsymbol{w}$, what first column $\boldsymbol{q}_1 = c\boldsymbol{w}$ would you choose ?

(b) The next column $\boldsymbol{q}_2$ can be any unit vector perpendicular to $\boldsymbol{q}_1$. To find $\boldsymbol{q}_3$, choose a solution $\boldsymbol{v} = (v_1, v_2, v_3)$ to the two equations $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{v} = 0$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{v} = 0$. *Why is there always a nonzero solution $\boldsymbol{v}$?*

26 Why is every solution $\boldsymbol{v}$ to $A\boldsymbol{v} = \mathbf{0}$ orthogonal to every row of A ?

27 Suppose $Q^{\mathrm{T}}Q = I$ but Q is not square. The matrix $P = QQ^{\mathrm{T}}$ is not I. But show that P is symmetric and $P^2 = P$. This is a **projection matrix**.

28 A 5 by 4 matrix Q can have $Q^{\mathrm{T}}Q = I$ but *it cannot possibly have* $QQ^{\mathrm{T}} = I$. Explain in words why the four equations $Q^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$ must have a nonzero solution $\boldsymbol{v}$. Then $\boldsymbol{v}$ is not the same as $QQ^{\mathrm{T}}\boldsymbol{v}$ and I is not the same as QQ^{T}.

Challenge Problems

29 Can you find a rotation matrix Q so that QDQ^{T} is a permutation ?

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \quad \text{equals} \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

30 Split an orthogonal matrix ($Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I$) into two rectangular submatrices:

$$Q = \begin{bmatrix} Q_1 \mid Q_2 \end{bmatrix} \quad \text{and} \quad Q^{\mathrm{T}}Q = \begin{bmatrix} Q_1^{\mathrm{T}}Q_1 & Q_1^{\mathrm{T}}Q_2\\ Q_2^{\mathrm{T}}Q_1 & Q_2^{\mathrm{T}}Q_2\end{bmatrix}$$

(a) What are those four blocks in $Q^{\mathrm{T}}Q = I$?

(b) $QQ^{\mathrm{T}} = Q_1Q_1^{\mathrm{T}} + Q_2Q_2^{\mathrm{T}} = I$ is column times row multiplication. Insert the diagonal matrix $D = \begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}$ and do the same multiplication for QDQ^{T}.

Note: The description of all symmetric orthogonal matrices S in (18) becomes $S = QDQ^{\mathrm{T}} = Q_1Q_1^{\mathrm{T}} - Q_2Q_2^{\mathrm{T}}$. This is exactly the reflection matrix $I - 2Q_2Q_2^{\mathrm{T}}$.

31 The real reason that the transpose "flips A across its main diagonal" is to obey this dot product law: $(A\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot (A^{\mathrm{T}}\boldsymbol{w})$. That rule $(A\boldsymbol{v})^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{w})$ **becomes integration by parts in calculus**, where $A = d/dx$ and $A^{\mathrm{T}} = -d/dx$.

(a) For 2 by 2 matrices, write out both sides (4 terms) and compare:

$$\left(\begin{bmatrix} a & \boldsymbol{b}\\ \boldsymbol{c} & d\end{bmatrix}\begin{bmatrix} v_1\\ v_2\end{bmatrix}\right) \cdot \begin{bmatrix} w_1\\ w_2\end{bmatrix} \quad \text{is equal to} \quad \begin{bmatrix} v_1\\ v_2\end{bmatrix} \cdot \left(\begin{bmatrix} a & \boldsymbol{c}\\ \boldsymbol{b} & d\end{bmatrix}\begin{bmatrix} w_1\\ w_2\end{bmatrix}\right).$$

(b) The rule $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ comes slowly but directly from part (a):

$$(AB)\,\boldsymbol{v} \cdot \boldsymbol{w} = A(B\boldsymbol{v}) \cdot \boldsymbol{w} = B\boldsymbol{v} \cdot A^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v} \cdot B^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{w}) = \boldsymbol{v} \cdot (B^{\mathrm{T}}A^{\mathrm{T}})\boldsymbol{w}$$

Steps 1 and 4 are the ______ law. Steps 2 and 3 are the dot product law.

32 How is a matrix $S = S^{\mathrm{T}}$ decided by its entries on and above the diagonal? How is Q with orthonormal columns decided by its entries *below* the diagonal? Together this matches the number of entries in an n by n matrix. So it is reasonable that every matrix can be factored into $A = SQ$ (like $re^{i\theta}$).

■ CHAPTER 4 NOTES ■

Important Question Where do the rules for matrix-matrix multiplication AB come from? *Answer* From matrix-vector multiplication $A\boldsymbol{v}$. The matrix AB is defined so that

$$AB \text{ times } \boldsymbol{v} \text{ equals } A \text{ times } B\boldsymbol{v}. \qquad \text{Then } AB \text{ times } C \text{ equals } A \text{ times } BC.$$

Key idea: Choose the special vector $\boldsymbol{v} = (1, 0, \ldots, 0)$. Then AB times this $\boldsymbol{v}$ is the first column of AB. And $B\boldsymbol{v}$ is the first column of B. *So column* 1 *of* AB *equals* A *times column* 1 *of* B. This was the AB rule from the start. Every other column of AB goes the same way, by moving the "1" in $\boldsymbol{v}$.

Thus $(AB)\boldsymbol{v} = A(B\boldsymbol{v})$. With several v's in a matrix C, this becomes $(AB)C = A(BC)$.

Elimination factors A into LU = (lower triangular) times (upper triangular).

The MATLAB command $[L,\ U] = \boldsymbol{lu}(A)$ will output L and U, unless there are row exchanges. L and U are a complete record of elimination on the left side of $A\boldsymbol{v} = \boldsymbol{b}$. The solution $\boldsymbol{v}$ comes from the right side $\boldsymbol{b}$ by solving the two triangular systems:

From b to c **Forward substitution**	$\boldsymbol{Lc = b}$	**From c to v** **Back substitution**	$\boldsymbol{Uv = c}$

Then $\boldsymbol{v}$ is the correct solution: $A\boldsymbol{v} = LU\boldsymbol{v} = L\boldsymbol{c} = \boldsymbol{b}$. The forward substitution is what happened to $\boldsymbol{b}$ as elimination went forward on $[\,A \ \ \boldsymbol{b}\,]$.

Second difference matrices have beautiful inverses and LU factors if the first diagonal entry is 1 instead of 2. Here is the 3 by 3 tridiagonal matrix T and its inverse:

$$\boldsymbol{T_{11} = 1} \qquad T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

One approach is Gauss-Jordan elimination on $[\,T \ \ I\,]$. That seems too mechanical. I would rather write T using first differences L and U. The inverses are **sum matrices** U^{-1} and L^{-1}:

$$T = \underbrace{\begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix}}_{\textbf{difference}} \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ & 1 & -1 \\ & & 1 \end{bmatrix}}_{\textbf{difference}} \qquad T^{-1} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}}_{\textbf{sum}} \underbrace{\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}}_{\textbf{sum}}$$

Question. (**4 by 4**) What are the pivots of T? What is its 4 by 4 inverse?

This book helps students understand and solve the most fundamental problems in differential equations and linear algebra.

Differential equations	**Matrix equations**
Continuous problems	Discrete problems
Systems in motion	Systems at rest
$dy/dt = Ay + q$	$Ay = b$ and $Ax = \lambda x$

$\mathbf{y_{particular}}$ + $\mathbf{y_{nullspace}}$ → $\mathbf{y_{complete}}$

$\mathbf{Ay_p = b}$ + $\mathbf{Ay_n = 0}$ → $\mathbf{A(y_p + y_n) = b}$

$\mathbf{dy_p/dt = Ay_p + q}$ + $\mathbf{dy_n/dt = Ay_n}$ → $\mathbf{dy/dt = Ay + q}$

You have conquered this course when you can solve these eight linear equations.

First order

$dy/dt = ay$

$dy/dt = ay + q$

Second order

$d^2y/dt^2 + B\,dy/dt + Cy = 0$

$d^2y/dt^2 + B\,dy/dt + Cy = q$

First order systems

$d\mathbf{y}/dt = A\mathbf{y}$

$d\mathbf{y}/dt = A\mathbf{y} + \mathbf{q}$

Second order systems

$d^2\mathbf{y}/dt^2 + S\mathbf{y} = \mathbf{0}$

$d^2\mathbf{y}/dt^2 + S\mathbf{y} = \mathbf{q}$

Advanced problems

Nonlinear	$dy/dt = f(t,y)$
Heat eqn	$\partial u/\partial t = \partial^2 u/\partial x^2$
Wave eqn	$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$

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