CONVEX OPTIMIZATION THEORY

Dimitri P. Bertsekas

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Contents

ABOUT THE AUTHOR

Dimitri Bertsekas studied Mechanical and Electrical Engineering at the National Technical University of Athens, Greece, and obtained his Ph.D. in system science from the Massachusetts Institute of Technology. He has held faculty positions with the Engineering-Economic Systems Department, Stanford University, and the Electrical Engineering Department of the University of Illinois, Urbana. Since 1979 he has been teaching at the Electrical Engineering and Computer Science Department of the Massachusetts Institute of Technology (M.I.T.), where he is currently McAfee Professor of Engineering.

His teaching and research spans several fields, including deterministic optimization, dynamic programming and stochastic control, large-scale and distributed computation, and data communication networks. He has authored or coauthored numerous research papers and fourteen books, several of which are used as textbooks in MIT classes, including "Nonlinear Programming," "Dynamic Programming and Optimal Control," "Data Networks," "Introduction to Probability," as well as the present book. He often consults with private industry and has held editorial positions in several journals.

Professor Bertsekas was awarded the INFORMS 1997 Prize for Research Excellence in the Interface Between Operations Research and Computer Science for his book "Neuro-Dynamic Programming" (co-authored with John Tsitsiklis), the 2000 Greek National Award for Operations Research, and the 2001 ACC John R. Ragazzini Education Award. In 2001, he was elected to the United States National Academy of Engineering.

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Preface

This book aims at an accessible, concise, and intuitive exposition of two related subjects that find broad practical application:

(a) Convex analysis, particularly as it relates to optimization.

(b) Duality theory for optimization and minimax problems, mainly within a convexity framework.

The focus on optimization is to derive conditions for existence of primal and dual optimal solutions for constrained problems such as

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \leq 0, \quad j = 1, \ldots, r. \end{aligned}$$

Other types of optimization problems, such as those arising in Fenchel duality, are also part of our scope. The focus on minimax is to derive conditions guaranteeing the equality

$$\inf_{x \in X} \sup_{z \in Z} \phi(x, z) = \sup_{z \in Z} \inf_{x \in X} \phi(x, z),$$

and the attainment of the "inf" and the "sup."

The treatment of convexity theory is fairly detailed. It touches upon nearly all major aspects of the subject, and it is sufficient for the development of the core analytical issues of convex optimization. The mathematical prerequisites are a first course in linear algebra and a first course in real analysis. A summary of the relevant material is provided in an appendix. Prior knowledge of linear and nonlinear optimization theory is not assumed, although it will undoubtedly be helpful in providing context and perspective. Other than this modest background, the development is self-contained, with rigorous proofs provided throughout.

We have aimed at a unified development of the strongest possible forms of duality with the most economic use of convexity theory. To this end, our analysis often departs from the lines of Rockafellar's classic 1970 book and other books that followed the Fenchel/Rockafellar formalism. For example, we treat differently closed set intersection theory and preservation of closure under linear transformations (Sections 1.4.2 and 1.4.3); we develop subdifferential calculus by using constrained optimization duality (Section 5.4.2); and we do not rely on concepts such as infimal convolution, image, polar sets and functions, bifunctions, and conjugate saddle functions. Perhaps our greatest departure is in duality theory itself: similar to Fenchel/Rockafellar, our development rests on Legendre/Fenchel conjugacy ideas, but is far more geometrical and visually intuitive.

Our duality framework is based on two simple geometrical problems: the *min common point problem* and the *max crossing point problem*. The salient feature of the min common/max crossing (MC/MC) framework is its highly visual geometry, through which all the core issues of duality theory become apparent and can be analyzed in a unified way. Our approach is to obtain a handful of broadly applicable theorems within the MC/MC framework, and then specialize them to particular types of problems (constrained optimization, Fenchel duality, minimax problems, etc). We address all duality questions (existence of duality gap, existence of dual optimal solutions, structure of the dual optimal solution set), and other issues (subdifferential theory, theorems of the alternative, duality gap estimates) in this way.

Fundamentally, the MC/MC framework is closely connected to the conjugacy framework, and owes its power and generality to this connection. However, the two frameworks offer complementary starting points for analysis and provide alternative views of the geometric foundation of duality: conjugacy emphasizes functional/algebraic descriptions, while MC/MC emphasizes set/epigraph descriptions. The MC/MC framework is simpler, and seems better suited for visualizing and investigating questions of strong duality and existence of dual optimal solutions. The conjugacy framework, with its emphasis on functional descriptions, is more suitable when mathematical operations on convex functions are involved, and the calculus of conjugate functions can be brought to bear for analysis or computation.

The book evolved from the earlier book of the author [BNO03] on the subject (coauthored with A. Nedić and A. Ozdaglar), but has different character and objectives. The 2003 book was quite extensive, was structured (at least in part) as a research monograph, and aimed to bridge the gap between convex and nonconvex optimization using concepts of nonsmooth analysis. By contrast, the present book is organized differently, has the character of a textbook, and concentrates exclusively on convex optimization. Despite the differences, the two books have similar style and level of mathematical sophistication, and share some material.

The chapter-by-chapter description of the book follows:

Chapter 1: This chapter develops all of the convex analysis tools that are needed for the development of duality theory in subsequent chapters. It covers basic algebraic concepts such as convex hulls and hyperplanes, and topological concepts such as relative interior, closure, preservation of closedness under linear transformations, and hyperplane separation. In addition, it develops subjects of special interest in duality and optimization, such as recession cones and conjugate functions.

Chapter 2: This chapter covers polyhedral convexity concepts: extreme points, the Farkas and Minkowski-Weyl theorems, and some of their applications in linear programming. It is not needed for the developments of subsequent chapters, and may be skipped at first reading.

Chapter 3: This chapter focuses on basic optimization concepts: types of minima, existence of solutions, and a few topics of special interest for duality theory, such as partial minimization and minimax theory.

Chapter 4: This chapter introduces the MC/MC duality framework. It discusses its connection with conjugacy theory, and it charts its applications to constrained optimization and minimax problems. It then develops broadly applicable theorems relating to strong duality and existence of dual optimal solutions.

Chapter 5: This chapter specializes the duality theorems of Chapter 4 to important contexts relating to linear programming, convex programming, and minimax theory. It also uses these theorems as an aid for the development of additional convex analysis tools, such as a powerful nonlinear version of Farkas' Lemma, subdifferential theory, and theorems of the alternative. A final section is devoted to nonconvex problems and estimates of the duality gap, with special focus on separable problems.

In aiming for brevity, I have omitted a number of topics that an instructor may wish for. One such omission is applications to specially structured problems; the book by Boyd and Vanderbergue [BoV04], as well as my book on parallel and distributed computation with John Tsitsiklis [BeT89] cover this material extensively (both books are available on line).

Another important omission is computational methods. However, I have written a long supplementary sixth chapter (over 100 pages), which covers the most popular convex optimization algorithms (and some new ones), and can be downloaded from the book's web page

http://www.athenasc.com/convexduality.html.

This chapter, together with a more comprehensive treatment of convex analysis, optimization, duality, and algorithms will be part of a more extensive textbook that I am currently writing. Until that time, the chapter will serve instructors who wish to cover convex optimization algorithms in addition to duality (as I do in my M.I.T. course). This is a "living" chapter that will be periodically updated. Its current contents are as follows:

Chapter 6 on Algorithms: 6.1. Problem Structures and Computational Approaches; 6.2. Algorithmic Descent; 6.3. Subgradient Methods; 6.4. Polyhedral Approximation Methods; 6.5. Proximal and Bundle Methods; 6.6. Dual Proximal Point Algorithms; 6.7. Interior Point Methods; 6.8. Approximate Subgradient Methods; 6.9. Optimal Algorithms and Complexity.

While I did not provide exercises in the text, I have supplied a substantial number of exercises (with detailed solutions) at the book's web page. The reader/instructor may also use the end-of-chapter problems (a total of 175) given in [BNO03], which have similar style and notation to the present book. Statements and detailed solutions of these problems can be downloaded from the book's web page and are also available on line at

http://www.athenasc.com/convexity.html.

The book may be used as a text for a theoretical convex optimization course; I have taught several variants of such a course at MIT and elsewhere over the last ten years. It may also be used as a supplementary source for nonlinear programming classes, and as a theoretical foundation for classes focused on convex optimization models (rather than theory).

The book has been structured so that the reader/instructor can use the material selectively. For example, the polyhedral convexity material of Chapter 2 can be omitted in its entirety, as it is not used in Chapters 3-5. Similarly, the material on minimax theory (Sections 3.4, 4.2.5, and 5.5) may be omitted; and if this is done, Sections 3.3 and 5.3.4, which use the tools of partial minimization, may be omitted. Also, Sections 5.4-5.7 are "terminal" and may each be omitted without effect on other sections.

A "minimal" self-contained selection, which I have used in my nonlinear programming class at MIT (together with the supplementary web-based Chapter 6 on algorithms), consists of the following:

- Chapter 1, except for Sections 1.3.3 and 1.4.1.
- Section 3.1.
- Chapter 4, except for Section 4.2.5.
- Chapter 5, except for Sections 5.2, 5.3.4, and 5.5-5.7.

This selection focuses on nonlinear convex optimization, and excludes all the material relating to polyhedral convexity and minimax theory.

I would like to express my thanks to several colleagues for their contributions to the book. My collaboration with Angelia Nedić and Asuman Ozdaglar on our 2003 book was important in laying the foundations of the present book. Huizhen (Janey) Yu read carefully early drafts of portions of the book, and offered several insightful suggestions. Paul Tseng contributed substantially through our joint research on set intersection theory, given in part in Section 1.4.2 (this research was motivated by earlier collaboration with Angelia Nedić). Feedback from students and colleagues, including Dimitris Bisias, Vivek Borkar, John Tsitsiklis, Mengdi Wang, and Yunjian Xu, is highly appreciated. Finally, I wish to thank the many outstanding students in my classes, who have been a continuing source of motivation and inspiration.

1

Basic Concepts of Convex Analysis

Contents

Convex sets and functions are very useful in optimization models, and have a rich structure that is convenient for analysis and algorithms. Much of this structure can be traced to a few fundamental properties. For example, each closed convex set can be described in terms of the hyperplanes that support the set, each point on the boundary of a convex set can be approached through the relative interior of the set, and each halfline belonging to a closed convex set still belongs to the set when translated to start at any point in the set.

Yet, despite their favorable structure, convex sets and their analysis are not free of anomalies and exceptional behavior, which cause serious difficulties in theory and applications. For example, contrary to affine and compact sets, some basic operations such as linear transformation and vector sum may not preserve the closedness of closed convex sets. This in turn complicates the treatment of some fundamental optimization issues, including the existence of optimal solutions and duality.

For this reason, it is important to be rigorous in the development of convexity theory and its applications. Our aim in this first chapter is to establish the foundations for this development, with a special emphasis on issues that are relevant to optimization.

1.1 CONVEX SETS AND FUNCTIONS

We introduce in this chapter some of the basic notions relating to convex sets and functions. This material permeates all subsequent developments in this book. Appendix A provides the definitions, notational conventions, and results from linear algebra and real analysis that we will need. We first define convex sets (cf. Fig. 1.1.1).

Definition 1.1.1: A subset C of $\Re^n$ is called *convex* if

$$\alpha x + (1-\alpha)y \in C, \qquad \forall\, x, y \in C,\ \forall\, \alpha \in [0,1].$$

Note that the empty set is by convention considered to be convex. Generally, when referring to a convex set, it will usually be apparent from the context whether this set can be empty, but we will often be specific in order to minimize ambiguities. The following proposition gives some operations that preserve convexity.

Proposition 1.1.1:

(a) The intersection $\cap_{i\in I} C_i$ of any collection $\{C_i \mid i \in I\}$ of convex sets is convex.

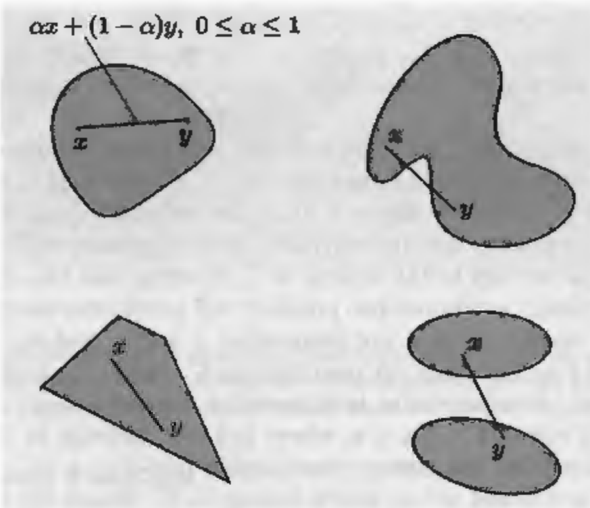


Figure 1.1.1. Illustration of the definition of a convex set. For convexity, linear interpolation between any two points of the set must yield points that lie within the set. Thus the sets on the left are convex, but the sets on the right are not.

(b) The vector sum $C_1 + C_2$ of two convex sets C_1 and C_2 is convex.

(c) The set λC is convex for any convex set C and scalar λ. Furthermore, if C is a convex set and λ_1, λ_2 are positive scalars,

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

(d) The closure and the interior of a convex set are convex.

(e) The image and the inverse image of a convex set under an affine function are convex.

Proof: The proof is straightforward using the definition of convexity. To prove part (a), we take two points x and y from $\cap_{i\in I} C_i$, and we use the convexity of C_i to argue that the line segment connecting x and y belongs to all the sets C_i, and hence, to their intersection.

Similarly, to prove part (b), we take two points of $C_1 + C_2$, which we represent as $x_1 + x_2$ and $y_1 + y_2$, with $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$. For any $\alpha \in [0, 1]$, we have

$$\alpha(x_1 + x_2) + (1-\alpha)(y_1 + y_2) = \big(\alpha x_1 + (1-\alpha)y_1\big) + \big(\alpha x_2 + (1-\alpha)y_2\big).$$

By convexity of C_1 and C_2, the vectors in the two parentheses of the right-hand side above belong to C_1 and C_2, respectively, so that their sum belongs to $C_1 + C_2$. Hence $C_1 + C_2$ is convex. The proof of part (c) is left as an exercise for the reader. The proof of part (e) is similar to the proof of part (b).

To prove part (d), let C be a convex set. Choose two points x and y from the closure of C, and sequences $\{x_k\} \subset C$ and $\{y_k\} \subset C$, such that $x_k \to x$ and $y_k \to y$. For any $\alpha \in [0,1]$, the sequence $\{\alpha x_k + (1-\alpha) y_k\}$, which belongs to C by the convexity of C, converges to $\alpha x + (1-\alpha)y$. Hence $\alpha x + (1-\alpha)y$ belongs to the closure of C, showing that the closure of C is convex. Similarly, we choose two points x and y from the interior of C, and we consider open balls that are centered at x and y, and have sufficiently small radius r so that they are contained in C. For any $\alpha \in [0,1]$, consider the open ball of radius r that is centered at $\alpha x + (1-\alpha)y$. Any point in this ball, say $\alpha x + (1-\alpha)y + z$, where $\|z\| < r$, belongs to C, because it can be expressed as the convex combination $\alpha(x+z) + (1-\alpha)(y+z)$ of the vectors $x+z$ and $y+z$, which belong to C. Hence the interior of C contains $\alpha x + (1-\alpha)y$ and is therefore convex. **Q.E.D.**

Special Convex Sets

We will often consider some special sets, which we now introduce. A *hyperplane* is a set specified by a single linear equation, i.e., a set of the form $\{x \mid a'x = b\}$, where a is a nonzero vector and b is a scalar. A *halfspace* is a set specified by a single linear inequality, i.e., a set of the form $\{x \mid a'x \le b\}$, where a is a nonzero vector and b is a scalar. It is clearly closed and convex. A set is said to be *polyhedral* if it is nonempty and it is the intersection of a finite number of halfspaces, i.e., if it has the form

$$\{x \mid a_j'x \le b_j,\ j = 1,\ldots,r\},$$

where $a_1,\ldots,a_r$ and $b_1,\ldots,b_r$ are some vectors in $\Re^n$ and scalars, respectively. A polyhedral set is convex and closed, being the intersection of halfspaces [cf. Prop. 1.1.1(a)].

A set C is said to be a *cone* if for all $x \in C$ and $\lambda > 0$, we have $\lambda x \in C$. A cone need not be convex and need not contain the origin, although the origin always lies in the closure of a nonempty cone (see Fig. 1.1.2). A *polyhedral cone* is a set of the form

$$C = \{x \mid a_j'x \le 0,\ j = 1,\ldots,r\},$$

where $a_1,\ldots,a_r$ are some vectors in $\Re^n$. A subspace is a special case of a polyhedral cone, which is in turn a special case of a polyhedral set.

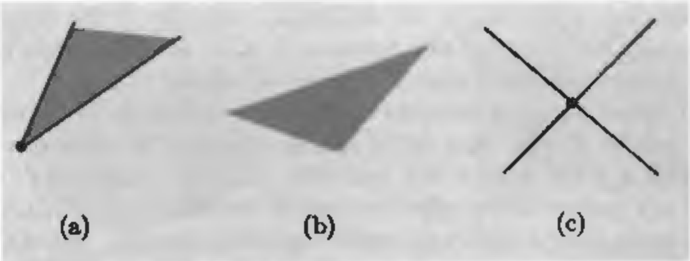


Figure 1.1.2. Illustration of convex and nonconvex cones. Cones (a) and (b) are convex, while cone (c), which consists of two lines passing through the origin, is not convex. Cone (a) is polyhedral. Cone (b) does not contain the origin.

1.1.1 Convex Functions

We now define a real-valued convex function (cf. Fig. 1.1.3).

Definition 1.1.2: Let C be a convex subset of $\Re^n$. We say that a function $f : C \mapsto \Re$ is *convex* if

$$f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y), \qquad \forall\ x, y \in C,\ \forall\ \alpha \in [0,1]. \tag{1.1}$$

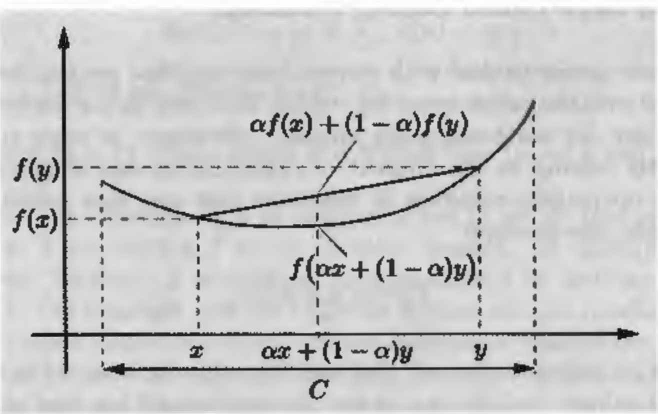


Figure 1.1.3. Illustration of the definition of a function $f : C \mapsto \Re$ that is convex. The linear interpolation $\alpha f(x) + (1-\alpha)f(y)$ overestimates the function value $f\big(\alpha x + (1-\alpha)y\big)$ for all $\alpha \in [0,1]$.

Note that, according to our definition, convexity of the domain C is a prerequisite for convexity of a function $f : C \mapsto \Re$. Thus when calling a function convex, we imply that its domain is convex.

We introduce some variants of the basic definition of convexity. A convex function $f : C \mapsto \Re$ is called *strictly convex* if the inequality (1.1) is strict for all $x, y \in C$ with $x \neq y$, and all $\alpha \in (0,1)$. A function $f : C \mapsto \Re$, where C is a convex set, is called *concave* if the function $(-f)$ is convex.

An example of a convex function is an affine function, one of the form $f(x) = a'x + b$, where $a \in \Re^n$ and $b \in \Re$; this is straightforward to verify using the definition of convexity. Another example is a norm $\|\cdot\|$, since by the triangle inequality, we have

$$\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\| = \alpha\|x\| + (1-\alpha)\|y\|,$$

for any $x, y \in \Re^n$, and $\alpha \in [0,1]$.

If $f : C \mapsto \Re$ is a function and γ is a scalar, the sets $\{x \in C \mid f(x) \leq \gamma\}$ and $\{x \in C \mid f(x) < \gamma\}$, are called *level sets* of f. If f is a convex function, then all its level sets are convex. To see this, note that if $x, y \in C$ are such that $f(x) \leq \gamma$ and $f(y) \leq \gamma$, then for any $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha)y \in C$, by the convexity of C, so

$$f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y) \leq \gamma,$$

by the convexity of f. A similar proof also shows that the level sets $\{x \in C \mid f(x) < \gamma\}$ are convex when f is convex. Note, however, that convexity of the level sets does not imply convexity of the function; for example, the scalar function $f(x) = \sqrt{|x|}$ has convex level sets but is not convex.

Extended Real-Valued Convex Functions

We generally prefer to deal with convex functions that are real-valued and are defined over the entire space $\Re^n$ (rather than over just a convex subset), because they are mathematically simpler. However, in some situations, prominently arising in the context of optimization and duality, we will encounter operations resulting in functions that can take infinite values. For example, the function

$$f(x) = \sup_{i \in I} f_i(x),$$

where I is an infinite index set, can take the value ∞ even if the functions f_i are real-valued, and the conjugate of a real-valued function often takes infinite values (cf. Section 1.6).

Furthermore, we will encounter functions f that are convex over a convex subset C and cannot be extended to functions that are real-valued and convex over the entire space $\Re^n$ [e.g., the function $f : (0,\infty) \mapsto \Re$

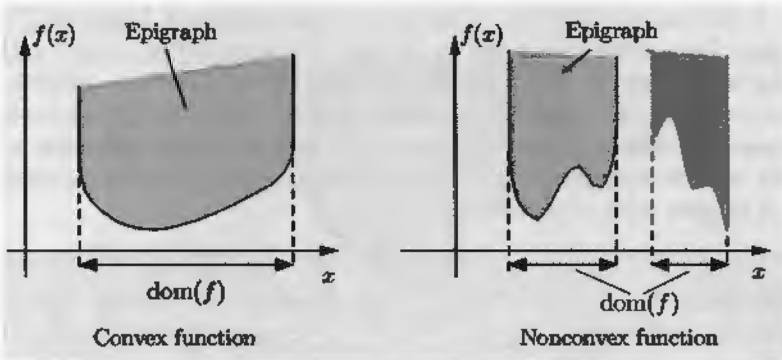


Figure 1.1.4. Illustration of the epigraphs and effective domains of extended real-valued convex and nonconvex functions.

defined by $f(x) = 1/x$]. In such situations, it may be convenient, instead of restricting the domain of f to the subset C where f takes real values, to extend the domain to all of $\Re^n$, but allow f to take infinite values.

We are thus motivated to introduce *extended real-valued* functions that can take the values $-\infty$ and ∞ at some points. Such functions can be characterized using the notion of epigraph, which we now introduce.

The *epigraph* of a function $f : X \mapsto [-\infty, \infty]$, where $X \subset \Re^n$, is defined to be the subset of $\Re^{n+1}$ given by

$$\text{epi}(f) = \big\{(x, w) \mid x \in X,\ w \in \Re,\ f(x) \leq w\big\}.$$

The *effective domain* of f is defined to be the set

$$\text{dom}(f) = \big\{x \in X \mid f(x) < \infty\big\}$$

(see Fig. 1.1.4). It can be seen that

$$\text{dom}(f) = \big\{x \mid \text{there exists } w \in \Re \text{ such that } (x, w) \in \text{epi}(f)\big\},$$

i.e., dom(f) is obtained by a projection of epi(f) on $\Re^n$ (the space of x). Note that if we restrict f to its effective domain, its epigraph remains unaffected. Similarly, if we enlarge the domain of f by defining $f(x) = \infty$ for $x \notin X$, the epigraph and the effective domain remain unaffected.

It is often important to exclude the degenerate case where f is identically equal to ∞ [which is true if and only if epi(f) is empty], and the case where the function takes the value $-\infty$ at some point [which is true if and only if epi(f) contains a vertical line]. We will thus say that f is *proper* if $f(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$, and we will say that f *improper* if it is not proper. In words, a function is proper if and only if its epigraph is nonempty and does not contain a vertical line.

A difficulty in defining extended real-valued convex functions f that can take both values $-\infty$ and ∞ is that the term $\alpha f(x) + (1 - \alpha)f(y)$ arising in our earlier definition for the real-valued case may involve the forbidden sum $-\infty + \infty$ (this, of course, may happen only if f is improper, but improper functions arise on occasion in proofs or other analyses, so we do not wish to exclude them *a priori*). The epigraph provides an effective way of dealing with this difficulty.

Definition 1.1.3: Let C be a convex subset of $\Re^n$. We say that an extended real-valued function $f : C \mapsto [-\infty, \infty]$ is *convex* if $\mathrm{epi}(f)$ is a convex subset of $\Re^{n+1}$.

It can be easily verified that, according to the above definition, convexity of f implies that its effective domain $\mathrm{dom}(f)$ and its level sets $\{x \in C \mid f(x) \leq \gamma\}$ and $\{x \in C \mid f(x) < \gamma\}$ are convex sets for all scalars γ. Furthermore, if $f(x) < \infty$ for all x, or $f(x) > -\infty$ for all x, then

$$f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y), \qquad \forall\, x, y \in C,\ \forall\, \alpha \in [0,1], \quad (1.2)$$

so the preceding definition is consistent with the earlier definition of convexity for real-valued functions.

By passing to epigraphs, we can use results about sets to infer corresponding results about functions (e.g., proving convexity). The reverse is also possible, through the notion of *indicator function* $\delta : \Re^n \mapsto (-\infty, \infty]$ of a set $X \subset \Re^n$, defined by

$$\delta(x \mid X) = \begin{cases} 0 & \text{if } x \in X, \\ \infty & \text{otherwise.} \end{cases}$$

In particular, a set is convex if and only if its indicator function is convex, and it is nonempty if and only if its indicator function is proper.

A convex function $f : C \mapsto (-\infty, \infty]$ is called *strictly convex* if the inequality (1.2) is strict for all $x, y \in \mathrm{dom}(f)$ with $x \neq y$, and all $\alpha \in (0, 1)$. A function $f : C \mapsto [-\infty, \infty]$, where C is a convex set, is called *concave* if the function $(-f) : C \mapsto [-\infty, \infty]$ is convex as per Definition 1.1.3.

Sometimes we will deal with functions that are defined over a (possibly nonconvex) domain C but are convex when restricted to a convex subset of their domain. The following definition formalizes this case.

Definition 1.1.4: Let C and X be subsets of $\Re^n$ such that C is nonempty and convex, and $C \subset X$. We say that an extended real-valued function $f : X \mapsto [-\infty, \infty]$ is *convex over* C if f becomes convex when the domain of f is restricted to C, i.e., if the function $\tilde{f} : C \mapsto [-\infty, \infty]$, defined by $\tilde{f}(x) = f(x)$ for all $x \in C$, is convex.

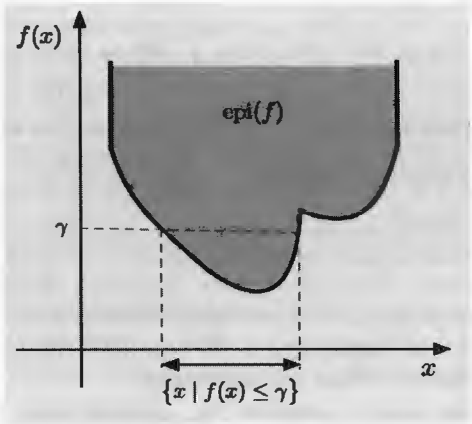


Figure 1.1.5. Visualization of the epigraph of a function in relation to its level sets. It can be seen that the level set $\{x \mid f(x) \leq \gamma\}$ can be identified with a translation of the intersection of epi(f) and the "slice" $\{(x, \gamma) \mid x \in \Re^n\}$, indicating that epi$(f)$ is closed if and only if all the level sets are closed.

By replacing the domain of an extended real-valued proper convex function with its effective domain, we can convert it to a real-valued function. In this way, we can use results stated in terms of real-valued functions, and we can also avoid calculations with ∞. Thus, nearly all the theory of convex functions can be developed without resorting to extended real-valued functions. The reverse is also true, namely that extended real-valued functions can be adopted as the norm; for example, this approach is followed by Rockafellar [Roc70]. We will adopt a flexible approach, and use both real-valued and extended real-valued functions, depending on the context.

1.1.2 Closedness and Semicontinuity

If the epigraph of a function $f : X \mapsto [-\infty, \infty]$ is a closed set, we say that f is a *closed* function. Closedness is related to the classical notion of lower semicontinuity. Recall that f is called *lower semicontinuous* at a vector $x \in X$ if

$$f(x) \leq \liminf_{k \to \infty} f(x_k)$$

for every sequence $\{x_k\} \subset X$ with $x_k \to x$. We say that f *is lower semicontinuous* if it is lower semicontinuous at each point x in its domain X. We say that f is *upper semicontinuous* if $-f$ is lower semicontinuous. These definitions are consistent with the corresponding definitions for real-valued functions [cf. Definition A.2.4(c)].

The following proposition connects closedness, lower semicontinuity, and closedness of the level sets of a function; see Fig. 1.1.5.

Proposition 1.1.2: For a function $f : \Re^n \mapsto [-\infty, \infty]$, the following are equivalent:

(i) The level set $V_\gamma = \{x \mid f(x) \leq \gamma\}$ is closed for every scalar γ.

(ii) f is lower semicontinuous.

(iii) $\text{epi}(f)$ is closed.

Proof: If $f(x) = \infty$ for all x, the result trivially holds. We thus assume that $f(x) < \infty$ for at least one $x \in \Re^n$, so that $\text{epi}(f)$ is nonempty and there exist level sets of f that are nonempty.

We first show that (i) implies (ii). Assume that the level set V_γ is closed for every scalar γ. Suppose, to arrive at a contradiction, that

$$f(\overline{x}) > \liminf_{k\to\infty} f(x_k)$$

for some $\overline{x}$ and sequence $\{x_k\}$ converging to $\overline{x}$, and let γ be a scalar such that

$$f(\overline{x}) > \gamma > \liminf_{k\to\infty} f(x_k).$$

Then, there exists a subsequence $\{x_k\}_\mathcal{K}$ such that $f(x_k) \leq \gamma$ for all $k \in \mathcal{K}$, so that $\{x_k\}_\mathcal{K} \subset V_\gamma$. Since V_γ is closed, $\overline{x}$ must also belong to V_γ, so $f(\overline{x}) \leq \gamma$, a contradiction.

We next show that (ii) implies (iii). Assume that f is lower semicontinuous over $\Re^n$, and let $(\overline{x}, \overline{w})$ be the limit of a sequence

$$\{(x_k, w_k)\} \subset \text{epi}(f).$$

Then we have $f(x_k) \leq w_k$, and by taking the limit as $k \to \infty$ and by using the lower semicontinuity of f at $\overline{x}$, we obtain

$$f(\overline{x}) \leq \liminf_{k\to\infty} f(x_k) \leq \overline{w}.$$

Hence, $(\overline{x}, \overline{w}) \in \text{epi}(f)$ and $\text{epi}(f)$ is closed.

We finally show that (iii) implies (i). Assume that $\text{epi}(f)$ is closed, and let $\{x_k\}$ be a sequence that converges to some $\overline{x}$ and belongs to V_γ for some scalar γ. Then $(x_k, \gamma) \in \text{epi}(f)$ for all k and $(x_k, \gamma) \to (\overline{x}, \gamma)$, so since $\text{epi}(f)$ is closed, we have $(\overline{x}, \gamma) \in \text{epi}(f)$. Hence, $\overline{x}$ belongs to V_γ, implying that this set is closed. **Q.E.D.**

For most of our development, we prefer to use the closedness notion, rather than lower semicontinuity. One reason is that contrary to closedness, lower semicontinuity is a domain-specific property. For example, the function $f : \Re \mapsto (-\infty, \infty]$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ \infty & \text{if } x \notin (0,1), \end{cases}$$

is neither closed nor lower semicontinuous; but if its domain is restricted to $(0,1)$ it becomes lower semicontinuous.

On the other hand, if a function $f : X \mapsto [-\infty, \infty]$ has a closed effective domain $\mathrm{dom}(f)$ and is lower semicontinuous at every $x \in \mathrm{dom}(f)$, then f is closed. We state this as a proposition. The proof follows from the argument we used to show that (ii) implies (iii) in Prop. 1.1.2.

Proposition 1.1.3: Let $f : X \mapsto [-\infty, \infty]$ be a function. If $\mathrm{dom}(f)$ is closed and f is lower semicontinuous at each $x \in \mathrm{dom}(f)$, then f is closed.

As an example of application of the preceding proposition, the indicator function of a set X is closed if and only if X is closed (the "if" part follows from the proposition, and the "only if" part follows using the definition of epigraph). More generally, if f_X is a function of the form

$$f_X(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases}$$

where $f : \Re^n \mapsto \Re$ is a continuous function, then it can be shown that f_X is closed if and only if X is closed.

We finally note that an improper closed convex function is very peculiar: it cannot take a finite value at any point, so it has the form

$$f(x) = \begin{cases} -\infty & \text{if } x \in \mathrm{dom}(f), \\ \infty & \text{if } x \notin \mathrm{dom}(f). \end{cases}$$

To see this, consider an improper closed convex function $f : \Re^n \mapsto [-\infty, \infty]$, and assume that there exists an x such that $f(x)$ is finite. Let $\overline{x}$ be such that $f(\overline{x}) = -\infty$ (such a point must exist since f is improper and f is not identically equal to ∞). Because f is convex, it can be seen that every point of the form

$$x_k = \frac{k-1}{k}x + \frac{1}{k}\overline{x}, \qquad \forall\, k = 1, 2, \ldots$$

satisfies $f(x_k) = -\infty$, while we have $x_k \to x$. Since f is closed, this implies that $f(x) = -\infty$, which is a contradiction. In conclusion, *a closed convex function that is improper cannot take a finite value anywhere.*

1.1.3 Operations with Convex Functions

We can verify the convexity of a given function in a number of ways. Several commonly encountered functions, such as affine functions and norms, are convex. An important type of convex function is a *polyhedral function*, which by definition is a proper convex function whose epigraph is a polyhedral set. Starting with some known convex functions, we can generate other convex functions by using some common operations that preserve convexity. Principal among these operations are the following:

(a) Composition with a linear transformation.

(b) Addition, and multiplication with a nonnegative scalar.

(c) Taking supremum.

(d) Taking partial minimum, i.e., minimizing with respect to z a function that is (jointly) convex in two vectors x and z.

The following three propositions deal with the first three cases, and Section 3.3 deals with the fourth.

Proposition 1.1.4: Let $f : \Re^m \mapsto (-\infty, \infty]$ be a given function, let A be an $m \times n$ matrix, and let $F : \Re^n \mapsto (-\infty, \infty]$ be the function

$$F(x) = f(Ax), \qquad x \in \Re^n.$$

If f is convex, then F is also convex, while if f is closed, then F is also closed.

Proof: Let f be convex. We use the definition of convexity to write for any $x, y \in \Re^n$ and $\alpha \in [0,1]$,

$$\begin{aligned} F\big(\alpha x + (1-\alpha)y\big) &= f\big(\alpha Ax + (1-\alpha)Ay\big) \\ &\leq \alpha f(Ax) + (1-\alpha) f(Ay) \\ &= \alpha F(x) + (1-\alpha)F(y). \end{aligned}$$

Hence F is convex.

Let f be closed. Then f is lower semicontinuous at every $x \in \Re^n$ (cf. Prop. 1.1.2), so for every sequence $\{x_k\}$ converging to x, we have

$$f(Ax) \leq \liminf_{k\to\infty} f(Ax_k),$$

or

$$F(x) \leq \liminf_{k\to\infty} F(x_k)$$

for all k. It follows that F is lower semicontinuous at every $x \in \Re^n$, and hence is closed by Prop. 1.1.2. **Q.E.D.**

The next proposition deals with sums of function and it is interesting to note that it can be viewed as a special case of the preceding proposition, which deals with compositions with linear transformations. The reason is that we may write a sum $F = f_1 + \cdots + f_m$ in the form $F(x) = f(Ax)$, where A is the matrix defined by $Ax = (x, \ldots, x)$, and $f : \Re^{mn} \mapsto (-\infty, \infty]$ is the function given by

$$f(x_1, \ldots, x_m) = f_1(x_1) + \cdots + f_m(x_m).$$

Proposition 1.1.5: Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be given functions, let $\gamma_1, \ldots, \gamma_m$ be positive scalars, and let $F : \Re^n \mapsto (-\infty, \infty]$ be the function

$$F(x) = \gamma_1 f_1(x) + \cdots + \gamma_m f_m(x), \qquad x \in \Re^n.$$

If $f_1, \ldots, f_m$ are convex, then F is also convex, while if $f_1, \ldots, f_m$ are closed, then F is also closed.

Proof: The proof follows closely the one of Prop. 1.1.4. **Q.E.D.**

Proposition 1.1.6: Let $f_i : \Re^n \mapsto (-\infty, \infty]$ be given functions for $i \in I$, where I is an arbitrary index set, and let $f : \Re^n \mapsto (-\infty, \infty]$ be the function given by

$$f(x) = \sup_{i \in I} f_i(x).$$

If f_i, $i \in I$, are convex, then f is also convex, while if f_i, $i \in I$, are closed, then f is also closed.

Proof: A pair (x, w) belongs to $\text{epi}(f)$ if and only if $f(x) \le w$, which is true if and only if $f_i(x) \le w$ for all $i \in I$, or equivalently $(x, w) \in \cap_{i \in I} \text{epi}(f_i)$. Therefore,

$$\text{epi}(f) = \cap_{i \in I} \text{epi}(f_i).$$

If the functions f_i are convex, the epigraphs $\text{epi}(f_i)$ are convex, so $\text{epi}(f)$ is convex, and f is convex. If the functions f_i are closed, the epigraphs $\text{epi}(f_i)$ are closed, so $\text{epi}(f)$ is closed, and f is closed. **Q.E.D.**

1.1.4 Characterizations of Differentiable Convex Functions

For once or twice differentiable functions, there are some additional criteria for verifying convexity, as we will now discuss. A useful alternative characterization of convexity for differentiable functions is given in the following proposition and is illustrated in Fig. 1.1.6.

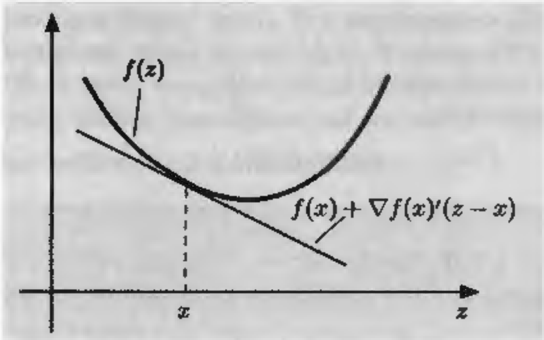


Figure 1.1.6. Characterization of convexity in terms of first derivatives. The condition $f(z) \geq f(x) + \nabla f(x)'(z-x)$ states that a linear approximation, based on the gradient, underestimates a convex function.

Proposition 1.1.7: Let C be a nonempty convex subset of $\Re^n$ and let $f : \Re^n \mapsto \Re$ be differentiable over an open set that contains C.

(a) f is convex over C if and only if

$$f(z) \geq f(x) + \nabla f(x)'(z-x), \qquad \forall\, x, z \in C. \tag{1.3}$$

(b) f is strictly convex over C if and only if the above inequality is strict whenever $x \neq z$.

Proof: The ideas of the proof are geometrically illustrated in Fig. 1.1.7. We prove (a) and (b) simultaneously. Assume that the inequality (1.3) holds. Choose any $x, y \in C$ and $\alpha \in [0,1]$, and let $z = \alpha x + (1-\alpha)y$. Using the inequality (1.3) twice, we obtain

$$f(x) \geq f(z) + \nabla f(z)'(x-z),$$

$$f(y) \geq f(z) + \nabla f(z)'(y-z).$$

We multiply the first inequality by α, the second by $(1-\alpha)$, and add them to obtain

$$\alpha f(x) + (1-\alpha)f(y) \geq f(z) + \nabla f(z)'\big(\alpha x + (1-\alpha)y - z\big) = f(z),$$

which proves that f is convex. If the inequality (1.3) is strict as stated in part (b), then if we take $x \neq y$ and $\alpha \in (0,1)$ above, the three preceding inequalities become strict, thus showing the strict convexity of f.

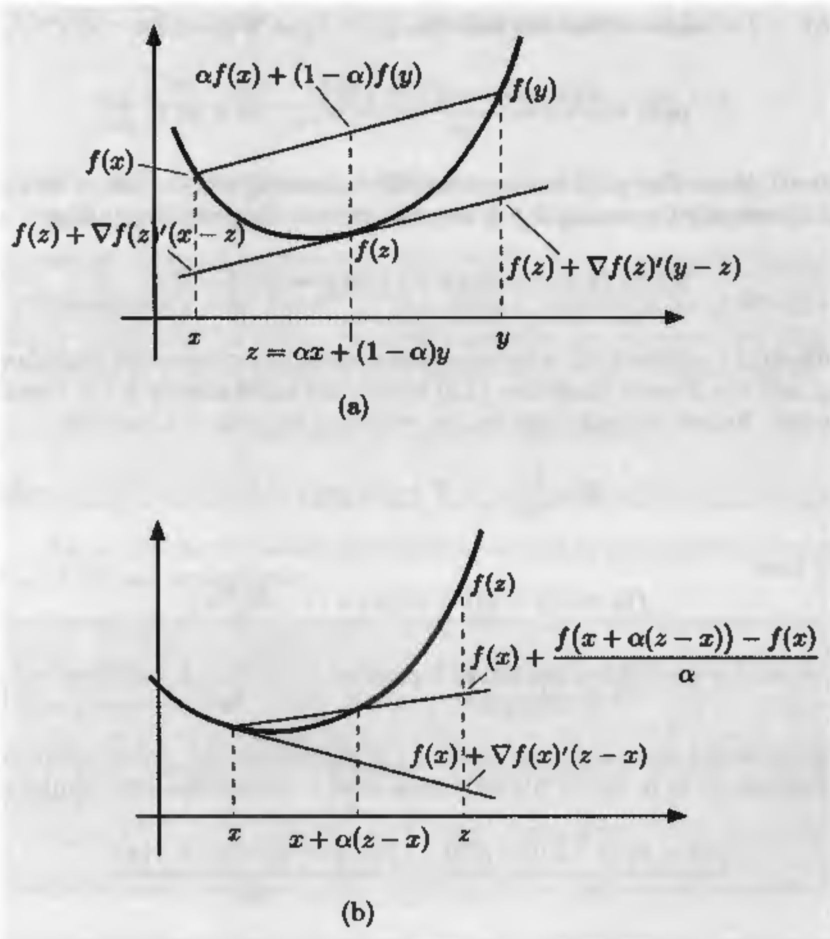


Figure 1.1.7. Geometric illustration of the ideas underlying the proof of Prop. 1.1.7. In figure (a), we linearly approximate f at $z = \alpha x + (1-\alpha)y$. The inequality (1.3) implies that

$$f(x) \geq f(z) + \nabla f(z)'(x - z),$$

$$f(y) \geq f(z) + \nabla f(z)'(y - z).$$

As can be seen from the figure, it follows that $\alpha f(x) + (1-\alpha)f(y)$ lies above $f(z)$, so f is convex.

In figure (b), we assume that f is convex, and from the figure's geometry, we note that

$$f(x) + \frac{f\big(x + \alpha(z - x)\big) - f(x)}{\alpha}$$

lies below $f(z)$, is monotonically nonincreasing as $\alpha \downarrow 0$, and converges to $f(x) + \nabla f(x)'(z - x)$. It follows that $f(z) \geq f(x) + \nabla f(x)'(z - x)$.

Conversely, assume that f is convex, let x and z be any vectors in C with $x \neq z$, and consider the function $g : (0, 1] \mapsto \Re$ given by

$$g(\alpha) = \frac{f\big(x + \alpha(z - x)\big) - f(x)}{\alpha}, \qquad \alpha \in (0, 1].$$

We will show that $g(\alpha)$ is monotonically increasing with α, and is strictly monotonically increasing if f is strictly convex. This will imply that

$$\nabla f(x)'(z - x) = \lim_{\alpha \downarrow 0} g(\alpha) \leq g(1) = f(z) - f(x),$$

with strict inequality if g is strictly monotonically increasing, thereby showing that the desired inequality (1.3) holds, and holds strictly if f is strictly convex. Indeed, consider any α_1, α_2, with $0 < \alpha_1 < \alpha_2 < 1$, and let

$$\overline{\alpha} = \frac{\alpha_1}{\alpha_2}, \qquad \overline{z} = x + \alpha_2(z - x). \tag{1.4}$$

We have

$$f\big(x + \overline{\alpha}(\overline{z} - x)\big) \leq \overline{\alpha} f(\overline{z}) + (1 - \overline{\alpha}) f(x),$$

or

$$\frac{f\big(x + \overline{\alpha}(\overline{z} - x)\big) - f(x)}{\overline{\alpha}} \leq f(\overline{z}) - f(x), \tag{1.5}$$

and the above inequalities are strict if f is strictly convex. Substituting the definitions (1.4) in Eq. (1.5), we obtain after a straightforward calculation

$$\frac{f\big(x + \alpha_1(z - x)\big) - f(x)}{\alpha_1} \leq \frac{f\big(x + \alpha_2(z - x)\big) - f(x)}{\alpha_2},$$

or

$$g(\alpha_1) \leq g(\alpha_2),$$

with strict inequality if f is strictly convex. Hence g is monotonically increasing with α, and strictly so if f is strictly convex. **Q.E.D.**

Note a simple consequence of Prop. 1.1.7(a): if $f : \Re^n \mapsto \Re$ is a differentiable convex function and $\nabla f(x^*) = 0$, then x^* minimizes f over $\Re^n$. This is a classical sufficient condition for unconstrained optimality, originally formulated (in one dimension) by Fermat in 1637. Similarly, from Prop. 1.1.7(a), we see that the condition

$$\nabla f(x^*)'(z - x^*) \geq 0, \qquad \forall\ z \in C,$$

implies that x^* minimizes a differentiable convex function f over a convex set C. This sufficient condition for optimality is also necessary. To see this, assume to arrive at a contradiction that x^* minimizes f over C and that $\nabla f(x^*)'(z - x^*) < 0$ for some $z \in C$. By differentiation, we have

$$\lim_{\alpha \downarrow 0} \frac{f\big(x^* + \alpha(z - x^*)\big) - f(x^*)}{\alpha} = \nabla f(x^*)'(z - x^*) < 0,$$

so $f\big(x^* + \alpha(z - x^*)\big)$ decreases strictly for sufficiently small $\alpha > 0$, contradicting the optimality of x^*. We state the conclusion as a proposition.

Proposition 1.1.8: Let C be a nonempty convex subset of $\Re^n$ and let $f : \Re^n \mapsto \Re$ be convex and differentiable over an open set that contains C. Then a vector $x^* \in C$ minimizes f over C if and only if

$$\nabla f(x^*)'(z - x^*) \geq 0, \qquad \forall\, z \in C.$$

Let us use the preceding optimality condition to prove a basic theorem of analysis and optimization.

Proposition 1.1.9: (Projection Theorem) Let C be a nonempty closed convex subset of $\Re^n$, and let z be a vector in $\Re^n$. There exists a unique vector that minimizes $\|z - x\|$ over $x \in C$, called the projection of z on C. Furthermore, a vector x^* is the projection of z on C if and only if

$$(z - x^*)'(x - x^*) \leq 0, \qquad \forall\, x \in C. \tag{1.6}$$

Proof: Minimizing $\|z - x\|$ is equivalent to minimizing the convex and differentiable function

$$f(x) = \tfrac{1}{2}\|z - x\|^2.$$

By Prop. 1.1.8, x^* minimizes f over C if and only if

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in C.$$

Since $\nabla f(x^*) = x^* - z$, this condition is equivalent to Eq. (1.6).

Minimizing f over C is equivalent to minimizing f over the compact set $C \cap \big\{\|z - x\| \leq \|z - w\|\big\}$, where w is any vector in C. By Weierstrass' Theorem (Prop. A.2.7), it follows that there exists a minimizing vector. To show uniqueness, let x_1^* and x_2^* be two minimizing vectors. Then by Eq. (1.6), we have

$$(z - x_1^*)'(x_2^* - x_1^*) \leq 0, \qquad (z - x_2^*)'(x_1^* - x_2^*) \leq 0.$$

Adding these two inequalities, we obtain

$$(x_2^* - x_1^*)'(x_2^* - x_1^*) = \|x_2^* - x_1^*\|^2 \le 0,$$

so $x_2^* = x_1^*$. **Q.E.D.**

For twice differentiable convex functions, there is another characterization of convexity, given by the following proposition.

Proposition 1.1.10: Let C be a nonempty convex subset of $\Re^n$ and let $f : \Re^n \mapsto \Re$ be twice continuously differentiable over an open set that contains C.

(a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then f is convex over C.

(b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C.

(c) If C is open and f is convex over C, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

Proof: (a) Using the mean value theorem (Prop. A.3.1), we have for all $x, y \in C$,

$$f(y) = f(x) + \nabla f(x)'(y - x) + \tfrac{1}{2}(y - x)'\nabla^2 f\big(x + \alpha(y - x)\big)(y - x)$$

for some $\alpha \in [0, 1]$. Therefore, using the positive semidefiniteness of $\nabla^2 f$, we obtain

$$f(y) \ge f(x) + \nabla f(x)'(y - x), \qquad \forall\, x, y \in C.$$

From Prop. 1.1.7(a), we conclude that f is convex over C.

(b) Similar to the proof of part (a), we have $f(y) > f(x) + \nabla f(x)'(y - x)$ for all $x, y \in C$ with $x \ne y$, and the result follows from Prop. 1.1.7(b).

(c) Assume, to obtain a contradiction, that there exist some $x \in C$ and some $z \in \Re^n$ such that $z'\nabla^2 f(x)z < 0$. Since C is open and $\nabla^2 f$ is continuous, we can choose z to have small enough norm so that $x + z \in C$ and $z'\nabla^2 f(x + \alpha z)z < 0$ for every $\alpha \in [0, 1]$. Then, using again the mean value theorem, we obtain $f(x + z) < f(x) + \nabla f(x)'z$, which, in view of Prop. 1.1.7(a), contradicts the convexity of f over C. **Q.E.D.**

If f is convex over a convex set C that is not open, $\nabla^2 f(x)$ may not be positive semidefinite at any point of C [take for example $n = 2$, $C = \big\{(x_1, 0) \mid x_1 \in \Re\big\}$, and $f(x) = x_1^2 - x_2^2$]. However, it can be shown that the conclusion of Prop. 1.1.10(c) also holds if C has nonempty interior instead of being open.

1.2 CONVEX AND AFFINE HULLS

We now discuss issues relating to the convexification of nonconvex sets. Let X be a nonempty subset of $\Re^n$. The *convex hull* of a set X, denoted conv(X), is the intersection of all convex sets containing X, and is a convex set by Prop. 1.1.1(a). A *convex combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where m is a positive integer, $x_1, \ldots, x_m$ belong to X, and $\alpha_1, \ldots, \alpha_m$ are scalars such that

$$\alpha_i \geq 0, \quad i = 1, \ldots, m, \qquad \sum_{i=1}^m \alpha_i = 1.$$

Note that a convex combination belongs to conv(X) (see the construction of Fig. 1.2.1). For any convex combination and function $f : \Re^n \mapsto \Re$ that is convex over conv(X), we have

$$f\left(\sum_{i=1}^m \alpha_i x_i\right) \leq \sum_{i=1}^m \alpha_i f(x_i). \tag{1.7}$$

This follows by using repeatedly the definition of convexity together with the construction of Fig. 1.2.1. The preceding relation is a special case of a relation known as *Jensen's inequality*, which finds wide use in applied mathematics and probability theory.

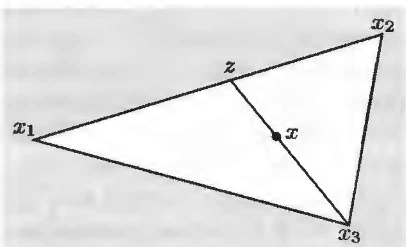


Figure 1.2.1. Construction of a convex combination of m vectors by forming a sequence of $m-1$ convex combinations of pairs of vectors (first combine two vectors, then combine the result with a third vector, etc). For example,

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = (\alpha_1 + \alpha_2)\left(\frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2\right) + \alpha_3 x_3,$$

so the convex combination $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ can be obtained by forming the convex combination

$$z = \frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2,$$

and then by forming the convex combination $x = (\alpha_1 + \alpha_2) z + \alpha_3 x_3$ as shown in the figure. This shows that a convex combination of vectors from a convex set belongs to the set, and that a convex combination of vectors from a nonconvex set belongs to the convex hull of the set.

It is straightforward to verify that the set of all convex combinations of elements of X is equal to $\mathrm{conv}(X)$. In particular, if X consists of a finite number of vectors $x_1, \ldots, x_m$, its convex hull is

$$\mathrm{conv}\big(\{x_1, \ldots, x_m\}\big) = \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; \alpha_i \geq 0,\, i = 1, \ldots, m,\, \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

Also, for any set S and linear transformation A, we have $\mathrm{conv}(AS) = A\,\mathrm{conv}(S)$. From this it follows that for any sets $S_1, \ldots, S_m$, we have $\mathrm{conv}(S_1 + \cdots + S_m) = \mathrm{conv}(S_1) + \cdots + \mathrm{conv}(S_m)$.

We recall that an affine set M in $\Re^n$ is a set of the form $x + S$, where x is some vector, and S is a subspace uniquely determined by M and called the *subspace parallel to* M. Alternatively, a set M is affine if it contains all the lines that pass through pairs of points $x, y \in M$ with $x \neq y$. If X is a subset of $\Re^n$, the *affine hull* of X, denoted $\mathrm{aff}(X)$, is the intersection of all affine sets containing X. Note that $\mathrm{aff}(X)$ is itself an affine set and that it contains $\mathrm{conv}(X)$. The dimension of $\mathrm{aff}(X)$ is defined to be the dimension of the subspace parallel to $\mathrm{aff}(X)$. It can be shown that

$$\mathrm{aff}(X) = \mathrm{aff}\big(\mathrm{conv}(X)\big) = \mathrm{aff}\big(\mathrm{cl}(X)\big).$$

For a convex set C, the *dimension* of C is defined to be the dimension of $\mathrm{aff}(C)$.

Given a nonempty subset X of $\Re^n$, a *nonnegative combination* of elements of X is a vector of the form $\sum_{i=1}^{m} \alpha_i x_i$, where m is a positive integer, $x_1, \ldots, x_m$ belong to X, and $\alpha_1, \ldots, \alpha_m$ are nonnegative scalars. If the scalars α_i are all positive, $\sum_{i=1}^{m} \alpha_i x_i$ is said to be a *positive combination*. The *cone generated by* X, denoted $\mathrm{cone}(X)$, is the set of all nonnegative combinations of elements of X. It is easily seen that $\mathrm{cone}(X)$ is a convex cone containing the origin, although it need not be closed even if X is compact, as shown in Fig. 1.2.2 [it can be proved that $\mathrm{cone}(X)$ is closed in special cases, such as when X is finite; see Section 1.4.3].

The following is a fundamental characterization of convex hulls (see Fig. 1.2.3).

Proposition 1.2.1: (Caratheodory's Theorem) Let X be a nonempty subset of $\Re^n$.

(a) Every nonzero vector from $\mathrm{cone}(X)$ can be represented as a positive combination of linearly independent vectors from X.

(b) Every vector from $\mathrm{conv}(X)$ can be represented as a convex combination of no more than $n + 1$ vectors from X.

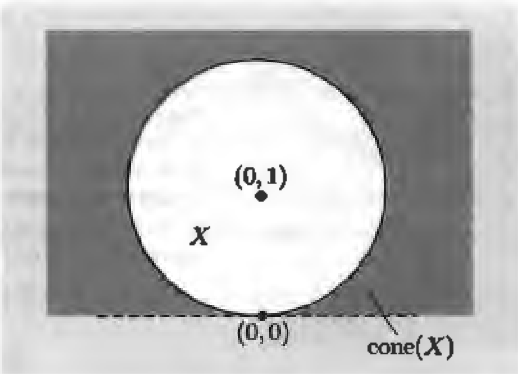


Figure 1.2.2. An example in $\Re^2$ where X is convex and compact, but cone(X) is not closed. Here

$$X = \{(x_1, x_2) \mid x_1^2 + (x_2 - 1)^2 \le 1\}, \quad \mathrm{cone}(X) = \{(x_1, x_2) \mid x_2 > 0\} \cup \{(0, 0)\}.$$

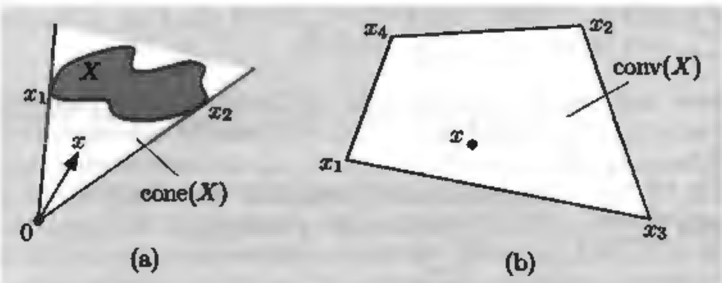


Figure 1.2.3. Illustration of Caratheodory's Theorem. In (a), X is a nonconvex set in $\Re^2$, and a point $x \in \mathrm{cone}(X)$ is represented as a positive combination of the two linearly independent vectors $x_1, x_2 \in X$. In (b), X consists of four points x_1, x_2, x_3, x_4 in $\Re^2$, and the point $x \in \mathrm{conv}(X)$ shown in the figure can be represented as a convex combination of the three vectors x_1, x_2, x_3. Note also that x can alternatively be represented as a convex combination of the vectors x_1, x_3, x_4, so the representation is not unique.

Proof: (a) Consider a vector $x \neq 0$ from cone(X), and let m be the smallest integer such that x has the form $\sum_{i=1}^m \alpha_i x_i$, where $\alpha_i > 0$ and $x_i \in X$ for all $i = 1, \ldots, m$. We argue by contradiction. If the vectors x_i are linearly dependent, there exist scalars $\lambda_1, \ldots, \lambda_m$, with $\sum_{i=1}^m \lambda_i x_i = 0$ and at least one λ_i is positive. Consider the linear combination $\sum_{i=1}^m (\alpha_i - \overline{\gamma}\lambda_i)x_i$, where $\overline{\gamma}$ is the largest γ such that $\alpha_i - \gamma\lambda_i \ge 0$ for all i. This combination provides a representation of x as a positive combination of fewer than m vectors of X – a contradiction. Therefore, $x_1, \ldots, x_m$ are linearly independent.

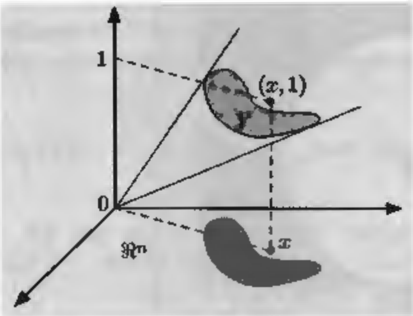


Figure 1.2.4. Illustration of the proof of Caratheodory's Theorem for convex hulls using the version of the theorem for generated cones. We consider the set $Y = \{(y,1) \mid y \in X\} \subset \Re^{n+1}$ and apply Prop. 1.2.1(a).

(b) We apply part (a) to the following subset of $\Re^{n+1}$:

$$Y = \{(y,1) \mid y \in X\}$$

(cf. Fig. 1.2.4). If $x \in \text{conv}(X)$, we have $x = \sum_{i=1}^{I} \gamma_i x_i$ for an integer $I > 0$ and scalars $\gamma_i > 0$, $i = 1, \dots, I$, with $1 = \sum_{i=1}^{I} \gamma_i$, so that $(x,1) \in \text{cone}(Y)$. By part (a), we have $(x,1) = \sum_{i=1}^{m} \alpha_i(x_i,1)$ for some scalars $\alpha_1, \dots, \alpha_m > 0$ and (at most $n+1$) linearly independent vectors $(x_1,1), \dots, (x_m,1)$. Thus, $x = \sum_{i=1}^{m} \alpha_i x_i$ and $1 = \sum_{i=1}^{m} \alpha_i$. **Q.E.D.**

Note that the proof of part (b) of Caratheodory's Theorem shows that if $m \geq 2$, the m vectors $x_1, \dots, x_m \in X$ used to represent a vector in $\text{conv}(X)$ may be chosen so that $x_2 - x_1, \dots, x_m - x_1$ are linearly independent [if $x_2 - x_1, \dots, x_m - x_1$ were linearly dependent, there exist $\lambda_2, \dots, \lambda_m$, not all 0, with $\sum_{i=2}^{m} \lambda_i(x_i - x_1) = 0$ so that by defining $\lambda_1 = -(\lambda_2 + \cdots + \lambda_m)$,

$$\sum_{i=1}^{m} \lambda_i(x_i,1) = 0,$$

contradicting the linear independence of $(x_1,1), \dots, (x_m,1)$].

Caratheodory's Theorem can be used to prove several other important results. An example is the following proposition.

Proposition 1.2.2: The convex hull of a compact set is compact.

Proof: Let X be a compact subset of $\Re^n$. To show that $\text{conv}(X)$ is compact, we take a sequence in $\text{conv}(X)$ and show that it has a convergent subsequence whose limit is in $\text{conv}(X)$. Indeed, by Caratheodory's Theorem, a sequence in $\text{conv}(X)$ can be expressed as $\left\{\sum_{i=1}^{n+1} \alpha_i^k x_i^k\right\}$, where for all k and i, $\alpha_i^k \geq 0$, $x_i^k \in X$, and $\sum_{i=1}^{n+1} \alpha_i^k = 1$. Since the sequence

$$\left\{(\alpha_1^k, \dots, \alpha_{n+1}^k, x_1^k, \dots, x_{n+1}^k)\right\}$$

is bounded, it has a limit point $\{(\alpha_1, \ldots, \alpha_{n+1}, x_1, \ldots, x_{n+1})\}$, which must satisfy $\sum_{i=1}^{n+1} \alpha_i = 1$, and $\alpha_i \geq 0$, $x_i \in X$ for all i. Thus, the vector $\sum_{i=1}^{n+1} \alpha_i x_i$, which belongs to $\text{conv}(X)$, is a limit point of the sequence $\left\{\sum_{i=1}^{n+1} \alpha_i^k x_i^k\right\}$, showing that $\text{conv}(X)$ is compact. **Q.E.D.**

Note that the convex hull of an unbounded closed set need not be closed. As an example, for the closed subset of $\Re^2$

$$X = \{(0,0)\} \cup \{(x_1, x_2) \mid x_1 x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\},$$

the convex hull is

$$\text{conv}(X) = \{(0,0)\} \cup \{(x_1, x_2) \mid x_1 > 0,\ x_2 > 0\},$$

which is not closed.

We finally note that just as one can convexify nonconvex sets through the convex hull operation, one can also convexify a nonconvex function by convexification of its epigraph. In fact, this can be done in a way that the optimality of the minima of the function is maintained (see Section 1.3.3).

1.3 RELATIVE INTERIOR AND CLOSURE

We now consider some generic topological properties of convex sets and functions. Let C be a nonempty convex subset of $\Re^n$. The closure of C, denoted $\text{cl}(C)$, is also a nonempty convex set [Prop. 1.1.1(d)]. The interior of C is also convex, but it may be empty. It turns out, however, that convexity implies the existence of interior points relative to the affine hull of C. This is an important property, which we now formalize.

Let C be a nonempty convex set. We say that x is a *relative interior point* of C if $x \in C$ and there exists an open sphere S centered at x such that $S \cap \text{aff}(C) \subset C$, i.e., x is an interior point of C relative to the affine hull of C. The set of all relative interior points of C is called the *relative interior of* C, and is denoted by $\text{ri}(C)$. The set C is said to be *relatively open* if $\text{ri}(C) = C$. The vectors in $\text{cl}(C)$ that are not relative interior points are said to be *relative boundary points* of C, and their collection is called the *relative boundary* of C.

For an example, let C be a line segment connecting two distinct points in the plane. Then $\text{ri}(C)$ consists of all points of C except for the two end points, and the relative boundary of C consists of the two end points. For another example, let C be an affine set. Then $\text{ri}(C) = C$ and the relative boundary of C is empty.

The most fundamental fact about relative interiors is given in the following proposition.

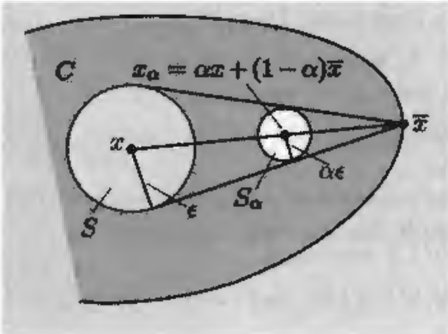


Figure 1.3.1. Proof of the Line Segment Principle for the case where $\overline{x} \in C$. Since $x \in \mathrm{ri}(C)$, there exists an open sphere $S = \{z \mid \|z - x\| < \epsilon\}$ such that $S \cap \mathrm{aff}(C) \subset C$. For all $\alpha \in (0, 1]$, let $x_\alpha = \alpha x + (1-\alpha)\overline{x}$ and let $S_\alpha = \{z \mid \|z - x_\alpha\| < \alpha\epsilon\}$. It can be seen that each point of $S_\alpha \cap \mathrm{aff}(C)$ is a convex combination of $\overline{x}$ and some point of $S \cap \mathrm{aff}(C)$. Therefore, by the convexity of C, $S_\alpha \cap \mathrm{aff}(C) \subset C$, implying that $x_\alpha \in \mathrm{ri}(C)$.

Proposition 1.3.1: (Line Segment Principle) Let C be a nonempty convex set. If $x \in \mathrm{ri}(C)$ and $\overline{x} \in \mathrm{cl}(C)$, then all points on the line segment connecting x and $\overline{x}$, except possibly $\overline{x}$, belong to $\mathrm{ri}(C)$.

Proof: For the case where $\overline{x} \in C$, the proof is given in Fig. 1.3.1. Consider the case where $\overline{x} \notin C$. To show that for any $\alpha \in (0, 1]$ we have $x_\alpha = \alpha x + (1-\alpha)\overline{x} \in \mathrm{ri}(C)$, consider a sequence $\{x_k\} \subset C$ that converges to $\overline{x}$, and let $x_{k,\alpha} = \alpha x + (1-\alpha)x_k$. Then as in Fig. 1.3.1, we see that $\{z \mid \|z - x_{k,\alpha}\| < \alpha\epsilon\} \cap \mathrm{aff}(C) \subset C$ for all k, where ϵ is such that the open sphere $S = \{z \mid \|z - x\| < \epsilon\}$ satisfies $S \cap \mathrm{aff}(C) \subset C$. Since $x_{k,\alpha} \to x_\alpha$, for large enough k, we have

$$\{z \mid \|z - x_\alpha\| < \alpha\epsilon/2\} \subset \{z \mid \|z - x_{k,\alpha}\| < \alpha\epsilon\}.$$

It follows that $\{z \mid \|z - x_\alpha\| < \alpha\epsilon/2\} \cap \mathrm{aff}(C) \subset C$, which shows that $x_\alpha \in \mathrm{ri}(C)$. **Q.E.D.**

A major consequence of the Line Segment Principle is given in the following proposition.

Proposition 1.3.2: (Nonemptiness of Relative Interior) Let C be a nonempty convex set. Then:

(a) $\mathrm{ri}(C)$ is a nonempty convex set, and has the same affine hull as C.

(b) If m is the dimension of $\text{aff}(C)$ and $m > 0$, there exist vectors $x_0, x_1, \ldots, x_m \in \text{ri}(C)$ such that $x_1 - x_0, \ldots, x_m - x_0$ span the subspace parallel to $\text{aff}(C)$.

Proof: (a) Convexity of $\text{ri}(C)$ follows from the Line Segment Principle (Prop. 1.3.1). By using a translation argument if necessary, we assume without loss of generality that $0 \in C$. Then $\text{aff}(C)$ is a subspace whose dimension will be denoted by m. To show that $\text{ri}(C)$ is nonempty, we will use a basis for $\text{aff}(C)$ to construct a relatively open set.

If the dimension m is 0, then C and $\text{aff}(C)$ consist of a single point, which is a unique relative interior point. If $m > 0$, we can find m linearly independent vectors $z_1, \ldots, z_m$ in C that span $\text{aff}(C)$; otherwise there would exist $r < m$ linearly independent vectors in C whose span contains C, contradicting the fact that the dimension of $\text{aff}(C)$ is m. Thus $z_1, \ldots, z_m$ form a basis for $\text{aff}(C)$.

Consider the set

$$X = \left\{ x \;\middle|\; x = \sum_{i=1}^{m} \alpha_i z_i,\ \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}$$

(see Fig. 1.3.2), and note that $X \subset C$ since C is convex. We claim that this set is open relative to $\text{aff}(C)$, i.e., for every vector $\overline{x} \in X$, there exists an open ball B centered at $\overline{x}$ such that $\overline{x} \in B$ and $B \cap \text{aff}(C) \subset X$. To see this, fix $\overline{x} \in X$ and let x be another vector in $\text{aff}(C)$. We have $\overline{x} = Z\overline{\alpha}$ and $x = Z\alpha$, where Z is the $n \times m$ matrix whose columns are the vectors $z_1, \ldots, z_m$, and $\overline{\alpha}$ and α are suitable m-dimensional vectors, which are unique since $z_1, \ldots, z_m$ form a basis for $\text{aff}(C)$. Since Z has linearly independent columns, the matrix $Z'Z$ is symmetric and positive definite, so for some positive scalar γ, which is independent of x and $\overline{x}$, we have

$$\|x - \overline{x}\|^2 = (\alpha - \overline{\alpha})' Z' Z (\alpha - \overline{\alpha}) \geq \gamma \|\alpha - \overline{\alpha}\|^2. \tag{1.8}$$

Since $\overline{x} \in X$, the corresponding vector $\overline{\alpha}$ lies in the open set

$$A = \left\{ (\alpha_1, \ldots, \alpha_m) \;\middle|\; \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}.$$

From Eq. (1.8), we see that if x lies in a suitably small ball centered at $\overline{x}$, the corresponding vector α lies in A, implying that $x \in X$. Hence X contains the intersection of $\text{aff}(C)$ and an open ball centered at $\overline{x}$, so X is open relative to $\text{aff}(C)$. It follows that all points of X are relative interior points of C, so that $\text{ri}(C)$ is nonempty. Also, since by construction,

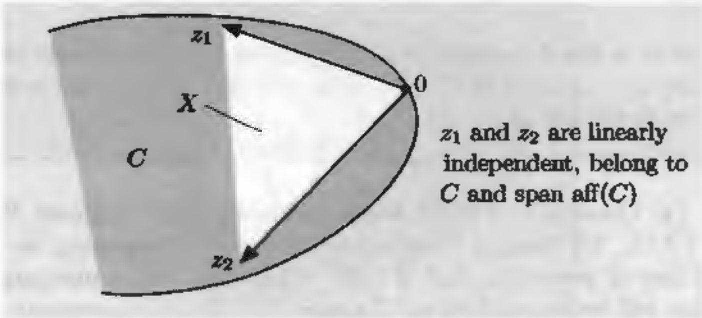


Figure 1.3.2. Construction of the relatively open set X in the proof of nonemptiness of the relative interior of a convex set C that contains the origin, assuming that $m > 0$. We choose m linearly independent vectors $z_1, \ldots, z_m \in C$, where m is the dimension of aff(C), and we let

$$X = \left\{ \sum_{i=1}^{m} \alpha_i z_i \;\middle|\; \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}.$$

Any point in X is shown to be a relative interior point of C.

$\text{aff}(X) = \text{aff}(C)$ and $X \subset \text{ri}(C)$, we see that $\text{ri}(C)$ and C have the same affine hull.

(b) Let x_0 be a relative interior point of C [there exists such a point by part (a)]. Translate C to $C - x_0$ (so that x_0 is translated to the origin), and consider vectors $z_1, \ldots, z_m \in C - x_0$ that span $\text{aff}(C - x_0)$, as in the proof of part (a). Let $\alpha \in (0, 1)$. Since $0 \in \text{ri}(C - x_0)$, by the Line Segment Principle (Prop. 1.3.1), we have $\alpha z_i \in \text{ri}(C - x_0)$ for all $i = 1, \ldots, m$. It follows that the vectors

$$x_i = x_0 + \alpha z_i, \qquad i = 1, \ldots, m,$$

are such that $x_1 - x_0, \ldots, x_m - x_0$ belong to $\text{ri}(C)$ and span $\text{aff}(C)$. **Q.E.D.**

Here is another useful consequence of the Line Segment Principle.

Proposition 1.3.3: (Prolongation Lemma) Let C be a nonempty convex set. A vector x is a relative interior point of C if and only if every line segment in C having x as one endpoint can be prolonged beyond x without leaving C [i.e., for every $\overline{x} \in C$, there exists a $\gamma > 0$ such that $x + \gamma(x - \overline{x}) \in C$].

Proof: If $x \in \text{ri}(C)$, the given condition clearly holds, using the definition of relative interior point. Conversely, let x satisfy the given condition, and let $\overline{x}$ be a point in $\mathrm{ri}(C)$ (by Prop. 1.3.2, there exists such a point). If $x = \overline{x}$, we are done, so assume that $x \neq \overline{x}$. By the given condition, there is a $\gamma > 0$ such that $y = x + \gamma(x - \overline{x}) \in C$, so that x lies strictly within the line segment connecting $\overline{x}$ and y. Since $\overline{x} \in \mathrm{ri}(C)$ and $y \in C$, by the Line Segment Principle (Prop. 1.3.1), it follows that $x \in \mathrm{ri}(C)$. **Q.E.D.**

We will see in the following chapters that the notion of relative interior is pervasive in convex optimization and duality theory. As an example, we provide an important characterization of the set of optimal solutions in the case where the cost function is concave.

Proposition 1.3.4: Let X be a nonempty convex subset of $\Re^n$, let $f : X \mapsto \Re$ be a concave function, and let X^* be the set of vectors where f attains a minimum over X, i.e.,

$$X^* = \left\{ x^* \in X \;\middle|\; f(x^*) = \inf_{x \in X} f(x) \right\}.$$

If X^* contains a relative interior point of X, then f must be constant over X, i.e., $X^* = X$.

Proof: Let x^* belong to $X^* \cap \mathrm{ri}(X)$, and let x be any vector in X. By the Prolongation Lemma (Prop. 1.3.3), there exists a $\gamma > 0$ such that

$$\hat{x} = x^* + \gamma(x^* - x)$$

belongs to X, implying that

$$x^* = \frac{1}{\gamma + 1}\hat{x} + \frac{\gamma}{\gamma + 1}x$$

(see Fig. 1.3.3). By the concavity of f, we have

$$f(x^*) \geq \frac{1}{\gamma + 1} f(\hat{x}) + \frac{\gamma}{\gamma + 1} f(x),$$

and using $f(\hat{x}) \geq f(x^*)$ and $f(x) \geq f(x^*)$, this shows that $f(x) = f(x^*)$. **Q.E.D.**

One consequence of the preceding proposition is that a linear cost function $f(x) = c'x$, with $c \neq 0$, cannot attain a minimum at some interior point of a convex constraint set, since such a function cannot be constant over an open sphere. This will be further discussed in Chapter 2, after we introduce the notion of an extreme point.

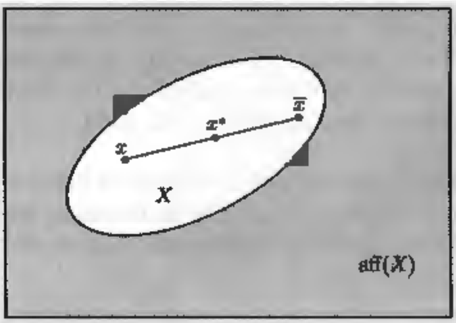


Figure 1.3.3. The idea of the proof of Prop. 1.3.4. If $x^* \in \mathrm{ri}(X)$ minimizes f over X and f is not constant over X, then there exists $x \in X$ such that $f(x) > f(x^*)$. By the Prolongation Lemma (Prop. 1.3.3), there exists $\overline{x} \in X$ such that x^* lies strictly between x and $\overline{x}$. Since f is concave and $f(x) > f(x^*)$, we must have $f(\overline{x}) < f(x^*)$ - a contradiction of the optimality of x^*.

1.3.1 Calculus of Relative Interiors and Closures

To deal with set operations such as intersection, vector sum, and linear transformation in convex analysis, we need tools for calculating the corresponding relative interiors and closures. These tools are provided in the next five propositions. Here is an informal summary of their content:

(a) Two convex sets have the same closure if and only if they have the same relative interior.

(b) Relative interior and closure commute with Cartesian product and inverse image under a linear transformation.

(c) Relative interior commutes with image under a linear transformation and vector sum, but closure does not.

(d) Neither closure nor relative interior commute with set intersection, unless the relative interiors of the sets involved have a point in common.

Proposition 1.3.5: Let C be a nonempty convex set.

(a) $\mathrm{cl}(C) = \mathrm{cl}\big(\mathrm{ri}(C)\big)$.

(b) $\mathrm{ri}(C) = \mathrm{ri}\big(\mathrm{cl}(C)\big)$.

(c) Let $\overline{C}$ be another nonempty convex set. Then the following three conditions are equivalent:

(i) C and $\overline{C}$ have the same relative interior.

(ii) C and $\overline{C}$ have the same closure.

(iii) $\mathrm{ri}(C) \subset \overline{C} \subset \mathrm{cl}(C)$.

Proof: (a) Since $\mathrm{ri}(C) \subset C$, we have $\mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}(C)$. Conversely, let

$\overline{x} \in \mathrm{cl}(C)$. We will show that $\overline{x} \in \mathrm{cl}\big(\mathrm{ri}(C)\big)$. Let x be any point in $\mathrm{ri}(C)$ [there exists such a point by Prop. 1.3.2(a)], and assume that $\overline{x} \neq x$ (otherwise we are done). By the Line Segment Principle (Prop. 1.3.1), we have $\alpha x + (1-\alpha)\overline{x} \in \mathrm{ri}(C)$ for all $\alpha \in (0,1]$. Thus, $\overline{x}$ is the limit of the sequence

$$\big\{(1/k)x + (1-1/k)\overline{x} \mid k \geq 1\big\}$$

that lies in $\mathrm{ri}(C)$, so $\overline{x} \in \mathrm{cl}\big(\mathrm{ri}(C)\big)$.

(b) The inclusion $\mathrm{ri}(C) \subset \mathrm{ri}\big(\mathrm{cl}(C)\big)$ follows from the definition of a relative interior point and the fact $\mathrm{aff}(C) = \mathrm{aff}\big(\mathrm{cl}(C)\big)$ (the proof of this is left for the reader). To prove the reverse inclusion, let $z \in \mathrm{ri}\big(\mathrm{cl}(C)\big)$. We will show that $z \in \mathrm{ri}(C)$. By Prop. 1.3.2(a), there exists an $x \in \mathrm{ri}(C)$. We may assume that $x \neq z$ (otherwise we are done). We use the Prolongation Lemma [Prop. 1.3.3, applied within the set $\mathrm{cl}(C)$] to choose $\gamma > 0$, with γ sufficiently close to 0 so that the vector $y = z + \gamma(z - x)$ belongs to $\mathrm{cl}(C)$. Then we have $z = (1-\alpha)x + \alpha y$ where $\alpha = 1/(\gamma+1) \in (0,1)$, so by the Line Segment Principle (Prop. 1.3.1, applied within the set C), we obtain $z \in \mathrm{ri}(C)$.

(c) If $\mathrm{ri}(C) = \mathrm{ri}(\overline{C})$, part (a) implies that $\mathrm{cl}(C) = \mathrm{cl}(\overline{C})$. Similarly, if $\mathrm{cl}(C) = \mathrm{cl}(\overline{C})$, part (b) implies that $\mathrm{ri}(C) = \mathrm{ri}(\overline{C})$. Thus, (i) and (ii) are equivalent. Also, (i), (ii), and the relation $\mathrm{ri}(\overline{C}) \subset \overline{C} \subset \mathrm{cl}(\overline{C})$ imply condition (iii). Finally, let condition (iii) hold. Then by taking closures, we have $\mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}(\overline{C}) \subset \mathrm{cl}(C)$, and by using part (a), we obtain $\mathrm{cl}(C) \subset \mathrm{cl}(\overline{C}) \subset \mathrm{cl}(C)$. Hence $\mathrm{cl}(\overline{C}) = \mathrm{cl}(C)$, i.e., (ii) holds. **Q.E.D.**

We now consider the image of a convex set C under a linear transformation A. Geometric intuition suggests that $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$, since spheres within C are mapped onto ellipsoids within the image $A \cdot C$ (relative to the corresponding affine hulls). This is shown in part (a) of the following proposition. However, the image of a closed convex set under a linear transformation is not closed [see part (b) of the following proposition], and this is a major source of analytical difficulty in convex optimization.

Proposition 1.3.6: Let C be a nonempty convex subset of $\Re^n$ and let A be an $m \times n$ matrix.

(a) We have $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$.

(b) We have $A \cdot \mathrm{cl}(C) \subset \mathrm{cl}(A \cdot C)$. Furthermore, if C is bounded, then $A \cdot \mathrm{cl}(C) = \mathrm{cl}(A \cdot C)$.

Proof: (a) For any set X, we have $A \cdot \mathrm{cl}(X) \subset \mathrm{cl}(A \cdot X)$, since if a sequence $\{x_k\} \subset X$ converges to some $x \in \mathrm{cl}(X)$ then the sequence $\{Ax_k\}$, which belongs to $A \cdot X$, converges to Ax, implying that $Ax \in \mathrm{cl}(A \cdot X)$. We use this fact and Prop. 1.3.5(a) to write

$$A\cdot \mathrm{ri}(C)\subset A\cdot C\subset A\cdot \mathrm{cl}(C)=A\cdot \mathrm{cl}\big(\mathrm{ri}(C)\big)\subset \mathrm{cl}\big(A\cdot \mathrm{ri}(C)\big).$$

Thus the convex set $A\cdot C$ lies between the convex set $A\cdot \mathrm{ri}(C)$ and the closure of that set, implying that the relative interiors of the sets $A\cdot C$ and $A\cdot \mathrm{ri}(C)$ are equal [Prop. 1.3.5(c)]. Hence $\mathrm{ri}(A\cdot C)\subset A\cdot \mathrm{ri}(C)$.

To show the reverse inclusion, we take any $z\in A\cdot \mathrm{ri}(C)$ and we show that $z\in \mathrm{ri}(A\cdot C)$. Let x be any vector in $A\cdot C$, and let $\overline{z}\in \mathrm{ri}(C)$ and $\overline{x}\in C$ be such that $A\overline{z}=z$ and $A\overline{x}=x$. By the Prolongation Lemma (Prop. 1.3.3), there exists a $\gamma>0$ such that the vector $\overline{y}=\overline{z}+\gamma(\overline{z}-\overline{x})$ belongs to C. Thus we have $A\overline{y}\in A\cdot C$ and $A\overline{y}=z+\gamma(z-x)$, so by the Prolongation Lemma, it follows that $z\in \mathrm{ri}(A\cdot C)$.

(b) By the argument given in part (a), we have $A\cdot \mathrm{cl}(C)\subset \mathrm{cl}(A\cdot C)$. To show the converse, assuming that C is bounded, choose any $z\in \mathrm{cl}(A\cdot C)$. Then, there exists a sequence $\{x_k\}\subset C$ such that $Ax_k\to z$. Since C is bounded, $\{x_k\}$ has a subsequence that converges to some $x\in \mathrm{cl}(C)$, and we must have $Ax=z$. It follows that $z\in A\cdot \mathrm{cl}(C)$. **Q.E.D.**

Note that if C is closed and convex but unbounded, the set $A\cdot C$ need not be closed [cf. part (b) of the preceding proposition]. For example, projection on the horizontal axis of the closed convex set

$$\big\{(x_1,x_2)\mid x_1>0,\ x_2>0,\ x_1x_2\geq 1\big\},$$

shown in Fig. 1.3.4, is the (nonclosed) halfline $\big\{(x_1,x_2)\mid x_1>0,\ x_2=0\big\}$.

Generally, the vector sum of sets $C_1,\ldots,C_m$ can be viewed as the result of the linear transformation $(x_1,\ldots,x_m)\mapsto x_1+\cdots+x_m$ on the Cartesian product $C_1\times\cdots\times C_m$. Thus, results involving linear transformations, such as the one of the preceding proposition, yield corresponding results for vector sums, such as the one of the following proposition.

Proposition 1.3.7: Let C_1 and C_2 be nonempty convex sets. We have

$$\mathrm{ri}(C_1+C_2)=\mathrm{ri}(C_1)+\mathrm{ri}(C_2),\qquad \mathrm{cl}(C_1)+\mathrm{cl}(C_2)\subset \mathrm{cl}(C_1+C_2).$$

Furthermore, if at least one of the sets C_1 and C_2 is bounded, then

$$\mathrm{cl}(C_1)+\mathrm{cl}(C_2)=\mathrm{cl}(C_1+C_2).$$

Proof: Consider the linear transformation $A:\Re^{2n}\mapsto\Re^n$ given by

$$A(x_1,x_2)=x_1+x_2,\qquad x_1,x_2\in\Re^n.$$

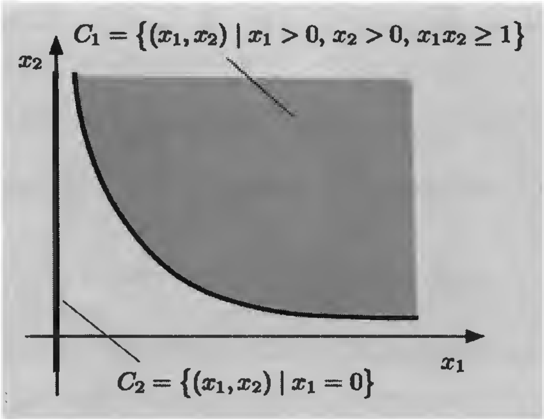


Figure 1.3.4. An example where the sum of two closed convex sets C_1 and C_2 is not closed. Here

$$C_1 = \{(x_1, x_2) \mid x_1 > 0,\, x_2 > 0,\, x_1x_2 \geq 1\}, \qquad C_2 = \{(x_1, x_2) \mid x_1 = 0\},$$

and $C_1 + C_2$ is the open halfspace $\{(x_1, x_2) \mid x_1 > 0\}$. Also the projection of the set C_1 on the horizontal axis is not closed.

The relative interior of the Cartesian product $C_1 \times C_2$ (viewed as a subset of $\Re^{2n}$) is $\mathrm{ri}(C_1) \times \mathrm{ri}(C_2)$ (the easy proof of this is left for the reader). Since

$$A(C_1 \times C_2) = C_1 + C_2,$$

from Prop. 1.3.6(a), we obtain $\mathrm{ri}(C_1 + C_2) = \mathrm{ri}(C_1) + \mathrm{ri}(C_2)$.

Similarly, the closure of $C_1 \times C_2$ is $\mathrm{cl}(C_1) \times \mathrm{cl}(C_2)$. From Prop. 1.3.6(b), we have

$$A \cdot \mathrm{cl}(C_1 \times C_2) \subset \mathrm{cl}\big(A \cdot (C_1 \times C_2)\big),$$

or equivalently, $\mathrm{cl}(C_1) + \mathrm{cl}(C_2) \subset \mathrm{cl}(C_1 + C_2)$.

To show the reverse inclusion, assuming that C_1 is bounded, let $x \in \mathrm{cl}(C_1 + C_2)$. Then there exist sequences $\{x_k^1\} \subset C_1$ and $\{x_k^2\} \subset C_2$ such that $x_k^1 + x_k^2 \to x$. Since $\{x_k^1\}$ is bounded, it follows that $\{x_k^2\}$ is also bounded. Thus, $\{(x_k^1, x_k^2)\}$ has a subsequence that converges to a vector (x^1, x^2), and we have $x^1 + x^2 = x$. Since $x^1 \in \mathrm{cl}(C_1)$ and $x^2 \in \mathrm{cl}(C_2)$, it follows that $x \in \mathrm{cl}(C_1) + \mathrm{cl}(C_2)$. Hence $\mathrm{cl}(C_1 + C_2) \subset \mathrm{cl}(C_1) + \mathrm{cl}(C_2)$. **Q.E.D.**

The requirement that at least one of the sets C_1 and C_2 be bounded is essential in the preceding proposition. This is illustrated by the example of Fig. 1.3.4.

Proposition 1.3.8: Let C_1 and C_2 be nonempty convex sets. We have

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \subset \mathrm{ri}(C_1 \cap C_2), \qquad \mathrm{cl}(C_1 \cap C_2) \subset \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2).$$

Furthermore, if the sets $\mathrm{ri}(C_1)$ and $\mathrm{ri}(C_2)$ have a nonempty intersection, then

$$\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2), \qquad \mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2).$$

Proof: Take any $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and any $y \in C_1 \cap C_2$. By the Prolongation Lemma (Prop. 1.3.3), it can be seen that the line segment connecting x and y can be prolonged beyond x by a small amount without leaving C_1 and also by another small amount without leaving C_2. Thus, by using the lemma again, it follows that $x \in \mathrm{ri}(C_1 \cap C_2)$, so that

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \subset \mathrm{ri}(C_1 \cap C_2).$$

Also, since the set $C_1 \cap C_2$ is contained in the closed set $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$, we have

$$\mathrm{cl}(C_1 \cap C_2) \subset \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2).$$

To show the reverse inclusions assuming that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, let $y \in \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$, and let $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$. By the Line Segment Principle (Prop. 1.3.1), $\alpha x + (1 - \alpha) y \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ for all $\alpha \in (0, 1]$ (see Fig. 1.3.5). Hence, y is the limit of a sequence $\alpha_k x + (1 - \alpha_k) y \subset \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ with $\alpha_k \to 0$, implying that $y \in \mathrm{cl}\big(\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)\big)$. Thus,

$$\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2) \subset \mathrm{cl}\big(\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)\big) \subset \mathrm{cl}(C_1 \cap C_2).$$

We showed earlier that $\mathrm{cl}(C_1 \cap C_2) \subset \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$, so equality holds throughout in the preceding relation, and therefore $\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Furthermore, the sets $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and $C_1 \cap C_2$ have the same closure. Therefore, by Prop. 1.3.5(c), they have the same relative interior, so that

$$\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}\big(\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)\big) \subset \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2).$$

We showed earlier the reverse inclusion, so $\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$. **Q.E.D.**

The requirement that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$ is essential in part (a) of the preceding proposition. As an example, consider the following subsets of the real line:

$$C_1 = \{x \mid x \geq 0\}, \qquad C_2 = \{x \mid x \leq 0\}.$$

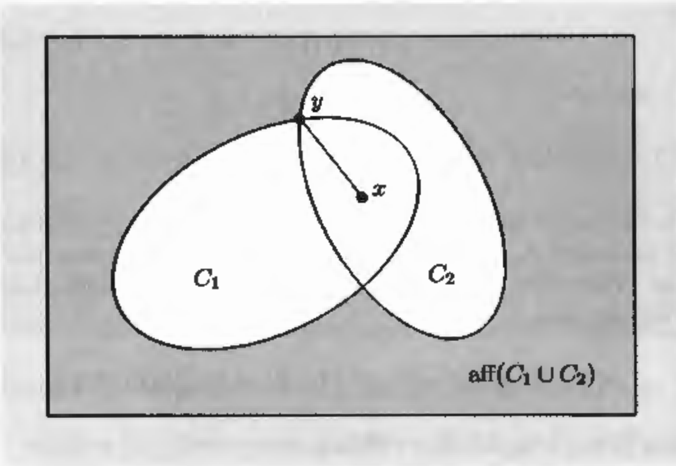


Figure 1.3.5. Construction used to show that

$$\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2) \subset \mathrm{cl}(C_1 \cap C_2),$$

assuming that there exists $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ (cf. Prop. 1.3.8). Any $y \in \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$ can be approached along the line segment of $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ connecting it with x, so it belongs to the closure of $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and hence also to $\mathrm{cl}(C_1 \cap C_2)$.

Then we have $\mathrm{ri}(C_1 \cap C_2) = \{0\} \neq \emptyset = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$. Also, consider the following subsets of the real line:

$$C_1 = \{x \mid x > 0\}, \qquad C_2 = \{x \mid x < 0\}.$$

Then we have $\mathrm{cl}(C_1 \cap C_2) = \emptyset \neq \{0\} = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$.

Proposition 1.3.9: Let C be a nonempty convex subset of $\Re^m$, and let A be an $m \times n$ matrix. If $A^{-1} \cdot \mathrm{ri}(C)$ is nonempty, then

$$\mathrm{ri}(A^{-1} \cdot C) = A^{-1} \cdot \mathrm{ri}(C), \qquad \mathrm{cl}(A^{-1} \cdot C) = A^{-1} \cdot \mathrm{cl}(C),$$

where A^{-1} denotes inverse image of the corresponding set under A.

Proof: Define the sets

$$D = \Re^n \times C, \qquad S = \big\{(x, Ax) \mid x \in \Re^n\big\},$$

and let T be the linear transformation that maps $(x, y) \in \Re^{n+m}$ into $x \in \Re^n$. We have

$$A^{-1} \cdot C = \{x \mid Ax \in C\} = T \cdot \big\{(x, Ax) \mid Ax \in C\big\} = T \cdot (D \cap S),$$

from which

$$\mathrm{ri}(A^{-1} \cdot C) = \mathrm{ri}\big(T \cdot (D \cap S)\big). \tag{1.9}$$

Similarly, we have

$$A^{-1} \cdot \mathrm{ri}(C) = \big\{x \mid Ax \in \mathrm{ri}(C)\big\} = T \cdot \big\{(x, Ax) \mid Ax \in \mathrm{ri}(C)\big\} = T \cdot \big(\mathrm{ri}(D) \cap S\big), \tag{1.10}$$

where the last equality holds because $\mathrm{ri}(D) = \Re^n \times \mathrm{ri}(C)$ (cf. Prop. 1.3.8). Since by assumption, $A^{-1} \cdot \mathrm{ri}(C)$ is nonempty, we see that $\mathrm{ri}(D) \cap S$ is nonempty. Therefore, using the fact $\mathrm{ri}(S) = S$, and Props. 1.3.6(a) and 1.3.8, it follows that

$$\mathrm{ri}\big(T \cdot (D \cap S)\big) = T \cdot \mathrm{ri}\big(D \cap S\big) = T \cdot \big(\mathrm{ri}(D) \cap S\big). \tag{1.11}$$

Combining Eqs. (1.9)-(1.11), we obtain

$$\mathrm{ri}(A^{-1} \cdot C) = A^{-1} \cdot \mathrm{ri}(C).$$

To show the second relation, note that

$$A^{-1} \cdot \mathrm{cl}(C) = \big\{x \mid Ax \in \mathrm{cl}(C)\big\} = T \cdot \big\{(x, Ax) \mid Ax \in \mathrm{cl}(C)\big\} = T \cdot \big(\mathrm{cl}(D) \cap S\big),$$

where the last equality holds because $\mathrm{cl}(D) = \Re^n \times \mathrm{cl}(C)$. Since $\mathrm{ri}(D) \cap S$ is nonempty and $\mathrm{ri}(S) = S$, it follows from Prop. 1.3.8 that

$$\mathrm{cl}(D) \cap S = \mathrm{cl}(D \cap S).$$

Using the last two relations and the continuity of T, we obtain

$$A^{-1} \cdot \mathrm{cl}(C) = T \cdot \mathrm{cl}(D \cap S) \subset \mathrm{cl}\big(T \cdot (D \cap S)\big),$$

which combined with Eq. (1.9) yields

$$A^{-1} \cdot \mathrm{cl}(C) \subset \mathrm{cl}(A^{-1} \cdot C).$$

To show the reverse inclusion, let $\overline{x}$ be a vector in $\mathrm{cl}(A^{-1} \cdot C)$. Then there exists a sequence $\{x_k\}$ converging to $\overline{x}$ such that $Ax_k \in C$ for all k. Since $\{x_k\}$ converges to $\overline{x}$, we see that $\{Ax_k\}$ converges to $A\overline{x}$, so that $A\overline{x} \in \mathrm{cl}(C)$, or equivalently, $\overline{x} \in A^{-1} \cdot \mathrm{cl}(C)$. **Q.E.D.**

We finally show a useful characterization of the relative interior of sets involving two variables. It generalizes the Cartesian product formula

$$\mathrm{ri}(C_1 \times C_2) = \mathrm{ri}(C_1) \times \mathrm{ri}(C_2)$$

for two convex sets $C_1 \subset \Re^n$ and $C_2 \in \Re^m$.

Proposition 1.3.10: Let C be a convex subset of $\Re^{n+m}$. For $x \in \Re^n$, denote

$$C_x = \{y \mid (x, y) \in C\},$$

and let

$$D = \{x \mid C_x \neq \emptyset\}.$$

Then

$$\mathrm{ri}(C) = \big\{(x, y) \mid x \in \mathrm{ri}(D),\ y \in \mathrm{ri}(C_x)\big\}.$$

Proof: Since D is the projection of C on the x-axis, from Prop. 1.3.6,

$$\mathrm{ri}(D) = \big\{x \mid \text{there exists } y \in \Re^m \text{ with } (x, y) \in \mathrm{ri}(C)\big\},$$

so that

$$\mathrm{ri}(C) = \cup_{x \in \mathrm{ri}(D)} \Big(M_x \cap \mathrm{ri}(C)\Big),$$

where $M_x = \big\{(x, y) \mid y \in \Re^m\big\}$. For every $x \in \mathrm{ri}(D)$, we have

$$M_x \cap \mathrm{ri}(C) = \mathrm{ri}(M_x \cap C) = \big\{(x, y) \mid y \in \mathrm{ri}(C_x)\big\},$$

where the first equality follows from Prop. 1.3.8. By combining the preceding two equations, we obtain the desired result. **Q.E.D.**

1.3.2 Continuity of Convex Functions

We now derive a basic continuity property of convex functions.

Proposition 1.3.11: If $f : \Re^n \mapsto \Re$ is convex, then it is continuous. More generally, if $f : \Re^n \mapsto (-\infty, \infty]$ is a proper convex function, then f, restricted to $\mathrm{dom}(f)$, is continuous over the relative interior of $\mathrm{dom}(f)$.

Proof: Restricting attention to the affine hull of $\mathrm{dom}(f)$ and using a transformation argument if necessary, we assume without loss of generality that the origin is an interior point of $\mathrm{dom}(f)$ and that the unit cube

$$X = \{x \mid \|x\|_\infty \leq 1\}$$

is contained in $\mathrm{dom}(f)$ (we use the norm $\|x\|_\infty = \max_{j \in \{1,\ldots,n\}} |x_j|$). It will suffice to show that f is continuous at 0, i.e., that for any sequence $\{x_k\} \subset \Re^n$ that converges to 0, we have $f(x_k) \to f(0)$.

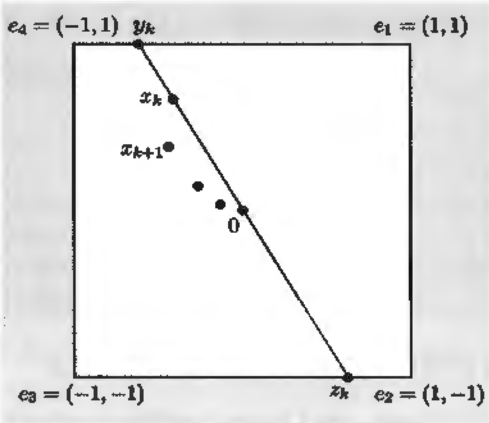


Figure 1.3.6. Construction for the proof of continuity of a real-valued convex function (cf. Prop. 1.3.11).

Let e_i, $i = 1, \ldots, 2^n$, be the corners of X, i.e., each e_i is a vector whose entries are either 1 or -1. It can be seen that any $x \in X$ can be expressed in the form $x = \sum_{i=1}^{2^n} \alpha_i e_i$, where each α_i is a nonnegative scalar and $\sum_{i=1}^{2^n} \alpha_i = 1$. Let $A = \max_i f(e_i)$. From Jensen's inequality [Eq. (1.7)], it follows that $f(x) \leq A$ for every $x \in X$.

For the purpose of proving continuity at 0, we can assume that $x_k \in X$ and $x_k \neq 0$ for all k. Consider the sequences $\{y_k\}$ and $\{z_k\}$ given by

$$y_k = \frac{x_k}{\|x_k\|_\infty}, \qquad z_k = -\frac{x_k}{\|x_k\|_\infty};$$

(cf. Fig. 1.3.6). Using the definition of a convex function for the line segment that connects y_k, x_k, and 0, we have

$$f(x_k) \leq \big(1 - \|x_k\|_\infty\big) f(0) + \|x_k\|_\infty f(y_k).$$

Since $\|x_k\|_\infty \to 0$ and $f(y_k) \leq A$ for all k, by taking the limit as $k \to \infty$, we obtain

$$\limsup_{k \to \infty} f(x_k) \leq f(0).$$

Using the definition of a convex function for the line segment that connects x_k, 0, and z_k, we have

$$f(0) \leq \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) + \frac{1}{\|x_k\|_\infty + 1} f(x_k)$$

and letting $k \to \infty$, we obtain

$$f(0) \leq \liminf_{k \to \infty} f(x_k).$$

Thus, $\lim_{k \to \infty} f(x_k) = f(0)$ and f is continuous at zero. **Q.E.D.**

Among other things, the proposition implies that a real-valued convex function is continuous and hence closed (cf. Prop. 1.1.2). We also have the following stronger result for the case of a function of one variable.

Proposition 1.3.12: If C is a closed interval of the real line, and $f : C \mapsto \Re$ is closed and convex, then f is continuous over C.

Proof: By the preceding proposition, f is continuous in the relative interior of C. To show continuity at a boundary point $\overline{x}$, let $\{x_k\} \subset C$ be a sequence that converges to $\overline{x}$, and write

$$x_k = \alpha_k x_0 + (1 - \alpha_k)\overline{x}, \qquad \forall\, k,$$

where $\{\alpha_k\}$ is a nonnegative sequence with $\alpha_k \to 0$. By convexity of f, we have for all k such that $\alpha_k \leq 1$,

$$f(x_k) \leq \alpha_k f(x_0) + (1 - \alpha_k) f(\overline{x}),$$

and by taking the limit as $k \to \infty$, we obtain

$$\limsup_{k \to \infty} f(x_k) \leq f(\overline{x}).$$

Consider the function $\tilde{f} : \Re \mapsto (-\infty, \infty]$, which takes the value $f(x)$ for $x \in C$ and ∞ for $x \notin C$, and note that it is closed (since it has the same epigraph as f), and hence lower semicontinuous (cf. Prop. 1.1.2). It follows that $f(\overline{x}) \leq \liminf_{k \to \infty} f(x_k)$, thus implying that $f(x_k) \to f(\overline{x})$, and that f is continuous at $\overline{x}$. **Q.E.D.**

1.3.3 Closures of Functions

In this section, we study operations that can transform a given function to a closed and/or convex function, while preserving much of its essential character. These operations play an important role in optimization and other contexts.

A nonempty subset E of $\Re^{n+1}$ is the epigraph of some function if for every $(\overline{x}, \overline{w}) \in E$, the set $\big\{w \mid (\overline{x}, w) \in E\big\}$ is either the real line or else it is a halfline that is bounded below and contains its (lower) endpoint. Then E is the epigraph of the function $f : D \mapsto [-\infty, \infty]$, where

$$D = \big\{x \mid \text{there exists } w \in \Re \text{ with } (x, w) \in E\big\},$$

and

$$f(x) = \inf\big\{w \mid (x, w) \in E\big\}, \qquad \forall\, x \in D$$

[the infimum is actually attained if $f(x)$ is finite]. Note that E is also the epigraph of other functions with different domain than f (but the same effective domain); for example, $\tilde{f} : \Re^n \mapsto [-\infty, \infty]$, where $\tilde{f}(x) = f(x)$ for $x \in D$ and $\tilde{f}(x) = \infty$ for $x \notin D$. If E is the empty set, it is the epigraph of the function that is identically equal to ∞.

The closure of the epigraph of a function $f : X \mapsto [-\infty, \infty]$ can be seen to be a legitimate epigraph of another function. This function, called the *closure of* f and denoted $\mathrm{cl}\, f : \Re^n \mapsto [-\infty, \infty]$, is given by†

$$(\mathrm{cl}\, f)(x) = \inf\big\{w \mid (x, w) \in \mathrm{cl}\big(\mathrm{epi}(f)\big)\big\}, \qquad x \in \Re^n.$$

When f is convex, the set $\mathrm{cl}\big(\mathrm{epi}(f)\big)$ is closed and convex [since the closure of a convex set is convex by Prop. 1.1.1(d)], implying that $\mathrm{cl}\, f$ is closed and convex since $\mathrm{epi}(\mathrm{cl}\, f) = \mathrm{cl}\big(\mathrm{epi}(f)\big)$ by definition.

The closure of the convex hull of the epigraph of f is the epigraph of some function, denoted $\check{\mathrm{cl}}\, f$ called the *convex closure of* f. It can be seen that $\check{\mathrm{cl}}\, f$ is the closure of the function $F : \Re^n \mapsto [-\infty, \infty]$ given by

$$F(x) = \inf\big\{w \mid (x, w) \in \mathrm{conv}\big(\mathrm{epi}(f)\big)\big\}, \qquad x \in \Re^n. \tag{1.12}$$

It is easily shown that F is convex, but it need not be closed and its domain may be strictly contained in $\mathrm{dom}(\check{\mathrm{cl}}\, f)$ (it can be seen though that the closures of the domains of F and $\check{\mathrm{cl}}\, f$ coincide).

From the point of view of optimization, an important property is that the minimal values of f, $\mathrm{cl}\, f$, F, and $\check{\mathrm{cl}}\, f$ coincide, as stated in the following proposition:

Proposition 1.3.13: Let $f : X \mapsto [-\infty, \infty]$ be a function. Then

$$\inf_{x \in X} f(x) = \inf_{x \in X} (\mathrm{cl}\, f)(x) = \inf_{x \in \Re^n} (\mathrm{cl}\, f)(x) = \inf_{x \in \Re^n} F(x) = \inf_{x \in \Re^n} (\check{\mathrm{cl}}\, f)(x),$$

where F is given by Eq. (1.12). Furthermore, any vector that attains the infimum of f over X also attains the infimum of $\mathrm{cl}\, f$, F, and $\check{\mathrm{cl}}\, f$.

† A note regarding the definition of closure: in Rockafellar [Roc70], p. 52, what we call "closure" of f is called the "lower semi-continuous hull" of f, and "closure" of f is defined somewhat differently (but denoted $\mathrm{cl}\, f$). Our definition of "closure" of f works better for our purposes, and results in a more streamlined analysis. It coincides with the one of [Roc70] when f is proper convex. For this reason the results of this section correspond to results in [Roc70] only in the case where the functions involved are proper convex. In Rockafellar and Wets [RoW98], p. 14, our "closure" of f is called the "lsc regularization" or "lower closure" of f, and is denoted by $\mathrm{cl}\, f$. Thus our notation is consistent with the one of [RoW98].

Proof: If epi(f) is empty, i.e., $f(x) = \infty$ for all x, the results trivially hold. Assume that epi(f) is nonempty, and let $f^* = \inf_{x\in\Re^n}(\mathrm{cl}\, f)(x)$. For any sequence $\{(\overline{x}_k, \overline{w}_k)\} \subset \mathrm{cl}\big(\mathrm{epi}(f)\big)$ with $\overline{w}_k \to f^*$, we can construct a sequence $\{(x_k, w_k)\} \subset \mathrm{epi}(f)$ such that $|w_k - \overline{w}_k| \to 0$, so that $w_k \to f^*$. Since $x_k \in X$, $f(x_k) \le w_k$, we have

$$\limsup_{k\to\infty} f(x_k) \le f^* \le (\mathrm{cl}\, f)(x) \le f(x), \qquad \forall\, x \in X,$$

so that

$$\inf_{x\in X} f(x) = \inf_{x\in X} (\mathrm{cl}\, f)(x) = \inf_{x\in\Re^n} (\mathrm{cl}\, f)(x).$$

Choose $\{(x_k, w_k)\} \subset \mathrm{conv}\big(\mathrm{epi}(f)\big)$ with $w_k \to \inf_{x\in\Re^n} F(x)$. Each (x_k, w_k) is a convex combination of vectors from epi(f), so that $w_k \ge \inf_{x\in X} f(x)$. Hence $\inf_{x\in\Re^n} F(x) \ge \inf_{x\in X} f(x)$. On the other hand, we have $F(x) \le f(x)$ for all $x \in X$, so it follows that $\inf_{x\in\Re^n} F(x) = \inf_{x\in X} f(x)$. Since $\check{\mathrm{cl}}\, f$ is the closure of F, it also follows (based on what was shown in the preceding paragraph) that $\inf_{x\in\Re^n}(\check{\mathrm{cl}}\, f)(x) = \inf_{x\in\Re^n} F(x)$.

We have $f(x) \ge (\mathrm{cl}\, f)(x)$ for all x, so if x^* attains the infimum of f,

$$\inf_{x\in\Re^n} (\mathrm{cl}\, f)(x) = \inf_{x\in X} f(x) = f(x^*) \ge (\mathrm{cl}\, f)(x^*),$$

showing that x^* attains the infimum of $\mathrm{cl}\, f$. Similarly, x^* attains the infimum of F and $\check{\mathrm{cl}}\, f$. **Q.E.D.**

The following is a characterization of closures and convex closures.

Proposition 1.3.14: Let $f : \Re^n \mapsto [-\infty, \infty]$ be a function.

(a) $\mathrm{cl}\, f$ is the greatest closed function majorized by f, i.e., if $g : \Re^n \mapsto [-\infty, \infty]$ is closed and satisfies $g(x) \le f(x)$ for all $x \in \Re^n$, then $g(x) \le (\mathrm{cl}\, f)(x)$ for all $x \in \Re^n$.

(b) $\check{\mathrm{cl}}\, f$ is the greatest closed and convex function majorized by f, i.e., if $g : \Re^n \mapsto [-\infty, \infty]$ is closed and convex, and satisfies $g(x) \le f(x)$ for all $x \in \Re^n$, then $g(x) \le (\check{\mathrm{cl}}\, f)(x)$ for all $x \in \Re^n$.

Proof: (a) Let $g : \Re^n \mapsto [-\infty, \infty]$ be closed and such that $g(x) \le f(x)$ for all x. Then epi$(f) \subset$ epi(g). Since $\mathrm{epi}(\mathrm{cl}\, f) = \mathrm{cl}\big(\mathrm{epi}(f)\big)$, we have that $\mathrm{epi}(\mathrm{cl}\, f)$ is the intersection of all closed sets $E \subset \Re^{n+1}$ with epi$(f) \subset E$, so that $\mathrm{epi}(\mathrm{cl}\, f) \subset \mathrm{epi}(g)$. It follows that $g(x) \le (\mathrm{cl}\, f)(x)$ for all $x \in \Re^n$.

(b) Similar to the proof of part (a). **Q.E.D.**

Working with the closure of a convex function is often useful because in some sense the closure "differs minimally" from the original. In particular, we can show that a convex function coincides with its closure on the relative interior of its domain. This and other properties of closures are derived in the following proposition.

Proposition 1.3.15: Let $f : \Re^n \mapsto [-\infty, \infty]$ be a convex function. Then:

(a) We have

$$\mathrm{cl}\big(\mathrm{dom}(f)\big) = \mathrm{cl}\big(\mathrm{dom}(\mathrm{cl}\, f)\big), \qquad \mathrm{ri}\big(\mathrm{dom}(f)\big) = \mathrm{ri}\big(\mathrm{dom}(\mathrm{cl}\, f)\big),$$

$$(\mathrm{cl}\, f)(x) = f(x), \qquad \forall\ x \in \mathrm{ri}\big(\mathrm{dom}(f)\big).$$

Furthermore, $\mathrm{cl}\, f$ is proper if and only if f is proper.

(b) If $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$, we have

$$(\mathrm{cl}\, f)(y) = \lim_{\alpha \downarrow 0} f\big(y + \alpha(x - y)\big), \qquad \forall\ y \in \Re^n.$$

Proof: (a) From Prop. 1.3.10, we have

$$\mathrm{ri}\big(\mathrm{epi}(f)\big) = \big\{(x, w) \mid x \in \mathrm{ri}\big(\mathrm{dom}(f)\big),\ f(x) < w\big\}, \tag{1.13}$$

$$\mathrm{ri}\big(\mathrm{epi}(\mathrm{cl}\, f)\big) = \big\{(x, w) \mid x \in \mathrm{ri}\big(\mathrm{dom}(\mathrm{cl}\, f)\big),\ (\mathrm{cl}\, f)(x) < w\big\}. \tag{1.14}$$

Since $\mathrm{epi}(f)$ and $\mathrm{epi}(\mathrm{cl}\, f)$ have the same closure, they have the same relative interior [Prop. 1.3.5(c)], i.e., the sets of Eqs. (1.13) and (1.14) are equal. Hence $\mathrm{dom}(f)$ and $\mathrm{dom}(\mathrm{cl}\, f)$ have the same relative interior and therefore also the same closure. Thus, the equality of the sets (1.13) and (1.14) yields

$$\begin{aligned} &\big\{(x, w) \mid x \in \mathrm{ri}\big(\mathrm{dom}(f)\big), f(x) < w\big\} \\ &\qquad\qquad = \big\{(x, w) \mid x \in \mathrm{ri}\big(\mathrm{dom}(f)\big),\ (\mathrm{cl}\, f)(x) < w\big\}, \end{aligned}$$

from which it follows that $f(x) = (\mathrm{cl}\, f)(x)$ for all $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$.

If $\mathrm{cl}\, f$ is proper, clearly f is proper. Conversely, if $\mathrm{cl}\, f$ is improper, then $(\mathrm{cl}\, f)(x) = -\infty$ for all $x \in \mathrm{dom}(\mathrm{cl}\, f)$ (cf. the discussion at the end of Section 1.1.2). Hence $(\mathrm{cl}\, f)(x) = -\infty$ for all $x \in \mathrm{ri}\big(\mathrm{dom}(\mathrm{cl}\, f)\big) = \mathrm{ri}\big(\mathrm{dom}(f)\big)$. Using what was just proved, it follows that $f(x) = (\mathrm{cl}\, f)(x) = -\infty$ for all $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$, implying that f is improper.

(b) Assume first $y \notin \mathrm{dom}(\mathrm{cl}\, f)$, i.e., $(\mathrm{cl}\, f)(y) = \infty$. Then, by the lower semicontinuity of $\mathrm{cl}\, f$, we have $(\mathrm{cl}\, f)(y_k) \to \infty$ for all sequences $\{y_k\}$ with $y_k \to y$, from which $f(y_k) \to \infty$, since $(\mathrm{cl}\, f)(y_k) \le f(y_k)$. Hence $(\mathrm{cl}\, f)(y) = \lim_{\alpha \downarrow 0} f\big(y + \alpha(x - y)\big) = \infty$.

Assume next that $y \in \mathrm{dom}(\mathrm{cl}\, f)$, and consider the function $g : [0,1] \mapsto \Re$ given by

$$g(\alpha) = (\mathrm{cl}\, f)\big(y + \alpha(x - y)\big).$$

For $\alpha \in (0,1]$, by the Line Segment Principle (Prop. 1.3.1), we have

$$y + \alpha(x - y) \in \mathrm{ri}\big(\mathrm{dom}(\mathrm{cl}\, f)\big),$$

so by part (a), $y + \alpha(x - y) \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$, and

$$g(\alpha) = (\mathrm{cl}\, f)\big(y + \alpha(x - y)\big) = f\big(y + \alpha(x - y)\big). \tag{1.15}$$

If $(\mathrm{cl}\, f)(y) = -\infty$, then $\mathrm{cl}\, f$ is improper and $(\mathrm{cl}\, f)(z) = -\infty$ for all $z \in \mathrm{dom}(\mathrm{cl}\, f)$, since an improper closed convex function cannot take a finite value at any point (cf. the discussion at the end of Section 1.1.2). Hence

$$f\big(y + \alpha(x - y)\big) = -\infty, \qquad \forall\ \alpha \in (0,1],$$

and the desired equation follows. If $(\mathrm{cl}\, f)(y) > -\infty$, then $(\mathrm{cl}\, f)(y)$ is finite, so $\mathrm{cl}\, f$ is proper and by part (a), f is also proper. It follows that the function g is real-valued, convex, and closed, and hence also continuous over $[0,1]$ (Prop. 1.3.12). By taking the limit in Eq. (1.15),

$$(\mathrm{cl}\, f)(y) = g(0) = \lim_{\alpha \downarrow 0} g(\alpha) = \lim_{\alpha \downarrow 0} f\big(y + \alpha(x - y)\big).$$

Q.E.D.

Note a corollary of part (a) of the preceding proposition: an improper convex function f takes the value $-\infty$ at all $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$, since its closure does (cf. the discussion at the end of Section 1.1.2).

Calculus of Closure Operations

We now characterize the closure of functions obtained by linear composition and summation of convex functions.

Proposition 1.3.16: Let $f : \Re^m \mapsto [-\infty, \infty]$ be a convex function and A be an $m \times n$ matrix such that the range of A contains a point in $\mathrm{ri}\big(\mathrm{dom}(f)\big)$. The function F defined by

$$F(x) = f(Ax),$$

is convex and

$$(\mathrm{cl}\, F)(x) = (\mathrm{cl}\, f)(Ax), \qquad \forall\ x \in \Re^n.$$

Proof: Let z be a point in the range of A that belongs to $\mathrm{ri}(\mathrm{dom}(f))$, and let y be such that $Ay = z$. Then, since $\mathrm{dom}(F) = A^{-1}\mathrm{dom}(f)$ and by Prop. 1.3.9, $\mathrm{ri}(\mathrm{dom}(F)) = A^{-1}\mathrm{ri}(\mathrm{dom}(f))$, we see that $y \in \mathrm{ri}(\mathrm{dom}(F))$. By using Prop. 1.3.15(b), we have for every $x \in \Re^n$,

$$(\mathrm{cl}\, F)(x) = \lim_{\alpha \downarrow 0} F\big(x + \alpha(y - x)\big) = \lim_{\alpha \downarrow 0} f\big(Ax + \alpha(Ay - Ax)\big) = (\mathrm{cl}\, f)(Ax).$$

Q.E.D.

The following proposition is essentially a special case of the preceding one (cf. the discussion in Section 1.1.3).

Proposition 1.3.17: Let $f_i : \Re^n \mapsto [-\infty, \infty]$, $i = 1, \ldots, m$, be convex functions such that

$$\cap_{i=1}^m \mathrm{ri}\big(\mathrm{dom}(f_i)\big) \neq \emptyset. \tag{1.16}$$

The function F defined by

$$F(x) = f_1(x) + \cdots + f_m(x),$$

is convex and

$$(\mathrm{cl}\, F)(x) = (\mathrm{cl}\, f_1)(x) + \cdots + (\mathrm{cl}\, f_m)(x), \qquad \forall\, x \in \Re^n.$$

Proof: We write F in the form $F(x) = f(Ax)$, where A is the matrix defined by $Ax = (x, \ldots, x)$, and $f : \Re^{mn} \mapsto (-\infty, \infty]$ is the function

$$f(x_1, \ldots, x_m) = f_1(x_1) + \cdots + f_m(x_m).$$

Since $\mathrm{dom}(F) = \cap_{i=1}^n \mathrm{dom}(f_i)$, Eq. (1.16) implies that

$$\cap_{i=1}^n \mathrm{ri}\big(\mathrm{dom}(f_i)\big) = \mathrm{ri}\big(\mathrm{dom}(F)\big) = \mathrm{ri}\big(A^{-1} \cdot \mathrm{dom}(f)\big) = A^{-1} \cdot \mathrm{ri}\big(\mathrm{dom}(f)\big)$$

(cf. Props. 1.3.8 and 1.3.9). Thus Eq. (1.16) is equivalent to the range of A containing a point in $\mathrm{ri}(\mathrm{dom}(f))$, so that $(\mathrm{cl}\, F)(x) = (\mathrm{cl}\, f)(x, \ldots, x)$ (cf. Prop. 1.3.16). Let $y \in \cap_{i=1}^n \mathrm{ri}(\mathrm{dom}(f_i))$, so that $(y, \ldots, y) \in \mathrm{ri}(\mathrm{dom}(f))$. Then, from Prop. 1.3.15(b), $(\mathrm{cl}\, F)(x) = \lim_{\alpha \downarrow 0} f_1\big(x + \alpha(y - x)\big) + \cdots + \lim_{\alpha \downarrow 0} f_m\big(x + \alpha(y - x)\big) = (\mathrm{cl}\, f_1)(x) + \cdots + (\mathrm{cl}\, f_m)(x)$. **Q.E.D.**

Note that the relative interior assumption (1.16) is essential. To see this, let f_1 and f_2 be the indicator functions of two convex sets C_1 and C_2 such that $\mathrm{cl}(C_1 \cap C_2) \neq \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$ (cf. the example following Prop. 1.3.8).

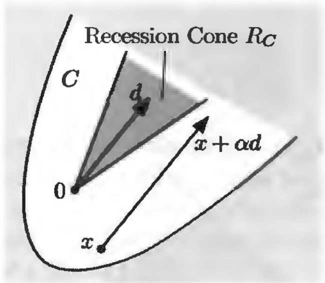


Figure 1.4.1. Illustration of the recession cone R_C of a convex set C. A direction of recession d has the property that $x + \alpha d \in C$ for all $x \in C$ and $\alpha \geq 0$.

1.4 RECESSION CONES

We will now develop some methodology to characterize the asymptotic behavior of convex sets and functions. This methodology is fundamental in several convex optimization contexts, including the issue of existence of optimal solutions, which will be discussed in Chapter 3.

Given a nonempty convex set C, we say that a vector d is a *direction of recession* of C if $x + \alpha d \in C$ for all $x \in C$ and $\alpha \geq 0$. Thus, d is a direction of recession of C if starting at any x in C and going indefinitely along d, we never cross the relative boundary of C to points outside C.

The set of all directions of recession is a cone containing the origin. It is called the *recession cone* of C and it is denoted by R_C (see Fig. 1.4.1). Thus $d \in R_C$ if $x+\alpha d \in C$ for all $x \in C$ and $\alpha \geq 0$. An important property of a *closed* convex set is that to test whether $d \in R_C$ it is enough to verify the property $x+\alpha d \in C$ for a *single* $x \in C$. This is part (b) of the following proposition.

Proposition 1.4.1: (Recession Cone Theorem) Let C be a nonempty closed convex set.

(a) The recession cone R_C is closed and convex.

(b) A vector d belongs to R_C if and only if there exists a vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.

Proof: (a) If d_1, d_2 belong to R_C and γ_1, γ_2 are positive scalars such that $\gamma_1 + \gamma_2 = 1$, we have for any $x \in C$ and $\alpha \geq 0$

$$x + \alpha(\gamma_1 d_1 + \gamma_2 d_2) = \gamma_1(x + \alpha d_1) + \gamma_2(x + \alpha d_2) \in C,$$

where the last inclusion holds because C is convex, and $x+\alpha d_1$ and $x+\alpha d_2$ belong to C by the definition of R_C. Hence $\gamma_1 d_1 + \gamma_2 d_2 \in R_C$, implying that R_C is convex.

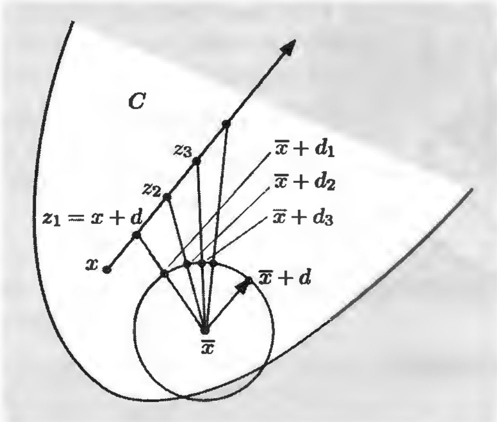


Figure 1.4.2. Construction for the proof of Prop. 1.4.1(b).

Let d be in the closure of R_C, and let $\{d_k\} \subset R_C$ be a sequence converging to d. For any $x \in C$ and $\alpha \geq 0$ we have $x + \alpha d_k \in C$ for all k, and because C is closed, $x + \alpha d \in C$. Hence $d \in R_C$, so R_C is closed.

(b) If $d \in R_C$, every vector $x \in C$ has the required property by the definition of R_C. Conversely, let d be such that there exists a vector $x \in C$ with $x + \alpha d \in C$ for all $\alpha \geq 0$. With no loss of generality, we assume that $d \neq 0$. We choose arbitrary $\overline{x} \in C$ and $\alpha > 0$, and we will show that $\overline{x} + \alpha d \in C$. In fact, it is sufficient to show that $\overline{x} + d \in C$, i.e., to assume that $\alpha = 1$, since the general case where $\alpha > 0$ can be reduced to the case where $\alpha = 1$ by replacing d with αd.

Let

$$z_k = x + kd, \qquad k = 1, 2, \ldots$$

and note that $z_k \in C$ for all k, by our choice of x and d. If $\overline{x} = z_k$ for some k, then $\overline{x} + d = x + (k+1)d$, which belongs to C and we are done. We thus assume that $\overline{x} \neq z_k$ for all k, and we define

$$d_k = \frac{z_k - \overline{x}}{\|z_k - \overline{x}\|}\|d\|, \qquad k = 1, 2, \ldots \tag{1.17}$$

so that $\overline{x} + d_k$ is the intersection of the surface of the sphere centered at $\overline{x}$ of radius $\|d\|$, and the halfline that starts at $\overline{x}$ and passes through z_k (see the construction of Fig. 1.4.2). We will now argue that $d_k \to d$, and that for large enough k, $\overline{x} + d_k \in C$, so using the closure of C, it follows that $\overline{x} + d \in C$.

Indeed, using the definition (1.17) of d_k, we have

$$\frac{d_k}{\|d\|} = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \cdot \frac{z_k - x}{\|z_k - x\|} + \frac{x - \overline{x}}{\|z_k - \overline{x}\|} = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \cdot \frac{d}{\|d\|} + \frac{x - \overline{x}}{\|z_k - \overline{x}\|}.$$

Because $\{z_k\}$ is an unbounded sequence,

$$\frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \to 1, \qquad \frac{x - \overline{x}}{\|z_k - \overline{x}\|} \to 0,$$

so by combining the preceding relations, we have $d_k \to d$. The vector $\overline{x}+d_k$ lies between $\overline{x}$ and z_k in the line segment connecting $\overline{x}$ and z_k for all k such that $\|z_k - \overline{x}\| \geq \|d\|$, so by the convexity of C, we have $\overline{x} + d_k \in C$ for all sufficiently large k. Since $\overline{x} + d_k \to \overline{x} + d$ and C is closed, it follows that $\overline{x} + d \in C$. **Q.E.D.**

It is essential to assume that the set C is closed in the preceding proposition. For an example where part (a) fails without this assumption, consider the set

$$C = \big\{(x_1, x_2) \mid 0 < x_1,\, 0 < x_2\big\} \cup \big\{(0,0)\big\}.$$

Its recession cone is equal to C, which is not closed. Part (b) also fails in this example, since for the direction $d = (1,0)$ we have $x + \alpha d \in C$ for all $\alpha \geq 0$ and all $x \in C$, except for $x = (0,0)$.

The following proposition gives some additional properties of recession cones.

Proposition 1.4.2: (Properties of Recession Cones) Let C be a nonempty closed convex set.

(a) R_C contains a nonzero direction if and only if C is unbounded.

(b) $R_C = R_{\mathrm{ri}(C)}$.

(c) For any collection of closed convex sets C_i, $i \in I$, where I is an arbitrary index set and $\cap_{i\in I} C_i \neq \emptyset$, we have

$$R_{\cap_{i\in I} C_i} = \cap_{i\in I} R_{C_i}.$$

(d) Let W be a compact and convex subset of $\Re^m$, and let A be an $m \times n$ matrix. The recession cone of the set

$$V = \{x \in C \mid Ax \in W\}$$

(assuming this set is nonempty) is $R_C \cap N(A)$, where $N(A)$ is the nullspace of A.

Proof: (a) Assuming that C is unbounded, we will show that R_C contains a nonzero direction (the reverse implication is clear). Choose any $x \in C$

and any unbounded sequence $\{z_k\} \subset C$. Consider the sequence $\{d_k\}$, where

$$d_k = \frac{z_k - x}{\|z_k - x\|},$$

and let d be a limit point of $\{d_k\}$ (compare with the construction of Fig. 1.4.2). Without loss of generality, assume that $\|z_k - x\|$ is monotonically increasing with k. For any fixed $\alpha \geq 0$, the vector $x + \alpha d_k$ lies between x and z_k in the line segment connecting x and z_k for all k such that $\|z_k - x\| \geq \alpha$. Hence by the convexity of C, we have $x + \alpha d_k \in C$ for all sufficiently large k. Since $x + \alpha d$ is a limit point of $\{x + \alpha d_k\}$ and C is closed, we have $x + \alpha d \in C$. Hence, using also Prop. 1.4.1(b), it follows that the nonzero vector d is a direction of recession.

(b) If $d \in R_{\mathrm{ri}(C)}$, then for a fixed $x \in \mathrm{ri}(C)$ and all $\alpha \geq 0$, we have $x + \alpha d \in \mathrm{ri}(C) \subset C$. Hence, by Prop. 1.4.1(b), we have $d \in R_C$. Conversely, if $d \in R_C$, then for any $x \in \mathrm{ri}(C)$, we have $x + \alpha d \in C$ for all $\alpha \geq 0$. It follows from the Line Segment Principle (Prop. 1.3.1) that $x + \alpha d \in \mathrm{ri}(C)$ for all $\alpha \geq 0$, so that d belongs to $R_{\mathrm{ri}(C)}$.

(c) By the definition of direction of recession, $d \in R_{\cap_{i \in I} C_i}$ implies that $x + \alpha d \in \cap_{i \in I} C_i$ for all $x \in \cap_{i \in I} C_i$ and all $\alpha \geq 0$. By Prop. 1.4.1(b), this in turn implies that $d \in R_{C_i}$ for all i, so that $R_{\cap_{i \in I} C_i} \subset \cap_{i \in I} R_{C_i}$. Conversely, by the definition of direction of recession, if $d \in \cap_{i \in I} R_{C_i}$ and $x \in \cap_{i \in I} C_i$, we have $x + \alpha d \in \cap_{i \in I} C_i$ for all $\alpha \geq 0$, so $d \in R_{\cap_{i \in I} C_i}$. Thus, $\cap_{i \in I} R_{C_i} \subset R_{\cap_{i \in I} C_i}$.

(d) Consider the closed convex set $\overline{V} = \{x \mid Ax \in W\}$, and choose some $x \in \overline{V}$. Then, by Prop. 1.4.1(b), $d \in R_{\overline{V}}$ if and only if $x + \alpha d \in \overline{V}$ for all $\alpha \geq 0$, or equivalently if and only if $A(x + \alpha d) \in W$ for all $\alpha \geq 0$. Since $Ax \in W$, the last statement is equivalent to $Ad \in R_W$. Thus, $d \in R_{\overline{V}}$ if and only if $Ad \in R_W$. Since W is compact, from part (a) we have $R_W = \{0\}$, so $R_{\overline{V}}$ is equal to $\{d \mid Ad = 0\}$, which is $N(A)$. Since $V = C \cap \overline{V}$, using part (c), we have $R_V = R_C \cap N(A)$. **Q.E.D.**

For an example where part (a) of the preceding proposition fails, consider the unbounded convex set

$$C = \{(x_1, x_2) \mid 0 \leq x_1 < 1,\ 0 \leq x_2\} \cup \{(1, 0)\}.$$

By using the definition, it can be verified that C has no nonzero directions of recession. It can also be verified that $(0, 1)$ is a direction of recession of $\mathrm{ri}(C)$, so part (b) also fails. Finally, by letting

$$D = \{(x_1, x_2) \mid -1 \leq x_1 \leq 0,\ 0 \leq x_2\},$$

it can be seen that $(0, 1) \in R_D$, so $R_{C \cap D} \neq R_C \cap R_D$ and part (c) fails as well.

Note that part (c) of the preceding proposition implies that if C and D are nonempty closed and convex sets such that $C \subset D$, then $R_C \subset R_D$. This can be seen by using part (c) to write $R_C = R_{C\cap D} = R_C \cap R_D$, from which we obtain $R_C \subset R_D$. This property can fail if the sets C and D are not closed; for example, if

$$C = \{(x_1, x_2) \mid 0 \leq x_1 < 1,\, 0 \leq x_2\}, \qquad D = C \cup \{(1,0)\},$$

then the vector $(0,1)$ is a direction of recession of C but not of D.

Lineality Space

A subset of the recession cone of a convex set C that plays an important role in a number of interesting contexts is its *lineality space*, denoted by L_C. It is defined as the set of directions of recession d whose opposite, $-d$, are also directions of recession:

$$L_C = R_C \cap (-R_C).$$

Thus $d \in L_C$ if and only if the entire line $\{x + \alpha d \mid \alpha \in \Re\}$ is contained in C for every $x \in C$.

The lineality space inherits several of the properties of the recession cone that we have shown (Props. 1.4.1 and 1.4.2). We collect these properties in the following proposition.

Proposition 1.4.3: (Properties of Lineality Space) Let C be a nonempty closed convex subset of $\Re^n$.

(a) L_C is a subspace of $\Re^n$.

(b) $L_C = L_{\mathrm{ri}(C)}$.

(c) For any collection of closed convex sets C_i, $i \in I$, where I is an arbitrary index set and $\cap_{i\in I} C_i \neq \emptyset$, we have

$$L_{\cap_{i\in I} C_i} = \cap_{i\in I} L_{C_i}.$$

(d) Let W be a compact and convex subset of $\Re^m$, and let A be an $m \times n$ matrix. The lineality space of the set

$$V = \{x \in C \mid Ax \in W\}$$

(assuming it is nonempty) is $L_C \cap N(A)$, where $N(A)$ is the nullspace of A.

Proof: (a) Let d_1 and d_2 belong to L_C, and let α_1 and α_2 be nonzero scalars. We will show that $\alpha_1 d_1 + \alpha_2 d_2$ belongs to L_C. Indeed, we have

$$\begin{aligned} \alpha_1 d_1 + \alpha_2 d_2 &= |\alpha_1|\big(\mathrm{sgn}(\alpha_1) d_1\big) + |\alpha_2|\big(\mathrm{sgn}(\alpha_2) d_2\big) \\ &= \big(|\alpha_1| + |\alpha_2|\big)\big(\alpha \overline{d}_1 + (1-\alpha)\overline{d}_2\big), \end{aligned} \tag{1.18}$$

where

$$\alpha = \frac{|\alpha_1|}{|\alpha_1| + |\alpha_2|}, \qquad \overline{d}_1 = \mathrm{sgn}(\alpha_1) d_1, \quad \overline{d}_2 = \mathrm{sgn}(\alpha_2) d_2,$$

and for a nonzero scalar s, we use the notation $\mathrm{sgn}(s) = 1$ or $\mathrm{sgn}(s) = -1$ depending on whether s is positive or negative, respectively. We now note that L_C is a convex cone, being the intersection of the convex cones R_C and $-R_C$. Hence, since $\overline{d}_1$ and $\overline{d}_2$ belong to L_C, any positive multiple of a convex combination of $\overline{d}_1$ and $\overline{d}_2$ belongs to L_C. It follows from Eq. (1.18) that $\alpha_1 d_1 + \alpha_2 d_2 \in L_C$.

(b) We have

$$L_{\mathrm{ri}(C)} = R_{\mathrm{ri}(C)} \cap \big(-R_{\mathrm{ri}(C)}\big) = R_C \cap (-R_C) = L_C,$$

where the second equality follows from Prop. 1.4.2(b).

(c) We have

$$\begin{aligned} L_{\cap_{i\in I} C_i} &= \big(R_{\cap_{i\in I} C_i}\big) \cap \big(-R_{\cap_{i\in I} C_i}\big) \\ &= \big(\cap_{i\in I} R_{C_i}\big) \cap \big(-\cap_{i\in I} R_{C_i}\big) \\ &= \cap_{i\in I}\big(R_{C_i} \cap (-R_{C_i})\big) \\ &= \cap_{i\in I} L_{C_i}, \end{aligned}$$

where the second equality follows from Prop. 1.4.2(c).

(d) We have

$$\begin{aligned} L_V &= R_V \cap (-R_V) \\ &= \big(R_C \cap N(A)\big) \cap \big((-R_C) \cap N(A)\big) \\ &= \big(R_C \cap (-R_C)\big) \cap N(A) \\ &= L_C \cap N(A), \end{aligned}$$

where the second equality follows from Prop. 1.4.2(d). **Q.E.D.**

Example 1.4.1: (Sets Specified by Linear and Convex Quadratic Inequalities)

Consider a nonempty set of the form

$$C = \{x \mid x'Qx + c'x + b \le 0\},$$

where Q is a symmetric positive semidefinite $n \times n$ matrix, c is a vector in $\Re^n$, and b is a scalar. A vector d is a direction of recession if and only if

$$(x + \alpha d)'Q(x + \alpha d) + c'(x + \alpha d) + b \leq 0, \qquad \forall\, \alpha > 0,\ x \in C,$$

or

$$x'Qx + c'x + b + \alpha(c + 2Qx)'d + \alpha^2 d'Qd \leq 0, \quad \forall\, \alpha > 0,\ x \in C. \tag{1.19}$$

Clearly, we cannot have $d'Qd > 0$, since then the left-hand side above would become arbitrarily large for a suitably large choice of α, so $d'Qd = 0$. Since Q is positive semidefinite, it can be written as $Q = M'M$ for some matrix M, so that we have $Md = 0$, implying that $Qd = 0$. It follows that Eq. (1.19) is equivalent to

$$x'Qx + c'x + b + \alpha c'd \leq 0, \qquad \forall\, \alpha > 0,\ x \in C,$$

which is true if and only if $c'd \leq 0$. Thus,

$$R_C = \{d \mid Qd = 0,\ c'd \leq 0\}.$$

Also, $L_C = R_C \cap (-R_C)$, so

$$L_C = \{d \mid Qd = 0,\ c'd = 0\}.$$

Consider now the case where C is nonempty and specified by any (possibly infinite) number of convex quadratic inequalities:

$$C = \{x \mid x'Q_jx + c_j'x + b_j \leq 0,\ j \in J\},$$

where J is some index set. Then using Props. 1.4.2(c) and 1.4.3(c), we have

$$R_C = \{d \mid Q_jd = 0,\ c_j'd \leq 0,\ \forall\, j \in J\},$$
$$L_C = \{d \mid Q_jd = 0,\ c_j'd = 0,\ \forall\, j \in J\}.$$

In particular, if C is a polyhedral set of the form

$$C = \{x \mid c_j'x + b_j \leq 0,\ j = 1, \ldots, r\},$$

we have

$$R_C = \{d \mid c_j'd \leq 0,\ j = 1, \ldots, r\}, \qquad L_C = \{d \mid c_j'd = 0,\ j = 1, \ldots, r\}.$$

Finally, let us prove a useful result that allows the decomposition of a convex set along a subspace of its lineality space (possibly the entire lineality space) and its orthogonal complement (see Fig. 1.4.3).

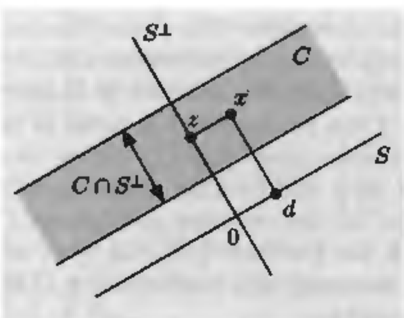


Figure 1.4.3. Illustration of the decomposition of a convex set C as

$$C = S + (C \cap S^\perp),$$

where S is a subspace contained in the lineality space L_C. A vector $x \in C$ is expressed as $x = d + z$ with $d \in S$ and $z \in C \cap S^\perp$, as shown.

Proposition 1.4.4: (Decomposition of a Convex Set) Let C be a nonempty convex subset of $\Re^n$. Then, for every subspace S that is contained in the lineality space L_C, we have

$$C = S + (C \cap S^\perp).$$

Proof: We can decompose $\Re^n$ as $S + S^\perp$, so for $x \in C$, let $x = d + z$ for some $d \in S$ and $z \in S^\perp$. Because $-d \in S \subset L_C$, the vector $-d$ is a direction of recession of C, so the vector $x - d$, which is equal to z, belongs to C, implying that $z \in C \cap S^\perp$. Thus, we have $x = d + z$ with $d \in S$ and $z \in C \cap S^\perp$ showing that $C \subset S + (C \cap S^\perp)$.

Conversely, if $x \in S + (C \cap S^\perp)$, then $x = d + z$ with $d \in S$ and $z \in C \cap S^\perp$. Thus, we have $z \in C$. Furthermore, because $S \subset L_C$, the vector d is a direction of recession of C, implying that $d + z \in C$. Hence $x \in C$, showing that $S + (C \cap S^\perp) \subset C$. **Q.E.D.**

In the special case where $S = L_C$ in Prop. 1.4.4, we obtain

$$C = L_C + (C \cap L_C^\perp). \tag{1.20}$$

Thus, C is the vector sum of two sets:

(1) The set L_C, which consists of the lines contained in C, translated to pass through the origin.

(2) The set $C \cap L_C^\perp$, which contains no lines; to see this, note that for any line $\{x + \alpha d \mid \alpha \in \Re\} \subset C \cap L_C^\perp$, we have $d \in L_C$ (since $x + \alpha d \in C$ for all $\alpha \in \Re$), so $d \perp (x + \alpha d)$ for all $\alpha \in \Re$, implying that $d = 0$.

Note that if $R_C = L_C$ and C is closed, the set $C \cap L_C^\perp$ contains no nonzero directions of recession, so it is compact [cf. Prop. 1.4.2(a)], and C can be decomposed into the sum of L_C and a compact set, as per Eq. (1.20).

1.4.1 Directions of Recession of a Convex Function

We will now develop a notion of direction of recession of a convex function. This notion is important in several contexts, including the existence of solutions of convex optimization problems, which will be discussed in Chapter 3. A key fact is that a convex function f can be described in terms of its epigraph, which is a convex set. The recession cone of epi(f) can be used to obtain the directions along which f does not increase monotonically. In particular, the "horizontal directions" in the recession cone of epi(f) correspond to the directions along which the level sets $\{x \mid f(x) \leq \gamma\}$ are unbounded. Along these directions, f is monotonically nonincreasing. This is the idea underlying the following proposition.

Proposition 1.4.5: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function and consider the level sets

$$V_\gamma = \{x \mid f(x) \leq \gamma\}, \qquad \gamma \in \Re.$$

Then:

(a) All the nonempty level sets V_γ have the same recession cone, denoted R_f, and given by

$$R_f = \{d \mid (d, 0) \in R_{\text{epi}(f)}\},$$

where $R_{\text{epi}(f)}$ is the recession cone of the epigraph of f.

(b) If one nonempty level set V_γ is compact, then all of these level sets are compact.

Proof: (a) Fix a γ such that V_γ is nonempty. Let S be the "γ-slice" of epi(f),

$$S = \{(x, \gamma) \mid f(x) \leq \gamma\},$$

and note that

$$S = \text{epi}(f) \cap \{(x, \gamma) \mid x \in \Re^n\}.$$

Using Prop. 1.4.2(c) [which applies since epi(f) is closed in view of the closedness of f], we have

$$R_S = R_{\text{epi}(f)} \cap \{(d, 0) \mid d \in \Re^n\} = \{(d, 0) \mid (d, 0) \in R_{\text{epi}(f)}\}.$$

From this equation and the fact $S = \{(x, \gamma) \mid x \in V_\gamma\}$, the desired formula for R_{V_γ} follows.

(b) From Prop. 1.4.2(a), a nonempty level set V_γ is compact if and only if the recession cone R_{V_γ} does not contain a nonzero direction. By part (a), all nonempty level sets V_γ have the same recession cone, so if one of them is compact, all of them are compact. **Q.E.D.**

Note that closedness of f is essential for the level sets V_γ to have a common recession cone, as per Prop. 1.4.5(a). The reader may verify this by using as an example the convex but not closed function $f : \Re^2 \mapsto (-\infty, \infty]$ given by

$$f(x_1, x_2) = \begin{cases} -x_1 & \text{if } x_1 > 0,\ x_2 \geq 0, \\ x_2 & \text{if } x_1 = 0,\ x_2 \geq 0, \\ \infty & \text{if } x_1 < 0 \text{ or } x_2 < 0. \end{cases}$$

Here, for $\gamma < 0$, we have $V_\gamma = \{(x_1, x_2) \mid x_1 \geq -\gamma,\ x_2 \geq 0\}$, so that $(0, 1) \in R_{V_\gamma}$, but $V_0 = \{(x_1, x_2) \mid x_1 > 0,\ x_2 \geq 0\} \cup \{(0, 0)\}$, so that $(0, 1) \notin R_{V_0}$.

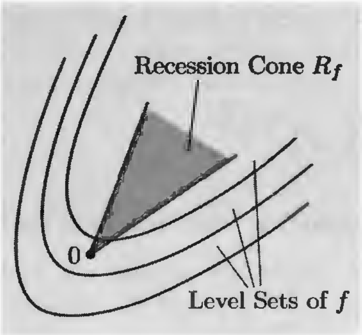


Figure 1.4.4. Illustration of the recession cone R_f of a closed proper convex function f. It is the (common) recession cone of the nonempty level sets of f.

For a closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, the (common) recession cone R_f of the nonempty level sets is called the *recession cone of* f (cf. Fig. 1.4.4). A vector $d \in R_f$ is called a *direction of recession of* f.

The most intuitive way to look at directions of recession of f is from a descent viewpoint: if we start at any $x \in \mathrm{dom}(f)$ and move indefinitely along a direction of recession, we must stay within each level set that contains x, or equivalently we must encounter exclusively points z with $f(z) \leq f(x)$. In words, *a direction of recession of* f *is a direction of continuous nonascent for* f. Conversely, if we start at some $x \in \mathrm{dom}(f)$ and while moving along a direction d, we encounter a point z with $f(z) > f(x)$, then d cannot be a direction of recession. By the convexity of the level sets of f, once we cross the relative boundary of a level set, we never cross it back again, and with a little thought, it can be seen that *a direction that is not a direction of recession of* f *is a direction of eventual continuous ascent of* f [see Figs. 1.4.5(e),(f)].

Constancy Space of a Convex Function

The lineality space of the recession cone R_f of a closed proper convex function f is denoted by L_f, and is the subspace of all $d \in \Re^n$ such that both d and $-d$ are directions of recession of f, i.e.,

$$L_f = R_f \cap (-R_f).$$

Equivalently, $d \in L_f$ if and only if both d and $-d$ are directions of recession of each of the nonempty level sets $\{x \mid f(x) \leq \gamma\}$ [cf. Prop. 1.4.5(a)]. In view of the convexity of f, which implies that f is monotonically nonincreasing along a direction of recession, we see that $d \in L_f$ if and only if

$$f(x + \alpha d) = f(x), \qquad \forall\, x \in \mathrm{dom}(f),\ \forall\, \alpha \in \Re.$$

Consequently, any $d \in L_f$ is called a *direction in which* f *is constant*, and L_f is called the *constancy space of* f.

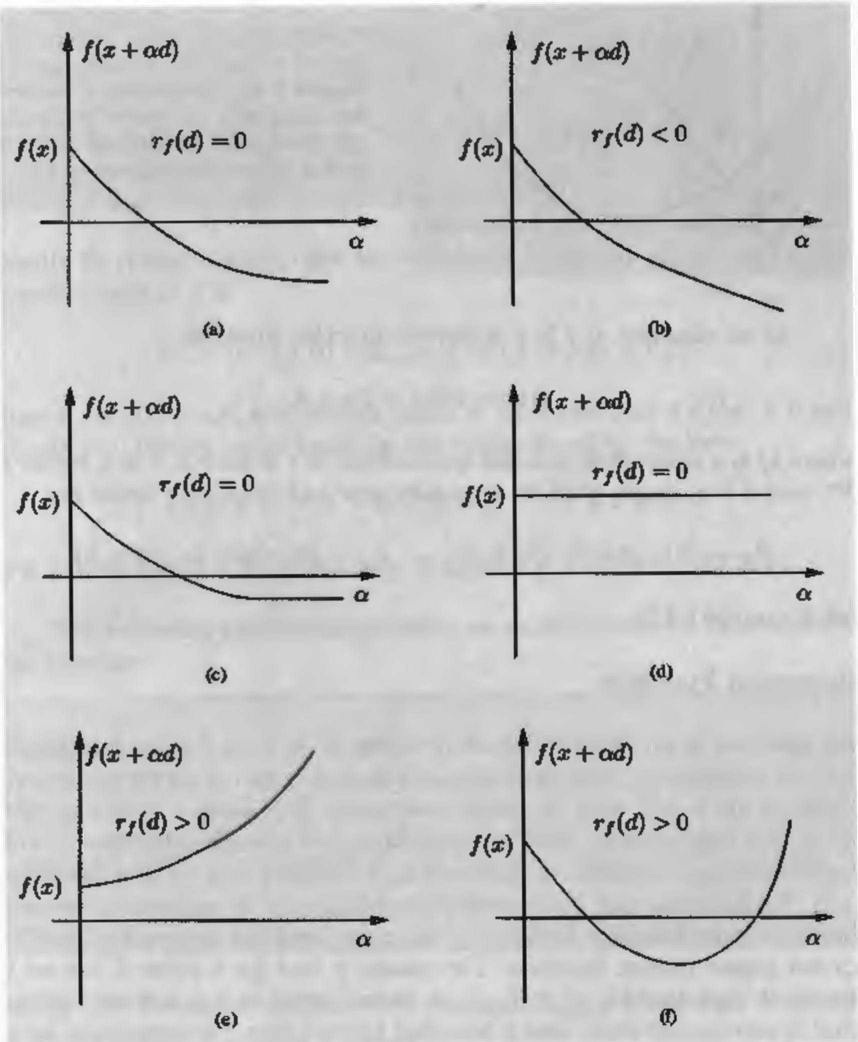


Figure 1.4.5. Ascent/descent behavior of a closed proper convex function starting at some $x \in \text{dom}(f)$ and moving along a direction d. If d is a direction of recession of f, there are two possibilities: either f decreases monotonically to a finite value or $-\infty$ [figures (a) and (b), respectively], or f reaches a value that is less or equal to $f(x)$ and stays at that value [figures (c) and (d)]. If d is not a direction of recession of f, then eventually f increases monotonically to ∞ [figures (e) and (f)], i.e., for some $\overline{\alpha} \geq 0$ and all $\alpha_1, \alpha_2 \geq \overline{\alpha}$ with $\alpha_1 < \alpha_2$, we have

$$f(x + \alpha_1 d) < f(x + \alpha_2 d).$$

This behavior is determined only by d, and is independent of the choice of x within $\text{dom}(f)$.

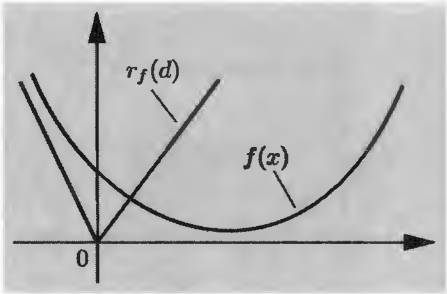


Figure 1.4.6. Illustration of the recession function r_f of a closed proper convex function f. Its epigraph is the recession cone of the epigraph of f.

As an example, if f is a quadratic function given by

$$f(x) = x'Qx + c'x + b,$$

where Q is a symmetric positive semidefinite $n \times n$ matrix, c is a vector in $\Re^n$, and b is a scalar, then its recession cone and constancy space are

$$R_f = \{d \mid Qd = 0,\ c'd \leq 0\}, \qquad L_f = \{d \mid Qd = 0,\ c'd = 0\}$$

(cf. Example 1.4.1).

Recession Function

We saw that if d is a direction of recession of f, then f is asymptotically nonincreasing along each halfline $x + \alpha d$, but in fact a stronger property holds: it turns out that *the asymptotic slope of f along d is independent of the starting point x.* The "asymptotic slope" of a closed proper convex function along a direction is expressed by a function that we now introduce.

We first note that the recession cone $R_{\mathrm{epi}(f)}$ of the epigraph of a closed proper convex function $f : \Re^n \mapsto (-\infty, \infty]$ is itself the epigraph of another closed proper convex function. The reason is that for a given d, the set of scalars w such that $(d, w) \in R_{\mathrm{epi}(f)}$ is either empty or it is a closed interval that is unbounded above and is bounded below (since f is proper and hence its epigraph does not contain a vertical line). Thus $R_{\mathrm{epi}(f)}$ is the epigraph of a proper function, which must be closed and convex [since f, epi(f), and $R_{\mathrm{epi}(f)}$ are all closed and convex]. This function is called the *recession function of f* and is denoted r_f, i.e.,

$$\mathrm{epi}(r_f) = R_{\mathrm{epi}(f)};$$

see Fig. 1.4.6.

The recession function can be used to characterize the recession cone and constancy space of the function, as in the following proposition (cf. Fig. 1.4.5).

Proposition 1.4.6: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function. Then the recession cone and constancy space of f are given in terms of its recession function by

$$R_f = \{d \mid r_f(d) \leq 0\}, \qquad L_f = \{d \mid r_f(d) = r_f(-d) = 0\}.$$

Proof: By Prop. 1.4.5(a) and the definition $\text{epi}(r_f) = R_{\text{epi}(f)}$ of r_f, the recession cone of f is

$$R_f = \{d \mid (d, 0) \in R_{\text{epi}(f)}\} = \{d \mid r_f(d) \leq 0\}.$$

Since $L_f = R_f \cap (-R_f)$, it follows that $d \in L_f$ if and only if $r_f(d) \leq 0$ and $r_f(-d) \leq 0$. On the other hand, by the convexity of r_f, we have

$$r_f(d) + r_f(-d) \geq 2r_f(0) = 0, \qquad \forall\, d \in \Re^n,$$

so it follows that $d \in L_f$ if and only if $r_f(d) = r_f(-d) = 0$. **Q.E.D.**

The following proposition provides an explicit formula for the recession function.

Proposition 1.4.7: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function. Then, for all $x \in \text{dom}(f)$ and $d \in \Re^n$,

$$r_f(d) = \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} = \lim_{\alpha \to \infty} \frac{f(x + \alpha d) - f(x)}{\alpha}. \tag{1.21}$$

Proof: By definition, we have that $(d, \nu) \in R_{\text{epi}(f)}$ if and only if for all $(x, w) \in \text{epi}(f)$,

$$(x + \alpha d, w + \alpha \nu) \in \text{epi}(f), \qquad \forall\, \alpha > 0,$$

or equivalently, $f(x + \alpha d) \leq f(x) + \alpha\nu$ for all $\alpha > 0$, which can be written as

$$\frac{f(x + \alpha d) - f(x)}{\alpha} \leq \nu, \qquad \forall\, \alpha > 0.$$

Hence

$$(d, \nu) \in R_{\text{epi}(f)} \qquad \text{if and only if} \qquad \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} \leq \nu,$$

for all $x \in \text{dom}(f)$. Since $R_{\text{epi}(f)}$ is the epigraph of r_f, this implies the first equality in Eq. (1.21).

From the convexity of f, we see that the ratio

$$\frac{f(x+\alpha d) - f(x)}{\alpha}$$

is monotonically nondecreasing as a function of α over the range $(0, \infty)$. This implies the second equality in Eq. (1.21). **Q.E.D.**

The last expression in Eq. (1.21) leads to the interpretation of $r_f(d)$ as the "asymptotic slope" of f along the direction d. In fact, for differentiable convex functions $f : \Re^n \mapsto \Re$, this interpretation can be made more precise: we have

$$r_f(d) = \lim_{\alpha \to \infty} \nabla f(x+\alpha d)' d, \qquad \forall\, x \in \Re^n,\ d \in \Re^n. \tag{1.22}$$

Indeed, for all x, d, and $\alpha > 0$, we have using Prop. 1.1.7(a),

$$\nabla f(x)'d \le \frac{f(x+\alpha d) - f(x)}{\alpha} \le \nabla f(x+\alpha d)'d,$$

so by taking the limit as $\alpha \to \infty$ and using Eq. (1.21), it follows that

$$\nabla f(x)'d \le r_f(d) \le \lim_{\alpha \to \infty} \nabla f(x+\alpha d)'d. \tag{1.23}$$

The left-hand side above holds for all x, so replacing x with $x + \alpha d$,

$$\nabla f(x+\alpha d)'d \le r_f(d), \qquad \forall\, \alpha > 0.$$

By taking the limit as $\alpha \to \infty$, we obtain

$$\lim_{\alpha \to \infty} \nabla f(x+\alpha d)'d \le r_f(d), \tag{1.24}$$

and by combining Eqs. (1.23) and (1.24), we obtain Eq. (1.22).

The calculation of recession functions can be facilitated by nice formulas for the sum and the supremum of closed convex functions. The following proposition deals with the case of a sum.

Proposition 1.4.8: **(Recession Function of a Sum)** Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be closed proper convex functions such that the function $f = f_1 + \cdots + f_m$ is proper. Then

$$r_f(d) = r_{f_1}(d) + \cdots + r_{f_m}(d), \qquad \forall\, d \in \Re^n. \tag{1.25}$$

Proof: Without loss of generality, assume that $m = 2$, and note that $f_1 + f_2$ is closed proper convex (cf. Prop. 1.1.5). By using Eq. (1.21), we have for all $x \in \text{dom}(f_1 + f_2)$ and $d \in \Re^n$,

$$\begin{aligned} r_{f_1+f_2}(d) &= \lim_{\alpha\to\infty} \left\{ \frac{f_1(x+\alpha d) - f_1(x)}{\alpha} + \frac{f_2(x+\alpha d) - f_2(x)}{\alpha} \right\} \\ &= \lim_{\alpha\to\infty} \left\{ \frac{f_1(x+\alpha d) - f_1(x)}{\alpha} \right\} + \lim_{\alpha\to\infty} \left\{ \frac{f_2(x+\alpha d) - f_2(x)}{\alpha} \right\} \\ &= r_{f_1}(d) + r_{f_2}(d), \end{aligned}$$

where the second equality holds because the limits involved exist. **Q.E.D.**

Note that for the formula (1.25) to hold, it is essential that f is proper, for otherwise its recession function is undefined. There is a similar result regarding the function

$$f(x) = \sup_{i\in I} f_i(x),$$

where I is an arbitrary index set, and $f_i : \Re^n \mapsto (-\infty, \infty]$, $i \in I$, are closed proper convex functions such that f is proper. In particular, we have

$$r_f(d) = \sup_{i\in I} r_{f_i}(d), \qquad d \in \Re^n. \tag{1.26}$$

To show this, we simply note that the epigraph of r_f is the recession cone of the epigraph of f, the intersection of the epigraphs of f_i. Thus, the epigraph of r_f is the intersection of the recession cones of the epigraphs of f_i by Prop. 1.4.2(c), which yields the formula (1.26).

1.4.2 Nonemptiness of Intersections of Closed Sets

The notions of recession cone and lineality space can be used to generalize some of the fundamental properties of compact sets to closed convex sets. One such property is that a sequence $\{C_k\}$ of nonempty and compact sets with $C_{k+1} \subset C_k$ for all k has nonempty and compact intersection [cf. Prop. A.2.4(h)]. Another property is that the image of a compact set under a linear transformation is compact [cf. Prop. A.2.6(d)]. These properties may not hold when the sets involved are closed but unbounded (cf. Fig. 1.3.4), and some additional conditions are needed for their validity. In this section we develop such conditions, using directions of recession and related notions. We focus on the case where the sets involved are convex, but the analysis generalizes to the nonconvex case (see [BeT07]).

To understand the significance of set intersection results, consider a sequence of nonempty closed sets $\{C_k\}$ in $\Re^n$ with $C_{k+1} \subset C_k$ for all k (such a sequence is said to be *nested*), and the question whether $\cap_{k=0}^{\infty} C_k$ is nonempty. Here are some of the contexts where this question arises:

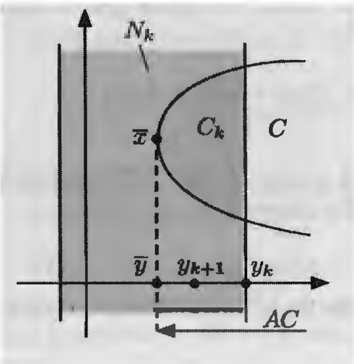


Figure 1.4.7. Set intersection argument to prove that the set AC closed when C is closed. Here A is the projection on the horizontal axis of points in the plane. For a sequence $\{y_k\} \subset AC$ that converges to some $\overline{y}$, in order to prove that $\overline{y} \in AC$, it is sufficient to prove that the intersection $\cap_{k=0}^{\infty} C_k$ is nonempty, where

$$C_k = C \cap N_k,$$

and

$$N_k = \{x \mid \|Ax - \overline{y}\| \leq \|y_k - \overline{y}\|\}.$$

(a) Does a function $f : \Re^n \mapsto \Re$ attain a minimum over a set X? This is true if and only if the intersection

$$\cap_{k=0}^{\infty} \{x \in X \mid f(x) \leq \gamma_k\}$$

is nonempty, where $\{\gamma_k\}$ is a scalar sequence with $\gamma_k \downarrow \inf_{x \in X} f(x)$.

(b) If C is a closed set and A is a matrix, is AC closed? To prove this, we may let $\{y_k\}$ be a sequence in AC that converges to some $\overline{y} \in \Re^n$, and then show that $\overline{y} \in AC$. If we introduce the sets $C_k = C \cap N_k$, where

$$N_k = \{x \mid \|Ax - \overline{y}\| \leq \|y_k - \overline{y}\|\},$$

it is sufficient to show that $\cap_{k=0}^{\infty} C_k$ is nonempty (see Fig. 1.4.7).

We will next consider a nested sequence $\{C_k\}$ of nonempty closed convex sets, and in the subsequent propositions, we will derive several alternative conditions under which the intersection $\cap_{k=0}^{\infty} C_k$ is nonempty. These conditions involve a variety of assumptions about the recession cones, the lineality spaces, and the structure of the sets C_k.

Asymptotic Sequences of Convex Sets

The following line of analysis actually extends to nonconvex closed sets (see [BeT07]). However, in this book we will restrict ourselves to set intersections involving only convex sets.

Our analysis revolves around sequences $\{x_k\}$ such that $x_k \in C_k$ for each k. An important fact is that $\cap_{k=0}^{\infty} C_k$ is empty if and only if every sequence of this type is unbounded. Thus the idea is to introduce assumptions that guarantee that not all such sequences are unbounded. In fact it will be sufficient to restrict attention to unbounded sequences that escape to ∞ along common directions of recession of the sets C_k, as in the following definition.

Definition 1.4.1: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. We say that $\{x_k\}$ is an *asymptotic sequence of* $\{C_k\}$ if $x_k \neq 0$, $x_k \in C_k$ for all k, and

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|},$$

where d is some nonzero common direction of recession of the sets C_k,

$$d \neq 0, \qquad d \in \cap_{k=0}^{\infty} R_{C_k}.$$

A special case is when all the sets C_k are equal. In particular, for a nonempty closed convex C, we say that $\{x_k\} \subset C$ is an asymptotic sequence of C if $\{x_k\}$ is asymptotic for the sequence $\{C_k\}$, where $C_k \equiv C$.

Note that given any unbounded sequence $\{x_k\}$ such that $x_k \in C_k$ for each k, there exists a subsequence $\{x_k\}_{k\in\mathcal{K}}$ that is asymptotic for the corresponding subsequence $\{C_k\}_{k\in\mathcal{K}}$. In fact, any limit point of $\{x_k/\|x_k\|\}$ is a common direction of recession of the sets C_k; this can be seen by using the proof argument of Prop. 1.4.1(b). Thus, asymptotic sequences are in a sense representative of unbounded sequences with $x_k \in C_k$ for each k.

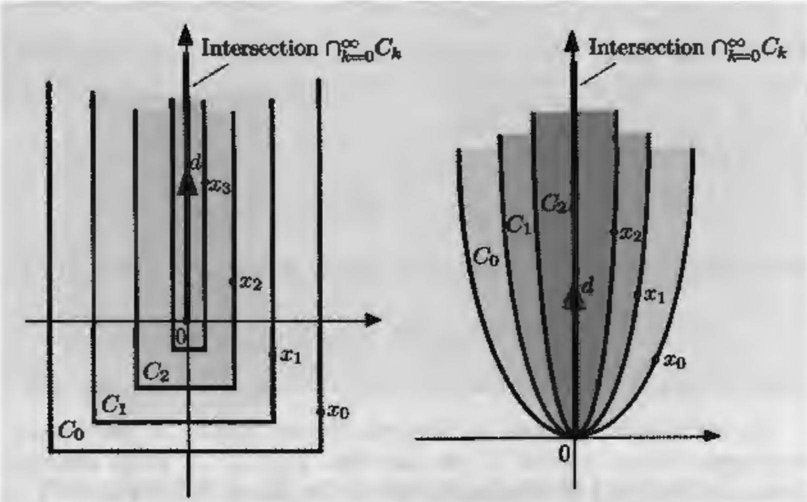
We now introduce a special type of set sequences that have favorable properties for our purposes.

Definition 1.4.2: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. We say that *an asymptotic sequence* $\{x_k\}$ *is retractive* if for the direction d corresponding to $\{x_k\}$ as per Definition 1.4.1, there exists an index $\overline{k}$ such that

$$x_k - d \in C_k, \qquad \forall\, k \geq \overline{k}.$$

We say that *the sequence* $\{C_k\}$ *is retractive* if all its asymptotic sequences are retractive. In the special case $C_k \equiv C$, we say that *the set* C *is retractive* if all its asymptotic sequences are retractive.

Retractive set sequences are those whose asymptotic sequences still belong to the corresponding sets C_k (for sufficiently large k) when shifted by $-d$, where d is any corresponding direction of recession. For an example, consider a nested sequence consisting of "cylindrical" sets in the plane, such as $C_k = \{(x^1, x^2) \mid |x^1| \leq 1/k\}$, whose asymptotic sequences $\{(x_k^1, x_k^2)\}$ are retractive: they satisfy $x_k^1 \to 0$, and either $x_k^2 \to \infty$ $[d = (0,1)]$ or $x_k^2 \to -\infty$ $[d = (0,-1)]$ (see also Fig. 1.4.8). Some important types of



(a) Retractive Set Sequence (b) Nonretractive Set Sequence

Figure 1.4.8. Illustration of retractive and nonretractive sequences in $\Re^2$. For both set sequences, the intersection is the vertical half line $\{x \mid x_2 \geq 0\}$, and the common directions of recession are of the form $(0, d_2)$ with $d_2 \geq 0$. For the example on the right, any unbounded sequence $\{x_k\}$ such that x_k is on the boundary of the set C_k is asymptotic but not retractive.

set sequences can be shown to be retractive. As an aid in this regard, we note that intersections and Cartesian products (involving a finite number of sets) preserve retractiveness, as can be easily seen from the definition. In particular, if $\{C_k^1\}, \ldots, \{C_k^r\}$ are retractive nested sequences of nonempty closed convex sets, the sequences $\{N_k\}$ and $\{T_k\}$ are retractive, where

$$N_k = C_k^1 \cap C_k^2 \cap \cdots \cap C_k^r, \qquad T_k = C_k^1 \times C_k^2 \times \cdots \times C_k^r, \qquad \forall\, k,$$

and we assume that all the sets N_k are nonempty.

The following proposition shows that a polyhedral set is retractive. Indeed, this is the most important type of retractive set for our purposes. Another retractive set of interest is the vector sum of a convex compact set and a polyhedral cone; we leave the proof of this for the reader. However, as a word of caution, we mention that a nonpolyhedral closed convex cone need not be retractive.

Proposition 1.4.9: A polyhedral set is retractive.

Proof: A closed halfspace is clearly retractive. A polyhedral set is the

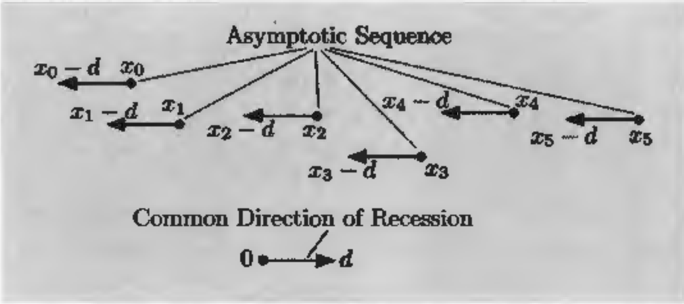


Figure 1.4.9. Geometric view of the proof idea of Prop. 1.4.10. An asymptotic sequence $\{x_k\}$ with corresponding direction of recession d eventually (for large k) gets closer to 0 when shifted by $-d$, so such a sequence cannot consist of the vectors of minimum norm from C_k without contradicting the retractiveness assumption.

nonempty intersection of a finite number of closed halfspaces, and set intersection preserves retractiveness. **Q.E.D.**

Set Intersection Theorems

The importance of retractive sequences is motivated by the following proposition.

Proposition 1.4.10: A retractive nested sequence of nonempty closed convex sets has nonempty intersection.

Proof: Let $\{C_k\}$ be the given sequence. For each k, let x_k be the vector of minimum norm in the closed set C_k (projection of the origin on C_k; cf. Prop. 1.1.9). The proof involves two ideas:

(a) The intersection $\cap_{k=0}^{\infty} C_k$ is empty if and only $\{x_k\}$ is unbounded, so there is a subsequence $\{x_k\}_{k\in\mathcal{K}}$ that is asymptotic.

(b) If a subsequence $\{x_k\}_{k\in\mathcal{K}}$ of minimum norm vectors of C_k is asymptotic with corresponding direction of recession d, then $\{x_k\}_{k\in\mathcal{K}}$ cannot be retractive, because x_k would eventually (for large k) get closer to 0 when shifted by $-d$ (see Fig. 1.4.9).

It will be sufficient to show that a subsequence $\{x_k\}_{k\in\mathcal{K}}$ is bounded. Then, since $\{C_k\}$ is nested, for each m, we have $x_k \in C_m$ for all $k \in \mathcal{K}$, $k \geq m$, and since C_m is closed, each of the limit points of $\{x_k\}_{k\in\mathcal{K}}$ will belong to each C_m and hence also to $\cap_{m=0}^{\infty} C_m$, thereby showing the result. Thus, we will prove the proposition by showing that there is no subsequence of $\{x_k\}$ that is unbounded.

Indeed, assume the contrary, let $\{x_k\}_{k\in\mathcal{K}}$ be a subsequence such that $\lim_{k\to\infty,\, k\in\mathcal{K}} \|x_k\| = \infty$, and let d be the limit of a subsequence $\{x_k/\|x_k\|\}_{k\in\overline{\mathcal{K}}}$, where $\overline{\mathcal{K}} \subset \mathcal{K}$. For each $k = 0, 1, \ldots$, define $z_k = x_m$, where m is the smallest index $m \in \overline{\mathcal{K}}$ with $k \leq m$. Then, since $z_k \in C_k$ for all k and $\lim_{k\to\infty}\{z_k/\|z_k\|\} = d$, we see that d is a common direction of recession of C_k [cf. the proof of Prop. 1.4.1(b)] and $\{z_k\}$ is an asymptotic sequence corresponding to d. Using the retractiveness assumption, let $\overline{k}$ be such that $z_k - d \in C_k$ for all $k \geq \overline{k}$. We have $d'z_k \to \infty$ since

$$\frac{d'z_k}{\|z_k\|} \to \|d\|^2 = 1,$$

so for all $k \geq \overline{k}$ with $2d'z_k > 1$, we obtain

$$\|z_k - d\|^2 = \|z_k\|^2 - (2d'z_k - 1) < \|z_k\|^2.$$

This is a contradiction, since for infinitely many k, z_k is the vector of minimum norm on C_k. **Q.E.D.**

For an example, consider the sequence $\{C_k\}$ of Fig. 1.4.8(a). Here the asymptotic sequences $\{(x_k^1, x_k^2)\}$ satisfy $x_k^1 \to 0$, $x_k^2 \to \infty$ and are retractive, and indeed the intersection $\cap_{k=0}^\infty C_k$ is nonempty. On the other hand, the condition for nonemptiness of $\cap_{k=0}^\infty C_k$ of the proposition is far from necessary, e.g., the sequence $\{C_k\}$ of Fig. 1.4.8(b) has nonempty intersection but is not retractive.

A simple example where the preceding proposition applies is a "cylindrical" set sequence, where $R_{C_k} \equiv L_{C_k} \equiv L$ for some subspace L. The following proposition gives an important extension.

Proposition 1.4.11: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. Denote

$$R = \cap_{k=0}^\infty R_{C_k}, \qquad L = \cap_{k=0}^\infty L_{C_k}.$$

(a) If $R = L$, then $\{C_k\}$ is retractive, and $\cap_{k=0}^\infty C_k$ is nonempty. Furthermore,

$$\cap_{k=0}^\infty C_k = L + \tilde{C},$$

where $\tilde{C}$ is some nonempty and compact set.

(b) Let X be a retractive closed convex set. Assume that all the sets $\overline{C}_k = X \cap C_k$ are nonempty, and that

$$R_X \cap R \subset L.$$

Then, $\{\overline{C}_k\}$ is retractive, and $\cap_{k=0}^\infty \overline{C}_k$ is nonempty.

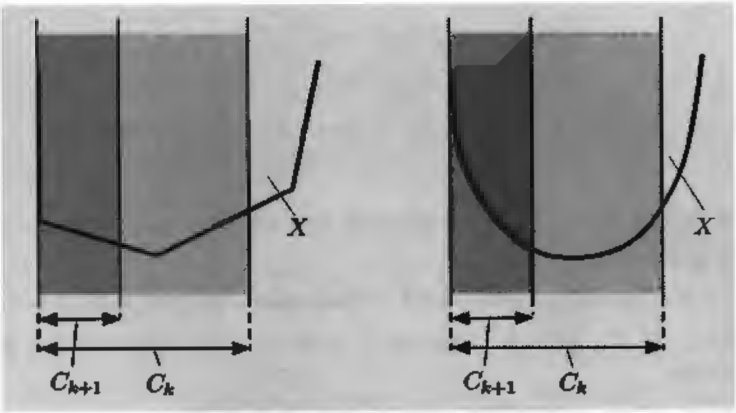


Figure 1.4.10. Illustration of the need to assume that X is retractive in Prop. 1.4.11(b). Here the intersection $\cap_{k=0}^{\infty} C_k$ is equal to the left vertical line. In the figure on the left, X is polyhedral, and the intersection $X \cap \left(\cap_{k=0}^{\infty} C_k\right)$ is nonempty. In the figure on the right, X is nonpolyhedral and nonretractive, and the intersection $X \cap \left(\cap_{k=0}^{\infty} C_k\right)$ is empty.

Proof: (a) The retractiveness of $\{C_k\}$ and consequent nonemptiness of $\cap_{k=0}^{\infty} C_k$ is the special case of part (b) where $X = \Re^n$. To show the given form of $\cap_{k=0}^{\infty} C_k$, we use the decomposition of Prop. 1.4.4, to obtain $\cap_{k=0}^{\infty} C_k = L + \tilde{C}$, where

$$\tilde{C} = (\cap_{k=0}^{\infty} C_k) \cap L^{\perp}.$$

The recession cone of $\tilde{C}$ is $R \cap L^{\perp}$, and since $R = L$, it is equal to $\{0\}$. Hence by Prop. 1.4.2(a), $\tilde{C}$ is compact.

(b) The common directions of recession of $\overline{C}_k$ are those in $R_X \cap R$, so by the hypothesis they must belong to L. Thus, for any asymptotic sequence $\{x_k\}$ of $\{\overline{C}_k\}$, corresponding to $d \in R_X \cap R$, we have $d \in L$, and hence $x_k - d \in C_k$ for all k. Since X is retractive, we also have $x_k - d \in X$ and hence $x_k - d \in \overline{C}_k$, for sufficiently large k. Hence $\{x_k\}$ is retractive, so $\{\overline{C}_k\}$ is retractive, and by Prop. 1.4.10, $\cap_{k=0}^{\infty} \overline{C}_k$ is nonempty. **Q.E.D.**

Figure 1.4.10 illustrates the need to assume that X is retractive in Prop. 1.4.11(b). The following is an important application of the preceding set intersection result.

Proposition 1.4.12: (Existence of Solutions of Convex Quadratic Programs) Let Q be a symmetric positive semidefinite $n \times n$ matrix, let c and $a_1, \ldots, a_r$ be vectors in $\Re^n$, and let $b_1, \ldots, b_r$ be scalars. Assume that the optimal value of the problem

$$\begin{aligned} &\text{minimize} \;\; x'Qx + c'x \\ &\text{subject to} \;\; a_j'x \le b_j, \quad j = 1, \ldots, r, \end{aligned}$$

is finite. Then the problem has at least one optimal solution.

Proof: Let f denote the cost function and let X be the polyhedral set of feasible solutions:

$$f(x) = x'Qx + c'x, \qquad X = \{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\}.$$

Let also f^* be the optimal value, let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and denote

$$\overline{C}_k = X \cap \{x \mid x'Qx + c'x \le \gamma_k\}.$$

We will use Prop. 1.4.11(b) to show that the set of optimal solutions, i.e., the intersection $\cap_{k=0}^{\infty} \overline{C}_k$, is nonempty. Indeed, let R_X be the recession cone of X, and let R and L be the common recession cone and lineality space of the sets $\{x \in \Re^n \mid x'Qx + c'x \le \gamma_k\}$ (i.e., the recession cone and constancy space of f). By Example 1.4.1, we have

$$R = \{d \mid Qd = 0,\ c'd \le 0\}, \qquad L = \{d \mid Qd = 0,\ c'd = 0\},$$

$$R_X = \{d \mid a_j'd \le 0,\ j = 1, \ldots, r\}.$$

If d is such that $d \in R_X \cap R$ but $d \notin L$, then

$$Qd = 0, \qquad c'd < 0, \qquad a_j'd \le 0, \quad j = 1, \ldots, r,$$

which implies that for any $x \in X$, we have $x + \alpha d \in X$ for all $\alpha \ge 0$, while $f(x + \alpha d) \to -\infty$ as $\alpha \to \infty$. This contradicts the finiteness of f^*, and shows that $R_X \cap R \subset L$. The nonemptiness of $\cap_{k=0}^{\infty} \overline{C}_k$ now follows from Prop. 1.4.11(b). **Q.E.D.**

1.4.3 Closedness Under Linear Transformations

The conditions just obtained regarding the nonemptiness of the intersection of a sequence of closed convex sets can be translated to conditions guaranteeing the closedness of the image, AC, of a closed convex set C under a linear transformation A. This is the subject of the following proposition.

Proposition 1.4.13: Let X and C be nonempty closed convex sets in $\Re^n$, and let A be an $m \times n$ matrix with nullspace denoted by $N(A)$. If X is a retractive closed convex set and

$$R_X \cap R_C \cap N(A) \subset L_C,$$

then $A(X \cap C)$ is a closed set.

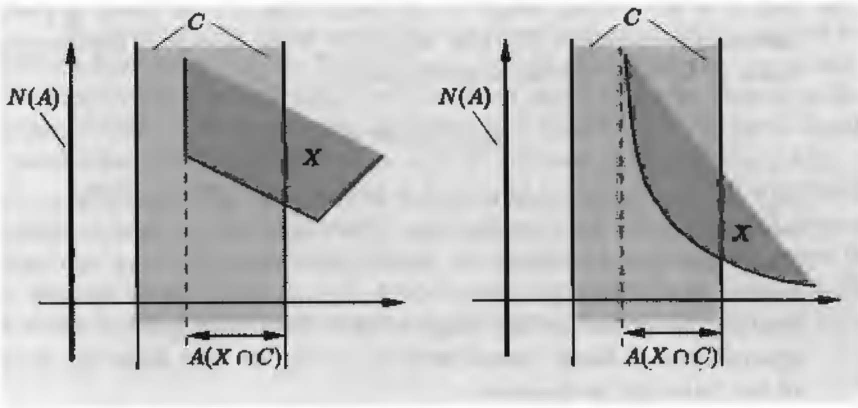


Figure 1.4.11. Illustration of the need to assume that the set X is retractive in Prop. 1.4.13. In both examples shown, the matrix A is the projection onto the horizontal axis, and its nullspace is the vertical axis. The condition $R_X \cap R_C \cap N(A) \subset L_C$ is satisfied. However, in the example on the right, X is not retractive, and the set $A(X \cap C)$ is not closed.

Proof: Let $\{y_k\}$ be a sequence in $A(X \cap C)$ converging to some $\overline{y}$. We will prove that $A(X \cap C)$ is closed by showing that $\overline{y} \in A(X \cap C)$. We introduce the sets

$$C_k = C \cap N_k,$$

where

$$N_k = \{x \mid \|Ax - \overline{y}\| \leq \|y_k - \overline{y}\|\},$$

(see Fig. 1.4.7). The sets C_k are closed and convex, and their (common) recession cones and lineality spaces are $R_C \cap N(A)$ and $L_C \cap N(A)$, respectively [cf. Props. 1.4.2(d) and 1.4.3(d)]. Therefore, by Prop. 1.4.11(b), the intersection $X \cap \left(\cap_{k=0}^{\infty} C_k\right)$ is nonempty. Every point x in this intersection is such that $x \in X \cap C$ and $Ax = \overline{y}$, showing that $\overline{y} \in A(X \cap C)$. **Q.E.D.**

Figure 1.4.11 illustrates the need for the assumptions of Prop. 1.4.13. The proposition has some interesting special cases:

(a) Let $C = \Re^n$ and let X be a polyhedral set. Then, $L_C = \Re^n$ and the assumption of Prop. 1.4.13 is automatically satisfied, so it follows that AX is closed. Thus *the image of a polyhedral set under a linear transformation is a closed set.* Simple as this result may seem, it is especially important in optimization. For example, as a special case, it yields that the cone generated by vectors $a_1, \ldots, a_r$ is a closed set, since it can be written as AC, where A is the matrix with columns $a_1, \ldots, a_r$ and C is the polyhedral set of all $(\alpha_1, \ldots, \alpha_r)$ with $\alpha_j \geq 0$ for all j. This fact is central in the proof of Farkas' Lemma, an important result given in Section 2.3.1.

(b) Let $X = \Re^n$. Then, Prop. 1.4.13 yields that AC *is closed if every direction of recession of* C *that belongs to* $N(A)$ *belongs to the lineality space of* C. This is true in particular if

$$R_C \cap N(A) = \{0\},$$

i.e., *there is no nonzero direction of recession of* C *that lies in the nullspace of* A. As a special case, this result can be used to obtain conditions that guarantee the closedness of the vector sum of closed convex sets. The idea is that the vector sum of a finite number of sets can be viewed as the image of their Cartesian product under a special type of linear transformation, as can be seen from the proof of the following proposition.

Proposition 1.4.14: Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of $\Re^n$ such that the equality $d_1 + \cdots + d_m = 0$ for some vectors $d_i \in R_{C_i}$ implies that $d_i \in L_{C_i}$ for all $i = 1, \ldots, m$. Then $C_1 + \cdots + C_m$ is a closed set.

Proof: Let C be the Cartesian product $C_1 \times \cdots \times C_m$. Then, C is closed convex, and its recession cone and lineality space are given by

$$R_C = R_{C_1} \times \cdots \times R_{C_m}, \qquad L_C = L_{C_1} \times \cdots \times L_{C_m}.$$

Let A be the linear transformation that maps $(x_1, \ldots, x_m) \in \Re^{mn}$ into $x_1 + \cdots + x_m \in \Re^n$. The null space of A is the set of all $(d_1, \ldots, d_m)$ such that $d_1 + \cdots + d_m = 0$. The intersection $R_C \cap N(A)$ consists of all $(d_1, \ldots, d_m)$ such that $d_1 + \cdots + d_m = 0$ and $d_i \in R_{C_i}$ for all i. By the given condition, every $(d_1, \ldots, d_m) \in R_C \cap N(A)$ is such that $d_i \in L_{C_i}$ for all i, implying that $(d_1, \ldots, d_m) \in L_C$. Thus, $R_C \cap N(A) \subset L_C$, and by Prop. 1.4.13, the set AC is closed. Since

$$AC = C_1 + \cdots + C_m,$$

the result follows. **Q.E.D.**

When specialized to just two sets, the above proposition implies that if C_1 and $-C_2$ are closed convex sets, then $C_1 - C_2$ is closed if there is no common nonzero direction of recession of C_1 and C_2, i.e.

$$R_{C_1} \cap R_{C_2} = \{0\}.$$

This is true in particular if either C_1 or C_2 is bounded, in which case either $R_{C_1} = \{0\}$ or $R_{C_2} = \{0\}$, respectively.

Some other conditions asserting the closedness of vector sums can be derived from Prop. 1.4.13. For example, we can show that the vector sum of a finite number of polyhedral sets is closed, since it can be viewed as the image of their Cartesian product (clearly a polyhedral set) under a linear transformation (in fact this vector sum is polyhedral; see Section 2.3.2).

Another useful result is that if X is a polyhedral set, and C is a closed convex set, then $X + C$ is closed if every direction of recession of X whose opposite is a direction of recession of C lies also in the lineality space of C (replace X and C by $X \times \Re^n$ and $\Re^n \times C$, respectively, in Prop. 1.4.13, and let A map Cartesian product to sum as in the proof of Prop. 1.4.14).

1.5 HYPERPLANES

Some of the most important principles in convexity and optimization, including duality, revolve around the use of hyperplanes, i.e., $(n-1)$-dimensional affine sets, which divide $\Re^n$ into two halfspaces. For example, we will see that a closed convex set can be characterized in terms of hyperplanes: it is equal to the intersection of all the halfspaces that contain it. In the next section, we will apply this fundamental result to a convex function via its epigraph, and obtain an important dual description, encoded by another convex function, called the conjugate of the original.

A *hyperplane* in $\Re^n$ is a set of the form $\{x \mid a'x = b\}$, where a is nonzero vector in $\Re^n$ and b is a scalar. If $\overline{x}$ is any vector in a hyperplane $H = \{x \mid a'x = b\}$, then we must have $a'\overline{x} = b$, so the hyperplane can be equivalently described as

$$H = \{x \mid a'x = a'\overline{x}\},$$

or

$$H = \overline{x} + \{x \mid a'x = 0\}.$$

Thus, H is an affine set that is parallel to the subspace $\{x \mid a'x = 0\}$. The vector a is orthogonal to this subspace, and consequently, a is called the *normal* vector of H; see Fig. 1.5.1.

The sets

$$\{x \mid a'x \geq b\}, \qquad \{x \mid a'x \leq b\},$$

are called the *closed halfspaces* associated with the hyperplane (also referred to as the *positive and negative halfspaces*, respectively). The sets

$$\{x \mid a'x > b\}, \qquad \{x \mid a'x < b\},$$

are called the *open halfspaces* associated with the hyperplane.

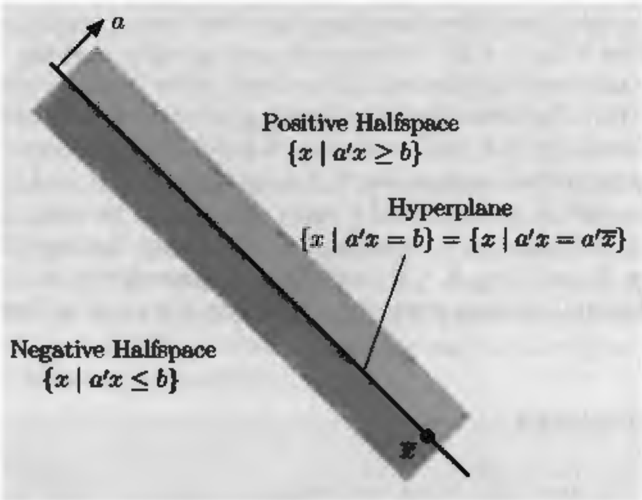


Figure 1.5.1. Illustration of the hyperplane $H = \{x \mid a'x = b\}$. If $\overline{x}$ is any vector in the hyperplane, then the hyperplane can be equivalently described as

$$H = \{x \mid a'x = a'\overline{x}\} = \overline{x} + \{x \mid a'x = 0\}.$$

The hyperplane divides the space into two halfspaces as illustrated.

1.5.1 Hyperplane Separation

We say that two sets C_1 and C_2 are *separated by a hyperplane* $H = \{x \mid a'x = b\}$ if each lies in a different closed halfspace associated with H, i.e., if either

$$a'x_1 \leq b \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2,$$

or

$$a'x_2 \leq b \leq a'x_1, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2.$$

We then also say that the hyperplane H *separates* C_1 and C_2, or that H is a *separating hyperplane* of C_1 and C_2. We use several different variants of this terminology. For example, the statement that two sets C_1 and C_2 *can be separated by a hyperplane* or that *there exists a hyperplane separating* C_1 and C_2, means that there exists a vector $a \neq 0$ such that

$$\sup_{x \in C_1} a'x \leq \inf_{x \in C_2} a'x;$$

[see Fig. 1.5.2(a)].

If a vector $\overline{x}$ belongs to the closure of a set C, a hyperplane that separates C and the singleton set $\{\overline{x}\}$ is said to be *supporting* C *at* $\overline{x}$. Thus

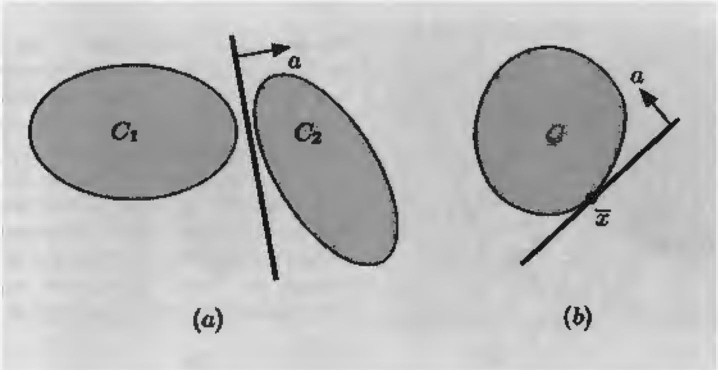


Figure 1.5.2. (a) Illustration of a hyperplane separating two sets C_1 and C_2. (b) Illustration of a hyperplane supporting a set C at a point $\overline{x}$ that belongs to the closure of C.

the statement that *there exists a supporting hyperplane of* C *at* $\overline{x}$ means that there exists a vector $a \neq 0$ such that

$$a'\overline{x} \leq a'x, \qquad \forall\, x \in C,$$

or equivalently, since $\overline{x}$ is a closure point of C,

$$a'\overline{x} = \inf_{x \in C} a'x.$$

As illustrated in Fig. 1.5.2(b), a supporting hyperplane of C is a hyperplane that "just touches" C.

We will prove several results regarding the existence of hyperplanes that separate two convex sets. Some of these results assert the existence of separating hyperplanes with special properties that will prove useful in various specialized contexts to be described later. The following proposition deals with the basic case where one of the two sets consists of a single vector. The proof is based on the Projection Theorem (Prop. 1.1.9) and is illustrated in Fig. 1.5.3.

Proposition 1.5.1: (Supporting Hyperplane Theorem) Let C be a nonempty convex subset of $\Re^n$ and let $\overline{x}$ be a vector in $\Re^n$. If $\overline{x}$ is not an interior point of C, there exists a hyperplane that passes through $\overline{x}$ and contains C in one of its closed halfspaces, i.e., there exists a vector $a \neq 0$ such that

$$a'\overline{x} \leq a'x, \qquad \forall\, x \in C. \tag{1.27}$$

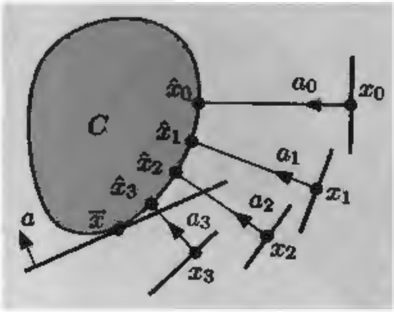


Figure 1.5.3. Illustration of the proof of the Supporting Hyperplane Theorem for the case where the vector $\overline{x}$ belongs to $\mathrm{cl}(C)$, the closure of C. We choose a sequence $\{x_k\}$ of vectors that do not belong to $\mathrm{cl}(C)$, with $x_k \to \overline{x}$, and we project x_k on $\mathrm{cl}(C)$. We then consider, for each k, the hyperplane that is orthogonal to the line segment connecting x_k and its projection $\hat{x}_k$, and passes through x_k. These hyperplanes "converge" to a hyperplane that supports C at $\overline{x}$.

Proof: Consider $\mathrm{cl}(C)$, the closure of C, which is a convex set by Prop. 1.1.1(d). Let $\{x_k\}$ be a sequence of vectors such that $x_k \to \overline{x}$ and $x_k \notin \mathrm{cl}(C)$ for all k; such a sequence exists because $\overline{x}$ does not belong to the interior of C and hence does not belong to the interior of $\mathrm{cl}(C)$ [cf. Prop. 1.3.5(b)]. If $\hat{x}_k$ is the projection of x_k on $\mathrm{cl}(C)$, we have by the optimality condition of the Projection Theorem (Prop. 1.1.9)

$$(\hat{x}_k - x_k)'(x - \hat{x}_k) \geq 0, \qquad \forall\, x \in \mathrm{cl}(C).$$

Hence we obtain for all $x \in \mathrm{cl}(C)$ and all k,

$$(\hat{x}_k - x_k)'x \geq (\hat{x}_k - x_k)'\hat{x}_k = (\hat{x}_k - x_k)'(\hat{x}_k - x_k) + (\hat{x}_k - x_k)'x_k \geq (\hat{x}_k - x_k)'x_k.$$

We can write this inequality as

$$a_k'x \geq a_k'x_k, \qquad \forall\, x \in \mathrm{cl}(C),\ \forall\, k, \tag{1.28}$$

where

$$a_k = \frac{\hat{x}_k - x_k}{\|\hat{x}_k - x_k\|}.$$

We have $\|a_k\| = 1$ for all k, so the sequence $\{a_k\}$ has a subsequence that converges to some $a \neq 0$. By considering Eq. (1.28) for all a_k belonging to this subsequence and by taking the limit as $k \to \infty$, we obtain Eq. (1.27). **Q.E.D.**

Note that if $\overline{x}$ is a closure point of C, then the hyperplane of the preceding proposition supports C at $\overline{x}$. Note also that if C has empty interior, then any vector $\overline{x}$ can be separated from C as in the proposition.

Proposition 1.5.2: (Separating Hyperplane Theorem) Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. If C_1 and C_2 are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2. \tag{1.29}$$

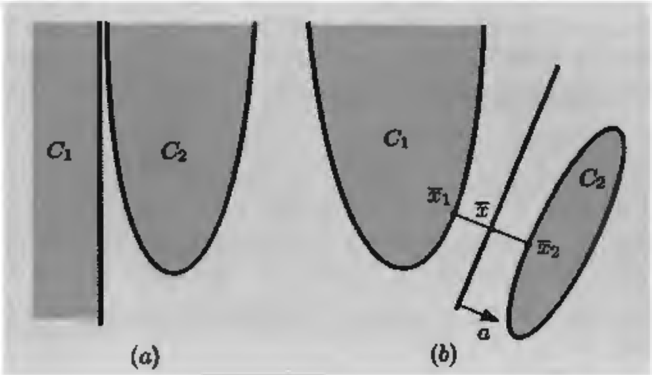


Figure 1.5.4. (a) An example of two disjoint convex sets that cannot be strictly separated. (b) Illustration of the construction of a strictly separating hyperplane.

Proof: Consider the convex set

$$C = C_2 - C_1 = \{x \mid x = x_2 - x_1,\ x_1 \in C_1,\ x_2 \in C_2\}.$$

Since C_1 and C_2 are disjoint, the origin does not belong to C, so by the Supporting Hyperplane Theorem (Prop. 1.5.1), there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \qquad \forall\ x \in C,$$

which is equivalent to Eq. (1.29). **Q.E.D.**

We next consider a stronger form of separation of two sets C_1 and C_2 in $\Re^n$. We say that a hyperplane $\{x \mid a'x = b\}$ *strictly separates C_1 and C_2* if it separates C_1 and C_2 while containing neither a point of C_1 nor a point of C_2, i.e.,

$$a'x_1 < b < a'x_2, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2,$$

or

$$a'x_2 < b < a'x_1, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2.$$

Clearly, C_1 and C_2 must be disjoint in order that they can be strictly separated. However, this is not sufficient to guarantee strict separation (see Fig. 1.5.4). The following proposition provides conditions that guarantee the existence of a strictly separating hyperplane.

Proposition 1.5.3: (Strict Separation Theorem) Let C_1 and C_2 be two disjoint nonempty convex sets. There exists a hyperplane that strictly separates C_1 and C_2 under any one of the following five conditions:

(1) $C_2 - C_1$ is closed.

(2) C_1 is closed and C_2 is compact.

(3) C_1 and C_2 are polyhedral.

(4) C_1 and C_2 are closed, and

$$R_{C_1} \cap R_{C_2} = L_{C_1} \cap L_{C_2},$$

where R_{C_i} and L_{C_i} denote the recession cone and the lineality space of C_i, $i = 1, 2$.

(5) C_1 is closed, C_2 is polyhedral, and $R_{C_1} \cap R_{C_2} \subset L_{C_1}$.

Proof: We will show the result under condition (1). The result will then follow under conditions (2)-(5), because these conditions imply condition (1) (see Prop. 1.4.14, and the discussion following its proof).

Assume that $C_2 - C_1$ is closed, and consider the vector of minimum norm (projection of the origin, cf. Prop. 1.1.9) in $C_2 - C_1$. This vector is of the form $\overline{x}_2 - \overline{x}_1$, where $\overline{x}_1 \in C_1$ and $\overline{x}_2 \in C_2$. Let

$$a = \frac{\overline{x}_2 - \overline{x}_1}{2}, \qquad \overline{x} = \frac{\overline{x}_1 + \overline{x}_2}{2}, \qquad b = a'\overline{x};$$

[cf. Fig. 1.5.4(b)]. Then, $a \neq 0$, since $\overline{x}_1 \in C_1$, $\overline{x}_2 \in C_2$, and C_1 and C_2 are disjoint. We will show that the hyperplane

$$\{x \mid a'x = b\}$$

strictly separates C_1 and C_2, i.e., that

$$a'x_1 < b < a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2. \tag{1.30}$$

To this end we note that $\overline{x}_1$ is the projection of $\overline{x}_2$ on $\mathrm{cl}(C_1)$ (otherwise there would exist a vector $x_1 \in C_1$ with $\|\overline{x}_2 - x_1\| < \|\overline{x}_2 - \overline{x}_1\|$ - a contradiction of the minimum norm property of $\overline{x}_2 - \overline{x}_1$). Thus, we have

$$(\overline{x}_2 - \overline{x}_1)'(x_1 - \overline{x}_1) \leq 0, \qquad \forall\, x_1 \in C_1,$$

or equivalently, since $\overline{x} - \overline{x}_1 = a$,

$$a'x_1 \leq a'\overline{x}_1 = a'\overline{x} + a'(\overline{x}_1 - \overline{x}) = b - \|a\|^2 < b, \qquad \forall\, x_1 \in C_1.$$

Thus, the left-hand side of Eq. (1.30) is proved. The right-hand side is proved similarly. **Q.E.D.**

Note that as a corollary of the preceding proposition, a closed set C can be strictly separated from a vector $\overline{x} \notin C$, i.e., from the singleton set $\{\overline{x}\}$. We will use this fact to provide the following important characterization of closed convex sets.

Proposition 1.5.4: The closure of the convex hull of a set C is the intersection of the closed halfspaces that contain C. In particular, a closed convex set is the intersection of the closed halfspaces that contain it.

Proof: Let H denote the intersection of all closed halfspaces that contain C. Since every closed halfspace containing C must also contain $\mathrm{cl}\big(\mathrm{conv}(C)\big)$, it follows that $H \supset \mathrm{cl}\big(\mathrm{conv}(C)\big)$.

To show the reverse inclusion, consider a vector $x \notin \mathrm{cl}\big(\mathrm{conv}(C)\big)$ and a hyperplane strictly separating x and $\mathrm{cl}\big(\mathrm{conv}(C)\big)$. The corresponding closed halfspace that contains $\mathrm{cl}\big(\mathrm{conv}(C)\big)$ does not contain x, so $x \notin H$. Hence $H \subset \mathrm{cl}\big(\mathrm{conv}(C)\big)$. **Q.E.D.**

1.5.2 Proper Hyperplane Separation

We now discuss a form of hyperplane separation, called *proper*, which turns out to be useful in some important optimization contexts, such as the duality theorems of Chapter 4 (Props. 4.4.1 and 4.5.1).

Let C_1 and C_2 be two subsets of $\Re^n$. We say that a hyperplane *properly separates* C_1 *and* C_2 if it separates C_1 and C_2, and does not fully contain both C_1 and C_2. Thus there exists a hyperplane that properly separates C_1 and C_2 if and only if there is a vector a such that

$$\sup_{x_1 \in C_1} a'x_1 \leq \inf_{x_2 \in C_2} a'x_2, \qquad \inf_{x_1 \in C_1} a'x_1 < \sup_{x_2 \in C_2} a'x_2;$$

(see Fig. 1.5.5). If C is a subset of $\Re^n$ and $\overline{x}$ is a vector in $\Re^n$, we say that a hyperplane *properly separates* C *and* $\overline{x}$ if it properly separates C and the singleton set $\{\overline{x}\}$.

Note that a convex set in $\Re^n$ that has nonempty interior (and hence has dimension n) cannot be fully contained in a hyperplane (which has dimension $n - 1$). Thus, in view of the Separating Hyperplane Theorem (Prop. 1.5.2), two disjoint convex sets one of which has nonempty interior can be properly separated. Similarly and more generally, two disjoint convex sets such that the affine hull of their union has dimension n can be properly separated. Figure 1.5.5(c) provides an example of two convex sets that cannot be properly separated.

The existence of a hyperplane that properly separates two convex sets is intimately tied to conditions involving the relative interiors of the sets.

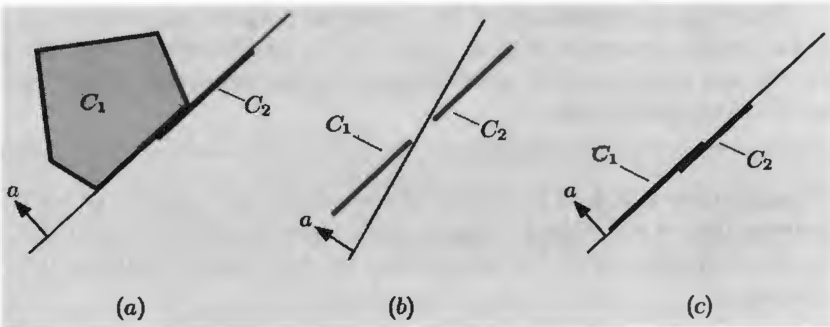


Figure 1.5.5. (a) and (b) Illustration of a properly separating hyperplanes. (c) Illustration of two convex sets that cannot be properly separated.

An important fact in this connection is that given a nonempty convex set C and a hyperplane H that contains C in one of its closed halfspaces, we have

$$C \subset H \qquad \text{if and only if} \qquad \mathrm{ri}(C) \cap H \neq \emptyset. \tag{1.31}$$

To see this, let H be of the form $\{x \mid a'x = b\}$ with $a'x \geq b$ for all $x \in C$. Then for a vector $\overline{x} \in \mathrm{ri}(C)$, we have $\overline{x} \in H$ if and only if $a'\overline{x} = b$, i.e., $a'x$ attains its minimum over C at $\overline{x}$. By Prop. 1.3.4, this is so if and only if $a'x = b$ for all $x \in C$, i.e., $C \subset H$.

The following propositions provide relative interior assumptions that guarantee the existence of properly separating hyperplanes.

> **Proposition 1.5.5: (Proper Separation Theorem)** Let C be a nonempty convex subset of $\Re^n$ and let $\overline{x}$ be a vector in $\Re^n$. There exists a hyperplane that properly separates C and $\overline{x}$ if and only if $\overline{x} \notin \mathrm{ri}(C)$.

Proof: Suppose that there exists a hyperplane H that properly separates C and $\overline{x}$. Then either $\overline{x} \notin H$, in which case $\overline{x} \notin C$ and $\overline{x} \notin \mathrm{ri}(C)$, or else $\overline{x} \in H$ and C is not contained in H, in which case by Eq. (1.31), $\mathrm{ri}(C) \cap H = \emptyset$, in which case again $\overline{x} \notin \mathrm{ri}(C)$.

Conversely, assume that $\overline{x} \notin \mathrm{ri}(C)$. To show the existence of a properly separating hyperplane, we consider two cases (see Fig. 1.5.6):

(a) $\overline{x} \notin \mathrm{aff}(C)$. In this case, since $\mathrm{aff}(C)$ is closed and convex, by the Strict Separation Theorem [Prop. 1.5.3 under condition (2)] there exists a hyperplane that separates $\{\overline{x}\}$ and $\mathrm{aff}(C)$ strictly, and hence also properly separates C and $\overline{x}$.

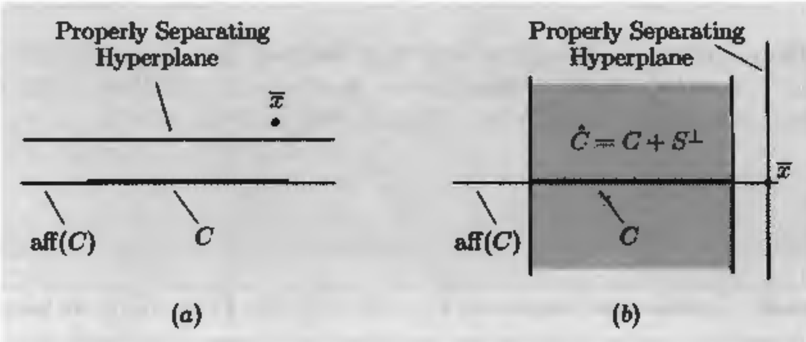


Figure 1.5.6. Illustration of the construction of a hyperplane that properly separates a convex set C and a point $\overline{x} \notin \mathrm{ri}(C)$ (cf. the proof of Prop. 1.5.5). In case (a), where $\overline{x} \notin \mathrm{aff}(C)$, the hyperplane is constructed as shown. In case (b), where $\overline{x} \in \mathrm{aff}(C)$, we consider the subspace S that is parallel to $\mathrm{aff}(C)$, we set $\hat{C} = C + S^{\perp}$, and we use the Supporting Hyperplane Theorem to separate $\overline{x}$ from $\hat{C}$ (Prop. 1.5.1).

(b) $\overline{x} \in \mathrm{aff}(C)$. In this case, let S be the subspace that is parallel to $\mathrm{aff}(C)$, and consider the set $\hat{C} = C + S^{\perp}$. From Prop. 1.3.7, we have $\mathrm{ri}(\hat{C}) = \mathrm{ri}(C) + S^{\perp}$, so that $\overline{x}$ is not an interior point of $\hat{C}$ [otherwise there must exist a vector $x \in \mathrm{ri}(C)$ such that $x - \overline{x} \in S^{\perp}$, which, since $x \in \mathrm{aff}(C)$, $\overline{x} \in \mathrm{aff}(C)$, and $x - \overline{x} \in S$, implies that $x - \overline{x} = 0$, thereby contradicting the hypothesis $\overline{x} \notin \mathrm{ri}(C)$]. By the Supporting Hyperplane Theorem (Prop. 1.5.1), it follows that there exists a vector $a \neq 0$ such that $a'x \geq a'\overline{x}$ for all $x \in \hat{C}$. Since $\hat{C}$ has nonempty interior, $a'x$ cannot be constant over $\hat{C}$, and

$$a'\overline{x} < \sup_{x \in \hat{C}} a'x = \sup_{x \in C,\, z \in S^{\perp}} a'(x+z) = \sup_{x \in C} a'x + \sup_{z \in S^{\perp}} a'z. \tag{1.32}$$

If we had $a'\overline{z} \neq 0$ for some $\overline{z} \in S^{\perp}$, we would also have

$$\inf_{\alpha \in \Re} a'(x + \alpha \overline{z}) = -\infty,$$

which contradicts the fact $a'(x+z) \geq a'\overline{x}$ for all $x \in C$ and $z \in S^{\perp}$. It follows that

$$a'z = 0, \qquad \forall\, z \in S^{\perp},$$

which when combined with Eq. (1.32), yields

$$a'\overline{x} < \sup_{x \in C} a'x.$$

Thus the hyperplane $\{x \mid a'x = a'\overline{x}\}$ properly separates C and $\overline{x}$. **Q.E.D.**

Proposition 1.5.6: (Proper Separation of Two Convex Sets) Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. There exists a hyperplane that properly separates C_1 and C_2 if and only if

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset.$$

Proof: Consider the convex set $C = C_2 - C_1$. By Prop. 1.3.7, we have

$$\mathrm{ri}(C) = \mathrm{ri}(C_2) - \mathrm{ri}(C_1),$$

so the assumption $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$ is equivalent to $0 \notin \mathrm{ri}(C)$. By using Prop. 1.5.5, it follows that there exists a hyperplane properly separating C and the origin, so we have

$$0 \le \inf_{x_1 \in C_1,\, x_2 \in C_2} a'(x_2 - x_1), \qquad 0 < \sup_{x_1 \in C_1,\, x_2 \in C_2} a'(x_2 - x_1),$$

if and only if $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$. This is equivalent to the desired assertion. **Q.E.D.**

The following proposition is a variant of Prop. 1.5.6. It shows that if C_2 is polyhedral and the slightly stronger condition $\mathrm{ri}(C_1) \cap C_2 = \emptyset$ holds, then there exists a properly separating hyperplane satisfying the extra restriction that it does not contain the nonpolyhedral set C_1 (rather than just the milder requirement that it does not contain either C_1 or C_2); see Fig. 1.5.7.

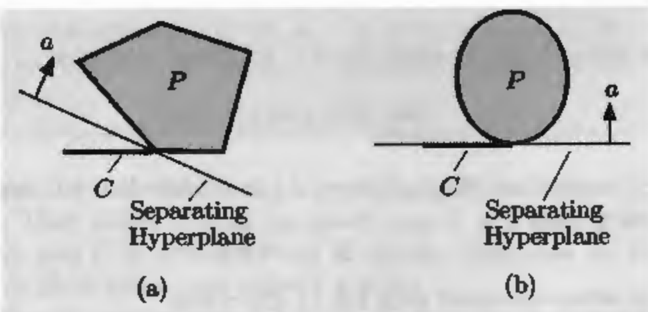


Figure 1.5.7. Illustration of the special proper separation property of a convex set C and a polyhedral set P, under the condition $\mathrm{ri}(C) \cap P = \emptyset$. In figure (a), the separating hyperplane can be chosen so that it does not contain C. If P is not polyhedral, as in figure (b), this may not be possible.

Proposition 1.5.7: (Polyhedral Proper Separation Theorem) Let C and P be two nonempty convex subsets of $\Re^n$ such that P is polyhedral. There exists a hyperplane that separates C and P, and does not contain C if and only if

$$\text{ri}(C) \cap P = \emptyset.$$

Proof: First, as a general observation, we recall from our discussion of proper separation that given a convex set X and a hyperplane H that contains X in one of its closed halfspaces, we have

$$X \subset H \qquad \text{if and only if} \qquad \text{ri}(X) \cap H \neq \emptyset; \tag{1.33}$$

cf. Eq. (1.31). We will use repeatedly this relation in the subsequent proof.

Assume that there exists a hyperplane H that separates C and P, and does not contain C. Then, by Eq. (1.33), H cannot contain a point in $\text{ri}(C)$, and since H separates C and P, we must have $\text{ri}(C) \cap P = \emptyset$.

Conversely, assume that $\text{ri}(C) \cap P = \emptyset$. We will show that there exists a separating hyperplane that does not contain C. Denote

$$D = P \cap \text{aff}(C).$$

If $D = \emptyset$, then since $\text{aff}(C)$ and P are polyhedral, the Strict Separation Theorem [cf. Prop. 1.5.3 under condition (3)] applies and shows that there exists a hyperplane H that separates $\text{aff}(C)$ and P strictly, and hence does not contain C.

We may thus assume that $D \neq \emptyset$. The idea now is to first construct a hyperplane that properly separates C and D, and then extend this hyperplane so that it suitably separates C and P. [If C had nonempty interior, the proof would be much simpler, since then $\text{aff}(C) = \Re^n$ and $D = P$.]

By assumption, we have $\text{ri}(C) \cap P = \emptyset$ implying that

$$\text{ri}(C) \cap \text{ri}(D) \subset \text{ri}(C) \cap \big(P \cap \text{aff}(C)\big) = \big(\text{ri}(C) \cap P\big) \cap \text{aff}(C) = \emptyset.$$

Hence, by Prop. 1.5.6, there exists a hyperplane H that properly separates C and D. Furthermore, H does not contain C, since if it did, H would also contain $\text{aff}(C)$ and hence also D, contradicting the proper separation property. Thus, C is contained in one of the closed halfspaces of H, but not in both. Let $\overline{C}$ be the intersection of $\text{aff}(C)$ and the closed halfspace of H that contains C; see Fig. 1.5.8. Note that H does not contain $\overline{C}$ (since H does not contain C), and by Eq. (1.33), we have $H \cap \text{ri}(\overline{C}) = \emptyset$, implying that

$$P \cap \text{ri}(\overline{C}) = \emptyset,$$

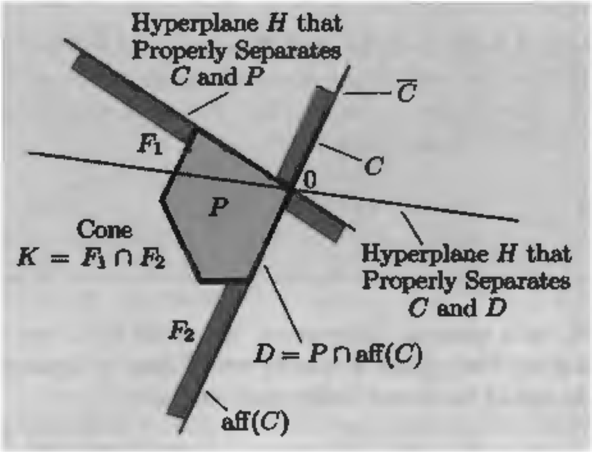


Figure 1.5.8. Illustration of the proof of Prop. 1.5.7 in the case where $D = P \cap \mathrm{aff}(C) \neq \emptyset$. The figure shows the construction of a hyperplane that properly separates C and P, and does not contain C, starting from the hyperplane H that properly separates C and D. In this two-dimensional example we have $M = \{0\}$, so $K = \mathrm{cone}(P) + M = \mathrm{cone}(P)$.

since P and $\overline{C}$ lie in the opposite closed halfspaces of H.

If $P \cap \overline{C} = \emptyset$, then by using again the Strict Separation Theorem [cf. Prop. 1.5.3 under condition (3)], we can construct a hyperplane that strictly separates P and $\overline{C}$. This hyperplane also strictly separates P and C, and we are done. We thus assume that $P \cap \overline{C} \neq \emptyset$, and by using a translation argument if necessary, we assume that

$$0 \in P \cap \overline{C},$$

as indicated in Fig. 1.5.8. The polyhedral set P can be represented as the intersection of halfspaces $\{x \mid a_j'x \leq b_j\}$ with $b_j \geq 0$ (since $0 \in P$) and with $b_j = 0$ for at least one j (since otherwise 0 would be in the interior of P, which is impossible since $0 \in H$ and P lies in a closed halfspace of H). Thus, we have

$$P = \{x \mid a_j'x \leq 0,\ j = 1, \ldots, m\} \cap \{x \mid a_j'x \leq b_j,\ j = m+1, \ldots, \overline{m}\},$$

for some integers $m \geq 1$ and $\overline{m} \geq m$, vectors a_j, and scalars $b_j > 0$.

Let M be the relative boundary of $\overline{C}$, i.e.,

$$M = H \cap \mathrm{aff}(C),$$

and consider the cone

$$K = \{x \mid a_j'x \leq 0,\ j = 1, \ldots, m\} + M.$$

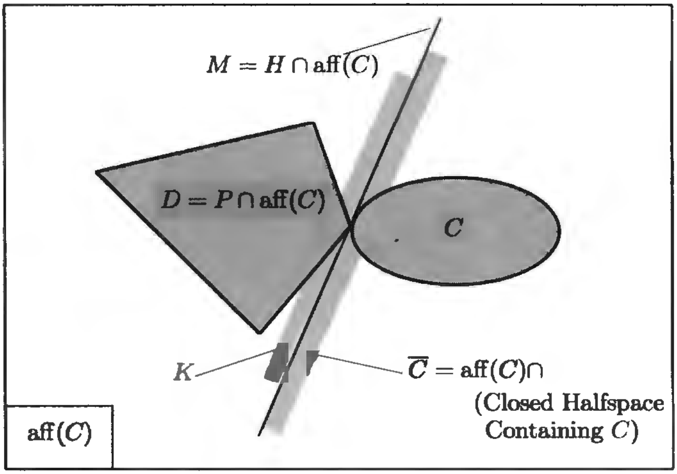


Figure 1.5.9. Illustration of the construction of the cone $K = \text{cone}(P) + M$ in the proof of Prop. 1.5.7.

Note that $K = \text{cone}(P) + M$ (see Figs. 1.5.8 and 1.5.9).

We claim that $K \cap \text{ri}(\overline{C}) = \emptyset$. The proof is by contradiction. If there exists $\overline{x} \in K \cap \text{ri}(\overline{C})$, then $\overline{x}$ can be expressed as $\overline{x} = \alpha w + v$ for some $\alpha > 0$, $w \in P$, and $v \in M$ [since $K = \text{cone}(P) + M$ and $0 \in P$], so that $(\overline{x}/\alpha) - (v/\alpha) \in P$. On the other hand, since $\overline{x} \in \text{ri}(\overline{C})$, $0 \in \overline{C}$, and M is a subset of the lineality space of $\overline{C}$ [and hence also of the lineality space of $\text{ri}(\overline{C})$], all vectors of the form $\overline{\alpha}\overline{x} + \overline{v}$, with $\overline{\alpha} > 0$ and $\overline{v} \in M$, belong to $\text{ri}(\overline{C})$, so in particular the vector $(\overline{x}/\alpha) - (v/\alpha)$ also belongs to $\text{ri}(\overline{C})$. This is a contradiction since $P \cap \text{ri}(\overline{C}) = \emptyset$, and it follows that $K \cap \text{ri}(\overline{C}) = \emptyset$.

The cone K is polyhedral (since it is the vector sum of two polyhedral sets), so it is the intersection of some closed halfspaces $F_1, \ldots, F_r$ that pass through 0 (cf. Fig. 1.5.8). Since $K = \text{cone}(P) + M$, each of these closed halfspaces contains M, the relative boundary of the set $\overline{C}$, and furthermore $\overline{C}$ is the closed half of a subspace. It follows that if any of the closed halfspaces $F_1, \ldots, F_r$ contains a vector in $\text{ri}(\overline{C})$, then that closed halfspace entirely contains $\overline{C}$. Hence, since K does not contain any point in $\text{ri}(\overline{C})$, at least one of $F_1, \ldots, F_r$, say F_1, does not contain any point in $\text{ri}(\overline{C})$ (cf. Fig. 1.5.8). Therefore, the hyperplane corresponding to F_1 contains no points of $\text{ri}(\overline{C})$, and hence also no points of $\text{ri}(C)$. Thus, this hyperplane does not contain C, while separating K and C. Since K contains P, this hyperplane also separates P and C. **Q.E.D.**

Note that in the preceding proof, it is essential to introduce M, the relative boundary of the set $\overline{C}$, and to define $K = \text{cone}(P) + M$. If instead we define $K = \text{cone}(P)$, then the corresponding halfspaces $F_1, \ldots, F_r$ may all intersect $\text{ri}(\overline{C})$, and the proof argument fails (see Fig. 1.5.9).

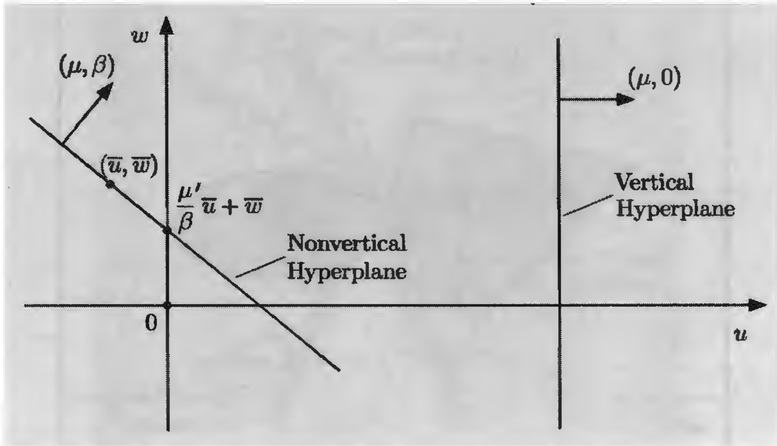


Figure 1.5.10. Illustration of vertical and nonvertical hyperplanes in $\Re^{n+1}$. A hyperplane with normal (μ,β) is nonvertical if $\beta \neq 0$, or, equivalently, if it intersects the $(n+1)$st axis at the unique point $\xi = (\mu/\beta)'\overline{u} + \overline{w}$, where $(\overline{u},\overline{w})$ is any vector on the hyperplane.

1.5.3 Nonvertical Hyperplane Separation

In the context of optimization, supporting hyperplanes are often used in conjunction with epigraphs of functions defined on $\Re^n$. Since the epigraph is a subset of $\Re^{n+1}$, we consider hyperplanes in $\Re^{n+1}$ and associate them with nonzero vectors of the form (μ,β), where $\mu \in \Re^n$ and $\beta \in \Re$. We say that such a hyperplane is *vertical* if $\beta = 0$.

Note that if a hyperplane with normal (μ,β) is nonvertical, then it crosses the $(n+1)$st axis (the axis associated with w) at a unique point. In particular, if $(\overline{u},\overline{w})$ is any vector on the hyperplane, the crossing point has the form $(0,\xi)$, where

$$\xi = \frac{\mu'}{\beta}\overline{u} + \overline{w},$$

since from the hyperplane equation, we have $(0,\xi)'(\mu,\beta) = (\overline{u},\overline{w})'(\mu,\beta)$. If the hyperplane is vertical, it either contains the entire $(n+1)$st axis, or else it does not cross it at all; see Fig. 1.5.10. Furthermore, a hyperplane H is vertical if and only if the recession cone of H, as well as the recession cones of the closed halfspaces associated with H, contain the $(n+1)$st axis.

Vertical lines in $\Re^{n+1}$ are sets of the form $\{(\overline{u},w) \mid w \in \Re\}$, where $\overline{u}$ is a fixed vector in $\Re^n$. If $f : \Re^n \mapsto (-\infty,\infty]$ is a proper convex function, then epi(f) cannot contain a vertical line, and it appears plausible that epi(f) is contained in a closed halfspace corresponding to some nonvertical hyperplane. We prove this fact in greater generality in the following proposition, which will be important for the development of duality.

Proposition 1.5.8: (Nonvertical Hyperplane Theorem) Let C be a nonempty convex subset of $\Re^{n+1}$ that contains no vertical lines. Let the vectors in $\Re^{n+1}$ be denoted by (u, w), where $u \in \Re^n$ and $w \in \Re$. Then:

(a) C is contained in a closed halfspace corresponding to a nonvertical hyperplane, i.e., there exist a vector $\mu \in \Re^n$, a scalar $\beta \neq 0$, and a scalar γ such that

$$\mu' u + \beta w \geq \gamma, \qquad \forall\ (u, w) \in C.$$

(b) If $(\overline{u}, \overline{w})$ does not belong to $\mathrm{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(\overline{u}, \overline{w})$ and C.

Proof: (a) Assume, to arrive at a contradiction, that every hyperplane containing C in one of its closed halfspaces is vertical. Then every hyperplane containing $\mathrm{cl}(C)$ in one of its closed halfspaces must also be vertical and its recession cone must contain the $(n+1)$st axis. By Prop. 1.5.4, $\mathrm{cl}(C)$ is the intersection of all closed halfspaces that contain it, so its recession cone contains the $(n+1)$st axis. Since the recession cones of $\mathrm{cl}(C)$ and $\mathrm{ri}(C)$ coincide [cf. Prop. 1.4.2(b)], for every $(\overline{u}, \overline{w}) \in \mathrm{ri}(C)$, the vertical line $\{(\overline{u}, w) \mid w \in \Re\}$ belongs to $\mathrm{ri}(C)$ and hence to C. This contradicts the assumption that C does not contain a vertical line.

(b) If $(\overline{u}, \overline{w}) \notin \mathrm{cl}(C)$, then there exists a hyperplane strictly separating $(\overline{u}, \overline{w})$ and $\mathrm{cl}(C)$ [cf. Prop. 1.5.3 under condition (2)]. If this hyperplane is nonvertical, since $C \subset \mathrm{cl}(C)$, we are done, so assume otherwise. Then, we have a nonzero vector $\overline{\mu}$ and a scalar $\overline{\gamma}$ such that

$$\overline{\mu}' u > \overline{\gamma} > \overline{\mu}' \overline{u}, \qquad \forall\ (u, w) \in \mathrm{cl}(C). \tag{1.34}$$

The idea now is to combine this vertical hyperplane with a suitable nonvertical hyperplane in order to construct a nonvertical hyperplane that strictly separates $(\overline{u}, \overline{w})$ from $\mathrm{cl}(C)$ (see Fig. 1.5.11).

Since, by assumption, C does not contain a vertical line, $\mathrm{ri}(C)$ also does not contain a vertical line. Since the recession cones of $\mathrm{cl}(C)$ and $\mathrm{ri}(C)$ coincide [cf. Prop. 1.4.2(b)], it follows that $\mathrm{cl}(C)$ does not contain a vertical line. Hence, by part (a), there exists a nonvertical hyperplane containing $\mathrm{cl}(C)$ in one of its closed halfspaces, so that for some (μ, β) and γ, with $\beta \neq 0$, we have

$$\mu' u + \beta w \geq \gamma, \qquad \forall\ (u, w) \in \mathrm{cl}(C).$$

By multiplying this relation with an $\epsilon > 0$ and combining it with Eq. (1.34), we obtain

$$(\overline{\mu} + \epsilon\mu)' u + \epsilon\beta w > \overline{\gamma} + \epsilon\gamma, \qquad \forall\ (u, w) \in \mathrm{cl}(C),\ \forall\ \epsilon > 0.$$

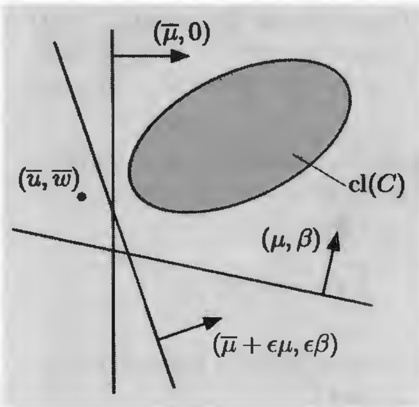


Figure 1.5.11. Construction of a strictly separating nonvertical hyperplane in the proof of Prop. 1.5.8(b).

Since $\overline{\gamma} > \overline{\mu}'\,\overline{u}$, there is a small enough ϵ such that

$$\overline{\gamma} + \epsilon\gamma > (\overline{\mu} + \epsilon\mu)'\overline{u} + \epsilon\beta\overline{w}.$$

From the above two relations, we obtain

$$(\overline{\mu} + \epsilon\mu)'u + \epsilon\beta w > (\overline{\mu} + \epsilon\mu)'\overline{u} + \epsilon\beta\overline{w}, \qquad \forall\ (u, w) \in \mathrm{cl}(C),$$

implying that there is a nonvertical hyperplane with normal $(\overline{\mu} + \epsilon\mu, \epsilon\beta)$ that strictly separates $(\overline{u}, \overline{w})$ and $\mathrm{cl}(C)$. Since $C \subset \mathrm{cl}(C)$, this hyperplane also strictly separates $(\overline{u}, \overline{w})$ and C. **Q.E.D.**

1.6 CONJUGATE FUNCTIONS

We will now develop a concept that is fundamental in convex optimization. This is the conjugacy transformation, which associates with any function f, a convex function, called the conjugate of f. The idea here is to describe f in terms of the affine functions that are majorized by f. When f is closed proper convex, we will show that the description is accurate and the transformation is symmetric, i.e., f can be recovered by taking the conjugate of the conjugate of f. The conjugacy transformation thus provides an alternative view of a convex function, which often reveals interesting properties, and is useful for analysis and computation.

Consider an extended real-valued function $f : \Re^n \mapsto [-\infty, \infty]$. The *conjugate function* of f is the function $f^\star : \Re^n \mapsto [-\infty, \infty]$ defined by

$$f^\star(y) = \sup_{x \in \Re^n} \big\{x'y - f(x)\big\}, \qquad y \in \Re^n. \tag{1.35}$$

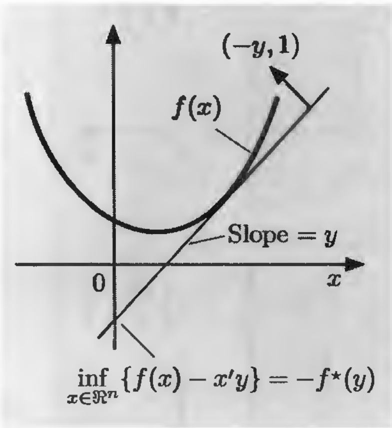


Figure 1.6.1. Visualization of the conjugate function

$$f^\star(y) = \sup_{x \in \Re^n} \big\{x'y - f(x)\big\}$$

of a function f. The crossing point of the vertical axis with the hyperplane that has normal $(-y, 1)$ and passes through a point $\big(\overline{x}, f(\overline{x})\big)$ on the graph of f is

$$f(\overline{x}) - \overline{x}'y.$$

Thus, the crossing point corresponding to the hyperplane that supports the epigraph of f is

$$\inf_{x \in \Re^n} \{f(x) - x'y\},$$

which by definition is equal to $-f^\star(y)$.

Figure 1.6.1 provides a geometrical interpretation of the definition.

Note that regardless of the structure of f, the conjugate $f^\star$ is a closed convex function, since it is the pointwise supremum of the collection of affine functions

$$x'y - f(x), \qquad \forall\ x \text{ such that } f(x) \text{ is finite},$$

(Prop. 1.1.6). Note also that $f^\star$ need not be proper, even if f is. We will show, however, that in the case where f is convex, $f^\star$ is proper if and only if f is.

Figure 1.6.2 shows some examples of conjugate functions. In this figure, all the functions are closed proper convex, and it can be verified that the conjugate of the conjugate yields the original function. This is a manifestation of a result that we will show shortly.

For a function $f : \Re^n \mapsto [-\infty, \infty]$, consider the conjugate of the conjugate function $f^\star$ (or *double conjugate*). It is denoted by $f^{\star\star}$, it is given by

$$f^{\star\star}(x) = \sup_{y \in \Re^n} \big\{y'x - f^\star(y)\big\}, \qquad x \in \Re^n,$$

and it can be constructed as shown in Fig. 1.6.3. As this figure suggests, and part (d) of the following proposition shows, by constructing $f^{\star\star}$, we typically obtain the convex closure of f [the function that has as epigraph the closure of the convex hull of epi(f); cf. Section 1.3.3]. In particular, part (c) of the proposition shows that if f is closed proper convex, then $f^{\star\star} = f$.

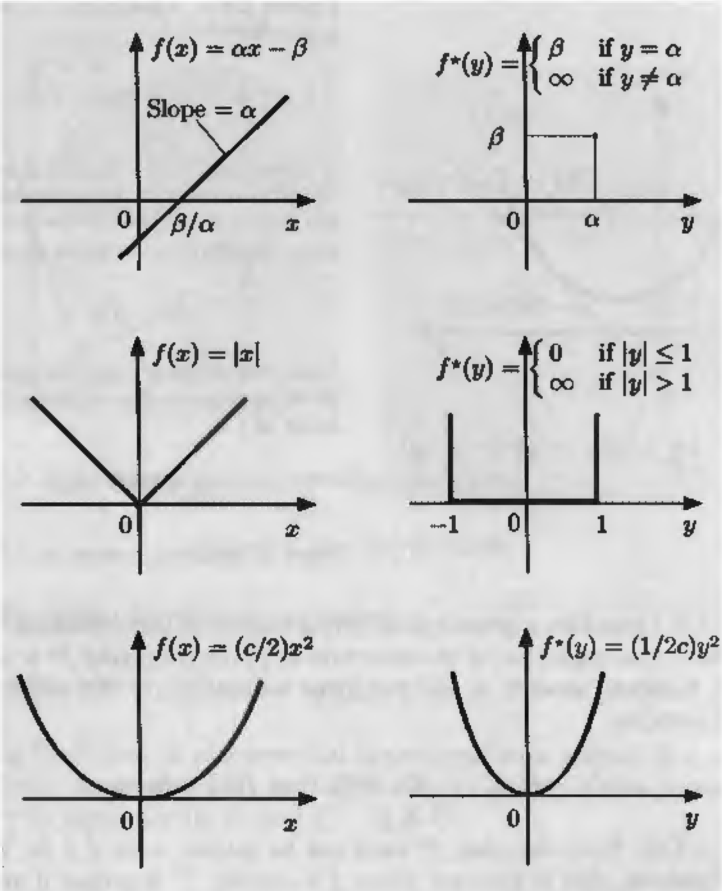


Figure 1.6.2. Some examples of conjugate functions. It can be verified that in each case, the conjugate of the conjugate is the original, i.e., the conjugates of the functions on the right are the corresponding functions on the left.

Proposition 1.6.1: (Conjugacy Theorem) Let $f : \Re^n \mapsto [-\infty, \infty]$ be a function, let $f^\star$ be its conjugate, and consider the double conjugate $f^{\star\star}$. Then:

(a) We have

$$f(x) \ge f^{\star\star}(x), \qquad \forall\, x \in \Re^n.$$

(b) If f is convex, then properness of any one of the functions f, $f^\star$, and $f^{\star\star}$ implies properness of the other two.

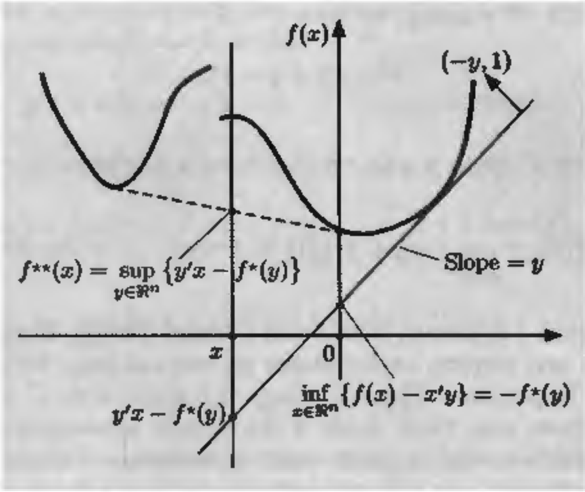


Figure 1.6.3. Visualization of the double conjugate (conjugate of the conjugate)

$$f^{**}(x) = \sup_{y\in\Re^n}\{y'x - f^\star(y)\}$$

of a function f, where $f^\star$ is the conjugate of f,

$$f^\star(y) = \sup_{x\in\Re^n}\{x'y - f(x)\}.$$

For each $x \in \Re^n$, we consider the vertical line in $\Re^{n+1}$ that passes through the point $(x, 0)$, and for each y in the effective domain of $f^\star$, we consider the crossing point of this line with the hyperplane with normal $(-y, 1)$ that supports the graph of f [and therefore passes through the point $(0, -f^\star(y))$]. This crossing point is $y'x - f^\star(y)$, so $f^{**}(x)$ is equal to the highest crossing level. As the figure indicates (and Prop. 1.6.1 shows), the double conjugate f^{**} is the convex closure of f (barring the exceptional case where the convex closure takes the value $-\infty$ at some point, in which case the figure above is not valid).

(c) If f is closed proper convex, then

$$f(x) = f^{**}(x), \qquad \forall\, x \in \Re^n.$$

(d) The conjugates of f and its convex closure $\check{\mathrm{cl}}\, f$ are equal. Furthermore, if $\check{\mathrm{cl}}\, f$ is proper, then

$$(\check{\mathrm{cl}}\, f)(x) = f^{**}(x), \qquad \forall\, x \in \Re^n.$$

Proof: (a) For all x and y, we have

$$f^\star(y) \geq x'y - f(x),$$

so that

$$f(x) \geq x'y - f^\star(y), \qquad \forall\ x, y \in \Re^n.$$

Hence

$$f(x) \geq \sup_{y \in \Re^n} \big\{x'y - f^\star(y)\big\} = f^{\star\star}(x), \qquad \forall\ x \in \Re^n.$$

(b) Assume that f is proper, in addition to being convex. Then its epigraph is nonempty and convex, and contains no vertical line. By applying the Nonvertical Hyperplane Theorem [Prop. 1.5.8(a)], with C being the set $\text{epi}(f)$, it follows that there exists a nonvertical hyperplane with normal $(y, 1)$ that contains $\text{epi}(f)$ in its positive halfspace. In particular, this implies the existence of a vector y and a scalar c such that

$$y'x + f(x) \geq c, \qquad \forall\ x \in \Re^n.$$

We thus obtain

$$f^\star(-y) = \sup_{x \in \Re^n} \big\{-y'x - f(x)\big\} \leq -c,$$

so that $f^\star$ is not identically equal to ∞. Also, by the properness of f, there exists a vector $\overline{x}$ such that $f(\overline{x})$ is finite. For every $y \in \Re^n$, we have

$$f^\star(y) \geq y'\overline{x} - f(\overline{x}),$$

so $f^\star(y) > -\infty$ for all $y \in \Re^n$. Thus, $f^\star$ is proper.

Conversely, assume that $f^\star$ is proper. The preceding argument shows that properness of $f^\star$ implies properness of its conjugate, $f^{\star\star}$, so that $f^{\star\star}(x) > -\infty$ for all $x \in \Re^n$. By part (a), $f(x) \geq f^{\star\star}(x)$, so $f(x) > -\infty$ for all $x \in \Re^n$. Also, f cannot be identically equal to ∞, since then by its definition, $f^\star$ would be identically equal to $-\infty$. Thus f is proper.

We have thus shown that a convex function is proper if and only if its conjugate is proper, and the result follows in view of the conjugacy relations between f, $f^\star$, and $f^{\star\star}$.

(c) We will apply the Nonvertical Hyperplane Theorem (Prop. 1.5.8), with C being the closed and convex set $\text{epi}(f)$, which contains no vertical line since f is proper. Let (x, γ) belong to $\text{epi}(f^{\star\star})$, i.e., $x \in \text{dom}(f^{\star\star})$, $\gamma \geq f^{\star\star}(x)$, and suppose, to arrive at a contradiction, that (x, γ) does not belong to $\text{epi}(f)$. Then by Prop. 1.5.8(b), there exists a nonvertical hyperplane with normal (y, ζ), where $\zeta \neq 0$, and a scalar c such that

$$y'z + \zeta w < c < y'x + \zeta\gamma, \qquad \forall\ (z, w) \in \text{epi}(f).$$

Since w can be made arbitrarily large, we have $\zeta < 0$, and without loss of generality, we can take $\zeta = -1$, so that

$$y'z - w < c < y'x - \gamma, \qquad \forall\ (z, w) \in \text{epi}(f).$$

Since $\gamma \geq f^{\star\star}(x)$ and $\big(z, f(z)\big) \in \text{epi}(f)$ for all $z \in \text{dom}(f)$, we obtain

$$y'z - f(z) < c < y'x - f^{\star\star}(x), \qquad \forall\ z \in \text{dom}(f).$$

Hence

$$\sup_{z \in \Re^n} \big\{y'z - f(z)\big\} \leq c < y'x - f^{\star\star}(x),$$

or

$$f^\star(y) < y'x - f^{\star\star}(x),$$

which contradicts the definition $f^{\star\star}(x) = \sup_{y \in \Re^n}\big\{y'x - f^\star(y)\big\}$. Thus, we have $\text{epi}(f^{\star\star}) \subset \text{epi}(f)$, which implies that $f(x) \leq f^{\star\star}(x)$ for all $x \in \Re^n$. This, together with part (a), shows that $f^{\star\star}(x) = f(x)$ for all x.

(d) Let $\check{f}^\star$ be the conjugate of $\check{\text{cl}}\, f$. For any y, $-f^\star(y)$ and $-\check{f}^\star(y)$ are the supremum crossing levels of the vertical axis with the hyperplanes with normal $(-y, 1)$ that contain the sets $\text{epi}(f)$ and $\text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$, respectively, in their positive closed halfspaces (cf. Fig. 1.6.1). Since the hyperplanes of this type are the same for the two sets, we have $f^\star(y) = \check{f}^\star(y)$ for all y. Thus, $f^{\star\star}$ is equal to the conjugate of $\check{f}^\star$, which is $\check{\text{cl}}\, f$ by part (c) when $\check{\text{cl}}\, f$ is proper. **Q.E.D.**

The properness assumptions on f and $\check{\text{cl}}\, f$ are essential for the validity of parts (c) and (d), respectively, of the preceding proposition. For an illustrative example, consider the closed convex (but improper) function

$$f(x) = \begin{cases} \infty & \text{if } x > 0, \\ -\infty & \text{if } x \leq 0. \end{cases} \tag{1.36}$$

We have $f = \check{\text{cl}}\, f$, and it can be verified that $f^\star(y) = \infty$ for all y and $f^{\star\star}(x) = -\infty$ for all x, so that $f \neq f^{\star\star}$ and $\check{\text{cl}}\, f \neq f^{\star\star}$.

For an example where f is proper (but not closed convex), while $\check{\text{cl}}\, f$ is improper and we have $\check{\text{cl}}\, f \neq f^{\star\star}$, let

$$f(x) = \begin{cases} \log(-x) & \text{if } x < 0, \\ \infty & \text{if } x \geq 0. \end{cases}$$

Then $\check{\text{cl}}\, f$ is equal to the function (1.36), and $\check{\text{cl}}\, f \neq f^{\star\star}$.

The exceptional behavior in the preceding example can be attributed to a subtle difference in the constructions of the conjugate function and the convex closure: while the conjugate functions $f^\star$ and $f^{\star\star}$ are defined exclusively in terms of nonvertical hyperplanes via the construction of Fig.

1.6.3, the epigraph of the convex closure $\check{\mathrm{cl}}\, f$ is defined in terms of nonvertical *and* vertical hyperplanes. This difference is inconsequential when there exists at least one nonvertical hyperplane containing the epigraph of f in one of its closed halfspaces [this is true in particular if $\check{\mathrm{cl}}\, f$ is proper; see Props. 1.5.8(a) and 1.6.1(d)]. The reason is that, in this case, the epigraph of $\check{\mathrm{cl}}\, f$ can equivalently be defined by using just nonvertical hyperplanes [this can be seen using Prop. 1.5.8(b)].

Example 1.6.1: (Indicator/Support Function Conjugacy)

Given a nonempty set X, consider the *indicator function* of X, defined by

$$\delta_X(x) = \begin{cases} 0 & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

The conjugate of δ_X is given by

$$\sigma_X(y) = \sup_{x \in X} y'x$$

and is called the *support function of* X (see Fig. 1.6.4). By the generic closedness and convexity properties of conjugate functions, σ_X is closed and convex. It is also proper since X is nonempty, so that $\sigma_X(0) = 0$ (an improper closed convex function cannot take finite values; cf. the discussion at the end of Section 1.1.2). Furthermore, the sets X, $\mathrm{cl}(X)$, $\mathrm{conv}(X)$, and $\mathrm{cl}\big(\mathrm{conv}(X)\big)$ all have the same support function [cf. Prop. 1.6.1(d)].

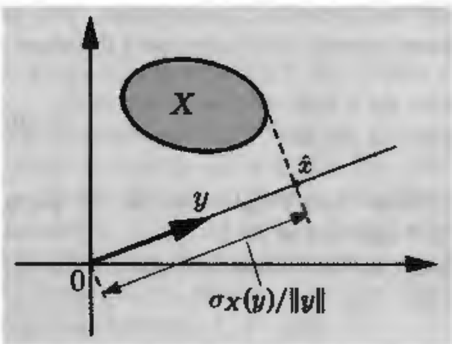


Figure 1.6.4. Visualization of the support function

$$\sigma_X(y) = \sup_{x \in X} y'x$$

of a set X. To determine the value $\sigma_X(y)$ for a given vector y, we project the set X on the line determined by y, and we find $\hat{x}$, the extreme point of projection in the direction y. Then

$$\sigma_X(y) = \|\hat{x}\| \cdot \|y\|.$$

Example 1.6.2: (Support Function of a Cone - Polar Cones)

Let C be a convex cone. By the preceding example, the conjugate of its indicator function δ_C is its support function,

$$\sigma_C(y) = \sup_{x \in C} y'x.$$

Since C is a cone, we see that

$$\sigma_C(y) = \begin{cases} 0 & \text{if } y'x \leq 0, \ \forall\, x \in C, \\ \infty & \text{otherwise.} \end{cases}$$

Thus the support/conjugate function σ_C is the indicator function δ_{C^*} of the cone

$$C^* = \{y \mid y'x \leq 0, \ \forall\, x \in C\}, \tag{1.37}$$

called the *polar cone of* C. It follows that the conjugate of σ_C is the indicator function of the polar cone of C^*, and by the Conjugacy Theorem [Prop. 1.6.1(c)] it is also $\mathrm{cl}\,\delta_C$. Thus the polar cone of C^* is $\mathrm{cl}(C)$. In particular, if C is closed, the polar of its polar is equal to the original. This is a special case of the *Polar Cone Theorem*, which will be discussed in more detail in Section 2.2.

A special case of particular interest in when $C = \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big)$, the cone generated by a finite set of vectors $a_1, \ldots, a_r$. Then it can be seen that

$$C^* = \{x \mid a_j'x \leq 0, \ j = 1, \ldots, r\}.$$

From this it follows that $(C^*)^* = C$ because C is a closed set [we have essentially shown this: $\mathrm{cone}\big(\{a_1, \ldots, a_r\}\big)$ is the image of the positive orthant $\{\alpha \mid \alpha \geq 0\}$ under the linear transformation that maps α to $\sum_{j=1}^r \alpha_j a_j$, and the image of any polyhedral set under a linear transformation is a closed set (see the discussion following the proof of Prop. 1.4.13)]. The assertion $(C^*)^* = C$ for the case $C = \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big)$ is known as Farkas' Lemma and will be discussed further in what follows (Sections 2.3 and 5.1).

Let us finally note that we may define the polar cone of any set C via Eq. (1.37). However, unless C is a cone, the support function of C will not be an indicator function, nor will the relation $(C^*)^* = C$ hold. Instead, we will show in Section 2.2 that in general we have $(C^*)^* = \mathrm{cl}\big(\mathrm{cone}(C)\big)$.

1.7 SUMMARY

In this section, we discuss how the material of this chapter is used in subsequent chapters. First, let us note that we have aimed to develop in this chapter the part of convexity theory that is needed for the optimization and duality analysis of Chapters 3-5, and no more. We have developed the basic principles of polyhedral convexity in Chapter 2 for completeness and a broader perspective, but this material is not needed for the mathematical development of Chapters 3-5. Despite the fact that we focus only on the essential, we still cover considerable ground, and it may help the reader to know how the various topics of this chapter connect with each other, and how they are used later. We thus provide a summary and guide on a section-by-section basis:

Section 1.1: The definitions and results on convexity and closure (Sections 1.1.1-1.1.3) are very basic and should be read carefully. The material of

Section 1.1.4 on differentiable convex functions, optimality conditions, and the projection theorem, is also basic. The reader may skip Prop. 1.1.10 on twice differentiable functions, which is used in nonessential ways later.

Section 1.2: The definitions of convex and affine hulls, and generated cones, as well as Caratheodory's theorem should also be read carefully. Proposition 1.2.2 on the compactness of the convex hull of a compact set is used only in the proof of Prop. 4.3.2, which is in turn used only in the specialized MC/MC framework of Section 5.7 on estimates of duality gap.

Section 1.3: The definitions and results on relative interior and closure up to and including Section 1.3.2 are pervasive in what follows. However, the somewhat tedious proofs of Props. 1.3.5-1.3.10 may be skipped at first reading. Similarly, the proof of continuity of a real-valued convex function (Prop. 1.3.11) is specialized and may be skipped. Regarding Section 1.3.3, it is important to understand the definitions, and gain intuition on closures and convex closures of functions, as they arise in the context of conjugacy. However, Props. 1.3.13-1.3.17 are used substantively later only for the development of minimax theory (Sections 3.4, 4.2.5, and 5.5) and the theory of directional derivatives (Section 5.4.4), which are themselves "terminal" and do not affect other sections.

Section 1.4: The material on directions of recession, up to and including Section 1.4.1, is very important for later developments, although the use of Props. 1.4.5-1.4.6 on recession functions is somewhat focused. In particular, Prop. 1.4.7 is used only to prove Prop. 1.4.8, and Props. 1.4.6-1.4.8 are used only for the development of existence of solutions criteria in Section 3.2. The set intersection analysis of Section 1.4.2, which may challenge some readers at first, is used for the development of some important theory: the existence of optimal solutions for linear and quadratic programs (Prop. 1.4.12), the criteria for preservations of closedness under linear transformations and vector sum (Props. 1.4.13 and 1.4.14), and the existence of solutions analysis of Chapter 3.

Section 1.5: The material on hyperplane separation is used extensively and should be read in its entirety. However, the long proof of the polyhedral proper separation theorem (Prop. 1.5.7, due to Rockafellar [Roc70], Th. 20.2) may be postponed for later. This theorem is important for our purposes. For example it is used (through Props. 4.5.1 and 4.5.2) to establish part of the Nonlinear Farkas' Lemma (Prop. 5.1.1), on which the constrained optimization and Fenchel duality theories rest.

Section 1.6: The material on conjugate functions and the Conjugacy Theorem (Prop. 1.6.1) is very basic. In our duality development, we use the theorem somewhat sparingly because we argue mostly in terms of the MC/MC framework of Chapter 4 and the Strong Duality Theorem (Prop. 4.3.1), which serves as an effective substitute.

2

Basic Concepts of Polyhedral Convexity

Contents

In this chapter, we discuss polyhedral sets, i.e., nonempty sets specified by systems of a finite number of affine inequalities

$$a_j'x \le b_j, \qquad j = 1, \ldots, r,$$

where $a_1, \ldots, a_r$ are vectors in $\Re^n$, and $b_1, \ldots, b_r$ are scalars. We start with the notion of an extreme point of a convex set. We then introduce a general duality relation between cones, and we specialize this relation to polyhedral cones and polyhedral sets. Then we apply the results to linear programming problems.†

2.1 EXTREME POINTS

A geometrically apparent property of a bounded polyhedral set in the plane is that it can be represented as the convex hull of a finite number of points, its "corner" points. This property has great analytical and algorithmic significance, and can be formalized through the notion of an extreme point, which we now introduce.

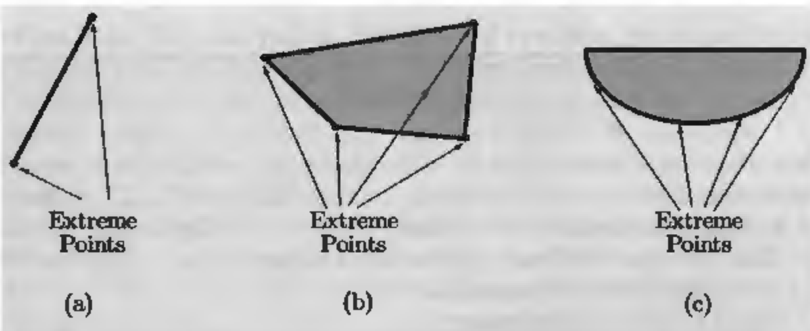


Figure 2.1.1. Illustration of extreme points of various convex sets in $\Re^2$. For the set in (c), the extreme points are the ones that lie on the circular arc.

Given a nonempty convex set C, a vector $x \in C$ is said to be an *extreme point* of C if it does not lie strictly between the endpoints of any line segment contained in the set, i.e., if there do not exist vectors $y \in C$ and $z \in C$, with $y \neq x$ and $z \neq x$, and a scalar $\alpha \in (0, 1)$ such that $x = \alpha y + (1 - \alpha)z$. It can be seen that an equivalent definition is that x cannot be expressed as a convex combination of some vectors of C, all of which are different from x.

† The material in this chapter is fundamental in itself, and is related to several of the topics treated later. However, the results of the chapter are not used directly in Chapters 3-5, and may be skipped by the reader.

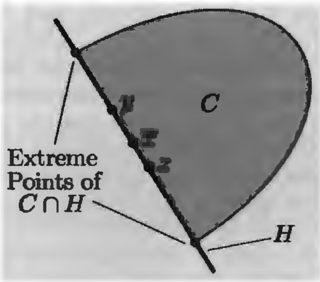


Figure 2.1.2. Construction to prove that the extreme points of $C \cap H$ are also extreme points of C (Prop. 2.1.1).

For some perspective, note that an interior point of a set cannot be an extreme point, so an open set has no extreme points. Also from the definition, we see that a convex cone may have at most one extreme point, the origin. We will show later in this section that a polyhedral set has at most a finite number of extreme points (possibly none). Moreover a concave (e.g., linear) function that attains a minimum over a polyhedral set with at least one extreme point, attains the minimum at some extreme point. Figure 2.1.1 illustrates the extreme points of various types of sets.

The following is a useful result for the analysis of extreme point-related issues. It is often used in proofs to obtain extreme points of convex sets by using the extreme points of lower dimensional subsets.

Proposition 2.1.1: Let C be a convex subset of $\Re^n$, and let H be a hyperplane that contains C in one of its closed halfspaces. Then the extreme points of $C \cap H$ are precisely the extreme points of C that belong to H.

Proof: Let $\overline{x}$ be an extreme point of $C \cap H$. To show that $\overline{x}$ is an extreme point of C, let $y \in C$ and $z \in C$ be such that $\overline{x} = \alpha y + (1-\alpha)z$ for some $\alpha \in (0,1)$ (see Fig. 2.1.2). We will show that $\overline{x} = y = z$. Indeed, since $\overline{x} \in H$, the closed halfspace containing C is of the form $\{x \mid a'x \geq a'\overline{x}\}$, where $a \neq 0$, and H is of the form $\{x \mid a'x = a'\overline{x}\}$. Thus, we have $a'y \geq a'\overline{x}$ and $a'z \geq a'\overline{x}$, which in view of $\overline{x} = \alpha y + (1-\alpha)z$, implies that $a'y = a'\overline{x}$ and $a'z = a'\overline{x}$. Therefore,

$$y \in C \cap H, \qquad z \in C \cap H.$$

Since $\overline{x}$ is an extreme point of $C \cap H$, it follows that $\overline{x} = y = z$, so that $\overline{x}$ is an extreme point of C.

Conversely, if $\overline{x} \in H$ and $\overline{x}$ is an extreme point of C, then $\overline{x}$ cannot lie strictly between two distinct points of C, and hence also strictly between two distinct points of $C \cap H$. It follows that $\overline{x}$ is an extreme point of $C \cap H$. **Q.E.D.**

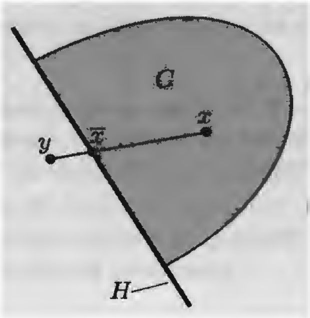


Figure 2.1.3. Construction used in the induction proof of Prop. 2.1.2 to show that if a closed convex set C does not contain a line, it must have an extreme point. Using the line segment connecting two points $x \in C$ and $y \notin C$, we obtain a relative boundary point $\overline{x}$ of C. The proof then reduces to asserting the existence of an extreme point of the lower-dimensional set $C \cap H$, where H is a supporting hyperplane passing through $\overline{x}$, which is also an extreme point of C by Prop. 2.1.1.

The following proposition characterizes the existence of an extreme point.

Proposition 2.1.2: A nonempty closed convex subset of $\Re^n$ has at least one extreme point if and only if it does not contain a line, i.e., a set of the form $\{x + \alpha d \mid \alpha \in \Re\}$, where x and d are vectors in $\Re^n$ with $d \neq 0$.

Proof: Let C be a convex set that has an extreme point x. Assume, to arrive at a contradiction, that C contains a line $\{\overline{x} + \alpha d \mid \alpha \in \Re\}$, where $\overline{x} \in C$ and $d \neq 0$. Then, by the Recession Cone Theorem [Prop. 1.4.1(b)] and the closedness of C, both d and $-d$ are directions of recession of C, so the line $\{x + \alpha d \mid \alpha \in \Re\}$ belongs to C. This contradicts the fact that x is an extreme point.

Conversely, we use induction on the dimension of the space to show that if C does not contain a line, it must have an extreme point. This is true for $\Re$, so assume it is true for $\Re^{n-1}$, where $n \geq 2$. We will show that any nonempty closed convex subset C of $\Re^n$, which does not contain a line, must have an extreme point. Since C does not contain a line, there must exist points $x \in C$ and $y \notin C$. The line segment connecting x and y intersects the relative boundary of C at some point $\overline{x}$, which belongs to C, since C is closed. Consider a hyperplane H passing through $\overline{x}$ and containing C in one of its closed halfspaces (such a hyperplane exists by Prop. 1.5.1; see Fig. 2.1.3). The set $C \cap H$ lies in an $(n-1)$-dimensional space and does not contain a line, since C does not. Hence, by the induction hypothesis, it must have an extreme point. By Prop. 2.1.1, this extreme point must also be an extreme point of C. **Q.E.D.**

As an example of application of the preceding proposition, consider a nonempty set of the form

$$\{x \mid x \in C,\ x \geq 0\},$$

where C is a closed convex set. Such sets arise commonly in optimization, and in view of the constraint $x \geq 0$, they do not contain a line. Hence, by Prop. 2.1.2, they always have at least one extreme point. The following is a useful generalization.

Proposition 2.1.3: Let C be a nonempty closed convex subset of $\Re^n$. Assume that for some $m \times n$ matrix A of rank n and some $b \in \Re^m$, we have

$$Ax \geq b, \qquad \forall\, x \in C.$$

Then C has at least one extreme point.

Proof: The set $\{x \mid Ax \geq b\}$ cannot contain a line $\{\overline{x} + \alpha d \mid \alpha \in \Re\}$, with $d \neq 0$; if it did, we would have

$$A\overline{x} + \alpha Ad \geq 0, \qquad \forall\, \alpha \in \Re,$$

a contradiction because $Ad \neq 0$, since A has rank n. Hence C does not contain a line, and the result follows from Prop. 2.1.2. **Q.E.D.**

Extreme Points of Polyhedral Sets

We recall the definition of a polyhedral set in $\Re^n$: it is a set that is nonempty and has the form

$$\{x \mid a_j'x \leq b_j,\ j = 1, \ldots, r\},$$

where $a_1, \ldots, a_r$ are vectors in $\Re^n$ and $b_1, \ldots, b_r$ are scalars.

The following proposition gives a characterization of extreme points of polyhedral sets that is central in the theory of linear programming (see Fig. 2.1.4). Parts (b) and (c) of the proposition are special cases of part (a), but we state them explicitly for completeness.

Proposition 2.1.4: Let P be a polyhedral subset of $\Re^n$.

(a) If P has the form

$$P = \{x \mid a_j'x \leq b_j,\ j = 1, \ldots, r\},$$

where $a_j \in \Re^n$, $b_j \in \Re$, $j = 1, \ldots, r$, then a vector $v \in P$ is an extreme point of P if and only if the set

$$A_v = \{a_j \mid a_j'v = b_j,\ j \in \{1, \ldots, r\}\}$$

contains n linearly independent vectors.

(b) If P has the form

$$P = \{x \mid Ax = b,\ x \geq 0\},$$

where A is an $m \times n$ matrix and b is a vector in $\Re^m$, then a vector $v \in P$ is an extreme point of P if and only if the columns of A corresponding to the nonzero coordinates of v are linearly independent.

(c) If P has the form

$$P = \{x \mid Ax = b,\ c \leq x \leq d\},$$

where A is $m \times n$ matrix, b is a vector in $\Re^m$, and c, d are vectors in $\Re^n$, then a vector $v \in P$ is an extreme point of P if and only if the columns of A corresponding to the coordinates of v that lie strictly between the corresponding coordinates of c and d are linearly independent.

Proof: (a) If the set A_v contains fewer than n linearly independent vectors, then the system of equations

$$a_j' w = 0, \qquad \forall\ a_j \in A_v,$$

has a nonzero solution, call it $\overline{w}$. For sufficiently small $\gamma > 0$, we have $v + \gamma\overline{w} \in P$ and $v - \gamma\overline{w} \in P$, thus showing that v is not an extreme point. Thus, if v is an extreme point, A_v must contain n linearly independent vectors.

Conversely, assume that A_v contains a subset $\bar{A}_v$ consisting of n linearly independent vectors. Suppose that for some $y \in P$, $z \in P$, and $\alpha \in (0,1)$, we have

$$v = \alpha y + (1 - \alpha) z.$$

Then, for all $a_j \in \bar{A}_v$, we have

$$b_j = a_j' v = \alpha a_j' y + (1 - \alpha) a_j' z \leq \alpha b_j + (1 - \alpha) b_j = b_j.$$

Thus, v, y, and z are all solutions of the system of n linearly independent equations

$$a_j' w = b_j, \qquad \forall\ a_j \in \bar{A}_v.$$

It follows that $v = y = z$, implying that v is an extreme point of P.

(b) We write P in terms of inequalities in the equivalent form

$$P = \{x \mid Ax \leq b,\ -Ax \leq -b,\ -x \leq 0\}.$$

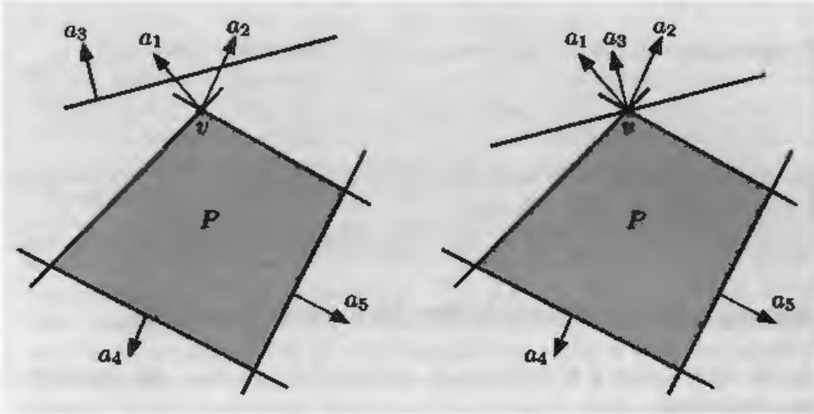


Figure 2.1.4. Illustration of extreme points of a 2-dimensional polyhedral set

$$P = \{x \mid a_j'x \leq b_j,\ j = 1, \ldots, r\},$$

[see Prop. 2.1.4(a)]. For a vector $v \in P$ to be an extreme point it is necessary and sufficient that the set of vectors

$$A_v = \{a_j \mid a_j'v = b_j,\ j \in \{1, \ldots, r\}\}$$

contains $n = 2$ linearly independent vectors. In the case on the left-hand side, the set A_v consists of the two linearly independent vectors a_1 and a_2. In the case on the right-hand side, the set A_v consists of the three vectors a_1, a_2, a_3, any pair of which are linearly independent.

Let S_v be the set of vectors of the form $(0, \ldots, 0, -1, 0, \ldots 0)$, where -1 is in a position corresponding to a zero coordinate of v. By applying the result of part (a) to the above inequality representation of P, we see that v is an extreme point if and only if the rows of A together with the vectors in S_v contain n linearly independent vectors. Let $\bar{A}$ be the matrix, which is the same as A except that all the columns corresponding to the zero coordinates of v are set to zero. It follows that v is an extreme point if and only if $\bar{A}$ contains $n - k$ linearly independent rows, where k is the number of vectors in S_v. The only (possibly) nonzero columns of $\bar{A}$ are the $n - k$ corresponding to nonzero coordinates of v, so these columns must be linearly independent. Since these columns are shared by $\bar{A}$ and A, the result follows.

(c) The proof is essentially the same as the proof of part (b). **Q.E.D.**

By combining Props. 2.1.3 and 2.1.4(a), we obtain the following characterization of the existence of an extreme point of a polyhedral set.

Proposition 2.1.5: A polyhedral set in $\Re^n$ of the form

$$\{x \mid a_j'x \le b_j,\ j = 1,\ldots,r\}$$

has an extreme point if and only if the set $\{a_j \mid j = 1,\ldots,r\}$ contains n linearly independent vectors.

Example 2.1.1: (Birkhoff-von Neumann Theorem)

In the space of $n \times n$ matrices $X = \{x_{ij} \mid i,j = 1,\ldots,n\}$, consider the polyhedral set

$$P = \left\{ X \;\middle|\; x_{ij} \ge 0,\ \sum_{j=1}^n x_{ij} = 1,\ \sum_{i=1}^n x_{ij} = 1,\ i,j = 1,\ldots,n \right\}.$$

Matrices in this set are called *doubly stochastic*, in view of the fact that both their rows and columns are probability distributions.

An important optimization problem that involves P is the *assignment problem*. Here, the elements of some finite set (say, n persons) are to be matched on a one-to-one basis, to the elements of another set (say, n objects), and x_{ij} is a variable taking value 0 or 1, and representing whether person i is or is not matched to object j. Thus a feasible solution to the assignment problem is a matrix that has a single 1 in each row and a single 1 in each column, with 0s in all other positions. Such a matrix is called a *permutation matrix*, since when multiplying a vector $x \in \Re^n$ it produces a vector whose coordinates are a permutation of the coordinates of x. The assignment problem is to minimize a linear cost of the form

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_{ij}$$

subject to X being a permutation matrix.

The Birkhoff-von Neumann Theorem states that *the extreme points of the polyhedral set of doubly stochastic matrices are the permutation matrices.* Thus if we minimize $\sum_{i=1}^n \sum_{j=1}^n a_{ij}x_{ij}$ subject to $X \in P$, using an algorithm that is guaranteed to find an optimal solution that is an extreme point (the simplex method, among others, offers this guarantee), we also solve the assignment problem. To prove the Birkhoff-von Neumann Theorem, we note that if X is a permutation matrix, and $Y, Z \in P$ are such that

$$x_{ij} = \alpha y_{ij} + (1-\alpha) z_{ij}, \qquad \forall\, i,j = 1,\ldots,n,$$

for some $\alpha \in (0,1)$, then since $y_{ij}, z_{ij} \in [0,1]$, we have $y_{ij} = z_{ij} = 0$ if $x_{ij} = 0$, and $y_{ij} = z_{ij} = 1$ if $x_{ij} = 1$. Thus, if

$$X = \alpha Y + (1-\alpha) Z,$$

we have $X = Y = Z$, implying that X is an extreme point of P.

Conversely, let X be an $n \times n$ matrix that is an extreme point of P. We will first show that there exists a row with a single nonzero component, necessarily a 1. Clearly, in view of the constraints $\sum_{j=1}^n x_{ij} = 1$, there cannot exist a row with all components equal to 0. If all rows have two or more nonzero components, then the number of zero components of X must be less or equal to $n(n-2)$. On the other hand, there are $2n$ equalities $\sum_{j=1}^n x_{ij} = 1$, $\sum_{i=1}^n x_{ij} = 1$, which must be satisfied, and at most $2n-1$ of these equalities are linearly independent, since adding the equalities $\sum_{j=1}^n x_{ij} = 1$ gives the same result as adding the equalities $\sum_{i=1}^n x_{ij} = 1$. Thus, by Prop. 2.1.4(b), there must be at most $2n-1$ nonzero components x_{ij} in X, or equivalently, there must be more than $n^2 - 2n$ inequalities $x_{ij} \geq 0$ that are satisfied as equalities, i.e., more than $n(n-2)$ components of X must be 0, a contradiction. Thus, there must exist a row, say row $\bar{\imath}$, with exactly one nonzero component, necessarily a 1, which must also be the only nonzero component of the corresponding column $\bar{\jmath}$.

We now argue that by Prop. 2.1.1, X is an extreme point of the polyhedral set $P \cap H$, where H is the hyperplane of matrices whose component at row $\bar{\imath}$ and column $\bar{\jmath}$ is equal to 1. Repeating the preceding argument with P replaced by $P \cap H$, we see that there must exist a second row of X with a single nonzero component. Continuing in this manner, by finding at each step a new row and column with a single nonzero component, we establish that all rows and columns of X have a single nonzero component, showing that X is a permutation matrix.

2.2 POLAR CONES

We will now consider an important type of cone that is associated with a nonempty set C, and provides a dual and equivalent description of C, when C is itself a closed cone. In particular, the *polar cone* of C, denoted C^*, is given by

$$C^* = \{y \mid y'x \leq 0, \ \forall\, x \in C\},$$

(see Fig. 2.2.1). We have already discussed briefly polar cones in Example 1.6.2.

Clearly, C^* is a cone, and being the intersection of a collection of closed halfspaces, it is closed and convex (regardless of whether C is closed and/or convex). If C is a subspace, it can be seen that C^* is equal to the orthogonal subspace $C^\perp$. The following proposition generalizes the equality $C = (C^\perp)^\perp$, which holds in the case where C is a subspace. Part (b) of the proposition was essentially proved in Example 1.6.2 using the Conjugacy Theorem (Prop. 1.6.1), and the conjugacy of indicator and support functions, but we will give here an alternative, more elementary proof that uses the Projection Theorem.

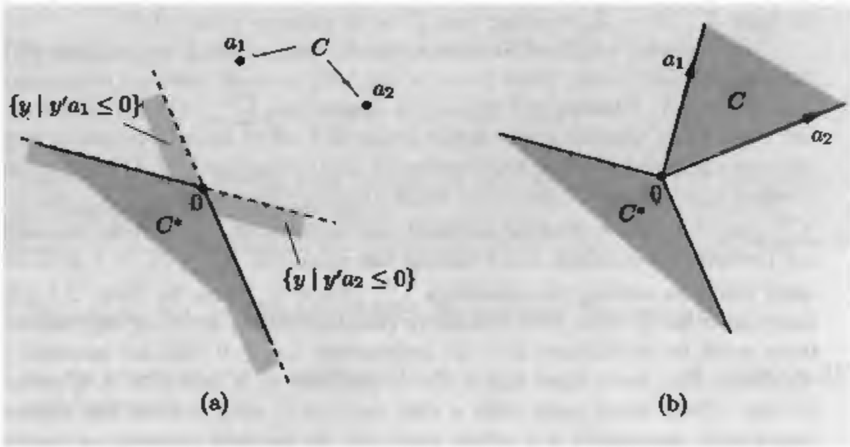


Figure 2.2.1. Illustration of the polar cone C^* of a subset C of $\Re^2$. In (a), C consists of just two points, a_1 and a_2, and C^* is the intersection of the two closed halfspaces $\{y \mid y'a_1 \le 0\}$ and $\{y \mid y'a_2 \le 0\}$. In (b), C is the convex cone

$$\{x \mid x = \mu_1 a_1 + \mu_2 a_2,\ \mu_1 \ge 0,\ \mu_2 \ge 0\},$$

and the polar cone C^* is the same as in case (a).

Proposition 2.2.1:

(a) For any nonempty set C, we have

$$C^* = \big(\mathrm{cl}(C)\big)^* = \big(\mathrm{conv}(C)\big)^* = \big(\mathrm{cone}(C)\big)^*.$$

(b) (*Polar Cone Theorem*) For any nonempty cone C, we have

$$(C^*)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big).$$

In particular, if C is closed and convex, we have $(C^*)^* = C$.

Proof: (a) For any two sets X and Y with $X \supset Y$, we have $X^* \subset Y^*$, from which it follows that $\big(\mathrm{cl}(C)\big)^* \subset C^*$. Conversely, if $y \in C^*$, then $y'x_k \le 0$ for all k and all sequences $\{x_k\} \subset C$, so that $y'x \le 0$ for all $x \in \mathrm{cl}(C)$. Hence $y \in \big(\mathrm{cl}(C)\big)^*$, implying that $C^* \subset \big(\mathrm{cl}(C)\big)^*$.

Also, since $\mathrm{conv}(C) \supset C$, we have $\big(\mathrm{conv}(C)\big)^* \subset C^*$. Conversely, if $y \in C^*$, then $y'x \le 0$ for all $x \in C$, so that $y'z \le 0$ for all z that are convex combinations of vectors $x \in C$. Hence, $y \in \big(\mathrm{conv}(C)\big)^*$ and $C^* \subset \big(\mathrm{conv}(C)\big)^*$. A nearly identical argument also shows that $C^* = \big(\mathrm{cone}(C)\big)^*$.

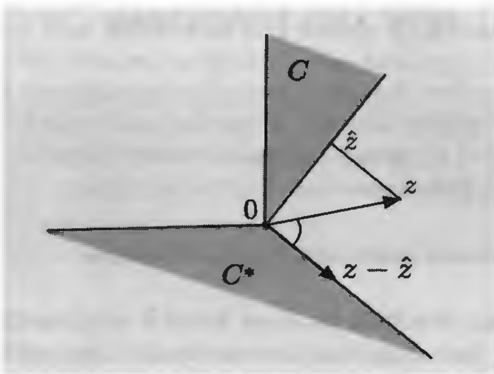


Figure 2.2.2. Illustration of the proof of the Polar Cone Theorem for the case where C is a closed convex cone. It is easily seen that $C \subset (C^*)^*$. To prove that $(C^*)^* \subset C$, we show that for any $z \in \Re^n$ and its projection on C, call it $\hat{z}$, we have $z - \hat{z} \in C^*$, so if $z \in (C^*)^*$, the geometry shown in the figure [(angle between z and $z - \hat{z}$) $< \pi/2$] is impossible, and we must have $z - \hat{z} = 0$, i.e., $z \in C$.

(b) We first show the result for the case where C is closed and convex. Indeed, if this is so, then for any $x \in C$, we have $x'y \leq 0$ for all $y \in C^*$, which implies that $x \in (C^*)^*$. Hence, $C \subset (C^*)^*$.

To prove that $(C^*)^* \subset C$, consider any $z \in \Re^n$, and its projection on C, denoted $\hat{z}$ (see Fig. 2.2.2). By the Projection Theorem (Prop. 1.1.9),

$$(z - \hat{z})'(x - \hat{z}) \leq 0, \qquad \forall\, x \in C.$$

By taking in this relation $x = 0$ and $x = 2\hat{z}$ (which belong to C since C is a closed cone), it is seen that

$$(z - \hat{z})'\hat{z} = 0.$$

Combining the last two relations, we obtain $(z - \hat{z})'x \leq 0$ for all $x \in C$. Therefore, $(z - \hat{z}) \in C^*$. Hence, if $z \in (C^*)^*$, we have $(z - \hat{z})'z \leq 0$, which when added to $-(z - \hat{z})'\hat{z} = 0$ yields $\|z - \hat{z}\|^2 \leq 0$. It follows that $z = \hat{z}$ and $z \in C$.

Using the result just shown for the case where C is closed and convex, it follows that

$$\Big(\big(\mathrm{cl}\big(\mathrm{conv}(C)\big)\big)^*\Big)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big).$$

By using part (a), we have

$$C^* = \big(\mathrm{conv}(C)\big)^* = \big(\mathrm{cl}\big(\mathrm{conv}(C)\big)\big)^*.$$

By combining the above two relations, we obtain $(C^*)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big)$. **Q.E.D.**

Note that for any nonempty set C, from part (a) of Prop. 2.2.1, we have $C^* = \big(\mathrm{cl}\big(\mathrm{cone}(C)\big)\big)^*$, so from part (b), $(C^*)^* = \mathrm{cl}\big(\mathrm{cone}(C)\big)$.

2.3 POLYHEDRAL SETS AND FUNCTIONS

One of the distinctive characteristics of polyhedral sets is that they can be described by a finite number of vectors and scalars. In this section, we focus on several alternative finite characterizations of polyhedral sets, starting with polyhedral cones and their polar cones.

2.3.1 Polyhedral Cones and Farkas' Lemma

We will introduce two different ways to view polyhedral cones, and use polarity to show that these two views are essentially equivalent. We recall that a polyhedral cone $C \subset \Re^n$ has the form

$$C = \{x \mid a_j'x \leq 0,\ j = 1, \ldots, r\},$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$, and r is a positive integer; see Fig. 2.3.1(a).

We say that a cone $C \subset \Re^n$ is *finitely generated*, if it is generated by a finite set of vectors, i.e., if it has the form

$$C = \operatorname{cone}\big(\{a_1, \ldots, a_r\}\big) = \left\{x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r\right\},$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$, and r is a positive integer; see Fig. 2.3.1(b).

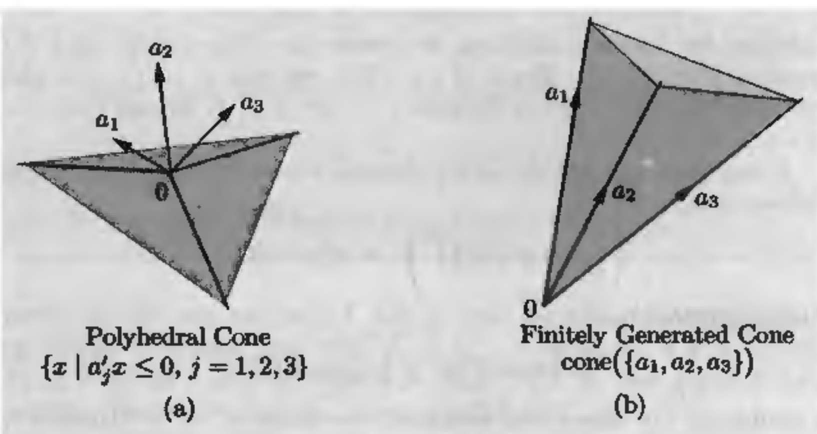


Figure 2.3.1. (a) Polyhedral cone defined by the inequality constraints $a_j'x \leq 0$, $j = 1, 2, 3$. (b) Cone generated by the vectors a_1, a_2, a_3.

It turns out that polyhedral and finitely generated cones are connected by a polarity relation, as shown in the following proposition.

Proposition 2.3.1: (Farkas' Lemma) Let $a_1, \ldots, a_r$ be vectors in $\Re^n$. Then, $\{x \mid a_j'x \leq 0,\ j = 1, \ldots, r\}$ and $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ are closed cones that are polar to each other.

Proof: The cone $\big\{x \mid a_j'x \leq 0,\ j = 1, \ldots, r\big\}$ is closed, being the intersection of closed halfspaces. To see that $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ is closed, note that it is the image of the positive orthant $\{\mu \mid \mu \geq 0\}$ under the linear transformation that maps μ to $\sum_{j=1}^r \mu_j a_j$, and that the image of any polyhedral set under a linear transformation is a closed set (see the discussion following the proof of Prop. 1.4.13).

To show that the two cones are polar to each other, we use Prop. 2.2.1(a) to write

$$\big(\{a_1, \ldots, a_r\}\big)^* = \Big(\text{cone}\big(\{a_1, \ldots, a_r\}\big)\Big)^*,$$

and we note that by the definition of polarity,

$$\big(\{a_1, \ldots, a_r\}\big)^* = \{x \mid a_j'x \leq 0,\ j = 1, \ldots, r\}.$$

Q.E.D.

Farkas' Lemma is often stated in an alternative (and equivalent) form that involves polyhedra specified by equality constraints. In particular, if $x, e_1, \ldots, e_m, a_1, \ldots, a_r$ are vectors in $\Re^n$, we have $x'y \leq 0$ for all $y \in \Re^n$ such that

$$y'e_i = 0, \quad \forall\ i = 1, \ldots, m, \qquad y'a_j \leq 0, \quad \forall\ j = 1, \ldots, r,$$

if and only if x can be expressed as

$$x = \sum_{i=1}^m \lambda_i e_i + \sum_{j=1}^r \mu_j a_j,$$

where λ_i and μ_j are some scalars with $\mu_j \geq 0$ for all j.

For the proof, define $a_{r+i} = e_i$ and $a_{r+m+i} = -e_i$, $i = 1, \ldots, m$. The result can be stated as $P^* = C$, where

$$P = \{y \mid y'a_j \leq 0,\ j = 1, \ldots, r + 2m\}, \qquad C = \text{cone}\big(\{a_1, \ldots, a_{r+2m}\}\big).$$

Since by Prop. 2.3.1, $P = C^*$, and C is closed and convex, we have $P^* = (C^*)^* = C$ by the Polar Cone Theorem [Prop. 2.2.1(b)].

2.3.2 Structure of Polyhedral Sets

We will now prove a major theorem in polyhedral convexity, which will lead to a useful representation of polyhedral sets. The proof is fairly intricate, but is visually intuitive. Alternative proofs can be found in [BNO03] and [Roc70].

Proposition 2.3.2: (Minkowski-Weyl Theorem) A cone is polyhedral if and only if it is finitely generated.

Proof: We first show that a finitely generated cone is polyhedral. Consider the cone generated by vectors $a_1, \ldots, a_r$ in $\Re^n$, and assume that these vectors span $\Re^n$, i.e., that n of these vectors are linearly independent. We will show that $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ is the intersection of halfspaces corresponding to hyperplanes/subspaces spanned by $(n-1)$ linearly independent vectors from $\{a_1, \ldots, a_r\}$. Since there is a finite number of such halfspaces, it will follow that $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ is polyhedral.

Indeed, if $\text{cone}\big(\{a_1, \ldots, a_r\}\big) = \Re^n$, it is polyhedral, and we are done, so assume otherwise. A vector b that does not belong to $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ can be strictly separated from it by a hyperplane [cf. Prop. 1.5.3 under condition (2)], so there exists $\xi \in \Re^n$ and $\gamma \in \Re$ such that $b'\xi > \gamma > x'\xi$ for all $x \in \text{cone}\big(\{a_1, \ldots, a_r\}\big)$. Since $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ is a cone, each of its vectors x must satisfy $0 \geq x'\xi$ (otherwise $x'\xi$ could be increased without bound), and since $0 \in \text{cone}\big(\{a_1, \ldots, a_r\}\big)$, we must have $\gamma > 0$. Thus,

$$b'\xi > \gamma > 0 \geq x'\xi, \qquad \forall\, x \in \text{cone}\big(\{a_1, \ldots, a_r\}\big).$$

It follows that the set

$$P_b = \{y \mid b'y \geq 1,\, a_j'y \leq 0,\, j = 1, \ldots, r\}$$

is nonempty (it contains ξ/γ – see Fig. 2.3.2). Since $\{a_1, \ldots, a_r\}$ contains n linearly independent vectors, by Prop. 2.1.5, P_b has at least one extreme point, denoted $\overline{y}$. By Prop. 2.1.4(a), there are two possibilities:

(1) We have $b'\overline{y} = 1$ and the set $\{a_j \mid a_j'\overline{y} = 0\}$ contains exactly $n-1$ linearly independent vectors.

(2) The set $\{a_j \mid a_j'\overline{y} = 0\}$ contains exactly n linearly independent vectors.

The second alternative is impossible since $\overline{y}$ is nonzero in view of $b'\overline{y} \geq 1$, while the first alternative implies that the hyperplane $\{x \mid \overline{y}'x = 0\}$, which is an $(n-1)$-dimensional subspace, is spanned by $n-1$ linearly independent vectors from $\{a_1, \ldots, a_r\}$, separates b and $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$,

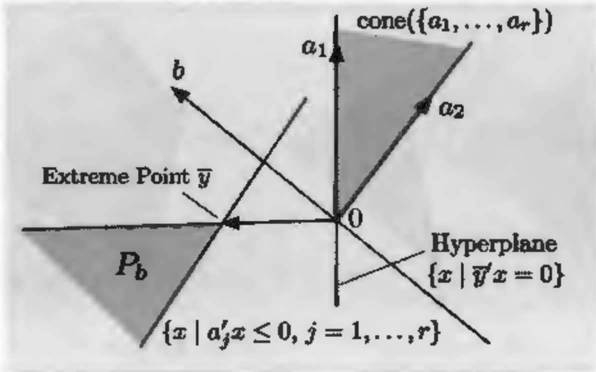


Figure 2.3.2. The polyhedral set

$$P_b = \{y \mid b'y \ge 1,\ a_j'y \le 0,\ j = 1,\ldots,r\},$$

corresponding to a vector $b \notin \text{cone}\big(\{a_1,\ldots,a_r\}\big)$, in the proof of the Minkowski-Weyl Theorem (cf. Prop. 2.3.2). It has a nonzero extreme point $\overline{y}$, which defines a halfspace $\{x \mid \overline{y}'x \le 0\}$ that contains $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$. These halfspaces are finite in number and their intersection defines $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$.

and does not contain b. By letting b range over all vectors that are not in $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$, it follows that $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$ is the intersection of some halfspaces corresponding to hyperplanes/subspaces spanned by $(n-1)$ linearly independent vectors from $\{a_1,\ldots,a_r\}$. Since there are only a finite number of such subspaces, $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$ is polyhedral.

In the case where $\{a_1,\ldots,a_r\}$ does not contain n linearly independent vectors, let S be the subspace spanned by $a_1,\ldots,a_r$. The vectors defining the finitely generated cone

$$S^\perp + \text{cone}\big(\{a_1,\ldots,a_r\}\big)$$

contain a linearly independent set, so by the result proved so far, this cone is a polyhedral set. By writing

$$\text{cone}\big(\{a_1,\ldots,a_r\}\big) = S \cap \big(S^\perp + \text{cone}\big(\{a_1,\ldots,a_r\}\big)\big),$$

we see that $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$ is the intersection of two polyhedral sets, so it is polyhedral.

Conversely, let $C = \{x \mid a_j'x \le 0,\ j = 1,\ldots,r\}$ be a polyhedral cone. We will show that C is finitely generated. Indeed, by Farkas' Lemma, C^* is equal to $\text{cone}\big(\{a_1,\ldots,a_r\}\big)$, which is polyhedral, as we have just shown. Thus, $C^* = \{x \mid c_j'x \le 0,\ j = 1,\ldots,\overline{r}\}$ for some vectors c_j. Applying Farkas' Lemma again, it follows that $C = \text{cone}\big(\{c_1,\ldots,c_{\overline{r}}\}\big)$. **Q.E.D.**

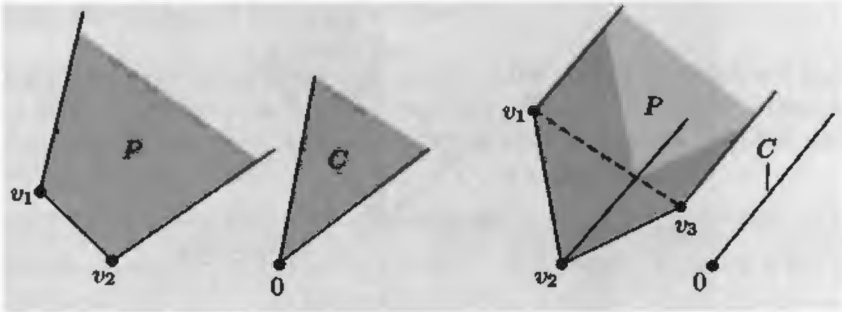


Figure 2.3.3. Examples of a Minkowski-Weyl representation of a two-dimensional and a three-dimensional polyhedral set P. It has the form

$$P = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + C,$$

where $v_1, \ldots, v_m$ are vectors and C is a finitely generated cone.

We now establish a fundamental result, showing that a polyhedral set can be represented as the sum of a finitely generated cone and the convex hull of a finite set of points (see Fig. 2.3.3).

Proposition 2.3.3: (Minkowski-Weyl Representation) A set P is polyhedral if and only if there is a nonempty finite set $\{v_1, \ldots, v_m\}$ and a finitely generated cone C such that $P = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + C$, i.e.,

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m,\ y \in C \right\}.$$

Proof: Assume that P is polyhedral. Then, it has the form

$$P = \big\{ x \mid a_j' x \leq b_j,\ j = 1, \ldots, r \big\},$$

for some $a_j \in \Re^n$, $b_j \in \Re$, $j = 1, \ldots, r$. Consider the polyhedral cone in $\Re^{n+1}$ given by

$$\hat{P} = \big\{ (x, w) \mid 0 \leq w,\ a_j' x \leq b_j w,\ j = 1, \ldots, r \big\}$$

(see Fig. 2.3.4), and note that

$$P = \big\{ x \mid (x, 1) \in \hat{P} \big\}.$$

By the Minkowski-Weyl Theorem (Prop. 2.3.2), $\hat{P}$ is finitely generated, so it has the form

$$\hat{P} = \left\{ (x, w) \;\middle|\; x = \sum_{j=1}^{m} \nu_j \tilde{v}_j,\ w = \sum_{j=1}^{m} \nu_j \tilde{d}_j,\ \nu_j \geq 0,\ j = 1, \ldots, m \right\},$$

for some $\tilde{v}_j \in \Re^n$, $\tilde{d}_j \in \Re$, $j = 1, \ldots, r$. Since $w \geq 0$ for all vectors $(x, w) \in \hat{P}$, we see that $\tilde{d}_j \geq 0$ for all j. Let

$$J^+ = \{j \mid \tilde{d}_j > 0\}, \qquad J^0 = \{j \mid \tilde{d}_j = 0\}.$$

By replacing, for all $j \in J^+$, $\nu_j \tilde{d}_j$ with μ_j and $\tilde{v}_j / \tilde{d}_j$ with v_j, and for all $j \in J^0$, ν_j with μ_j and $\tilde{v}_j$ with v_j, we obtain the equivalent description

$$\hat{P} = \left\{ (x, w) \;\middle|\; x = \sum_{j \in J^+ \cup J^0} \mu_j v_j,\ w = \sum_{j \in J^+} \mu_j,\ \mu_j \geq 0,\ j \in J^+ \cup J^0 \right\}. \tag{2.1}$$

Since $P = \big\{x \mid (x, 1) \in \hat{P}\big\}$, we obtain

$$P = \left\{ x \;\middle|\; x = \sum_{j \in J^+ \cup J^0} \mu_j v_j,\ \sum_{j \in J^+} \mu_j = 1,\ \mu_j \geq 0,\ j \in J^+ \cup J^0 \right\}. \tag{2.2}$$

Thus, P is the vector sum of $\mathrm{conv}\big(\{v_j \mid j \in J^+\}\big)$ and the finitely generated cone

$$\left\{ \sum_{j \in J^0} \mu_j v_j \;\middle|\; \mu_j \geq 0,\ j \in J^0 \right\}.$$

To prove that the vector sum of $\mathrm{conv}\big(\{v_1, \ldots, v_m\}\big)$ and a finitely generated cone is a polyhedral set, we reverse the preceding argument: starting from Eq. (2.2), we express P as $\big\{x \mid (x, 1) \in \hat{P}\big\}$, where $\hat{P}$ is the finitely generated cone of Eq. (2.1). We then use the Minkowski-Weyl Theorem to assert that this cone is polyhedral, and we finally construct a polyhedral set description using the equation $P = \big\{x \mid (x, 1) \in \hat{P}\big\}$. **Q.E.D.**

As indicated by the examples of Fig. 2.3.3, the finitely generated cone C in a Minkowski-Weyl representation of a polyhedral set P is just the recession cone of P. Let us now use the Minkowski-Weyl representation to show that the major algebraic operations on polyhedral sets preserve their polyhedral character.

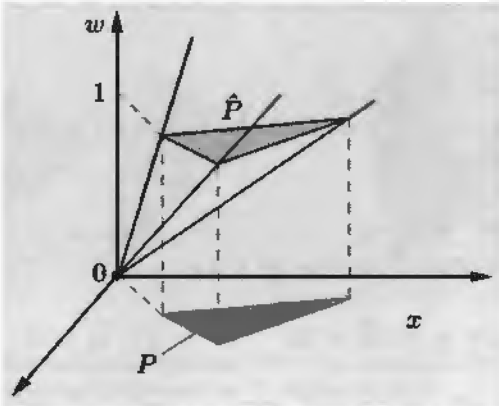


Figure 2.3.4. Illustration of the proof of the Minkowski-Weyl representation. The cone of $\Re^{n+1}$

$$\hat{P} = \{(x, w) \mid 0 \le w,\ a_j'x \le b_j w,\ j = 1, \ldots, r\},$$

is derived from the polyhedral set

$$P = \{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\}.$$

Proposition 2.3.4: (Algebraic Operations on Polyhedral Sets)

(a) The intersection of polyhedral sets is polyhedral, if it is nonempty.

(b) The Cartesian product of polyhedral sets is polyhedral.

(c) The image of a polyhedral set under a linear transformation is a polyhedral set.

(d) The vector sum of two polyhedral sets is polyhedral.

(e) The inverse image of a polyhedral set under a linear transformation is polyhedral.

Proof: Parts (a) and (b) are evident from the definition of a polyhedral set. To show part (c), let the polyhedral set P be represented as

$$P = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big),$$

and let A be a matrix. We have

$$AP = \mathrm{conv}\big(\{Av_1, \ldots, Av_m\}\big) + \mathrm{cone}\big(\{Aa_1, \ldots, Aa_r\}\big).$$

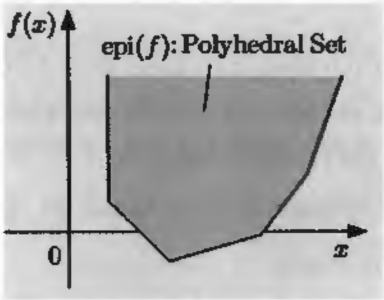


Figure 2.3.5. Illustration of a polyhedral function. By definition, the function must be proper, and its epigraph must be a polyhedral set.

It follows that AP has a Minkowski-Weyl representation, and by Prop. 2.3.3, it is polyhedral. Part (d) follows from part (c) since $P_1 + P_2$ can be viewed as the image of the polyhedral set $P_1 \times P_2$ under the linear transformation $(x_1, x_2) \mapsto (x_1 + x_2)$.

To prove part (e) note that the inverse image of a polyhedral set

$$\{y \mid a_j'y \leq b_j,\, j = 1, \ldots, r\}$$

under the linear transformation A is the set

$$\{x \mid a_j'Ax \leq b_j,\, j = 1, \ldots, r\},$$

which is clearly polyhedral. **Q.E.D.**

2.3.3 Polyhedral Functions

Polyhedral sets can also be used to define functions with polyhedral structure. In particular, we say that a function $f : \Re^n \mapsto (-\infty, \infty]$ is *polyhedral* if its epigraph is a polyhedral set in $\Re^{n+1}$; see Fig. 2.3.5. Note that a polyhedral function f is, by definition, closed, convex, and also proper [since f cannot take the value $-\infty$, and epi(f) is closed, convex, and nonempty (based on our convention that only nonempty sets can be polyhedral)]. The following proposition provides a useful representation of polyhedral functions.

Proposition 2.3.5: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a convex function. Then f is polyhedral if and only if dom(f) is a polyhedral set and

$$f(x) = \max_{j=1,\ldots,m} \{a_j'x + b_j\}, \qquad \forall\, x \in \text{dom}(f),$$

where a_j are vectors in $\Re^n$, b_j are scalars, and m is a positive integer.

Proof: If f has the representation given above, then epi(f) is written as

$$\text{epi}(f) = \{(x,w) \mid x \in \text{dom}(f)\} \cap \{(x,w) \mid a_j'x + b_j \le w,\ j = 1,\ldots,m\}.$$

Since the two sets in the right-hand side are polyhedral, their intersection, epi(f), [which is nonempty since dom(f) is polyhedral and hence nonempty] is also polyhedral. Hence f is polyhedral.

Conversely, if f is polyhedral, its epigraph is a polyhedral set, and can be represented as

$$\{(x,w) \mid a_j'x + b_j \le c_j w,\ j = 1,\ldots,r\},$$

where a_j are some vectors in $\Re^n$, and b_j and c_j are some scalars. Since for any $(x,w) \in \text{epi}(f)$, we have $(x, w+\gamma) \in \text{epi}(f)$ for all $\gamma \ge 0$, it follows that $c_j \ge 0$, so by normalizing if necessary, we may assume without loss of generality that either $c_j = 0$ or $c_j = 1$. If $c_j = 0$ for all j, then f would not be proper, contradicting the fact that a polyhedral function is proper. Hence we must have, for some m with $1 \le m \le r$, $c_j = 1$ for $j = 1,\ldots,m$, and $c_j = 0$ for $j = m+1,\ldots,r$, i.e.,

$$\text{epi}(f) = \{(x,w) \mid a_j'x + b_j \le w,\ j = 1,\ldots,m,\ a_j'x + b_j \le 0,\ j = m+1,\ldots,r\}.$$

Thus the effective domain of f is the polyhedral set

$$\text{dom}(f) = \{x \mid a_j'x + b_j \le 0,\ j = m+1,\ldots,r\},$$

and we have

$$f(x) = \max_{j=1,\ldots,m} \{a_j'x + b_j\}, \qquad \forall\, x \in \text{dom}(f).$$

Q.E.D.

Some common operations on polyhedral functions, such as sum and linear composition preserve their polyhedral character as shown by the following two propositions.

> **Proposition 2.3.6:** The sum of two polyhedral functions f_1 and f_2, such that $\text{dom}(f_1) \cap \text{dom}(f_2) \neq \emptyset$, is a polyhedral function.

Proof: By Prop. 2.3.5, dom(f_1) and dom(f_2) are polyhedral sets in $\Re^n$, and

$$f_1(x) = \max\{a_1'x + b_1, \ldots, a_m'x + b_m\}, \qquad \forall\, x \in \text{dom}(f_1),$$

$$f_2(x) = \max\{\overline{a}_1'x + \overline{b}_1, \ldots, \overline{a}_{\overline{m}}'x + \overline{b}_{\overline{m}}\}, \qquad \forall\, x \in \mathrm{dom}(f_2),$$

where a_i and $\overline{a}_i$ are vectors in $\Re^n$, and b_i and $\overline{b}_i$ are scalars. The domain of $f_1 + f_2$ is $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, which is polyhedral since $\mathrm{dom}(f_1)$ and $\mathrm{dom}(f_2)$ are polyhedral. Furthermore, for all $x \in \mathrm{dom}(f_1 + f_2)$,

$$\begin{aligned} f_1(x) + f_2(x) &= \max\{a_1'x + b_1, \ldots, a_m'x + b_m\} \\ &\quad + \max\{\overline{a}_1'x + \overline{b}_1, \ldots, \overline{a}_{\overline{m}}'x + \overline{b}_{\overline{m}}\} \\ &= \max_{1 \le i \le m,\ 1 \le j \le \overline{m}} \{a_i'x + b_i + \overline{a}_j'x + \overline{b}_j\} \\ &= \max_{1 \le i \le m,\ 1 \le j \le \overline{m}} \{(a_i + \overline{a}_j)'x + (b_i + \overline{b}_j)\}. \end{aligned}$$

Therefore, by Prop. 2.3.5, $f_1 + f_2$ is polyhedral. **Q.E.D.**

Proposition 2.3.7: If A is a matrix and g is a polyhedral function such that $\mathrm{dom}(g)$ contains a point in the range of A, the function f given by $f(x) = g(Ax)$ is polyhedral.

Proof: Since $g : \Re^m \mapsto (-\infty, \infty]$ is polyhedral, by Prop. 2.3.5, $\mathrm{dom}(g)$ is a polyhedral set in $\Re^m$ and g is given by

$$g(y) = \max\{a_1'y + b_1, \ldots, a_m'y + b_m\}, \qquad \forall\, y \in \mathrm{dom}(g),$$

for some vectors a_i in $\Re^m$ and scalars b_i. We have

$$\mathrm{dom}(f) = \{x \mid f(x) < \infty\} = \{x \mid g(Ax) < \infty\} = \{x \mid Ax \in \mathrm{dom}(g)\}.$$

Thus, $\mathrm{dom}(f)$ is the inverse image of the polyhedral set $\mathrm{dom}(g)$ under the linear transformation A. By the assumption that $\mathrm{dom}(g)$ contains a point in the range of A, it follows that $\mathrm{dom}(f)$ is nonempty, while $\mathrm{dom}(f)$ is polyhedral. Furthermore, for all $x \in \mathrm{dom}(f)$, we have

$$\begin{aligned} f(x) &= g(Ax) \\ &= \max\{a_1'Ax + b_1, \ldots, a_m'Ax + b_m\} \\ &= \max\{(A'a_1)'x + b_1, \ldots, (A'a_m)'x + b_m\}. \end{aligned}$$

Thus, by Prop. 2.3.5, the function f is polyhedral. **Q.E.D.**

2.4 POLYHEDRAL ASPECTS OF OPTIMIZATION

Polyhedral convexity plays a very important role in optimization. One reason is that many practical problems can be readily formulated in terms of polyhedral sets and functions. Another reason is that for polyhedral constraint sets and/or linear cost functions, it is often possible to show stronger optimization results than those available for general convex constraint sets and/or general cost functions. We have seen a few such instances so far. In particular:

(1) A linear (or, more generally, concave) function that is not constant over a convex constraint set C, can only attain its minimum at a relative boundary point of C (Prop. 1.3.4).

(2) A linear (or, more generally, convex quadratic) function that is bounded below over a nonempty polyhedral set C, attains a minimum over C (Prop. 1.4.12).

In this section, we explore some further consequences of polyhedral convexity in optimization, with an emphasis on linear programming (the minimization of a linear function over a polyhedral set). In particular, we prove one of the fundamental linear programming results: if a linear function f attains a minimum over a polyhedral set C that has at least one extreme point, then f attains a minimum at some extreme point of C (as well as possibly at some other nonextreme points). This is a special case of the following more general result, which holds when f is concave, and C is closed and convex.

Proposition 2.4.1: Let C be a closed convex subset of $\Re^n$ that has at least one extreme point. A concave function $f : C \mapsto \Re$ that attains a minimum over C attains the minimum at some extreme point of C.

Proof: Let x^* minimize f over C. If $x^* \in \text{ri}(C)$ [see Fig. 2.4.1(a)], by Prop. 1.3.4, f must be constant over C, so it attains a minimum at an extreme point of C (since C has at least one extreme point by assumption). If $x^* \notin \text{ri}(C)$, then by Prop. 1.5.5, there exists a hyperplane H_1 properly separating x^* and C. Since $x^* \in C$, H_1 must contain x^*, so by the proper separation property, H_1 cannot contain C, and it follows that the intersection $C \cap H_1$ has dimension smaller than the dimension of C.

If $x^* \in \text{ri}(C \cap H_1)$ [see Fig. 2.4.1(b)], then f must be constant over $C \cap H_1$, so it attains a minimum at an extreme point of $C \cap H_1$ (since C contains an extreme point, it does not contain a line by Prop. 2.1.2, and hence $C \cap H_1$ does not contain a line, which implies that $C \cap H_1$ has an extreme point). By Prop. 2.1.1, this optimal extreme point is also an extreme point of C. If $x^* \notin \text{ri}(C \cap H_1)$, there exists a hyperplane H_2 properly separating x^* and $C \cap H_1$. Again, since $x^* \in C \cap H_1$, H_2 contains x^*, so it cannot contain $C \cap H_1$, and it follows that the intersection $C \cap H_1 \cap H_2$ has dimension smaller than the dimension of $C \cap H_1$.

If $x^* \in \text{ri}(C \cap H_1 \cap H_2)$ [see Fig. 2.4.1(c)], then f must be constant

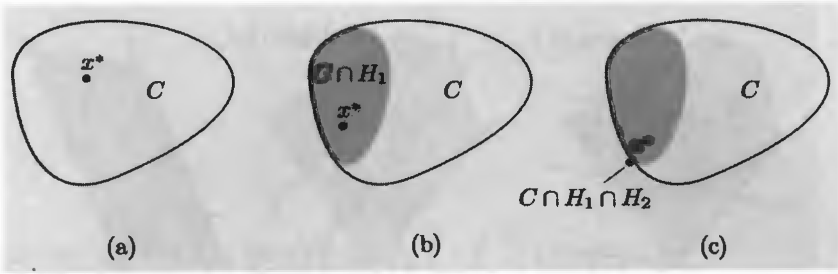


Figure 2.4.1. An outline of the argument of the proof of Prop. 2.4.1 for a 3-dimensional set C.

over $C \cap H_1 \cap H_2$, etc. Since with each new hyperplane, the dimension of the intersection of C with the generated hyperplanes is reduced, this process will be repeated at most n times, until x^* is a relative interior point of some set $C \cap H_1 \cap \cdots \cap H_k$, at which time an extreme point of $C \cap H_1 \cap \cdots \cap H_k$ will be obtained. Through a reverse argument, repeatedly applying Prop. 2.1.1, it follows that this extreme point is an extreme point of C. **Q.E.D.**

We now specialize the preceding result to the case of a linear program, where the cost function f is linear.

Proposition 2.4.2: (Fundamental Theorem of Linear Programming) Let P be a polyhedral set that has at least one extreme point. A linear function that is bounded below over P attains a minimum at some extreme point of P.

Proof: Since the cost function is bounded below over P, it attains a minimum (Prop. 1.4.12). The result now follows from Prop. 2.4.1. **Q.E.D.**

Figure 2.4.2 illustrates the possibilities for a linear programming problem. There are two cases:

(a) The constraint set P contains an extreme point (equivalently, by Prop. 2.1.2, P does not contain a line). In this case, the linear cost function is either unbounded below over P or else it attains a minimum at an extreme point of P. For an example, let $P = [0, \infty)$. Then, the minimum of $1 \cdot x$ over P is attained at the extreme point 0, the minimum of $0 \cdot x$ over P is attained at every point of P, including the extreme point 0, and the minimum of $-1 \cdot x$ over P is not attained while the cost is unbounded below.

(b) The constraint set P does not contain an extreme point (equivalently,

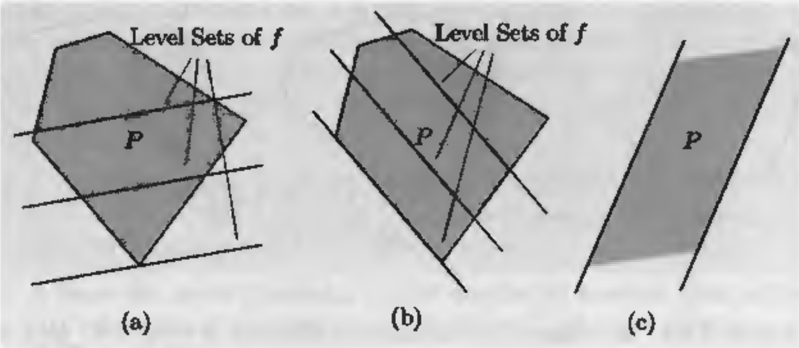


Figure 2.4.2. Illustration of the fundamental theorem of linear programming. In (a) and (b), the constraint set P has at least one extreme point and the linear cost function f is bounded over P. Then f either attains a minimum at a unique extreme point as in (a), or it attains a minimum at one or more extreme points as well as at an infinite number of nonextreme points as in (b). In (c), the constraint set has no extreme points because it contains a line (cf. Prop. 2.1.2), and the linear cost function is either unbounded below over P or attains a minimum at an infinite set of (nonextreme) points whose lineality space is equal to the lineality space of P.

by Prop. 2.1.2, P contains a line). In this case, L_P, the lineality space of P, is a subspace of dimension greater or equal to one. If the linear cost function f is bounded below over P, it attains a minimum by Prop. 1.4.12. However, because every direction in L_P must be a direction along which f is constant (otherwise f would be unbounded below over P), the set of minima is a polyhedral set whose lineality space is equal to L_P. Therefore, the set of minima must be unbounded. For an example, let $P = \Re$. Then, the minimum of $1 \cdot x$ over P is not attained and the cost is unbounded below, while the set of minima of $0 \cdot x$ over P is P.

The theory of polyhedral sets and their extreme points can be used for the development of linear programming algorithms, such as the simplex method and other related methods. These algorithms are outside our scope, so we refer to standard textbooks, such as Dantzig [Dan63], Chvatal [Chv83], and Bertsimas and Tsitsiklis [BeT97].

3

Basic Concepts of Convex Optimization

Contents

In this chapter, we introduce some basic concepts of convex optimization and minimax theory, with a special focus on the question of existence of optimal solutions.

3.1 CONSTRAINED OPTIMIZATION

Let us consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

where $f : \Re^n \mapsto (-\infty, \infty]$ is a function and X is a nonempty subset of $\Re^n$. Any vector $x \in X \cap \text{dom}(f)$ is said to be a *feasible solution* of the problem (we also use the terms *feasible vector* or *feasible point*). If there is at least one feasible solution, i.e., $X \cap \text{dom}(f) \neq \emptyset$, we say that the problem is *feasible*; otherwise we say that the problem is *infeasible*. Thus, when f is extended real-valued, we view only the points in $X \cap \text{dom}(f)$ as candidates for optimality, and we view $\text{dom}(f)$ as an implicit constraint set. Furthermore, feasibility of the problem is equivalent to $\inf_{x \in X} f(x) < \infty$.

We say that a vector x^* is a *minimum of f over X* if

$$x^* \in X \cap \text{dom}(f), \qquad \text{and} \qquad f(x^*) = \inf_{x \in X} f(x).$$

We also call x^* a *minimizing point* or *minimizer* or *global minimum of f over X*. Alternatively, we say that f *attains a minimum over X at x^**, and we indicate this by writing

$$x^* \in \arg\min_{x \in X} f(x).$$

If x^* is known to be the unique minimizer of f over X, with slight abuse of notation, we also write

$$x^* = \arg\min_{x \in X} f(x).$$

We use similar terminology for maxima, i.e., a vector $x^* \in X$ such that $f(x^*) = \sup_{x \in X} f(x)$ is said to be a *maximum of f over X* if x^* is a minimum of $(-f)$ over X, and we indicate this by writing

$$x^* \in \arg\max_{x \in X} f(x).$$

If $X = \Re^n$ or if the domain of f is the set X (instead of $\Re^n$), we also call x^* a (global) minimum or (global) maximum of f (without the qualifier "over X").

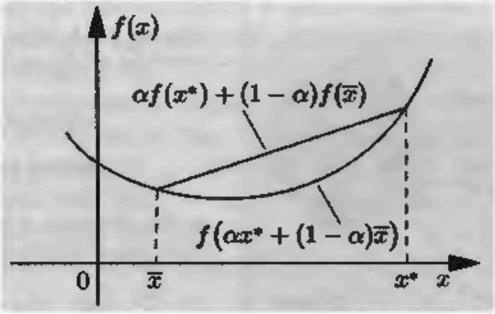


Figure 3.1.1. Illustration of why local minima of convex functions are also global (cf. Prop. 3.1.1). Given x^* and $\overline{x}$ with $f(\overline{x}) < f(x^*)$, every point of the form

$$x_\alpha = \alpha x^* + (1-\alpha)\overline{x}, \qquad \alpha \in (0,1),$$

satisfies $f(x_\alpha) < f(x^*)$. Thus x^* cannot be a local minimum that is not global.

Local Minima

Often in optimization problems we have to deal with a weaker form of minimum, one that is optimum only when compared with points that are "nearby." In particular, given a subset X of $\Re^n$ and a function $f : \Re^n \mapsto (-\infty, \infty]$, we say that a vector x^* is a *local minimum of f over X* if $x^* \in X \cap \text{dom}(f)$ and there exists some $\epsilon > 0$ such that

$$f(x^*) \leq f(x), \qquad \forall\, x \in X \text{ with } \|x - x^*\| < \epsilon.$$

If $X = \Re^n$ or the domain of f is the set X (instead of $\Re^n$), we also call x^* a local minimum of f (without the qualifier "over X"). A local minimum x^* is said to be *strict* if there is no other local minimum within some open sphere centered at x^*. Local maxima are defined similarly.

In practical applications we are typically interested in global minima, yet we have to contend with local minima because of the inability of many optimality conditions and algorithms to distinguish between the two types of minima. This can be a major practical difficulty, but an important implication of convexity of f and X is that all local minima are also global, as shown in the following proposition and in Fig. 3.1.1.

Proposition 3.1.1: If X is a convex subset of $\Re^n$ and $f : \Re^n \mapsto (-\infty, \infty]$ is a convex function, then a local minimum of f over X is also a global minimum. If in addition f is strictly convex, then there exists at most one global minimum of f over X.

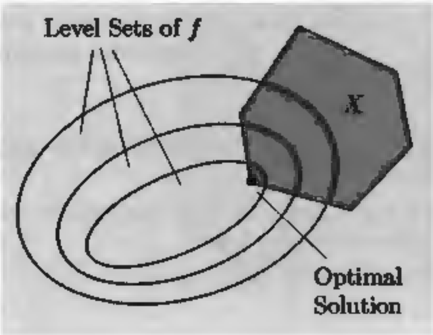


Figure 3.2.1. View of the set of optimal solutions of the problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && x \in X, \end{aligned}$$

as the intersection of all the nonempty level sets of the form

$$\{x \in X \mid f(x) \leq \gamma\}, \quad \gamma \in \Re.$$

Proof: Let f be convex, and assume to arrive at a contradiction, that x^* is a local minimum of f over X that is not global (see Fig. 3.1.1). Then, there must exist an $\overline{x} \in X$ such that $f(\overline{x}) < f(x^*)$. By convexity, for all $\alpha \in (0, 1)$,

$$f\big(\alpha x^* + (1 - \alpha)\overline{x}\big) \leq \alpha f(x^*) + (1 - \alpha) f(\overline{x}) < f(x^*).$$

Thus, f has strictly lower value than $f(x^*)$ at every point on the line segment connecting x^* with $\overline{x}$, except at x^*. Since X is convex, the line segment belongs to X, thereby contradicting the local minimality of x^*.

Let f be strictly convex, let x^* be a global minimum of f over X, and let x be a point in X with $x \neq x^*$. Then the midpoint $y = (x+x^*)/2$ belongs to X since X is convex, and by strict convexity, $f(y) < 1/2\big(f(x) + f(x^*)\big)$, while by the optimality of x^*, we have $f(x^*) \leq f(y)$. These two relations imply that $f(x^*) < f(x)$, so x^* is the unique global minimum. **Q.E.D.**

3.2 EXISTENCE OF OPTIMAL SOLUTIONS

A basic question in optimization problems is whether an optimal solution exists. It can be seen that the set of minima of a real-valued function f over a nonempty set X, call it X^*, is equal to the intersection of X and the level sets of f that have a common point with X:

$$X^* = \cap_{k=0}^{\infty} \{x \in X \mid f(x) \leq \gamma_k\},$$

where $\{\gamma_k\}$ is any scalar sequence with $\gamma_k \downarrow \inf_{x \in X} f(x)$ (see Fig. 3.2.1).

From this characterization of X^*, it follows that the set of minima is nonempty and compact if the sets

$$\{x \in X \mid f(x) \leq \gamma\},$$

are compact (since the intersection of a nested sequence of nonempty and compact sets is nonempty and compact). This is the essence of the classical theorem of Weierstrass (Prop. A.2.7), which states that a continuous function attains a minimum over a compact set. We will provide a more general version of this theorem, and to this end, we introduce some terminology.

We say that a function $f : \Re^n \mapsto (-\infty, \infty]$ is *coercive* if for every sequence $\{x_k\}$ such that $\|x_k\| \to \infty$, we have $\lim_{k\to\infty} f(x_k) = \infty$. Note that as a consequence of the definition, if $\mathrm{dom}(f)$ is bounded, then f is coercive. Furthermore, all the nonempty level sets of a coercive function are bounded.

Proposition 3.2.1: (Weierstrass' Theorem) Consider a closed proper function $f : \Re^n \mapsto (-\infty, \infty]$, and assume that any one of the following three conditions holds:

(1) $\mathrm{dom}(f)$ is bounded.

(2) There exists a scalar $\overline{\gamma}$ such that the level set

$$\{x \mid f(x) \le \overline{\gamma}\}$$

is nonempty and bounded.

(3) f is coercive.

Then the set of minima of f over $\Re^n$ is nonempty and compact.

Proof: It is sufficient to show that each of the three conditions implies that the nonempty level sets $V_\gamma = \{x \mid f(x) \le \gamma\}$ are compact for all $\gamma \le \overline{\gamma}$, where $\overline{\gamma}$ is such that $V_{\overline{\gamma}}$ is nonempty and compact, and then use the fact that the set of minima of f is the intersection of its nonempty level sets. (Note that f is assumed proper, so it has some nonempty level sets.) Since f is closed, its level sets are closed (cf. Prop. 1.1.2). It is evident that under each of the three conditions the level sets are also bounded for γ less or equal to some $\overline{\gamma}$, so they are compact. **Q.E.D.**

The most common application of Weierstrass' Theorem is when we want to minimize a real-valued function $f : \Re^n \mapsto \Re$ over a nonempty set X. Then, by applying the proposition to the extended real-valued function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases}$$

we see that the set of minima of f over X is nonempty and compact if X is closed, f is lower semicontinuous at each $x \in X$ (implying, by Prop. 1.1.3, that $\tilde{f}$ is closed), and one of the following conditions holds:

(1) X is bounded.

(2) Some set $\{x \in X \mid f(x) \leq \overline{\gamma}\}$ is nonempty and bounded.

(3) $\tilde{f}$ is coercive, or equivalently, for every sequence $\{x_k\} \subset X$ such that $\|x_k\| \to \infty$, we have $\lim_{k\to\infty} f(x_k) = \infty$.

The following is essentially Weierstrass' Theorem specialized to convex functions.

Proposition 3.2.2: Let X be a closed convex subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed convex function with $X \cap \text{dom}(f) \neq \emptyset$. The set of minima of f over X is nonempty and compact if and only if X and f have no common nonzero direction of recession.

Proof: Let $f^* = \inf_{x\in X} f(x)$ and note that $f^* < \infty$ since $X \cap \text{dom}(f) \neq \emptyset$. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and consider the sets

$$V_k = \{x \mid f(x) \leq \gamma_k\}.$$

Then the set of minima of f over X is

$$X^* = \cap_{k=1}^{\infty}(X \cap V_k).$$

The sets $X \cap V_k$ are nonempty and have $R_X \cap R_f$ as their common recession cone, which is also the recession cone of X^*, when $X^* \neq \emptyset$ [cf. Props. 1.4.5, 1.4.2(c)]. It follows using Prop. 1.4.2(a) that X^* is nonempty and compact if and only if $R_X \cap R_f = \{0\}$. **Q.E.D.**

If X and f of the above proposition have a common direction of recession, then either the optimal solution set is empty [take for example, $X = \Re$ and $f(x) = e^x$] or else it is nonempty and unbounded [take for example, $X = \Re$ and $f(x) = \max\{0, x\}$]. Here is another result that addresses an important special case where the set of minima is compact.

Proposition 3.2.3: (Existence of Solution, Sum of Functions) Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be closed proper convex functions such that the function $f = f_1 + \cdots + f_m$ is proper. Assume that the recession function of a single function f_i satisfies $r_{f_i}(d) = \infty$ for all $d \neq 0$. Then the set of minima of f is nonempty and compact.

Proof: By Prop. 3.2.2, the set of minima of f is nonempty and compact if and only if $R_f = \{0\}$, which by Prop. 1.4.6, is true if and only if $r_f(d) > 0$ for all $d \neq 0$. The result now follows from Prop. 1.4.8. **Q.E.D.**

As an example of application of the preceding proposition, if one of the functions f_i is a positive definite quadratic function, the set of minima of the sum f is nonempty and compact. In fact in this case f has a unique minimum because the positive definite quadratic is strictly convex, which makes f strictly convex.

The next proposition addresses cases where the optimal solution set is not compact.

Proposition 3.2.4: (Existence of Solution, Noncompact Level Sets) Let X be a closed convex subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed convex function with $X \cap \mathrm{dom}(f) \neq \emptyset$. The set of minima of f over X, denoted X^*, is nonempty under any one of the following two conditions:

(1) $R_X \cap R_f = L_X \cap L_f$.

(2) $R_X \cap R_f \subset L_f$ and X is a polyhedral set.

Furthermore, under condition (1),

$$X^* = \tilde{X} + (L_X \cap L_f),$$

where $\tilde{X}$ is some nonempty and compact set.

Proof: Let $f^* = \inf_{x \in X} f(x)$ and note that $f^* < \infty$ since $X \cap \mathrm{dom}(f) \neq \emptyset$. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, consider the level sets

$$V_k = \{x \mid f(x) \leq \gamma_k\},$$

and note that

$$X^* = \cap_{k=1}^{\infty} (X \cap V_k).$$

Let condition (1) hold. The sets $X \cap V_k$ are nonempty, closed, convex, and nested. Furthermore, they have the same recession cone, $R_X \cap R_f$, and the same lineality space $L_X \cap L_f$, while by assumption, $R_X \cap R_f = L_X \cap L_f$. By Prop. 1.4.11(a), it follows that X^* is nonempty and has the form

$$X^* = \tilde{X} + (L_X \cap L_f),$$

where $\tilde{X}$ is some nonempty compact set.

Let condition (2) hold. The sets V_k are nested and $X \cap V_k$ is nonempty for all k. Furthermore, all the sets V_k have the same recession cone, R_f, and the same lineality space, L_f, while by assumption, $R_X \cap R_f \subset L_f$, and X is polyhedral and hence retractive (cf. Prop. 1.4.9). By Prop. 1.4.11(b), it follows that X^* is nonempty. **Q.E.D.**

Note that in the special case $X = \Re^n$, conditions (1) and (2) of Prop. 3.2.4 coincide. Figure 3.2.3(b) provides a counterexample showing that if X is nonpolyhedral, the condition

$$R_X \cap R_f \subset L_f$$

is not sufficient to guarantee the existence of optimal solutions or even the finiteness of f^*. This counterexample also shows that the cost function may be bounded below and attain a minimum over any closed halfline contained in the constraint set, and yet it may not attain a minimum over the entire set. Recall, however, that in the special cases of linear and quadratic programming problems, boundedness from below of the cost function over the constraint set guarantees the existence of an optimal solution (cf. Prop. 1.4.12).

3.3 PARTIAL MINIMIZATION OF CONVEX FUNCTIONS

In our development of duality and minimax theory we will often encounter functions obtained by minimizing other functions partially, i.e., with respect to some of their variables. It is then useful to be able to deduce properties of the function obtained, such as convexity and closedness, from corresponding properties of the original.

There is an important geometric relation between the epigraph of a given function and the epigraph of its partially minimized version: except for some boundary points, *the latter is obtained by projection from the former* [see part (b) of the following proposition, and Fig. 3.3.1]. This is the key to understanding the properties of partially minimized functions.

Proposition 3.3.1: Consider a function $F : \Re^{n+m} \mapsto (-\infty, \infty]$ and the function $f : \Re^n \mapsto [-\infty, \infty]$ defined by

$$f(x) = \inf_{z \in \Re^m} F(x, z).$$

Then:

(a) If F is convex, then f is also convex.

(b) We have

$$P\big(\text{epi}(F)\big) \subset \text{epi}(f) \subset \text{cl}\Big(P\big(\text{epi}(F)\big)\Big), \tag{3.1}$$

where $P(\cdot)$ denotes projection on the space of (x, w), i.e., for any subset S of $\Re^{n+m+1}$, $P(S) = \big\{(x, w) \mid (x, z, w) \in S\big\}$.

Proof: (a) If $\text{epi}(f) = \emptyset$, i.e., $f(x) = \infty$ for all $x \in \Re^n$, then $\text{epi}(f)$ is convex, so f is convex. Assume that $\text{epi}(f) \neq \emptyset$, and let $(\overline{x}, \overline{w})$ and $(\tilde{x}, \tilde{w})$

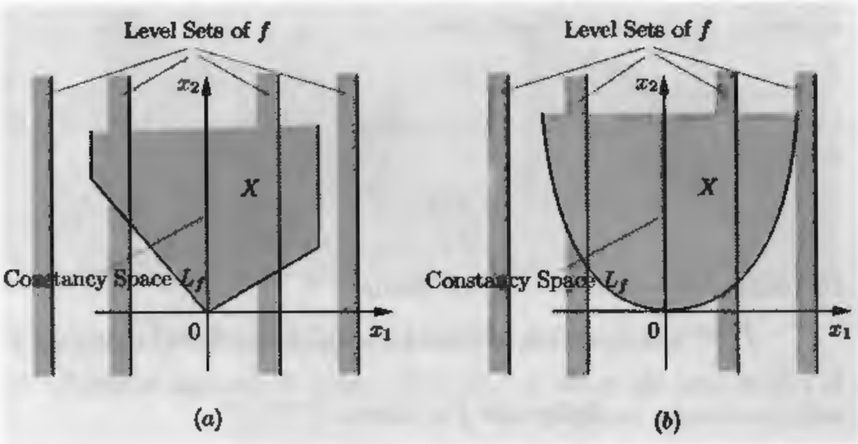


Figure 3.2.3. Illustration of the issues regarding existence of an optimal solution assuming $R_X \cap R_f \subset L_f$, i.e., that every common direction of recession of X and f is a direction in which f is constant [cf. Prop. 3.2.4 under condition (2)].

In both problems illustrated in (a) and (b) the cost function is

$$f(x_1, x_2) = e^{x_1}.$$

In the problem of (a), the constraint set X is the polyhedral set shown in the figure, while in the problem of (b), X is specified by a quadratic inequality:

$$X = \left\{(x_1, x_2) \mid x_1^2 \leq x_2\right\},$$

as shown in the figure. In both cases we have

$$R_X = \left\{(d_1, d_2) \mid d_1 = 0,\ d_2 \geq 0\right\},$$

$$R_f = \left\{(d_1, d_2) \mid d_1 \leq 0,\ d_2 \in \Re\right\}, \qquad L_f = \left\{(d_1, d_2) \mid d_1 = 0,\ d_2 \in \Re\right\},$$

so that $R_X \cap R_f \subset L_f$.

In the problem of (a) it can be seen that an optimal solution exists. In the problem of (b), however, we have $f(x_1, x_2) > 0$ for all (x_1, x_2), while for $x_1 = -\sqrt{x_2}$ where $x_2 \geq 0$, we have $(x_1, x_2) \in X$ with

$$\lim_{x_2 \to \infty} f\left(-\sqrt{x_2}, x_2\right) = \lim_{x_2 \to \infty} e^{-\sqrt{x_2}} = 0,$$

implying that $f^* = 0$. Thus f cannot attain the minimum value f^* over X. Note that f attains a minimum over the intersection of any line with X.

If in the problem of (b) the cost function were instead $f(x_1, x_2) = x_1$, we would still have $R_X \cap R_f \subset L_f$ and f would still attain a minimum over the intersection of any line with X, but it can be seen that $f^* = -\infty$. If the constraint set were instead $X = \left\{(x_1, x_2) \mid |x_1| \leq x_2\right\}$, which is polyhedral, we would still have $f^* = -\infty$, but then the condition $R_X \cap R_f \subset L_f$ would be violated.

be two points in epi(f). Then $f(\overline{x}) < \infty$, $f(\tilde{x}) < \infty$, and there exist sequences $\{\overline{z}_k\}$ and $\{\tilde{z}_k\}$ such that

$$F(\overline{x}, \overline{z}_k) \to f(\overline{x}), \qquad F(\tilde{x}, \tilde{z}_k) \to f(\tilde{x}).$$

Using the definition of f and the convexity of F, we have for all $\alpha \in [0, 1]$ and k,

$$\begin{aligned} f\big(\alpha\overline{x} + (1-\alpha)\tilde{x}\big) &\le F\big(\alpha\overline{x} + (1-\alpha)\tilde{x}, \alpha\overline{z}_k + (1-\alpha)\tilde{z}_k\big) \\ &\le \alpha F(\overline{x}, \overline{z}_k) + (1-\alpha)F(\tilde{x}, \tilde{z}_k). \end{aligned}$$

By taking the limit as $k \to \infty$, we obtain

$$f\big(\alpha\overline{x} + (1-\alpha)\tilde{x}\big) \le \alpha f(\overline{x}) + (1-\alpha)f(\tilde{x}) \le \alpha\overline{w} + (1-\alpha)\tilde{w}.$$

It follows that the point $\alpha(\overline{x}, \overline{w}) + (1-\alpha)(\tilde{x}, \tilde{w})$ belongs to epi(f). Thus epi(f) is convex, implying that f is convex.

(b) To show the left-hand side of Eq. (3.1), let $(x, w) \in P\big(\text{epi}(F)\big)$, so that there exists $\overline{z}$ such that $(x, \overline{z}, w) \in \text{epi}(F)$, or equivalently $F(x, \overline{z}) \le w$. Then

$$f(x) = \inf_{z \in \Re^m} F(x, z) \le w,$$

implying that $(x, w) \in \text{epi}(f)$.

To show the right-hand side, note that for any $(x, w) \in \text{epi}(f)$ and every k, there exists a z_k such that

$$(x, z_k, w + 1/k) \in \text{epi}(F),$$

so that $(x, w + 1/k) \in P\big(\text{epi}(F)\big)$ and $(x, w) \in \text{cl}\Big(P\big(\text{epi}(F)\big)\Big)$. **Q.E.D.**

Among other things, part (b) of the preceding proposition asserts that if F is closed, and if the projection operation preserves closedness of its epigraph, then partial minimization of F yields a closed function. Note also a connection between closedness of $P\big(\text{epi}(F)\big)$ and the attainment of the infimum of $F(x, z)$ over z. As illustrated in Fig. 3.3.1, for a fixed x, $F(x, z)$ attains a minimum over z if and only if $\big(x, f(x)\big)$ belongs to $P\big(\text{epi}(F)\big)$. Thus if $P\big(\text{epi}(F)\big)$ is closed, $F(x, z)$ attains a minimum over z for all x such that $f(x)$ is finite.

We now provide criteria guaranteeing that closedness is preserved under partial minimization, while simultaneously the partial minimum is attained.

Proposition 3.3.2: Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider the function f given by

$$f(x) = \inf_{z \in \Re^m} F(x, z), \qquad x \in \Re^n.$$

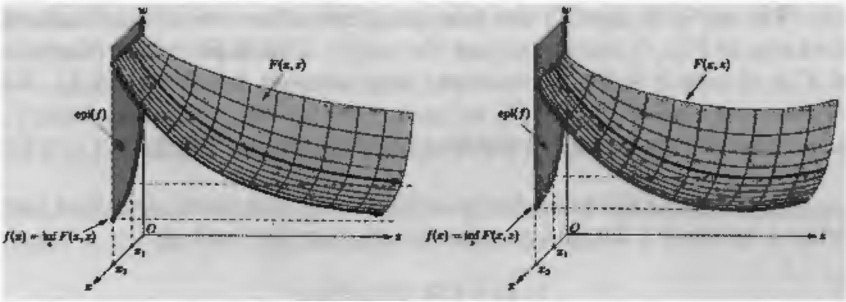


Figure 3.3.1. Illustration of partial minimization and the relation

$$P\big(\mathrm{epi}(F)\big) \subset \mathrm{epi}(f) \subset \mathrm{cl}\Big(P\big(\mathrm{epi}(F)\big)\Big).$$

In the figure on the right, where equality holds throughout in the above relation, the minimum over z is attained for all x such that $f(x)$ is finite. In the figure on the left the minimum over z is not attained and $P\big(\mathrm{epi}(F)\big)$ is not equal to epi(f) because it is missing the points $\big(x, f(x)\big)$.

Assume that for some $\overline{x} \in \Re^n$ and $\overline{\gamma} \in \Re$ the set

$$\big\{z \mid F(\overline{x}, z) \leq \overline{\gamma}\big\}$$

is nonempty and compact. Then f is closed proper convex. Furthermore, for each $x \in \mathrm{dom}(f)$, the set of minima in the definition of $f(x)$ is nonempty and compact.

Proof: We first note that by Prop. 3.3.1(a), f is convex. By Prop. 1.4.5(a), the recession cone of F has the form

$$R_F = \big\{(d_x, d_z) \mid (d_x, d_z, 0) \in R_{\mathrm{epi}(F)}\big\}.$$

The (common) recession cone of the nonempty level sets of $F(\overline{x}, \cdot)$ has the form

$$\big\{d_z \mid (0, d_z) \in R_F\big\},$$

and by the compactness hypothesis, using also Props. 1.4.2(a) and 1.4.5(b), it consists of just the origin. Thus there exists no vector $d_z \neq 0$ such that $(0, d_z, 0) \in R_{\mathrm{epi}(F)}$. Equivalently, there is no nonzero vector in $R_{\mathrm{epi}(F)}$ that belongs to the nullspace of $P(\cdot)$, where $P(\cdot)$ denotes projection on the space of (x, w). Therefore by the closedness of epi(F) and Prop. 1.4.13, $P\big(\mathrm{epi}(F)\big)$ is a closed set, and from the relation (3.1) it follows that f is closed.

For any $x \in \text{dom}(f)$ the (common) recession cone of the nonempty level sets of $F(x, \cdot)$ consists of just the origin. Therefore the set of minima of $F(x, z)$ over $z \in \Re^m$ is nonempty and compact (cf. Prop. 3.2.2). Furthermore, for all $x \in \text{dom}(f)$, we have $f(x) > -\infty$, and since $\text{dom}(f)$ is nonempty (it contains $\overline{x}$), it follows that f is proper. **Q.E.D.**

Note that as the preceding proof indicates, the assumption that there exists a vector $\overline{x} \in \Re^n$ and a scalar $\overline{\gamma}$ such that the level set

$$\{z \mid F(\overline{x}, z) \leq \overline{\gamma}\}$$

is nonempty and compact is equivalent to assuming that all the nonempty level sets of the form $\{z \mid F(x, z) \leq \gamma\}$ are compact. This is so because all these sets have the same recession cone, namely $\{d_z \mid (0, d_z) \in R_F\}$.

A simple but useful special case of the preceding proposition is the following.

Proposition 3.3.3: Let X and Z be nonempty convex sets of $\Re^n$ and $\Re^m$, respectively, let $F : X \times Z \mapsto \Re$ be a closed convex function, and assume that Z is compact. Then the function f given by

$$f(x) = \inf_{z \in Z} F(x, z), \qquad x \in X,$$

is a real-valued convex function over X.

Proof: Apply Prop. 3.3.2 with the partially minimized function being equal to $F(x, z)$ for $x \in X$ and $z \in Z$, and ∞ otherwise. **Q.E.D.**

The following is a slight generalization of Prop. 3.3.2.

Proposition 3.3.4: Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider the function f given by

$$f(x) = \inf_{z \in \Re^m} F(x, z), \qquad x \in \Re^n.$$

Assume that for some $\overline{x} \in \Re^n$ and $\overline{\gamma} \in \Re$ the set

$$\{z \mid F(\overline{x}, z) \leq \overline{\gamma}\}$$

is nonempty and its recession cone is equal to its lineality space. Then f is closed proper convex. Furthermore, for each $x \in \text{dom}(f)$, the set of minima in the definition of $f(x)$ is nonempty.

Proof: The proof is nearly identical to the one of Prop. 3.3.2, using Props. 1.4.13 and 3.2.4. **Q.E.D.**

3.4 SADDLE POINT AND MINIMAX THEORY

Let us consider a function $\phi : X \times Z \mapsto \Re$, where X and Z are nonempty subsets of $\Re^n$ and $\Re^m$, respectively. We wish to either

$$\begin{aligned} &\text{minimize} && \sup_{z \in Z} \phi(x, z) \\ &\text{subject to} && x \in X \end{aligned}$$

or

$$\begin{aligned} &\text{maximize} && \inf_{x \in X} \phi(x, z) \\ &\text{subject to} && z \in Z. \end{aligned}$$

Our main interest will be to derive conditions guaranteeing that

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z), \tag{3.2}$$

and that the infima and the suprema above are attained. Minimax problems are encountered in several important contexts.

One major context is *zero sum games*. In the simplest such game there are two players: the first may choose one out of n moves and the second may choose one out of m moves. If moves i and j are selected by the first and the second player, respectively, the first player gives a specified amount a_{ij} to the second. The objective of the first player is to minimize the amount given to the other player, and the objective of the second player is to maximize this amount. The players use mixed strategies, whereby the first player selects a probability distribution $x = (x_1, \ldots, x_n)$ over his n possible moves and the second player selects a probability distribution $z = (z_1, \ldots, z_m)$ over his m possible moves. Since the probability of selecting i and j is $x_i z_j$, the expected amount to be paid by the first player to the second is $\sum_{i,j} a_{ij} x_i z_j$ or $x'Az$, where A is the $n \times m$ matrix with components a_{ij}. If each player adopts a worst case viewpoint, whereby he optimizes his choice against the worst possible selection by the other player, the first player must minimize $\max_z x'Az$ and the second player must maximize $\min_x x'Az$. The main result, a special case of a theorem we will prove in Chapter 5, is that these two optimal values are equal, implying that there is an amount that can be meaningfully viewed as the value of the game for its participants.

Another major context is *duality theory for inequality-constrained problems* of the form

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && x \in X, \quad g(x) \le 0, \end{aligned} \tag{3.3}$$

where $f : \Re^n \mapsto \Re$, $g_j : \Re^n \mapsto \Re$ are given functions, and X is a nonempty subset of $\Re^n$. A common approach here is to introduce a vector $\mu = (\mu_1, \ldots, \mu_r) \in \Re^r$, consisting of multipliers for the inequality constraints, and to form the *Lagrangian function*

$$L(x, \mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x).$$

We may then form a dual problem of optimization over μ of a cost function defined in terms of the Lagrangian. In particular, we will introduce later the dual problem

$$\begin{aligned} &\text{maximize} \quad \inf_{x \in X} L(x, \mu) \\ &\text{subject to} \quad \mu \geq 0. \end{aligned}$$

Note that the original problem (3.3) can also be written as

$$\begin{aligned} &\text{minimize} \quad \sup_{\mu \geq 0} L(x, \mu) \\ &\text{subject to} \quad x \in X \end{aligned}$$

[if x violates any of the constraints $g_j(x) \leq 0$, we have $\sup_{\mu \geq 0} L(x, \mu) = \infty$, and if it does not, we have $\sup_{\mu \geq 0} L(x, \mu) = f(x)$]. A major question is whether there is no duality gap, i.e., whether the optimal primal and dual values are equal. This is so if and only if

$$\sup_{\mu \geq 0} \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \sup_{\mu \geq 0} L(x, \mu), \tag{3.4}$$

where L is the Lagrangian function.

In Section 5.5 we will prove a classical result, von Neumann's Saddle Point Theorem, which guarantees the minimax equality (3.2), as well as the attainment of the infima and suprema, assuming convexity/concavity assumptions on ϕ, and compactness assumptions on X and Z. Unfortunately, von Neumann's Theorem is not fully adequate for the development of constrained optimization duality theory, because compactness of Z and to some extent compactness of X turn out to be restrictive assumptions [for example Z corresponds to the set $\{\mu \mid \mu \geq 0\}$ in Eq. (3.4), which is not compact].

A first observation regarding the potential validity of the minimax equality (3.2) is that we always have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z), \tag{3.5}$$

[for every $\overline{z} \in Z$, write

$$\inf_{x \in X} \phi(x, \overline{z}) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

and take the supremum over $\overline{z} \in Z$ of the left-hand side]. We refer to this relation as the *minimax inequality*. It is sufficient to show the reverse inequality in order for the minimax equality (3.2) to hold. However, special conditions are required for the reverse inequality to be true.

Saddle Points

The following definition formalizes pairs of vectors that attain the infimum and the supremum in the minimax equality (3.2).

Definition 3.4.1: A pair of vectors $x^* \in X$ and $z^* \in Z$ is called a *saddle point* of ϕ if

$$\phi(x^*, z) \leq \phi(x^*, z^*) \leq \phi(x, z^*), \qquad \forall\, x \in X,\ \forall\, z \in Z.$$

Note that (x^*, z^*) is a saddle point if and only if $x^* \in X$, $z^* \in Z$, and

$$\sup_{z \in Z} \phi(x^*, z) = \phi(x^*, z^*) = \inf_{x \in X} \phi(x, z^*), \tag{3.6}$$

i.e., if "x^* minimizes against z^*" and "z^* maximizes against x^*". We have the following characterization of a saddle point.

Proposition 3.4.1: A pair (x^*, z^*) is a saddle point of ϕ if and only if the minimax equality (3.2) holds, and x^* is an optimal solution of the problem

$$\begin{aligned} &\text{minimize} \quad \sup_{z \in Z} \phi(x, z) \\ &\text{subject to} \quad x \in X, \end{aligned} \tag{3.7}$$

while z^* is an optimal solution of the problem

$$\begin{aligned} &\text{maximize} \quad \inf_{x \in X} \phi(x, z) \\ &\text{subject to} \quad z \in Z. \end{aligned} \tag{3.8}$$

Proof: Suppose that x^* is an optimal solution of problem (3.7) and z^* is an optimal solution of problem (3.8). Then we have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \phi(x, z^*) \leq \phi(x^*, z^*) \leq \sup_{z \in Z} \phi(x^*, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z).$$

If the minimax equality [cf. Eq. (3.2)] holds, then equality holds throughout above, so that

$$\sup_{z \in Z} \phi(x^*, z) = \phi(x^*, z^*) = \inf_{x \in X} \phi(x, z^*),$$

i.e., (x^*, z^*) is a saddle point of ϕ [cf. Eq. (3.6)].

Conversely, if (x^*, z^*) is a saddle point, then using Eq. (3.6), we have

$$\inf_{x\in X}\sup_{z\in Z}\phi(x,z) \le \sup_{z\in Z}\phi(x^*,z) = \phi(x^*,z^*) = \inf_{x\in X}\phi(x,z^*) \le \sup_{z\in Z}\inf_{x\in X}\phi(x,z). \tag{3.9}$$

Combined with the minimax inequality (3.5), this relation shows that the minimax equality (3.2) holds. Therefore, equality holds throughout in Eq. (3.9), which implies that x^* and z^* are optimal solutions of problems (3.7) and (3.8), respectively. **Q.E.D.**

Note a simple consequence of Prop. 3.4.1: *assuming the minimax equality (3.2) holds*, the set of saddle points is the Cartesian product $X^* \times Z^*$, where X^* and Z^* are the sets of optimal solutions of problems (3.7) and (3.8), respectively. In other words x^* and z^* can be *independently* chosen from the sets X^* and Z^* to form a saddle point. Note also that if the minimax equality (3.2) does not hold, there is no saddle point, even if the sets X^* and Z^* are nonempty.

To obtain saddle points using Prop. 3.4.1, we may calculate the "sup" and "inf" functions appearing in the proposition, then minimize and maximize them, respectively, and obtain the corresponding sets of minima X^* and maxima Z^* [cf. Eqs. (3.7) and (3.8)]. If the optimal values are equal (i.e., $\inf_x \sup_z \phi = \sup_z \inf_x \phi$), the set of saddle points is $X^* \times Z^*$. Otherwise, there are no saddle points. The following example illustrates this process.

Example 3.4.1:

Consider the case where

$$\phi(x,z) = \tfrac{1}{2}x'Qx + x'z - \tfrac{1}{2}z'Rz, \qquad X = Z = \Re^n,$$

and Q and R are symmetric invertible matrices. If Q and R are positive definite, a straightforward calculation shows that

$$\sup_{z\in\Re^n}\phi(x,z) = \tfrac{1}{2}x'(Q+R^{-1})x, \qquad \inf_{x\in\Re^n}\phi(x,z) = -\tfrac{1}{2}z'(Q^{-1}+R)z.$$

Therefore, $\inf_x \sup_z \phi(x,z) = \sup_z \inf_x \phi(x,z) = 0$, we have

$$X^* = Z^* = \{0\},$$

and it follows that $(0,0)$ is the unique saddle point.

Assume now that Q is not positive semidefinite, but R is positive definite and such that $Q + R^{-1}$ is positive definite. Then it can be seen that

$$\sup_{z\in\Re^n}\phi(x,z) = \tfrac{1}{2}x'(Q+R^{-1})x, \qquad \inf_{x\in\Re^n}\phi(x,z) = -\infty, \quad \forall\, z \in \Re^n.$$

Here, the sets X^* and Z^* are nonempty ($X^* = \{0\}$ and $Z^* = \Re^n$). However, we have $0 = \inf_x \sup_z \phi(x,z) > \sup_z \inf_x \phi(x,z) = -\infty$, and there are no saddle points.

4

Geometric Duality Framework

Contents

Duality in optimization is often viewed as a manifestation of the fundamental description of a closed convex set as the intersection of all closed halfspaces containing the set (cf. Prop. 1.5.4). When specialized to the epigraph of a function $f : \Re^n \mapsto [-\infty, \infty]$, this description leads to the formalism of the conjugate function of f:

$$f^\star(y) = \sup_{x \in \Re^n} \big\{x'y - f(x)\big\}. \tag{4.1}$$

Indeed, if f is closed proper convex, by the Conjugacy Theorem (Prop. 1.6.1), we have $f(x) = f^{\star\star}(x) = \sup_{y \in \Re^n}\big\{x'y - f^\star(y)\big\}$, so that

$$\begin{aligned} \text{epi}(f) &= \big\{(x, w) \mid f(x) \leq w\big\} \\ &= \Big\{(x, w) \mid \sup_{y \in \Re^n} \big\{x'y - f^\star(y)\big\} \leq w\Big\} \\ &= \cap_{y \in \Re^n}\big\{(x, w) \mid x'y - w \leq f^\star(y)\big\}. \end{aligned}$$

Thus the conjugate $f^\star$ defines epi(f) as the intersection of closed halfspaces.

In this chapter, we focus on a geometric framework for duality analysis, referred to as *min common/max crossing* (MC/MC for short). It is inspired by the above description of epigraphs, and it is related to conjugacy, but it does not involve an algebraic definition such as Eq. (4.1). For this reason it is simpler and better suited for geometric visualization and analysis in many important convex optimization contexts.

4.1 MIN COMMON/MAX CROSSING DUALITY

Our framework aims to capture the most essential characteristics of duality in two simple geometrical problems, defined by a nonempty subset M of $\Re^{n+1}$.

(a) *Min Common Point Problem*: Consider all vectors that are common to M and the $(n+1)$st axis. We want to find one whose $(n+1)$st component is minimum.

(b) *Max Crossing Point Problem*: Consider nonvertical hyperplanes that contain M in their corresponding "upper" closed halfspace, i.e., the closed halfspace whose recession cone contains the vertical halfline $\big\{(0, w) \mid w \geq 0\big\}$ (see Fig. 4.1.1). We want to find the maximum crossing point of the $(n+1)$st axis with such a hyperplane.

We refer to the two problems as the min common/max crossing (MC/MC) framework, and we will show that it can be used to develop much of the core theory of convex optimization in a unified way.

Figure 4.1.1 suggests that the optimal value of the max crossing problem is no larger than the optimal value of the min common problem, and

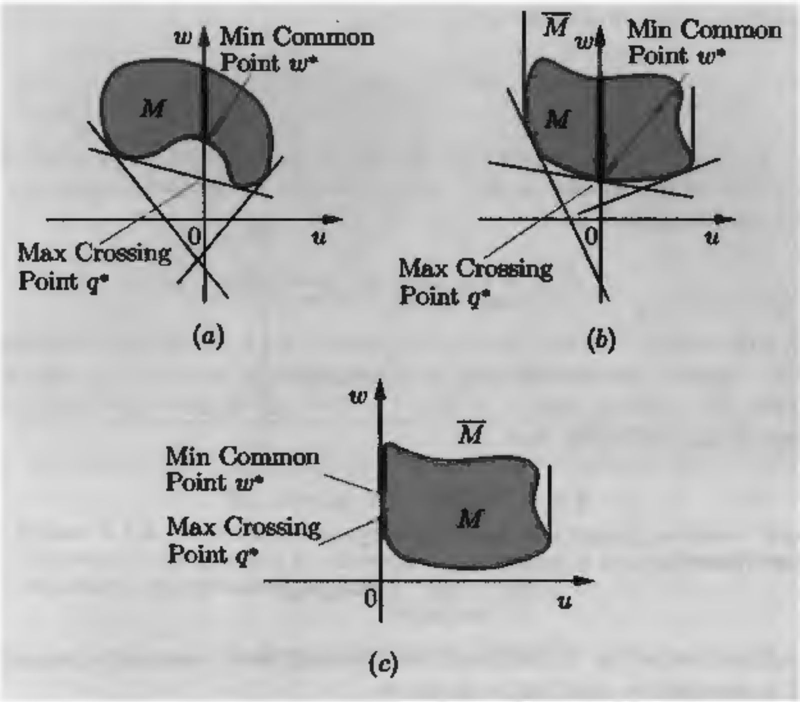


Figure 4.1.1. Illustration of the optimal values of the min common and max crossing problems. In (a), the two optimal values are not equal. In (b), when M is "extended upwards" along the $(n+1)$st axis, it yields the set

$$\begin{aligned}\overline{M} &= M + \big\{(0,w) \mid w \geq 0\big\} \\ &= \big\{(u,w) \mid \text{there exists } \overline{w} \text{ with } \overline{w} \leq w \text{ and } (u,\overline{w}) \in M\big\},\end{aligned}$$

which is convex and admits a nonvertical supporting hyperplane passing through $(0, w^*)$. As a result, the two optimal values are equal. In (c), the set $\overline{M}$ is convex but not closed, and there are points $(0, \overline{w})$ on the vertical axis with $\overline{w} < w^*$ that lie in the closure of $\overline{M}$. Here q^* is equal to the minimum such value of $\overline{w}$, and we have $q^* < w^*$.

that under favorable circumstances the two optimal values are equal. In this chapter, we will provide conditions that guarantee equality of the optimal values and the existence of optimal solutions. In the next chapter we will use these conditions to analyze a variety of duality-related issues.

Mathematically, the min common problem is

$$\begin{aligned}&\text{minimize} \quad w \\ &\text{subject to} \quad (0, w) \in M.\end{aligned}$$

Its optimal value is denoted by w^*, i.e.,

$$w^* = \inf_{(0,w)\in M} w.$$

To describe mathematically the max crossing problem, we recall that a nonvertical hyperplane in $\Re^{n+1}$ is specified by its normal vector $(\mu, 1) \in \Re^{n+1}$, and a scalar ξ as

$$H_{\mu,\xi} = \big\{(u, w) \mid w + \mu' u = \xi\big\}.$$

Such a hyperplane crosses the $(n+1)$st axis at $(0, \xi)$. For M to be contained in the "upper" closed halfspace that corresponds to $H_{\mu,\xi}$ [the one that contains the vertical halfline $\big\{(0, w) \mid w \geq 0\big\}$ in its recession cone], it is necessary and sufficient that

$$\xi \leq w + \mu' u, \qquad \forall\ (u, w) \in M,$$

or equivalently

$$\xi \leq \inf_{(u,w)\in M} \{w + \mu' u\}.$$

For a fixed normal $(\mu, 1)$, the maximum crossing level ξ over all hyperplanes $H_{\mu,\xi}$ is denoted by $q(\mu)$ and is given by

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\}; \tag{4.2}$$

(see Fig. 4.1.2). The max crossing problem is to maximize over all $\mu \in \Re^n$ the maximum crossing level corresponding to μ, i.e.,

$$\begin{aligned} &\text{maximize} \quad q(\mu) \\ &\text{subject to} \quad \mu \in \Re^n. \end{aligned}$$

We also refer to this as the *dual problem*, we denote by q^* its optimal value,

$$q^* = \sup_{\mu\in\Re^n} q(\mu),$$

and we refer to $q(\mu)$ as the *crossing* or *dual* function.

We will show shortly that we always have $q^* \leq w^*$; we refer to this as *weak duality*. When $q^* = w^*$, we say that *strong duality holds* or that *there is no duality gap*.

Note that both w^* and q^* remain unaffected if M is replaced by its "upwards extension"

$$\begin{aligned} \overline{M} &= M + \big\{(0, w) \mid w \geq 0\big\} \\ &= \big\{(u, w) \mid \text{there exists } \overline{w} \text{ with } \overline{w} \leq w \text{ and } (u, \overline{w}) \in M\big\} \end{aligned} \tag{4.3}$$

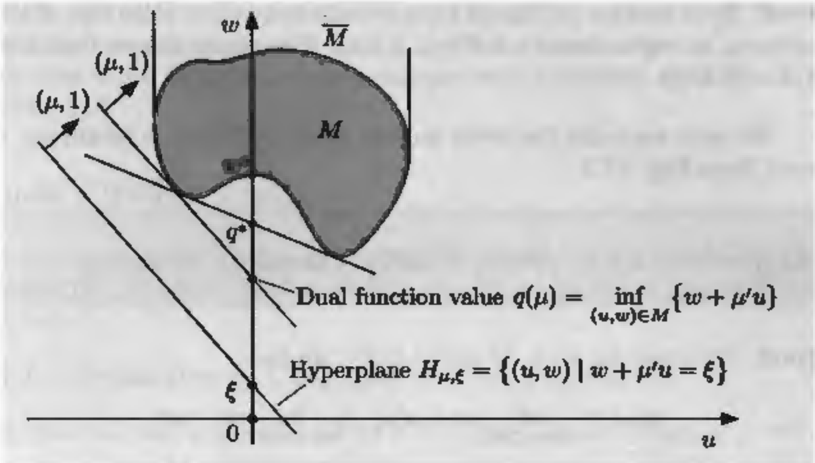


Figure 4.1.2. Mathematical specification of the max crossing problem. When considering crossing points by nonvertical hyperplanes, it is sufficient to restrict attention to hyperplanes of the form

$$H_{\mu,\xi} = \big\{(u, w) \mid w + \mu' u = \xi\big\},$$

that have normal $(\mu, 1)$ and cross the vertical axis at a point ξ with $\xi \le w + \mu' u$ for all $(u, w) \in M$, or equivalently $\xi \le \inf_{(u,w)\in M}\{w + \mu' u\}$. For a given $\mu \in \Re^n$, the highest crossing level is

$$q(\mu) = \inf_{(u,w)\in M}\{w + \mu' u\}.$$

To find the max crossing point q^* we must maximize $q(\mu)$ over $\mu \in \Re^n$.

(cf. Fig. 4.1.1). It is often more convenient to work with $\overline{M}$ because in many cases of interest $\overline{M}$ is convex while M is not. However, on occasion M has some interesting properties (such as compactness) that are masked when passing to $\overline{M}$, in which case it may be preferable to work with M.

Note also that we do not exclude the possibility that either w^* or q^* (or both) are infinite. In particular, we have $w^* = \infty$ if the min common problem has no feasible solution $[M \cap \{(0, w) \mid w \in \Re\} = \emptyset]$. Similarly, we have $q^* = -\infty$ if the max crossing problem has no feasible solution, which occurs in particular if $\overline{M}$ contains a vertical line, i.e., a set of the form $\{(x, w) \mid w \in \Re\}$ for some $x \in \Re^n$.

We now establish some generic properties of the MC/MC framework.

Proposition 4.1.1: The dual function q is concave and upper semicontinuous.

Proof: By definition [cf. Eq. (4.2)], q is the infimum of a collection of affine functions, so $-q$ is closed (cf. Prop. 1.1.6). The result follows from Prop. 1.1.2. **Q.E.D.**

We next establish the weak duality property, which is intuitively apparent from Fig. 4.1.1.

> **Proposition 4.1.2: (Weak Duality Theorem)** We have $q^* \leq w^*$.

Proof: For every $(u, w) \in M$ and $\mu \in \Re^n$, we have

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} \leq \inf_{(0,w)\in M} w = w^*,$$

so by taking the supremum of the left-hand side over $\mu \in \Re^n$, we obtain $q^* \leq w^*$. **Q.E.D.**

As Fig. 4.1.2 indicates, the feasible solutions of the max crossing problem are restricted by the horizontal directions of recession of $\overline{M}$. This is the essence of the following proposition.

> **Proposition 4.1.3:** Assume that the set
>
> $$\overline{M} = M + \{(0, w) \mid w \geq 0\}$$
>
> is convex. Then the set of feasible solutions of the max crossing problem, $\{\mu \mid q(\mu) > -\infty\}$, is contained in the cone
>
> $$\{\mu \mid \mu' d \geq 0 \text{ for all } d \text{ with } (d, 0) \in R_{\overline{M}}\},$$
>
> where $R_{\overline{M}}$ is the recession cone of $\overline{M}$.

Proof: Let $(\overline{u}, \overline{w}) \in \overline{M}$. If $(d, 0) \in R_{\overline{M}}$, then $(\overline{u} + \alpha d, \overline{w}) \in \overline{M}$ for all $\alpha \geq 0$, so that for all $\mu \in \Re^n$,

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} = \inf_{(u,w)\in \overline{M}} \{w + \mu' u\} \leq \overline{w} + \mu' \overline{u} + \alpha \mu' d, \quad \forall\, \alpha \geq 0.$$

Thus, if $\mu' d < 0$, we must have $q(\mu) = -\infty$, implying that $\mu' d \geq 0$ for all μ with $q(\mu) > -\infty$. **Q.E.D.**

As an example, consider the case where $\overline{M}$ is the vector sum of a convex set and the nonnegative orthant of $\Re^{n+1}$. Then it can be seen that the set $\{d \mid (d,0) \in R_{\overline{M}}\}$ contains the nonnegative orthant of $\Re^n$, so the preceding proposition implies that $\mu \geq 0$ for all μ such that $q(\mu) > -\infty$. This case arises in optimization problems with inequality constraints (see Section 5.3).

4.2 SOME SPECIAL CASES

We will now consider some examples illustrating the use of the MC/MC framework. In all of these examples the set M is the epigraph of some function.

4.2.1 Connection to Conjugate Convex Functions

Consider the case where the set M is the epigraph of a function $p : \Re^n \mapsto [-\infty, \infty]$. Then M coincides with its upwards extension $\overline{M}$ of Eq. (4.3) (cf. Fig. 4.1.1), and the min common value is

$$w^* = p(0).$$

The dual function q of Eq. (4.2) is given by

$$q(\mu) = \inf_{(u,w)\in \text{epi}(p)} \{w + \mu' u\} = \inf_{\{(u,w)\mid p(u)\leq w\}} \{w + \mu' u\},$$

and finally

$$q(\mu) = \inf_{u\in\Re^m} \{p(u) + \mu' u\}. \tag{4.4}$$

Thus, we have $q(\mu) = -p^\star(-\mu)$, where $p^\star$ is the conjugate of p:

$$p^\star(\mu) = \sup_{u\in\Re^n} \{\mu' u - p(u)\}.$$

Furthermore,

$$q^* = \sup_{\mu\in\Re^n} q(\mu) = \sup_{\mu\in\Re^n} \{0\cdot(-\mu) - p^\star(-\mu)\} = p^{\star\star}(0),$$

where $p^{\star\star}$ is the conjugate of $p^\star$ (double conjugate of p); see Fig. 4.2.1. In particular, if $p = p^{\star\star}$ (e.g., if p is closed proper convex), then $w^* = q^*$.

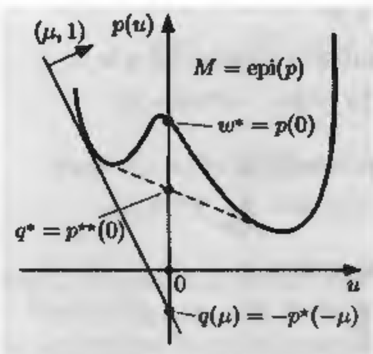


Figure 4.2.1. Illustration of the MC/MC framework for the case where the set M is the epigraph of a function $p : \Re^n \mapsto [-\infty, \infty]$ We have

$$q(\mu) = -p^\star(-\mu),$$

$$w^* = p(0), \qquad q^* = p^{\star\star}(0),$$

where $p^{\star\star}$ is the double conjugate of p. If p is closed proper convex, by the Conjugacy Theorem (cf. Prop. 1.6.1), we have $p = p^{\star\star}$, so $w^* = q^*$.

Note that there is a symmetry between the min common and max crossing problems: if we consider the min common problem defined by the epigraph of $-q$, the corresponding dual function is $-p^{\star\star}$ [or $-p$ when p is closed proper and convex, by the Conjugacy Theorem (cf. Prop. 1.6.1)]. Moreover, by Prop. 1.6.1(d), the min common problems corresponding to $M = \text{epi}(p)$ and $M = \text{epi}(\check{\text{cl}}\, p)$ have the same dual function, namely $q(\mu) = -p^\star(-\mu)$. Furthermore, if $\check{\text{cl}}\, p$ is proper, then $\check{\text{cl}}\, p = p^{\star\star}$, so the min common problem corresponding to $M = \text{epi}(p^{\star\star})$ also has the same dual function.

4.2.2 General Optimization Duality

Consider the problem of minimizing a function $f : \Re^n \mapsto [-\infty, \infty]$. We introduce a function $F : \Re^{n+r} \mapsto [-\infty, \infty]$ of the pair (x, u), which satisfies

$$f(x) = F(x, 0), \qquad \forall\, x \in \Re^n. \tag{4.5}$$

Let the function $p : \Re^r \mapsto [-\infty, \infty]$ be defined by

$$p(u) = \inf_{x \in \Re^n} F(x, u). \tag{4.6}$$

For a helpful insight, we may view u as a *perturbation*, and $p(u)$ as the optimal value of an optimization problem whose cost function is perturbed by u. When $u = 0$, the perturbed problem coincides with the original problem of minimizing f. Note that p is defined by a partial minimization, so its closedness may be checked using the theory of Section 3.3.

Consider the MC/MC framework with

$$M = \text{epi}(p).$$

The min common value w^* is the minimal value of f, since

$$w^* = p(0) = \inf_{x \in \Re^n} F(x, 0) = \inf_{x \in \Re^n} f(x).$$

By Eq. (4.4), the dual function is

$$q(\mu) = \inf_{u \in \Re^r} \big\{ p(u) + \mu' u \big\} = \inf_{(x,u) \in \Re^{n+r}} \big\{ F(x, u) + \mu' u \big\}, \tag{4.7}$$

and the max crossing problem is

$$\begin{aligned} &\text{maximize} \quad q(\mu) \\ &\text{subject to} \quad \mu \in \Re^r. \end{aligned}$$

Note that from Eq. (4.7), an alternative expression for q is

$$q(\mu) = -\sup_{(x,u) \in \Re^{n+r}} \big\{ -\mu' u - F(x, u) \big\} = -F^\star(0, -\mu),$$

where $F^\star$ is the conjugate of F, viewed as a function of (x, u). Since

$$q^* = \sup_{\mu \in \Re^r} q(\mu) = -\inf_{\mu \in \Re^r} F^\star(0, -\mu) = -\inf_{\mu \in \Re^r} F^\star(0, \mu),$$

the strong duality relation $w^* = q^*$ can be written as

$$\inf_{x \in \Re^n} F(x, 0) = -\inf_{\mu \in \Re^r} F^\star(0, \mu).$$

4.2.3 Optimization with Inequality Constraints

Different choices of perturbation structure and function F, as in Eqs. (4.5) and (4.6), yield corresponding MC/MC frameworks and dual problems. An example of this type, to be considered in detail later, is minimization with inequality constraints:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \quad g(x) \leq 0, \end{array} \tag{4.8}$$

where X is a nonempty subset of $\Re^n$, $f : X \mapsto \Re$ is a given function, and $g(x) = \big(g_1(x), \ldots, g_r(x)\big)$ with $g_j : X \mapsto \Re$ being given functions. We introduce a "perturbed constraint set" of the form

$$C_u = \big\{x \in X \mid g(x) \leq u\big\}, \qquad u \in \Re^r, \tag{4.9}$$

and the function

$$F(x, u) = \begin{cases} f(x) & \text{if } x \in C_u, \\ \infty & \text{otherwise,} \end{cases}$$

which satisfies the condition $F(x, 0) = f(x)$ for all $x \in C_0$ [cf. Eq. (4.5)].

The function p of Eq. (4.6) is given by

$$p(u) = \inf_{x \in \Re^n} F(x, u) = \inf_{x \in X,\, g(x) \leq u} f(x), \tag{4.10}$$

and is known as the *primal function* or *perturbation function* (see Fig. 4.2.2). It captures the essential structure of the constrained minimization problem, relating to duality and other properties, such as sensitivity (the change in optimal cost as a result of changes in constraint level).

Consider now the MC/MC framework corresponding to $M = \text{epi}(p)$. From Eq. (4.4) [or Eq. (4.7)],

$$\begin{aligned} q(\mu) &= \inf_{u \in \Re^r} \big\{p(u) + \mu' u\big\} \\ &= \inf_{x \in X,\, g(x) \leq u} \big\{f(x) + \mu' u\big\} \\ &= \begin{cases} \inf_{x \in X} \big\{f(x) + \mu' g(x)\big\} & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{4.11}$$

Later, in Section 5.3, we will view q as the standard dual function, obtained by minimizing over $x \in X$ the function $f(x) + \mu' g(x)$, which is known as the *Lagrangian function* (compare also with the discussion of duality and its connection with minimax theory in Section 3.4).

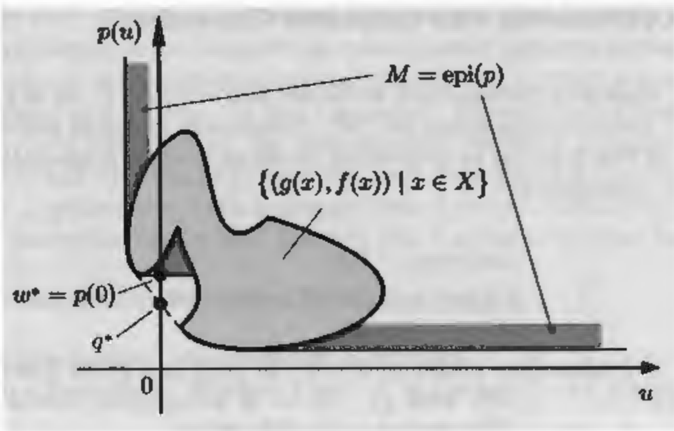


Figure 4.2.2. Illustration of the perturbation function p of Eq. (4.10). Note that epi(p) is the sum of the positive orthant with the set $\{(g(x), f(x)) \mid x \in X\}$.

Example 4.2.1: (Linear Programming Duality)

Consider the linear program

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad a_j'x \geq b_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $c \in \Re^n$, $a_j \in \Re^n$, and $b_j \in \Re$, $j = 1, \ldots, r$. For $\mu \geq 0$, the corresponding dual function has the form

$$q(\mu) = \inf_{x \in \Re^n} \left\{ c'x + \sum_{j=1}^{r} \mu_j (b_j - a_j'x) \right\} = \begin{cases} b'\mu & \text{if } \sum_{j=1}^{r} a_j \mu_j = c, \\ -\infty & \text{otherwise,} \end{cases}$$

while for all other $\mu \in \Re^r$, $q(\mu) = -\infty$ [cf. Eq. (4.11)]. Thus the dual problem is

$$\begin{aligned} &\text{maximize} \quad b'\mu \\ &\text{subject to} \quad \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0. \end{aligned}$$

4.2.4 Augmented Lagrangian Duality

In connection with the inequality-constrained problem (4.8), let us consider the function

$$F_c(x, u) = \begin{cases} f(x) + \frac{c}{2}\|u\|^2 & \text{if } x \in C_u, \\ \infty & \text{otherwise,} \end{cases}$$

where C_u is the perturbed constraint set (4.9), and c is a positive scalar. Then instead of the perturbation function p of Eq. (4.10), we have

$$p_c(u) = \inf_{x\in\Re^n} F_c(x,u) = \inf_{x\in X,\, g(x)\le u} \left\{ f(x) + \frac{c}{2}\|u\|^2 \right\}$$

[yet $p(0) = p_c(0)$, so the min common value is left unchanged]. The corresponding dual function is

$$q_c(\mu) = \inf_{u\in\Re^r} \{p_c(u) + \mu' u\} = \inf_{x\in X,\, g(x)\le u} \left\{ f(x) + \mu' u + \frac{c}{2}\|u\|^2 \right\}.$$

For a fixed $x \in X$, we find the infimum above by separately minimizing, over each component u_j of u, the one-dimensional quadratic function $\mu_j u_j + \frac{c}{2}(u_j)^2$ subject to the constraint $g_j(x) \le u_j$. The minimum is attained at

$$g_j^+(x,\mu,c) = \max\left\{ -\frac{\mu_j}{c}, g_j(x) \right\}, \qquad j = 1,\ldots,r,$$

and substituting into the expression for q_c, we obtain

$$q_c(\mu) = \inf_{x\in X} \left\{ f(x) + \mu' g^+(x,\mu,c) + \frac{c}{2}\left\|g^+(x,\mu,c)\right\|^2 \right\}, \qquad \mu \in \Re^r,$$

where $g^+(x,\mu,c)$ is the vector with components $g_j^+(x,\mu,c)$, $j = 1,\ldots,r$.

The expression within braces in the right-hand side above is known as the *augmented Lagrangian function*. It plays a central role in important practical algorithms that aim to maximize the "penalized" dual function $q_c(\mu)$ over μ (we refer to the monograph [Ber82] and standard nonlinear programming textbooks). Note that [in contrast with the corresponding expression for q in Eq. (4.11)], q_c *is often real-valued* (for example, when f and g_j are continuous functions, and X is a compact set), and in some important contexts it turns out to be differentiable. Based on this property, augmented Lagrangian functions provide a form of regularization of the dual problem that is significant in algorithmic practice.

4.2.5 Minimax Problems

Consider a function $\phi : X \times Z \mapsto \Re$, where X and Z are nonempty subsets of $\Re^n$ and $\Re^m$, respectively. As in Section 3.4, we wish to either

$$\begin{aligned} &\text{minimize} && \sup_{z\in Z} \phi(x,z) \\ &\text{subject to} && x \in X \end{aligned}$$

or

$$\begin{aligned} &\text{maximize} && \inf_{x\in X} \phi(x,z) \\ &\text{subject to} && z \in Z. \end{aligned}$$

An important question is whether the minimax equality

$$\sup_{z\in Z}\inf_{x\in X}\phi(x,z) = \inf_{x\in X}\sup_{z\in Z}\phi(x,z), \tag{4.12}$$

holds, and whether the infimum and the supremum above are attained. This is significant in a zero sum game context, as well as in optimization duality theory; cf. Section 3.4.

We introduce the function $p : \Re^m \mapsto [-\infty,\infty]$ given by

$$p(u) = \inf_{x\in X}\sup_{z\in Z}\big\{\phi(x,z) - u'z\big\}, \qquad u \in \Re^m, \tag{4.13}$$

which can be viewed as a perturbation function. It characterizes how the "infsup" of the function ϕ changes when the linear perturbation term $u'z$ is subtracted from ϕ. We consider the MC/MC framework with

$$M = \text{epi}(p),$$

so that the min common value is

$$w^* = p(0) = \inf_{x\in X}\sup_{z\in Z}\phi(x,z). \tag{4.14}$$

We will show that the max crossing value q^* is equal to the "supinf" value of a function derived from ϕ via a convexification operation.

We recall that given a set $X \subset \Re^n$ and a function $f : X \mapsto [-\infty,\infty]$, the convex closure of f, denoted by $\check{\text{cl}}\, f$, is the function whose epigraph is the closure of the convex hull of the epigraph of f (cf. Section 1.3.3). The *concave closure of* f, denoted by $\hat{\text{cl}}\, f : \Re^n \mapsto [-\infty,\infty]$, is the opposite of the convex closure of $-f$, i.e.,

$$\hat{\text{cl}}\, f = -\check{\text{cl}}\,(-f).$$

It is the smallest concave and upper semicontinuous function that majorizes f, i.e., $\hat{\text{cl}}\, f \le g$ for any $g : X \mapsto [-\infty,\infty]$ that is concave and upper semicontinuous with $g \ge f$ (Prop. 1.3.14). Note that by Prop. 1.3.13, we have

$$\sup_{x\in X} f(x) = \sup_{x\in X}(\hat{\text{cl}}\, f)(x) = \sup_{x\in \text{conv}(X)}(\hat{\text{cl}}\, f)(x) = \sup_{x\in\Re^n}(\hat{\text{cl}}\, f)(x). \tag{4.15}$$

The following proposition derives the form of the dual function, under a mild assumption on $(\hat{\text{cl}}\,\phi)(x,\cdot)$, the concave closure of the function $\phi(x,\cdot)$.

Proposition 4.2.1: Let X and Z be nonempty subsets of $\Re^n$ and $\Re^m$, respectively, and let $\phi : X \times Z \mapsto \Re$ be a function. Assume that $(-\hat{\text{cl}}\,\phi)(x,\cdot)$ is proper for all $x \in X$, and consider the MC/MC framework corresponding to $M = \text{epi}(p)$, where p is given by Eq. (4.13). Then the dual function is given by

$$q(\mu) = \inf_{x\in X}(\hat{\text{cl}}\,\phi)(x,\mu), \qquad \forall\, \mu \in \Re^m. \tag{4.16}$$

Proof: Let us write

$$p(u) = \inf_{x \in X} p_x(u),$$

where

$$p_x(u) = \sup_{z \in Z} \{\phi(x,z) - u'z\}, \qquad x \in X, \tag{4.17}$$

and note that

$$\inf_{u \in \Re^m} \{p_x(u) + u'\mu\} = - \sup_{u \in \Re^m} \{u'(-\mu) - p_x(u)\} = -p_x^\star(-\mu), \tag{4.18}$$

where $p_x^\star$ is the conjugate of p_x. From Eq. (4.17), we see that (except for a sign change of its argument) p_x is the conjugate of $(-\phi)(x,\cdot)$ (restricted to Z), so from the Conjugacy Theorem [Prop. 1.6.1(d)], using also the properness assumption on $(-\hat{\mathrm{cl}}\,\phi)(x,\cdot)$, we have

$$p_x^\star(-\mu) = -(\hat{\mathrm{cl}}\,\phi)(x,\mu). \tag{4.19}$$

By using Eq. (4.4) for q, and Eqs. (4.18), (4.19), we obtain for all $\mu \in \Re^m$,

$$\begin{aligned} q(\mu) &= \inf_{u \in \Re^m} \{p(u) + u'\mu\} \\ &= \inf_{u \in \Re^m} \inf_{x \in X} \{p_x(u) + u'\mu\} \\ &= \inf_{x \in X} \inf_{u \in \Re^m} \{p_x(u) + u'\mu\} \\ &= \inf_{x \in X} \{ - p_x^\star(-\mu)\} \\ &= \inf_{x \in X} (\hat{\mathrm{cl}}\,\phi)(x,\mu), \end{aligned} \tag{4.20}$$

which is the desired relation. **Q.E.D.**

Note that without the properness assumption on $(-\hat{\mathrm{cl}}\,\phi)(x,\cdot)$ for all $x \in X$ the conclusion of Prop. 4.2.1 may fail, since then Eq. (4.19) may not hold. The key conclusion from the proposition is that, *while w^* is equal to the "infsup" value of ϕ, the dual value q^* is equal to the "supinf" value of the concave closure $\hat{\mathrm{cl}}\,\phi$.* In particular, we may observe the following:

(a) In general, we have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x,z) \le q^* \le w^* = \inf_{x \in X} \sup_{z \in Z} \phi(x,z), \tag{4.21}$$

where the first inequality follows using the calculation

$$\begin{aligned} q(\mu) &= \inf_{u \in \Re^m} \{p(u) + u'\mu\} \\ &= \inf_{u \in \Re^m} \inf_{x \in X} \sup_{z \in Z} \{\phi(x,z) + u'(\mu - z)\} \\ &\ge \inf_{x \in X} \phi(x,\mu), \end{aligned}$$

[cf. Eq. (4.4)], and taking supremum over $\mu \in Z$ (the inequality above is obtained by setting $z = \mu$), the second inequality is the weak duality relation, and the last equality is Eq. (4.14). Therefore *the minimax equality (4.12) always implies the strong duality relation $q^* = w^*$.*

(b) If

$$\phi(x, z) = (\hat{\mathrm{cl}}\,\phi)(x, z), \qquad \forall\ x \in X,\ z \in Z,$$

as in the case where $-\phi(x, \cdot)$ is closed and convex for all $x \in X$, then using Prop. 4.2.1 and the properness of $(-\hat{\mathrm{cl}}\,\phi)(x, \cdot)$ for all $x \in X$, we have

$$q^* = \sup_{z \in \Re^m} q(z) = \sup_{z \in \Re^m} \inf_{x \in X} (\hat{\mathrm{cl}}\,\phi)(x, z) = \sup_{z \in Z} \inf_{x \in X} \phi(x, z).$$

In view of Eq. (4.21), it follows that *if $\phi = \hat{\mathrm{cl}}\,\phi$, the minimax equality (4.12) is equivalent to the strong duality relation $q^* = w^*$.*

(c) From Eq. (4.15), we have $\sup_{z \in Z} \phi(x, z) = \sup_{z \in \Re^m} (\hat{\mathrm{cl}}\,\phi)(x, z)$, so

$$w^* = \inf_{x \in X} \sup_{z \in \Re^m} (\hat{\mathrm{cl}}\,\phi)(x, z).$$

Furthermore, assuming that $(-\hat{\mathrm{cl}}\,\phi)(x, \cdot)$ is proper for all $x \in X$ so that Prop. 4.2.1 applies, we have

$$q^* = \sup_{z \in \Re^m} \inf_{x \in X} (\hat{\mathrm{cl}}\,\phi)(x, z).$$

Thus, *w^* and q^* are the "infsup" and "supinf" values of* $\hat{\mathrm{cl}}\,\phi$, respectively.

(d) If $(\hat{\mathrm{cl}}\,\phi)(\cdot, z)$ is closed and convex for each z, then the "infsup" and "supinf" values of the convex/concave function $\hat{\mathrm{cl}}\,\phi$ will ordinarily be expected to be equal (see Section 5.5). In this case we will have strong duality ($q^* = w^*$), but this will not necessarily imply that the minimax equality (4.12) holds, because strict inequality may hold in the first relation of Eq. (4.21). If $(\hat{\mathrm{cl}}\,\phi)(\cdot, z)$ is not closed and convex for some z, then the strong duality relation $q^* = w^*$ should not ordinarily be expected. The size of the duality gap $w^* - q^*$ will depend on the difference between $(\hat{\mathrm{cl}}\,\phi)(\cdot, z)$ and its convex closure, and may be investigated in some cases by using methods to be presented in Section 5.7.

Example 4.2.2: (Finite Set Z)

Let

$$\phi(x, z) = z' f(x),$$

where $f : X \mapsto \Re^m$, X is a subset of $\Re^n$, and $f(x)$ is viewed as a column vector whose components are functions $f_j : X \mapsto \Re$, $j = 1, \ldots, m$. Suppose that Z is the finite set $Z = \{e_1, \ldots, e_m\}$, where e_j is the jth unit vector (the jth column of the $m \times m$ identity matrix). Then we have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \max\left\{ \inf_{x \in X} f_1(x), \ldots, \inf_{x \in X} f_m(x) \right\},$$

and

$$w^* = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) = \inf_{x \in X} \max\{f_1(x), \ldots, f_m(x)\}.$$

Let $\overline{Z}$ denote the unit simplex in $\Re^m$ (the convex hull of Z). We have

$$(\hat{\mathrm{cl}}\, \phi)(x, z) = \begin{cases} z' f(x), & \text{if } z \in \overline{Z}, \\ -\infty, & \text{if } z \notin \overline{Z}, \end{cases}$$

and by Prop. 4.2.1,

$$q^* = \sup_{z \in Z} \inf_{x \in X} (\hat{\mathrm{cl}}\, \phi)(x, z) = \sup_{z \in \overline{Z}} \inf_{x \in X} z' f(x).$$

Note that it can easily happen that $\sup_{z \in Z} \inf_{x \in X} \phi(x, z) < q^*$. For example, if $X = [0, 1]$, $f_1(x) = x$, and $f_2(x) = 1 - x$, it is straightforward to verify that

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = 0 < \frac{1}{2} = q^*.$$

If X is compact and $f_1, \ldots, f_m$ are continuous convex functions (as in the preceding example), we may show that $q^* = w^*$ by using results to be presented in Section 5.5. On the other hand, if $f_1, \ldots, f_m$ are not convex, we may have $q^* < w^*$.

Under certain conditions, q^* can be associated with a special minimax problem derived from the original by introducing mixed (randomized) strategies. This is so in the following classical game context.

Example 4.2.3: (Finite Zero Sum Games)

Consider a minimax problem where the sets X and Z are finite:

$$X = \{d_1 \ldots, d_n\}, \qquad Z = \{e_1, \ldots, e_m\},$$

where d_i is the ith column of the $n \times n$ identity matrix, and e_j is the jth column of the $m \times m$ identity matrix. Let

$$\phi(x, z) = x' A z,$$

where A is an $n \times m$ matrix. This corresponds to the classical game context, discussed at the beginning of Section 3.4, where upon selection of $x = d_i$ and $z = e_j$, the payoff is the ijth component of A. Let $\overline{X}$ and $\overline{Z}$ be the unit simplexes in $\Re^n$ and $\Re^m$, respectively. Then it can be seen, similar to the preceding example, that $q^* = \max_{z \in \overline{Z}} \min_{x \in X} x' A z$, and using theory to be developed in Section 5.5, it can be shown that

$$q^* = \max_{z \in \overline{Z}} \min_{x \in \overline{X}} x' A z = \min_{x \in \overline{X}} \max_{z \in \overline{Z}} x' A z.$$

Note that $\overline{X}$ and $\overline{Z}$ can be interpreted as sets of mixed strategies, so q^* is the value of the corresponding mixed strategy game.

4.3 STRONG DUALITY THEOREM

We will now establish conditions for strong duality to hold in the MC/MC framework, i.e., $q^* = w^*$. To avoid degenerate cases, we will often exclude the case $w^* = \infty$, which corresponds to an infeasible min common problem.

An important point, around which much of our analysis revolves, is that when w^* is finite, the vector $(0, w^*)$ is a closure point of the set $\overline{M}$ of Eq. (4.3), so *if we assume that* $\overline{M}$ *is closed convex and admits a nonvertical supporting hyperplane at* $(0, w^*)$, *then we have* $q^* = w^*$ *while the optimal values* q^* *and* w^* *are attained.* Between the "unfavorable" case where $q^* < w^*$, and the "most favorable" case where $q^* = w^*$ while the optimal values q^* and w^* are attained, there are several intermediate cases.

The following proposition provides a necessary and sufficient condition for $q^* = w^*$, but does not address the attainment of the optimal values. Aside from convexity of $\overline{M}$, it requires that points on the vertical axis that lie below w^* cannot be approached through a sequence from M; this has the flavor of a "lower semicontinuity" property for M at $(0, w^*)$.

Proposition 4.3.1: (MC/MC Strong Duality) Consider the min common and max crossing problems, and assume the following:

(1) Either $w^* < \infty$, or else $w^* = \infty$ and M contains no vertical lines.

(2) The set
$$\overline{M} = M + \{(0, w) \mid w \geq 0\}$$
is convex.

Then, we have $q^* = w^*$ if and only if for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k\to\infty} w_k$.

Proof: Assume that $w^* \leq \liminf_{k\to\infty} w_k$ for all $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$. We will show that $q^* = w^*$. The proof first deals with the easier cases where $w^* = -\infty$ and $w^* = \infty$, and then focuses on the case where w^* is finite. In the latter case, the key idea is that the assumption $w^* \leq \liminf_{k\to\infty} w_k$ implies that $(0, w^* - \epsilon)$ is not a closure point of $\overline{M}$ for any $\epsilon > 0$, so [by Prop. 1.5.8(b)] it can be strictly separated from $\overline{M}$ by a nonvertical hyperplane, which must cross the vertical axis between $w^* - \epsilon$ and w^*. This implies that $w^* - \epsilon < q^* \leq w^*$, and as $\epsilon \downarrow 0$, $q^* = w^*$.

Consider the case $w^* = -\infty$. Then, by the Weak Duality Theorem (Prop. 4.1.2), we also have $q^* = -\infty$, so the conclusion trivially follows. Consider next the case where $w^* = \infty$ and M contains no vertical lines. Since $\liminf_{k\to\infty} w_k = w^* = \infty$ for all $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, it follows that the vertical axis of $\Re^{n+1}$ contains no closure points of $\overline{M}$. Hence by the Nonvertical Hyperplane Theorem [Prop. 1.5.8(b)], for any vector

$(0, w) \in \Re^{n+1}$, there exists a nonvertical hyperplane strictly separating $(0, w)$ and $\overline{M}$. The crossing point of this hyperplane with the vertical axis lies between w and q^*, so $w < q^*$ for all $w \in \Re$, which implies that $q^* = w^* = \infty$.

Consider finally the case where w^* is a real number. We first show by contradiction that for any $\epsilon > 0$, $(0, w^* - \epsilon) \notin \mathrm{cl}(\overline{M})$. If this were not so, i.e., $(0, w^* - \epsilon)$ is a closure point of $\overline{M}$ for some $\epsilon > 0$, there would exist a sequence $\{(u_k, \overline{w}_k)\} \subset \overline{M}$ converging to $(0, w^* - \epsilon)$. In view of the definition of $\overline{M}$, this implies the existence of another sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$ and $w_k \leq \overline{w}_k$ for all k, so that

$$\liminf_{k \to \infty} w_k \leq w^* - \epsilon,$$

which contradicts the assumption $w^* \leq \liminf_{k \to \infty} w_k$.

We next argue by contradiction that $\overline{M}$ does not contain any vertical lines. If this were not so, by convexity of $\overline{M}$, the direction $(0, -1)$ would be a direction of recession of $\mathrm{cl}(\overline{M})$. Because $(0, w^*) \in \mathrm{cl}(\overline{M})$ [since $(0, w^*)$ is the "infimum" common point of $\overline{M}$ and the vertical axis], it follows that the entire halfline $\{(0, w^* - \epsilon) \mid \epsilon \geq 0\}$ belongs to $\mathrm{cl}(\overline{M})$ [cf. Prop. 1.4.1(a)], contradicting what was shown earlier.

Since $\overline{M}$ does not contain any vertical lines and the vector $(0, w^* - \epsilon)$ does not belong to $\mathrm{cl}(\overline{M})$ for any $\epsilon > 0$, by Prop. 1.5.8(b), it follows that there exists a nonvertical hyperplane strictly separating $(0, w^* - \epsilon)$ and $\overline{M}$. This hyperplane crosses the $(n+1)$st axis at a unique vector $(0, \xi)$, which must lie between $(0, w^* - \epsilon)$ and $(0, w^*)$, i.e., $w^* - \epsilon < \xi \leq w^*$. Furthermore, ξ cannot exceed the optimal value q^* of the max crossing problem, which together with weak duality ($q^* \leq w^*$), implies that

$$w^* - \epsilon < q^* \leq w^*.$$

Since ϵ can be arbitrarily small, it follows that $q^* = w^*$.

Conversely, assume $q^* = w^*$ and consider a sequence $\{(u_k, w_k)\} \subset M$ such that $u_k \to 0$. We will show that $w^* \leq \liminf_{k \to \infty} w_k$. Indeed we have

$$q(\mu) = \inf_{(u,w) \in M} \{w + \mu' u\} \leq w_k + \mu' u_k, \qquad \forall\ k, \quad \forall\ \mu \in \Re^n,$$

and by taking the limit as $k \to \infty$, we obtain $q(\mu) \leq \liminf_{k \to \infty} w_k$. Hence

$$w^* = q^* = \sup_{\mu \in \Re^n} q(\mu) \leq \liminf_{k \to \infty} w_k.$$

Q.E.D.

For an example where assumption (1) of the preceding proposition is violated and $q^* < w^*$, let M consist of a vertical line that does not pass through the origin. Then we have $q^* = -\infty$, $w^* = \infty$, and yet the condition $w^* \leq \liminf_{k\to\infty} w_k$, for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, trivially holds.

An important corollary of Prop. 4.3.1 is that *if* $M = \text{epi}(p)$ *where* $p : \Re^n \mapsto [-\infty, \infty]$ *is a convex function with* $p(0) = w^* < \infty$, *then we have* $q^* = w^*$ *if and only if* p *is lower semicontinuous at 0.*

We will now consider cases where M (rather than $\overline{M}$) is convex, and where M has some compactness structure but may not be convex.

Proposition 4.3.2: Consider the MC/MC framework, assuming that $w^* < \infty$.

(a) Let M be closed and convex. Then $q^* = w^*$. Furthermore, the function

$$p(u) = \inf\{w \mid (u, w) \in M\}, \qquad u \in \Re^n,$$

is convex and its epigraph is the set

$$\overline{M} = M + \{(0, w) \mid w \geq 0\}.$$

If in addition $-\infty < w^*$, then p is closed and proper.

(b) q^* is equal to the optimal value of the min common problem corresponding to $\text{cl}\big(\text{conv}(M)\big)$.

(c) If M is of the form

$$M = \tilde{M} + \{(u, 0) \mid u \in C\},$$

where $\tilde{M}$ is a compact set and C is a closed convex set, then q^* is equal to the optimal value of the min common problem corresponding to $\text{conv}(M)$.

Proof: (a) Since M is closed, for each $u \in \text{dom}(p)$, the infimum defining $p(u)$ is either $-\infty$ or else it is attained. In view of the definition of $\overline{M}$, this implies that $\overline{M}$ is the epigraph of p. Furthermore $\overline{M}$ is convex, being the vector sum of two convex sets, so p is convex.

If $w^* = -\infty$, then $q^* = w^*$ by weak duality. It will suffice to show that if $w^* > -\infty$, then p is closed and proper, since by Prop. 4.3.1, this will also imply that $q^* = w^*$. We note that $(0, -1)$ is not a direction of recession of M (since $w^* > -\infty$). Since $\overline{M}$ is the vector sum of M and $\{(0, w) \mid w \geq 0\}$, this implies, using Prop. 1.4.14, that $\overline{M}$ is closed as well as convex. Since $\overline{M} = \text{epi}(p)$, it follows that p is closed. Since w^* is finite, p is also proper (an improper closed convex function cannot take finite values; cf. the discussion at the end of Section 1.1.2).

(b) The max crossing value q^* is the same for M and $\mathrm{cl}\big(\mathrm{conv}(M)\big)$, since the closed halfspaces containing M are the ones that contain $\mathrm{cl}\big(\mathrm{conv}(M)\big)$ (cf. Prop. 1.5.4). Since $\mathrm{cl}\big(\mathrm{conv}(M)\big)$ is closed and convex, the result follows from part (a).

(c) The max crossing value is the same for M and $\mathrm{conv}(M)$, since the closed halfspaces containing M are the ones that contain $\mathrm{conv}(M)$. It can be seen that

$$\mathrm{conv}(M) = \mathrm{conv}(\tilde{M}) + \big\{(u,0) \mid u \in C\big\}.$$

Since $\tilde{M}$ is compact, $\mathrm{conv}(\tilde{M})$ is also compact (cf. Prop. 1.2.2), so $\mathrm{conv}(M)$ is the vector sum of two closed convex sets one of which is compact. Hence, by Prop. 1.4.14, $\mathrm{conv}(M)$ is closed, and the result follows from part (a). **Q.E.D.**

The preceding proposition involves a few subtleties. First note that in part (a), if M is closed and convex, but $w^* = -\infty$, then p is convex, but it need not be closed. For an example in $\Re^2$, consider the closed and convex set

$$M = \big\{(u,w) \mid w \le -1/(1-|u|),\ |u| < 1\big\}.$$

Then,

$$\overline{M} = \big\{(u,w) \mid |u| < 1\big\},$$

so $\overline{M}$ is convex but not closed, implying that p (which is $-\infty$ for $|u| < 1$ and ∞ otherwise) is not closed.

Also note that if M is closed but violates the assumptions of Prop. 4.3.2(c), the min common values corresponding to $\mathrm{conv}(M)$ and $\mathrm{cl}\big(\mathrm{conv}(M)\big)$ may differ. For an example in $\Re^2$, consider the set

$$M = \big\{(0,0)\big\} \cup \big\{(u,w) \mid u > 0,\ w \le -1/u\big\}.$$

Then

$$\mathrm{conv}(M) = \big\{(0,0)\big\} \cup \big\{(u,w) \mid u > 0,\ w < 0\big\}.$$

We have $q^* = -\infty$ but the min common value corresponding to $\mathrm{conv}(M)$ is $w^* = 0$, while the min common value corresponding to $\mathrm{cl}\big(\mathrm{conv}(M)\big)$ is equal to $-\infty$ [consistent with part (b)].

4.4 EXISTENCE OF DUAL OPTIMAL SOLUTIONS

We now discuss the nonemptiness and the structure of the optimal solution set of the max crossing problem. The following proposition, in addition to the equality $q^* = w^*$, guarantees the attainment of the maximum crossing point under appropriate assumptions. Key among these assumptions is that 0 is a relative interior point of the projection of M (or $\overline{M}$) on the horizontal plane [cf. Fig. 4.1.1(b)]. A subsequent proposition will show that for compactness (as well as nonemptiness) of the set of max crossing hyperplanes, it is necessary and sufficient that 0 be an interior point of the projection of M on the horizontal plane.

Proposition 4.4.1: (MC/MC Existence of Max Crossing Solutions) Consider the MC/MC framework and assume the following:

(1) $-\infty < w^*$.

(2) The set
$$\overline{M} = M + \big\{(0, w) \mid w \geq 0\big\}$$
is convex.

(3) The origin is a relative interior point of the set
$$D = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \overline{M}\big\}.$$

Then $q^* = w^*$, and there exists at least one optimal solution of the max crossing problem.

Proof: Condition (3) implies that the vertical axis contains a point of M, so that $w^* < \infty$. Hence in view of condition (1), w^* is finite.

We note that $(0, w^*) \notin \text{ri}(\overline{M})$, since by Prop. 1.3.10,
$$\text{ri}(\overline{M}) = \big\{(u, w) \mid u \in \text{ri}(D),\ \overline{w} < w \text{ for some } (u, \overline{w}) \in \overline{M}\big\},$$
and
$$w^* = \inf_{(0, \overline{w}) \in \overline{M}} \overline{w}.$$
Therefore, by the Proper Separation Theorem (Prop. 1.5.5), there exists a hyperplane that passes through $(0, w^*)$, contains $\overline{M}$ in one of its closed halfspaces, but does not fully contain $\overline{M}$, i.e., there exists (μ, β) such that
$$\beta w^* \leq \mu' u + \beta w, \qquad \forall\ (u, w) \in \overline{M}, \tag{4.22}$$
$$\beta w^* < \sup_{(u,w) \in \overline{M}} \{\mu' u + \beta w\}. \tag{4.23}$$
Since for any $(\overline{u}, \overline{w}) \in M$, the set $\overline{M}$ contains the halfline $\big\{(\overline{u}, w) \mid \overline{w} \leq w\big\}$, it follows from Eq. (4.22) that $\beta \geq 0$. If $\beta = 0$, then from Eq. (4.22), we have
$$0 \leq \mu' u, \qquad \forall\ u \in D.$$

Thus, the linear function $\mu' u$ attains its minimum over the set D at 0, which is a relative interior point of D by condition (3). Since D is convex, being the projection on the space of u of the set $\overline{M}$, which is convex by condition (2), it follows from Prop. 1.3.4 that $\mu' u$ is constant over D, i.e.,

$$\mu' u = 0, \qquad \forall\, u \in D.$$

This, however, contradicts Eq. (4.23). Therefore, we must have $\beta > 0$, and by appropriate normalization if necessary, we can assume that $\beta = 1$. From Eq. (4.22), we then obtain

$$w^* \leq \inf_{(u,w)\in \overline{M}} \{\mu' u + w\} \leq \inf_{(u,w)\in M} \{\mu' u + w\} = q(\mu) \leq q^*.$$

Since $q^* \leq w^*$, by the Weak Duality Theorem (Prop. 4.1.2), equality holds throughout in the above relation, and we must have $q(\mu) = q^* = w^*$. Thus μ is an optimal solution of the max crossing problem. **Q.E.D.**

Note that if $w^* = -\infty$, by weak duality we have $q^* \leq w^*$, so that $q^* = w^* = -\infty$. This means that $q(\mu) = -\infty$ for all $\mu \in \Re^n$, and that the dual problem is infeasible. The following proposition supplements the preceding one, and characterizes the optimal solution set of the max crossing problem.

Proposition 4.4.2: Let the assumptions of Prop. 4.4.1 hold. Then Q^*, the set of optimal solutions of the max crossing problem, has the form

$$Q^* = \big(\text{aff}(D)\big)^{\perp} + \tilde{Q},$$

where $\tilde{Q}$ is a nonempty, convex, and compact set. In particular, Q^* is compact if and only if the origin is an interior point of D.

Proof: By Prop. 4.4.1, q^* is finite and Q^* is nonempty. Since q is concave and upper semicontinuous (cf. Prop. 4.1.1), and $Q^* = \{\mu \mid q(\mu) \geq q^*\}$, it follows that Q^* is convex and closed. We will first show that the recession cone R_{Q^*} and the lineality space L_{Q^*} of Q^* are both equal to $\big(\text{aff}(D)\big)^{\perp}$ [note here that aff(D) is a subspace since it contains the origin]. The proof of this is based on the generic relation $L_{Q^*} \subset R_{Q^*}$ and the following two relations

$$\big(\text{aff}(D)\big)^{\perp} \subset L_{Q^*}, \qquad R_{Q^*} \subset \big(\text{aff}(D)\big)^{\perp},$$

which we show next.

Let d be a vector in $\big(\text{aff}(D)\big)^{\perp}$, so that $d'u = 0$ for all $u \in D$. For any vector $\mu \in Q^*$ and any scalar α, we then have

$$q(\mu + \alpha d) = \inf_{(u,w)\in \overline{M}} \{(\mu + \alpha d)' u + w\} = \inf_{(u,w)\in \overline{M}} \{\mu' u + w\} = q(\mu),$$

so that $\mu + \alpha d \in Q^*$. Hence $d \in L_{Q^*}$, and it follows that $\big(\mathrm{aff}(D)\big)^\perp \subset L_{Q^*}$.

Let d be a vector in R_{Q^*}, so that for any $\mu \in Q^*$ and $\alpha \geq 0$,

$$q(\mu + \alpha d) = \inf_{(u,w)\in \overline{M}} \big\{ (\mu + \alpha d)'u + w \big\} = q^*.$$

Since $0 \in \mathrm{ri}(D)$, for any $u \in \mathrm{aff}(D)$, there exists a positive scalar γ such that the vectors γu and $-\gamma u$ are in D. By the definition of D, there exist scalars w^+ and w^- such that the pairs $(\gamma u, w^+)$ and $(-\gamma u, w^-)$ are in $\overline{M}$. Using the preceding equation, it follows that for any $\mu \in Q^*$, we have

$$(\mu + \alpha d)'(\gamma u) + w^+ \geq q^*, \qquad \forall\ \alpha \geq 0,$$

$$(\mu + \alpha d)'(-\gamma u) + w^- \geq q^*, \qquad \forall\ \alpha \geq 0.$$

If $d'u \neq 0$, then for sufficiently large $\alpha \geq 0$, one of the preceding two relations will be violated. Thus we must have $d'u = 0$, showing that $d \in \big(\mathrm{aff}(D)\big)^\perp$ and implying that

$$R_{Q^*} \subset \big(\mathrm{aff}(D)\big)^\perp.$$

This relation, together with the generic relation $L_{Q^*} \subset R_{Q^*}$ and the relation $\big(\mathrm{aff}(D)\big)^\perp \subset L_{Q^*}$ proved earlier, shows that

$$\big(\mathrm{aff}(D)\big)^\perp \subset L_{Q^*} \subset R_{Q^*} \subset \big(\mathrm{aff}(D)\big)^\perp.$$

Therefore

$$L_{Q^*} = R_{Q^*} = \big(\mathrm{aff}(D)\big)^\perp.$$

We now use the decomposition result of Prop. 1.4.4, to assert that

$$Q^* = L_{Q^*} + (Q^* \cap L_{Q^*}^\perp).$$

Since $L_{Q^*} = \big(\mathrm{aff}(D)\big)^\perp$, we obtain

$$Q^* = \big(\mathrm{aff}(D)\big)^\perp + \tilde{Q},$$

where $\tilde{Q} = Q^* \cap \mathrm{aff}(D)$. Furthermore, by Prop. 1.4.2(c), we have

$$R_{\tilde{Q}} = R_{Q^*} \cap R_{\mathrm{aff}(D)}.$$

Since $R_{Q^*} = \big(\mathrm{aff}(D)\big)^\perp$, as shown earlier, and $R_{\mathrm{aff}(D)} = \mathrm{aff}(D)$, the recession cone $R_{\tilde{Q}}$ consists of the zero vector only, implying that the set $\tilde{Q}$ is compact.

From the formula $Q^* = \big(\mathrm{aff}(D)\big)^\perp + \tilde{Q}$, it follows that Q^* is compact if and only if $\big(\mathrm{aff}(D)\big)^\perp = \{0\}$, or equivalently $\mathrm{aff}(D) = \Re^n$. Since 0 is a relative interior point of D by assumption, this is equivalent to 0 being an interior point of D. **Q.E.D.**

4.5 DUALITY AND POLYHEDRAL CONVEXITY

We now strengthen the result on the existence of max crossing solutions (Prop. 4.4.1) for a special case where the upwards extension

$$\overline{M} = M + \{(0, w) \mid w \geq 0\}$$

of the set M has partially polyhedral structure. In particular, we will consider the case where $\overline{M}$ is a vector sum of the form

$$\overline{M} = \tilde{M} - \{(u, 0) \mid u \in P\}, \tag{4.24}$$

where $\tilde{M}$ is convex and P is polyhedral. Then the corresponding set

$$D = \{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \overline{M}\},$$

can be written as

$$D = \tilde{D} - P,$$

where

$$\tilde{D} = \{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \tilde{M}\}. \tag{4.25}$$

To understand the nature of the following proposition, we note that from Props. 4.4.1 and 4.4.2, assuming that $-\infty < w^*$, we have:

(a) $q^* = w^*$ and Q^* is nonempty, provided that $0 \in \text{ri}(D)$, which is equivalent to

$$\text{ri}(\tilde{D}) \cap \text{ri}(P) \neq \emptyset, \tag{4.26}$$

since, by Prop. 1.3.7, $\text{ri}(D) = \text{ri}(\tilde{D}) - \text{ri}(P)$.

(b) $q^* = w^*$, and Q^* is nonempty and compact, provided that $0 \in \text{int}(D)$, which is true in particular if either

$$\text{int}(\tilde{D}) \cap P \neq \emptyset,$$

or

$$\tilde{D} \cap \text{int}(P) \neq \emptyset.$$

The following proposition shows in part that when P is polyhedral, these results can be strengthened, and in particular, the condition $\text{ri}(\tilde{D}) \cap \text{ri}(P) \neq \emptyset$ [Eq. (4.26)] can be replaced by the condition

$$\text{ri}(\tilde{D}) \cap P \neq \emptyset.$$

The proof is similar to the proofs of Props. 4.4.1 and 4.4.2, but uses the Polyhedral Proper Separation Theorem (Prop. 1.5.7) to exploit the polyhedral structure of P.

Proposition 4.5.1: Consider the MC/MC framework, and assume the following:

(1) $-\infty < w^*$.

(2) The set $\overline{M}$ has the form

$$\overline{M} = \tilde{M} - \{(u, 0) \mid u \in P\},$$

where $\tilde{M}$ and P are convex sets.

(3) Either $\text{ri}(\tilde{D}) \cap \text{ri}(P) \neq \emptyset$, or P is polyhedral and $\text{ri}(\tilde{D}) \cap P \neq \emptyset$, where $\tilde{D}$ is the set given by Eq. (4.25).

Then $q^* = w^*$, and Q^*, the set of optimal solutions of the max crossing problem, is a nonempty subset of R_P^*, the polar cone of the recession cone of P. Furthermore, Q^* is compact if $\text{int}(\tilde{D}) \cap P \neq \emptyset$.

Proof: We assume that P is polyhedral and $\text{ri}(\tilde{D}) \cap P \neq \emptyset$. The proof for the case where P is just convex and $\text{ri}(\tilde{D}) \cap \text{ri}(P) \neq \emptyset$ is similar: we just use the Proper Separation Theorem (Prop. 1.5.6) in place of the Polyhedral Proper Separation Theorem (Prop. 1.5.7) in the following argument. We consider the sets

$$C_1 = \{(u, v) \mid v > w \text{ for some } (u, w) \in \tilde{M}\},$$

$$C_2 = \{(u, w^*) \mid u \in P\},$$

(cf. Fig. 4.5.1).

It can be seen that C_1 and C_2 are nonempty and convex, and C_2 is polyhedral. Furthermore, C_1 and C_2 are disjoint. To see this, note that if $\overline{u}$ is such that $\overline{u} \in P$ and there exists $(\overline{u}, w)$ with $w^* > w$, then $(0, w) \in \overline{M}$, which is impossible since w^* is the min common value. Therefore, by Prop. 1.5.7, there exists a hyperplane that separates C_1 and C_2, and does not contain C_1, i.e., a vector $(\overline{\mu}, \beta)$ such that

$$\beta w^* + \overline{\mu}' z \leq \beta v + \overline{\mu}' u, \qquad \forall\ (u, v) \in C_1,\ \forall\ z \in P, \tag{4.27}$$

$$\inf_{(u,v) \in C_1} \{\beta v + \overline{\mu}' u\} < \sup_{(u,v) \in C_1} \{\beta v + \overline{\mu}' u\}. \tag{4.28}$$

From Eq. (4.27), since $(0, 1)$ is a direction of recession of C_1, we see that $\beta \geq 0$. If $\beta = 0$, then for a vector $\overline{u} \in \text{ri}(\tilde{D}) \cap P$, Eq. (4.27) yields

$$\overline{\mu}' \overline{u} \leq \inf_{u \in \tilde{D}} \overline{\mu}' u,$$

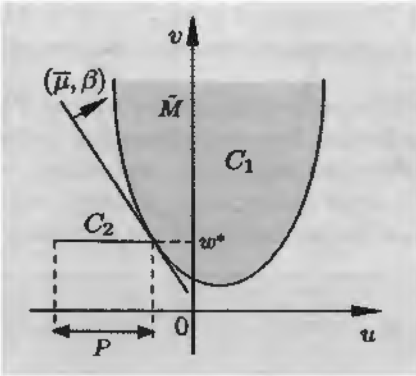


Figure 4.5.1. Illustration of the sets

$$C_1 = \Big\{(u,v) \mid v > w \text{ for some } (u,w) \in \tilde{M}\Big\}, \qquad C_2 = \Big\{(u,w^*) \mid u \in P\Big\},$$

and the hyperplane separating them in the proof of Prop. 4.5.1.

so $\overline{u}$ attains the minimum of the linear function $\overline{\mu}'u$ over $\tilde{D}$, and from Prop. 1.3.4, it follows that $\overline{\mu}'u$ is constant over $\tilde{D}$. On the other hand, from Eq. (4.28) we have $\inf_{u\in\tilde{D}} \overline{\mu}'u < \sup_{u\in\tilde{D}} \overline{\mu}'u$, a contradiction. Hence $\beta > 0$, and by normalizing $(\overline{\mu},\beta)$ if necessary, we may assume that $\beta = 1$.

Thus, from Eq. (4.27), we have

$$w^* + \overline{\mu}'z \leq \inf_{(u,v)\in C_1} \{v + \overline{\mu}'u\}, \qquad \forall\, z \in P, \tag{4.29}$$

which in particular implies that $\overline{\mu}'d \leq 0$ for all $d \in R_P$. Hence $\overline{\mu} \in R_P^*$. From Eq. (4.29), we also obtain

$$\begin{aligned} w^* &\leq \inf_{(u,v)\in C_1,\, z\in P} \big\{v + \overline{\mu}'(u-z)\big\} \\ &= \inf_{(u,v)\in \tilde{M} - \{(z,0)\mid z\in P\}} \{v + \overline{\mu}'u\} \\ &= \inf_{(u,v)\in \overline{M}} \{v + \overline{\mu}'u\} \\ &= q(\overline{\mu}). \end{aligned}$$

Using the weak duality relation $q^* \leq w^*$, we have $q(\overline{\mu}) = q^* = w^*$.

To show that $Q^* \subset R_P^*$, note that for all μ, we have

$$q(\mu) = \inf_{(u,w)\in\overline{M}} \{w + \mu'u\} = \inf_{(u,w)\in\tilde{M},\, z\in P} \{w + \mu'(u-z)\},$$

so $q(\mu) = -\infty$ if $\mu'd > 0$ for some $d \in R_P$. Hence if $\mu \in Q^*$, we must have $\mu'd \leq 0$ for all $d \in R_P$, or $\mu \in R_P^*$.

The proof that Q^* is compact if $\mathrm{int}(\tilde{D}) \cap P \neq \emptyset$ is similar to the one of Prop. 4.4.2 (see the discussion preceding the proposition). **Q.E.D.**

We now discuss an interesting special case of Prop. 4.5.1. It corresponds to $\tilde{M}$ being a linearly transformed epigraph of a convex function f, and P being a polyhedral set such as the nonpositive orthant. This special case applies to constrained optimization duality, and will be used in the proof of the Nonlinear Farkas' Lemma given in Section 5.1.

Proposition 4.5.2: Consider the MC/MC framework, and assume that:

(1) $-\infty < w^*$.

(2) The set $\overline{M}$ is defined in terms of a polyhedral set P, an $r \times n$ matrix A, a vector $b \in \Re^r$, and a convex function $f : \Re^n \mapsto (-\infty, \infty]$ as follows:

$$\overline{M} = \big\{(u, w) \mid Ax - b - u \in P \text{ for some } (x, w) \in \mathrm{epi}(f)\big\}.$$

(3) There is a vector $\overline{x} \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$ such that $A\overline{x} - b \in P$.

Then $q^* = w^*$ and Q^*, the set of optimal solutions of the max crossing problem, is a nonempty subset of R_P^*, the polar cone of the recession cone of P. Furthermore, Q^* is compact if the matrix A has rank r and there is a vector $\overline{x} \in \mathrm{int}\big(\mathrm{dom}(f)\big)$ such that $A\overline{x} - b \in P$.

Proof: Let

$$\tilde{M} = \big\{(Ax - b, w) \mid (x, w) \in \mathrm{epi}(f)\big\}.$$

The following calculation relates $\tilde{M}$ and $\overline{M}$:

$$\begin{aligned}
\tilde{M} - \big\{(z, 0) \mid \ z \in P\big\} & \\
&= \big\{(u, w) \mid u = Ax - b - z \text{ for some } (x, w) \in \mathrm{epi}(f) \text{ and } z \in P\big\} \\
&= \big\{(u, w) \mid Ax - b - u \in P \text{ for some } (x, w) \in \mathrm{epi}(f)\big\} \\
&= \overline{M}.
\end{aligned}$$

Thus the framework of Prop. 4.5.1 applies. Furthermore, the set $\tilde{D}$ of Eq. (4.25) is given by

$$\tilde{D} = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \tilde{M}\big\} = \big\{Ax - b \mid x \in \mathrm{dom}(f)\big\}.$$

Hence, the relative interior assumption (3) implies that the corresponding relative interior assumption of Prop. 4.5.1 is satisfied. Furthermore, if A

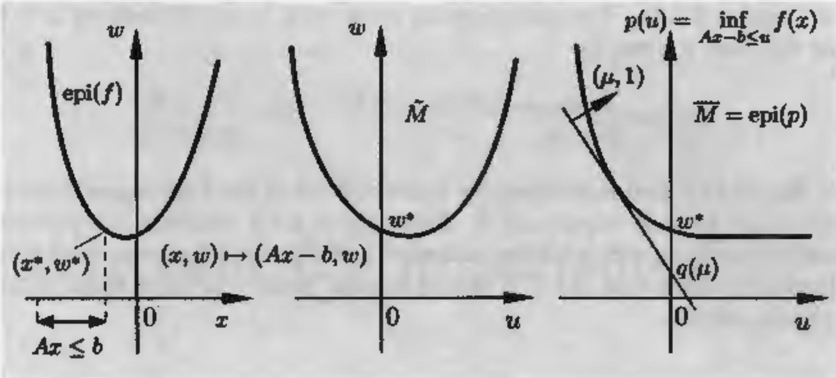


Figure 4.5.2. The MC/MC framework for minimizing $f(x)$ subject to $Ax \le b$. The set $\overline{M}$ is

$$\overline{M} = \Big\{(u, w) \mid Ax - b \le u \text{ for some } (x, w) \in \text{epi}(f)\Big\}$$

(cf. Prop. 4.5.2) and can be written as a vector sum:

$$\overline{M} = \tilde M + \Big\{(u, 0) \mid u \ge 0\Big\},$$

where $\tilde M$ is obtained by transformation of epi(f),

$$\tilde M = \Big\{(Ax - b, w) \mid (x, w) \in \text{epi}(f)\Big\}.$$

Also $\overline{M}$ is the epigraph of the perturbation function $p(u) = \inf_{Ax-b\le u} f(x)$.

has rank r, we have $A\overline{x} - b \in \text{int}(\tilde D)$ for all $\overline{x} \in \text{int}\big(\text{dom}(f)\big)$. The result follows from the conclusion of Prop. 4.5.1. **Q.E.D.**

The special case where P is the negative orthant

$$P = \{u \mid u \le 0\},$$

is noteworthy as it corresponds to a major convex optimization model. To see this, consider the minimization of a convex function $f : \Re^n \mapsto (-\infty, \infty]$ subject to $Ax \le b$, and note that the set $\overline{M}$ in Prop. 4.5.2 is equal to the epigraph of the perturbation function

$$p(u) = \inf_{Ax-b\le u} f(x)$$

(see Fig. 4.5.2). The min common value, $p(0)$, is the optimal value of the problem

$$w^* = p(0) = \inf_{Ax\le b} f(x)$$

(cf. Section 4.2.3). The max crossing problem is to maximize over $\mu \in \Re^r$ the function q given by

$$q(\mu) = \begin{cases} \inf_{x\in\Re^n}\{f(x) + \mu'(Ax - b)\} & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

[cf. Eq. (4.11)], and is obtained by maximization of the Lagrangian function $f(x) + \mu'(Ax - b)$ when $\mu \geq 0$. Proposition 4.5.2 provides the principal duality result for this problem, namely that assuming the existence of $\overline{x} \in \text{ri}\big(\text{dom}(f)\big)$ such that $A\overline{x} \leq b$, strong duality holds and there exists a dual optimal solution.

4.6 SUMMARY

Several propositions regarding the structure of the MC/MC framework were given in this chapter, and it may be helpful to discuss how they are used in specific duality contexts. These are:

(1) Propositions 4.3.1 and 4.3.2, which assert the strong duality property $q^* = w^*$ under convexity of $\overline{M}$, and various conditions on M that involve a "lower semicontinuity structure," and/or closure and compactness. Note that these propositions say nothing about existence of optimal primal or dual solutions.

(2) Propositions 4.4.1-4.5.2, which under various relative interior conditions, assert (in addition to $q^* = w^*$) that the max crossing solution set is nonempty and describe some of its properties, such as the structure of its recession cone and/or its compactness.

Proposition 4.3.1 is quite general and will be used to assert strong duality in Section 5.3.4 (convex programming without relative interior conditions - Prop. 5.3.7), and Section 5.5.1 (minimax problems - Prop. 5.5.1). Proposition 4.3.2 is more specialized, and assumes some structure for the set M (rather than $\overline{M}$). It is helpful in providing perspective, and it is used in Section 5.7 for analysis of the size of the duality gap.

Propositions 4.4.1 and 4.4.2 are also quite general, and are used throughout Chapter 5. For example, they are used to prove part (a) of the Nonlinear Farkas' Lemma (Prop. 5.1.1) on which much of convex programming duality rests (Section 5.3). Propositions 4.5.1 and 4.5.2 are more specialized, but still broadly applicable. They assume a polyhedral structure that is relevant to duality theory for problems involving linear constraints (cf. the problem discussed at the end of Section 4.5). In particular, they are used to prove part (b) of the Nonlinear Farkas' Lemma (and by extension the Fenchel and conic duality theorems of Section 5.3), as well as various results in subdifferential calculus (Section 5.4) and (implicitly) in theorems of the alternative (Section 5.6).

5

Duality and Optimization

Contents

In this chapter we develop many of the basic analytical results of constrained optimization and minimax theory, including duality, and optimality conditions. We also develop the core of subdifferential theory, as well as theorems of the alternative. These lead to additional optimization-related results, such as subgradient-based optimality conditions, sensitivity interpretations of dual optimal solutions, and conditions for compactness of the solution set of a linear program. We finally consider nonconvex optimization and minimax problems, and discuss estimates of the duality gap. The MC/MC duality results of the preceding chapter are the principal tools.

5.1 NONLINEAR FARKAS' LEMMA

We will first prove a nonlinear version of Farkas' Lemma that captures the essence of convex programming duality. The lemma involves a nonempty convex set $X \subset \Re^n$, and functions $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$. We denote $g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, and assume the following.

Assumption 5.1.1: The functions f and g_j, $j = 1, \ldots, r$, are convex, and

$$f(x) \geq 0, \qquad \forall\, x \in X \text{ with } g(x) \leq 0.$$

The Nonlinear Farkas' Lemma asserts, under some conditions, that there exists a nonvertical hyperplane that passes through the origin and contains the set

$$\big\{\big(g(x), f(x)\big) \mid x \in X\big\}$$

in its positive halfspace. Figure 5.1.1 provides a geometric interpretation and suggests the strong connection with the MC/MC framework.

Proposition 5.1.1: (Nonlinear Farkas' Lemma) Let Assumption 5.1.1 hold and let Q^* be the subset of $\Re^r$ given by

$$Q^* = \big\{\mu \mid \mu \geq 0,\ f(x) + \mu' g(x) \geq 0,\ \forall\, x \in X\big\}.$$

Assume that one of the following two conditions holds:

(1) There exists $\overline{x} \in X$ such that $g_j(\overline{x}) < 0$ for all $j = 1, \ldots, r$.

(2) The functions g_j, $j = 1, \ldots, r$, are affine, and there exists $\overline{x} \in \mathrm{ri}(X)$ such that $g(\overline{x}) \leq 0$.

Then Q^* is nonempty, and under condition (1) it is also compact.

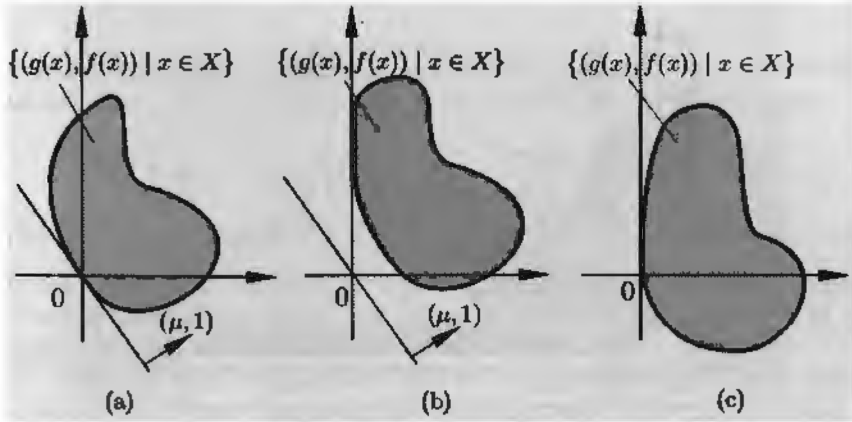


Figure 5.1.1. Geometrical interpretation of the Nonlinear Farkas' Lemma. Assuming that $f(x) \geq 0$ for all $x \in X$ with $g(x) \leq 0$, the lemma asserts the existence of a nonvertical hyperplane in $\Re^{r+1}$, with normal $(\mu, 1)$, that passes through the origin and contains the set

$$\big\{\big(g(x), f(x)\big) \mid x \in X\big\}$$

in its positive halfspace. Figures (a) and (b) show examples where such a hyperplane exists, and figure (c) shows an example where it does not. In Fig. (a) there exists a point $\overline{x} \in X$ with $g(\overline{x}) < 0$.

Proof: Let condition (1) hold. We consider the MC/MC framework corresponding to the subset of $\Re^{r+1}$ given by

$$M = \big\{(u, w) \mid \text{there exists } x \in X \text{ such that } g(x) \leq u,\ f(x) \leq w\big\}$$

(cf. Fig. 5.1.2). We will apply Props. 4.4.1 and 4.4.2 to assert that the set of max crossing hyperplanes is nonempty and compact. To this end, we will verify that the assumptions of these propositions are satisfied. In particular, we will show that:

(i) The optimal value w^* of the corresponding min common problem,

$$w^* = \inf\big\{w \mid (0, w) \in M\big\},$$

satisfies $-\infty < w^*$.

(ii) The set

$$\overline{M} = M + \big\{(0, w) \mid w \geq 0\big\},$$

is convex. (Note here that $\overline{M} = M$.)

(iii) The set

$$D = \big\{u \mid \text{there exists } w \in \Re \text{ such that } (u, w) \in \overline{M}\big\}$$

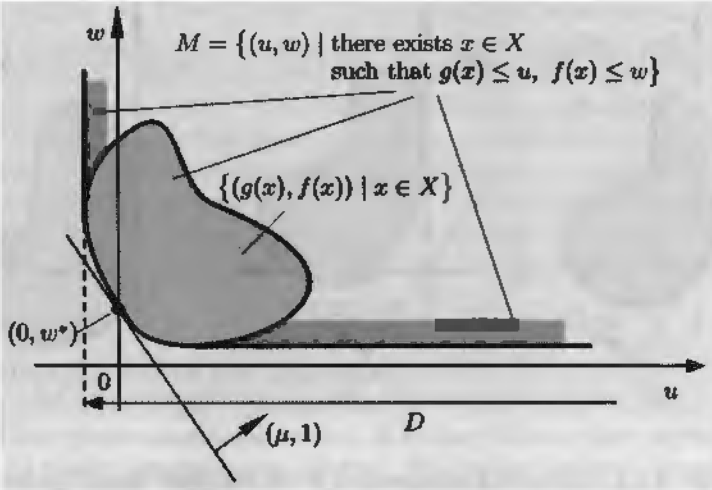


Figure 5.1.2. Illustration of the sets used in the MC/MC framework of the proof of Prop. 5.1.1 under condition (1). We have

$$M = \overline{M} = \Big\{(u,w) \mid \text{there exists } x \in X \text{ such that } g(x) \le u,\ f(x) \le w\Big\}$$

and

$$\begin{aligned} D &= \Big\{u \mid \text{ there exists } w \in \Re \text{ such that } (u,w) \in \overline{M}\Big\} \\ &= \Big\{u \mid \text{ there exists } x \in X \text{ such that } g(x) \le u\Big\}. \end{aligned}$$

The existence of $\overline{x} \in X$ such that $g(\overline{x}) < 0$ is equivalent to $0 \in \text{int}(D)$. Note that M is convex, even though the set $\Big\{\big(g(x), f(x)\big) \mid x \in X\Big\}$ need not be [take for example $X = \Re$, $f(x) = x$, $g(x) = x^2$].

contains the origin in its interior.

To show (i), note that since $f(x) \ge 0$ for all $x \in X$ with $g(x) \le 0$, we have $w \ge 0$ for all $(0,w) \in M$, so that $w^* \ge 0$.

To show (iii), note that D can also be written as

$$D = \Big\{u \mid \text{ there exists } x \in X \text{ such that } g(x) \le u\Big\}.$$

Since D contains the set $g(\overline{x}) + \{u \mid u \ge 0\}$, the condition $g(\overline{x}) < 0$ for some $\overline{x} \in X$, implies that $0 \in \text{int}(D)$.

There remains to show (ii), i.e., that the set $\overline{M}$ is convex. Since $\overline{M} = M$, we will prove that M is convex. To this end, we consider vectors $(u,w) \in M$ and $(\tilde{u}, \tilde{w}) \in M$, and we show that their convex combinations lie in M. By the definition of M, for some $x \in X$ and $\tilde{x} \in X$, we have

$$f(x) \le w, \qquad g_j(x) \le u_j, \quad \forall\, j = 1, \ldots, r,$$

$$f(\tilde{x}) \leq \tilde{w}, \qquad g_j(\tilde{x}) \leq \tilde{u}_j, \quad \forall\, j = 1, \ldots, r.$$

For any $\alpha \in [0,1]$, we multiply these relations with α and $1-\alpha$, respectively, and add them. By using the convexity of f and g_j for all j, we obtain

$$f\big(\alpha x + (1-\alpha)\tilde{x}\big) \leq \alpha f(x) + (1-\alpha) f(\tilde{x}) \leq \alpha w + (1-\alpha)\tilde{w},$$

$$g_j\big(\alpha x+(1-\alpha)\tilde{x}\big) \leq \alpha g_j(x)+(1-\alpha)g_j(\tilde{x}) \leq \alpha u_j+(1-\alpha)\tilde{u}_j, \quad \forall\, j = 1, \ldots, r.$$

By convexity of X, we have $\alpha x + (1-\alpha)\tilde{x} \in X$ for all $\alpha \in [0,1]$, so the preceding inequalities imply that the convex combination $\alpha(u,w) + (1-\alpha)(\tilde{u},\tilde{w})$ belongs to M, showing that M is convex.

Thus our assumptions imply that all the assumptions of Props. 4.4.1 and 4.4.2 hold. From Prop. 4.4.1, we obtain $w^* = q^* = \sup_\mu q(\mu)$, where q is the dual function,

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} = \begin{cases} \inf_{x\in X}\big\{f(x) + \mu' g(x)\big\} & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

From Prop. 4.4.2, the optimal solution set $\tilde{Q} = \big\{\mu \mid q(\mu) \geq w^*\big\}$ is nonempty and compact. Furthermore, according to the definition of Q^*, we have

$$Q^* = \big\{\mu \mid \mu \geq 0,\ f(x) + \mu' g(x) \geq 0,\ \forall\, x \in X\big\} = \big\{\mu \mid q(\mu) \geq 0\big\}.$$

Thus Q^* and $\tilde{Q}$ are level sets of the closed proper convex function $-q$, with $Q^* \supset \tilde{Q}$ (in view of $w^* \geq 0$). Since $\tilde{Q}$ is nonempty and compact, so is Q^* [cf. Prop. 1.4.5(b)].

Let condition (2) hold, and let the constraint $g(x) \leq 0$ be written as

$$Ax - b \leq 0,$$

where A is an $r \times n$ matrix and b is a vector in $\Re^n$. We introduce a MC/MC framework with polyhedral structure and apply Prop. 4.5.2, with P being the nonpositive orthant and the set $\overline{M}$ defined by

$$\overline{M} = \big\{(u,w) \mid Ax - b - u \leq 0, \text{ for some } (x,w) \in \text{epi}(\tilde{f})\big\},$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

Since by assumption, $f(x) \geq 0$ for all $x \in X$ with $Ax - b \leq 0$, the min common value satisfies

$$w^* = \inf_{Ax-b\leq 0} \tilde{f}(x) \geq 0.$$

By Prop. 4.5.2 and the discussion following its proof, there exists $\mu \geq 0$ such that

$$q^* = q(\mu) = \inf_{x \in \Re^n} \{\tilde{f}(x) + \mu'(Ax - b)\}.$$

Since $q^* = w^* \geq 0$, it follows that $\tilde{f}(x) + \mu'(Ax - b) \geq 0$ for all $x \in \Re^n$, or $f(x) + \mu'(Ax - b) \geq 0$ for all $x \in X$, so $\mu \in Q^*$. **Q.E.D.**

By selecting f and g_j to be linear, and X to be the entire space in the Nonlinear Farkas' Lemma, we obtain a version of Farkas' Lemma (cf. Section 2.3.1) as a special case.

Proposition 5.1.2: (Linear Farkas' Lemma) Let A be an $m \times n$ matrix and c be a vector in $\Re^m$.

(a) The system $Ay = c$, $y \geq 0$ has a solution if and only if

$$A'x \leq 0 \quad \Rightarrow \quad c'x \leq 0. \tag{5.1}$$

(b) The system $Ay \geq c$ has a solution if and only if

$$A'x = 0,\ x \geq 0 \quad \Rightarrow \quad c'x \leq 0.$$

Proof: (a) If $y \in \Re^n$ is such that $Ay = c$, $y \geq 0$, then $y'A'x = c'x$ for all $x \in \Re^m$, which implies Eq. (5.1). Conversely, let us apply the Nonlinear Farkas' Lemma under condition (2) with $f(x) = -c'x$, $g(x) = A'x$, and $X = \Re^m$. We see that the relation (5.1) implies the existence of $\mu \geq 0$ such that

$$-c'x + \mu'A'x \geq 0, \qquad \forall\ x \in \Re^m,$$

or equivalently $(A\mu - c)'x \geq 0$ for all $x \in \Re^m$, or $A\mu = c$.

(b) This part follows by writing the system $Ay \geq c$ in the equivalent form

$$Ay^+ - Ay^- - z = c, \qquad y^+ \geq 0,\ y^- \geq 0,\ z \geq 0,$$

and by applying part (a). **Q.E.D.**

5.2 LINEAR PROGRAMMING DUALITY

We will now derive one of the most important results in optimization: the linear programming duality theorem. Consider the problem

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad a_j'x \geq b_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $c \in \Re^n$, $a_j \in \Re^n$, and $b_j \in \Re$, $j = 1, \ldots, r$. We refer to this as the *primal problem*. We consider the *dual problem*

$$\begin{aligned} &\text{maximize } b'\mu \\ &\text{subject to } \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0, \end{aligned}$$

which was derived from the MC/MC duality framework in Section 4.2.3. We denote the primal and dual optimal values by f^* and q^*, respectively.

We will show that $f^* = q^*$, assuming that the problem is feasible. While we may argue in terms of the MC/MC framework and the duality theory of Chapter 4, we will use instead the Linear Farkas' Lemma (Prop. 5.1.2), which captures the essential argument. The first step in the proof is to assert that weak duality holds, i.e., $q^* \leq f^*$ (cf. Prop. 4.1.2). [We can also prove this relation with a simple argument: if x and μ are feasible solutions of the primal and dual problems, we have $b'\mu \leq c'x$. Indeed,

$$b'\mu = \sum_{j=1}^{r} b_j \mu_j + \left(c - \sum_{j=1}^{r} a_j \mu_j\right)' x = c'x + \sum_{j=1}^{r} \mu_j (b_j - a_j' x) \leq c'x, \tag{5.2}$$

where the inequality follows from the feasibility of x and μ. By taking supremum of the left-hand side over all feasible μ and infimum of the right-hand side over all feasible x, we obtain $q^* \leq f^*$.] The second step is to demonstrate the existence of feasible vectors x^* and μ^* such that $b'\mu^* = c'x^*$, assuming f^* or q^* is finite. This will follow from Farkas' Lemma, as geometrically illustrated in Fig. 5.2.1.

Proposition 5.2.1: (Linear Programming Duality Theorem)

(a) If either f^* or q^* is finite, then $f^* = q^*$ and both the primal and the dual problem have optimal solutions.

(b) If $f^* = -\infty$, then $q^* = -\infty$.

(c) If $q^* = \infty$, then $f^* = \infty$.

Proof: (a) If f^* is finite, by Prop. 1.4.12, there exists a primal optimal solution x^*. Let

$$J = \{ j \in \{1, \ldots, r\} \mid a_j' x^* = b_j \}.$$

We claim that

$$c'y \geq 0, \qquad \forall \; y \text{ such that } a_j' y \geq 0 \text{ for all } j \in J. \tag{5.3}$$

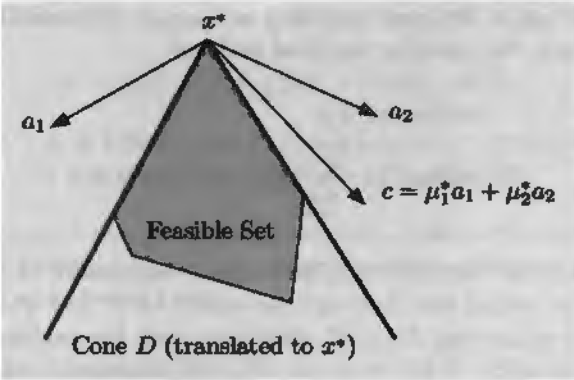


Figure 5.2.1. Illustration of the use of Farkas' Lemma to prove the linear programming duality theorem. Let x^* be a primal optimal solution, and let $J = \{j \mid a_j' x^* = b_j\}$. Then, we have $c'y \geq 0$ for all y in the cone of "feasible directions"

$$D = \{y \mid a_j' y \geq 0, \ \forall\, j \in J\}$$

(cf. Prop. 1.1.8). By Farkas' Lemma [Prop. 5.1.2(a)], this implies that c can be expressed in terms of some scalars $\mu_j^* \geq 0$ as

$$c = \sum_{j=1}^{r} \mu_j^* a_j, \qquad \mu_j^* \geq 0, \ \forall\, j \in J, \qquad \mu_j^* = 0, \ \forall\, j \notin J.$$

Taking inner product with x^*, we obtain $c'x^* = b'\mu^*$, which in view of $q^* \leq f^*$, shows that $q^* = f^*$ and that μ^* is optimal.

Indeed, each y such that $a_j' y \geq 0$ for all $j \in J$ is a feasible direction at x^*, in the sense that $x^* + \alpha y$ is a feasible point for all $\alpha > 0$ that are sufficiently small. It follows that the inequality $c'y < 0$ would violate the optimality of x^*, thus showing Eq. (5.3).

By Farkas' Lemma [Prop. 5.1.2(a)], Eq. (5.3) implies that

$$c = \sum_{j=1}^{r} \mu_j^* a_j,$$

for some $\mu^* \in \Re^r$ with

$$\mu_j^* \geq 0, \ \forall\, j \in J, \qquad \mu_j^* = 0, \ \forall\, j \notin J.$$

Thus, μ^* is a dual-feasible solution, and taking inner product with x^*, and using the fact $a_j' x^* = b_j$ for $j \in J$, and $\mu_j^* = 0$ for $j \notin J$, we obtain

$$c'x^* = \sum_{j=1}^{r} \mu_j^* a_j{}' x^* = \sum_{j=1}^{r} \mu_j^* b_j = b'\mu^*.$$

This, together with Eq. (5.2), implies that $q^* = f^*$ and that μ^* is optimal.

If q^* is finite, again by Prop. 1.4.12, there exists a dual optimal solution μ^*, and a similar argument proves the result.

(b) If $f^* = -\infty$, the inequality $q^* \le f^*$ implies that $q^* = -\infty$.

(c) If $q^* = \infty$, the inequality $q^* \le f^*$ implies that $f^* = \infty$. **Q.E.D.**

The one possibility left open by the proposition is to have $f^* = \infty$ and $q^* = -\infty$ (both primal and dual problems are infeasible). It turns out that this is possible: a trivial example is the infeasible scalar problem $\min_{0 \cdot x \ge 1} x$ whose dual is the infeasible scalar problem $\max_{0 \cdot \mu = 1,\, \mu \ge 0} \mu$.

Another result, related to the duality theorem, is the following necessary and sufficient condition for primal and dual optimality.

Proposition 5.2.2: (Linear Programming Optimality Conditions) A pair of vectors (x^*, μ^*) form a primal and dual optimal solution pair if and only if x^* is primal-feasible, μ^* is dual-feasible, and

$$\mu_j^*(b_j - a_j' x^*) = 0, \qquad \forall\, j = 1, \ldots, r. \tag{5.4}$$

Proof: If x^* is primal-feasible and μ^* is dual-feasible, then [cf. Eq. (5.2)]

$$b'\mu^* = \sum_{j=1}^{r} b_j \mu_j^* + \left(c - \sum_{j=1}^{r} a_j \mu_j^* \right)' x^* = c'x^* + \sum_{j=1}^{r} \mu_j^*(b_j - a_j' x^*). \tag{5.5}$$

Thus, if Eq. (5.4) holds, we have $b'\mu^* = c'x^*$, so Eq. (5.2) implies that x^* is a primal optimal solution and μ^* is a dual optimal solution.

Conversely, if (x^*, μ^*) form a primal and dual optimal solution pair, then x^* is primal-feasible, μ^* is dual-feasible, and by the duality theorem [Prop. 5.2.1(a)], we have $b'\mu^* = c'x^*$. From Eq. (5.5), we obtain Eq. (5.4). **Q.E.D.**

The condition (5.4) is known as *complementary slackness*. For a primal-feasible vector x^* and a dual-feasible vector μ^*, complementary slackness can be written in several equivalent ways, such as:

$$a_j' x^* > b_j \quad \Rightarrow \quad \mu_j^* = 0, \qquad \forall\, j = 1, \ldots, r,$$

$$\mu_j^* > 0 \quad \Rightarrow \quad a_j' x^* = b_j, \qquad \forall\, j = 1, \ldots, r.$$

5.3 CONVEX PROGRAMMING DUALITY

We will now derive duality results and optimality conditions for convex programming, i.e., problems with convex cost function and constraints.

5.3.1 Strong Duality Theorem – Inequality Constraints

We first focus on the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \;\; x \in X, \quad g(x) \le 0, \end{aligned} \tag{5.6}$$

where X is a convex set, $g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, and $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$, are convex functions. We refer to this as the *primal problem*. The duality results of this section assume that the problem is feasible, i.e., $f^* < \infty$, where f^* is the optimal value:

$$f^* = \inf_{x \in X,\, g(x) \le 0} f(x).$$

We will explore the connection with the dual problem derived from the MC/MC framework of Section 4.2.3, by using the Nonlinear Farkas' Lemma (Prop. 5.1.1) as our primary analytical tool.

Consider the Lagrangian function

$$L(x, \mu) = f(x) + \mu' g(x), \qquad x \in X,\ \mu \in \Re^r,$$

and the function q given by

$$q(\mu) = \begin{cases} \inf_{x \in X} L(x, \mu) & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise.} \end{cases}$$

We refer to q as the *dual function* and to the problem

$$\begin{aligned} &\text{maximize} \quad q(\mu) \\ &\text{subject to} \quad \mu \in \Re^r, \end{aligned}$$

as the *dual problem*. The dual optimal value is

$$q^* = \sup_{\mu \in \Re^r} q(\mu).$$

Note the weak duality relation $q^* \le f^*$ (cf. Prop. 4.1.2).

To investigate the question of existence of optimal solutions of the dual problem, we may use the MC/MC general results of Sections 4.3-4.5. However, for convex programming, the essential nature of these results is captured in the Nonlinear Farkas' Lemma (Prop. 5.1.1), which we will use instead in our analysis. To see the connection, assume that the optimal value f^* is finite. Then, we have

$$0 \le f(x) - f^*, \qquad \forall\, x \in X \text{ with } g(x) \le 0,$$

so by replacing $f(x)$ by $f(x) - f^*$ and by applying the Nonlinear Farkas' Lemma (assuming that one of the two conditions of the lemma holds), we see that the set

$$Q^* = \{\mu \mid \mu^* \geq 0,\, 0 \leq f(x) - f^* + \mu^{*\prime} g(x),\, \forall\, x \in X\},$$

is nonempty [and also compact if condition (1) of the lemma is satisfied]. The vectors $\mu^* \in Q^*$ are precisely those for which $\mu^* \geq 0$ and

$$f^* \leq \inf_{x \in X} \{f(x) + \mu^{*\prime} g(x)\} = q(\mu^*) \leq q^*.$$

Using the weak duality relation $q^* \leq f^*$, we see that $f^* = q(\mu^*) = q^*$ if and only if $\mu^* \in Q^*$, i.e., Q^* coincides with the set of optimal solutions of the dual problem. We state the conclusion as a proposition.

Proposition 5.3.1: (Convex Programming Duality - Existence of Dual Optimal Solutions) Consider the problem (5.6). Assume that f^* is finite, and that one of the following two conditions holds:

(1) There exists $\overline{x} \in X$ such that $g_j(\overline{x}) < 0$ for all $j = 1, \ldots, r$.

(2) The functions g_j, $j = 1, \ldots, r$, are affine, and there exists $\overline{x} \in \mathrm{ri}(X)$ such that $g(\overline{x}) \leq 0$.

Then $q^* = f^*$ and the set of optimal solutions of the dual problem is nonempty. Under condition (1) this set is also compact.

The interior condition (1) in the preceding proposition is known in the nonlinear programming literature as the *Slater condition.*

5.3.2 Optimality Conditions

The following proposition gives necessary and sufficient conditions for optimality, which generalize the linear programming conditions of Prop. 5.2.2.

Proposition 5.3.2: (Optimality Conditions) Consider the problem (5.6). There holds $q^* = f^*$, and (x^*, μ^*) are a primal and dual optimal solution pair if and only if x^* is feasible, $\mu^* \geq 0$, and

$$x^* \in \arg\min_{x \in X} L(x, \mu^*), \qquad \mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r. \tag{5.7}$$

Proof: If $q^* = f^*$, and x^* and μ^* are primal and dual optimal solutions,

$$f^* = q^* = q(\mu^*) = \inf_{x \in X} L(x, \mu^*) \leq L(x^*, \mu^*) = f(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) \leq f(x^*).$$

where the last inequality follows since $\mu_j^* \geq 0$ and $g_j(x^*) \leq 0$ for all j. Hence equality holds throughout above and Eq. (5.7) holds.

Conversely, if x^* is feasible, $\mu^* \geq 0$, and Eq. (5.7) is satisfied, then

$$q(\mu^*) = \inf_{x \in X} L(x, \mu^*) = L(x^*, \mu^*) = f(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) = f(x^*).$$

In view of the weak duality relation $q^* \leq f^*$, it follows that $q^* = f^*$, x^* is primal optimal, and μ^* is dual optimal. **Q.E.D.**

Note that the preceding proof makes no use of the convexity assumptions on f, g, or X, and indeed Prop. 5.3.2 holds without these assumptions. On the other hand the proposition is meaningful only when $q^* = f^*$, which can rarely be guaranteed without convexity assumptions. The condition $\mu_j^* g_j(x^*) = 0$ is known as *complementary slackness* and generalizes the corresponding linear programming condition (5.4).

When using Prop. 5.3.2 as a sufficient optimality condition, it is important to realize that for $x^* \in X$ to be primal optimal, it is not sufficient that it attain the minimum of $L(x, \mu^*)$ over X for some dual optimal solution μ^*; this minimum may be attained by some x^* that are infeasible [violate the constraints $g(x) \leq 0$] or by some x^* that are feasible, but are not optimal (these must violate complementary slackness).

Example 5.3.1: (Quadratic Programming Duality)

Consider the quadratic programming problem

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2} x' Q x + c' x \\ &\text{subject to} \quad Ax \leq b, \end{aligned}$$

where Q is a symmetric positive definite $n \times n$ matrix, A is an $r \times n$ matrix, b is a vector in $\Re^r$, and c is a vector in $\Re^n$. This is problem (5.6) with

$$f(x) = \tfrac{1}{2} x' Q x + c' x, \qquad g(x) = Ax - b, \qquad X = \Re^n.$$

Assuming the problem is feasible, it has a unique optimal solution x^*, since the cost function is strictly convex and coercive.

The dual function is

$$q(\mu) = \inf_{x \in \Re^n} L(x, \mu) = \inf_{x \in \Re^n} \left\{ \tfrac{1}{2} x' Q x + c' x + \mu'(Ax - b) \right\}.$$

The infimum is attained for $x = -Q^{-1}(c + A'\mu)$, and after substituting this expression in the preceding relation, a straightforward calculation yields

$$q(\mu) = -\tfrac{1}{2} \mu' A Q^{-1} A' \mu - \mu'(b + AQ^{-1}c) - \tfrac{1}{2} c' Q^{-1} c.$$

The dual problem, after changing the minus sign to convert the maximization to a minimization and dropping the constant $\frac{1}{2}c'Q^{-1}c$, can be written as

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\mu' P\mu + t'\mu \\ &\text{subject to} \quad \mu \geq 0, \end{aligned}$$

where

$$P = AQ^{-1}A', \qquad t = b + AQ^{-1}c.$$

Note that the dual problem has simpler constraints than the primal. Furthermore, if the row dimension r of A is smaller than its column dimension n, the dual problem is defined on a space of smaller dimension than the primal, and this can be algorithmically significant.

By applying Prop. 5.3.1 under condition (2), we see that $f^* = q^*$ and that the dual problem has an optimal dual solution (the relative interior assumption of the proposition is trivially satisfied since $X = \Re^n$).

According to the optimality conditions of Prop. 5.3.2, (x^*, μ^*) is a primal and dual optimal solution pair if and only if $Ax^* \leq b$, $\mu^* \geq 0$, and the two conditions of Eq. (5.7) hold. The first of these conditions [x^* minimizes $L(x, \mu^*)$ over $x \in \Re^n$] yields

$$x^* = -Q^{-1}(c + A'\mu^*).$$

The second is the complementary slackness condition $(Ax^* - b)'\mu^* = 0$, encountered in linear programming (cf. Prop. 5.2.2). It can be written as

$$\mu_j^* > 0 \quad \Rightarrow \quad a_j'x^* = b_j, \qquad \forall\, j = 1, \ldots, r,$$

where a_j' is the jth row of A, and b_j is the jth component of b.

5.3.3 Partially Polyhedral Constraints

The preceding analysis for the inequality-constrained problem (5.6) can be refined by making more specific assumptions regarding available polyhedral structure in the constraint functions and the abstract constraint set X. We first consider some of the simplest cases where there is a mixture of polyhedral and nonpolyhedral constraints. We then provide a general duality theorem that covers simultaneously a broad variety of structures.

Consider an extension of problem (5.6) where there are additional linear equality constraints:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \quad g(x) \leq 0, \quad Ax = b, \end{aligned} \tag{5.8}$$

where X is a convex set, $g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$, are convex functions, A is an $m \times n$ matrix, and

$b \in \Re^m$. We can deal with this problem by simply converting the constraint $Ax = b$ to the equivalent set of linear inequality constraints

$$Ax \le b, \qquad -Ax \le -b, \tag{5.9}$$

with corresponding dual variables $\lambda^+ \ge 0$ and $\lambda^- \ge 0$. The Lagrangian function is

$$f(x) + \mu' g(x) + (\lambda^+ - \lambda^-)'(Ax - b),$$

and by introducing a dual variable

$$\lambda = \lambda^+ - \lambda^- \tag{5.10}$$

with no sign restriction, it can be written as

$$L(x, \mu, \lambda) = f(x) + \mu' g(x) + \lambda'(Ax - b).$$

The dual problem is

$$\begin{aligned} &\text{maximize} \quad q(\mu, \lambda) \equiv \inf_{x \in X} L(x, \mu, \lambda) \\ &\text{subject to} \quad \mu \ge 0, \ \lambda \in \Re^m. \end{aligned}$$

In the special case of a problem with just linear equality constraints:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \quad Ax = b, \end{aligned} \tag{5.11}$$

the Lagrangian function is

$$L(x, \lambda) = f(x) + \lambda'(Ax - b),$$

and the dual problem is

$$\begin{aligned} &\text{maximize} \quad q(\lambda) \equiv \inf_{x \in X} L(x, \lambda) \\ &\text{subject to} \quad \lambda \in \Re^m. \end{aligned}$$

The following two propositions are obtained by transcription of Prop. 5.3.1 [under condition (2) which deals with linear constraints] and Prop. 5.3.2, using the transformations (5.9), (5.10). The straightforward details are left for the reader.

Proposition 5.3.3: (Convex Programming - Linear Equality Constraints) Consider problem (5.11).

(a) Assume that f^* is finite and that there exists $\overline{x} \in \mathrm{ri}(X)$ such that $A\overline{x} = b$. Then $f^* = q^*$ and there exists at least one dual optimal solution.

(b) There holds $f^* = q^*$, and (x^*, λ^*) are a primal and dual optimal solution pair if and only if x^* is feasible and

$$x^* \in \arg\min_{x \in X} L(x, \lambda^*). \tag{5.12}$$

Proposition 5.3.4: (Convex Programming - Linear Equality and Inequality Constraints) Consider problem (5.8).

(a) Assume that f^* is finite, that the functions g_j are linear, and that there exists $\overline{x} \in \mathrm{ri}(X)$ such that $A\overline{x} = b$ and $g(\overline{x}) \leq 0$. Then $q^* = f^*$ and there exists at least one dual optimal solution.

(b) There holds $f^* = q^*$, and (x^*, μ^*, λ^*) are a primal and dual optimal solution pair if and only if x^* is feasible, $\mu^* \geq 0$, and

$$x^* \in \arg\min_{x \in X} L(x, \mu^*, \lambda^*), \qquad \mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r.$$

The following is an extension of part (a) of the preceding proposition to the case where the inequality constraints may be nonlinear.

Proposition 5.3.5: (Convex Programming - Linear Equality and Nonlinear Inequality Constraints) Consider problem (5.8). Assume that f^* is finite, that there exists $\overline{x} \in X$ such that $A\overline{x} = b$ and $g(\overline{x}) < 0$, and that there exists $\tilde{x} \in \mathrm{ri}(X)$ such that $A\tilde{x} = b$. Then $q^* = f^*$ and there exists at least one dual optimal solution.

Proof: By applying Prop. 5.3.1 under condition (1), we see that there exists $\mu^* \geq 0$ such that

$$f^* = \inf_{x \in X,\, Ax = b} \big\{ f(x) + \mu^{*\prime} g(x) \big\}.$$

By applying Prop. 5.3.3 to the minimization problem above, we see that there exists λ^* such that

$$f^* = \inf_{x \in X} \big\{ f(x) + \mu^{*\prime} g(x) + \lambda^{*\prime}(Ax - b) \big\} = \inf_{x \in X} L(x, \mu^*, \lambda^*) = q(\lambda^*, \mu^*).$$

From weak duality we have $q(\mu^*, \lambda^*) \leq q^* \leq f^*$, so it follows that $q^* = f^*$ and that (μ^*, λ^*) is a dual optimal solution. **Q.E.D.**

The following example illustrates how we can have $q^* < f^*$ in convex programming.

Example 5.3.2: (Strong Duality Counterexample)

Consider the two-dimensional problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x_1 = 0, \qquad x \in X = \{x \mid x \geq 0\}, \end{aligned}$$

where

$$f(x) = e^{-\sqrt{x_1 x_2}}, \qquad \forall\, x \in X.$$

Here it can be verified that f is convex (its Hessian is positive definite in the interior of X). Since for feasibility we must have $x_1 = 0$, we see that $f^* = 1$. On the other hand, the dual function is

$$q(\lambda) = \inf_{x \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + \lambda x_1 \right\} = \begin{cases} 0 & \text{if } \lambda \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

since when $\lambda \geq 0$, the expression in braces above is nonnegative for $x \geq 0$ and can approach zero by taking $x_1 \to 0$ and $x_1 x_2 \to \infty$. It follows that $q^* = 0$. Thus, there is a duality gap, $f^* - q^* = 1$. Here the relative interior assumption of Prop. 5.3.3(a) is violated.

For an example where $q^* = f^*$ but there exists no dual optimal solution, take

$$X = \Re, \qquad f(x) = x, \qquad g(x) = x^2.$$

Then $x^* = 0$ is the only feasible/optimal solution, and we have

$$q(\mu) = \inf_{x \in \Re} \{x + \mu x^2\} = -\frac{1}{4\mu}, \qquad \forall\, \mu > 0,$$

and $q(\mu) = -\infty$ for $\mu \leq 0$, so that $q^* = f^* = 0$. However, there is no $\mu^* \geq 0$ such that $q(\mu^*) = q^* = 0$. This is a typical constrained optimization situation where there are no Lagrange multipliers (as usually defined in nonlinear programming; see e.g., [Ber99]), because some form of "regularity" condition fails to hold. We will revisit this example in Section 5.3.4.

Mixtures of Constraints

We finally consider a more refined polyhedral structure that allows an essentially arbitrary mixture of polyhedral and nonpolyhedral constraints. In particular, we consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \quad g(x) \leq 0, \quad Ax = b, \end{aligned} \tag{5.13}$$

where X is the intersection of a polyhedral set P and a convex set C,

$$X = P \cap C,$$

$g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, the functions $f : \Re^n \mapsto \Re$ and $g_j : \Re^n \mapsto \Re$, $j = 1, \ldots, r$, are defined over $\Re^n$, A is an $m \times n$ matrix, and $b \in \Re^m$.

We will assume that some of the functions g_j are polyhedral, i.e., each is specified as the maximum of a finite number of linear functions (cf.

Section 2.3.3). We will also assume that f and g_j are convex over C (rather than just X). This stronger convexity assumption is significant for taking advantage of the (partially) polyhedral character of X (for example, it fails to hold in Example 5.3.2, with a duality gap resulting).

Proposition 5.3.6: (Convex Programming - Mixed Polyhedral and Nonpolyhedral Constraints) Consider problem (5.13). Assume that f^* is finite and that for some $\overline{r}$ with $1 \leq \overline{r} \leq r$, the functions g_j, $j = 1, \ldots, \overline{r}$, are polyhedral, and the functions f and g_j, $j = \overline{r}+1, \ldots, r$, are convex over C. Assume further that:

(1) There exists a vector $\tilde{x} \in \mathrm{ri}(C)$ in the set

$$\tilde{P} = P \cap \{x \mid Ax = b,\ g_j(x) \leq 0,\ j = 1, \ldots, \overline{r}\}.$$

(2) There exists $\overline{x} \in \tilde{P} \cap C$ such that $g_j(\overline{x}) < 0$ for all $j = \overline{r}+1, \ldots, r$.

Then $q^* = f^*$ and there exists at least one dual optimal solution.

Proof: Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in \tilde{P} \cap C, \quad g_j(x) \leq 0,\ j = \overline{r}+1, \ldots, r, \end{aligned}$$

which is equivalent to problem (5.13). We use Prop. 5.3.1 and the assumption (2) to assert that there exist $\mu_j^* \geq 0$, $j = \overline{r}+1, \ldots, r$, such that

$$f^* = \inf_{x \in \tilde{P} \cap C} \left\{ f(x) + \sum_{j=\overline{r}+1}^{r} \mu_j^* g_j(x) \right\}. \tag{5.14}$$

We next consider the minimization problem in the preceding equation, and we introduce explicit representations for the polyhedral set P and the polyhedral functions $g_j(x) \leq 0$, $j = 1, \ldots, \overline{r}$, in terms of linear inequalities and linear functions:

$$P = \{x \mid e_{i0}' x \leq d_{i0},\ i = 1, \ldots, m_0\},$$

$$g_j(x) = \max_{i=1,\ldots,m_j} \{e_{ij}' x - d_{ij}\}, \qquad j = 1, \ldots, \overline{r},$$

where e_{ij} are some vectors in $\Re^n$ and d_{ij} are corresponding scalars. We write Eq. (5.14) as

$$f^* = \inf_{x \in C,\ Ax = b,\ e_{ij}' x - d_{ij} \leq 0,\ j = 0, \ldots, \overline{r},\ i = 1, \ldots, m_j} \left\{ f(x) + \sum_{j=\overline{r}+1}^{r} \mu_j^* g_j(x) \right\}. \tag{5.15}$$

We now use the assumption (1) and Prop. 5.3.4 in connection with the minimization problem in the right-hand side of the preceding equation, to assert that there exist a vector λ^* and scalars $\nu_{ij}^* \geq 0$ such that

$$
\begin{aligned}
f^* &= \inf_{x \in C} \left\{ f(x) + \sum_{j=\bar{r}+1}^{r} \mu_j^* g_j(x) + \lambda^{*\prime}(Ax - b) + \sum_{j=0}^{\bar{r}} \sum_{i=1}^{m_j} \nu_{ij}^* (e_{ij}' x - d_{ij}) \right\} \\
&\leq \inf_{x \in C \cap P} \left\{ f(x) + \sum_{j=\bar{r}+1}^{r} \mu_j^* g_j(x) + \lambda^{*\prime}(Ax - b) + \sum_{j=0}^{\bar{r}} \sum_{i=1}^{m_j} \nu_{ij}^* (e_{ij}' x - d_{ij}) \right\}
\end{aligned}
$$

(the inequality follows since we are taking infimum over a subset of C). Since for all $x \in P$, we have $\nu_{i0}^*(e_{i0}'x - d_{i0}) \leq 0$, it follows that

$$
\begin{aligned}
f^* &\leq \inf_{x \in C \cap P} \left\{ f(x) + \sum_{j=\bar{r}+1}^{r} \mu_j^* g_j(x) + \lambda^{*\prime}(Ax - b) + \sum_{j=1}^{\bar{r}} \sum_{i=1}^{m_j} \nu_{ij}^* (e_{ij}' x - d_{ij}) \right\} \\
&\leq \inf_{x \in C \cap P} \left\{ f(x) + \sum_{j=\bar{r}+1}^{r} \mu_j^* g_j(x) + \lambda^{*\prime}(Ax - b) + \sum_{j=1}^{\bar{r}} \left(\sum_{i=1}^{m_j} \nu_{ij}^* \right) g_j(x) \right\} \\
&= \inf_{x \in C \cap P} \left\{ f(x) + \mu^{*\prime} g(x) + \lambda^{*\prime}(Ax - b) \right\} \\
&= q(\mu^*, \lambda^*),
\end{aligned}
$$

where $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$ with

$$
\mu_j^* = \sum_{i=1}^{m_j} \nu_{ij}^*, \qquad j = 1, \ldots, \bar{r}.
$$

In view of the weak duality relation $q(\mu^*, \lambda^*) \leq q^* \leq f^*$, it follows that $q^* = f^*$ and that (μ^*, λ^*) is a dual optimal solution. **Q.E.D.**

Note that the preceding proposition contains as special cases Props. 5.3.3(a), 5.3.4(a), and 5.3.5.

5.3.4 Duality and Existence of Optimal Primal Solutions

Our approach so far for establishing strong duality in constrained optimization has been based on the Nonlinear Farkas' Lemma. It provides assumptions guaranteeing that the dual problem has an optimal solution (even if there may be no primal optimal solution; cf. Prop. 5.3.1). We will now give an alternative approach, which under some compactness assumptions, guarantees strong duality and that there exists an optimal primal solution (even if there may be no dual optimal solution).

We focus on the convex programming problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \quad g(x) \leq 0, \end{aligned} \tag{5.16}$$

where X is a convex set, $g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, and $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$, are convex functions. We consider the MC/MC framework with $M = \text{epi}(p)$, where p is the perturbation function

$$p(u) = \inf_{x \in X,\, g(x) \leq u} f(x)$$

(cf. Section 4.2.3). From Prop. 4.3.1, we know that strong duality holds if p is closed proper convex (see also Fig. 4.2.1). With this in mind, let

$$F(x, u) = \begin{cases} f(x) & \text{if } x \in X, \ g(x) \leq u, \\ \infty & \text{otherwise,} \end{cases}$$

and note that

$$p(u) = \inf_{x \in \Re^n} F(x, u),$$

so p is obtained from F by partial minimization. By using the corresponding analysis of Section 3.3, we obtain the following.

Proposition 5.3.7: (Convex Programming Duality - Existence of Primal Optimal Solutions) Assume that the problem (5.16) is feasible, that the convex functions f and g_j are closed, and that the function

$$F(x, 0) = \begin{cases} f(x) & \text{if } g(x) \leq 0, \ x \in X, \\ \infty & \text{otherwise,} \end{cases}$$

has compact level sets. Then $f^* = q^*$ and the set of optimal solutions of the primal problem is nonempty and compact.

Proof: From Prop. 3.3.2 applied to the function F, the partial minimum function p is convex and closed. From Prop. 4.3.1, it follows that $f^* = q^*$. Furthermore, since $F(x, 0)$ has compact level sets, the set of minima of $F(x, 0)$, which is equal to the set of optimal solutions of the primal problem, is nonempty and compact. **Q.E.D.**

The compactness assumption of the preceding proposition is satisfied, in particular, if either X is compact, or if X is closed and f has compact level sets. More generally, it is satisfied if X is closed, and X, f, and g_j, $j = 1, \ldots, r$, have no common nonzero direction of recession. The proposition, however, does not guarantee the existence of a dual optimal solution, as illustrated by the following example.

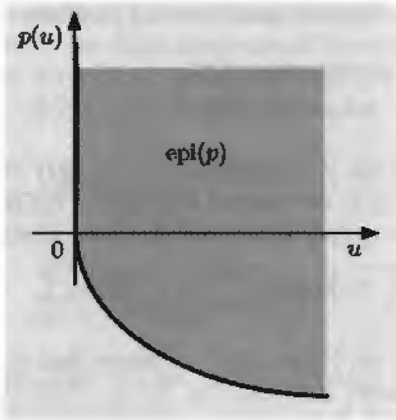


Figure 5.3.1. The perturbation function p for Example 5.3.3:

$$p(u) = \inf_{x^2 \leq u} x = \begin{cases} -\sqrt{u} & \text{if } u \geq 0, \\ \infty & \text{if } u < 0. \end{cases}$$

Here p is lower semicontinuous at 0 and there is no duality gap. However, there is no dual optimal solution.

Example 5.3.3: (Nonexistence of Dual Optimal Solutions)

Consider the one-dimensional problem considered following Example 5.3.2, where

$$f(x) = x, \qquad g(x) = x^2, \qquad X = \Re.$$

The perturbation function is shown in Fig. 5.3.1. It is convex and closed. Also there is a unique primal optimal solution $x^* = 0$ and there is no duality gap, consistent with the preceding proposition (the compactness assumption is satisfied). The dual function is

$$q(\mu) = \inf_{x \in \Re} \{x + \mu x^2\} = \begin{cases} -1/(4\mu) & \text{if } \mu > 0, \\ -\infty & \text{if } \mu \leq 0. \end{cases}$$

Thus there is no dual optimal solution, something that is also apparent from Fig. 5.3.1.

A result that is related to Prop. 5.3.7 is that if x^* is the *unique* minimum over X of the Lagrangian $L(\cdot, \mu^*)$ for some dual optimal solution μ^*, and if X, $F(\cdot, 0)$, and $L(\cdot, \mu^*)$ are closed, then x^* is the unique primal optimal solution. The reason is that $L(x, \mu^*) \leq F(x, 0)$ for all $x \in X$, which implies that $F(\cdot, 0)$ has compact level sets, so there exists a primal optimal solution. Since by Prop. 5.3.2, all primal optimal solutions minimize $L(\cdot, \mu^*)$ over X, it follows that x^* is the unique primal optimal solution.

5.3.5 Fenchel Duality

We will now analyze another important optimization framework, which can be embedded within the convex programming framework discussed so far. Consider the problem

$$\begin{aligned} &\text{minimize} \quad f_1(x) + f_2(Ax) \\ &\text{subject to} \quad x \in \Re^n, \end{aligned} \tag{5.17}$$

where A is an $m \times n$ matrix, $f_1 : \Re^n \mapsto (-\infty, \infty]$ and $f_2 : \Re^m \mapsto (-\infty, \infty]$ are closed convex functions, and we assume that there exists a feasible solution. We convert it to the following equivalent problem in the variables $x_1 \in \Re^n$ and $x_2 \in \Re^m$:

$$\begin{aligned} &\text{minimize} \quad f_1(x_1) + f_2(x_2) \\ &\text{subject to} \quad x_1 \in \text{dom}(f_1), \quad x_2 \in \text{dom}(f_2), \qquad x_2 = Ax_1. \end{aligned} \tag{5.18}$$

We can view this as a convex programming problem with the linear equality constraint $x_2 = Ax_1$ [cf. problem (5.11)]. The dual function is

$$\begin{aligned} q(\lambda) &= \inf_{x_1 \in \text{dom}(f_1),\ x_2 \in \text{dom}(f_2)} \big\{ f_1(x_1) + f_2(x_2) + \lambda'(x_2 - Ax_1) \big\} \\ &= \inf_{x_1 \in \Re^n} \big\{ f_1(x_1) - \lambda' A x_1 \big\} + \inf_{x_2 \in \Re^n} \big\{ f_2(x_2) + \lambda' x_2 \big\} . \end{aligned}$$

The dual problem, after a sign change to convert it to a minimization problem, takes the form

$$\begin{aligned} &\text{minimize} \quad f_1^\star(A'\lambda) + f_2^\star(-\lambda) \\ &\text{subject to} \quad \lambda \in \Re^m, \end{aligned} \tag{5.19}$$

where $f_1^\star$ and $f_2^\star$ are the conjugate functions of f_1 and f_2, respectively:

$$f_1^\star(\lambda) = \sup_{x \in \Re^n} \big\{ \lambda' x - f_1(x) \big\}, \quad f_2^\star(\lambda) = \sup_{x \in \Re^n} \big\{ \lambda' x - f_2(x) \big\}, \qquad \lambda \in \Re^n.$$

Note that the primal and dual problems have a similar/symmetric form. Figure 5.3.2 illustrates the duality between problems (5.17) and (5.19).

We now apply Prop. 5.3.3 to the primal problem (5.18) and obtain the following proposition.

Proposition 5.3.8: (Fenchel Duality)

(a) If f^* is finite and $\big(A \cdot \text{ri}(\text{dom}(f_1))\big) \cap \text{ri}\big(\text{dom}(f_2)\big) \neq \emptyset$, then $f^* = q^*$ and there exists at least one dual optimal solution.

(b) There holds $f^* = q^*$, and (x^*, λ^*) is a primal and dual optimal solution pair if and only if

$$x^* \in \arg\min_{x \in \Re^n} \big\{ f_1(x) - x' A' \lambda^* \big\} \ \text{ and } \ Ax^* \in \arg\min_{z \in \Re^n} \big\{ f_2(z) + z' \lambda^* \big\}. \tag{5.20}$$

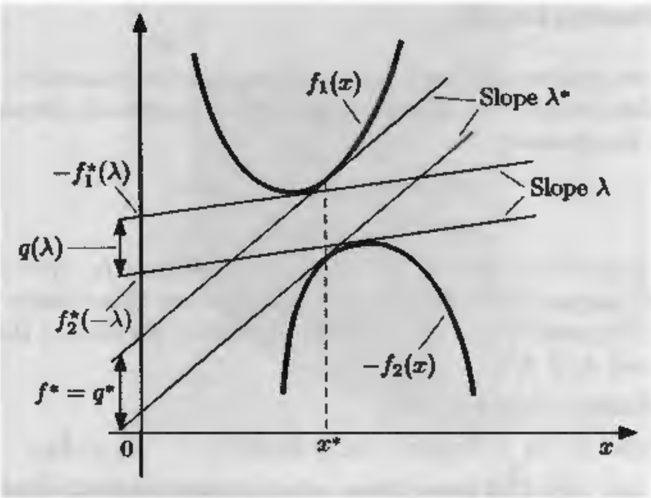


Figure 5.3.2. Illustration of Fenchel duality for the case where A is the identity matrix. The dual function value $q(\lambda)$ is constructed as in the figure. As λ changes, the dual value $q(\lambda) = -\big(f_1^*(\lambda) + f_2^*(-\lambda)\big)$ reaches its maximum at a vector λ^* such that corresponding hyperplanes support the epigraphs of f_1 and f_2 at a common point x^*, which is a primal optimal solution.

Proof: (a) Using the Cartesian product formula for relative interiors (cf. Prop. 1.3.10), it follows that the relative interior assumption of Prop. 5.3.3(a), applied to problem (5.18), is satisfied if for some $\overline{x}_1 \in \mathrm{ri}\big(\mathrm{dom}(f_1)\big)$ and $\overline{x}_2 \in \mathrm{ri}\big(\mathrm{dom}(f_2)\big)$ we have $\overline{x}_2 = A\overline{x}_1$. This is equivalent to our relative interior assumption, so Prop. 5.3.3(a) can be used, yielding the desired result.

(b) Similarly, we apply Prop. 5.3.3(b) to problem (5.18). **Q.E.D.**

Note that in the case where $\mathrm{dom}(f_1) = \Re^n$, $\mathrm{dom}(f_2) = \Re^m$, and f_1 and f_2 are differentiable, the optimality condition (5.20) is equivalent to

$$A'\lambda^* = \nabla f_1(x^*), \qquad \lambda^* = -\nabla f_2(Ax^*);$$

see Fig. 5.3.2. This condition will be generalized to the nondifferentiable case, by using subgradients (see Section 5.4.1, following the Conjugate Subgradient Theorem).

By reversing the roles of the (symmetric) primal and dual problems, and by applying the preceding conditions to the dual problem (5.19), we can obtain alternative criteria for strong duality. In particular, if q^* is finite and $\mathrm{ri}\big(\mathrm{dom}(f_1^*)\big) \cap \big(A' \cdot \mathrm{ri}\big(-\mathrm{dom}(f_2^*)\big)\big) \neq \emptyset$, then $f^* = q^*$ and there exists at least one primal optimal solution.

Finally, we note that a more refined Fenchel duality theorem can be obtained if there is polyhedral structure in f_1 and f_2. For example, if f_1 is polyhedral, the relative interior condition of Prop. 5.3.8(a) can be replaced by the weaker condition $A \cdot \mathrm{dom}(f_1) \cap \mathrm{ri}\big(\mathrm{dom}(f_2)\big) \neq \emptyset$. This follows by applying Prop. 5.3.6 instead of Prop. 5.3.3(a) in the preceding proof. More generally, the same can be shown if f_1 is of the form

$$f_1(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases}$$

where X is a polyhedral set and $\tilde{f} : \Re^n \mapsto \Re$ is a function that is convex over a convex set C with $X \subset \mathrm{ri}(C)$. Similarly, if f_1 and f_2 are both polyhedral, the relative interior condition is unnecessary. In this case the problem (5.17) is equivalent to a linear program for which strong duality holds and a dual optimal solution exists under just the condition that f^* is finite (cf. Prop. 5.2.1).

5.3.6 Conic Duality

Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \ \ x \in C, \end{aligned} \tag{5.21}$$

where $f : \Re^n \mapsto (-\infty, \infty]$ is a closed proper convex function and C is a closed convex cone in $\Re^n$. This is known as a *conic program*, and some of its special cases (semidefinite programming, second order cone programming) have many practical applications, for which we refer to the literature, e.g., [BeN01], [BoV04].

We apply Fenchel duality with A equal to the identity and the definitions

$$f_1(x) = f(x), \qquad f_2(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

The corresponding conjugates are

$$f_1^\star(\lambda) = \sup_{x \in \Re^n} \big\{ \lambda' x - f(x) \big\}, \qquad f_2^\star(\lambda) = \sup_{x \in C} \lambda' x = \begin{cases} 0 & \text{if } \lambda \in C^*, \\ \infty & \text{if } \lambda \notin C^*, \end{cases}$$

where

$$C^* = \{ \lambda \mid \lambda' x \leq 0, \ \forall \ x \in C \}$$

is the polar cone of C (note that $f_2^\star$ is the support function of C; cf. Example 1.6.1). The dual problem [cf. Eq. (5.19)] is

$$\begin{aligned} &\text{minimize} \quad f^\star(\lambda) \\ &\text{subject to} \ \ \lambda \in \hat{C}, \end{aligned} \tag{5.22}$$

where $f^\star$ is the conjugate of f and $\hat{C}$ is the negative polar cone (also called the *dual cone* of C):

$$\hat{C} = -C^* = \{ \lambda \mid \lambda' x \geq 0, \ \forall \ x \in C \}.$$

Note the symmetry between primal and dual problems. The strong duality relation $f^* = q^*$ can be written as

$$\inf_{x \in C} f(x) = - \inf_{\lambda \in \hat{C}} f^\star(\lambda).$$

The following proposition translates the conditions of Prop. 5.3.8 for asserting that there is no duality gap and the dual problem has an optimal solution.

Proposition 5.3.9: (Conic Duality Theorem) Assume that the optimal value of the primal conic problem (5.21) is finite, and that $\mathrm{ri}\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}(C) \neq \emptyset$. Then, there is no duality gap and that the dual problem (5.22) has an optimal solution.

Using the symmetry of the primal and dual problems, we also obtain that there is no duality gap and the primal problem (5.21) has an optimal solution if the optimal value of the dual conic problem (5.22) is finite and $\mathrm{ri}\big(\mathrm{dom}(f^\star)\big) \cap \mathrm{ri}(\hat{C}) \neq \emptyset$. It is also possible to exploit polyhedral structure in f and/or C, using Prop. 5.3.6 (cf. the discussion at the end of the preceding section). Furthermore, we may derive primal and dual optimality conditions using Prop. 5.3.8(b).

5.4 SUBGRADIENTS AND OPTIMALITY CONDITIONS

In this section we introduce the notion of a subgradient of a convex function at a point. Subgradients serve as a substitute for gradients when the function is nondifferentiable: like gradients in differentiable cost minimization, they enter in optimality conditions and find wide use in algorithms.

We will develop a connection of subgradients with the MC/MC framework. In particular, we will show that the subdifferential can be identified with the set of max crossing hyperplanes in a suitable MC/MC framework. Through this connection we will obtain some of the basic results of subdifferential theory by using the MC/MC theory of Chapter 4, the Nonlinear Farkas' Lemma (cf. Prop. 5.1.1), and the constrained optimization duality theory (cf. Prop. 5.3.6).

Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function. We say that a vector $g \in \Re^n$ is a *subgradient* of f at a point $x \in \mathrm{dom}(f)$ if

$$f(z) \geq f(x) + g'(z - x), \qquad \forall\, z \in \Re^n. \tag{5.23}$$

The set of all subgradients of f at x is called the *subdifferential of f at x* and is denoted by $\partial f(x)$. By convention, $\partial f(x)$ is considered empty for all $x \notin \mathrm{dom}(f)$. Generally, $\partial f(x)$ is closed and convex, since based

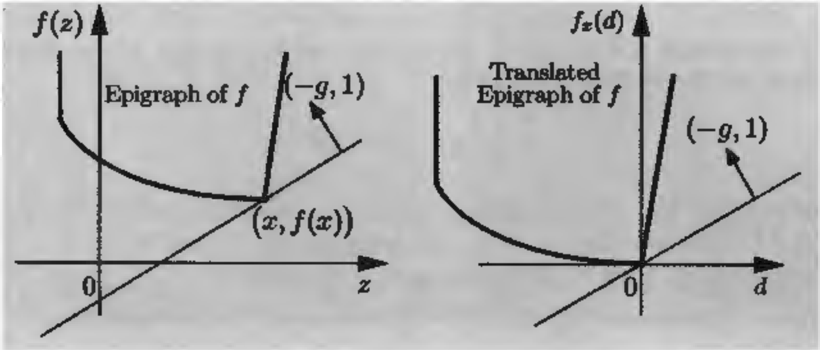


Figure 5.4.1. Illustration of the subdifferential $\partial f(x)$ of a convex function f and its connection with the MC/MC framework. The subgradient inequality (5.23) can be written as

$$f(z) - z'g \geq f(x) - x'g, \qquad \forall\, z \in \Re^n. \tag{5.24}$$

Thus, g is a subgradient of f at x if and only if the hyperplane in $\Re^{n+1}$ that has normal $(-g, 1)$ and passes through $\big(x, f(x)\big)$ supports the epigraph of f, as shown in the left-hand side figure.

The right-hand side figure shows that $\partial f(x)$ is the set of max crossing solutions in the MC/MC framework where M is the epigraph of f_x, the x-translation of f.

on the subgradient inequality (5.23), it is the intersection of a collection of closed halfspaces. Note that we restrict attention to proper functions (subgradients are not useful and make no sense for improper functions).

As Fig. 5.4.1 illustrates, g is a subgradient of f at x if and only if the hyperplane in $\Re^{n+1}$ that has normal $(-g, 1)$ and passes through $\big(x, f(x)\big)$ supports the epigraph of f. From this geometric view, it is evident that there is a strong connection with the MC/MC framework. In particular, for any $x \in \text{dom}(f)$, consider the *x-translation of f*, which is the function f_x, whose epigraph is the epigraph of f translated so that $\big(x, f(x)\big)$ is moved to the origin of $\Re^{n+1}$:

$$f_x(d) = f(x + d) - f(x), \qquad d \in \Re^n.$$

Then the subdifferential $\partial f(x)$ is the set of all max crossing solutions for the MC/MC framework corresponding to the set

$$M = \text{epi}(f_x) = \text{epi}(f) - \big\{\big(x, f(x)\big)\big\} \tag{5.25}$$

(cf. the right-hand side of Fig. 5.4.1). Based on this fact, we can use the MC/MC theory to obtain results regarding the existence of subgradients, as in the following proposition.

Proposition 5.4.1: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function. For every $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$,

$$\partial f(x) = S^\perp + G,$$

where S is the subspace that is parallel to the affine hull of $\mathrm{dom}(f)$, and G is a nonempty convex and compact set. In particular, if $x \in \mathrm{int}\big(\mathrm{dom}(f)\big)$, then $\partial f(x)$ is nonempty and compact.

Proof: The result follows by applying Props. 4.4.1 and 4.4.2 to the set M given by Eq. (5.25). **Q.E.D.**

It follows from the preceding proposition that *if f is real-valued, then $\partial f(x)$ is nonempty and compact for all $x \in \Re^n$*. If f is extended real-valued, $\partial f(x)$ can be unbounded, and it can be empty not only for $x \notin \mathrm{dom}(f)$, but also for some x in the boundary of $\mathrm{dom}(f)$. As an example, consider the function

$$f(x) = \begin{cases} -\sqrt{x} & \text{if } 0 \le x \le 1, \\ \infty & \text{otherwise.} \end{cases}$$

Its subdifferential is

$$\partial f(x) = \begin{cases} -\frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ \big[-\frac{1}{2}, \infty\big) & \text{if } x = 1, \\ \emptyset & \text{if } x \le 0 \text{ or } 1 < x, \end{cases}$$

so it is empty or unbounded at boundary points within $\mathrm{dom}(f)$ (the points 0 and 1, respectively).

An important property is that *if f is differentiable at some $x \in \mathrm{int}\big(\mathrm{dom}(f)\big)$, its gradient $\nabla f(x)$ is the unique subgradient at x.* Indeed, by Prop. 1.1.7(a), $\nabla f(x)$ satisfies the subgradient inequality (5.23), so it is a subgradient at x. To show uniqueness, note that if g is a subgradient at x, we have

$$f(x) + \alpha g'd \le f(x + \alpha d) = f(x) + \alpha \nabla f(x)'d + o(|\alpha|), \quad \forall\, \alpha \in \Re,\ d \in \Re^n.$$

By letting $d = \nabla f(x) - g$, we obtain

$$0 \le \alpha\big(\nabla f(x) - g\big)'d + o(|\alpha|) = \alpha \big\|\nabla f(x) - g\big\|^2 + o(|\alpha|).$$

In particular, we have

$$\big\|\nabla f(x) - g\big\|^2 \le -\frac{o(|\alpha|)}{\alpha}, \qquad \forall\, \alpha < 0,$$

Taking $\alpha \uparrow 0$, we obtain $\nabla f(x) - g = 0$.

Let us also consider another important special case, where f is the indicator function of a convex set.

Example 5.4.1: (Subdifferential of an Indicator Function)

Let us derive the subdifferential of the indicator function of a nonempty convex set C:

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

For all $x \notin C$, we have $\partial\delta_C(x) = \emptyset$, by convention. For $x \in C$, we have $g \in \partial\delta_C(x)$ if and only if

$$\delta_C(z) \geq \delta_C(x) + g'(z - x), \qquad \forall\, z \in C,$$

or equivalently $g'(z-x) \leq 0$ for all $z \in C$. For $x \in C$, the set of all g satisfying this relation is called the *normal cone of* C *at* x and is denoted by $N_C(x)$:

$$N_C(x) = \big\{g \mid g'(z - x) \leq 0,\ \forall\, z \in C\big\}.$$

Thus the normal cone $N_C(x)$ is the polar cone of $C-\{x\}$, the set C translated so that x is moved to the origin (see Fig. 5.4.2).

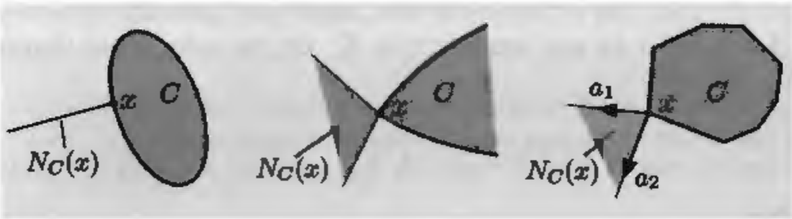


Figure 5.4.2. Illustration of the normal cone of a convex set C at a point $x \in C$, i.e., the subdifferential of the indicator function of C at x. For $x \in \text{int}(C)$ we have $N_C(x) = \{0\}$, while for $x \notin \text{int}(C)$, the normal cone contains at least one half line, as in the examples shown. In the case of the polyhedral set on the right, $N_C(x)$ is the cone generated by the normal vectors to the hyperplanes that correspond to the active inequalities at x.

Finally, let us show an important property of real-valued convex functions.

Proposition 5.4.2: (Subdifferential Boundedness and Lipschitz Continuity) Let $f : \Re^n \mapsto \Re$ be a real-valued convex function, and let X be a nonempty compact subset of $\Re^n$.

(a) The set $\cup_{x\in X}\partial f(x)$ is nonempty and bounded.

(b) The function f is Lipschitz continuous over X, i.e., there exists a scalar L such that

$$\big|f(x) - f(z)\big| \leq L\,\|x - z\|, \qquad \forall\, x, z \in X.$$

Proof: (a) Nonemptiness follows from Prop. 5.4.1. To prove boundedness, assume the contrary, so that there exists a sequence $\{x_k\} \subset X$, and an unbounded sequence $\{g_k\}$ with

$$g_k \in \partial f(x_k), \qquad 0 < \|g_k\| < \|g_{k+1}\|, \quad k = 0, 1, \ldots .$$

We denote $d_k = g_k / \|g_k\|$. Since $g_k \in \partial f(x_k)$, we have

$$f(x_k + d_k) - f(x_k) \geq g_k' d_k = \|g_k\|.$$

Since both $\{x_k\}$ and $\{d_k\}$ are bounded, they contain convergent subsequences. We assume without loss of generality that $\{x_k\}$ and $\{d_k\}$ converge to some vectors. Therefore, by the continuity of f (cf. Prop. 1.3.11), the left-hand side of the preceding relation is bounded. Hence the right-hand side is also bounded, thereby contradicting the unboundedness of $\{g_k\}$.

(b) Let x and z be any two points in X. By the subgradient inequality (5.23), we have

$$f(x) + g'(z - x) \leq f(z), \qquad \forall\ g \in \partial f(x),$$

so that

$$f(x) - f(z) \leq \|g\| \cdot \|x - z\|, \qquad \forall\ g \in \partial f(x).$$

By part (a), $\cup_{y \in X} \partial f(y)$ is bounded, so that for some constant $L > 0$, we have

$$\|g\| \leq L, \qquad \forall\ g \in \partial f(y), \quad \forall\ y \in X, \tag{5.26}$$

and therefore,

$$f(x) - f(z) \leq L\ \|x - z\|.$$

By exchanging the roles of x and z, we similarly obtain

$$f(z) - f(x) \leq L\ \|x - z\|,$$

and by combining the preceding two relations, we see that

$$\big|f(x) - f(z)\big| \leq L\ \|x - z\|,$$

showing that f is Lipschitz continuous over X. **Q.E.D.**

Note that the proof of part (b) shows how to determine the Lipschitz constant L: it is the maximum subgradient norm, over all subgradients in $\cup_{x \in X} \partial f(x)$ [cf. Eq. (5.26)].

5.4.1 Subgradients of Conjugate Functions

We will now derive an important relation between the subdifferentials of a proper convex function $f : \Re^n \mapsto (-\infty, \infty]$ and its conjugate $f^\star$. Using the definition of conjugacy, we have

$$x'y \leq f(x) + f^\star(y), \qquad \forall\, x \in \Re^n,\ y \in \Re^n.$$

This is known as the *Fenchel inequality*. A pair (x, y) satisfies this inequality as an equation if and only if x attains the supremum in the definition

$$f^\star(y) = \sup_{z \in \Re^n} \big\{ y'z - f(z) \big\}.$$

Pairs of this type are connected with the subdifferentials of f and $f^\star$, as shown in the following proposition, and illustrated in Fig. 5.4.3.

Proposition 5.4.3: (Conjugate Subgradient Theorem) Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function and let $f^\star$ be its conjugate. The following two relations are equivalent for a pair of vectors (x, y):

(i) $x'y = f(x) + f^\star(y)$.

(ii) $y \in \partial f(x)$.

If in addition f is closed, the relations (i) and (ii) are equivalent to

(iii) $x \in \partial f^\star(y)$.

Proof: A pair (x, y) satisfies (i) if and only if x attains the supremum in the definition $f^\star(y) = \sup_{z \in \Re^n} \big\{ y'z - f(z) \big\}$, and by Eq. (5.24), this is true if and only if $y \in \partial f(x)$. This shows that (i) and (ii) are equivalent. If in addition f is closed, then by the Conjugacy Theorem [Prop. 1.6.1(c)], f is equal to the conjugate of $f^\star$, so by using the equivalence just shown with the roles of f and $f^\star$ reversed, we see that (i) is equivalent to (iii). **Q.E.D.**

Note that the closure assumption in condition (iii) of the Conjugate Subgradient Theorem is necessary, because by the Conjugacy Theorem [Prop. 1.6.1(d)], the conjugate of $f^\star$ is $\mathrm{cl}\, f$, so the relation $x \in \partial f^\star(y)$ implies that

$$x'y = (\mathrm{cl}\, f)(x) + f^\star(y)$$

[by the equivalence of conditions (i) and (ii)]. On the other hand, for $x \in \partial f^\star(y)$, we may have $(\mathrm{cl}\, f)(x) < f(x)$ and hence $x'y < f(x) + f^\star(y)$

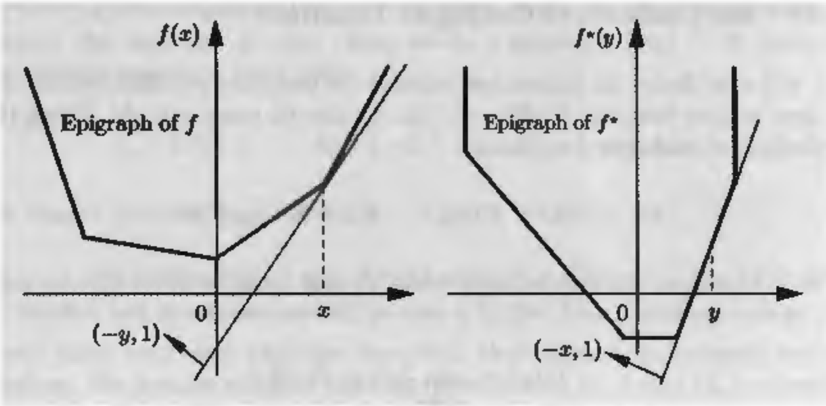


Figure 5.4.3. Illustration of a pair (x, y) satisfying Fenchel's inequality as an equation, i.e., $x'y = f(x) + f^*(y)$. If f is closed, this equality is equivalent to

$$y \in \partial f(x) \qquad \text{and} \qquad x \in \partial f^*(y).$$

Since x minimizes a proper convex function f if and only if $0 \in \partial f(x)$ [a consequence of the subgradient inequality (5.23)], a corollary is that the set of minima of a closed proper convex f is $\partial f^*(0)$, while the set of minima of f^* is $\partial f(0)$.

[for example, take f to be the indicator function of the interval $(-1, 1)$, $f^*(y) = |y|$, $x = 1$, and $y = 0$].

For an application of the Conjugate Subgradient Theorem, note that the necessary and sufficient optimality condition (5.20) in the Fenchel Duality Theorem can be equivalently written as

$$A'\lambda^* \in \partial f_1(x^*), \qquad \lambda^* \in -\partial f_2(Ax^*)$$

(cf. Fig. 5.3.2).

The following proposition gives some useful corollaries of the Conjugate Subgradient Theorem:

Proposition 5.4.4: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function and let f^* be its conjugate.

(a) f^* is differentiable at a vector $y \in \operatorname{int}(\operatorname{dom}(f^*))$ if and only if the supremum of $x'y - f(x)$ over $x \in \Re^n$ is uniquely attained.

(b) The set of minima of f is given by

$$\arg\min_{x \in \Re^n} f(x) = \partial f^*(0),$$

Proof: Both parts follow from the fact

$$\arg\max_{x\in\Re^n}\big\{x'y - f(x)\big\} = \partial f^\star(y),$$

which is a consequence of Prop. 5.4.3. **Q.E.D.**

Proposition 5.4.4(a) can be used to characterize differentiability properties of $f^\star$ in terms of strict convexity properties of f, which guarantee the unique attainment of the supremum of $x'y - f(x)$ over $x \in \Re^n$. This is the basis for the Legendre transformation, a precursor to the conjugacy transformation (see [Roc70], §26). Proposition 5.4.4(b) shows that the set of minima of f is nonempty and compact if and only if $0 \in \operatorname{int}\big(\operatorname{dom}(f^\star)\big)$ (cf. Prop. 5.4.1).

The following example applies the Conjugate Subgradient Theorem to establish an important interpretation of dual optimal solutions within the MC/MC framework.

Example 5.4.2: (Sensitivity Interpretation of Dual Optimal Solutions)

Consider the MC/MC framework for the case where the set M is the epigraph of a function $p : \Re^n \mapsto [-\infty, \infty]$. Then the dual function is

$$q(\mu) = \inf_{u\in\Re^m}\big\{p(u) + \mu'u\big\} = -p^\star(-\mu),$$

where $p^\star$ is the conjugate of p (cf. Section 4.2.1). Assume that p is proper convex, and that strong duality holds, i.e.,

$$p(0) = w^* = q^* = \sup_{\mu\in\Re^m}\big\{-p^\star(-\mu)\big\}.$$

Let Q^* be the set of dual optimal solutions, i.e.,

$$Q^* = \big\{\mu^* \mid p(0) + p^\star(-\mu^*) = 0\big\}.$$

Then it follows from Prop. 5.4.3 that $\mu^* \in Q^*$ if and only if $-\mu^* \in \partial p(0)$, i.e.,

$$Q^* = -\partial p(0).$$

This leads to various sensitivity interpretations of dual optimal solutions (assuming strong duality holds). The most interesting case is when p is convex and differentiable at 0, in which case $-\nabla p(0)$ is equal to the unique dual optimal solution μ^*. As an example, for the constrained optimization problem

$$\begin{aligned} &\text{minimize} \quad f(x)\\ &\text{subject to} \quad x \in X, \quad g(x) \le 0, \end{aligned}$$

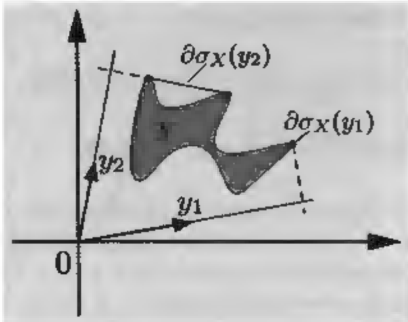


Figure 5.4.4. For a fixed y, the subdifferential $\partial\sigma_X(y)$ of the support function of X is the set of maxima of $y'x$ over $x \in \mathrm{cl}\big(\mathrm{conv}(X)\big)$ (cf. Example 5.4.3).

of Section 5.3, where

$$p(u) = \inf_{x\in X,\, g(x)\le u} f(x),$$

we have

$$\mu_j^* = -\frac{\partial p(0)}{\partial u_j}, \qquad j = 1,\ldots,r,$$

so μ_j^* *is the rate of improvement of the optimal primal cost as the* j*th constraint* $g_j(x) \le 0$ *is violated.*

Example 5.4.3: (Subdifferential of a Support Function)

Let us derive the subdifferential of the support function σ_X of a nonempty set X at a vector $\overline{y}$. Note that σ_X is closed proper convex, since it is the conjugate of the indicator function of X; cf. Example 1.6.1. To calculate $\partial\sigma_X(\overline{y})$, we introduce the closed proper convex function

$$r(y) = \sigma_X(y + \overline{y}), \qquad y \in \Re^n,$$

and we note that $\partial\sigma_X(\overline{y}) = \partial r(0)$. The conjugate of r is

$$r^*(x) = \sup_{y\in\Re^n} \big\{ y'x - \sigma_X(y+\overline{y}) \big\},$$

or

$$r^*(x) = \sup_{y\in\Re^n} \big\{ (y+\overline{y})'x - \sigma_X(y+\overline{y}) \big\} - \overline{y}'x,$$

and finally

$$r^*(x) = \delta(x) - \overline{y}'x,$$

where δ is the indicator function of $\mathrm{cl}\big(\mathrm{conv}(X)\big)$ (cf. Example 1.6.1). Letting $r = f^*$ in Prop. 5.4.4(b), we see that $\partial r(0)$ is the set of minima of $\delta(x) - \overline{y}'x$, or equivalently $\partial\sigma_X(\overline{y})$ is the set of maxima of $\overline{y}'x$ over $x \in \mathrm{cl}\big(\mathrm{conv}(X)\big)$ (see Fig. 5.4.4).

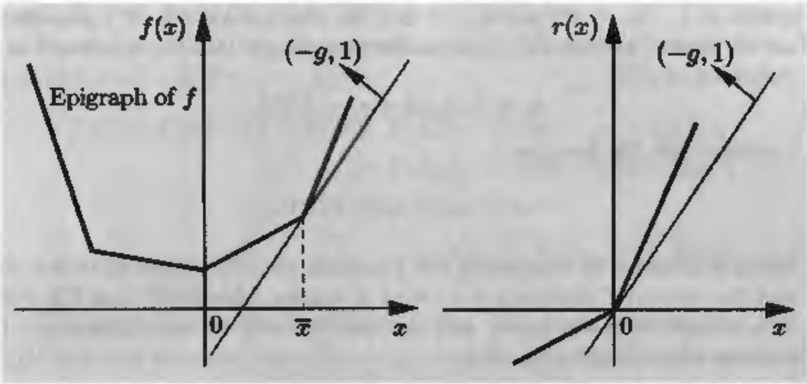


Figure 5.4.5. Construction used to derive the subdifferential at $\overline{x}$ of the function

$$f(x) = \max\{a_1'x + b_1, \ldots, a_r'x + b_r\},$$

shown on the left (cf. Example 5.4.4). Consider the function on the right,

$$r(x) \doteq \max\left\{a_j'x \mid j \in A_{\overline{x}}\right\},$$

with $A_{\overline{x}} = \left\{j \mid a_j'\overline{x} + b_j = f(\overline{x})\right\}$. We claim that $\partial f(\overline{x}) = \partial r(0)$, from which we obtain $\partial f(\overline{x}) = \text{conv}\big(\{a_j \mid j \in A_{\overline{x}}\}\big)$ [since r is the support function of the finite set $\{a_j \mid j \in A_{\overline{x}}\}$; cf. Example 5.4.3].

To verify that $\partial f(\overline{x}) = \partial r(0)$, note that for all $g \in \partial r(0)$, we have

$$f(x) - f(\overline{x}) \geq r(x - \overline{x}) \geq r(0) + g'(x - \overline{x}) = g'(x - \overline{x}), \qquad \forall\, x \in \Re^n.$$

It follows that $g \in \partial f(\overline{x})$ and $\partial r(0) \subset \partial f(\overline{x})$.

Conversely, let $g \in \partial f(\overline{x})$. Then for x sufficiently close to $\overline{x}$, we have $f(x) - f(\overline{x}) = r(x - \overline{x})$, so the inequality $f(x) - f(\overline{x}) \geq g'(x - \overline{x})$ implies that

$$r(x - \overline{x}) \geq r(0) + g'(x - \overline{x}).$$

From the definition of r, it follows that

$$r(x - \overline{x}) \geq r(0) + g'(x - \overline{x}), \qquad \forall\, x \in \Re^n,$$

so $g \in \partial r(0)$ and $\partial f(\overline{x}) \subset \partial r(0)$.

Example 5.4.4: (Subdifferential of a Real-Valued Polyhedral Function)

Let us derive the subdifferential of a real-valued polyhedral function of the form

$$f(x) = \max\{a_1'x + b_1, \ldots, a_r'x + b_r\},$$

where $a_1, \ldots, a_r \in \Re^n$ and $b_1, \ldots, b_r \in \Re$. For a fixed $\overline{x} \in \Re^n$, consider the set of "active" indices at x, that is, the ones that attain the maximum in the definition of $f(\overline{x})$:

$$A_{\overline{x}} = \big\{ j \mid a_j'\overline{x} + b_j = f(\overline{x}) \big\}.$$

Consider also the function

$$r(x) = \max\big\{ a_j'x \mid j \in A_{\overline{x}} \big\},$$

which is obtained by translating epi(f) so that $\big(\overline{x}, f(\overline{x})\big)$ is moved to the origin and the "inactive" functions $a_j'x + b_j$, $j \notin A_{\overline{x}}$, are "discarded" (see Fig. 5.4.5). It is evident from the figure, and can also be easily shown algebraically (see the caption of Fig. 5.4.5), that

$$\partial f(\overline{x}) = \partial r(0).$$

We now note that r is the support function of the finite set $\{a_j \mid j \in A_{\overline{x}}\}$, so from the result of Example 5.4.3, it follows that $\partial r(0)$ is the convex hull of this set. Thus,

$$\partial f(\overline{x}) = \mathrm{conv}\big(\{a_j \mid j \in A_{\overline{x}}\}\big).$$

5.4.2 Subdifferential Calculus

We will now generalize some of the basic theorems of ordinary differentiation by using the convex programming results of Section 5.3 (and by extension the Nonlinear Farkas' Lemma and the MC/MC framework). The following proposition generalizes the differentiation rule

$$\nabla F(x) = A'\nabla f(Ax)$$

for the function $F(x) = f(Ax)$, where f is differentiable.

Proposition 5.4.5: (Chain Rule) Let $f : \Re^m \mapsto (-\infty, \infty]$ be a convex function, let A be an $m \times n$ matrix, and assume that the function F given by

$$F(x) = f(Ax)$$

is proper. Then

$$\partial F(x) \supset A'\partial f(Ax), \qquad \forall\, x \in \Re^n.$$

Furthermore, if either f is polyhedral or else the range of A contains a point in the relative interior of dom(f), we have

$$\partial F(x) = A'\partial f(Ax), \qquad \forall\, x \in \Re^n.$$

Proof: If $x \notin \text{dom}(F)$, then $\partial F(x) = A'\partial f(Ax) = \emptyset$. For any $x \in \text{dom}(F)$, if $d \in A'\partial f(Ax)$, there exists a $g \in \partial f(Ax)$ such that $d = A'g$. We have for all $z \in \Re^n$,

$$\begin{aligned} F(z) - F(x) - (z - x)'d &= f(Az) - f(Ax) - (z - x)'A'g \\ &= f(Az) - f(Ax) - (Az - Ax)'g \\ &\geq 0, \end{aligned}$$

where the inequality follows since $g \in \partial f(Ax)$. Hence $d \in \partial F(x)$, and we have $\partial F(x) \supset A'\partial f(Ax)$.

To prove the reverse inclusion under the given assumption, we let $d \in \partial F(x)$ and we show that $d \in A'\partial f(Ax)$ by viewing x as the solution of an optimization problem defined by d. Indeed, we have

$$F(z) \geq F(x) + (z - x)'d \geq 0, \qquad \forall\ z \in \Re^n,$$

or

$$f(Az) - z'd \geq f(Ax) - x'd, \qquad \forall\ z \in \Re^n.$$

Thus (Ax, x) solves the following optimization problem in the variables (y, z):

$$\begin{aligned} &\text{minimize} \quad f(y) - z'd \\ &\text{subject to} \quad y \in \text{dom}(f), \quad Az = y. \end{aligned} \tag{5.27}$$

If f is polyhedral, dom(f) is polyhedral and f can be replaced in the above problem by a real-valued polyhedral function, so that we can use Prop. 5.3.6. If instead the range of A contains a point in $\text{ri}\big(\text{dom}(f)\big)$, we use Prop. 5.3.3. In either case, we conclude that there is no duality gap, and that there exists a dual optimal solution λ, such that

$$(Ax, x) \in \arg \min_{y \in \Re^m,\, z \in \Re^n} \big\{ f(y) - z'd + \lambda'(Az - y) \big\}$$

[cf. Eq. (5.12)]. Since the minimization over z is unconstrained, we must have $d = A'\lambda$, thus obtaining

$$Ax \in \arg \min_{y \in \Re^m} \big\{ f(y) - \lambda'y \big\},$$

or

$$f(y) \geq f(Ax) + \lambda'(y - Ax), \qquad \forall\ y \in \Re^m.$$

Hence $\lambda \in \partial f(Ax)$, so that $d = A'\lambda \in A'\partial f(Ax)$. It follows that $\partial F(x) \subset A'\partial f(Ax)$. **Q.E.D.**

From the preceding proof, it can be seen that the assumption that f is polyhedral in Prop. 5.4.5 may be replaced by the weaker assumption that for some function $\tilde{f} : \Re^n \mapsto \Re$ and polyhedral set X, f is of form

$$f(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases} \tag{5.28}$$

where $\tilde{f}$ is convex over a convex set C with $X \subset \mathrm{ri}(C)$ [replace f with $\tilde{f}$ and $\mathrm{dom}(f)$ with $C \cap X$ in Eq. (5.27), and use Prop. 5.3.6].

As a special case of Prop. 5.4.5, we obtain the following.

Proposition 5.4.6: (Subdifferential of Sum of Functions) Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be convex functions, and assume that the function $F = f_1 + \cdots + f_m$ is proper. Then

$$\partial F(x) \supset \partial f_1(x) + \cdots + \partial f_m(x), \qquad \forall\, x \in \Re^n.$$

Furthermore, if $\cap_{i=1}^m \mathrm{ri}\big(\mathrm{dom}(f_i)\big) \neq \emptyset$, we have

$$\partial F(x) = \partial f_1(x) + \cdots + \partial f_m(x), \qquad \forall\, x \in \Re^n.$$

More generally, the same is true if for some $\overline{m}$ with $1 \le \overline{m} \le m$, the functions f_i, $i = 1, \ldots, \overline{m}$, are polyhedral and

$$\Big(\cap_{i=1}^{\overline{m}} \mathrm{dom}(f_i)\Big) \cap \Big(\cap_{i=\overline{m}+1}^{m} \mathrm{ri}\big(\mathrm{dom}(f_i)\big)\Big) \neq \emptyset. \qquad (5.29)$$

Proof: We can write F in the form $F(x) = f(Ax)$, where A is the matrix defined by $Ax = (x, \ldots, x)$, and $f : \Re^{mn} \mapsto (-\infty, \infty]$ is the function

$$f(x_1, \ldots, x_m) = f_1(x_1) + \cdots + f_m(x_m).$$

If $\cap_{i=1}^m \mathrm{ri}\big(\mathrm{dom}(f_i)\big) \neq \emptyset$, the range of A contains a point in $\mathrm{ri}\big(\mathrm{dom}(f)\big)$ (cf. the proof of Prop. 1.3.17). The result then follows from Prop. 5.4.5. If the weaker assumption (5.29) holds, a refinement of the proof of Prop. 5.4.5 is needed. In particular, the optimization problem (5.27) takes the form

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^m \big(f_i(y_i) - z_i' d_i\big) \\ &\text{subject to} \quad y_i \in \mathrm{dom}(f_i), \quad z_i = y_i, \quad i = 1, \ldots, m. \end{aligned}$$

By using Prop. 5.3.6 and the assumption (5.29) it follows that strong duality holds and there exists a dual optimal solution for this problem. The proof then proceeds similar to the one of Prop. 5.4.5. **Q.E.D.**

For an example illustrating the need for the assumption (5.29), let $f_1(x) = -\sqrt{x}$ for $x \ge 0$ and $f_1(x) = \infty$ for $x < 0$, and let $f_2(x) = f_1(-x)$. Then $\partial f_1(0) = \partial f_2(0) = \emptyset$, but $\partial(f_1 + f_2)(0) = \Re$. We finally note a slight extension of the preceding proposition: instead of assuming that the functions f_i, $i = 1, \ldots, \overline{m}$, are polyhedral, we may assume that they are of the form (5.28), and the same line of proof applies.

5.4.3 Optimality Conditions

It can be seen from the definition of subgradient that a vector x^* minimizes f over $\Re^n$ if and only if $0 \in \partial f(x^*)$. We will now generalize this condition to constrained problems.

Proposition 5.4.7: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function, let X be a nonempty convex subset of $\Re^n$, and assume that one of the following four conditions holds:

(1) $\mathrm{ri}\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}(X) \neq \emptyset$.

(2) f is polyhedral and $\mathrm{dom}(f) \cap \mathrm{ri}(X) \neq \emptyset$.

(3) X is polyhedral and $\mathrm{ri}\big(\mathrm{dom}(f)\big) \cap X \neq \emptyset$.

(4) f and X are polyhedral, and $\mathrm{dom}(f) \cap X \neq \emptyset$.

Then, a vector x^* minimizes f over X if and only if there exists $g \in \partial f(x^*)$ such that $-g$ belongs to the normal cone $N_X(x^*)$, or equivalently,

$$g'(x - x^*) \geq 0, \qquad \forall\, x \in X. \tag{5.30}$$

Proof: The problem can be written as

$$\begin{aligned} &\text{minimize} \quad f(x) + \delta_X(x) \\ &\text{subject to} \quad x \in \Re^n, \end{aligned}$$

where δ_X is the indicator function of X. We have that x^* is an optimal solution if and only if

$$0 \in \partial(f + \delta_X)(x^*) = \partial f(x^*) + \partial\delta_X(x^*) = \partial f(x^*) + N_X(x^*),$$

where the first equality follows from the assumptions (1)-(4) and Prop. 5.4.6, and the second equality follows from the form of subdifferential of δ_X derived in Example 5.4.1. Thus x^* is an optimal solution if and only if there exists $g \in \partial f(x^*)$ such that $-g \in N_X(x^*)$. **Q.E.D.**

When f is real-valued, the relative interior condition (1) of the preceding proposition is automatically satisfied [we have $\mathrm{dom}(f) = \Re^n$]. If in addition, f is differentiable, the optimality condition (5.30) reduces to the one of Prop. 1.1.8:

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

Also, conditions (2) and (4) can be weakened: f can be assumed to be of the form (5.28), instead of being polyhedral.

5.4.4 Directional Derivatives

Optimization algorithms are often based on iterative cost function improvement. This process is often guided by the directional derivative of the cost function. In the case of a proper convex function $f : \Re^n \mapsto (-\infty, \infty]$, the directional derivative at any $x \in \mathrm{dom}(f)$ in a direction $d \in \Re^n$, is defined by

$$f'(x; d) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha}. \tag{5.31}$$

An important fact here is that the ratio in Eq. (5.31) is monotonically nonincreasing as $\alpha \downarrow 0$, so that the limit above is well-defined (see Fig. 5.4.6). To verify this, note that for any $\overline{\alpha} > 0$, the convexity of f implies that for all $\alpha \in (0, \overline{\alpha})$,

$$f(x + \alpha d) \le \frac{\alpha}{\overline{\alpha}} f(x + \overline{\alpha} d) + \left(1 - \frac{\alpha}{\overline{\alpha}}\right) f(x) = f(x) + \frac{\alpha}{\overline{\alpha}}\big(f(x + \overline{\alpha} d) - f(x)\big),$$

so that

$$\frac{f(x + \alpha d) - f(x)}{\alpha} \le \frac{f(x + \overline{\alpha} d) - f(x)}{\overline{\alpha}}, \qquad \forall\, \alpha \in (0, \overline{\alpha}). \tag{5.32}$$

Thus the limit in Eq. (5.31) is well-defined (as a real number, or ∞, or $-\infty$) and an alternative definition of $f'(x; d)$ is

$$f'(x; d) = \inf_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha}, \qquad d \in \Re^n. \tag{5.33}$$

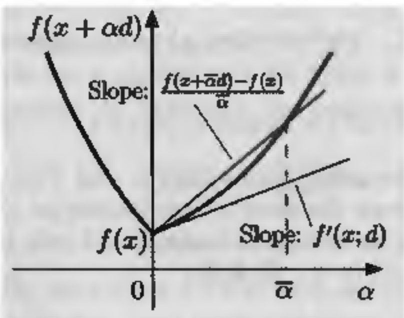


Figure 5.4.6. Illustration of the directional derivative of a convex function f. The ratio

$$\frac{f(x + \alpha d) - f(x)}{\alpha}$$

is monotonically nonincreasing as $\alpha \downarrow 0$ and converges to $f'(x; d)$ [cf. Eqs. (5.32), (5.33)].

It can be shown that $f'(x; \cdot)$ *is convex* for all $x \in \mathrm{dom}(f)$. For this, it is sufficient to prove that the set of all (d, w) such that $f'(x; d) < w$ is convex, and the verification is straightforward using the convexity of f and Eq. (5.33). If $x \in \mathrm{int}\big(\mathrm{dom}(f)\big)$, we have $f'(x; d) < \infty$ and $f'(x; -d) < \infty$ [cf. Eq. (5.33)], so the convexity of $f'(x; \cdot)$ implies that

$$0 = f'(x; 0) \le \frac{1}{2} f'(x; -d) + \frac{1}{2} f'(x; d),$$

or

$$-f'(x;-d) \le f'(x;d), \qquad \forall\, x \in \operatorname{int}\big(\operatorname{dom}(f)\big),\ d \in \Re^n.$$

This inequality, combined with $f'(x;d) < \infty$ and $f'(x;-d) < \infty$, shows that

$$-\infty < f'(x;d) < \infty, \qquad \forall\, x \in \operatorname{int}\big(\operatorname{dom}(f)\big),\ d \in \Re^n,$$

i.e., $f'(x;\cdot)$ is real-valued. More generally, the same argument shows that *$f'(x;d)$ is a real number for all $x \in \operatorname{ri}\big(\operatorname{dom}(f)\big)$ and all d in the subspace that is parallel to* $\operatorname{aff}\big(\operatorname{dom}(f)\big)$.

The directional derivative is related to the support function of the subdifferential $\partial f(x)$, as indicated in the following proposition.

Proposition 5.4.8: (Support Function of the Subdifferential) Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function, and let $(\operatorname{cl} f')(x;\cdot)$ be the closure of the directional derivative $f'(x;\cdot)$.

(a) For all $x \in \operatorname{dom}(f)$ such that $\partial f(x)$ is nonempty, $(\operatorname{cl} f')(x;\cdot)$ is the support function of $\partial f(x)$.

(b) For all $x \in \operatorname{ri}\big(\operatorname{dom}(f)\big)$, $f'(x;\cdot)$ is closed and it is the support function of $\partial f(x)$.

Proof: (a) Let us fix $x \in \operatorname{dom}(f)$. From the subgradient inequality, we have

$$g \in \partial f(x) \quad \text{if and only if} \quad \frac{f(x+\alpha d) - f(x)}{\alpha} \ge g'd, \quad \forall\, d \in \Re^n,\ \alpha > 0.$$

Therefore, using the equivalent definition (5.33) of directional derivative, we obtain

$$g \in \partial f(x) \qquad \text{if and only if} \qquad f'(x;d) \ge g'd,\ \forall\, d \in \Re^n. \tag{5.34}$$

Let δ be the conjugate function of $f'(x;\cdot)$:

$$\delta(y) = \sup_{d \in \Re^n} \big\{ d'y - f'(x;d) \big\}.$$

Since for any $\gamma > 0$, we have [using Eq. (5.33)]

$$f'(x;\gamma d) = \gamma f'(x;d),$$

it follows that

$$\gamma\delta(y) = \sup_{d \in \Re^n} \big\{ \gamma d'y - \gamma f'(x;d) \big\} = \sup_{d \in \Re^n} \big\{ (\gamma d)'y - f'(x;\gamma d) \big\},$$

and finally

$$\delta(y) = \gamma\delta(y), \qquad \forall\, \gamma > 0.$$

This implies that δ takes only the values 0 and ∞, so it is the indicator function of some closed convex set, call it Y. In fact, we have

$$\begin{aligned} Y &= \{y \mid \delta(y) \leq 0\} \\ &= \left\{ y \mid \sup_{d\in\Re^n} \{d'y - f'(x;d)\} \right\} \\ &= \{y \mid d'y \leq f'(x;d),\ \forall\, d \in \Re^n\}. \end{aligned}$$

Combining this with Eq. (5.34), we obtain $Y = \partial f(x)$.

In conclusion, the conjugate of the convex function $f'(x;\cdot)$ is the indicator function of $\partial f(x)$, which is a proper function. By the Conjugacy Theorem (Prop. 1.6.1), $f'(x;\cdot)$ is also proper, and using the conjugacy between indicator and support function of a closed convex set (cf. Example 1.6.1), it follows that $(\mathrm{cl}\, f')(x;\cdot)$ is the support function of $\partial f(x)$.

(b) For $x \in \mathrm{ri}\big(\mathrm{dom}(f)\big)$, by the definition (5.31), $f'(x;d)$ is finite if and only if d belongs to the subspace S that is parallel to $\mathrm{aff}\big(\mathrm{dom}(f)\big)$. Thus, S is equal to both the domain of $f'(x;\cdot)$ and its relative interior. Hence, by Prop. 1.3.15, $f'(x;\cdot)$ coincides with $(\mathrm{cl}\, f')(x;\cdot)$, and the result follows from part (a), since $\partial f(x) \neq \emptyset$. **Q.E.D.**

Generally, for all $x \in \mathrm{dom}(f)$ for which $\partial f(x) \neq \emptyset$, we have

$$f'(x;d) \geq \sup_{g\in\partial f(x)} d'g, \qquad \forall\, d \in \Re^n;$$

this follows from the argument of the preceding proof [cf. Eq. (5.34)]. However, strict inequality may hold for x in the relative boundary of $\mathrm{dom}(f)$, even if $\partial f(x) \neq \emptyset$. As a result, it is essential to use $(\mathrm{cl}\, f')(x;\cdot)$ rather than $f'(x;\cdot)$ in Prop. 5.4.8(a). For example, consider the function of two variables

$$f(x_1,x_2) = \begin{cases} 0 & \text{if } x_1^2 + (x_2-1)^2 \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

Then the subdifferential at the origin is

$$\partial f(0) = \{x \mid x_1 = 0,\ x_2 \leq 0\},$$

while for $d = (1,0)$, we have

$$f'(0;d) = \infty > 0 = \sup_{g\in\partial f(0)} d'g.$$

What is happening here is that $f'(0;\cdot)$ is not closed and does not coincide with $(\mathrm{cl}\, f')(0;\cdot)$ [the support function of $\partial f(0)$ by the preceding proposition]. Thus, we have

$$\infty = f'(0;d) > (\mathrm{cl}\, f')(0;d) = \sup_{g\in\partial f(0)} d'g = 0.$$

Example 5.4.5: (Directional Derivative and Subdifferential of the Max Function)

Let us derive the directional derivative of the function

$$f(x) = \max\big\{f_1(x), \dots, f_r(x)\big\},$$

where $f_j : \Re^n \mapsto \Re$, $j = 1, \dots, r$, are convex functions. For $x \in \Re^n$, let

$$A_x = \big\{j \mid f_j(x) = f(x)\big\}.$$

For any $x, d \in \Re^n$, and $\alpha > 0$, we have

$$\frac{f(x+\alpha d) - f(x)}{\alpha} \geq \frac{f_j(x+\alpha d) - f_j(x)}{\alpha}, \qquad \forall\ j \in A_x,$$

so by taking the limit as $\alpha \downarrow 0$, we obtain

$$f'(x;d) \geq f_j'(x;d), \qquad \forall\ j \in A_x. \tag{5.35}$$

Consider a sequence $\{\alpha_k\}$ with $\alpha_k \downarrow 0$, and let $x_k = x + \alpha_k d$. For each k, let $\overline{j}$ be an index such that $\overline{j} \in A_{x_k}$ for infinitely many k, and by restricting attention to the corresponding subsequence, assume without loss of generality that $\overline{j} \in A_{x_k}$ for all k. Then,

$$f_{\overline{j}}(x_k) \geq f_j(x_k), \qquad \forall\ k,\ j,$$

and by taking the limit as $k \to \infty$, and using the continuity of f_j, we have

$$f_{\overline{j}}(x) \geq f_j(x), \qquad \forall\ j.$$

It follows that $\overline{j} \in A_x$, so that

$$f'(x;d) = \lim_{k\to\infty} \frac{f(x+\alpha_k d) - f(x)}{\alpha_k} = \lim_{k\to\infty} \frac{f_{\overline{j}}(x+\alpha_k d) - f_{\overline{j}}(x)}{\alpha_k} = f_{\overline{j}}'(x;d).$$

Combining this relation with Eq. (5.35), we obtain

$$f'(x;d) = \max\big\{f_j'(x;d) \mid j \in A_x\big\}, \qquad \forall\ x, d \in \Re^n.$$

Since by Prop. 5.4.8(b), $f_j'(x;\cdot)$ are the support functions of $\partial f_j(x)$, the preceding equation shows that $f'(x;\cdot)$ is the support function of $\cup_{j\in A_x} \partial f_j(x)$, and hence also of the closure of $\mathrm{conv}\big(\cup_{j\in A_x} \partial f_j(x)\big)$. By Prop. 5.4.1, the sets $\partial f_j(x)$ are compact, so that $\cup_{j\in A_x}\partial f_j(x)$ is compact, and hence also by Prop. 1.2.2, $\mathrm{conv}\big(\cup_{j\in A_x} \partial f_j(x)\big)$ is compact. On the other hand, by Prop. 5.4.8(b), $f'(x;\cdot)$ is also the support function of $\partial f(x)$. We thus conclude that

$$\partial f(x) = \mathrm{conv}\big(\cup_{j\in A_x} \partial f_j(x)\big).$$

Let us finally note that in convex optimization algorithms, directional derivatives and subgradients arise typically in contexts where f is real-valued. In this case there are no anomalies: the directional derivatives are real-valued, and they are the support functions of the corresponding nonempty and compact subdifferentials.

5.5 MINIMAX THEORY

We will now prove theorems regarding the validity of the minimax equality and the existence of saddle points by specializing the MC/MC theorems of Chapter 4. We will assume throughout this section the following.

Assumption 5.5.1: (Convexity/Concavity and Closedness) X and Z are nonempty convex subsets of $\Re^n$ and $\Re^m$, respectively, and $\phi : X \times Z \mapsto \Re$ is a function such that $\phi(\cdot, z) : X \mapsto \Re$ is convex and closed for each $z \in Z$, and $-\phi(x, \cdot) : Z \mapsto \Re$ is convex and closed for each $x \in X$.

The analysis will revolve around the function

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x, z) - u'z\}, \qquad u \in \Re^m, \tag{5.36}$$

whose epigraph defines the set M used in the MC/MC framework (cf. Section 4.2.5). Note that under Assumption 5.5.1, p is convex. The reason is that p is obtained by partial minimization,

$$p(u) = \inf_{x \in \Re^n} F(x, u),$$

where

$$F(x, u) = \begin{cases} \sup_{z \in Z} \{\phi(x, z) - u'z\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases}$$

is a convex function, since it is the supremum of a collection of convex functions (cf. Props. 1.1.6 and 3.3.1).

5.5.1 Minimax Duality Theorems

We will use the MC/MC Strong Duality Theorem (Prop. 4.3.1) to prove the following proposition.

Proposition 5.5.1: Assume that the function p of Eq. (5.36) satisfies either $p(0) < \infty$, or else $p(0) = \infty$ and $p(u) > -\infty$ for all $u \in \Re^m$. Then

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

if and only if p is lower semicontinuous at $u = 0$.

Proof: The proof consists of showing that with an appropriate selection of the set M, the assumptions of the proposition are essentially equivalent to the corresponding assumptions of the MC/MC strong duality theorem (Prop. 4.3.1).

We choose the set M in the MC/MC strong duality theorem to be the epigraph of p,

$$M = \overline{M} = \big\{(u, w) \mid u \in \Re^m,\ p(u) \leq w\big\},$$

which is convex in view of the convexity of p noted earlier. Thus, condition (2) of the theorem is satisfied.

From the definition of p, we have

$$w^* = p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z).$$

It follows that the assumption $p(0) < \infty$, or else $p(0) = \infty$ and $p(u) > -\infty$ for all u, is equivalent to condition (1) of the MC/MC strong duality theorem.

Finally, with the preceding definition of M, the condition of lower semicontinuity at $u = 0$, i.e.,

$$p(0) \leq \liminf_{k \to \infty} p(u_k)$$

for all $\{u_k\}$ with $u_k \to 0$, is equivalent to $w^* \leq \liminf_{k \to \infty} w_k$ for every sequence $\big\{(u_k, w_k)\big\} \subset M$. Thus, by the conclusion of the MC/MC strong duality theorem, the lower semicontinuity at $u = 0$ holds if and only if $q^* = w^*$, which is in turn equivalent to the minimax equality (see the discussion following Prop. 4.2.1). **Q.E.D.**

The finiteness assumptions on p in the preceding proposition are essential. As an example, consider the case where x and z are scalars and

$$\Phi(x, z) = x + z, \qquad X = \{x \mid x \leq 0\}, \qquad Z = \{z \mid z \geq 0\}.$$

We have

$$p(u) = \inf_{x \leq 0} \sup_{z \geq 0} \{x + z - uz\} = \begin{cases} \infty & \text{if } u < 1, \\ -\infty & \text{if } u \geq 1, \end{cases}$$

so p is closed convex, but the finiteness assumptions of the proposition are violated, and we also have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = -\infty < \infty = \inf_{x \in X} \sup_{z \in Z} \phi(x, z).$$

We will now use the MC/MC existence of solutions results of Section 4.4 (cf. Props. 4.4.1 and 4.4.2) to derive conditions for the attainment of the supremum in the minimax equality. What we need is an assumption that 0 lies in the relative interior or the interior of $\mathrm{dom}(p)$ and that $p(0) > -\infty$.

We then obtain the following result, which also asserts that the supremum in the minimax equality is attained (this follows from the corresponding attainment assertions of Props. 4.4.1 and 4.4.2).

Proposition 5.5.2: Assume that $0 \in \mathrm{ri}\big(\mathrm{dom}(p)\big)$ and $p(0) > -\infty$. Then

$$\sup_{z\in Z}\inf_{x\in X}\phi(x,z) = \inf_{x\in X}\sup_{z\in Z}\phi(x,z),$$

and the supremum over Z in the left-hand side is finite and is attained. Furthermore, the set of $z \in Z$ attaining this supremum is compact if and only if 0 lies in the interior of $\mathrm{dom}(p)$.

The preceding two minimax theorems indicate that the properties of the function p around $u = 0$ are critical to guarantee the minimax equality. Here is an illustrative example:

Example 5.5.1:

Let

$$X = \big\{x \in \Re^2 \mid x \geq 0\big\}, \quad Z = \{z \in \Re \mid z \geq 0\}, \quad \phi(x,z) = e^{-\sqrt{x_1x_2}} + zx_1,$$

which can be shown to satisfy the convexity/concavity and closedness Assumption 5.5.1. (This is essentially the constrained minimization Example 5.3.2, recast as a minimax problem.) For all $z \geq 0$, we have

$$\inf_{x\geq 0}\phi(x,z) = \inf_{x\geq 0}\big\{e^{-\sqrt{x_1x_2}} + zx_1\big\} = 0,$$

since the expression in braces is nonnegative for $x \geq 0$ and can approach zero by taking $x_1 \to 0$ and $x_1x_2 \to \infty$. Hence,

$$\sup_{z\geq 0}\inf_{x\geq 0}\phi(x,z) = 0.$$

We also have for all $x \geq 0$,

$$\sup_{z\geq 0}\phi(x,z) = \sup_{z\geq 0}\big\{e^{-\sqrt{x_1x_2}} + zx_1\big\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0. \end{cases}$$

Hence,

$$\inf_{x\geq 0}\sup_{z\geq 0}\phi(x,z) = 1,$$

so

$$\inf_{x\geq 0}\sup_{z\geq 0}\phi(x,z) > \sup_{z\geq 0}\inf_{x\geq 0}\phi(x,z).$$

Here, the function p is given by

$$p(u) = \inf_{x \geq 0} \sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z(x_1 - u) \right\} = \begin{cases} \infty & \text{if } u < 0, \\ 1 & \text{if } u = 0, \\ 0 & \text{if } u > 0. \end{cases}$$

Thus, p is not lower semicontinuous at 0, the assumptions of Props. 5.5.1 are violated, and the minimax equality does not hold.

The following example illustrates how the minimax equality may hold, while the supremum over $z \in Z$ is not attained because the relative interior assumption of Prop. 5.5.2 is not satisfied.

Example 5.5.2:

Let

$$X = \Re, \qquad Z = \{z \in \Re \mid z \geq 0\}, \qquad \phi(x, z) = x + zx^2,$$

which satisfy Assumption 5.5.1. (This is essentially the constrained minimization Example 5.3.3, recast as a minimax problem.) For all $z \geq 0$, we have

$$\inf_{x \in \Re} \phi(x, z) = \inf_{x \in \Re} \left\{ x + zx^2 \right\} = \begin{cases} -1/(4z) & \text{if } z > 0, \\ -\infty & \text{if } z = 0. \end{cases}$$

Hence,

$$\sup_{z \geq 0} \inf_{x \in \Re} \phi(x, z) = 0.$$

We also have for all $x \in \Re$,

$$\sup_{z \geq 0} \phi(x, z) = \sup_{z \geq 0} \left\{ x + zx^2 \right\} = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \neq 0. \end{cases}$$

Hence,

$$\inf_{x \in \Re} \sup_{z \geq 0} \phi(x, z) = 0,$$

and the minimax equality holds. However, the problem

$$\begin{aligned} &\text{maximize} \quad \inf_{x \in \Re} \phi(x, z) \\ &\text{subject to} \quad z \in Z \end{aligned}$$

does not have an optimal solution. Here we have

$$F(x, u) = \sup_{z \geq 0} \{x + zx^2 - uz\} = \begin{cases} x & \text{if } x^2 \leq u, \\ \infty & \text{if } x^2 > u, \end{cases}$$

and

$$p(u) = \inf_{x \in \Re} \sup_{z \geq 0} F(x, u) = \begin{cases} -\sqrt{u} & \text{if } u \geq 0, \\ \infty & \text{if } u < 0. \end{cases}$$

It can be seen that $0 \notin \mathrm{ri}\big(\mathrm{dom}(p)\big)$, thus violating the assumption of Prop. 5.5.2.

5.5.2 Saddle Point Theorems

We will now use the two minimax theorems just derived (Props. 5.5.1 and 5.5.2) to obtain more specific conditions for the validity of the minimax equality and the existence of saddle points. Again, Assumption 5.5.1 is assumed throughout. The preceding analysis has underscored the importance of the function

$$p(u) = \inf_{x \in \Re^n} F(x, u), \tag{5.37}$$

where

$$F(x, u) = \begin{cases} \sup_{z \in Z} \{\phi(x, z) - u'z\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases} \tag{5.38}$$

and suggests a two-step process to ascertain the validity of the minimax equality and the existence of a saddle point:

(1) Show that p is closed and convex, thereby showing that the minimax equality holds by using Prop. 5.5.1.

(2) Verify that the infimum of $\sup_{z \in Z} \phi(x, z)$ over $x \in X$, and the supremum of $\inf_{x \in X} \phi(x, z)$ over $z \in Z$ are attained, thereby showing that the set of saddle points is nonempty (cf. Prop. 3.4.1).

Step (1) requires two types of assumptions:

(a) The convexity/concavity and closedness Assumption 5.5.1. This guarantees that F is convex and closed (being the pointwise supremum over $z \in Z$ of closed convex functions), and also guarantees that p is convex.

(b) Conditions that guarantee that the partial minimization in the definition of p preserves closedness, so that p is also closed.

Step (2) requires that either Weierstrass' Theorem can be applied, or else that some other suitable condition for existence of an optimal solution is satisfied (cf. Section 3.3). Fortunately, conditions that guarantee that the partial minimization in the definition of F preserves closedness as in (b), also guarantee the existence of corresponding optimal solutions.

As an example of this line of analysis, we obtain the following classical result.

Proposition 5.5.3: (Classical Saddle Point Theorem) Let the sets X and Z be compact. Then the set of saddle points of ϕ is nonempty and compact.

Proof: We note that F is convex and closed. Using the convexity of F and Prop. 3.3.1, we see that the function p of Eq. (5.37) is convex. Using the compactness of Z, F is real-valued over $X \times \Re^m$, and from the compactness of X and Prop. 3.3.3, it follows that p is also real-valued and therefore continuous. Hence, the minimax equality holds by Prop. 5.5.1.

Finally, the function $\sup_{z \in Z} \phi(x, z)$ is equal to $F(x, 0)$, so it is closed, and the set of its minima over $x \in X$ is nonempty and compact by Weierstrass' Theorem. Similarly the set of maxima of the function $\inf_{x \in X} \phi(x, z)$ over $z \in Z$ is nonempty and compact. From Prop. 3.4.1 it follows that the set of saddle points is nonempty and compact. **Q.E.D.**

We will now derive alternative and more general saddle point theorems, using a similar line of analysis. To formulate these theorems, we consider the functions $t : \Re^n \mapsto (-\infty, \infty]$ and $r : \Re^m \mapsto (-\infty, \infty]$ given by

$$t(x) = \begin{cases} \sup_{z \in Z} \phi(x, z) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases}$$

and

$$r(z) = \begin{cases} -\inf_{x \in X} \phi(x, z) & \text{if } z \in Z, \\ \infty & \text{if } z \notin Z. \end{cases}$$

Note that by Assumption 5.5.1, t is closed and convex, being the supremum of closed and convex functions. Furthermore, since $t(x) > -\infty$ for all x, we have

$$t \text{ is proper if and only if } \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty.$$

Similarly, the function r is closed and convex, and

$$r \text{ is proper if and only if } -\infty < \sup_{z \in Z} \inf_{x \in X} \phi(x, z).$$

The next two propositions provide conditions for the minimax equality to hold. These propositions are subsequently used to prove results about nonemptiness and compactness of the set of saddle points.

Proposition 5.5.4: Assume that t is proper and that the level sets $\{x \mid t(x) \leq \gamma\}$, $\gamma \in \Re$, are compact. Then

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

and the infimum over X in the right-hand side above is attained at a set of points that is nonempty and compact.

Proof: The function p is defined by partial minimization of the function F of Eq. (5.38), i.e.,

$$p(u) = \inf_{x \in \Re^n} F(x, u)$$

[cf. Eq. (5.37)]. We note that

$$t(x) = F(x, 0),$$

so F is proper, since t is proper and F is closed [by the Recession Cone Theorem (Prop. 1.4.1), $\text{epi}(F)$ contains a vertical line if and only if $\text{epi}(t)$ contains a vertical line]. Furthermore, the compactness assumption on the level sets of t can be translated to the compactness assumption of Prop. 3.3.2 (with 0 playing the role of the vector $\overline{x}$). It follows from the result of that proposition that p is closed and proper, and that $p(0)$ is finite. By Prop. 5.5.1, it follows that the minimax equality holds. Finally, the infimum over X in the right-hand side of the minimax equality is attained at the set of minima of t, which is nonempty and compact since t is proper and has compact level sets. **Q.E.D.**

Example 5.5.3:

Let us show that

$$\min_{\|x\|\le 1} \max_{z\in S+C} x'z = \max_{z\in S+C} \min_{\|x\|\le 1} x'z,$$

where S is a subspace, and C is a nonempty, convex, and compact subset of $\Re^n$. By defining

$$X = \{x \mid \|x\| \le 1\}, \qquad Z = S + C,$$

$$\phi(x, z) = x'z, \qquad \forall\, (x, z) \in X \times Z,$$

we see that Assumption 5.5.1 is satisfied, so we can apply Prop. 5.5.4. We have

$$\begin{aligned} t(x) &= \begin{cases} \sup_{z\in S+C} x'z & \text{if } \|x\| \le 1, \\ \infty & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sup_{z\in C} x'z & \text{if } x \in S^\perp,\ \|x\| \le 1, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\sup_{z\in C} x'z$, viewed as a function of x over $\Re^n$, is continuous, the level sets of t are compact. It follows from Prop. 5.5.4 that the minimax equality holds. It also turns out that a saddle point exists. We will show this shortly, after we develop some additional machinery.

Proposition 5.5.5: Assume that t is proper, and that the recession cone and the constancy space of t are equal. Then

$$\sup_{z\in Z} \inf_{x\in X} \phi(x, z) = \inf_{x\in X} \sup_{z\in Z} \phi(x, z)$$

and the infimum over X in the right-hand side above is attained.

Proof: The proof is similar to the one of Prop. 5.5.4. We use Prop. 3.3.4 in place of Prop. 3.3.2. **Q.E.D.**

By combining the preceding two propositions, we obtain conditions for existence of a saddle point.

Proposition 5.5.6: Assume that either t is proper or r is proper.

(a) If the level sets $\{x \mid t(x) \leq \gamma\}$ and $\{z \mid r(z) \leq \gamma\}$, $\gamma \in \Re$, of t and r are compact, the set of saddle points of ϕ is nonempty and compact.

(b) If the recession cones of t and r are equal to the constancy spaces of t and r, respectively, the set of saddle points of ϕ is nonempty.

Proof: We assume that t is proper. If instead r is proper, we reverse the roles of x and z.

(a) From Prop. 5.5.4, it follows that the minimax equality holds, and that the infimum over X of $\sup_{z \in Z} \phi(x, z)$ is finite and is attained at a nonempty and compact set. Therefore,

$$-\infty < \sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty,$$

and we can reverse the roles of x and z, and apply Prop. 5.5.4 again to show that the supremum over Z of $\inf_{x \in X} \phi(x, z)$ is attained at a nonempty and compact set.

(b) The proof is similar to the one of part (a), except that we use Prop. 5.5.5 instead of Prop. 5.5.4. **Q.E.D.**

Example 5.5.2: (continued)

To illustrate the difference between Props. 5.5.4 and 5.5.6, let

$$X = \Re, \qquad Z = \{z \in \Re \mid z \geq 0\}, \qquad \phi(x, z) = x + zx^2.$$

A straightforward calculation yields

$$t(x) = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \neq 0, \end{cases} \qquad r(z) = \begin{cases} \frac{1}{4z} & \text{if } z > 0, \\ \infty & \text{if } z \leq 0. \end{cases}$$

Thus, t satisfies the assumptions of Prop. 5.5.4 and the minimax equality holds, but r violates the assumptions of Prop. 5.5.6 and there is no saddle point, since the supremum over z of $-r(z) = \inf_{x \in \Re} \Phi(x, z)$ is not attained.

Example 5.5.3: (continued)

Let

$$X = \{x \mid \|x\| \le 1\}, \qquad Z = S + C, \qquad \phi(x,z) = x'z,$$

where S is a subspace, and C is a nonempty, convex, and compact subset of $\Re^n$. We have

$$t(x) = \begin{cases} \sup_{z\in S+C} x'z & \text{if } \|x\| \le 1, \\ \infty & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \sup_{z\in C} x'z & \text{if } x \in S^\perp,\ \|x\| \le 1, \\ \infty & \text{otherwise,} \end{cases}$$

$$r(z) = \begin{cases} \sup_{\|x\|\le 1} -x'z & \text{if } z \in S + C, \\ \infty & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \|z\| & \text{if } z \in S + C, \\ \infty & \text{otherwise.} \end{cases}$$

Since the level sets of both t and r are compact, it follows from Prop. 5.5.6(a) that the minimax equality holds, and that the set of saddle points is nonempty and compact.

The compactness of the level sets of t can be guaranteed by simpler sufficient conditions. In particular, the level sets $\{x \mid t(x) \le \gamma\}$ are compact if any one of the following two conditions holds:

(1) The set X is compact [since $\{x \mid t(x) \le \gamma\}$ is closed, by the closedness of t, and is contained in X].

(2) For some $\overline{z} \in Z$, $\overline{\gamma} \in \Re$, the set

$$\{x \in X \mid \phi(x, \overline{z}) \le \overline{\gamma}\}$$

is nonempty and compact [since then all sets $\{x \in X \mid \phi(x,\overline{z}) \le \gamma\}$ are compact, and a nonempty level set $\{x \mid t(x) \le \gamma\}$ is contained in $\{x \in X \mid \phi(x,\overline{z}) \le \gamma\}$].

Furthermore, any one of above two conditions also guarantees that r is proper; for example under condition (2), the infimum over $x \in X$ in the relation

$$r(\overline{z}) = -\inf_{x\in X} \phi(x,\overline{z})$$

is attained by Weierstrass' Theorem, so that $r(\overline{z}) < \infty$.

By a symmetric argument, we also see that the level sets of r are compact under any one of the following two conditions:

(1) The set Z is compact.

(2) For some $\overline{x} \in X$, $\overline{\gamma} \in \Re$, the set

$$\{z \in Z \mid \phi(\overline{x}, z) \ge \overline{\gamma}\}$$

is nonempty and compact.

Furthermore, any one of these conditions guarantees that t is proper.

Thus, by combining the preceding discussion and Prop. 5.5.6(a), we obtain the following result, which generalizes the classical saddle point theorem (Prop. 5.5.3), and provides sufficient conditions for the existence of a saddle point.

Proposition 5.5.7: (Saddle Point Theorem) The set of saddle points of ϕ is nonempty and compact under any one of the following conditions:

(1) X and Z are compact.

(2) Z is compact, and for some $\overline{z} \in Z$, $\overline{\gamma} \in \Re$, the level set

$$\{x \in X \mid \phi(x, \overline{z}) \leq \overline{\gamma}\}$$

is nonempty and compact.

(3) X is compact, and for some $\overline{x} \in X$, $\overline{\gamma} \in \Re$, the level set

$$\{z \in Z \mid \phi(\overline{x}, z) \geq \overline{\gamma}\}$$

is nonempty and compact.

(4) For some $\overline{x} \in X$, $\overline{z} \in Z$, $\overline{\gamma} \in \Re$, the level sets

$$\{x \in X \mid \phi(x, \overline{z}) \leq \overline{\gamma}\}, \qquad \{z \in Z \mid \phi(\overline{x}, z) \geq \overline{\gamma}\},$$

are nonempty and compact.

Proof: From the discussion preceding the proposition, it is seen that under Assumption 5.5.1, t and r are proper, and the level sets of t and r are compact. The result follows from Prop. 5.5.6(a). **Q.E.D.**

5.6 THEOREMS OF THE ALTERNATIVE

Theorems of the alternative are important tools in optimization, which address the feasibility (possibly strict) of affine inequalities. We will show that these theorems can be viewed as special cases of MC/MC duality (cf. Props. 4.4.1, 4.4.2). This connection is demonstrated in the proofs of some classical theorems.

One of them is Gordan's Theorem, which dates to 1873. In its classical version it states that there exists a vector $x \in \Re^n$ such that

$a_1'x < 0, \ldots, a_r'x < 0$, if and only if $\text{cone}\big(\{a_1, \ldots, a_r\}\big)$ does not contain a line (both vectors d and $-d$ for some $d \neq 0$); see alternatives (i) and (ii) in the following proposition, for $b = 0$. We provide a slight extension of this theorem.

Proposition 5.6.1: (Gordan's Theorem) Let A be an $m \times n$ matrix and b be a vector in $\Re^m$. The following are equivalent:

(i) There exists a vector $x \in \Re^n$ such that

$$Ax < b.$$

(ii) For every vector $\mu \in \Re^m$,

$$A'\mu = 0, \quad b'\mu \leq 0, \quad \mu \geq 0 \qquad \Rightarrow \qquad \mu = 0.$$

(iii) Any polyhedral set of the form

$$\{\mu \mid A'\mu = c,\, b'\mu \leq d,\, \mu \geq 0\}, \tag{5.39}$$

where $c \in \Re^n$ and $d \in \Re$, is compact.

Proof: We will show that (i) and (ii) are equivalent, and then that (ii) and (iii) are equivalent. The equivalence of (i) and (ii) is geometrically evident, once the proper MC/MC framework is considered (see Fig. 5.6.1). For completeness we provide the (somewhat lengthy) details. Consider the set

$$M = \big\{(u, w) \mid w \geq 0,\, Ax - b \leq u \text{ for some } x \in \Re^n\big\},$$

its projection on the u axis

$$D = \{u \mid Ax - b \leq u \text{ for some } x \in \Re^n\},$$

and the corresponding MC/MC framework. Let w^* and q^* be the min common and max crossing values, respectively. Clearly, the system $Ax \leq b$ has a solution if and only if $w^* = 0$. Also, if (ii) holds, we have

$$A'\mu = 0, \quad \mu \geq 0 \qquad \Rightarrow \qquad b'\mu \geq 0,$$

which by the linear version of Farkas' Lemma (Prop. 5.1.2), implies that the system $Ax \leq b$ has a solution. In conclusion, both (i) and (ii) imply that the system $Ax \leq b$ has a solution, which is in turn equivalent to $w^* = q^* = 0$. Thus in proving the equivalence of (i) and (ii), we may assume that the system $Ax \leq b$ has a solution and $w^* = q^* = 0$.

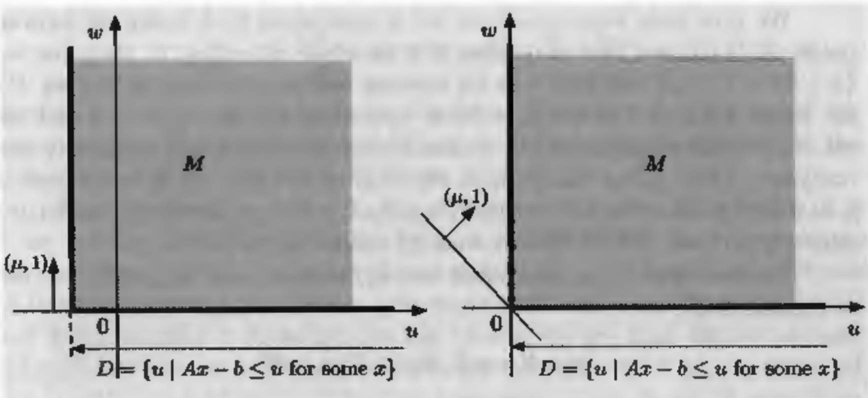


Figure 5.6.1. MC/MC framework for showing the equivalence of conditions (i) and (ii) of Gordan's Theorem. We consider the set

$$M = \Big\{(u, w) \mid w \ge 0,\ Ax - b \le u \text{ for some } x \in \Re^n\Big\},$$

its projection on the u axis

$$D = \{u \mid Ax - b \le u \text{ for some } x \in \Re^n\},$$

and the corresponding MC/MC framework. As shown by the figure on the left, condition (i) of Gordan's Theorem is equivalent to $0 \in \text{int}(D)$, and is also equivalent to 0 (the horizontal hyperplane) being the unique optimal solution of the max crossing problem. It is seen that the latter condition can be written as

$$\mu \ge 0, \quad 0 \le \mu'(Ax - b) + w, \quad \forall\, x \in \Re^n,\ w \ge 0 \qquad \Rightarrow \qquad \mu = 0,$$

which is equivalent to condition (ii). In the figure on the right both conditions (i) and (ii) are violated.

The cost function of the max crossing problem is

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\}.$$

Since M contains vectors (u, w) with arbitrarily large components [for each $(u, w) \in M$, we have $(\overline{u}, \overline{w}) \in M$ for all $\overline{u} \ge u$ and $\overline{w} \ge w$], it follows that $q(\mu) = -\infty$ for all μ that are not in the nonnegative orthant, and we have

$$q(\mu) = \begin{cases} \inf_{Ax-b\le u} \mu' u & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise,} \end{cases}$$

or equivalently,

$$q(\mu) = \begin{cases} \inf_{x\in\Re^n} \mu'(Ax - b) & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{5.40}$$

We now note that condition (i) is equivalent to 0 being an interior point of D (to see this note that if $\overline{x}$ satisfies $A\overline{x} - b < 0$, then the set $\{u \mid A\overline{x} - b \leq u\}$ contains 0 in its interior and is contained in the set D). By Prop. 4.4.2, it follows that (i) is equivalent to $w^* = q^* = 0$ and the set of optimal solutions of the max crossing problem being nonempty and compact. Thus, using the form (5.40) of q, we see that (i) is equivalent to $\mu = 0$ being the only $\mu \geq 0$ satisfying $q(\mu) \geq 0$ or equivalently, satisfying $A'\mu = 0$ and $\mu' b \leq 0$. It follows that (i) is equivalent to (ii).

To show that (ii) is equivalent to (iii), we note that the recession cone of the set (5.39) is

$$\{\mu \mid A'\mu = 0,\ b'\mu \leq 0,\ \mu \geq 0\}.$$

Thus (ii) states that the recession cone of the set (5.39) consists of just the origin, which is equivalent to (iii); cf. Prop. 1.4.2(a). **Q.E.D.**

There are several theorems of the alternative involving strict inequalities. Among the ones involving linear constraints, the following is the most general.

Proposition 5.6.2: (Motzkin's Transposition Theorem) Let A and B be $p \times n$ and $q \times n$ matrices, and let $b \in \Re^p$ and $c \in \Re^q$ be vectors. The system

$$Ax < b, \qquad Bx \leq c$$

has a solution if and only if for all $\mu \in \Re^p$ and $\nu \in \Re^q$, with $\mu \geq 0$, $\nu \geq 0$, the following two conditions hold:

$$A'\mu + B'\nu = 0 \quad \Rightarrow \quad b'\mu + c'\nu \geq 0, \tag{5.41}$$

$$A'\mu + B'\nu = 0,\ \mu \neq 0 \quad \Rightarrow \quad b'\mu + c'\nu > 0. \tag{5.42}$$

Proof: Consider the set

$$M = \big\{(u, w) \mid w \geq 0,\ Ax - b \leq u \text{ for some } x \text{ with } Bx \leq c\big\},$$

its projection on the u axis

$$D = \{u \mid Ax - b \leq u \text{ for some } x \text{ with } Bx \leq c\},$$

and the corresponding MC/MC framework. The cost function of the max crossing problem is

$$q(\mu) = \begin{cases} \inf_{(u,w)\in M}\{w + \mu' u\} & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

or

$$q(\mu) = \begin{cases} \inf_{Ax-b\le u,\, Bx\le c} \mu' u & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{5.43}$$

Similar to the proof of Gordan's Theorem (Prop. 5.6.1), and using the linear version of Farkas' Lemma (Prop. 5.1.2), we may assume that the system $Ax \le b$, $Bx \le c$ has a solution.

Now the system $Ax < b$, $Bx \le c$ has a solution if and only if 0 is an interior point of D, so by Prop. 4.4.2, it follows that the system $Ax < b$, $Bx \le c$ has a solution if and only if the set of optimal solutions of the max crossing problem is nonempty and compact. Since $q(0) = 0$ and $\sup_{\mu\in\Re^m} q(\mu) = 0$, using also Eq. (5.43), we see that the set of dual optimal solutions is nonempty and compact if and only if $q(\mu) < 0$ for all $\mu \ge 0$ with $\mu \ne 0$ [if $q(\mu) = 0$ for some nonzero $\mu \ge 0$, it can be seen from Eq. (5.43) that we must have $q(\gamma\mu) = 0$ for all $\gamma > 0$]. We will complete the proof by showing that these conditions hold if and only if conditions (5.41) and (5.42) hold for all $\mu \ge 0$ and $\nu \ge 0$.

We calculate the infimum of the linear program in (x, u) in the right-hand side of Eq. (5.43) by using duality, and simplifying the dual problem. In particular, we have for all $\mu \ge 0$, after a straightforward calculation,

$$q(\mu) = \inf_{Ax-b\le u,\, Bx\le c} \mu' u = \sup_{A'\mu+B'\nu=0,\, \nu\ge 0} (-b'\mu - c'\nu).$$

(In these relations, μ is fixed; the infimum is over (x, u), and the supremum is over ν.) Using this equation, we see that $q(0) = 0$ and $q(\mu) < 0$ for all $\mu \ge 0$ with $\mu \ne 0$ if and only if conditions (5.41) and (5.42) hold for all $\mu \ge 0$ and $\nu \ge 0$. **Q.E.D.**

Let us derive another interesting theorem of the alternative. Like the Gordan and Motzkin Theorems, it can be proved by means of a suitable MC/MC formulation. We will give an alternative proof that uses Motzkin's theorem. (Actually, Motzkin's Theorem, being the most general theorem of the alternative that involves linear equations and inequalities, can also be similarly used to prove Gordan's Theorem. However, the proof we gave, based on the connection with the MC/MC framework, is more intuitive.)

Proposition 5.6.3: (Stiemke's Transposition Theorem) Let A be an $m \times n$ matrix, and let c be a vector in $\Re^m$. The system

$$Ax = c, \qquad x > 0$$

has a solution if and only if

$$A'\mu \ge 0 \text{ and } c'\mu \le 0 \qquad \Rightarrow \qquad A'\mu = 0 \text{ and } c'\mu = 0.$$

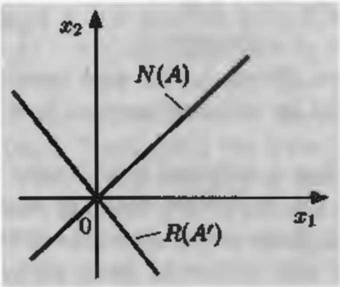


Figure 5.6.2. Geometric interpretation of Stiemke's Transposition Theorem for the case $c = 0$. The nullspace $N(A)$ of A meets the interior of the nonnegative orthant if and only if the range $R(A')$ of A' meets the nonnegative orthant at just the origin.

Proof: We first consider the case where $c = 0$, i.e., we show that the system $Ax = 0$, $x > 0$ has a solution if and only if

$$A'\mu \geq 0 \qquad \Rightarrow \qquad A'\mu = 0.$$

Indeed, we can equivalently write the system $Ax = 0$, $x > 0$ in the form

$$-x < 0, \qquad Ax \leq 0, \qquad -Ax \leq 0,$$

so that Motzkin's Theorem can be applied with the identifications $A \sim -I$, $B \sim [A \;\; -A]$, $b \sim 0$, $c \sim 0$. We obtain that the system has a solution if and only if the system

$$-y + A'z^+ - A'z^- = 0, \qquad y \geq 0, \qquad z^+ \geq 0, \qquad z^- \geq 0,$$

has no solution with $y \neq 0$, or equivalently, by introducing the transformation $\mu = z^+ - z^-$, if and only if the system

$$A'\mu \geq 0, \qquad A'\mu \neq 0$$

has no solution.

We now assume that $c \neq 0$, and we note that the system $Ax = c$, $x > 0$ has a solution if and only if the system

$$(A \;\; -c)\begin{pmatrix} x \\ z \end{pmatrix} = 0, \qquad x > 0,\ z > 0,$$

has a solution. By applying the result already shown for the case $c = 0$, we see that the above system has a solution if and only if

$$\begin{pmatrix} A' \\ -c' \end{pmatrix}\mu \geq 0 \qquad \Rightarrow \qquad \begin{pmatrix} A' \\ -c' \end{pmatrix}\mu = 0,$$

or equivalently if and only if

$$A'\mu \geq 0 \text{ and } c'\mu \leq 0 \qquad \Rightarrow \qquad A'\mu = 0 \text{ and } c'\mu = 0.$$

Q.E.D.

A geometric interpretation of Stiemke's Theorem for the case $c = 0$ is given in Fig. 5.6.2.

Compactness of the Solution Set of a Linear Program

The theorems of Gordan and Stiemke (Props. 5.6.1 and 5.6.3) can be used to provide necessary and sufficient conditions for the compactness of the primal and the dual optimal solution sets of linear programs (cf. Example 4.2.1). We say that the primal linear program

$$\begin{aligned} &\text{minimize } c'x \\ &\text{subject to } a_j'x \geq b_j, \quad j = 1, \ldots, r, \end{aligned} \tag{5.44}$$

is strictly feasible if there exists a primal-feasible vector $x \in \Re^n$ with $a_j'x > b_j$ for all $j = 1, \ldots, r$. Similarly, we say that the dual linear program

$$\begin{aligned} &\text{maximize } b'\mu \\ &\text{subject to } \sum_{j=1}^{r} a_j \mu_j = c, \quad \mu \geq 0, \end{aligned} \tag{5.45}$$

is strictly feasible if there exists a dual-feasible vector μ with $\mu > 0$. We have the following proposition. Part (b) requires that the matrix defining the constraints has rank n, the dimension of the primal vector x.

Proposition 5.6.4: Consider the primal and dual linear programs [cf. Eqs. (5.44), (5.45)], and assume that their common optimal value is finite. Then:

(a) The dual optimal solution set is compact if and only if the primal problem is strictly feasible.

(b) Assuming that the set $\{a_1, \ldots, a_r\}$ contains n linearly independent vectors, the primal optimal solution set is compact if and only if the dual problem is strictly feasible.

Proof: We first note that, by the duality theorem (Prop. 5.2.1), the finiteness of the optimal value of the primal problem implies that the dual optimal solution set, denoted D^*, is nonempty.

(a) The dual optimal solution set is

$$D^* = \{\mu \mid \mu_1 a_1 + \cdots + \mu_r a_r = c, \ \mu_1 b_1 + \cdots + \mu_r b_r \geq q^*, \ \mu \geq 0\},$$

The result now follows from the equivalence of conditions (i) and (iii) in Gordan's Theorem (Prop. 5.6.1).

(b) The primal optimal solution set is

$$P^* = \{x \mid c'x \leq f^*, \ a_j'x \geq b_j, \ j = 1, \ldots, r\}.$$

The dual problem is strictly feasible if there exists $\mu > 0$ such that $A\mu = c$, where A is the matrix with columns a_j, $j = 1, \ldots, r$. From Stiemke's Transposition Theorem (Prop. 5.6.3), we see that this is true if and only if every vector d in the recession cone of P^*, which is

$$R_{P^*} = \{d \mid c'd \le 0,\, a_j'd \ge 0,\, j = 1, \ldots, r\},$$

satisfies $c'd = 0$ and $a_j'd = 0$ for all j, or just $a_j'd = 0$ for all j (since c is a linear combination of $a_1, \ldots, a_r$, by the feasibility of the dual problem). Since the set $\{a_1, \ldots, a_r\}$ contains n linearly independent vectors, it follows that R_{P^*} consists of just the origin. Hence, by Prop. 1.4.2(a), it follows that P^* is compact. **Q.E.D.**

For an example where the dual problem is strictly feasible, but the primal optimal solution set is not compact, consider the problem

$$\begin{aligned} &\text{minimize} \;\; x_1 + x_2 \\ &\text{subject to} \;\; x_1 + x_2 \ge 1. \end{aligned}$$

The dual problem,

$$\begin{aligned} &\text{maximize} \;\; \mu \\ &\text{subject to} \;\; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mu \ge 0, \end{aligned}$$

is strictly feasible, but the primal problem has an unbounded optimal solution set. The difficulty here is that the linear independence assumption of Prop. 5.6.4(b) is violated (since $n = 2$ but $r = 1$).

5.7 NONCONVEX PROBLEMS

In this section we focus on the MC/MC framework in the absence of the convex structure that guarantees strong duality. We aim to estimate the duality gap. Consider first the case where the set M is the epigraph of a function $p : \Re^n \mapsto [-\infty, \infty]$. Then $w^* = p(0)$, and as discussed in Section 4.2.1, $q(\mu) = -p^\star(-\mu)$, where $p^\star$ is the conjugate of p. Thus

$$q^* = \sup_{\mu \in \Re^n} q(\mu) = \sup_{\mu \in \Re^n} \big\{ 0 \cdot (-\mu) - p^\star(-\mu) \big\} = p^{\star\star}(0), \tag{5.46}$$

where $p^{\star\star}$ is the conjugate of $p^\star$ (double conjugate of p). The duality gap is

$$w^* - q^* = p(0) - p^{\star\star}(0),$$

and can be interpreted as a "measure of nonconvexity" of p at 0.

Consider next the case where M has the form

$$M = \tilde{M} + \{(u, 0) \mid u \in C\},$$

where $\tilde{M}$ is a compact set and C is a closed convex set, or more generally, where $\mathrm{conv}(M)$ is closed and $q^* > -\infty$. Then, by Prop. 4.3.2(c), the duality gap $w^* - q^*$ is equal to $w^* - \overline{w}^*$, where w^* and $\overline{w}^*$ are the min common values corresponding to the sets M and $\mathrm{conv}(M)$.

In either case, we see that estimates of the duality gap should depend on how much M differs from its convex hull along the vertical axis. A special case where interesting estimates of this type can be obtained arises in separable optimization problems, as we now discuss.

5.7.1 Duality Gap in Separable Problems

Suppose that x consists of m components $x_1, \ldots, x_m$ of corresponding dimensions $n_1, \ldots, n_m$, and the problem has the form

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^m f_i(x_i) \\ &\text{subject to} \quad \sum_{i=1}^m g_i(x_i) \le 0, \quad x_i \in X_i, \ \ i = 1, \ldots, m, \end{aligned} \tag{5.47}$$

where $f_i : \Re^{n_i} \mapsto \Re$ and $g_i : \Re^{n_i} \mapsto \Re^r$ are given functions, and X_i are given subsets of $\Re^{n_i}$. An important nonconvex instance is when the functions f_i and g_i are linear, and the sets X_i are finite, in which case we obtain a discrete/integer programming problem.

We call a problem of the form (5.47) *separable*. Its salient feature is that the minimization involved in the calculation of the dual function

$$q(\mu) = \inf_{\substack{x_i \in X_i \\ i=1,\ldots,m}} \left\{ \sum_{i=1}^m \big(f_i(x_i) + \mu' g_i(x_i) \big) \right\} = \sum_{i=1}^m q_i(\mu),$$

is decomposed into the m simpler minimizations

$$q_i(\mu) = \inf_{x_i \in X_i} \big\{ f_i(x_i) + \mu' g_i(x_i) \big\}, \qquad i = 1, \ldots, m.$$

These minimizations are often conveniently done either analytically or computationally, in which case the dual function can be easily evaluated.

When the cost and/or the constraints are not convex, the separable structure is helpful in another, somewhat unexpected way. In particular, in this case *the duality gap turns out to be relatively small and can often be shown to diminish to zero relative to the optimal primal value as the number*

m of separable terms increases. As a result, one can often obtain a near-optimal primal solution, starting from a dual optimal solution. In integer programming problems, this may obviate the need for computationally intensive procedures such as branch-and-bound.

The small duality gap is a consequence of the structure of the set of achievable constraint-cost pairs

$$S = \{(g(x), f(x)) \mid x \in X\},$$

where

$$g(x) = \sum_{i=1}^{m} g_i(x_i), \qquad f(x) = \sum_{i=1}^{m} f_i(x_i).$$

Note that this set plays a central role in the proof of the Nonlinear Farkas' Lemma (see Fig. 5.1.2). In the case of a separable problem, it can be written as a vector sum of m sets, one for each separable term, i.e.,

$$S = S_1 + \cdots + S_m,$$

where

$$S_i = \{(g_i(x_i), f_i(x_i)) \mid x_i \in X_i\}.$$

A key fact is that generally, a set that is the vector sum of a large number of possibly nonconvex but roughly similar sets "tends to be convex" in the sense that any vector in its convex hull can be closely approximated by a vector in the set. As a result, the duality gap tends to be relatively small. The analytical substantiation is based on the following theorem, which roughly states that at most $r+1$ out of the m convex sets "contribute to the nonconvexity" of their vector sum, regardless of the value of m.

Proposition 5.7.1: (Shapley-Folkman Theorem) Let S_i, $i = 1, \ldots, m$, be nonempty subsets of $\Re^{r+1}$, with $m > r+1$, and let $S = S_1 + \cdots + S_m$. Then every vector $s \in \text{conv}(S)$ can be represented as $s = s_1 + \cdots + s_m$, where $s_i \in \text{conv}(S_i)$ for all $i = 1, \ldots, m$, and $s_i \notin S_i$ for at most $r+1$ indices i.

Proof: We clearly have $\text{conv}(S) = \text{conv}(S_1) + \cdots + \text{conv}(S_m)$ (since convex combinations commute with linear transformations). Thus any $s \in \text{conv}(S)$ can be written as $s = \sum_{i=1}^{m} y_i$ with $y_i \in \text{conv}(S_i)$, so that $y_i = \sum_{j=1}^{t_i} a_{ij} y_{ij}$ for some $a_{ij} > 0$, $\sum_{j=1}^{t_i} a_{ij} = 1$, and $y_{ij} \in S_i$. Consider the following vectors of $\Re^{r+1+m}$:

$$z = \begin{pmatrix} s \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad z_{1j} = \begin{pmatrix} y_{1j} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \cdots \qquad z_{mj} = \begin{pmatrix} y_{mj} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

so that $z = \sum_{i=1}^{m} \sum_{j=1}^{t_i} a_{ij} z_{ij}$. We now view z as a vector in the cone generated by the vectors z_{ij}, and use Caratheodory's Theorem [Prop. 1.2.1(a)] to write $z = \sum_{i=1}^{m} \sum_{j=1}^{t_i} b_{ij} z_{ij}$ for some nonnegative scalars b_{ij}, at most $r + 1 + m$ of which are strictly positive. Focusing on the first component of z, we have

$$s = \sum_{i=1}^{m} \sum_{j=1}^{t_i} b_{ij} y_{ij}, \qquad \sum_{j=1}^{t_i} b_{ij} = 1, \quad \forall\, i = 1, \ldots, m.$$

Let $s_i = \sum_{j=1}^{t_i} b_{ij} y_{ij}$, so that $s = s_1 + \cdots + s_m$ with $s_i \in \text{conv}(S_i)$ for all i. For each $i = 1, \ldots, m$, at least one of $b_{i1}, \ldots, b_{it_i}$ must be positive, so there are at most only $r + 1$ additional coefficients b_{ij} that can be positive (since at most $r + 1 + m$ of the b_{ij} are positive). It follows that for at least $m - r - 1$ indices i, we have $b_{ik} = 1$ for some k and $b_{ij} = 0$ for all $j \neq k$. For these indices, we have $s_i \in S_i$. **Q.E.D.**

Let us use the Shapley-Folkman Theorem to estimate the duality gap in the case of a linear program with integer 0-1 constraints:

$$\begin{aligned} &\text{minimize } \sum_{i=1}^{m} c_i x_i \\ &\text{subject to } \sum_{i=1}^{m} a_{ji} x_i \leq b_j, \quad j = 1, \ldots, r, \\ &\qquad x_i = 0 \text{ or } 1, \quad i = 1, \ldots, m. \end{aligned} \tag{5.48}$$

Let f^* and q^* denote the optimal primal and dual values, respectively. Note that the "relaxed" linear program, where the integer constraints are replaced by $x_i \in [0, 1]$, $i = 1, \ldots, m$, has the same dual, so its optimal value is q^*. The set S can be written as

$$S = S_1 + \cdots + S_m - (b, 0),$$

where $b = (b_1, \ldots, b_r)$ and each S_i consists of just two elements corresponding to $x_i = 0, 1$, i.e.,

$$S_i = \big\{ (0, \ldots, 0, 0), (a_{1i}, \ldots, a_{ri}, c_i) \big\}.$$

Thus, S consists of a total of 2^m points. A natural idea is to solve the "relaxed" program, and then try to suitably "round" the fractional (i.e., noninteger) components of the relaxed optimal solution to an integer, thereby obtaining a suboptimal solution of the original integer program (5.48).

Let us denote

$$\gamma = \max_{i=1,\ldots,m} |c_i|, \qquad \delta = \max_{i=1,\ldots,m} \delta_i,$$

where

$$\delta_i = \begin{cases} 0 & \text{if } a_{1i},\dots,a_{ri} \geq 0, \\ 0 & \text{if } a_{1i},\dots,a_{ri} \leq 0, \\ \max_{j=1\dots,r} |a_{ji}| & \text{otherwise.} \end{cases} \tag{5.49}$$

Note that δ_i is an upper bound to the maximum amount of constraint violation that can result when a fractional variable $x_i \in (0,1)$ is rounded suitably (up or down). The following proposition shows what can be achieved with such a rounding procedure.†

Proposition 5.7.2: Assume that the relaxed version of the linear/integer problem (5.48) is feasible. Then there exists $\overline{x} = (\overline{x}_1,\dots,\overline{x}_m)$, with $\overline{x}_i \in \{0,1\}$ for all i, which violates each of the inequality constraints of problem (5.48) by at most $(r+1)\delta$, and has cost that is at most $q^* + (r+1)\gamma$.

Proof: We note that the relaxed problem, being a feasible linear program with compact constraint set, has an optimal solution with optimal value q^*. Since

$$\text{conv}(S) = \text{conv}(S_1) + \cdots + \text{conv}(S_m) - (b,0),$$

we see that conv(S) is the set of constraint-cost pairs of the relaxed problem, so it contains a vector of the form (u^*, q^*) with $u^* \leq 0$. By the Shapley-Folkman Theorem, there is an index subset I with at most $r+1$ elements such that (u^*, q^*) can be written as

$$u^* = \sum_{i\in I} \overline{u}_i + \sum_{i\notin I} u_i, \qquad q^* = \sum_{i\in I} \overline{w}_i + \sum_{i\notin I} w_i,$$

where $(\overline{u}_i, \overline{w}_i) \in \text{conv}(S_i)$ for $i \in I$ and $(u_i, w_i) \in S_i$ for $i \notin I$. Each pair (u_i, w_i), $i \notin I$ corresponds to an integer component $\overline{x}_i \in \{0,1\}$. Each pair $(\overline{u}_i, \overline{w}_i)$, $i \in I$ may be replaced/rounded by one of the two elements (u_i, w_i) of S_i, yielding again an integer component $\overline{x}_i \in \{0,1\}$, with an increase in cost of at most γ and an increase of the level of each inequality constraint of at most δ_i. We thus obtain an integer vector $\overline{x}$ that violates each inequality constraint by at most $(r+1)\delta$, and has cost that is at most $q^* + (r+1)\gamma$. **Q.E.D.**

The preceding proof also indicates the rounding mechanism for obtaining the vector $\overline{x}$ in the proposition. In practice, the simplex method

† A slightly stronger bound [$r\gamma$ and $r\delta$ in place of $(r+1)\gamma$ and $(r+1)\delta$, respectively] can be shown with an alternative proof that uses the theory of the simplex method. However, the proof given here generalizes to the case where f_i and g_i are nonlinear (see the subsequent Prop. 5.7.4).

provides a relaxed problem solution with no more than r noninteger components, which are then rounded as indicated in the proof to obtain $\overline{x}$. Note that $\overline{x}$ may not be feasible, and indeed it is possible that the relaxed problem is feasible, while the original integer problem is not. For example the constraints $x_1 - x_2 \leq -1/2$, $x_1 + x_2 \leq 1/2$ are feasible for the relaxed problem but not for the original integer problem.

On the other hand if for each j, each of the constraint coefficients $a_{j1}, \ldots, a_{jm}$ is either 0 or has the same sign, we have $\delta = 0$ and a feasible solution that is $(r+1)\gamma$-optimal can be found. Assuming this condition, let us consider now a sequence of similar problems where the number of inequality constraints r is kept constant, but the dimension m grows to infinity. Assuming that

$$\beta_1 m \leq |f^*| \leq \beta_2 m$$

for some $\beta_1, \beta_2 > 0$, we see that the rounding error $(r+1)\gamma$ (which bounds the duality gap $f^* - q^*$) diminishes in relation to f^*, i.e., its ratio to f^* tends to 0 as $m \to \infty$. In particular, we have

$$\lim_{m \to \infty} \frac{f^* - q^*}{f^*} \to 0.$$

We will now generalize the line of analysis of the preceding proposition to the nonlinear separable problem (5.47) and obtain similar results. In particular, under assumptions that parallel the condition $\delta = 0$ discussed earlier, we will prove that the duality gap satisfies

$$f^* - q^* \leq (r+1) \max_{i=1,\ldots,m} \gamma_i,$$

where for each i, γ_i is a nonnegative scalar that depends on the functions f_i, g_i, and the set X_i. This suggests that as $m \to \infty$, the duality gap diminishes relative to f^* as $m \to \infty$. We first derive a general estimate for the case where the set M in the MC/MC framework is a vector sum, and then we specialize to the case of the separable problem.

Proposition 5.7.3: Consider the MC/MC framework corresponding to a set $M \subset \Re^{n+1}$ given by

$$M = M_1 + \cdots + M_m,$$

where the sets M_i have the form

$$M_i = \tilde{M}_i + \big\{(u, 0) \mid u \in C_i\big\}, \qquad i = 1, \ldots, m,$$

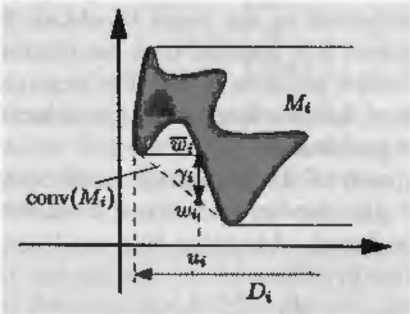


Figure 5.7.1. Geometric interpretation of the bound coefficient γ_i in Prop. 5.7.3 for the case where

$$M_i = \tilde{M}_i + \big\{(u,0) \mid u \in C_i\big\}$$

and

$$C_i = \{u \mid u \geq 0\}.$$

and for each i, $\tilde{M}_i$ is compact and C_i is convex, and such that $C_1 + \cdots + C_m$ is closed. For $i = 1, \ldots, m$, let

$$D_i = \big\{u_i \mid \text{there exists } w_i \text{ such that } (u_i, w_i) \in M_i\big\},$$

and assume that for each vector $(u_i, w_i) \in \text{conv}(M_i)$, we have $u_i \in D_i$. Let

$$\gamma_i = \sup_{u_i \in D_i} \inf\big\{\overline{w}_i - w_i \mid (u_i, \overline{w}_i) \in M_i,\ (u_i, w_i) \in \text{conv}(M_i)\big\}.$$

Then assuming that w^* and q^* are finite, we have

$$w^* - q^* \leq (n+1) \max_{i=1,\ldots,m} \gamma_i.$$

Proof: Note that the meaning of γ_i is that we can approximate any vector $(u_i, w_i) \in \text{conv}(M_i)$ with a vector $(u_i, \overline{w}_i) \in M_i$, while incurring an increase $\overline{w}_i - w_i$ of the vertical component w_i that is at most $\gamma_i + \epsilon$, where ϵ is arbitrarily small (cf. Fig. 5.7.1).

The idea of the proof is to use the Shapley-Folkman Theorem to approximate a vector $(u_1 + \cdots + u_m, w_1 + \cdots + w_m) \in \text{conv}(M)$ with a vector $(u_1 + \cdots + u_m, \overline{w}_1 + \cdots + \overline{w}_m) \in M$ such that $\overline{w}_i = w_i$ for all except at most $n+1$ indexes i for which $\overline{w}_i - w_i \leq \gamma_i + \epsilon$.

By the result of Prop. 4.3.2(c), q^* is finite and is equal to the optimal value of the MC/MC problem corresponding to

$$\text{conv}(M) = \text{conv}(\tilde{M}_1) + \cdots + \text{conv}(\tilde{M}_m) + \big\{(u,0) \mid u \in C_1 + \cdots + C_m\big\}.$$

Since $\tilde{M}_i$ is compact, so is $\text{conv}(\tilde{M}_i)$ (cf. Prop. 1.2.2). Since $C_1 + \cdots + C_m$ is closed, $\text{conv}(M)$ is closed (cf. Prop. 1.4.14), and it follows that the vector $(0, q^*)$ belongs to $\text{conv}(M)$. By the Shapley-Folkman Theorem (Prop.

5.7.1), there is an index subset I with at most $n+1$ elements such that

$$0 = \sum_{i=1}^{m} u_i, \qquad q^* = \sum_{i \in I} w_i + \sum_{i \notin I} \overline{w}_i,$$

where $(u_i, w_i) \in \text{conv}(M_i)$ for $i \in I$ and $(u_i, \overline{w}_i) \in M_i$ for $i \notin I$. By assumption, for any $\epsilon > 0$ and $i \in I$, there exist vectors of the form $(u_i, \overline{w}_i) \in M_i$ with $\overline{w}_i \leq w_i + \gamma_i + \epsilon$. Hence, $(0, \overline{w}) \in M$, where

$$\overline{w} = \sum_{i \in I} \overline{w}_i + \sum_{i \notin I} \overline{w}_i.$$

Thus, $w^* \leq \overline{w} = \sum_{i \in I} \overline{w}_i + \sum_{i \notin I} \overline{w}_i$, and it follows that

$$w^* \leq \sum_{i \in I} \overline{w}_i + \sum_{i \notin I} \overline{w}_i \leq \sum_{i \in I} (w_i + \gamma_i + \epsilon) + \sum_{i \notin I} \overline{w}_i \leq q^* + (n+1) \max_{i=1,\ldots,m} (\gamma_i + \epsilon).$$

By taking $\epsilon \downarrow 0$, the result follows. **Q.E.D.**

We now apply the preceding proposition to the general separable problem of this section.

Proposition 5.7.4: Consider the separable problem (5.47). Assume that the sets

$$\tilde{M}_i = \big\{ \big(g_i(x_i), f_i(x_i) \big) \mid x_i \in X_i \big\}, \qquad i = 1, \ldots, m,$$

are nonempty and compact, and that for each i, given any $x_i \in \text{conv}(X_i)$, there exists $\tilde{x}_i \in X_i$ such that

$$g_i(\tilde{x}_i) \leq (\check{\text{cl}}\, g_i)(x_i), \tag{5.50}$$

where $\check{\text{cl}}\, g_i$ is the function whose components are the convex closures of the corresponding components of g_i. Then

$$f^* - q^* \leq (r+1) \max_{i=1,\ldots,m} \gamma_i,$$

where

$$\gamma_i = \sup\big\{ \tilde{f}_i(x_i) - (\check{\text{cl}}\, f_i)(x_i) \mid x_i \in \text{conv}(X_i) \big\}, \tag{5.51}$$

$\check{\text{cl}}\, f_i$ is the convex closure of f_i, and $\tilde{f}_i$ is given by

$$\tilde{f}_i(x_i) = \inf\big\{ f_i(\tilde{x}_i) \mid g_i(\tilde{x}_i) \leq (\check{\text{cl}}\, g_i)(x_i),\ \tilde{x}_i \in X_i \big\}, \quad \forall\, x_i \in \text{conv}(X_i).$$

Note that Eq. (5.50) is a vector (rather than scalar) inequality, so it is not automatically satisfied. In the context of the integer programming problem (5.48), this inequality is satisfied if for each i, all the coefficients $a_{1i}, \ldots, a_{ri}$ have the same sign [cf. Eq. (5.49)]. It is also satisfied if X_i is convex and the components of g_i are convex over X_i. We will prove Prop. 5.7.4 by first showing the following lemma, which can be visualized with the aid of Fig. 5.7.1:

Lemma 5.7.1: Let $h : \Re^n \mapsto \Re^r$ and $\ell : \Re^n \mapsto \Re$ be given functions, and let X be a nonempty set. Denote

$$\tilde{M} = \{(h(x), \ell(x)) \mid x \in X\}, \qquad M = \tilde{M} + \{(u, 0) \mid u \geq 0\}.$$

Then every $(u, w) \in \mathrm{conv}(M)$ satisfies

$$(\check{\mathrm{cl}}\, h)(x) \leq u, \qquad (\check{\mathrm{cl}}\, \ell)(x) \leq w,$$

for some $x \in \mathrm{conv}(X)$, where $\check{\mathrm{cl}}\, \ell$ is the convex closure of ℓ, and $\check{\mathrm{cl}}\, h$ is the function whose components are the convex closures of the corresponding components of h.

Proof: Any $(u, w) \in \mathrm{conv}(M)$ can be written as

$$(u, w) = \sum_{j=1}^{s} \alpha_j \cdot (u_j, w_j),$$

where s is some positive integer and

$$\sum_{j=1}^{s} \alpha_j = 1, \qquad \alpha_j \geq 0, \quad (u_j, w_j) \in M, \quad j = 1, \ldots, r.$$

By the definition of M, there exists $x_j \in X$ such that $h(x_j) \leq u_j, \ell(x_j) = w_j$. Let $x = \sum_{j=1}^{s} \alpha_j x_j$, so that $x \in \mathrm{conv}(X)$. Then

$$(\check{\mathrm{cl}}\, h)(x) \leq \sum_{j=1}^{s} \alpha_j (\check{\mathrm{cl}}\, h)(x_j) \leq \sum_{j=1}^{s} \alpha_j h(x_j) \leq u,$$

$$(\check{\mathrm{cl}}\, \ell)(x) \leq \sum_{j=1}^{s} \alpha_j (\check{\mathrm{cl}}\, \ell)(x_j) \leq \sum_{j=1}^{s} \alpha_j \ell(x_j) = w,$$

where we used the definition of convex closure and Prop. 1.3.14. **Q.E.D.**

Proof of Prop. 5.7.4: Denote

$$M_i = \tilde{M}_i + \{(u, 0) \mid u \geq 0\},$$

and consider the MC/MC framework corresponding to

$$M = M_1 + \cdots + M_m.$$

By applying Prop. 4.3.2(c) with $C = \{u \mid u \geq 0\}$, we have

$$w^* = \inf_{(0,w)\in M} w, \qquad q^* = \inf_{(0,w)\in \mathrm{conv}(M)} w.$$

We will now apply the result of Prop. 5.7.3 to derive the desired bound.

To this end, we use Lemma 5.7.1 to assert that for each $i = 1, \ldots, m$ and vector $(u_i, w_i) \in \mathrm{conv}(M_i)$, there exists $x_i \in \mathrm{conv}(X_i)$ satisfying

$$(\check{\mathrm{cl}}\, g_i)(x_i) \leq u_i, \qquad (\check{\mathrm{cl}}\, f_i)(x_i) \leq w_i.$$

Since from the definition (5.51) of γ_i, we have $\tilde{f}_i(x_i) \leq (\check{\mathrm{cl}}\, f_i)(x_i) + \gamma_i$, we obtain

$$\tilde{f}_i(x_i) \leq w_i + \gamma_i.$$

It follows from the definition of $\tilde{f}_i$ that there exists $\tilde{x}_i \in X_i$ such that

$$g_i(\tilde{x}_i) \leq (\check{\mathrm{cl}}\, g_i)(x_i) \leq u_i, \qquad f_i(\tilde{x}_i) \leq w_i + \gamma_i.$$

Hence $\big(u_i, f_i(\tilde{x}_i)\big) \in M_i$, and by letting $\overline{w}_i = f_i(\tilde{x}_i)$, we have that for each vector $(u_i, w_i) \in \mathrm{conv}(M_i)$, there exists a vector of the form $(u_i, \overline{w}_i) \in M_i$ with $\overline{w}_i - w_i \leq \gamma_i$. By applying Prop. 5.7.3, we obtain

$$w^* - q^* \leq (r + 1) \max_{i=1,\ldots,m} \gamma_i.$$

Q.E.D.

As an example, consider the special case of the integer programming problem (5.48), where f_i and g_i are linear, and $X_i = \{0, 1\}$. Then the assumption (5.50) is equivalent to $\delta_i = 0$ [cf. Eq. (5.49)], and Props. 5.7.2 and 5.7.4 give the same estimate. For further discussion and visualization of the duality gap estimate of Prop. 5.7.4, we refer to [Ber82], Section 5.6.1. The estimate suggests that many nonconvex/integer separable problems with a fixed number of constraints become easier to solve as their dimension increases, because the corresponding duality gap diminishes.

5.7.2 Duality Gap in Minimax Problems

Consider a minimax problem involving a function $\phi : X \times Z \mapsto \Re$ defined over nonempty sets X and Z. We saw in Sections 4.2.5 and 5.5 that minimax theory can be developed within the context of the MC/MC framework involving the set

$$M = \mathrm{epi}(p),$$

where $p : \Re^m \mapsto [-\infty, \infty]$ is the function given by

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x, z) - u'z\}, \qquad u \in \Re^m.$$

Let us assume that $(-\hat{\mathrm{cl}}\,\phi)(x, \cdot)$ is proper for all $x \in X$, so that the corresponding dual function is

$$q(\mu) = \inf_{x \in X} (\hat{\mathrm{cl}}\,\phi)(x, \mu)$$

(cf. Prop. 4.2.1). Then we have

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \sup_{z \in Z} \inf_{x \in X} (\hat{\mathrm{cl}}\,\phi)(x, z) = \sup_{\mu \in \Re^m} q(\mu) = q^*.$$

Furthermore, $q^* = p^{\star\star}(0)$ [cf. Eq. (5.46)], so that

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq q^* = p^{\star\star}(0) \leq p(0) = w^* = \inf_{x \in X} \sup_{z \in Z} \phi(x, z).$$

Thus the gap between "infsup" and "supinf" can be decomposed into the sum of two terms:

$$\inf_{x \in X} \sup_{z \in Z} \phi(x, z) - \sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \overline{G} + \underline{G},$$

where

$$\overline{G} = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) - p^{\star\star}(0) = p(0) - p^{\star\star}(0),$$

$$\underline{G} = q^* - \sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \sup_{z \in \Re^m} \inf_{x \in X} (\hat{\mathrm{cl}}\,\phi)(x, z) - \sup_{z \in Z} \inf_{x \in X} \phi(x, z).$$

The term $\overline{G}$ is the duality gap $w^* - q^*$ of the MC/MC framework. It can be attributed to the lack of convexity and/or closure of p, and in view of the definition of p, it can also be attributed to lack of convexity and/or closure of ϕ with respect to x. The term $\underline{G}$ can be attributed to the lack of concavity and/or upper semicontinuity of ϕ with respect to z.

In cases where ϕ has a separable form in x, and ϕ is concave and upper semicontinuous with respect to z, we have $\underline{G} = 0$, while the Shapley-Folkman Theorem can be used to estimate $\overline{G}$, similar to Prop. 5.7.4. Similarly, we can estimate the duality gap if ϕ is separable in z, and ϕ is convex and lower semicontinuous with respect to x, or if ϕ is separable in both x and z.

APPENDIX A: Mathematical Background

In this appendix, we list some basic definitions, notational conventions, and results from linear algebra and real analysis. We assume that the reader is familiar with these subjects, so no proofs are given. For additional related material, we refer to textbooks such as Hoffman and Kunze [HoK71], Lancaster and Tismenetsky [LaT85], and Strang [Str76] (linear algebra), and Ash [Ash72], Ortega and Rheinboldt [OrR70], and Rudin [Rud76] (real analysis).

Set Notation

If X is a set and x is an element of X, we write $x \in X$. A set can be specified in the form $X = \{x \mid x \text{ satisfies } P\}$, as the set of all elements satisfying property P. The union of two sets X_1 and X_2 is denoted by $X_1 \cup X_2$, and their intersection by $X_1 \cap X_2$. The symbols $\exists$ and $\forall$ have the meanings "there exists" and "for all," respectively. The empty set is denoted by $\emptyset$.

The set of real numbers (also referred to as scalars) is denoted by $\Re$. The set $\Re$ augmented with $+\infty$ and $-\infty$ is called the *set of extended real numbers*. We write $-\infty < x < \infty$ for all real numbers x, and $-\infty \leq x \leq \infty$ for all extended real numbers x. We denote by $[a, b]$ the set of (possibly extended) real numbers x satisfying $a \leq x \leq b$. A rounded, instead of square, bracket denotes strict inequality in the definition. Thus $(a, b]$, $[a, b)$, and (a, b) denote the set of all x satisfying $a < x \leq b$, $a \leq x < b$, and $a < x < b$, respectively. Furthermore, we use the natural extensions of the rules of arithmetic: $x \cdot 0 = 0$ for every extended real number x, $x \cdot \infty = \infty$ if $x > 0$, $x \cdot \infty = -\infty$ if $x < 0$, and $x + \infty = \infty$ and $x - \infty = -\infty$ for every scalar x. The expression $\infty - \infty$ is meaningless and is never allowed to occur.

Inf and Sup Notation

The *supremum* of a nonempty set X of scalars, denoted by $\sup X$, is defined as the smallest scalar y such that $y \geq x$ for all $x \in X$. If no such scalar exists, we say that the supremum of X is ∞. Similarly, the *infimum* of X, denoted by $\inf X$, is defined as the largest scalar y such that $y \leq x$ for all $x \in X$, and is equal to $-\infty$ if no such scalar exists. For the empty set, we use the convention

$$\sup \emptyset = -\infty, \qquad \inf \emptyset = \infty.$$

If $\sup X$ is equal to a scalar $\overline{x}$ that belongs to the set X, we say that $\overline{x}$ is the *maximum point* of X and we write $\overline{x} = \max X$. Similarly, if $\inf X$ is equal to a scalar $\overline{x}$ that belongs to the set X, we say that $\overline{x}$ is the *minimum point* of X and we write $\overline{x} = \min X$. Thus, when we write $\max X$ (or $\min X$) in place of $\sup X$ (or $\inf X$, respectively), we do so just for emphasis: we indicate that it is either evident, or it is known through earlier analysis, or it is about to be shown that the maximum (or minimum, respectively) of the set X is attained at one of its points.

Vector Notation

We denote by $\Re^n$ the set of n-dimensional real vectors. For any $x \in \Re^n$, we use x_i to indicate its ith *coordinate*, also called its ith *component*. Vectors in $\Re^n$ will be viewed as column vectors, unless the contrary is explicitly stated. For any $x \in \Re^n$, x' denotes the transpose of x, which is an n-dimensional row vector. The *inner product* of two vectors $x, y \in \Re^n$ is defined by $x'y = \sum_{i=1}^n x_i y_i$. Two vectors $x, y \in \Re^n$ satisfying $x'y = 0$ are called *orthogonal*.

If x is a vector in $\Re^n$, the notations $x > 0$ and $x \geq 0$ indicate that all components of x are positive and nonnegative, respectively. For any two vectors x and y, the notation $x > y$ means that $x - y > 0$. The notations $x \geq y$, $x < y$, etc., are to be interpreted accordingly.

Function Notation

If f is a function, we use the notation $f : X \mapsto Y$ to indicate the fact that f is defined on a nonempty set X (its *domain*) and takes values in a set Y (its *range*). Thus when using the notation $f : X \mapsto Y$, we implicitly assume that X is nonempty. If $f : X \mapsto Y$ is a function, and U and V are subsets of X and Y, respectively, the set $\{f(x) \mid x \in U\}$ is called the *image* or *forward image of U under f*, and the set $\{x \in X \mid f(x) \in V\}$ is called the *inverse image of V under f*.

A.1 LINEAR ALGEBRA

If X is a set and λ is a scalar, we denote by λX the set $\{\lambda x \mid x \in X\}$. If X_1 and X_2 are two subsets of $\Re^n$, we denote by $X_1 + X_2$ the set

$$\{x_1 + x_2 \mid x_1 \in X_1,\, x_2 \in X_2\},$$

which is referred to as the *vector sum of* X_1 *and* X_2. We use a similar notation for the sum of any finite number of subsets. In the case where one of the subsets consists of a single vector $\overline{x}$, we simplify this notation as follows:

$$\overline{x} + X = \{\overline{x} + x \mid x \in X\}.$$

We also denote by $X_1 - X_2$ the set

$$\{x_1 - x_2 \mid x_1 \in X_1,\, x_2 \in X_2\}.$$

Given sets $X_i \subset \Re^{n_i}$, $i = 1, \ldots, m$, the *Cartesian product* of the X_i, denoted by $X_1 \times \cdots \times X_m$, is the set

$$\big\{(x_1, \ldots, x_m) \mid x_i \in X_i,\, i = 1, \ldots, m\big\},$$

which is viewed as a subset of $\Re^{n_1 + \cdots + n_m}$.

Subspaces and Linear Independence

A nonempty subset S of $\Re^n$ is called a *subspace* if $ax + by \in S$ for every $x, y \in S$ and every $a, b \in \Re$. An *affine set* in $\Re^n$ is a translated subspace, i.e., a set X of the form $X = \overline{x} + S = \{\overline{x} + x \mid x \in S\}$, where $\overline{x}$ is a vector in $\Re^n$ and S is a subspace of $\Re^n$, called the *subspace parallel to* X. Note that there can be only one subspace S associated with an affine set in this manner. [To see this, let $X = x + S$ and $X = \overline{x} + \overline{S}$ be two representations of the affine set X. Then, we must have $x = \overline{x} + \overline{s}$ for some $\overline{s} \in \overline{S}$ (since $x \in X$), so that $X = \overline{x} + \overline{s} + S$. Since we also have $X = \overline{x} + \overline{S}$, it follows that $S = \overline{S} - \overline{s} = \overline{S}$.] A nonempty set X is a subspace if and only if it contains the origin, and every line that passes through any pair of its points that are distinct, i.e., it contains 0 and all points $\alpha x + (1 - \alpha)y$, where $\alpha \in \Re$ and $x, y \in X$ with $x \neq y$. Similarly X is affine if and only if it contains every line that passes through any pair of its points that are distinct. The *span* of a finite collection $\{x_1, \ldots, x_m\}$ of elements of $\Re^n$, denoted by $\text{span}(x_1, \ldots, x_m)$, is the subspace consisting of all vectors y of the form $y = \sum_{k=1}^m \alpha_k x_k$, where each α_k is a scalar.

The vectors $x_1, \ldots, x_m \in \Re^n$ are called *linearly independent* if there exists no set of scalars $\alpha_1, \ldots, \alpha_m$, at least one of which is nonzero, such that $\sum_{k=1}^m \alpha_k x_k = 0$. An equivalent definition is that $x_1 \neq 0$, and for every $k > 1$, the vector x_k does not belong to the span of $x_1, \ldots, x_{k-1}$.

If S is a subspace of $\Re^n$ containing at least one nonzero vector, a *basis* for S is a collection of vectors that are linearly independent and whose span is equal to S. Every basis of a given subspace has the same number of vectors. This number is called the *dimension* of S. By convention, the subspace $\{0\}$ is said to have dimension zero. Every subspace of nonzero dimension has a basis that is orthogonal (i.e., any pair of distinct vectors from the basis is orthogonal). The *dimension of an affine set* $\overline{x} + S$ is the dimension of the corresponding subspace S. An $(n-1)$-dimensional affine set is called a *hyperplane*. It is a set specified by a single linear equation, i.e., a set of the form $\{x \mid a'x = b\}$, where $a \neq 0$ and $b \in \Re$.

Given any set X, the set of vectors that are orthogonal to all elements of X is a subspace denoted by $X^\perp$:

$$X^\perp = \{y \mid y'x = 0,\ \forall\ x \in X\}.$$

If S is a subspace, $S^\perp$ is called the *orthogonal complement* of S. Any vector x can be uniquely decomposed as the sum of a vector from S and a vector from $S^\perp$. Furthermore, we have $(S^\perp)^\perp = S$.

Matrices

For any matrix A, we use A_{ij}, $[A]_{ij}$, or a_{ij} to denote its ijth component. The *transpose* of A, denoted by A', is defined by $[A']_{ij} = a_{ji}$. For any two matrices A and B of compatible dimensions, the transpose of the product matrix AB satisfies $(AB)' = B'A'$. The inverse of a square and invertible A is denoted A^{-1}.

If X is a subset of $\Re^n$ and A is an $m \times n$ matrix, then the *image of X under A* is denoted by AX (or $A \cdot X$ if this enhances notational clarity):

$$AX = \{Ax \mid x \in X\}.$$

If Y is a subset of $\Re^m$, the *inverse image of Y under A* is denoted by $A^{-1}Y$:

$$A^{-1}Y = \{x \mid Ax \in Y\}.$$

Let A be a square matrix. We say that A is *symmetric* if $A' = A$. A symmetric matrix has real eigenvalues and an orthogonal set of corresponding real eigenvectors. We say that A is *diagonal* if $[A]_{ij} = 0$ whenever $i \neq j$. We use I to denote the identity matrix (this is the diagonal matrix whose diagonal components are equal to 1).

A symmetric $n \times n$ matrix A is called *positive definite* if $x'Ax > 0$ for all $x \in \Re^n$, $x \neq 0$. It is called *positive semidefinite* if $x'Ax \geq 0$ for all $x \in \Re^n$. Throughout this book, the notion of positive definiteness applies exclusively to symmetric matrices. Thus whenever we say that a matrix is positive (semi)definite, we implicitly assume that the matrix is symmetric, although we usually add the term "symmetric" for clarity. A positive semidefinite matrix A can be written as $A = M'M$ for some matrix M. A symmetric matrix is positive definite (or semidefinite) if and only if its eigenvalues are positive (nonnegative, respectively).

Let A be an $m \times n$ matrix. The *range space* of A, denoted by $R(A)$, is the set of all vectors $y \in \Re^m$ such that $y = Ax$ for some $x \in \Re^n$. The *nullspace* of A, denoted by $N(A)$, is the set of all vectors $x \in \Re^n$ such that $Ax = 0$. It is seen that the range space and the null space of A are subspaces. The *rank* of A is the dimension of the range space of A. The rank of A is equal to the maximal number of linearly independent columns of A, and is also equal to the maximal number of linearly independent rows of A. The matrix A and its transpose A' have the same rank. We say that A has *full rank*, if its rank is equal to $\min\{m, n\}$. This is true if and only if either all the rows of A are linearly independent, or all the columns of A are linearly independent. A symmetric matrix is positive (semi)definite if and only if its eignevalues are positive (nonnegative, respectively)

The range space of an $m \times n$ matrix A is equal to the orthogonal complement of the nullspace of its transpose, i.e., $R(A) = N(A')^\perp$. Another way to state this result is that given vectors $a_1, \ldots, a_n \in \Re^m$ (the columns of A) and a vector $x \in \Re^m$, we have $x'y = 0$ for all y such that $a_i'y = 0$ for all i if and only if $x = \lambda_1 a_1 + \cdots + \lambda_n a_n$ for some scalars $\lambda_1, \ldots, \lambda_n$. This is a special case of Farkas' Lemma, an important result for constrained optimization, which will be discussed in Section 2.3.1.

A function $f : \Re^n \mapsto \Re$ is said to be *affine* if it has the form $f(x) = a'x + b$ for some $a \in \Re^n$ and $b \in \Re$. Similarly, a function $f : \Re^n \mapsto \Re^m$ is said to be *affine* if it has the form $f(x) = Ax + b$ for some $m \times n$ matrix A and some $b \in \Re^m$. If $b = 0$, f is said to be a *linear function* or *linear transformation*. Sometimes, with slight abuse of terminology, an equation or inequality involving a linear function, such as $a'x = b$ or $a'x \leq b$, is referred to as a *linear equation or inequality*, respectively.

A.2 TOPOLOGICAL PROPERTIES

Definition A.2.1: A *norm* $\|\cdot\|$ on $\Re^n$ is a function that assigns a scalar $\|x\|$ to every $x \in \Re^n$ and that has the following properties:

(a) $\|x\| \geq 0$ for all $x \in \Re^n$.

(b) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for every scalar α and every $x \in \Re^n$.

(c) $\|x\| = 0$ if and only if $x = 0$.

(d) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \Re^n$ (this is referred to as the *triangle inequality*).

The *Euclidean norm* of a vector $x = (x_1, \ldots, x_n)$ is defined by

$$\|x\| = (x'x)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

We will use the Euclidean norm almost exclusively in this book. In particular, *in the absence of a clear indication to the contrary,* $\|\cdot\|$ *will denote the Euclidean norm.* The Schwarz inequality states that for any two vectors x and y, we have

$$|x'y| \leq \|x\| \cdot \|y\|,$$

with equality holding if and only if $x = \alpha y$ for some scalar α. The Pythagorean Theorem states that for any two vectors x and y that are orthogonal, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Sequences

A sequence $\{x_k \mid k = 1, 2, \ldots\}$ (or $\{x_k\}$ for short) of scalars is said to *converge* if there exists a scalar x such that for every $\epsilon > 0$ we have $|x_k - x| < \epsilon$ for every k greater than some integer K (that depends on ϵ). The scalar x is said to be the *limit* of $\{x_k\}$, and the sequence $\{x_k\}$ is said to *converge to* x; symbolically, $x_k \to x$ or $\lim_{k\to\infty} x_k = x$. If for every scalar b there exists some K (that depends on b) such that $x_k \geq b$ for all $k \geq K$, we write $x_k \to \infty$ and $\lim_{k\to\infty} x_k = \infty$. Similarly, if for every scalar b there exists some integer K such that $x_k \leq b$ for all $k \geq K$, we write $x_k \to -\infty$ and $\lim_{k\to\infty} x_k = -\infty$. Note, however, that implicit in any of the statements "$\{x_k\}$ converges" or "the limit of $\{x_k\}$ exists" or "$\{x_k\}$ has a limit" is that the limit of $\{x_k\}$ is a scalar.

A scalar sequence $\{x_k\}$ is said to be *bounded above* (respectively, *below*) if there exists some scalar b such that $x_k \leq b$ (respectively, $x_k \geq b$) for all k. It is said to be *bounded* if it is bounded above and bounded below. The sequence $\{x_k\}$ is said to be monotonically *nonincreasing* (respectively, *nondecreasing*) if $x_{k+1} \leq x_k$ (respectively, $x_{k+1} \geq x_k$) for all k. If $x_k \to x$ and $\{x_k\}$ is monotonically nonincreasing (nondecreasing), we also use the notation $x_k \downarrow x$ ($x_k \uparrow x$, respectively).

Proposition A.2.1: Every bounded and monotonically nonincreasing or nondecreasing scalar sequence converges.

Note that a monotonically nondecreasing sequence $\{x_k\}$ is either bounded, in which case it converges to some scalar x by the above proposition, or else it is unbounded, in which case $x_k \to \infty$. Similarly, a monotonically nonincreasing sequence $\{x_k\}$ is either bounded and converges, or it is unbounded, in which case $x_k \to -\infty$.

Given a scalar sequence $\{x_k\}$, let

$$y_m = \sup\{x_k \mid k \geq m\}, \qquad z_m = \inf\{x_k \mid k \geq m\}.$$

The sequences $\{y_m\}$ and $\{z_m\}$ are nonincreasing and nondecreasing, respectively, and therefore have a limit whenever $\{x_k\}$ is bounded above or is bounded below, respectively (Prop. A.2.1). The limit of y_m is denoted by $\limsup_{k\to\infty} x_k$, and is referred to as the *upper limit* of $\{x_k\}$. The limit of z_m is denoted by $\liminf_{k\to\infty} x_k$, and is referred to as the *lower limit* of $\{x_k\}$. If $\{x_k\}$ is unbounded above, we write $\limsup_{k\to\infty} x_k = \infty$, and if it is unbounded below, we write $\liminf_{k\to\infty} x_k = -\infty$.

Proposition A.2.2: Let $\{x_k\}$ and $\{y_k\}$ be scalar sequences.

(a) We have

$$\inf\{x_k \mid k \geq 0\} \leq \liminf_{k\to\infty} x_k \leq \limsup_{k\to\infty} x_k \leq \sup\{x_k \mid k \geq 0\}.$$

(b) $\{x_k\}$ converges if and only if

$$-\infty < \liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k < \infty.$$

Furthermore, if $\{x_k\}$ converges, its limit is equal to the common scalar value of $\liminf_{k\to\infty} x_k$ and $\limsup_{k\to\infty} x_k$.

(c) If $x_k \leq y_k$ for all k, then

$$\liminf_{k\to\infty} x_k \leq \liminf_{k\to\infty} y_k, \qquad \limsup_{k\to\infty} x_k \leq \limsup_{k\to\infty} y_k.$$

(d) We have

$$\liminf_{k\to\infty} x_k + \liminf_{k\to\infty} y_k \leq \liminf_{k\to\infty}(x_k + y_k),$$

$$\limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \geq \limsup_{k\to\infty}(x_k + y_k).$$

A sequence $\{x_k\}$ of vectors in $\Re^n$ is said to converge to some $x \in \Re^n$ if the ith component of x_k converges to the ith component of x for every i. We use the notations $x_k \to x$ and $\lim_{k\to\infty} x_k = x$ to indicate convergence for vector sequences as well. The sequence $\{x_k\}$ is called bounded if each of its corresponding component sequences is bounded. It can be seen that $\{x_k\}$ is bounded if and only if there exists a scalar c such that $\|x_k\| \leq c$ for all k. An infinite subset of a sequence $\{x_k\}$ is called a *subsequence* of $\{x_k\}$. Thus a subsequence can itself be viewed as a sequence, and can be represented as a set $\{x_k \mid k \in \mathcal{K}\}$, where $\mathcal{K}$ is an infinite subset of positive integers (the notation $\{x_k\}_{\mathcal{K}}$ will also be used).

A vector $x \in \Re^n$ is said to be a *limit point* of a sequence $\{x_k\}$ if there exists a subsequence of $\{x_k\}$ that converges to x.† The following is a classical result that will be used often.

Proposition A.2.3: (Bolzano-Weierstrass Theorem) A bounded sequence in $\Re^n$ has at least one limit point.

$o(\cdot)$ Notation

For a function $h : \Re^n \mapsto \Re^m$ we write $h(x) = o\big(\|x\|^p\big)$, where p is a positive integer, if

$$\lim_{k\to\infty} \frac{h(x_k)}{\|x_k\|^p} = 0,$$

for all sequences $\{x_k\}$ such that $x_k \to 0$ and $x_k \neq 0$ for all k.

Closed and Open Sets

We say that x is a *closure point of a subset* X of $\Re^n$ if there exists a sequence $\{x_k\} \subset X$ that converges to x. The *closure* of X, denoted cl(X), is the set of all closure points of X.

Definition A.2.2: A subset X of $\Re^n$ is called *closed* if it is equal to its closure. It is called *open* if its complement, $\{x \mid x \notin X\}$, is closed. It is called *bounded* if there exists a scalar c such that $\|x\| \leq c$ for all $x \in X$. It is called *compact* if it is closed and bounded.

Given $x^* \in \Re^n$ and $\epsilon > 0$, the sets $\big\{x \mid \|x - x^*\| < \epsilon\big\}$ and $\big\{x \mid \|x - x^*\| \leq \epsilon\big\}$ are called an *open sphere* and a *closed sphere* centered at

† Some authors prefer the alternative term "cluster point" of a sequence, and use the term "limit point of a set S" to indicate a point $\overline{x}$ such that $\overline{x} \notin S$ and there exists a sequence $\{x_k\} \subset S$ that converges to $\overline{x}$. With this terminology, $\overline{x}$ is a cluster point of a sequence $\{x_k \mid k = 1, 2, \ldots\}$ if and only if $(\overline{x}, 0)$ is a limit point of the set $\big\{(x_k, 1/k) \mid k = 1, 2, \ldots\big\}$. Our use of the term "limit point" of a sequence is quite popular in optimization and should not lead to any confusion.

x^*. Sometimes the terms *open ball* and *closed ball* are used, respectively. A consequence of the definitions, is that a subset X of $\Re^n$ is open if and only if for every $x \in X$ there is an open sphere that is centered at x and is contained in X. A *neighborhood* of a vector x is an open set containing x.

Definition A.2.3: We say that x is an *interior point* of a subset X of $\Re^n$ if there exists a neighborhood of x that is contained in X. The set of all interior points of X is called the *interior* of X, and is denoted by $\text{int}(X)$. A vector $x \in \text{cl}(X)$ which is not an interior point of X is said to be a *boundary point* of X. The set of all boundary points of X is called the *boundary* of X.

Proposition A.2.4:

(a) The union of a finite collection of closed sets is closed.

(b) The intersection of any collection of closed sets is closed.

(c) The union of any collection of open sets is open.

(d) The intersection of a finite collection of open sets is open.

(e) A set is open if and only if all of its elements are interior points.

(f) Every subspace of $\Re^n$ is closed.

(g) A set X is compact if and only if every sequence of elements of X has a subsequence that converges to an element of X.

(h) If $\{X_k\}$ is a sequence of nonempty and compact sets such that $X_{k+1} \subset X_k$ for all k, then the intersection $\cap_{k=0}^{\infty} X_k$ is nonempty and compact.

The topological properties of sets in $\Re^n$, such as being open, closed, or compact, do not depend on the norm being used. This is a consequence of the following proposition.

Proposition A.2.5: (Norm Equivalence Property)

(a) For any two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\Re^n$, there exists a scalar c such that

$$\|x\| \leq c\|x\|', \qquad \forall\, x \in \Re^n.$$

(b) If a subset of $\Re^n$ is open (respectively, closed, bounded, or compact) with respect to some norm, it is open (respectively, closed, bounded, or compact) with respect to all other norms.

Continuity

Let $f : X \mapsto \Re^m$ be a function, where X is a subset of $\Re^n$, and let x be a vector in X. If there exists a vector $y \in \Re^m$ such that the sequence $\{f(x_k)\}$ converges to y for every sequence $\{x_k\} \subset X$ such that $\lim_{k\to\infty} x_k = x$, we write $\lim_{z\to x} f(z) = y$. If there exists a vector $y \in \Re^m$ such that the sequence $\{f(x_k)\}$ converges to y for every sequence $\{x_k\} \subset X$ such that $\lim_{k\to\infty} x_k = x$ and $x_k \le x$ (respectively, $x_k \ge x$) for all k, we write $\lim_{z\uparrow x} f(z) = y$ [respectively, $\lim_{z\downarrow x} f(z)$].

Definition A.2.4: Let X be a subset of $\Re^n$.

(a) A function $f : X \mapsto \Re^m$ is called *continuous* at a vector $x \in X$ if $\lim_{z\to x} f(z) = f(x)$.

(b) A function $f : X \mapsto \Re^m$ is called *right-continuous* (respectively, *left-continuous*) at a vector $x \in X$ if $\lim_{z\downarrow x} f(z) = f(x)$ [respectively, $\lim_{z\uparrow x} f(z) = f(x)$].

(c) A real-valued function $f : X \mapsto \Re$ is called *upper semicontinuous* (respectively, *lower semicontinuous*) at a vector $x \in X$ if $f(x) \ge \limsup_{k\to\infty} f(x_k)$ [respectively, $f(x) \le \liminf_{k\to\infty} f(x_k)$] for every sequence $\{x_k\} \subset X$ that converges to x.

If $f : X \mapsto \Re^m$ is continuous at every vector in a subset of its domain X, we say that f *is continuous over that subset*. If $f : X \mapsto \Re^m$ is continuous at every vector in its domain X, we say that f *is continuous* (without qualification). We use similar terminology for right-continuous, left-continuous, upper semicontinuous, and lower semicontinuous functions.

Proposition A.2.6:

(a) Any vector norm on $\Re^n$ is a continuous function.

(b) Let $f : \Re^m \mapsto \Re^p$ and $g : \Re^n \mapsto \Re^m$ be continuous functions. The composition $f \cdot g : \Re^n \mapsto \Re^p$, defined by $(f \cdot g)(x) = f\big(g(x)\big)$, is a continuous function.

(c) Let $f : \Re^n \mapsto \Re^m$ be continuous, and let Y be an open (respectively, closed) subset of $\Re^m$. Then the inverse image of Y, $\{x \in \Re^n \mid f(x) \in Y\}$, is open (respectively, closed).

(d) Let $f : \Re^n \mapsto \Re^m$ be continuous, and let X be a compact subset of $\Re^n$. Then the image of X, $\{f(x) \mid x \in X\}$, is compact.

If $f : \Re^n \mapsto \Re$ is a continuous function and $X \subset \Re^n$ is compact, by Prop. A.2.6(b), the sets

$$V_\gamma = \{x \in X \mid f(x) \leq \gamma\}$$

are nonempty and compact for all $\gamma \in \Re$ with $\gamma > f^*$, where

$$f^* = \inf_{x \in X} f(x).$$

Since the set of minima of f is the intersection of the nonempty and compact sets V_{γ_k} for any sequence $\{\gamma_k\}$ with $\gamma_k \downarrow f^*$ and $\gamma_k > f^*$ for all k, it follows from Prop. A.2.4(h) that the set of minima is nonempty. This proves the classical theorem of Weierstrass, which is discussed further and generalized considerably in Section 3.2.

Proposition A.2.7: (Weierstrass' Theorem for Continuous Functions) A continuous function $f : \Re^n \mapsto \Re$ attains a minimum over any compact subset of $\Re^n$.

A.3 DERIVATIVES

Let $f : \Re^n \mapsto \Re$ be some function, fix some $x \in \Re^n$, and consider the expression

$$\lim_{\alpha \to 0} \frac{f(x + \alpha e_i) - f(x)}{\alpha},$$

where e_i is the ith unit vector (all components are 0 except for the ith component which is 1). If the above limit exists, it is called the ith *partial derivative* of f at the vector x and it is denoted by $(\partial f / \partial x_i)(x)$ or $\partial f(x)/\partial x_i$ (x_i in this section will denote the ith component of the vector x). Assuming all of these partial derivatives exist, the *gradient* of f at x is defined as the column vector

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}.$$

For any $d \in \Re^n$, we define the one-sided *directional derivative* of f at a vector x in the direction d by

$$f'(x;d) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha},$$

provided that the limit exists.

If the directional derivative of f at a vector x exists in all directions and $f'(x;d)$ is a linear function of d, we say that f is *differentiable* at x. It can be seen that f is differentiable at x if and only if the gradient $\nabla f(x)$ exists and satisfies $\nabla f(x)'d = f'(x;d)$ for all $d \in \Re^n$, or equivalently

$$f(x + \alpha d) = f(x) + \alpha \nabla f(x)'d + o(|\alpha|), \qquad \forall\, \alpha \in \Re.$$

The function f is called *differentiable over a subset S of $\Re^n$* if it is differentiable at every $x \in S$. The function f is called *differentiable* (without qualification) if it is differentiable at all $x \in \Re^n$.

If f is differentiable over an open set S and $\nabla f(\cdot)$ is continuous at all $x \in S$, f is said to be *continuously differentiable over S*. It can then be shown that for any $x \in S$ and norm $\|\cdot\|$,

$$f(x + d) = f(x) + \nabla f(x)'d + o(\|d\|), \qquad \forall\, d \in \Re^n.$$

If each one of the partial derivatives of a function $f : \Re^n \mapsto \Re$ is a continuously differentiable function of x over an open set S, we say that f is *twice continuously differentiable* over S. We then denote by

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

the ith partial derivative of $\partial f / \partial x_j$ at a vector $x \in \Re^n$. The *Hessian* of f at x, denoted by $\nabla^2 f(x)$, is the matrix whose components are the above second derivatives. The matrix $\nabla^2 f(x)$ is symmetric. The following theorem will be useful to us.

Proposition A.3.1: (Mean Value Theorem) Let $f : \Re^n \mapsto \Re$ be differentiable over an open sphere S, let x be a vector in S, and let d be such that $x + d \in S$. Then there exists an $\alpha \in [0,1]$ such that

$$f(x + d) = f(x) + \nabla f(x + \alpha d)'d.$$

If in addition f is twice continuously differentiable over S, there exists an $\alpha \in [0,1]$ such that

$$f(x + d) = f(x) + \nabla f(x)'d + \tfrac{1}{2} d' \nabla^2 f(x + \alpha d) d.$$

Notes and Sources

There is a very extensive literature on convex analysis and optimization and it is beyond our scope to give a complete bibliography. We are providing instead a brief historical account and list some of the main textbooks in the field.

Among early classical works on convexity, we mention Caratheodory [Car11], Minkowski [Min11], and Steinitz [Ste13], [Ste14], [Ste16]. In particular, Caratheodory gave the theorem on convex hulls that carries his name, while Steinitz developed the theory of relative interiors and recession cones. Minkowski is credited with initiating the theory of hyperplane separation of convex sets and the theory of support functions (a precursor to conjugate convex functions). Furthermore, Minkowski and Farkas (whose work, published in Hungarian, spans a 30-year period starting around 1894), are credited with laying the foundations of polyhedral convexity.

The work of Fenchel was instrumental in launching the modern era of convex analysis, when the subject came to a sharp focus thanks to its rich applications in optimization and game theory. In his 1951 lecture notes [Fen51], Fenchel laid the foundations of convex duality theory, and together with related works by von Neumann [Neu28], [Neu37] on saddle points and game theory, and Kuhn and Tucker on nonlinear programming [KuT51], inspired much subsequent work on convexity and its connections with optimization. Furthermore, Fenchel developed several of the topics that are fundamental in our exposition, such as the theory of conjugate convex functions (introduced earlier in a more limited form by Legendre), and the theory of subgradients.

There are several books that relate to both convex analysis and optimization. The book by Rockafellar [Roc70], widely viewed as the classic convex analysis text, contains a detailed development. It has been a major influence to subsequent convex optimization books, including the present work. The book by Rockafellar and Wets [RoW98] is an extensive treatment of "variational analysis," a broad spectrum of topics that integrate classical analysis, convexity, and optimization of both convex and nonconvex (possibly nonsmooth) functions. Stoer and Witzgall [StW70] discuss similar topics as Rockafellar [Roc70] but less comprehensively. Ekeland and Temam [EkT76], and Zalinescu [Zal02] develop the subject in infinite dimensional spaces. Hiriart-Urruty and Lemarechal [HiL93] emphasize algorithms for dual and nondifferentiable optimization. Rockafellar [Roc84] focuses on convexity and duality in network optimization, and an important generalization, called monotropic programming. Bertsekas [Ber98] also gives a detailed coverage of this material, which owes much to the early work of Minty [Min60] on network optimization. Schrijver [Sch86] provides an extensive account of polyhedral convexity with applications to integer programming and combinatorial optimization, and gives many historical references. Bonnans and Shapiro [BoS00] emphasize sensitivity analysis and discuss infinite dimensional problems as well. Borwein and Lewis [BoL00] develop many of the concepts in Rockafellar and Wets [RoW98], but more succinctly. The author's earlier book with Nedić and Ozdaglar [BNO03] also straddles the boundary between convex and variational analysis. Ben-Tal and Nemirovski [BeN01] focus on conic and semidefinite programming [see also the 2005 class notes by Nemirovski (on line)]. Auslender and Teboulle [AuT03] emphasize the question of existence of solutions for convex as well as nonconvex optimization problems, and related issues in duality theory and variational inequalities. Boyd and Vanderbergue [BoV04] discuss many applications of convex optimization.

We also note a few books that focus on the geometry and other properties of convex sets, but have limited connection with duality and optimization: Bonnesen and Fenchel [BoF34], Eggleston [Egg58], Klee [Kle63], Valentine [Val64], Grunbaum [Gru67], Webster [Web94], and Barvinok [Bar02].

The MC/MC framework was initially developed by the author in joint research with A. Nedić and A. Ozdaglar, which is described in the book [BNO03]. The present account is improved and more comprehensive. In particular, it contains more streamlined proofs and some new results, particularly in connection with minimax problems (Sections 4.2.5 and 5.7.2), and nonconvex problems (Section 5.7, which generalizes the work on duality gap estimates in [Ber82], Section 5.6.1).

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