Chapter 5

Vector Spaces and Subspaces

5.1 The Column Space of a Matrix

To a newcomer, matrix calculations involve a lot of numbers. To you, they involve vectors. The columns of $A\boldsymbol{v}$ and AB are linear combinations of n vectors—the columns of A. This chapter moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual columns, we look at "spaces" of vectors. Without seeing ***vector spaces*** and their ***subspaces***, you haven't understood everything about $A\boldsymbol{v} = \boldsymbol{b}$.

Since this chapter goes a little deeper, it may seem a little harder. That is natural. We are looking inside the calculations, to find the mathematics. The author's job is to make it clear. Section 5.5 will present the "*Fundamental Theorem of Linear Algebra*."

We begin with the most important vector spaces. They are denoted by $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$, ... Each space $\mathbf{R}^n$ consists of a whole collection of vectors. $\mathbf{R}^5$ contains all column vectors with five components. This is called "5-dimensional space."

DEFINITION ***The space $\mathbf{R}^n$ consists of all column vectors $\boldsymbol{v}$ with n components.***

The components of $\boldsymbol{v}$ are real numbers, which is the reason for the letter $\mathbf{R}$. When the n components are complex numbers, $\boldsymbol{v}$ lies in the space $\mathbf{C}^n$.

The vector space $\mathbf{R}^2$ is represented by the usual xy plane. Each vector $\boldsymbol{v}$ in $\mathbf{R}^2$ has two components. The word "*space*" asks us to think of all those vectors—the whole plane. Each vector gives the x and y coordinates of a point in the plane: $\boldsymbol{v} = (x, y)$.

Similarly the vectors in $\mathbf{R}^3$ correspond to points (x, y, z) in three-dimensional space. The one-dimensional space $\mathbf{R}^1$ is a line (like the x axis). As before, we print vectors as a column between brackets, or along a line using commas and parentheses:

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \text{ is in } \mathbf{R}^2, \quad (1,1,0,1,1) \text{ is in } \mathbf{R}^5, \quad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \text{ is in } \mathbf{C}^2.$$

The great thing about linear algebra is that it deals easily with five-dimensional space. We don't draw the vectors, we just need the five numbers (or n numbers).

To multiply $\boldsymbol{v}$ by 7, multiply every component by 7. Here 7 is a "scalar." To add vectors in $\mathbf{R}^5$, add them a component at a time: five additions. The two essential vector operations go on *inside the vector space*, and they produce ***linear combinations***:

We can add any vectors in $\mathbf{R}^n$, ***and we can multiply any vector*** v ***by any scalar*** c.

"Inside the vector space" means that ***the result stays in the space***: This is crucial.

If $\boldsymbol{v}$ is in $\mathbf{R}^4$ with components $1, 0, 0, 1$, then $2\boldsymbol{v}$ is the vector in $\mathbf{R}^4$ with components $2, 0, 0, 2$. (In this case 2 is the scalar.) A whole series of properties can be verified in $\mathbf{R}^n$. The commutative law is $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$; the distributive law is $c(\boldsymbol{v} + \boldsymbol{w}) = c\boldsymbol{v} + c\boldsymbol{w}$. Every vector space has a unique "zero vector" satisfying $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$. Those are three of the eight conditions listed in the Chapter 5 Notes.

These eight conditions are required of every vector space. There are vectors other than column vectors, and there are vector spaces other than $\mathbf{R}^n$. All vector spaces have to obey the eight reasonable rules.

A real vector space is a set of "vectors" *together with rules for vector addition and multiplication by real numbers.* The addition and the multiplication must produce vectors that are in the space. And the eight conditions must be satisfied (which is usually no problem). You need to see three vector spaces other than $\mathbf{R}^n$:

$\mathbf{M}$	The vector space of ***all real*** 2 ***by*** 2 ***matrices.***
$\mathbf{Y}$	The vector space of ***all solutions*** $y(t)$ to $Ay'' + By' + Cy = 0$.
$\mathbf{Z}$	The vector space that consists only of a ***zero vector***.

In $\mathbf{M}$ the "vectors" are really matrices. In $\mathbf{Y}$ the vectors are functions of t, like $y = e^{st}$. In $\mathbf{Z}$ the only addition is $\mathbf{0} + \mathbf{0} = \mathbf{0}$. In each space we can add: matrices to matrices, functions to functions, zero vector to zero vector. We can multiply a matrix by 4 or a function by 4 or the zero vector by 4. The result is still in $\mathbf{M}$ or $\mathbf{Y}$ or $\mathbf{Z}$.

The space $\mathbf{R}^4$ is four-dimensional, and so is the space $\mathbf{M}$ of 2 by 2 matrices. Vectors in those spaces are determined by four numbers. The solution space $\mathbf{Y}$ is two-dimensional, because second order differential equations have two independent solutions. Section 5.4 will pin down those key words, *independence of vectors* and *dimension of a space*.

The space $\mathbf{Z}$ is zero-dimensional (by any reasonable definition of dimension). It is the smallest possible vector space. We hesitate to call it $\mathbf{R}^0$, which means no components—you might think there was no vector. *The vector space* $\mathbf{Z}$ *contains exactly one vector.* No space can do without that zero vector. Each space has its own zero vector—the zero matrix, the zero function, the vector $(0, 0, 0)$ in $\mathbf{R}^3$.

Subspaces

At different times, we will ask you to think of matrices and functions as vectors. But at all times, the vectors that we need most are ordinary column vectors. They are vectors with n components—but *maybe not all* of the vectors with n components. There are important vector spaces ***inside*** $\mathbf{R}^n$. Those are ***subspaces*** of $\mathbf{R}^n$.

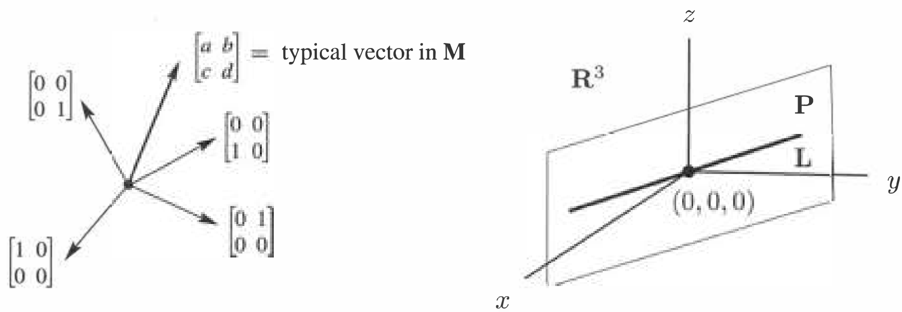


Figure 5.1: "4-dimensional" matrix space $\mathbf{M}$. 3 subspaces of $\mathbf{R^3}$: plane $\mathbf{P}$, line $\mathbf{L}$, point $\mathbf{Z}$.

Start with the usual three-dimensional space $\mathbf{R}^3$. Choose a plane through the origin $(0, 0, 0)$. ***That plane is a vector space in its own right.*** If we add two vectors in the plane, their sum is in the plane. If we multiply an in-plane vector by 2 or -5, it is still in the plane. A plane in three-dimensional space is not $\mathbf{R}^2$ (even if it looks like $\mathbf{R}^2$). The vectors have three components and they belong to $\mathbf{R}^3$. The plane $\mathbf{P}$ is a vector space ***inside*** $\mathbf{R}^3$.

This illustrates one of the most fundamental ideas in linear algebra. The plane going through $(0, 0, 0)$ is a ***subspace*** of the full vector space $\mathbf{R}^3$.

DEFINITION A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: ***If*** $\boldsymbol{v}$ ***and*** $\boldsymbol{w}$ ***are vectors in the subspace and*** c ***is any scalar, then***

(i) $\boldsymbol{v} + \boldsymbol{w}$ is in the subspace and **(ii)** $c\boldsymbol{v}$ is in the subspace.

In other words, the set of vectors is "closed" under addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication $c\boldsymbol{v}$ (and $d\boldsymbol{w}$). Those operations leave us in the subspace. We can also subtract, because $-\boldsymbol{w}$ is in the subspace and its sum with $\boldsymbol{v}$ is $\boldsymbol{v} - \boldsymbol{w}$. In short, ***all linear combinations*** $c\boldsymbol{v} + d\boldsymbol{w}$ ***stay in the subspace***.

First fact: ***Every subspace contains the zero vector***. The plane in $\mathbf{R}^3$ has to go through $(0, 0, 0)$. We mention this separately, for extra emphasis, but it follows directly from rule **(ii)**. Choose $c = 0$, and the rule requires $0\boldsymbol{v}$ to be in the subspace.

Planes that don't contain the origin fail those tests. When $\boldsymbol{v}$ is on such a plane, $-\boldsymbol{v}$ and $0\boldsymbol{v}$ are *not* on the plane. A plane that misses the origin is not a subspace.

Lines through the origin are also subspaces. When we multiply by 5, or add two vectors on the line, we stay on the line. But the line must go through $(0, 0, 0)$.

Another subspace is all of $\mathbf{R}^3$. The whole space is a subspace (*of itself*). That is a fourth subspace in the figure. Here is a list of all the possible subspaces of $\mathbf{R}^3$:

($\mathbf{L}$) Any line through $(0, 0, 0)$ ($\mathbf{R}^3$) The whole space
($\mathbf{P}$) Any plane through $(0, 0, 0)$ ($\mathbf{Z}$) The single vector $(0, 0, 0)$

If we try to keep only *part* of a plane or line, the requirements for a subspace don't hold. Look at these examples in $\mathbf{R}^2$.

Example 1 Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane). The vector $(2, 3)$ is included but $(-2, -3)$ is not. So rule (**ii**) is violated when we try to multiply by $c = -1$. ***The quarter-plane is not a subspace.***

Example 2 Include also the vectors whose components are both negative. Now we have two quarter-planes. Requirement (**ii**) is satisfied; we can multiply by any c. But rule (**i**) now fails. The sum of $\boldsymbol{v} = (2, 3)$ and $\boldsymbol{w} = (-3, -2)$ is $(-1, 1)$, which is outside the quarter-planes. ***Two quarter-planes don't make a subspace.***

Rules (**i**) and (**ii**) involve vector addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication by scalars like c and d. The rules can be combined into a single requirement—*the rule for subspaces*:

A subspace containing v and w must contain all linear combinations $cv + dw$.

Example 3 Inside the vector space $\mathbf{M}$ of all 2 by 2 matrices, here are two subspaces:

$$(\mathbf{U}) \text{ All upper triangular matrices } \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \qquad (\mathbf{D}) \text{ All diagonal matrices } \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Add any two matrices in $\mathbf{U}$, and the sum is in $\mathbf{U}$. Add diagonal matrices, and the sum is diagonal. In this case $\mathbf{D}$ is also a subspace of $\mathbf{U}$! The zero matrix alone is also a subspace, when a, b, and d all equal zero.

For a smaller subspace of diagonal matrices, we could require $a = d$. The matrices are multiples of the identity matrix I. These aI form a "line of matrices" in $\mathbf{M}$ and $\mathbf{U}$ and $\mathbf{D}$.

Is the matrix I a subspace by itself? Certainly not. Only the zero matrix is. Your mind will invent more subspaces of 2 by 2 matrices—write them down for Problem 6.

The Column Space of A

The most important subspaces are tied directly to a matrix A. We are trying to solve $A\boldsymbol{v} = \boldsymbol{b}$. If A is not invertible, the system is solvable for some $\boldsymbol{b}$ and not solvable for other $\boldsymbol{b}$. We want to describe the good right sides $\boldsymbol{b}$—the vectors that *can* be written as A times $\boldsymbol{v}$. Those $\boldsymbol{b}'s$ form the "*column space*" of A.

Remember that $A\boldsymbol{v}$ is a combination of the columns of A. To get every possible $\boldsymbol{b}$, we use every possible $\boldsymbol{v}$. Start with the columns of A, and ***take all their linear combinations. This produces the column space of A.*** It contains not just the n columns of A!

DEFINITION ***The column space consists of all combinations of the columns.***

The combinations are all possible vectors $A\boldsymbol{v}$. They fill the column space $\boldsymbol{C}(A)$.

This column space is crucial to the whole book, and here is why. ***To solve $A\boldsymbol{v} = \boldsymbol{b}$ is to express $\boldsymbol{b}$ as a combination of the columns. The right side $\boldsymbol{b}$ has to be in the column space*** produced by A on the left side. If $\boldsymbol{b}$ is not in $\boldsymbol{C}(A)$, $A\boldsymbol{v} = \boldsymbol{b}$ has no solution.

The system $Av = b$ is solvable if and only if b is in the column space of A.

When $\boldsymbol{b}$ is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution $\boldsymbol{v}$ to the system $A\boldsymbol{v} = \boldsymbol{b}$.

Suppose A is an m by n matrix. Its columns have m components (not n). So the columns belong to $\mathbf{R}^m$. ***The column space of A is a subspace of*** $\mathbf{R}^m$ ***(not*** $\mathbf{R}^n$**).** The set of all column combinations $A\boldsymbol{x}$ satisfies rules (**i**) and (**ii**) for a subspace: When we add linear combinations or multiply by scalars, we still produce combinations of the columns. The word "subspace" is always justified *by taking all linear combinations.*

Here is a 3 by 2 matrix A, whose column space is a subspace of $\mathbf{R}^3$. The column space of A is a plane in Figure 5.2.

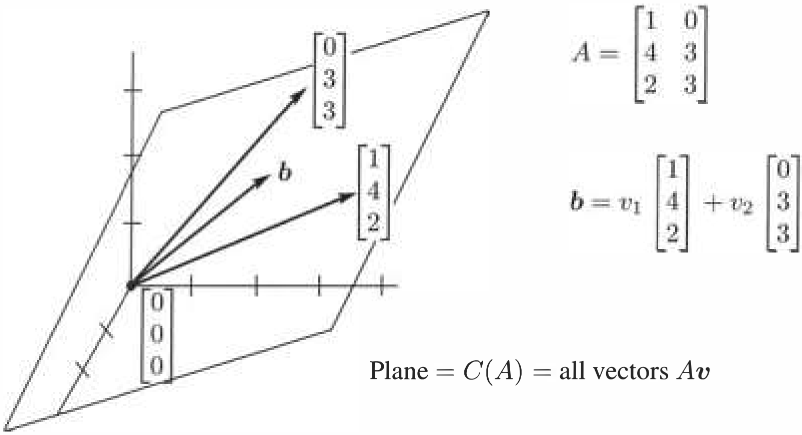


Figure 5.2: The column space $\boldsymbol{C}(A)$ is a plane containing the two columns of A. $A\boldsymbol{v} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is on that plane. Then $\boldsymbol{b}$ is a combination of the columns.

We drew one particular $\boldsymbol{b}$ (a combination of the columns). This $\boldsymbol{b} = A\boldsymbol{v}$ lies on the plane. The plane has zero thickness, so most right sides $\boldsymbol{b}$ in $\mathbf{R}^3$ are *not* in the column space. For most $\boldsymbol{b}$ there is no solution to our 3 equations in 2 unknowns.

Of course $(0, 0, 0)$ is in the column space. The plane passes through the origin. There is certainly a solution to $A\boldsymbol{v} = \mathbf{0}$. That solution, always available, is $\boldsymbol{v} =$ _____.

To repeat, the attainable right sides $\boldsymbol{b}$ are exactly the vectors in the column space. One possibility is the first column itself—take $v_1 = 1$ and $v_2 = 0$. Another combination is the second column—take $v_1 = 0$ and $v_2 = 1$. The new level of understanding is to see *all* combinations—the whole subspace is generated by those two columns.

Notation The column space of A is denoted by $\boldsymbol{C}(A)$. Start with the columns and take all their linear combinations. We might get the whole $\mathbf{R}^m$ or only a small subspace.

Important Instead of columns in $\mathbf{R}^m$, we could start with any set of vectors in a vector space $\mathbf{V}$. To get a subspace $\mathbf{SS}$ of $\mathbf{V}$, we take *all combinations* of the vectors in that set:

$$\begin{aligned} \mathbf{S} &= \text{set of vectors } \boldsymbol{s} \text{ in } \mathbf{V} \text{ (S is probably } \textit{not} \text{ a subspace)} \\ \mathbf{SS} &= \text{all combinations of vectors in S (SS } \textit{is} \text{ a subspace)} \end{aligned}$$

$$\mathbf{SS} = \boxed{\text{all } c_1\boldsymbol{s}_1 + \cdots + c_N\boldsymbol{s}_N} = \boxed{\textbf{the subspace of V "spanned" by S}}$$

When $\mathbf{S}$ is the set of columns, $\mathbf{SS}$ is the column space. When there is only one nonzero vector $\boldsymbol{v}$ in $\mathbf{S}$, the subspace $\mathbf{SS}$ is the line through $\boldsymbol{v}$. *Always* $\mathbf{SS}$ *is the smallest subspace containing* $\mathbf{S}$. This is a fundamental way to create subspaces and we will come back to it.

The subspace SS is the "span" of S, containing all combinations of vectors in S.

Example 4 Describe the column spaces (they are subspaces of $\mathbf{R}^2$) for these matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution The column space of I is the *whole space* $\mathbf{R}^2$. Every vector is a combination of the columns of I. In vector space language, $\boldsymbol{C}(I)$ equals $\mathbf{R}^2$.

The column space of A is only a line. The second column $(2,4)$ is a multiple of the first column $(1,2)$. Those vectors are different, but our eye is on vector ***spaces***. The column space contains $(1,2)$ and $(2,4)$ and all other vectors $(c, 2c)$ along that line. The equation $A\boldsymbol{v} = \boldsymbol{b}$ is only solvable when $\boldsymbol{b}$ is on the line.

For the third matrix (with three columns) the column space $\boldsymbol{C}(B)$ is all of $\mathbf{R}^2$. Every $\boldsymbol{b}$ is attainable. The vector $\boldsymbol{b} = (5,4)$ is column 2 plus column 3, so $\boldsymbol{v}$ can be $(0,1,1)$. The same vector $(5,4)$ is also 2(column 1) + column 3, so another possible $\boldsymbol{v}$ is $(2,0,1)$. This matrix has the same column space as I—any $\boldsymbol{b}$ is allowed. But now $\boldsymbol{v}$ has extra components and $A\boldsymbol{v} = \boldsymbol{b}$ has more solutions—more combinations that give $\boldsymbol{b}$.

The next section creates the *nullspace* $\boldsymbol{N}(A)$, to describe all the solutions of $A\boldsymbol{v} = \mathbf{0}$. This section created the column space $\boldsymbol{C}(A)$, to describe all the attainable right sides $\boldsymbol{b}$.

■ REVIEW OF THE KEY IDEAS ■

1. $\mathbf{R}^n$ contains all column vectors with n real components.

2. $\mathbf{M}$ (2 by 2 matrices) and $\mathbf{Y}$ (functions) and $\mathbf{Z}$ (zero vector alone) are vector spaces.

3. A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all their combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

4. The combinations of the columns of A form the ***column space*** $\boldsymbol{C}(A)$. Then the column space is "spanned" by the columns.

5. $A\boldsymbol{v} = \boldsymbol{b}$ has a solution exactly when $\boldsymbol{b}$ is in the column space of A.

WORKED EXAMPLES

5.1 A We are given three different vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$. Construct a matrix so that the equations $A\boldsymbol{v} = \boldsymbol{b}_1$ and $A\boldsymbol{v} = \boldsymbol{b}_2$ are solvable, but $A\boldsymbol{v} = \boldsymbol{b}_3$ is *not* solvable. How can you decide if this is possible? How could you construct A?

Solution We want to have $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in the column space of A. Then $A\boldsymbol{v} = \boldsymbol{b}_1$ and $A\boldsymbol{v} = \boldsymbol{b}_2$ will be solvable. *The quickest way is to make $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ the two columns of A.* Then the solutions are $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{v} = (0, 1)$.

Also, we don't want $A\boldsymbol{v} = \boldsymbol{b}_3$ to be solvable. So don't make the column space any larger! Keeping only the columns $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, the question is: *Do we already have $\boldsymbol{b}_3$?*

$$\text{Is } A\boldsymbol{v} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \boldsymbol{b}_3 \text{ solvable?} \qquad \text{Is } \boldsymbol{b}_3 \text{ a combination of } \boldsymbol{b}_1 \text{ and } \boldsymbol{b}_2\text{?}$$

If the answer is *no*, we have the desired matrix A. If $\boldsymbol{b}_3$ *is* a combination of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, then it is *not possible* to construct A. The column space $\boldsymbol{C}(A)$ will have to contain $\boldsymbol{b}_3$.

5.1 B Describe a subspace $\mathbf{S}$ of each vector space $\mathbf{V}$, and then a subspace $\mathbf{SS}$ of $\mathbf{S}$.

$$\begin{aligned} \mathbf{V}_3 &= \text{all combinations of } (1,1,0,0) \text{ and } (1,1,1,0) \text{ and } (1,1,1,1) \\ \mathbf{V}_2 &= \text{all vectors } \boldsymbol{v} \text{ perpendicular to } \boldsymbol{u} = (1,2,1), \text{ so } \boldsymbol{u} \cdot \boldsymbol{v} = 0 \\ \mathbf{V}_4 &= \text{all solutions } y(x) \text{ to the equation } d^4y/dx^4 = 0 \end{aligned}$$

Describe each $\mathbf{V}$ two ways: (1) *All combinations of* .... (2) *All solutions of* ....

Solution $\mathbf{V}_3$ starts with three vectors. A subspace $\mathbf{S}$ comes from all combinations of the first two vectors $(1,1,0,0)$ and $(1,1,1,0)$. A subspace $\mathbf{SS}$ of $\mathbf{S}$ comes from all multiples $(c, c, 0, 0)$ of the first vector. So many possibilities.

A subspace $\mathbf{S}$ of $\mathbf{V}_2$ is the line through $(1, -1, 1)$. This line is perpendicular to $\boldsymbol{u}$. The zero vector $\boldsymbol{z} = (0,0,0)$ is in $\mathbf{S}$. The smallest subspace $\mathbf{SS}$ is $\mathbf{Z}$.

$\mathbf{V}_4$ contains all cubic polynomials $y = a + bx + cx^2 + dx^3$, with $d^4y/dx^4 = 0$. The quadratic polynomials (without an x^3 term) give a subspace $\mathbf{S}$. The linear polynomials are one choice of $\mathbf{SS}$. The constants $y = a$ could be $\mathbf{SSS}$.

In all three parts we could take $\mathbf{S} = \mathbf{V}$ itself, and $\mathbf{SS} =$ the zero subspace $\mathbf{Z}$.

Each $\mathbf{V}$ can be described as *all combinations of* .... and as *all solutions of* ....:

$$\begin{aligned} \mathbf{V}_3 &= \text{all combinations of the 3 vectors} & \mathbf{V}_3 &= \text{all solutions of } v_1 - v_2 = 0. \\ \mathbf{V}_2 &= \text{all combinations of } (1,0,-1) \text{ and } (1,-1,1) & \mathbf{V}_2 &= \text{all solutions of } \boldsymbol{u} \cdot \boldsymbol{v} = 0. \\ \mathbf{V}_4 &= \text{all combinations of } 1, x, x^2, x^3 & \mathbf{V}_4 &= \text{all solutions to } d^4y/dx^4 = 0. \end{aligned}$$

Problem Set 5.1

Questions 1–10 are about the "subspace requirements": $v + w$ and cv (and then all linear combinations $cv + dw$) stay in the subspace.

1 One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbf{R}^2$ for which $\boldsymbol{v} + \boldsymbol{w}$ stays in the set but $\frac{1}{2}\boldsymbol{v}$ may be outside.

(b) A set of vectors in $\mathbf{R}^2$ (other than two quarter-planes) for which every $c\boldsymbol{v}$ stays in the set but $\boldsymbol{v} + \boldsymbol{w}$ may be outside.

2 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces ?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1, 4, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \le b_2 \le b_3$.

3 Describe the smallest subspace of the matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4 Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

5 Let $\mathbf{P}_0$ be the plane through $(0, 0, 0)$ parallel to the previous plane $\mathbf{P}$. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

6 The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2 by 2 diagonal matrices.

7 (a) The intersection of two planes through $(0, 0, 0)$ is probably a _____ but it could be a _____ . It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a _____ but it could be a _____ .

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, prove that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of the vectors that lie in both subspaces. *Check the requirements on $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$.*

8 Suppose $\mathbf{P}$ is a plane through $(0, 0, 0)$ and $\mathbf{L}$ is a line through $(0, 0, 0)$. The smallest vector space $\mathbf{P} + \mathbf{L}$ containing both $\mathbf{P}$ and $\mathbf{L}$ is either _____ or _____ .

9 (a) Show that the set of *invertible* matrices in $\mathbf{M}$ is not a subspace.

(b) Show that the set of *singular* matrices in $\mathbf{M}$ is not a subspace.

10 True or false (check addition in each case by an example):

(a) The symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

Questions 11–19 are about column spaces $C(A)$ and the equation $Av = b$.

11 Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

12 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

13 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. Which matrices have the same column space? Which have the same *row space*?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

14 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

15 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{v} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger? Then it is the same for A and $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$.

16 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

17 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, then $\boldsymbol{b} + \boldsymbol{b}^*$ is also in $\boldsymbol{C}(A)$.

18 If A is any 5 by 5 invertible matrix, then its column space is ______. Why?

19 True or false (with a counterexample if false):

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

20 Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.

21 If the 9 by 12 system $A\boldsymbol{v} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\boldsymbol{C}(A)$ must be ______.

Challenge Problems

22 Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$. The **sum** $\mathbf{S} + \mathbf{T}$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in $\mathbf{S}$ and a vector $\boldsymbol{t}$ in $\mathbf{T}$. Then $\mathbf{S} + \mathbf{T}$ is a vector space.

If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and all vectors from $\mathbf{T}$. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

23 If $\mathbf{S}$ is the column space of A and $\mathbf{T}$ is $\boldsymbol{C}(B)$, then $\mathbf{S} + \mathbf{T}$ is the column space of what matrix M? The columns of A and B and M are all in $\mathbf{R}^m$. (I don't think $A + B$ is always a correct M.)

24 Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (this has extra columns) have the same column space. But find a square matrix with $\boldsymbol{C}(A^2)$ smaller than $\boldsymbol{C}(A)$.

25 An n by n matrix has $\boldsymbol{C}(A) = \mathbf{R}^n$ exactly when A is an ______ matrix.

5.2 The Nullspace of A : Solving $Av = 0$

This section is about the subspace containing all solutions to $A\boldsymbol{v} = \mathbf{0}$. The m by n matrix A can be square or rectangular. *One immediate solution is* $\boldsymbol{v} = \mathbf{0}$. For invertible matrices this is the only solution. For other matrices, not invertible, there are nonzero solutions to $A\boldsymbol{v} = \mathbf{0}$. *Each solution* $\boldsymbol{v}$ *belongs to the nullspace of* $\boldsymbol{N}(A)$.

Elimination will find all solutions and identify this very important subspace.

> ***The nullspace of*** A ***consists of all solutions to*** $A\boldsymbol{v} = \mathbf{0}$. **These vectors $\boldsymbol{v}$ are in $\mathbf{R}^n$.**

Check that the solution vectors form a subspace. Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ are in the nullspace, so that $A\boldsymbol{v} = \mathbf{0}$ and $A\boldsymbol{w} = \mathbf{0}$. The rules of matrix multiplication give $A(\boldsymbol{v} + \boldsymbol{w}) = \mathbf{0} + \mathbf{0}$. The rules also give $A(c\boldsymbol{v}) = c\mathbf{0}$. The right sides are still zero. Therefore $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$ are also in the nullspace $\boldsymbol{N}(A)$. Since we can add and multiply without leaving the nullspace, it is a subspace.

The solution vectors $\boldsymbol{v}$ have n components. They are vectors in $\mathbf{R}^n$, so *the nullspace* $\boldsymbol{N}(A)$ *is a subspace of* $\mathbf{R}^n$. The column space $\boldsymbol{C}(A)$ is a subspace of $\mathbf{R}^m$.

If the right side $\boldsymbol{b}$ is not zero, the solutions of $A\boldsymbol{v} = \boldsymbol{b}$ do *not* form a subspace. The vector $\boldsymbol{v} = \mathbf{0}$ is only a solution if $\boldsymbol{b} = \mathbf{0}$. When the set of solutions does not include $\boldsymbol{v} = \mathbf{0}$, it cannot be a subspace. Section 5.3 will show how the solutions to $A\boldsymbol{v} = \boldsymbol{b}$ (if there are any solutions) are shifted away from the origin by one particular solution $\boldsymbol{v}_p$.

Example 1 $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = [1\ \ 2\ \ 3]$. This equation $A\boldsymbol{v} = \mathbf{0}$ produces a plane through the origin $(0,0,0)$. The plane is a subspace of $\mathbf{R}^3$, and *it is the nullspace of* A.

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example 2 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular !

Solution Apply elimination to the linear equations $A\boldsymbol{v} = \mathbf{0}$:

$$\begin{aligned} v_1 + 2v_2 &= 0 \\ 3v_1 + 6v_2 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} v_1 + 2v_2 &= 0 \\ 0 &= 0 \end{aligned}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $v_1 + 2v_2 = 0$ is the same as the line $3v_1 + 6v_2 = 0$. That line is the nullspace $\boldsymbol{N}(A)$. It contains all solutions $\boldsymbol{v} = (v_1, v_2)$.

To describe this line of solutions, here is an efficient way. Choose one point on the line (one "***special solution***"). Then all points on the line are multiples of this one. We choose the second component to be $v_2 = 1$ (a special choice). From the equation $v_1 + 2v_2 = 0$, the first component must be $v_1 = -2$. The special solution $\boldsymbol{s}$ is $(-2, 1)$:

> **Special solution** The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $\boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

This is the best way to describe the nullspace, by computing special solutions to $A\boldsymbol{v} = \mathbf{0}$.

The nullspace consists of all combinations of the special solutions.

The plane $x + 2y + 3z = 0$ in Example 1 had *two* special solutions:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has the special solutions } \boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Those vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ lie on the plane $x + 2y + 3z = 0$, which is the nullspace of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. All vectors on the plane are combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

Notice what is special about $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. They have ones and zeros in the last two components. *Those components are "free" and we choose them specially as* 1 *and* 0. Then the first components -2 and -3 are determined by the equation $A\boldsymbol{v} = \mathbf{0}$.

The first column of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ contains the *pivot*, so the first component $\boldsymbol{v}_1$ is *not free*. The free components correspond to columns without pivots. This description of special solutions will be completed after one more example.

The special choice (one or zero) is only for the free variables in the special solutions.

Example 3 Describe the nullspaces $\boldsymbol{N}(A), \boldsymbol{N}(B), \boldsymbol{N}(C)$ of these three matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $A\boldsymbol{v} = \mathbf{0}$ has only the zero solution $\boldsymbol{v} = \mathbf{0}$. *The nullspace is* $\mathbf{Z}$. It contains only the single point $\boldsymbol{v} = \mathbf{0}$ in $\mathbf{R}^2$. This comes from elimination:

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} v_1 = 0 \\ v_2 = 0 \end{bmatrix}.$$

A is invertible. There are no special solutions. All columns of this A have pivots.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $B\boldsymbol{v} = \mathbf{0}$ again require $\boldsymbol{v} = \mathbf{0}$. The last two equations would also force $\boldsymbol{v} = \mathbf{0}$. When we add extra equations, the nullspace certainly cannot become larger. The extra rows impose more conditions on the vectors $\boldsymbol{v}$ in the nullspace.

The rectangular matrix C is different. It has extra columns instead of extra rows. The solution vector $\boldsymbol{v}$ has *four* components. Elimination will produce pivots in the first two columns of C, but the last two columns are "free". *They don't have pivots*:

2 pivot columns
2 free columns

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$
$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

pivot columns **free columns**

For the free variables v_3 and v_4, we make special choices of ones and zeros. First $v_3 = 1$, $v_4 = 0$ and second $v_3 = 0$, $v_4 = 1$. Then the pivot variables v_1 and v_2 are determined.

Solve $Uv = \mathbf{0}$ to get two special solutions in the nullspace of C (and U).

Special solutions s_1 **and** s_2

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{pivot} \\ \leftarrow \quad \text{variables} \\ \leftarrow \text{free} \\ \leftarrow \quad \text{variables} \end{array}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular U ! We can continue to make this matrix simpler, in two ways:

1. ***Produce zeros above the pivots.*** **Eliminate upward.**
2. ***Produce ones in the pivots.*** **Divide the whole row by its pivot.**

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same. This nullspace becomes easiest to see when we reach the ***reduced row echelon form*** R. It has I in the pivot columns, when row 2 is divided by 2:

Reduced form $\boldsymbol{R}$

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$
$$\uparrow \ \uparrow$$

Now the pivot columns contain $\boldsymbol{I}$

I subtracted row 2 of U from row 1, and then multiplied row 2 by $\frac{1}{2}$. The original two equations have simplified to $x_1 + 2x_3 = 0$ and $x_2 + 2x_4 = 0$.

The first special solution is still $s_1 = (-2, 0, 1, 0)$. All special solutions are unchanged. Special solutions are much easier to find from the reduced system $Rv = \mathbf{0}$.

Before moving to m by n matrices A and their nullspaces $\boldsymbol{N}(A)$ and special solutions, allow me to repeat one comment. For many matrices, the only solution to $Av = \mathbf{0}$ is $v = \mathbf{0}$. Their nullspaces $\boldsymbol{N}(A) = \mathbf{Z}$ contain only that zero vector. The only combination of the columns that produces $b = \mathbf{0}$ is then the "zero combination" or "trivial combination". The solution is trivial (just $v = \mathbf{0}$) but the idea is not trivial.

This case of a zero nullspace $\mathbf{Z}$ is of the greatest importance. It says that the columns of A are **independent**. No combination of columns gives the zero vector (except the zero combination). All columns have pivots, and no columns are free. You will see this idea of independence again . . .

Solving $Av = 0$ by Elimination

This is important. ***A is rectangular and we still use elimination***. We solve m equations in n unknowns. After A is simplified to U or to R, we read off the solution (or solutions). Remember the two stages (forward and back) in solving $Av = \mathbf{0}$:

1. **Elimination** takes A to a triangular U (or its reduced form R).

2. **Back substitution** in $U\boldsymbol{v} = \mathbf{0}$ or $R\boldsymbol{v} = \mathbf{0}$ produces $\boldsymbol{v}$.

You will notice a difference in back substitution, when A and U have fewer than n pivots. *We are allowing all matrices in this chapter*, not just the nice ones (which are square matrices with inverses).

Pivots are still nonzero. The columns below the pivots are still zero. But it might happen that a column has no pivot. That free column doesn't stop the calculation. ***Go on to the next column***. The first example is a 3 by 4 matrix with two pivots:

$$\textbf{Elimination on} \qquad A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}.$$

Certainly $a_{11} = 1$ is the first pivot. Clear out the 2 and 3 below that pivot:

$$A \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \qquad \begin{matrix} \\ (\text{subtract } 2 \times \text{ row } 1) \\ (\text{subtract } 3 \times \text{ row } 1) \end{matrix}$$

The second column has a zero in the pivot position. We look below the zero for a nonzero entry, ready to do a row exchange. *The entry below that position is also zero.* Elimination can do nothing with the second column. This signals trouble, which we expect anyway for a rectangular matrix. There is no reason to quit, and we go on to the third column.

The second pivot is 4 (but it is in the third column). Subtracting row 2 from row 3 clears out that third column below the pivot. **The pivot columns are** 1 **and** 3:

$$\textbf{Triangular } \boldsymbol{U} \qquad U = \begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \textit{Only two pivots} \\ \textit{The last equation} \\ \textit{became } 0 = 0 \end{matrix}$$

The fourth column also has a zero in the pivot position—but nothing can be done. There is no row below it to exchange, and forward elimination is complete. The matrix has three rows, four columns, and *only two pivots*. The third equation in $A\boldsymbol{v} = \mathbf{0}$ is the sum of the first two. It is automatically satisfied $(0 = 0)$ when the first two equations are satisfied. Elimination reveals the inner truth about $A\boldsymbol{v} = \mathbf{0}$. Soon we push on from U to R.

Now comes back substitution, to find all solutions to $U\boldsymbol{v} = \mathbf{0}$. With four unknowns and only two pivots, there are many solutions. The question is how to write them all down. A good method is to separate the ***pivot variables*** from the ***free variables***.

P	The *pivot* variables are v_1 and v_3.	**Columns 1 and 3 contain pivots.**
F	The *free* variables are v_2 and v_4.	**Columns 2 and 4 have no pivots.**

The free variables v_2 and v_4 can be given any values whatsoever. Then back substitution finds the pivot variables v_1 and v_3. (In Chapter 2 no variables were free. When A is invertible, all variables are pivot variables.) The simplest choices for the free variables are ones and zeros. Those choices give the *special solutions*.

Special solutions to $v_1 + v_2 + 2v_3 + 3v_4 = 0$ and $4v_3 + 4v_4 = 0$

- Set $v_2 = 1$ and $v_4 = 0$. By back substitution $v_3 = 0$. Then $v_1 = -1$.
- Set $v_2 = 0$ and $v_4 = 1$. By back substitution $v_3 = -1$. Then $v_1 = -1$.

These special solutions solve $U\boldsymbol{v} = \boldsymbol{0}$ and therefore $A\boldsymbol{v} = \boldsymbol{0}$. They are in the nullspace. The good thing is that *every solution is a combination of the special solutions.*

Complete solution to $A\boldsymbol{v} = \boldsymbol{0}$

$$\boldsymbol{v} = \underbrace{v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\textbf{special}} + \underbrace{v_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\textbf{special}} = \underbrace{\begin{bmatrix} -v_2 - v_4 \\ v_2 \\ -v_4 \\ v_4 \end{bmatrix}}_{\textbf{complete}}. \qquad (1)$$

Please look again at that answer. It is the main goal of this section. The vector $\boldsymbol{s}_1 = (-1, 1, 0, 0)$ is the special solution when $v_2 = 1$ and $v_4 = 0$. The second special solution has $v_2 = 0$ and $v_4 = 1$. ***All solutions are linear combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.*** The special solutions are in the nullspace $\boldsymbol{N}(A)$, and their combinations fill the whole nullspace.

There is a special solution for each free variable. If no variables are free—this means all n columns have pivots—then the only solution to $U\boldsymbol{v} = \boldsymbol{0}$ and $A\boldsymbol{v} = \boldsymbol{0}$ is the trivial solution $\boldsymbol{v} = \boldsymbol{0}$. With no free variables, the nullspace is $\mathbf{Z}$.

Example 4 Find the nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$.

The second column of U has no pivot. So v_2 is free. The special solution has $v_2 = 1$. Back substitution into $9v_3 = 0$ gives $v_3 = 0$. Then $v_1 + 5v_2 = 0$ or $v_1 = -5$. The solutions to $U\boldsymbol{v} = \boldsymbol{0}$ are multiples of one special solution $\boldsymbol{s}_1$:

$$\boldsymbol{v} = c \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

The nullspace of U is a line in $\mathbf{R}^3$.
It contains multiples of the special solution $\boldsymbol{s}_1 = (-5, 1, 0)$.
One variable is free.

The matrix $\boldsymbol{R}$ has zeros above and below the pivots, and ones in the pivots. By continuing elimination on U, the 7 is removed and the pivot changes from 9 to 1. The final result will be the **reduced row echelon form $\boldsymbol{R}$**:

$$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} \mathbf{1} & 5 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} = \mathsf{rref}(U).$$

Echelon Matrices

Forward elimination goes from A to U. It acts by row operations, including row exchanges. It goes on to the next column when no pivot is available in the current column. The m by n "staircase" U is an ***echelon matrix***.

Here is a 4 by 7 echelon matrix with the three pivots $\boldsymbol{p}$ highlighted in boldface:

$$U = \begin{bmatrix} \boldsymbol{p} & \boldsymbol{x} & x & x & x & \boldsymbol{x} & x \\ 0 & \boldsymbol{p} & x & x & x & \boldsymbol{x} & x \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{p} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Three pivot variables	v_1, v_2, v_6
Four free variables	v_3, v_4, v_5, v_7
Four special solutions in $\boldsymbol{N}(U)$	
$\boldsymbol{R}$ **will have** $\boldsymbol{p} = 1$ **and bold** $\boldsymbol{x} = 0$	

Question What are the column space and the nullspace for this matrix?

Answer The columns have four components so they lie in $\mathbf{R}^4$. (Not in $\mathbf{R}^3$!) The fourth component of every column is zero. *The column space $\boldsymbol{C}(U)$ consists of all vectors of the form* $(b_1, b_2, b_3, 0)$. For those vectors we can solve $U\boldsymbol{v} = \boldsymbol{b}$ by back substitution. These vectors $\boldsymbol{b}$ are all possible combinations of the seven columns.

The nullspace $\boldsymbol{N}(U)$ is a subspace of $\mathbf{R}^7$. The solutions to $U\boldsymbol{v} = \mathbf{0}$ are all the combinations of the four special solutions—*one for each free variable*:

1. Columns $3, 4, 5, 7$ have no pivots. The free variables are v_3, v_4, v_5, v_7.

2. Set one free variable to 1 and set the other free variables to zero.

3. Solve $U\boldsymbol{v} = \mathbf{0}$ for the pivot variables v_1, v_2, v_6 to get a special solution.

The nonzero rows of an echelon matrix go down in a staircase pattern. The pivots are the first nonzero entries in those rows. There is a column of zeros below every pivot.

The Counting Theorem

Counting the pivots leads to an extremely important theorem. Suppose A has more columns than rows. ***With $n > m$ there is at least one free variable.*** The system $A\boldsymbol{v} = \mathbf{0}$ has at least one special solution. This solution is *not zero*!

Suppose $A\boldsymbol{v} = \mathbf{0}$ has more unknowns than equations ($\boldsymbol{n} > \boldsymbol{m}$, more columns than rows). Then there are **nonzero solutions** in $\boldsymbol{N}(A)$. There must be free columns, without pivots.

A short wide matrix ($n > m$) always has nonzero vectors in its nullspace. There must be at least $n - m$ free variables, since the number of pivots cannot exceed m. (The matrix only has m rows, and a row never has two pivots.) Of course a row might have *no* pivot—which means an extra free variable. But here is the point: When there is a free variable, it can be set to 1. Then the equation $A\boldsymbol{v} = \mathbf{0}$ has a nonzero solution.

To repeat: There are at most m pivots. With $n > m$, the system $A\boldsymbol{v} = \mathbf{0}$ has a nonzero solution. Actually there are infinitely many solutions, since any multiple $c\boldsymbol{v}$ is also a solution. The nullspace contains at least a line of solutions. With two free variables, there will be two special solutions and the nullspace will be even larger.

The nullspace is a subspace. Its "dimension" is the number of special solutions. This central idea—the ***dimension*** of a subspace—is defined and explained in this chapter.

Dimension of $\boldsymbol{C}(A)$	=	**rank**	of matrix	=	number of pivot columns
Dimension of $\boldsymbol{N}(A)$	=	**nullity**	of matrix	=	number of free columns.

Counting Theorem with n columns ***Rank r plus nullity $n - r$ equals n.***

The Reduced Row Echelon Matrix R

From an echelon matrix U we go one more step. Continue with a 3 by 4 example:

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can divide the second row by 4. Then both pivots equal 1. *We can subtract* 2 *times this new row* $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ *from the row above.* ***The reduced row echelon matrix R has zeros above the pivots as well as below*:**

Reduced row echelon matrix $\quad R = \mathrm{rref}(A) = \begin{bmatrix} \mathbf{1} & 1 & \mathbf{0} & 1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad$ **Pivot rows contain *I***

R has 1's as pivots. Zeros above pivots come from upward elimination.

Important *If A is invertible, its reduced row echelon form is the identity matrix $R = I$.* This is the ultimate in row reduction. Of course the nullspace is then **Z**.

The zeros in R make it easy to find the special solutions (the same as before):

1. Set $v_2 = 1$ and $v_4 = 0$. Solve $R\boldsymbol{v} = \mathbf{0}$. Then $v_1 = -1$ and $v_3 = 0$.

Those numbers -1 and 0 are sitting in column 2 of R (with plus signs).

2. Set $v_2 = 0$ and $v_4 = 1$. Solve $R\boldsymbol{v} = \mathbf{0}$. Then $v_1 = -1$ and $v_3 = -1$.

Those numbers -1 and -1 are sitting in column 4 (with plus signs).

By reversing signs we can read off the special solutions directly from R. The nullspace $\boldsymbol{N}(A) = \boldsymbol{N}(U) = \boldsymbol{N}(R)$ contains all combinations of the special solutions:

$$\boldsymbol{v} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (\textbf{\textit{complete solution of }} A\boldsymbol{v} = \mathbf{0}).$$

The next section of the book moves firmly from U to the row reduced form R. The MATLAB command $[\,R, \mathit{pivcol}\,] = \mathrm{rref}(A)$ produces R and a list of the pivot columns.

■ REVIEW OF THE KEY IDEAS ■

1. The nullspace $\boldsymbol{N}(A)$ is a subspace of $\mathbf{R}^n$. It contains all solutions to $A\boldsymbol{v} = \mathbf{0}$.

2. Elimination produces an echelon matrix U, and then a row reduced R (pivots $= 1$).

3. Every free column of U or R leads to a special solution. The free variable equals 1 and the other free variables equal 0. Back substitution solves $A\boldsymbol{v} = \mathbf{0}$.

4. The complete solution to $A\boldsymbol{v} = \mathbf{0}$ is a combination of the special solutions.

5. A has at least one free column and one special solution if $n > m$: $\boldsymbol{N}(A)$ is not $\boldsymbol{Z}$.

6. The count of pivot columns and free columns is $r + (n - r) = n$.

■ WORKED EXAMPLES ■

3.2 A Create a 3 by 4 matrix R whose special solutions to $R\boldsymbol{v} = 0$ are $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$:

$$\boldsymbol{s}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix} \qquad \begin{array}{c} \text{pivot columns 1 and 3} \\ \text{free variables } v_2 \text{ and } v_4 \end{array}$$

Describe all matrices A with this nullspace $\boldsymbol{N}(A) =$ combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

Solution The reduced matrix R has pivots $= 1$ in columns 1 and 3. There is no third pivot, so the third row of R is all zeros. The free columns 2 and 4 will be combinations of the pivot columns:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad R\boldsymbol{s}_1 = \mathbf{0} \quad \text{and} \quad R\boldsymbol{s}_2 = \mathbf{0}.$$

The entries $3, 2, 6$ in R are the negatives of $-3, -2, -6$ in the special solutions!

R is only one matrix (one possible A) with the required nullspace. We could do any elementary operations on R—exchange rows, multiply a row by any $c \neq 0$, subtract any multiple of one row from another. ***R* can be multiplied** (*on the left*) **by any invertible matrix, without changing its nullspace.**

Every 3 by 4 matrix has at least one special solution. *These matrices have two.*

3.2 B Find the special solutions and the *complete solutions* to $A\boldsymbol{v} = \mathbf{0}$ and $A_2\boldsymbol{v} = \mathbf{0}$:

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} A & A \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 & 6 \\ 1 & 2 & 1 & 2 \end{bmatrix}.$$

Which are the pivot columns? Which are the free variables? What is R in each case?

Solution $A\boldsymbol{v} = \mathbf{0}$ has one special solution $\boldsymbol{s} = (-2, 1)$. The line of all $c\boldsymbol{s}$ is the complete solution. The first column of A is its pivot column, and v_2 is the free variable:

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \to R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} A & A \end{bmatrix} \to R_2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that R_2 has only one pivot column (the first column). All the variables v_2, v_3, v_4 are free. There are three special solutions to $A_2\,\boldsymbol{v} = \mathbf{0}$ (and also $R_2\,\boldsymbol{v} = \mathbf{0}$):

$$\boldsymbol{s}_1 = (-2, 1, 0, 0) \quad \boldsymbol{s}_2 = (-1, 0, 1, 0) \quad \boldsymbol{s}_3 = (-2, 0, 0, 1) \quad \textbf{Complete } \boldsymbol{v} = c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2 + c_3\boldsymbol{s}_3.$$

With r pivots, A has $n - r$ free variables and $A\boldsymbol{v} = \mathbf{0}$ has $n - r$ special solutions.

Problem Set 5.2

Questions 1–4 and 5–8 are about the matrices in Problems 1 and 5.

1 Reduce these matrices to their ordinary echelon forms U:

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

2 For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

3 By combining the special solutions in Problem 2, describe every solution to $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when there are no ______.

4 By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false*: The nullspace of R equals the nullspace of U.

5 By row operations reduce this new A and B to triangular echelon form U. Write down a 2 by 2 lower triangular L such that $B = LU$.

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}.$$

6 For the same A and B, find the special solutions to $A\boldsymbol{v} = \mathbf{0}$ and $B\boldsymbol{v} = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is ______.

7 In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors $\boldsymbol{v}$ that satisfy those equations as combinations of the special solutions.

8 Reduce the echelon forms U in Problem 5 to R. For each R draw a box around the identity matrix that is in the pivot rows and pivot columns.

Questions 9–17 are about free variables and pivot variables.

9 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

10 Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

11 Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot columns are

(a) 2, 4, 5

(b) 1, 3, 6, 7

(c) 4 and 6.

12 Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix R so that the free columns are

(a) 2, 4, 5, 6

(b) 1, 3, 6, 7, 8.

13 Suppose column 4 of a 3 by 5 matrix is all zero. Then v_4 is certainly a ______ variable. The special solution for this variable is the vector $\boldsymbol{s} =$ ______.

14 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then ______ is a free variable. Find the special solution for this variable.

15 Suppose an m by n matrix has r pivots. The number of special solutions is ______. The nullspace contains only $\boldsymbol{v} = \mathbf{0}$ when $r =$ ______. The column space is all of $\mathbf{R}^m$ when $r =$ ______.

16 The nullspace of a 5 by 5 matrix contains only $\boldsymbol{v} = \mathbf{0}$ when the matrix has ____ pivots. The column space is $\mathbf{R}^5$ when there are ____ pivots. Explain why.

17 The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which are the free variables? The special solutions are $(3, 1, 0)$ and ____.

18 (Recommended) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

19 Prove that U and $A = LU$ have the same nullspace when L is invertible:

$$\text{If } U\boldsymbol{v} = \mathbf{0} \text{ then } LU\boldsymbol{v} = \mathbf{0}. \quad \text{If } LU\boldsymbol{v} = \mathbf{0}, \text{ how do you know } U\boldsymbol{v} = \mathbf{0}?$$

20 Suppose column 1 + column 3 + column 5 $= \mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Questions 21–28 ask for matrices (if possible) with specific properties.

21 Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

22 Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

23 Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

24 Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

25 Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.

26 Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.

27 Why does no 3 by 3 matrix have a nullspace that equals its column space?

28 (Important) If $AB = 0$ then the column space of B is contained in the ____ of A. Give an example of A and B.

29 The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be ____. What reduced form R is virtually certain if the random A is 4 by 3?

30 Show by example that these three statements are generally *false*:

(a) A and A^{T} have the same nullspace.

(b) A and A^{T} have the same free variables.

(c) If R is the reduced form of A then R^{T} is the reduced form of A^{T}.

31 If the nullspace of A consists of all multiples of $\boldsymbol{v} = (2,1,0,1)$, how many pivots appear in U? What is R?

32 If the special solutions to $R\boldsymbol{v} = \mathbf{0}$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \end{bmatrix} \quad \text{(empty 3 by 1).}$$

33 (a) What are the five 2 by 2 reduced echelon matrices R whose entries are all 0's and 1's?

(b) What are the eight 1 by 3 matrices containing only 0's and 1's? Are all eight of them reduced echelon matrices R?

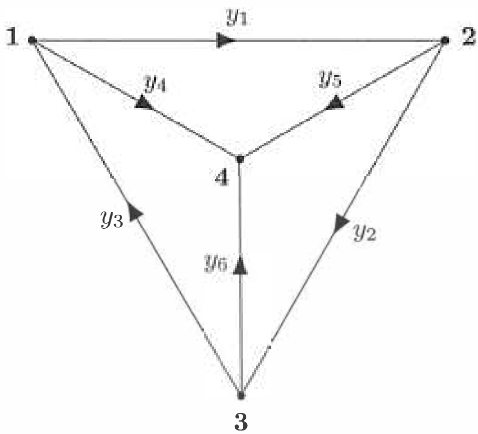
34 Explain why A and $-A$ always have the same reduced echelon form R.

Challenge Problems

35 If A is 4 by 4 and invertible, describe all vectors in the nullspace of the 4 by 8 matrix $B = [A \ \ A]$.

36 How is the nullspace $\boldsymbol{N}(C)$ related to the spaces $\boldsymbol{N}(A)$ and $\boldsymbol{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

37 Kirchhoff's Law says that *current in* = *current out* at every node. This network has six currents $y_1, \ldots, y_6$ (the arrows show the positive direction, each y_i could be positive or negative). Find the four equations $A\boldsymbol{y} = \mathbf{0}$ for Kirchhoff's Law at the four nodes. Reduce to $U\boldsymbol{y} = \mathbf{0}$. Find three special solutions in the nullspace of A.



5.3 The Complete Solution to $Av = b$

To solve $Av = b$ by elimination, include b as a new column next to the n columns of A. This "augmented matrix" is $\begin{bmatrix} A & b \end{bmatrix}$. When the steps of elimination operate on A (the left side of the equations), they also operate on the right side b. So we always keep correct equations, and they become simple to solve.

There are still r pivot columns and $n - r$ free columns in A. Each free column still gives a special solution to $Av = 0$. The new question is to find a *particular solution* v_p with $Av_p = b$. That solution will exist unless elimination leads to an impossible equation (a zero row on the left side, a nonzero number on the right side). Then back substitution finds v_p. **Every solution to $Av = b$ has the form $v_p + v_n$.**

In the process of elimination, we discover the **rank** of A. This is the number of pivots. The rank is also the number of nonzero rows after elimination. We start with m equations $Av = 0$, but *the true number of equations is the rank* r. We don't want to count repeated rows, or rows that are combinations of previous rows, or zero rows. You will soon see that r *counts the number of independent rows*. And the great fact, still to prove and explain, is that the rank r *also counts the number of independent columns*:

number of **pivots** = number of **independent rows** = number of **independent columns**.

This is part of the Fundamental Theorem of Linear Algebra in Section 5.5.

An example of $Av = b$ will make the possibilities clear.

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \quad \begin{matrix} \text{has the} \\ \text{augmented} \\ \text{matrix} \end{matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 1 & 3 & 1 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} A & b \end{bmatrix}.$$

The augmented matrix is just $\begin{bmatrix} A & b \end{bmatrix}$. When we apply the usual elimination steps to A and b, all the equations stay correct. Those steps produce R and d.

In this example we subtract row 1 from row 3 and then subtract row 2 from row 3. This produces a *row of zeros* in R, and it changes b to a new right side $d = (1, 6, 0)$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ \mathbf{0} \end{bmatrix} \quad \begin{matrix} \text{has the} \\ \text{augmented} \\ \text{matrix} \end{matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R & d \end{bmatrix}.$$

That very last zero is crucial. The third equation has become $0 = 0$, and we are safe. *The equations can be solved.* In the original matrix A, the first row plus the second row equals the third row. If the equations are consistent, this must be true on the right side of the equations also ! The all-important property on the right side was $1 + 6 = 7$.

Here are the same augmented matrices for any vector $b = (b_1, b_2, b_3)$:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} = \begin{bmatrix} R & d \end{bmatrix}$$

Now we get $0 = 0$ in the third equation provided $b_3 - b_1 - b_2 = 0$. This is $b_1 + b_2 = b_3$. The example satisfied this requirement with $1 + 6 = 7$. You see how elimination on $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ brings out the test on $\boldsymbol{b}$ for $A\boldsymbol{v} = \boldsymbol{b}$ to be solvable.

One Particular Solution

For an easy solution $\boldsymbol{v_p}$, *choose the free variables to be* $v_2 = v_4 = 0$. Then the two nonzero equations give the two pivot variables $v_1 = 1$ and $v_3 = 6$. Our particular solution to $A\boldsymbol{v} = \boldsymbol{b}$ (and also $R\boldsymbol{v} = \boldsymbol{d}$) is $\boldsymbol{v_p} = (1, 0, 6, 0)$. This particular solution is my favorite: *free variables are zero, pivot variables come from* $\boldsymbol{d}$. The method always works.

For $R\boldsymbol{v} = \boldsymbol{d}$ to have a solution, zero rows in R must also be zero in $\boldsymbol{d}$.
When I is in the pivot rows and columns of R, the pivot variables are in $\boldsymbol{d}$:

$$R\boldsymbol{v}_p = \boldsymbol{d} \qquad \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 \\ \mathbf{0} & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{6} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{6} \\ 0 \end{bmatrix} \qquad \begin{array}{l} \textbf{Pivot variables } \mathbf{1, 6} \\ \textbf{Free variables } 0, 0 \end{array}$$

Notice how we *choose* the free variables (as zero) and *solve* for the pivot variables. After the row reduction to R, those steps are quick. When the free variables are zero, the pivot variables for $\boldsymbol{v}_p$ are already seen in the right side vector $\boldsymbol{d}$.

$\boldsymbol{v}_{\textbf{particular}}$	***The particular solution $\boldsymbol{v_p}$ solves***	$A\boldsymbol{v}_p = \boldsymbol{b}$
$\boldsymbol{v}_{\textbf{nullspace}}$	***The $n - r$ special solutions solve***	$A\boldsymbol{v}_n = \boldsymbol{0}$.

That particular solution to $A\boldsymbol{v} = \boldsymbol{b}$ and $R\boldsymbol{v} = \boldsymbol{d}$ is $(1, 0, 6, 0)$. The two special (null) solutions to $R\boldsymbol{v} = \boldsymbol{0}$ come from the two free columns of R, by reversing signs of 3, 2, and 4. ***Please notice the form I use for the complete solution $\boldsymbol{v}_p + \boldsymbol{v}_n$ to $A\boldsymbol{v} = \boldsymbol{b}$:***

Complete solution
one $\boldsymbol{v}_p$
many $\boldsymbol{v}_n$

$$\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Question Suppose A is a square invertible matrix, $m = n = r$. What are $\boldsymbol{v}_p$ and $\boldsymbol{v}_n$?
Answer If A^{-1} exists, the particular solution is the one and *only* solution $\boldsymbol{v} = A^{-1}\boldsymbol{b}$. There are no special solutions or free variables. $R = I$ has no zero rows. The only vector in the nullspace is $\boldsymbol{v}_n = \boldsymbol{0}$. The complete solution is $\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n = A^{-1}\boldsymbol{b} + \boldsymbol{0}$.

This was the situation in Chapter 4. We didn't mention the nullspace in that chapter. $\boldsymbol{N}(A)$ contained only the zero vector. Reduction goes from $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1}\boldsymbol{b} \end{bmatrix}$. The original $A\boldsymbol{v} = \boldsymbol{b}$ is reduced all the way to $\boldsymbol{v} = A^{-1}\boldsymbol{b}$ which is $\boldsymbol{d}$. This is a special case here, but square invertible matrices are the ones we see most often in practice. So they got their own chapter at the start of linear algebra.

For small examples we can reduce $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} R & d \end{bmatrix}$. For a large matrix, MATLAB does it better. One particular solution (not necessarily ours) is $A\backslash b$ from the backslash command. Here is an example with *full column rank*. Both columns have pivots.

Example 1 Find the condition on (b_1, b_2, b_3) for $Av = b$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This condition puts b in the column space of A. Find the complete $v = v_p + v_n$.

Solution Use the augmented matrix, with its extra column b. Subtract row 1 of $\begin{bmatrix} A & b \end{bmatrix}$ from row 2, and add 2 times row 1 to row 3 to reach $\begin{bmatrix} R & d \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix}.$$

The last equation is $0 = 0$ provided $b_3 + b_1 + b_2 = 0$. This is the condition that puts b in the column space; then $Av = b$ will be solvable. The rows of A add to the zero row. So for consistency (these are equations!) the entries of b must also add to zero. This example has no free variables since $n - r = 2 - 2$. Therefore no special solutions. The rank is $r = n$ so the only null solution is $v_n = \mathbf{0}$. The unique particular solution to $Av = b$ and $Rv = d$ is at the top of the augmented column d:

$$\textbf{Only one solution} \qquad v = v_p + v_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $b_3 + b_1 + b_2$ is not zero, there is *no* solution to $Av = b$ (v_p doesn't exist).

This example is typical of an extremely important case: A has *full column rank*. Every column has a pivot. *The rank is* $r = n$. The matrix is tall and thin ($m \geq n$). Elimination puts I at the top, when A is reduced to R with rank n:

$$\textbf{Full column rank} \quad R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} \tag{1}$$

There are no free columns or free variables. The nullspace is $\boldsymbol{Z}$.

We will collect together the different ways of recognizing this type of matrix.

Every matrix A with full column rank $(r = n)$ has all these properties:

1. All columns of A are pivot columns. They are independent.
2. There are no free variables or special solutions.
3. Only the zero vector $v = \mathbf{0}$ solves $Av = \mathbf{0}$ and is in the nullspace $N(A)$.
4. If $Av = b$ has a solution (it might not) then it has only *one solution*.

In the essential language of the next section, A has ***independent columns*** if $r = n$. $A\boldsymbol{v} = \mathbf{0}$ only happens when $\boldsymbol{v} = \mathbf{0}$. Eventually we will add one more fact to the list: *The square matrix $A^{\mathrm{T}}A$ is invertible when the columns are independent.*

In Example 1 the nullspace of A (and R) has shrunk to the zero vector. The solution to $A\boldsymbol{v} = \boldsymbol{b}$ is *unique* (if it exists). There will be $m - n$ (here $3 - 2$) zero rows in R. There are $m - n$ conditions on $\boldsymbol{b}$ to have $0 = 0$ in those rows. Then $\boldsymbol{b}$ is in the column space.
With full column rank, $A\boldsymbol{v} = \boldsymbol{b}$ has *one* solution or *no* solution: $m > n$ is overdetermined.

The Complete Solution

The other extreme case is full row rank. Now $A\boldsymbol{v} = \boldsymbol{b}$ has *one or infinitely many* solutions. In this case A must be *short and wide* ($m \le n$). A matrix has ***full row rank*** if $r = m$ ("*independent rows*"). Every row has a pivot, and here is an example.

Example 2 There are $n = 3$ unknowns but only $m = 2$ equations:

$$\textbf{Full row rank} \qquad \begin{aligned} x + y + z &= 3 \\ x + 2y - z &= 4 \end{aligned} \qquad (\text{rank } r = m = 2)$$

These are two planes in xyz space. The planes are not parallel so they intersect in a line. This line of solutions is exactly what elimination will find. ***The particular solution will be one point on the line. Adding the nullspace vectors $\boldsymbol{v}_n$ will move us along the line.*** Then $\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n$ gives the whole line of solutions.

We find $\boldsymbol{v}_p$ and $\boldsymbol{v}_n$ by elimination on $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$. Subtract row 1 from row 2 and then subtract row 2 from row 1:

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{3} \\ 1 & 2 & -1 & \mathbf{4} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & \mathbf{3} \\ 0 & 1 & -2 & \mathbf{1} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & \mathbf{2} \\ 0 & 1 & -2 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}.$$

The particular solution has free variable $\boldsymbol{v}_3 = 0$. The special solution has $\boldsymbol{v}_3 = 1$:

$\boldsymbol{v}_{\textbf{particular}}$ comes directly from $\boldsymbol{d}$ on the right side: $\boldsymbol{v}_p = (2, 1, 0)$
$\boldsymbol{s}$ comes from the third column (free column) of R: $\boldsymbol{s} = (-3, 2, 1)$

It is wise to check that $\boldsymbol{v}_p$ and $\boldsymbol{s}$ satisfy the original equations $A\boldsymbol{v}_p = \boldsymbol{b}$ and $A\boldsymbol{s} = \mathbf{0}$:

$$\begin{aligned} 2+1 &= 3 \\ 2+2 &= 4 \end{aligned} \qquad\qquad \begin{aligned} -3+2+1 &= 0 \\ -3+4-1 &= 0 \end{aligned}$$

The nullspace solution $\boldsymbol{v}_n$ is any multiple of $\boldsymbol{s}$. It moves along the line of solutions, starting at $\boldsymbol{v}_{\textbf{particular}}$. ***Please notice again how to write the answer***:

$$\textbf{Complete solution} \qquad \boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \boldsymbol{v}_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

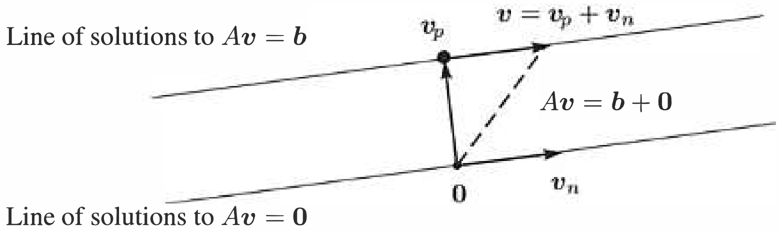


Figure 5.3: Complete solution = *one* particular solution + *all* nullspace solutions.

The line of solutions is drawn in Figure 5.3. Any point on the line *could* have been chosen as the particular solution; we chose the point with $v_3 = 0$.

The particular solution is *not* multiplied by an arbitrary constant! The special solution is, and you understand why.

Now we summarize this short wide case of *full row rank*. If $m < n$ the equations $Av = b$ are **underdetermined** (they have many solutions if they have one).

Every matrix A with ***full row rank*** $(r = m)$ **has all these properties:**

1. All m rows have pivots, and R has no zero rows.
2. $Av = b$ has a solution for every right side b.
3. The column space is the whole space $\mathbf{R}^m$.
4. There are $n - r = n - m$ special solutions in the nullspace of A.

In this case with m pivots, the rows are "***linearly independent***." We are more than ready for the idea of linear independence, as soon as we summarize the four possibilities—which depend on the rank. Notice how r, m, n are the critical numbers.

The four possibilities for linear equations depend on the rank r.

$\boldsymbol{r = m}$	and	$\boldsymbol{r = n}$	*Square and invertible*	$Av = b$	has 1 solution
$\boldsymbol{r = m}$	and	$r < n$	*Short and wide*	$Av = b$	has ∞ solutions
$r < m$	and	$\boldsymbol{r = n}$	*Tall and thin*	$Av = b$	has 0 or 1 solution
$r < m$	and	$r < n$	*Not full rank*	$Av = b$	has 0 or ∞ solutions

The reduced R will fall in the same category as the matrix A. They have the same rank.

In case the pivot columns happen to come first, we can display these four possibilities for R. For $Rv = d$ and $Av = b$ to be solvable, d must end in $m - r$ zeros.

Four types	$R = \begin{bmatrix} I \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$r = m = n$	$r = m < n$	$r = n < m$	$r < m, r < n$

Cases 1 and 2 have full row rank $r = m$. Cases 1 and 3 have full column rank $r = n$. Case 4 is the most general in theory and it is the least common in practice.

■ REVIEW OF THE KEY IDEAS ■

1. The rank r is the number of pivots. The reduced matrix R has $m-r$ zero rows.
2. $Av = b$ is solvable if and only if the last $m-r$ equations in $Rv = d$ are $0=0$.
3. One particular solution v_p has all free variables equal to zero.
4. The r pivot variables are determined after the $n-r$ free variables are chosen.
5. Full column rank $r=n$ means no free variables: one solution or no solution.
6. Full row rank $r=m$ means one solution if $m=n$ or infinitely many if $m<n$.

■ WORKED EXAMPLES ■

5.3 A This question connects elimination (**pivot columns and back substitution**) to **column space-nullspace-rank-solvability** (the full picture). A is 3 by 4 with rank 2:

$$Av=b \quad \text{is} \quad \begin{aligned} v_1 + 2v_2 + 3v_3 + \ \ 5v_4 &= \mathbf{b_1} \\ 2v_1 + 4v_2 + 8v_3 + 12v_4 &= \mathbf{b_2} \\ 3v_1 + 6v_2 + 7v_3 + 13v_4 &= \mathbf{b_3} \end{aligned}$$

1. Reduce $[\,A \ \ b\,]$ to $[\,U \ \ c\,]$, so that $Av=b$ becomes a triangular system $Uv=c$.
2. Find the condition on b_1, b_2, b_3 for $Av=b$ to have a solution.
3. Describe the column space of A. Which plane in $\mathbf{R}^3$ is the column space?
4. Describe the nullspace of A. What are the special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Av=(0,6,-6)$ and then the complete solution.

Solution

1. The multipliers in elimination are 2 and 3 and -1. They take $[\,A \ \ b\,]$ into $[\,U \ \ c\,]$.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 2 & 4 & 8 & 12 & \mathbf{b_2} \\ 3 & 6 & 7 & 13 & \mathbf{b_3} \end{bmatrix} \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 0 & 0 & 2 & 2 & \mathbf{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \mathbf{b_3 - 3b_1} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 0 & 0 & 2 & 2 & \mathbf{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \mathbf{b_3 + b_2 - 5b_1} \end{array}\right]$$

2. The last equation shows the solvability condition $\mathbf{b_3 + b_2 - 5b_1} = 0$. Then $0=0$.
3. **First description**: The column space is the plane containing all combinations of the pivot columns $(1,2,3)$ and $(3,8,7)$. Those columns are in A, not in U or R. **Second description**: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Av=b$ solvable. *All columns of A pass this test* $b_3 + b_2 - 5b_1 = 0$. *This is the equation for the plane in the first description of the column space.*
4. The special solutions have free variables $v_2 = 1, v_4 = 0$ and then $v_2 = 0, v_4 = 1$: $s_1 = (-2,1,0,0)$ and $s_2 = (-2,0,-1,1)$. The nullspace contains all $c_1 s_1 + c_2 s_2$.

5. One particular solution $\boldsymbol{v}_p$ has free variables = zero. Back substitute in $U\boldsymbol{v} = \boldsymbol{c}$:

Particular solution to $A\boldsymbol{v}_p = \boldsymbol{b} = (0, 6, -6)$
This vector $\boldsymbol{b}$ **satisfies** $b_3 + b_2 - 5b_1 = 0$
The complete solution is $\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n$.

$$\boldsymbol{v}_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

5.3 B Find the complete solution $\boldsymbol{v} = \boldsymbol{v}_p + \boldsymbol{v}_n$ by forward elimination on $[A \;\; \boldsymbol{b}]$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}.$$

Find numbers y_1, y_2, y_3 so that y_1 (row 1) + y_2 (row 2) + y_3 (row 3) = ***zero row***.

Check that $\boldsymbol{b} = (4, 2, 10)$ satisfies the condition $y_1b_1 + y_2b_2 + y_3b_3 = 0$. Why is this the condition for the equations to be solvable and $\boldsymbol{b}$ to be in the column space?

Solution Forward elimination on $[A \;\; \boldsymbol{b}]$ produces a zero row in $[U \;\; \boldsymbol{c}]$. The third equation becomes $0 = 0$. The equations are consistent (and solvable because $0 = 0$):

$$\begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 2 & 4 & 4 & 8 & \mathbf{2} \\ 4 & 8 & 6 & 8 & \mathbf{10} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}.$$

Columns 1 and 3 contain pivots. The variables $\boldsymbol{v}_2$ and $\boldsymbol{v}_4$ are free. If $\boldsymbol{v}_2 = \boldsymbol{v}_4 = 0$ we can solve (back substitution) for the particular solution $\boldsymbol{v}_p = (7, 0, -3, 0)$. The 7 and -3 appear again if elimination continues all the way to the row reduced $[R \;\; \boldsymbol{d}]$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 1 & 4 & \mathbf{-3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -4 & \mathbf{7} \\ 0 & 0 & 1 & 4 & \mathbf{-3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}.$$

For the nullspace part $\boldsymbol{v}_n$ with $\boldsymbol{b} = \boldsymbol{0}$, set the free variables $\boldsymbol{v}_2, \boldsymbol{v}_4$ to $1, 0$ and also $0, 1$:

$$\textbf{Special solutions} \qquad \boldsymbol{s}_1 = (-2, \; 1, \; 0, \; 0) \text{ and } \quad \boldsymbol{s}_2 = (4, \; 0, \; -4, \; 1)$$

Then the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$ (and $R\boldsymbol{v} = \boldsymbol{d}$) is $\boldsymbol{v}_{\text{complete}} = \boldsymbol{v}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

The rows of A produced the zero row from $2(\text{row } 1) + (\text{row } 2) - (\text{row } 3) = (0, 0, 0, 0)$. Thus $\boldsymbol{y} = (2, 1, -1)$. The same combination for $\boldsymbol{b} = (4, 2, 10)$ gives $2(4) + (2) - (10) = 0$. Combinations that give $\boldsymbol{y}^{\mathrm{T}}A$ = zero must also give $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ = zero. *Otherwise no solution.*

Later we will say this in different words: $\boldsymbol{y} = (2, 1, -1)$, **is in the nullspace of** A^{T}. Then $\boldsymbol{y}$ *will be* perpendicular to every $\boldsymbol{b}$ in the column space of A. I am looking ahead...

Problem Set 5.3

1 (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $A\boldsymbol{v} = \boldsymbol{b}$ to be solvable. Together these two conditions put $\boldsymbol{b}$ into the _____ space.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

Questions 3–15 are about the solution of $A\boldsymbol{v} = \boldsymbol{b}$. Follow the steps in the text to $\boldsymbol{v}_p$ and $\boldsymbol{v}_n$. Start from the augmented matrix $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$.

3 Write the complete solution as $\boldsymbol{v}_p$ plus any multiple of $\boldsymbol{s}$ in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Under what condition on b_1, b_2, b_3 is this system solvable? Include $\boldsymbol{b}$ as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find $\boldsymbol{v}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

7 Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination y_1(row 1) + y_2(row 2) + y_3(row 3) gives the zero row ?

8 Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero ?

(a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$.

9 In Worked Example **5.3 A**, combine the pivot columns of A with the numbers -9 and 3 in the particular solution v_p. What is that linear combination and why ?

10 Construct a 2 by 3 system $Av = b$ with particular solution $v_p = (2, 4, 0)$ and null (homogeneous) solution v_n = any multiple of $(1, 1, 1)$.

11 Why can't a 1 by 3 system have $v_p = (2, 4, 0)$ and v_n = any multiple of $(1, 1, 1)$?

12 (a) If $Av = b$ has two solutions v_1 and v_2, find two solutions to $Av = 0$.

(b) Then find another solution to $Av = b$.

13 Explain why these are all false :

(a) The complete solution is any linear combination of v_p and v_n.

(b) A system $Av = b$ has at most one particular solution.

(c) The solution v_p with all free variables zero is the shortest solution (minimum length $\|v\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution v_n in the nullspace.

14 Suppose column 5 has no pivot. Then v_5 is a _____ variable. The zero vector (is) (is not) the only solution to $Av = 0$. If $Av = b$ has a solution, then it has _____ solutions.

15 Suppose row 3 has no pivot. Then that row is _____. The equation $Rv = d$ is only solvable provided _____. The equation $Av = b$ (*is*) (*is not*) (*might not be*) solvable.

Questions 16–21 are about matrices of "full rank" $r = m$ or $r = n$.

16 The largest possible rank of a 3 by 5 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $Av = b$ (*always exists*) (*is unique*). The column space of A is _____. An example is $A =$ _____.

17 The largest possible rank of a 6 by 4 matrix is ______. Then there is a pivot in every ______ of U and R. The solution to $A\boldsymbol{v} = \boldsymbol{b}$ (*always exists*) (*is unique*). The nullspace of A is ______. An example is $A =$ ______.

18 Find by elimination the rank of A and also the rank of A^{T}:

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \quad \text{(rank depends on } q\text{)}.$$

19 Find the rank of A and also of $A^{\mathrm{T}}A$ and also of AA^{T}:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

20 Reduce A to its echelon form U. Then find a triangular L so that $A = LU$.

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}.$$

21 Find the complete solution in the form $\boldsymbol{v}_p + \boldsymbol{v}_n$ to these full rank systems:

(a) $x + y + z = 4$ (b) $\begin{aligned} x + y + z &= 4 \\ x - y + z &= 4. \end{aligned}$

22 If $A\boldsymbol{v} = \boldsymbol{b}$ has infinitely many solutions, why is it impossible for $A\boldsymbol{v} = \boldsymbol{B}$ (new right side) to have only one solution? Could $A\boldsymbol{v} = \boldsymbol{B}$ have no solution?

23 Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

24 Give examples of matrices A for which the number of solutions to $A\boldsymbol{v} = \boldsymbol{b}$ is

(a) 0 or 1, depending on $\boldsymbol{b}$

(b) ∞, regardless of $\boldsymbol{b}$

(c) 0 or ∞, depending on $\boldsymbol{b}$

(d) 1, regardless of $\boldsymbol{b}$.

25 Write down all known relations between r and m and n if $Av = b$ has

(a) no solution for some b

(b) infinitely many solutions for every b

(c) exactly one solution for some b, no solution for other b

(d) exactly one solution for every b.

Questions 26–33 are about Gauss-Jordan elimination (upwards as well as downwards) and the reduced echelon matrix R.

26 Continue elimination from U to R. Divide rows by pivots so the new pivots are all 1. Then produce zeros *above* those pivots to reach R:

$$U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

27 Suppose U is square with n pivots (an invertible matrix). *Explain why* $R = I$.

28 Apply Gauss-Jordan elimination to $Uv = \mathbf{0}$ and $Uv = c$. Reach $Rv = \mathbf{0}$ and $Rv = d$:

$$\begin{bmatrix} U & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $Rv = \mathbf{0}$ to find v_n (its free variable is $v_2 = 1$). Solve $Rv = d$ to find v_p (its free variable is $v_2 = 0$).

29 Apply Gauss-Jordan elimination to reduce to $Rv = \mathbf{0}$ and $Rv = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{9} \\ 0 & 0 & 2 & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}.$$

Solve $Uv = \mathbf{0}$ or $Rv = \mathbf{0}$ to find v_n (free variable $= 1$). What are the solutions to $Rv = d$?

30 Reduce to $Uv = c$ (Gaussian elimination) and then $Rv = d$ (Gauss-Jordan):

$$Av = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution v_p and all homogeneous (null) solutions v_n.

31 Find matrices A and B with the given property or explain why you can't:

(a) The only solution of $A\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) The only solution of $B\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

32 Reduce $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ to $\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$ and find the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

33 The complete solution to $A\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find A.

Challenge Problems

34 Suppose you know that the 3 by 4 matrix A has the vector $\boldsymbol{s} = (2, 3, 1, 0)$ as the only special solution to $A\boldsymbol{v} = \mathbf{0}$.

(a) What is the *rank* of A and the complete solution to $A\boldsymbol{v} = \mathbf{0}$?

(b) What is the exact row reduced echelon form R of A? Good question.

(c) How do you know that $A\boldsymbol{v} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$?

35 If you have this information about the solutions to $A\boldsymbol{v} = \boldsymbol{b}$ for a specific $\boldsymbol{b}$, what does that tell you about the *shape* of A (m and n)? And possibly about $\boldsymbol{r}$ and $\boldsymbol{b}$.

1. There is exactly one solution.
2. All solutions to $A\boldsymbol{v} = \boldsymbol{b}$ have the form $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
3. There are no solutions.
4. All solutions to $A\boldsymbol{v} = \boldsymbol{b}$ have the form $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
5. There are infinitely many solutions.

36 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $C\boldsymbol{v} = \boldsymbol{b}$ have the same (complete) solutions for every $\boldsymbol{b}$. Is it true that $A = C$?

5.4 Independence, Basis and Dimension

This important section is about the true size of a subspace. There are n columns in an m by n matrix. But the true "dimension" of the column space is not necessarily n. The dimension is measured by counting *independent columns*—and we have to say what that means. We will see that **the true dimension of the column space is the rank $\boldsymbol{r}$**.

The idea of independence applies to any vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ in any vector space. Most of this section concentrates on the subspaces that we know and use—especially the column space and the nullspace of A. In the last part we also study "vectors" that are not column vectors. They can be matrices, or solutions to differential equations. They can be linearly independent (or dependent). First come the key examples using column vectors.

The goal is to understand a ***basis*** : **independent vectors that "span the space"**.

Any basis Each vector in the space is a unique combination of the basis vectors.

We are at the heart of our subject, and we cannot go on without a basis. The four essential ideas in this section (with first hints at their meaning) are:

1. Independent vectors	(*no extra vectors*)
2. Spanning a space	(*their combinations produce the whole space*)
3. Basis for a space	(*independent and spanning : not too many or too few*)
4. Dimension of a space	(*the number of vectors in each and every basis*)

Bases for Important Spaces

Here are three examples to show you what a basis looks like (before the definition). A basis is a set of vectors that perfectly describes all vectors in the space. Take all combinations of the basis vectors to get every vector in the space.

1. *Basis for the column space of A*

A natural choice is the r pivot columns. Their combinations yield all columns.

2. *Basis for the nullspace of A*

A natural choice is the set of $n - r$ special solutions to $A\boldsymbol{v} = \mathbf{0}$.

3. *Basis for the space of null solutions to* $Ay'' + By' + Cy = 0$

A natural choice is the pair of solutions $y_1 = e^{s_1 t}$ and $y_2 = e^{s_2 t}$. These exponents s_1 and s_2 satisfy $As^2 + Bs + C = 0$, so y_1 and y_2 solve the differential equation.

If s is a double root of the quadratic, then $y_2 = te^{st}$ can be the second member of the basis. (Always two y's for a linear second order equation.) All other solutions are combinations of y_1 and y_2. Then y_1 and y_2 **span** the solution space.

The dimension of a space is easy. Just count the number of basis vectors:

Column space	**Nullspace**	**Solution space**
Dimension r	Dimension $n - r$	Dimension 2

Those bases were natural choices. They are not at all the only bases. A space has *many different bases.* The column space of this matrix A is the whole space $\mathbf{R}^2$.

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 5 & 9 \end{bmatrix} \qquad \textbf{Bases for } C(A)$$

1. Pivot columns 1 and 2
2. Columns 1 and 3, or columns 2 and 3
3. Any independent $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbf{R}^2$

The vectors $(1, 0)$ and $(0, 1)$ are a perfectly good basis for the column space of this A.

Linear Independence

Our first definition of independence is not so conventional, but you are ready for it.

DEFINITION The columns of A are **linearly independent** when the only solution to $A\boldsymbol{v} = \mathbf{0}$ is $\boldsymbol{v} = \mathbf{0}$. ***No combination*** $A\boldsymbol{v}$ ***of the columns is the zero vector, except*** $\boldsymbol{v} = \mathbf{0}$.

The columns are independent when the nullspace $\boldsymbol{N}(A)$ contains only the zero vector. Let me illustrate linear independence (and dependence) with three vectors in $\mathbf{R}^3$:

1. If three vectors are *not* in the same plane, they are independent. No combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ in Figure 5.4 gives zero except the combination $0\,\boldsymbol{u}_1 + 0\,\boldsymbol{u}_2 + 0\,\boldsymbol{u}_3$.

2. If three vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are *in the same plane*, they are dependent.

This idea of independence applies to 7 vectors in 12-dimensional space. If they are the columns of A, and independent, the nullspace only contains $\boldsymbol{v} = \mathbf{0}$. None of the vectors is a combination of the other six vectors.

Now we express the same idea in different words. The following definition of independence will apply to any sequence of vectors in any vector space. When the vectors are the columns of A, the two definitions say exactly the same thing.

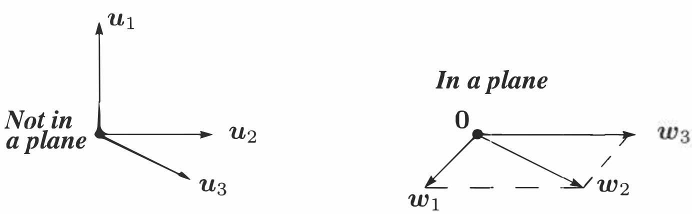


Figure 5.4: Independent vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$. Only $0\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + 0\boldsymbol{u}_3$ gives the vector $\mathbf{0}$. Dependent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$. The combination $\boldsymbol{w}_1 - \boldsymbol{w}_2 + \boldsymbol{w}_3$ is $(0, 0, 0)$.

DEFINITION The sequence of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ is ***linearly independent*** **if the only combination that gives the zero vector is** $0\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \cdots + 0\boldsymbol{u}_n$.

$$x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + \cdots + x_n\boldsymbol{u}_n = \mathbf{0} \quad \text{only happens when all } x\text{'s are zero.} \tag{1}$$

If a combination gives $\mathbf{0}$, when the x's are not all zero, the vectors are *dependent*.

Correct language: "The sequence of vectors is linearly independent." *Acceptable shortcut*: "The vectors are independent." *Not acceptable*: "The matrix is independent."

A sequence of vectors is either dependent or independent. They can be combined to give the zero vector (with nonzero x's) or they can't. So the key question is: Which combinations of the vectors give zero? We begin with some small examples in $\mathbf{R}^2$:

(a) The vectors $(1, 0)$ and $(1, 0.00001)$ are independent.

(b) The vectors $(1, 1)$ and $(-1, -1)$ on the same line through $(0, 0)$ are *dependent*.

(c) The vectors $(1, 1)$ and $(0, 0)$ are *dependent* because of the zero vector.

(d) In $\mathbf{R}^2$, any three vectors (a, b) and (c, d) and (e, f) are *dependent*.

The columns of A are dependent exactly when *there is a nonzero vector in the nullspace*.

If one of the $\boldsymbol{u}$'s is the zero vector, independence has no chance. Why not?

Three vectors in $\mathbf{R}^2$ cannot be independent! The matrix A with those three columns must have a free variable and then a special solution $A\boldsymbol{s} = \mathbf{0}$. The nullspace is larger than $\mathbf{Z}$. For three vectors in $\mathbf{R}^3$, we put them in a matrix and try to solve $A\boldsymbol{v} = \mathbf{0}$.

Example 1 The columns of this A are dependent. The nonzero vector $\boldsymbol{v}$ has $A\boldsymbol{v} = \mathbf{0}$.

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank is only $r = 2$. *Independent columns produce full column rank* $r = n$.

In that matrix the rows are also dependent. Row 1 minus row 3 is the zero row. For a *square matrix*, we will show that dependent columns imply dependent rows.

Question How to find that solution to $A\boldsymbol{v} = \mathbf{0}$? The systematic way is elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{reduces to } R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution $\boldsymbol{v} = (-3, 1, 1)$ was exactly the special solution. It shows how the free column (column 3) is a combination of the pivot columns. That kills independence!

Full column rank n. The columns of A are independent when the rank is $r = n$: n pivots and no free variables. Only $\boldsymbol{v} = \mathbf{0}$ is in the nullspace.

Dependent columns if $n > m$. Suppose seven columns have five components each ($m = 5$ is less than $n = 7$). Then the columns *must be dependent*. Any seven vectors from $\mathbf{R}^5$ are dependent. The rank of A cannot be larger than 5. There cannot be more than five pivots in five rows. $A\boldsymbol{v} = \mathbf{0}$ has at least $7 - 5 = 2$ free variables, so it has nonzero solutions—which means that the columns are dependent.

Any set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n > m$.

This type of matrix has more columns than rows—it is short and wide. The columns are certainly dependent if $n > m$, because $A\boldsymbol{v} = \mathbf{0}$ has a nonzero solution. Elimination will reveal the r pivot columns. *Those r pivot columns are independent.*

Note Another way to describe linear dependence is this: "*One vector is a combination of the other vectors.*" That sounds clear. Why don't we say this? Our definition was longer: "*Some combination gives the zero vector, other than the trivial combination with every* $v = 0$." Our definition doesn't pick out one particular vector as guilty.

All columns of A are treated the same. We look at $A\boldsymbol{v} = \mathbf{0}$, and it has a nonzero solution or it hasn't. In the end that is better than asking if the last column (or the first, or a column in the middle) is a combination of the others.

Spanning a Subspace

The first subspace in this book was the column space. Starting with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, the subspace was filled out by including all their $\boldsymbol{v}$ combinations $v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$. *The column space consists of all combinations $A\boldsymbol{v}$ of the columns.* We now introduce the single word "span" to describe this: The column space is ***spanned*** by the columns.

DEFINITION A set of vectors ***spans*** a space if their linear combinations *fill* the space.

The columns of a matrix span its column space. They might be dependent.

Example 2 $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full two-dimensional space $\mathbf{R}^2$.

Example 3 $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbf{R}^2$.

Example 4 $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbf{R}^2$. So does $\boldsymbol{w}_1$ alone.

Think of two vectors coming out from $(0, 0, 0)$ in 3-dimensional space. Generally they span a plane. Your mind fills in that plane by taking linear combinations. Mathematically you know other possibilities: two vectors could span a line, three vectors could span all of $\mathbf{R}^3$, or they could span only a plane or a line or $\mathbf{Z}$.

It is possible that three vectors span only a line in $\mathbf{R}^5$, or ten vectors span only a plane. They are certainly not independent!

The columns span the column space. Here is a new subspace—*spanned by the rows. **The combinations of the rows produce the "row space".***

DEFINITION The ***row space*** of a matrix is the subspace of $\mathbf{R}^n$ spanned by the rows.

The row space of A is $C(A^{\mathrm{T}})$. It is the column space of A^{T}.

The rows of an m by n matrix have n components. They are vectors in $\mathbf{R}^n$—or they would be if they were written as column vectors. There is a quick way to fix that: *Transpose the matrix.* Instead of the rows of A, look at the columns of A^{T}. Same numbers, but now in the column space of A^{T}. This row space $C(A^{\mathrm{T}})$ is a subspace of $\mathbf{R}^n$.

Example 5 The column space of A is a plane. The row space is all of $\mathbf{R}^2$.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}. \text{ Here } m = 3 \text{ and } n = 2.$$

The row space is spanned by the three rows of A (which are columns of A^{T}). The columns are in $\mathbf{R}^m$ spanning the column space. Same numbers, different vectors, different spaces.

A Basis for a Vector Space

Two vectors can't span all of $\mathbf{R}^3$, even if they are independent. Four vectors can't be independent, even if they span $\mathbf{R}^3$. We want ***enough independent vectors to span the space*** (and not more). A "***basis***" is just right.

DEFINITION A ***basis*** for a vector space is a sequence of vectors with two properties:

The basis vectors are linearly independent and they span the space.

This combination of properties is fundamental to linear algebra. Every vector $\boldsymbol{u}$ in the space is a combination of the basis vectors, because they span the space. More than that, the combination that produces $\boldsymbol{u}$ is *unique*, because the basis vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are independent:

There is one and only one way to write $\boldsymbol{u}$ as a combination of the basis vectors.

Reason: Suppose $\boldsymbol{u} = a_1\boldsymbol{u}_1 + \cdots + a_n\boldsymbol{u}_n$ and also $\boldsymbol{u} = b_1\boldsymbol{u}_1 + \cdots + b_n\boldsymbol{u}_n$. By subtraction $(a_1 - b_1)\boldsymbol{u}_1 + \cdots + (a_n - b_n)\boldsymbol{u}_n$ is the zero vector. From the independence of the $\boldsymbol{u}$'s, each $a_i - b_i = 0$. Hence $a_i = b_i$, and there are not two ways to produce $\boldsymbol{u}$.

Example 6 The columns of the identity matrix I are the "*standard basis*" for $\mathbf{R}^n$.

$$\text{The basis vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{are independent. They span } \mathbf{R}^2.$$

Everybody thinks of this basis first. The vector $\boldsymbol{i}$ goes across and $\boldsymbol{j}$ goes straight up. The columns of the 3 by 3 identity matrix are the standard basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ for $\mathbf{R}^3$.

Now we find many other bases (infinitely many). The basis is not unique!

Example 7 (Important) The columns of *every invertible n by n matrix* give a basis for $\mathbf{R}^n$:

$$\begin{array}{ll} \textbf{Invertible matrix} & \\ \text{Independent columns} & A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ \text{Column space is } \mathbf{R}^3 & \end{array} \qquad \begin{array}{ll} \textbf{Singular matrix} & \\ \text{Dependent columns} & B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}. \\ \text{Column space} \neq \mathbf{R}^3 & \end{array}$$

The only solution to $A\boldsymbol{v} = \mathbf{0}$ is $\boldsymbol{v} = A^{-1}\mathbf{0} = \mathbf{0}$. The columns are independent. They span the whole space $\mathbf{R}^n$—because every vector $\boldsymbol{b}$ is a combination of the columns. $A\boldsymbol{v} = \boldsymbol{b}$ can always be solved by $\boldsymbol{v} = A^{-1}\boldsymbol{b}$. Do you see how everything comes together for invertible matrices? Here it is in one sentence:

> The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a ***basis for*** $\mathbf{R}^n$ exactly when they are ***the columns of an n by n invertible matrix***. The vector space $\mathbf{R}^n$ has infinitely many different bases.

When the columns are dependent, we keep only the *pivot columns*—the first two columns of B above, with its two pivots. They are independent and they span the column space.

> ***The pivot columns of A are a basis for its column space.*** The pivot rows are a basis for the row space. The pivot rows of the reduced R are also a basis for the row space.

Example 8 This matrix is not invertible. Its columns are not a basis for anything!

$$\begin{array}{l} \textbf{One pivot column} \\ \textbf{One pivot row } (r = 1) \end{array} \qquad A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Column 1 of A is the pivot column. That column alone is a basis for its column space. Column 1 of R is **not** a basis for the column space of A. That column $(1, 0)$ in R does not even belong to the column space of A. Elimination changes column spaces. (But the *dimension* remains the same: here dimension $= 1$.)

The row space of A *is the same* as the row space of R. It contains $(2, 4)$ and $(1, 2)$ and all other multiples of those vectors. As always, there are infinitely many bases to choose from. One natural choice is to pick the nonzero rows of R (rows with a pivot). So this matrix A with rank one has only one vector in the basis:

$$\text{Basis for the column space: } \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \qquad \text{Basis for the row space: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Example 9 Find bases for the column and row spaces of this rank two matrix:

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1 and 3 are the pivot columns. They are a basis for the column space (of R!). The vectors in that column space all have the form $\boldsymbol{b} = (x, y, 0)$. This space is the "$xy$ plane" inside the full xyz space. That plane is not $\mathbf{R}^2$, it is a subspace of $\mathbf{R}^3$. Columns 2 and 3 are also a basis for the same column space. Which pairs of columns of R are *not* a basis for its column space?

The row space of R is a subspace of $\mathbf{R}^4$. The simplest basis for that row space is the two nonzero rows of R. The third row (the zero vector) is in the row space too. But it is *not in a basis* for the row space. The basis vectors must be independent.

Question Given five vectors in $\mathbf{R}^7$, ***how do you find a basis for the space they span?***

First answer Make them the rows of A, and eliminate to find the nonzero rows of R.
Second answer Put the five vectors into the columns of A. Eliminate to find the pivot columns (of A not R). Could another basis have more vectors, or fewer? This question has a good answer: *No*! ***All bases for a vector space contain the same*** *number* ***of vectors***.

Dimension of a Vector Space

The number of vectors, in any and every basis, is the "dimension" of the space.

We have to prove what was stated above. There are many choices for the basis vectors, but the *number of basis vectors* doesn't change.

If $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ are both bases for the same vector space, then $m = n$.

Proof Suppose that there are more $\boldsymbol{w}$'s than $\boldsymbol{u}$'s. From $n > m$ we want to reach a contradiction. The $\boldsymbol{u}$'s are a basis, so $\boldsymbol{w}_1$ must be a combination of the $\boldsymbol{u}$'s. If $\boldsymbol{w}_1$ equals $a_{11}\boldsymbol{u}_1 + \cdots + a_{m1}\boldsymbol{u}_m$, this is the first column of a matrix multiplication UA:

Each w is a combination of the u's

$$\begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix} = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} = UA.$$

We don't know each number a_{ij}, but we know the shape of A (it is m by n). The second vector $\boldsymbol{w}_2$ is also a combination of the $\boldsymbol{u}$'s. The coefficients in that combination fill the second column of A. The key is that A has a row for every $\boldsymbol{u}$ and a column for every $\boldsymbol{w}$. A is a short wide matrix, since $n > m$. So $A\boldsymbol{v} = \mathbf{0}$ ***has a nonzero solution***.

$A\boldsymbol{v} = \mathbf{0}$ gives $UA\boldsymbol{v} = \mathbf{0}$ which is $W\boldsymbol{v} = \mathbf{0}$. *A combination of the $\boldsymbol{w}$'s gives zero*! Then the $\boldsymbol{w}$'s could not be a basis—our assumption $n > m$ is **not possible** for two bases.

If $m > n$ we exchange the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. This completes the proof that $m = n$.

The number of basis vectors depends on the space—not on a particular basis. The number is the same for every basis, and it counts the "degrees of freedom" in the space. The dimension of the space $\mathbf{R}^n$ is n. We now introduce the important word ***dimension*** for other vector spaces too.

DEFINITION The ***dimension of a space*** is the ***number of vectors*** in every basis.

This matches our intuition. The line through $\boldsymbol{u} = (1, 5, 2)$ has dimension one. It is a subspace with this one vector $\boldsymbol{u}$ in its basis. Perpendicular to that line is the plane $x + 5y + 2z = 0$. This plane has dimension 2. To prove it, we find a basis $(-5, 1, 0)$ and $(-2, 0, 1)$. The dimension is 2 because the basis contains two vectors.

The plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix}$, which has two free variables. Our basis vectors $(-5, 1, 0)$ and $(-2, 0, 1)$ are the "special solutions" to $A\boldsymbol{v} = \mathbf{0}$. The $n - r$ special solutions give *a basis for the nullspace*, so the dimension of $\boldsymbol{N}(A)$ is $n - r$.

Note about the language of linear algebra We never say "the rank of a space" or "the dimension of a basis" or "the basis of a matrix". Those terms have no meaning. It is the ***dimension of the column space*** that equals the ***rank of the matrix.***

Bases for Matrix Spaces and Function Spaces

The words "independence" and "basis" and "dimension" are not at all restricted to column vectors. We can ask whether three matrices A_1, A_2, A_3 are independent. When they are in the space of all 3 by 4 matrices, some combination might give the zero matrix. We can also ask the dimension of the full 3 by 4 matrix space. (It is 12.)

In differential equations, $d^2y/dx^2 = y$ has a space of solutions. One basis is $y = e^x$ and $y = e^{-x}$. Counting the basis functions gives the dimension 2 for the space of all solutions. (The dimension is 2 because of the second derivative.)

Matrix spaces and function spaces may look a little strange after $\mathbf{R}^n$. But in some way, you haven't got the ideas of basis and dimension straight until you can apply them to "vectors" other than column vectors.

Example 10 Find a basis for the space of 3 by 3 symmetric matrices.

The basis vectors will be matrices! We need enough to span the space (then every $A = A^{\mathrm{T}}$ is a combination). The matrices must be independent (combinations don't give the zero matrix). Here is one basis for the symmetric matrices (many other bases).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

You could write every $A = A^{\mathrm{T}}$ as a combination of those six matrices. What coefficients would produce 1, 4, 5 and 4, 2, 8 and 5, 8, 9 in the rows? There is only one way to do this. The six matrices are independent. The *dimension* of symmetric matrix space (3 by 3 matrices) is **6**.

To push this further, think about the space of all n by n matrices. One possible basis uses matrices that have only a single nonzero entry (that entry is 1). There are n^2 positions for that 1, so there are n^2 basis matrices:

The dimension of the whole n by n matrix space is n^2.

The dimension of the subspace of *upper triangular* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$.

The dimension of the subspace of *diagonal* matrices is n.

The dimension of the subspace of *symmetric* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$ (why?).

Function spaces The equations $d^2y/dt^2 = 0$ and $d^2y/dt^2 = -y$ and $d^2y/dt^2 = y$ involve the second derivative. In calculus we solve to find the functions $y(t)$:

$$\begin{array}{ll} y'' = 0 & \text{is solved by any linear function } y = ct + d \\ y'' = -y & \text{is solved by any combination } y = c\sin t + d\cos t \\ y'' = y & \text{is solved by any combination } y = ce^t + de^{-t}. \end{array}$$

That solution space for $y'' = -y$ has two basis functions: $\sin t$ and $\cos t$. The space for $y'' = 0$ has t and 1. It is the "nullspace" of the second derivative! The dimension is 2 in each case (these are second-order equations). We are finding the null solutions y_n.

The solutions of $y'' = 2$ don't form a subspace—the right side $b = 2$ is not zero. A particular solution is $y = t^2$. The complete solution is $y = y_p + y_n = t^2 + ct + d$.

That complete solution is one particular solution plus any function in the nullspace. A linear differential equation is like a linear matrix equation $A\boldsymbol{v} = \boldsymbol{b}$. But we solve it by calculus instead of linear algebra.

We end here with the space $\mathbf{Z}$ that contains only the zero vector. The dimension of this space is *zero*. ***The empty set*** (containing no vectors) ***is a basis for*** $\mathbf{Z}$. We can never allow the zero vector into a basis, because then linear independence is lost.

■ REVIEW OF THE KEY IDEAS ■

1. The columns of A are ***independent*** if $\boldsymbol{v} = \mathbf{0}$ is the only solution to $A\boldsymbol{v} = \mathbf{0}$.

2. The vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ ***span*** a space if their combinations fill that space. Spanning vectors can be dependent or independent.

3. ***A basis consists of linearly independent vectors that span the space.*** Every vector in the space is a *unique* combination of the basis vectors.

4. All bases for a space have the same number of vectors. This number of vectors in a basis is the ***dimension*** of the space.

5. The ***pivot columns*** are one basis for the column space. The dimension is the rank r.

6. The $n - r$ special solutions will be seen as a basis for the nullspace.

■ WORKED EXAMPLES ■

5.4 A Start with the vectors $\boldsymbol{u}_1 = (1, 2, 0)$ and $\boldsymbol{u}_2 = (2, 3, 0)$. **(a)** Are they linearly independent? **(b)** Are they a basis for any space? **(c)** What space $\mathbf{V}$ do they span? **(d)** What is the dimension of $\mathbf{V}$? **(e)** Which matrices A have $\mathbf{V}$ as their column space? **(f)** Which matrices have $\mathbf{V}$ as their nullspace?

Solution

(a) $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are independent—the only combination to give $\mathbf{0}$ is $0\boldsymbol{u}_1 + 0\boldsymbol{u}_2$.

(b) Yes, they are a basis for the space they span.

(c) That space $\mathbf{V}$ contains all vectors $(x, y, 0)$. It is the xy plane in $\mathbf{R}^3$.

(d) The dimension of $\mathbf{V}$ is 2 since the basis contains two vectors.

(e) This $\mathbf{V}$ is the column space of any 3 by n matrix A of rank 2, if row 3 is all zero. In particular A could just have columns $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

(f) This $\mathbf{V}$ is the nullspace of any m by 3 matrix B of rank 1, if every row has the form $(0, 0, c)$. In particular take $B = [0\ \ 0\ \ 1]$. Then $B\boldsymbol{u}_1 = \mathbf{0}$ and $B\boldsymbol{u}_2 = \mathbf{0}$.

5.4 B (***Important example***) Suppose $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ is a basis for $\mathbf{R}^n$ and the n by n matrix A is invertible. Show that $A\boldsymbol{u}_1, \ldots, A\boldsymbol{u}_n$ is also a basis for $\mathbf{R}^n$.

Solution In *matrix language*: Put the basis vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ in the columns of an invertible(!) matrix U. Then $A\boldsymbol{u}_1, \ldots, A\boldsymbol{u}_n$ are the columns of AU. Since A and U are invertible, so is AU and its columns give a basis.

In *vector language*: Suppose $c_1 A\boldsymbol{u}_1 + \cdots + c_n A\boldsymbol{u}_n = \mathbf{0}$. This is $A\boldsymbol{v} = \mathbf{0}$ with $\boldsymbol{v} = c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n$. Multiply by A^{-1} to reach $\boldsymbol{v} = \mathbf{0}$. Linear independence of the $\boldsymbol{u}$'s forces all $c_i = 0$. This shows that the $A\boldsymbol{u}$'s are independent.

To show that the $A\boldsymbol{u}$'s span $\mathbf{R}^n$, solve $c_1 A\boldsymbol{u}_1 + \cdots + c_n A\boldsymbol{u}_n = \boldsymbol{b}$. This is the same as $c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n = A^{-1}\boldsymbol{b}$. Since the $\boldsymbol{u}$'s are a basis, this must be solvable for all $\boldsymbol{b}$.

Problem Set 5.4

Questions 1–10 are about linear independence and linear dependence.

1 Show that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are independent but $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ are dependent:

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{u}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 + c_4\boldsymbol{u}_4 = \mathbf{0}$ or $A\boldsymbol{c} = \mathbf{0}$. The $\boldsymbol{u}$'s go in the columns of A.

2 (Recommended) Find the largest possible number of independent vectors among

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{u}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{u}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{u}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $U\boldsymbol{v} = \mathbf{0}$ is $\boldsymbol{v} = \mathbf{0}$. Then the upper triangular U has independent columns.

5 Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

6 Choose three independent columns of U and A. Then make two other choices.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

7 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the differences $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$ are *dependent*. Find a combination of the $\boldsymbol{v}$'s that gives zero. Which singular matrix gives $[\, \boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \;\; \boldsymbol{v}_3 \,] = [\, \boldsymbol{w}_1 \;\; \boldsymbol{w}_2 \;\; \boldsymbol{w}_3 \,]\, A$?

8 If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the sums $\boldsymbol{v}_1 = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 + \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$ are *independent*. (Write $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$ in terms of the $\boldsymbol{w}$'s. Find and solve equations for the c's, to show they are zero.)

9 Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because ______.

(b) The two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ will be dependent if ______.

(c) The vectors $\boldsymbol{u}_1$ and $(0,0,0)$ are dependent because ______.

10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Questions 11–14 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors, to find the space they span.

11 Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1,1,-1)$ and $(-1,-1,1)$

(b) the three vectors $(0,1,1)$ and $(1,1,0)$ and $(0,0,0)$

(c) all vectors in $\mathbf{R}^3$ with whole number components

(d) all vectors with positive components.

12 The vector $\boldsymbol{b}$ is in the subspace spanned by the columns of A when ______ has a solution. The vector $\boldsymbol{c}$ is in the row space of A when ______ has a solution.

True or false: If the zero vector is in the row space, the rows are dependent.

13 Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of A (b) column space of U (c) row space of A (d) row space of U:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

14 $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ are combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$. Write $\boldsymbol{v}$ and $\boldsymbol{w}$ as combinations of $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$. The two pairs of vectors ______ the same space. When are they a basis for the same space?

Questions 15–25 are about the requirements for a basis.

15 If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent, the space they span has dimension ______. These vectors are a ______ for that space. If the vectors are the columns of an m by n matrix, then m is ______ than n. If $m = n$, that matrix is ______.

16 Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do) (do not) (might not) span $\mathbf{R}^4$.

(b) Those vectors (are) (are not) (might be) linearly independent.

(c) Any four of those vectors (are) (are not) (might be) a basis for $\mathbf{R}^4$.

17 Find three different bases for the column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$. Then find two different bases for the row space of U.

18 Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$.

(d) The column space and the nullspace of I $(4 \text{ by } 4)$.

19 The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then? *Looking ahead*: The rank r counts the number of _____ columns.

20 Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

21 Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $A\boldsymbol{v} = \mathbf{0}$ has only the solution $\boldsymbol{v} = \mathbf{0}$ because _____ .

(b) If $\boldsymbol{b}$ is in $\mathbf{R}^5$ then $A\boldsymbol{v} = \boldsymbol{b}$ is solvable because the basis vectors _____ $\mathbf{R}^5$.

Conclusion: A is invertible. Its rank is 5. Its rows are also a basis for $\mathbf{R}^5$.

22 Suppose $\mathbf{S}$ is a 5-dimensional subspace of $\mathbf{R}^6$. True or false (example if false):

(a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

23 U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

24 True or false (give a good reason):

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

25 For which numbers c and d do these matrices have rank 2 ?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Questions 26–28 are about spaces where the "vectors" are matrices.

26 Find a basis (and the dimension) for these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

27 Construct six linearly independent 3 by 3 echelon matrices $U_1, \ldots, U_6$. What space of 3 by 3 matrices do they span?

28 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

Questions 29–32 are about spaces where the "vectors" are functions.

29 (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

30 The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace $\boldsymbol{S}$ with $y(0) = 0$. What is the dimension of $\boldsymbol{S}$?

31 Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$ (b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

32 Suppose y_1, y_2, y_3 are three different functions of x. The space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

33 Find a basis for the space $\mathbf{S}$ of vectors (a, b, c, d) with $a + c + d = 0$ and also for the space $\mathbf{T}$ with $a + b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

34 Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$

(b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$

(c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$

(d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$

35 Suppose A is 5 by 4 with rank 4. Show that $A\boldsymbol{v} = \boldsymbol{b}$ has no solution when the 5 by 5 matrix $[\, A \;\; \boldsymbol{b} \,]$ is invertible. Show that $A\boldsymbol{v} = \boldsymbol{b}$ is solvable when $[\, A \;\; \boldsymbol{b} \,]$ is singular.

36 (a) Find a basis for all solutions to $d^4y/dx^4 = y(x)$.

(b) Find a particular solution to $d^4y/dx^4 = y(x) + 1$. Find the complete solution.

Challenge Problems

37 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 =$ zero matrix, and prove that each $c_i = 0$.)

38 Intersections and sums have $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$. Start with a basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ for the intersection $\mathbf{V} \cap \mathbf{W}$. Extend with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$ to a basis for $\mathbf{V}$, and separately with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$ to a basis for $\mathbf{W}$. Prove that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s and $\boldsymbol{w}$'s together are ***independent***. The dimensions have $(r + s) + (r + t) = (r) + (r + s + t)$ as desired.

39 Inside $\mathbf{R}^n$, suppose dimension $(\mathbf{V})$ + dimension $(\mathbf{W}) > n$. Why is some nonzero vector in both $\mathbf{V}$ and $\mathbf{W}$? Start with bases $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_q$, $p + q > n$.

40 Suppose A is 10 by 10 and $A^2 = 0$ (zero matrix): A times each column of A is $\mathbf{0}$. This means that the column space of A is contained in the ____. If A has rank r, those subspaces have dimension $r \leq 10 - r$. So the rank of A is $r \leq 5$, if $A^2 = 0$.

5.5 The Four Fundamental Subspaces

The figure on this page is the *big picture of linear algebra*. The Four Fundamental Subspaces are in position: Two orthogonal subspaces in $\mathbf{R}^n$ and two in $\mathbf{R}^m$. For any $\boldsymbol{b}$ in the column space, the complete solution to $A\boldsymbol{v} = \boldsymbol{b}$ has one particular solution $\boldsymbol{v}_p$ in the row space, plus any $\boldsymbol{v}_n$ in the nullspace.

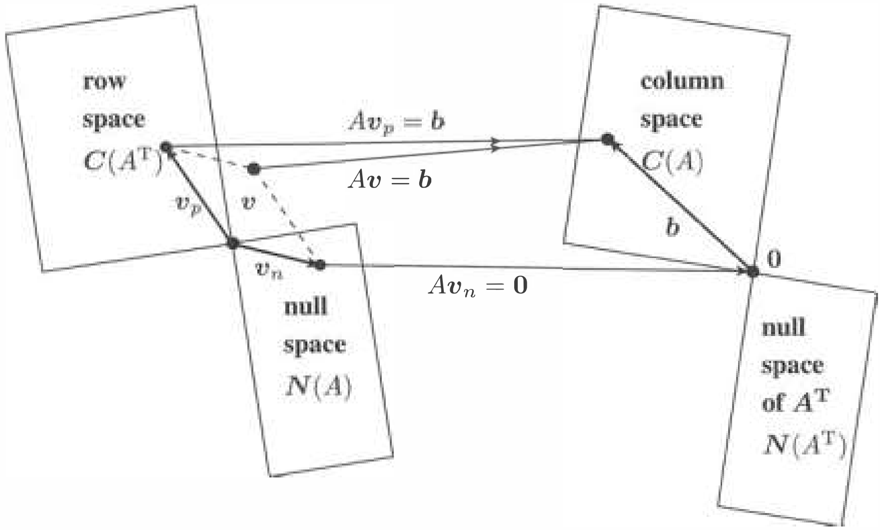


Figure 5.5: The Four Fundamental Subspaces. The complete solution $\boldsymbol{v}_p + \boldsymbol{v}_n$ to $A\boldsymbol{v} = \boldsymbol{b}$.

The main theorem in this chapter connects ***rank*** and ***dimension***. The ***rank*** of a matrix is the number of pivots. The ***dimension*** of a subspace is the number of vectors in a basis. We count pivots or we count basis vectors. *The rank of A reveals the dimensions of all four fundamental subspaces.* Here are the subspaces, including the new one.

Two subspaces come directly from A, and the other two come from A^{T}:

Four Fundamental Subspaces		**Dimensions**
1. The ***row space*** $\boldsymbol{C}(A^{\mathrm{T}})$	Subspace of $\mathbf{R}^n$.	r
2. The ***column space*** $\boldsymbol{C}(A)$	Subspace of $\mathbf{R}^m$.	r
3. The ***nullspace*** $\boldsymbol{N}(A)$	Subspace of $\mathbf{R}^n$.	$n - r$
4. The ***left nullspace*** $\boldsymbol{N}(A^{\mathrm{T}})$	Subspace of $\mathbf{R}^m$. This is our new space.	$m - r$

In this book the column space and nullspace came first. We know $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$ pretty well. Now the other two subspaces come forward. The row space contains all combinations of the rows. *This is the column space of A^{T}.*

For the left nullspace we solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$—that system is n by m. *This is the nullspace* $\boldsymbol{N}(A^{\mathrm{T}})$. The vectors $\boldsymbol{y}$ go on the *left* side of A when we transpose to get $\boldsymbol{y}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$. The matrices A and A^{T} are usually different. So are their column spaces and their nullspaces. But those spaces are connected in an absolutely beautiful way.

Part 1 of the Fundamental Theorem finds the dimensions of the four subspaces. One fact stands out: ***The row space and column space have the same dimension*** r. This is the rank of the matrix. The other important fact involves the two nullspaces:

$\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ have dimensions $n - r$ and $m - r$, to make up the full n and m.

Part 2 of the Fundamental Theorem will describe how the four subspaces fit together (two in $\mathbf{R}^n$ and two in $\mathbf{R}^m$). That completes the "right way" to understand every $A\boldsymbol{v} = \boldsymbol{b}$. Stay with it—you are doing real mathematics.

The Four Subspaces for R

Suppose A is reduced to its row echelon form R. For that special form, the four subspaces are easy to identify. We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (two of them don't change) as we look back at A. The main point will be that *the four dimensions are the same for A and R.*

As a specific 3 by 5 example, look at the four subspaces for this echelon matrix R:

$$\begin{matrix} m = 3 \\ n = 5 \\ r = 2 \end{matrix} \qquad \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \textbf{pivot rows } 1 \textbf{ and } 2 \\ \\ \textbf{pivot columns } 1 \textbf{ and } 4 \end{matrix}$$

The rank of this matrix R is $r = 2$ (*two pivots*). Take the four subspaces in order.

1. The row space of R has dimension 2, matching the rank.

Reason: The first two rows are a basis. The row space contains combinations of all three rows, but the third row (the zero row) adds nothing new. So rows 1 and 2 span the row space. $\boldsymbol{C}(R^{\mathrm{T}})$.

The pivot rows 1 and 2 are independent. That is obvious for this example, and it is always true. If we look only at the pivot columns, we see the r by r identity matrix. There is no way to combine its rows to give the zero row (except by the combination with all coefficients zero). So the r pivot rows are a basis for the row space.

The dimension of the row space is the rank r. The nonzero rows of R form a basis.

2. The column space of R also has dimension $r = 2$, matching the rank.

Reason: The pivot columns 1 and 4 form a basis for $\boldsymbol{C}(R)$. They are independent because they start with the r by r identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coefficients zero). And they also span the column space. Every other (free) column is a combination of the pivot columns.

The combinations we need are revealed by the three special solutions:

Column 2 is 3 times column 1. The special solution is $(-3, 1, 0, 0, 0)$.

Column 3 is 5 times column 1. The special solution is $(-5, 0, 1, 0, 0,)$.

Column 5 is 7 (column 1) $+\,2$ (column 4). That solution is $(-7, 0, 0, -2, 1)$.

The pivot columns are independent, and they span $\boldsymbol{C}(R)$, so they are a basis for $\boldsymbol{C}(R)$.

The dimension of the column space is the rank r***. The pivot columns form a basis.***

3. The nullspace has dimension $n - r = 5 - 2$. There are $n - r = 3$ free variables. v_2, v_3, v_5 are free (no pivots in those columns). They yield the three special solutions $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, $\boldsymbol{s}_5$ to $R\boldsymbol{v} = \mathbf{0}$. Set a free variable to 1, and solve for the pivot variables v_1 and v_4.

$$\boldsymbol{s}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{s}_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{s}_5 = \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$R\boldsymbol{v} = \mathbf{0}$ has the complete solution $\boldsymbol{v} = v_2\boldsymbol{s}_2 + v_3\boldsymbol{s}_3 + v_5\boldsymbol{s}_5$

There is a special solution for each free variable. With n variables and r pivot variables, that leaves $n - r$ free variables and special solutions. $\boldsymbol{N}(R)$ has dimension $n - r$.

The nullspace has dimension $n - r$***. The special solutions form a basis.***

The special solutions are independent, because they contain the identity matrix in rows 2, 3, 5. All solutions are combinations of special solutions, $\boldsymbol{v} = v_2\boldsymbol{s}_2 + v_3\boldsymbol{s}_3 + v_5\boldsymbol{s}_5$, because this puts v_2, v_3 and v_5 in the correct positions. Then the pivot variables v_1 and v_4 are totally determined by the equations $R\boldsymbol{v} = \mathbf{0}$.

4. The nullspace of R^{T} (*the left nullspace of R*) has dimension $m - r = 3 - 2$.

Reason: The equation $R^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ looks for combinations of the columns of R^{T} (*the rows of* R) that produce zero. You see why y_1 and y_2 must be zero, and y_3 *is free*.

Left nullspace

$$\begin{array}{r} y_1\,[1,\ 3,\ 5,\ 0,\ 7] \\ +y_2\,[0,\ 0,\ 0,\ 1,\ 2] \\ +y_3\,[0,\ 0,\ 0,\ 0,\ 0] \\ \hline [0\ \ 0\ \ y_3\,]R = [0,\ 0,\ 0,\ 0,\ 0] \end{array} \tag{1}$$

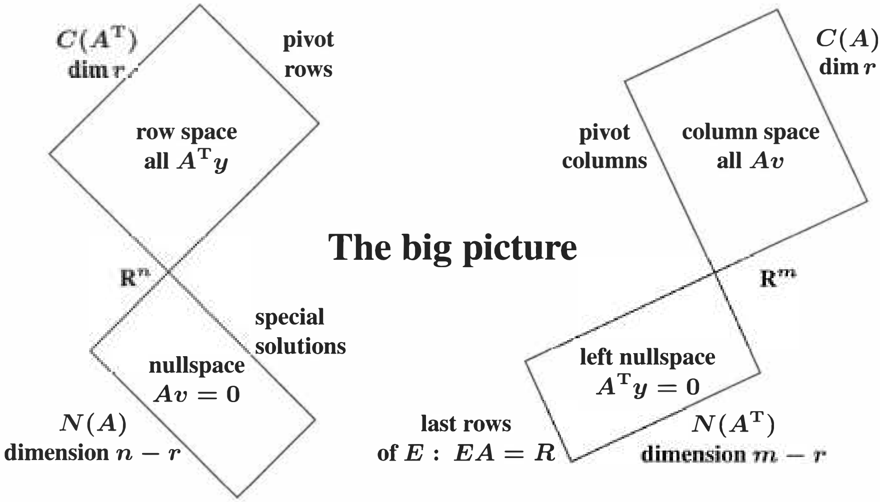


Figure 5.6: Bases and dimensions of the Four Fundamental Subspaces.

In all cases R ends with $m - r$ zero rows. Every combination of these $m - r$ rows gives zero. These are the *only* combinations of the rows of R that give zero, because the r pivot rows are linearly independent. The left nullspace of R contains all these solutions $\boldsymbol{y} = (0, \ldots, 0, y_{r+1}, \ldots, y_m)$ to $R^T\boldsymbol{y} = \mathbf{0}$.

***If** A **is** m **by** n **of rank** r**, its left nullspace has dimension** $m - r$.*

This subspace came fourth, and it completes the picture of linear algebra.

In** $\mathbf{R}^n$ **the row space and nullspace have dimensions r and $n - r$ (adding to n).
In** $\mathbf{R}^m$ **the column space and left nullspace have dimensions r and $m - r$ (total m).

So far this is proved for echelon matrices R. Figure 5.6 shows the same for A.

The Four Subspaces for A

We have a job still to do. ***The subspace dimensions for A are the same as for R.*** The job is to explain why. A is now any matrix that reduces to $R = \text{rref}(A)$.

$$\textbf{This } A \textbf{ reduces to } R \qquad A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \qquad \text{Notice } C(A) \neq C(R) \qquad (2)$$

An elimination matrix takes A to R. The big picture (Figure 5.6) applies to both. The invertible matrix E is the product of the elementary matrices that reduce A to R:

$$\textbf{A to R and back} \qquad EA = R \quad \text{and} \quad A = E^{-1}R \qquad (3)$$

1 ***A has the same row space as R.* Same dimension r and same basis.**

Reason: Every row of A is a combination of the rows of R. Also every row of R is a combination of the rows of A. Elimination changes rows, but not row *spaces*.

Since A has the same row space as R, we can choose the first r rows of R as a basis. *The first r rows of A could be dependent*. The good r rows of A end up as pivot rows.

2 ***The column space of A has dimension r.*** The r pivot columns of A are a basis.

The number of independent columns equals the number of independent rows.

Wrong reason: "A and R have the same column space." This is false. The columns of R often end in zeros. The columns of A don't often end in zeros. The column spaces can be different! But their *dimensions* are the same—both equal to r.

Right reason: The ***same combinations*** of the columns are zero (or nonzero) for A and R. Say that another way: $A\boldsymbol{v} = \mathbf{0}$ *exactly when* $R\boldsymbol{v} = \mathbf{0}$. Pivot columns are independent.

We have just given one proof of the first great theorem of linear algebra: **Row rank equals column rank**. This was easy for R, and the ranks are the same for A. The Chapter 5 Notes propose three direct proofs not using R.

3 ***A has the same nullspace as R.* Same dimension $n - r$ and same basis.**

Reason: The elimination steps don't change the solutions. The special solutions are a basis for this nullspace (as we always knew). There are $n - r$ free variables, so the dimension of the nullspace is $n - r$. Notice that $r + (n - r)$ equals n:

$$\boxed{(\textbf{dimension of column space}) + (\textbf{dimension of nullspace}) = \textbf{dimension of } \mathbf{R}^n.}$$

That beautiful fact is the **Counting Theorem.** Now apply it also to A^{T}.

4 ***The left nullspace of A*** (the nullspace of A^{T}) ***has dimension $m - r$.***

Reason: A^{T} is just as good a matrix as A. When we know the dimensions for every A, we also know them for A^{T}. Its column space was proved to have dimension r. Since A^{T} is n by m, the "whole space" is now $\mathbf{R}^m$. The counting rule for A was $r + (n - r) = n$. The counting rule for A^{T} is $r + (m - r) = m$. We have all details of the main theorem:

***Fundamental Theorem of Linear Algebra*, Part 1**

The column space and row space both have dimension r.

The nullspaces have dimensions $n - r$ and $m - r$.

By concentrating on *spaces* of vectors, not on individual numbers or vectors, we get these clean rules. You will soon take them for granted. But for an 11 by 17 matrix with 187 nonzero entries, I don't think most people would see why these facts are true:

Two key facts

$$\begin{aligned} &\text{dimension of } \boldsymbol{C}(A) = \text{dimension of } \boldsymbol{C}(A^{\mathrm{T}}) = \text{rank of } A \\ &\text{dimension of } \boldsymbol{C}(A) + \text{dimension of } \boldsymbol{N}(A) = 17. \end{aligned}$$

Example 1 $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ has $m = 1$ and $n = 3$ and rank $r = 1$.

The row space is a line in $\mathbf{R}^3$. The nullspace is the plane $Av = x + 2y + 3z = 0$. This plane has dimension 2 (which is $3 - 1$). The dimensions add to $\mathbf{1} + \mathbf{2} = \mathbf{3}$.

The columns of this 1 by 3 matrix are in $\mathbf{R}^1$. The column space is all of $\mathbf{R}^1$. The left nullspace contains only the zero vector. The only solution to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ is $\boldsymbol{y} = \mathbf{0}$, no other multiple of $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ gives the zero row. Thus $\boldsymbol{N}(A^{\mathrm{T}})$ is $\mathbf{Z}$, the zero space with dimension 0 (which is $m - r$). In $\mathbf{R}^m$ the dimensions add to $\mathbf{1} + \mathbf{0} = \mathbf{1}$.

Example 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has $m = 2$ and $n = 3$ and rank $r = 1$.

The row space is the same line through $(1, 2, 3)$. The nullspace must be the same plane $x + 2y + 3z = 0$. The dimensions of those two spaces still add to n: $1 + 2 = 3$.

All columns are multiples of the first column $(1, 2)$. Twice the first row minus the second row is the zero row. Therefore $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has the solution $\boldsymbol{y} = (2, -1)$. The column space and left nullspace are **perpendicular lines** in $\mathbf{R}^2$. Dimensions add to m: $1 + 1 = 2$.

$$\text{Column space} = \text{line through} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{Left nullspace} = \text{line through} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

If A has three equal rows, its rank is _____. What are two of the $\boldsymbol{y}$'s in its left nullspace?

The $\boldsymbol{y}$'s in the left nullspace combine with the rows to give the zero row.

Matrices of Rank One

Those examples had rank $r = 1$—and rank one matrices are special. We can describe them all. You will see again that dimension of row space = dimension of column space. When $r = 1$, every row is a multiple of the same row $\boldsymbol{r}^{\mathrm{T}}$:

$$\boldsymbol{A} = \boldsymbol{c}\boldsymbol{r}^{\mathbf{T}} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{is} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \boldsymbol{r}^{\mathrm{T}}.$$

A column times a row (4 by 1 times 1 by 3) produces a matrix (4 by 3). All rows are multiples of the row $\boldsymbol{r}^{\mathrm{T}} = (1, 2, 3)$. All columns are multiples of the first column $\boldsymbol{c} = (1, 2, -3, 0)$. The row space is a line in $\mathbf{R}^n$, and the column space is a line in $\mathbf{R}^m$.

Every rank one matrix has the special form $A = \boldsymbol{c}\boldsymbol{r}^{\mathrm{T}}$ = column times row.

All columns are multiples of $\boldsymbol{c}$. All rows are multiples of $\boldsymbol{r}^{\mathrm{T}}$. *The nullspace is the plane perpendicular to $\boldsymbol{r}$.* ($A\boldsymbol{v} = \mathbf{0}$ means that $\boldsymbol{c}(\boldsymbol{r}^{\mathrm{T}}\boldsymbol{v}) = \mathbf{0}$ and then $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{v} = 0$.) This **perpendicularity** of the subspaces will become Part 2 of the Fundamental Theorem.

A column vector $\boldsymbol{c}$ times a row vector $\boldsymbol{r}^{\mathrm{T}}$ is often called an *outer product*. The inner product $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{c}$ is a number, the outer product $\boldsymbol{c}\boldsymbol{r}^{\mathrm{T}}$ is a matrix.

Perpendicular Subspaces

Look at the equation $A\boldsymbol{v} = \mathbf{0}$. This says that $\boldsymbol{v}$ is in the nullspace of A. **It also says that $\boldsymbol{v}$ is perpendicular to every row of $\boldsymbol{A}$.** The first row multiplies $\boldsymbol{v}$ to give the first zero in $A\boldsymbol{v} = \mathbf{0}$:

$$A\boldsymbol{v} = \begin{bmatrix} \textbf{row 1} \\ \cdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{v} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 3 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{-3} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector $\boldsymbol{v} = (1, -3, 2)$ in the nullspace is perpendicular to the first row $(1, 1, 1)$. Their dot product is $1 - 3 + 2 = 0$. That vector $\boldsymbol{v}$ is also perpendicular to the rows $(3, 1, 0)$ and $(0, 2, 3)$—because of the zeros on the right hand side. The dot product of every row and every $\boldsymbol{v}$ is zero.

Every $\boldsymbol{v}$ in the nullspace is perpendicular to the whole row space. It is perpendicular to each row and it is perpendicular to all combinations of rows. We have found new words to describe the nullspace of A:

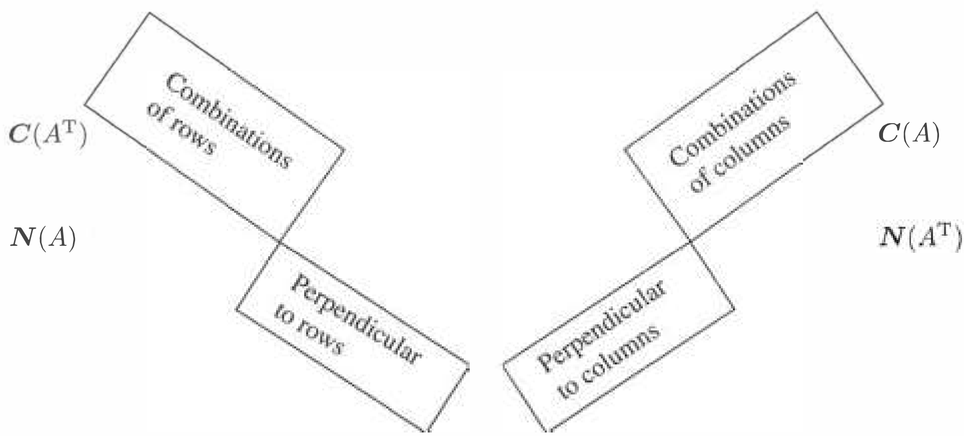
$\boldsymbol{N}(\boldsymbol{A})$ contains all vectors $\boldsymbol{v}$ that a perpendicular to the row space of $\boldsymbol{A}$.

These two fundamental subspaces $\boldsymbol{N}(A)$ and $\boldsymbol{R}(A^{\mathrm{T}})$ now have a *position in space*. They are "orthogonal subspaces" like the xy plane and the z axis in $\boldsymbol{R}^3$. Tilt that picture and you still have orthogonal subspaces. Their dimensions 2 and 1 still add to 3: the dimension of the whole space. For any matrix, the r-dimensional row space is perpendicular to the $(n-r)$-dimensional nullspace. If that matrix is A^{T} instead of A, we have subspaces of $\boldsymbol{R}^m$.

(In $\boldsymbol{R}^n$) All solutions to $A\boldsymbol{v} = \mathbf{0}$ are perpendicular to all *rows* of A.
(In $\boldsymbol{R}^m$) All solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are perpendicular to all *columns* of A.

If A is square and invertible, the two nullspaces are just $\boldsymbol{Z}$: only the zero vector. The row and column spaces are the whole space. These are the extreme in perpendicular subspaces: everything and nothing. No, *not nothing*, the zero vector is perpendicular to everything.

Let me draw the big picture using this new insight of perpendicular subspaces.



This perpendicularity is Part 2 of the Fundamental Theorem of Linear Algebra. We use a new symbol $\boldsymbol{S}^{\perp}$ (*called* $\boldsymbol{S}$ *perp*) for all vectors that are orthogonal to the subspace $\boldsymbol{S}$.

$$\textbf{Fundamental Theorem, Part 2}: \boldsymbol{N}(A) = \boldsymbol{C}(A^{\mathrm{T}})^{\perp} \text{ and } \boldsymbol{N}(A^{\mathrm{T}}) = \boldsymbol{C}(A)^{\perp}.$$

We know we have *all* perpendicular vectors (not just some of them, like 2 lines in space). The dimensions r and $n-r$ add to the full dimension n. For a line and plane in $\boldsymbol{R}^3$: (Line in space)$^{\perp}$ = (Plane in space) and $1+2=3$.

Here is Problem 37 in the problem set: Explain why $(\boldsymbol{S}^{\perp})^{\perp} = \boldsymbol{S}$.

■ REVIEW OF THE KEY IDEAS ■

1. The r pivot rows of R are a basis for the row spaces of R and A (same space).
2. The r pivot columns of A (not R) are a basis for its column space $\boldsymbol{C}(A)$.
3. The $n-r$ special solutions are a basis for the nullspaces of A and R (same space).
4. The last $m-r$ rows of I are a basis for the left nullspace of R.
5. The last $m-r$ rows of E are a basis for the left nullspace of A, if $EA = R$.
6. $\boldsymbol{R}(A^{\mathrm{T}})$ is perpendicular to $\boldsymbol{N}(A)$. And $\boldsymbol{C}(A)$ is perpendicular to $\boldsymbol{N}(A^{\mathrm{T}})$.

■ WORKED EXAMPLES ■

5.5 A Find bases and dimensions for all four fundamental subspaces if you know that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E^{-1}R.$$

By changing only *one number* in R, change the dimensions of all four subspaces.

Solution This matrix has pivots in columns 1 and 3. Its rank is $r = 2$.

Row space	Basis $(1,3,0,5)$ and $(0,0,1,6)$ from R. Dimension 2.
Column space	Basis $(1,2,5)$ and $(0,1,0)$ from E^{-1} (and A). Dimension 2.
Nullspace	Basis $(-3,1,0,0)$ and $(-5,0,-6,1)$ from R. Dimension 2.
Nullspace of A^{T}	Basis $(-5,0,1)$ from row 3 of E. Dimension $3-2=1$.

We need to comment on that left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$. $EA = R$ says that the last row of E combines the three rows of A into the zero row of R. So that last row of E is a basis vector for the left nullspace. If R had *two* zero rows, then the last *two* rows of E would be a basis. (Just like elimination, $\boldsymbol{y}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$ combines rows of A to give zero rows in R.)

To change all these dimensions we need to change the rank r. The way to do that is to change the zero row of R. **The best entry to change is R_{34} in the corner**.

5.5 B How can you put four 1's into a 5 by 6 matrix of zeros, so that its *row space* has dimension 1? Describe all the ways to make its *column space* have dimension 1. Describe all the ways to make the dimension of its *nullspace* $\boldsymbol{N}(A)$ as small as possible. How would you make the *sum of the dimensions of all four subspaces small*?

Solution The rank is 1 if the four 1's go into the same row, or into the same column. They can also go into *two rows and two columns* (so $a_{ii} = a_{ij} = a_{ji} = a_{jj} = 1$). Since the column space and row space always have the same dimension, this answers the first two questions: The smallest dimension is 1.

The nullspace has its smallest possible dimension $6 - 4 = 2$ when the rank is $r = 4$. To achieve rank 4, the 1's must go into four different rows and columns.

You can't do anything about the sum $r + (n - r) + r + (m - r) = n + m$. It will be $6 + 5 = 11$ no matter how the 1's are placed. The sum is 11 even if there aren't any 1's...

If all the other entries of A are 2's instead of 0's, how do these answers change?

Problem Set 5.5

1 (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

2 Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Construct a matrix with the required property or explain why this is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(c) Dimension of nullspace $= 1 +$ dimension of left nullspace.

(d) Left nullspace contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, row space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

5 If $\boldsymbol{V}$ is the subspace spanned by $(1,1,1)$ and $(2,1,0)$, find a matrix A that has $\boldsymbol{V}$ as its row space. Find a matrix B that has $\boldsymbol{V}$ as its nullspace.

6 Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

7 Suppose the 3 by 3 matrix A is invertible. Write down bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [\, A \;\; A \,]$.

8 What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \end{bmatrix}.$$

9 Which subspaces are the same for these matrices of different sizes?

(a) $[\, A \,]$ and $\begin{bmatrix} A \\ A \end{bmatrix}$ (b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

Prove that all three of those matrices have the *same rank* r.

10 If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces ? What if the matrix is 3 by 5?

11 (Important) A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{v} = \boldsymbol{b}$ has *no solution*.

(a) What are all inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ has solutions other than $\boldsymbol{y} = \boldsymbol{0}$?

12 Construct a matrix with $(1,0,1)$ and $(1,2,0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

13 True or false (with a reason or a counterexample):

(a) If $m = n$ then the row space of A equals the column space.

(b) The matrices A and $-A$ share the same four subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

14 Without computing A, find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

15 If you exchange the first two rows of A, which of the four subspaces stay the same? If $\boldsymbol{v} = (1, 2, 3, 4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix.

16 Explain why $\boldsymbol{v} = (1, 0, -1)$ *cannot be a row of A and also in the nullspace.*

17 Describe the four subspaces of $\boldsymbol{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

18 (Left nullspace) Add the extra column $\boldsymbol{b}$ and reduce A to echelon form:

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^{T} and which vectors are in the nullspace of A?

19 Following the method of Problem 18, reduce A to echelon form and look at the zero rows. The $\boldsymbol{b}$ column tells which combinations you have taken of the rows:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$$

From the $\boldsymbol{b}$ column after elimination, read off $m - r$ basis vectors in the left nullspace. Those $\boldsymbol{y}$'s are combinations of rows that give zero rows.

20 (a) Find the solutions to $A\boldsymbol{v} = \boldsymbol{0}$. Check that $\boldsymbol{v}$ is are perpendicular to the rows:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = ER.$$

(b) How many independent solutions to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$? Why is $\boldsymbol{y}^{\mathrm{T}}$ the last row of E^{-1}?

21 Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if _____ or if _____.

(d) Compute A and its rank if $\boldsymbol{u} = \boldsymbol{z} = (1, 0, 0)$ and $\boldsymbol{v} = \boldsymbol{w} = (0, 0, 1)$.

22 Construct $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ whose column space has basis $(1, 2, 4), (2, 2, 1)$ and whose row space has basis $(1, 0), (1, 1)$. Write A as (3 by 2) times (2 by 2).

23 Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that $A =$ (3 by 2) (2 by 3) cannot be invertible?

24 (Important) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ is solvable when $\boldsymbol{d}$ is in which of the four subspaces? The solution $\boldsymbol{y}$ is unique when the _____ contains only the zero vector.

25 True or false (with a reason or a counterexample):

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space of A.

26 (***Rank of $AB \le$ ranks of A and B***) If $AB = C$, the rows of C are combinations of the rows of _____. So the rank of C is not greater than the rank of _____. Since $B^{\mathrm{T}}A^{\mathrm{T}} = C^{\mathrm{T}}$, the rank of C is also not greater than the rank of _____.

27 If a, b, c are given with $a \neq 0$, how would you choose d so that $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

28 Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & \text{four zero rows} & & & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and the left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

29 Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that rank $(A) = 2$ but neither side passed up a winning move?

Problems 30-33 are about perpendicularity of the fundamental subspaces (two perpendicular pairs.)

30 The floor and a wall of your room are *not* perpendicular subspaces in $\boldsymbol{R}^3$. *Why not*? I am extending the floor and wall to be planes in $\boldsymbol{R}^3$.

31 Explain why every $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is perpendicular to every column of A.

32 Suppose $\boldsymbol{P}$ is the plane of vectors $\boldsymbol{R}^4$ satisfying $v_1 + v_2 + v_3 + v_4 = 0$. Find a basis for $\boldsymbol{P}^{\perp}$. Find a matrix A with $\boldsymbol{N}(A) = \boldsymbol{P}$.

33 Why can't A have $(1, 4, 5)$ in its row space and $(4, 5, 1)$ in its nullspace?

Challenge Problems

34 If $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ is a 2 by 2 matrix of rank 1, redraw Figure 5.6 to show clearly the Four Fundamental Subspaces in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$. If another matrix B produces those same four subspaces, what is the exact relation of B to A?

35 **M** is the 9-dimensional space of 3 by 3 matrices. Multiply every matrix X by A:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Which matrices X lead to AX = zero matrix?

(b) Which matrices have the form AX for some matrix X?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of $\boldsymbol{M}$? Why do the dimensions add to $(n - r) + r = 9$?

36 Suppose the m by n matrices A and B lead to *the same four subspaces*. If both matrices are already in row reduced echelon form, prove that F must equal G:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}.$$

37 For any subspace $\boldsymbol{S}$ of $\boldsymbol{R}^n$, why is $(\boldsymbol{S}^{\perp})^{\perp} = \boldsymbol{S}$? "If $\boldsymbol{S}^{\perp}$ contains all vectors perpendicular to $\boldsymbol{S}$, then $\boldsymbol{S}$ contains all vectors perpendicular to $\boldsymbol{S}^{\perp}$." Dimensions add to n.

38 If $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{0}$ then $A\boldsymbol{v} = \boldsymbol{0}$. Reason: This $A\boldsymbol{v}$ is in the nullspace of A^{T}. Every $A\boldsymbol{v}$ is in the column space of A (*why*?). Those spaces are perpendicular, and only $A\boldsymbol{v} = \boldsymbol{0}$ can be perpendicular to itself. *So $A^{\mathrm{T}}A$ has the same nullspace as A.*

5.6 Graphs and Networks

Over the years I have seen one model so often, and I found it so basic and useful, that I always put it first. The model consists of ***nodes connected by edges***. This is called a ***graph***.

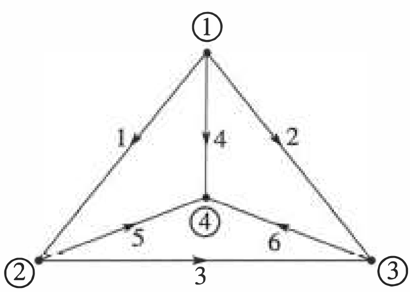
Graphs of the usual kind display functions $f(x)$. Graphs of this node-edge kind lead to matrices. This section is about the ***incidence matrix*** of a graph—which tells how the n nodes are connected by the m edges. Normally $m > n$, there are more edges than nodes.

Every entry of an incidence matrix is 0 or 1 or -1. This continues to hold during elimination. All pivots and multipliers are ± 1. Then the echelon matrix R after elimination also contains $0, 1, -1$. So do the special solutions ! All four subspaces have basis vectors with these exceptionally simple components. The matrices are not concocted for a textbook, they come from a model that is absolutely essential in pure and applied mathematics.

For these incidence matrices, the four fundamental subspaces have meaning and importance. Up to now, I have created small matrix examples to show the column space and nullspace. I was claiming that all four subspaces need to be understood, but you wouldn't know their importance from such small examples. Now comes the chance to learn about the most valuable models in discrete mathematics—graphs and their matrices.

Graphs and Incidence Matrices

Figure 5.7 displays a *graph* with $m = 6$ edges and $n = 4$ nodes. Its incidence matrix will be 6 by 4. This matrix A tells which nodes are connected by which edges. The entries -1 and $+1$ also tell the direction of each arrow. *The first row* $-1, 1, 0, 0$ *of* A (the incidence matrix) *shows that the first edge goes from node* 1 *to node* 2.



$$\begin{array}{cc} & \textbf{node} \\ & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ A = & \begin{bmatrix} \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{1} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \quad \textbf{edge}$$

Figure 5.7: Complete graph with $m = 6$ edges and $n = 4$ nodes. Edge 1 gives row 1.

Row numbers in A are edge numbers on the graph. Column numbers are node numbers. This particular graph is *complete*—every pair of nodes is connected by an edge. You can write down A immediately by looking at the graph. The graph and the matrix have the same information.

If edge 6 is removed from the graph, row 6 is removed from the matrix. The constant vector $(1, 1, 1, 1)$ is still in the nullspace of A. Our goal is to understand all four of the fundamental subspaces coming from A.

The Nullspace and Row Space

For the nullspace we solve $A\boldsymbol{v} = \mathbf{0}$. By writing down those m equations we see that A is a **difference matrix**:

$$A\boldsymbol{v} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_1 \\ v_3 - v_2 \\ v_4 - v_1 \\ v_4 - v_2 \\ v_4 - v_3 \end{bmatrix}. \tag{1}$$

The numbers v_1, v_2, v_3, v_4 can represent ***voltages*** at the nodes. Then $A\boldsymbol{v}$ gives the ***voltage differences*** across the six edges. It is these differences that make currents flow.

The nullspace contains the solutions to $A\boldsymbol{v} = \mathbf{0}$. All six voltage differences are zero. This means: All four voltages are *equal*. **Every $\boldsymbol{v}$ in the nullspace is a constant vector** $\boldsymbol{v} = (c, c, c, c)$. The nullspace of A is a line in $\mathbf{R}^n$. Its dimension is $n - r = 1$, so $\boldsymbol{r} = \mathbf{3}$.

Counting Theorem $\quad \boldsymbol{r} + (\boldsymbol{n} - \boldsymbol{r}) = \mathbf{3} + \mathbf{1} = \mathbf{4} =$ **count of columns.**

We can raise or lower all voltages by the same c, without changing the voltage *differences*. There is an "arbitrary constant" in $\boldsymbol{v}$. For functions, we can raise or lower $f(x)$ by any constant amount C, without changing its derivative.

Calculus adds an arbitrary constant "$+C$" to indefinite integrals. Graph theory adds (c, c, c, c) to the voltages. Linear algebra adds any vector $\boldsymbol{v}_n$ in the nullspace to one particular solution of $A\boldsymbol{v} = \boldsymbol{b}$.

The **row space** of A is also a subspace of $\mathbf{R}^4$. Every row adds to zero, because -1 cancels $+1$ in each row. Then every combination of the rows also adds to zero. This is just saying that $\boldsymbol{v} = (c, c, c, c)$ in the nullspace is orthogonal to every vector in the row space.

For any connected graph with n nodes, the situation is the same. The vectors $\boldsymbol{v} = (c, \ldots, c)$ fill the nullspace in $\mathbf{R}^n$. All rows are orthogonal to $\boldsymbol{v}$; their components add to zero. **The row space $\boldsymbol{C}(A^{\mathrm{T}})$ has dimension** $n - 1$. This is the rank of A.

The Column Space and Left Nullspace

The ***column space*** contains all combinations of the four columns. We expect three independent columns, since the rank is $r = n - 1 = 3$. The first three columns are independent (so are any three). But the four columns add to the zero vector, which says again that $(1, 1, 1, 1)$ is in the nullspace. ***How can we tell if a particular vector b is in the column space of an incidence matrix***?

First answer Apply elimination to $A\boldsymbol{v} = \boldsymbol{b}$. On the left side, some combinations of rows will give zero rows. Then the same combination of b's on the right side must be zero! Here is the first combination that elimination will discover:

$$\text{Row 1} - \text{Row 2} + \text{Row 3} = \textit{Zero row}. \text{ The right side } \boldsymbol{b} \text{ needs } b_1 - b_2 + b_3 = 0. \tag{2}$$

Since A has $m = 6$ rows and its rank is $r = 3$, elimination leads to $6 - 3$ zero rows in the reduced matrix R. There will be *three tests* for the vector $\boldsymbol{b}$ to lie in the column space. Elimination will lead to *three conditions* on $\boldsymbol{b}$ for $A\boldsymbol{v} = \boldsymbol{b}$ to be solvable.

I want to find those conditions in a better way. The graph has three small loops.

Second answer using loops $A\boldsymbol{v}$ contains differences in v's. If we add differences around a closed loop in the graph, the cancellation leaves zero. Around the big triangle formed by edges $1, 3, -2$ (the arrow goes backward on edge 2) the differences cancel out:

Around a loop $$(v_2 - v_1) + (v_3 - v_2) - (v_3 - v_1) = 0.$$

The components of $A\boldsymbol{v}$ add to zero around every loop. When $\boldsymbol{b}$ is in the column space of A, then $A\boldsymbol{v} = \boldsymbol{b}$. The vector $\boldsymbol{b}$ must obey the voltage law:

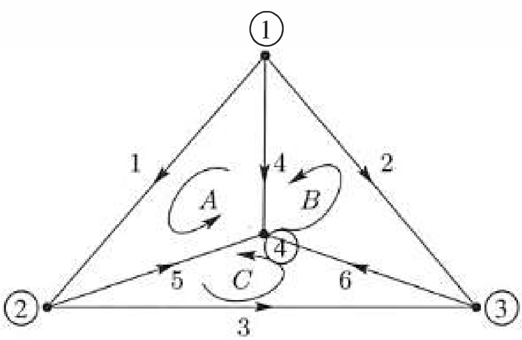
KVL ***Kirchhoff's Voltage Law*** (on a typical loop) $\qquad b_1 + b_3 - b_2 = 0.$

By testing all the loops, we decide whether $\boldsymbol{b}$ is in the column space. $A\boldsymbol{v} = \boldsymbol{b}$ can be solved exactly when the components of $\boldsymbol{b}$ satisfy all the same dependencies as the rows of A. Then KVL is satisfied, elimination leads to $0 = 0$, and $A\boldsymbol{v} = \boldsymbol{b}$ is consistent.

Question I can see four loops in the graph, three small and one large. We are only expecting three tests, not four, for $\boldsymbol{b}$ to be in $\boldsymbol{C}(A)$. What is the explanation?

Answer Those four loops are not independent. If you combine the small loops in Figure 5.8, you get the large loop. So the tests from the small loops combine to give the test from the large loop. We only have to test KVL on the small loops.

We have described the column space of A in two ways. First, $\boldsymbol{C}(A)$ contains all combinations of the columns (and $n - 1$ columns are enough , the nth column is dependent). Second, $\boldsymbol{C}(A)$ contains all vectors $\boldsymbol{b}$ that satisfy the Voltage Law. Around every loop the components of $\boldsymbol{b}$ add to zero. We will now see that this is requiring $\boldsymbol{b}$ to be orthogonal to every vector $\boldsymbol{y}$ in the nullspace of A^{T}. $\boldsymbol{C}(A)$ *is orthogonal to the left nullspace* $\boldsymbol{N}(A^{\mathrm{T}})$.



	Voltage laws
Loop A	$b_1 - b_4 + b_5 = 0$
Loop B	$b_4 - b_6 - b_2 = 0$
Loop C	$b_3 + b_6 - b_5 = 0$

Big loop $A + B + C$: $b_1 - b_2 + b_3 = 0$

Figure 5.8: Loops reveal the column space of A and the nullspace of A^{T} and the tests on $\boldsymbol{b}$.

$\boldsymbol{N}(A^{\mathrm{T}})$ contains all solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Its dimension is $m - r = 6 - 3$: **three $\boldsymbol{y}$'s.**

$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (3)$$

The true number of equations is $r = 3$ and not $n = 4$. Reason : The four equations add to $0 = 0$. The fourth equation follows automatically from the first three.

What do the equations mean? The first equation says that $-y_1 - y_2 - y_4 = 0$. ***The net flow into node 1 is zero***. The fourth equation says that $y_4 + y_5 + y_6 = 0$. ***Flow into the node minus flow out is zero***. These equations are famous and fundamental:

Kirchhoff's Current Law	$A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$	***Flow in equals flow out at each node.***

This law deserves first place among the equations of applied mathematics. It expresses "*conservation*" and "*continuity*" and "*balance*." Nothing is lost, nothing is gained. When currents or forces are balanced, the equation to solve is $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Notice the beautiful fact that the matrix in this balance equation is the transpose of the incidence matrix A.

What are the actual solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$? The currents must balance themselves. The easiest way is to **flow around a loop**. If a unit of current goes around the big triangle (forward on edge 1, forward on 3, backward on 2), the vector is $\boldsymbol{y} = (1, -1, 1, 0, 0, 0)$. This satisfies $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. *Every loop current is a solution to Kirchhoff's Current Law.*

Around the loop, flow in equals flow out at every node. The smaller loop A goes forward on edge 1, forward on 5, back on 4. Then $\boldsymbol{y} = (1, 0, 0, -1, 1, 0)$ will have $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. **Each loop in the graph gives a vector $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$.**

We expect three independent $\boldsymbol{y}$'s, since $6 - 3 = 3$. The three small loops in the graph are independent. The big triangle seems to give a fourth $\boldsymbol{y}$, but it is the sum of flows around the small loops. The small loops A, B, C give a basis $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ for the nullspace of A^{T}.

Solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$
Big loop
from three
small loops

$$\boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 = \underset{A}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + \underset{B}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} + \underset{C}{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}} = \underset{A+B+C}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

Summary The m by n incidence matrix A comes from a connected graph with n nodes and m edges. The row space and column space have dimension $r = n - 1 = \text{rank of } A$. The nullspaces of A and A^{T} have dimension 1 and $m - r = m - n + 1$:

1 The constant vectors $(c, c, \ldots, c)$ make up the nullspace $\boldsymbol{N}(A)$.

2 There are $r = n - 1$ independent rows, from $n - 1$ edges with no loops (a tree).

3 ***Voltage law gives*** $\boldsymbol{C}(A)$: The components of $A\boldsymbol{v}$ add to zero around every loop.

4 ***Current law*** $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $\boldsymbol{N}(A^{\mathrm{T}})$ from currents on $m - r$ independent loops.

For every graph in a plane, linear algebra yields ***Euler's formula*** :

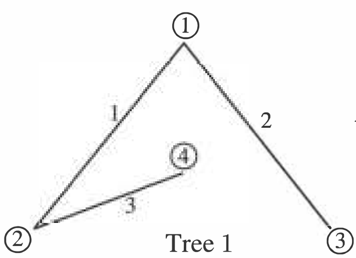
$$\boxed{(\textit{\textbf{number of nodes}}) - (\textit{\textbf{number of edges}}) + (\textit{\textbf{number of small loops}}) = \mathbf{1}.}$$

This is $(\boldsymbol{n}) - (\boldsymbol{m}) + (\boldsymbol{m} - \boldsymbol{n} + \mathbf{1}) = \mathbf{1}$. The graph in our example has $4 - 6 + 3 = 1$.

A single triangle has (3 nodes) $-$ (3 edges) $+$ (1 loop). On a 10-node tree with 9 edges and no loops, Euler's count is $10 - 9 + 0 = 1$. All planar graphs lead to the answer 1.

Trees

A tree is a graph with no loops. Figure 5.9 shows two trees with $n = 4$ nodes. These graphs (and all our graphs) are *connected*: Between every two nodes there is a path of edges, so the graph doesn't break into separate pieces. The tree must have $m = n - 1$ edges, to connect all n nodes. The rank of the incidence matrix is also $r = n - 1$. Then the number of loops in a tree is confirmed as $m - r = 0$ (no loops).



$$A_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

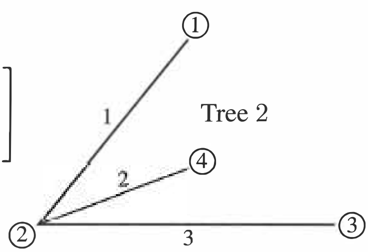


Figure 5.9: Two trees with $n = 4$ nodes and $m = 3$ edges. The rank of A_1 is $r = m$.

The incidence matrix A of a tree has *independent rows*. In fact the three rows of A_1 are three independent rows $1, 2, 5$ of the previous 6 by 4 matrix (for the complete graph).

That original graph contains 16 different trees.

The Adjacency Matrix and the Graph Laplacian

The adjacency matrix W is square. With n nodes in the graph, this matrix is n by n. If there is an edge from node i to node j, then $W_{ij} = 1$. If no edge, then $W_{ij} = 0$. Since our edges go both ways, W is symmetric. The diagonal entries are zero.

All information about the graph is in the adjacency matrix W, except the numbering and arrow directions of the edges.

There are m 1's above the diagonal of W, and also below. Section 7.5 will study the **graph Laplacian matrix** $\boldsymbol{A^{\mathrm{T}}A}$ (A is the incidence matrix) and find this formula:

$$\boxed{\textbf{Graph Laplacian} \qquad \boldsymbol{A^{\mathrm{T}}A = D - W} = (\textbf{degree matrix}) - (\textbf{adjacency matrix}).}$$

The diagonal matrix D tells the "degree" of every node. This is the number of edges that go in or out of that node. Here are W and $A^{\mathrm{T}}A$ for the complete graph with six edges.

$$\textbf{Adjacency } \boldsymbol{W} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad \textbf{Graph Laplacian } \boldsymbol{A^{\mathrm{T}}A} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Every row of $A^{\mathrm{T}}A$ adds to zero. The degree 3 on the diagonal cancels the -1's off the diagonal. The vector $(1, 1, 1, 1)$ in the nullspace of A is also in the nullspace of $A^{\mathrm{T}}A$.

Challenge Reconstruct a graph with arrows from A and a graph without arrows from W.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

■ REVIEW OF THE KEY IDEAS ■

1. The n nodes and m edges of a graph give n columns and m rows in A.

2. Each row of the incidence matrix A has -1 and 1 (start and end of that edge).

3. **Voltage Law for $\boldsymbol{C}(A)$:** The components of $A\boldsymbol{v}$ add to zero around any loop.

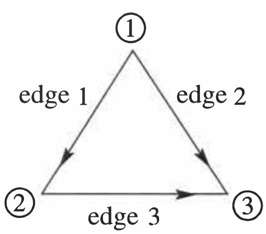
4. **Current Law for $\boldsymbol{N}(A^{\mathrm{T}})$:** $A^{\mathrm{T}}\boldsymbol{y} =$ (flow in) minus (flow out) $=$ zero at every node.

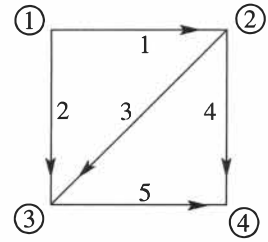
5. Rank of $A = n - 1$. Then $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ for the currents $\boldsymbol{y}$ around $m - n + 1$ small loops.

6. The adjacency matrix W and the graph Laplacian $A^{\mathrm{T}}A$ are symmetric n by n.

Problem Set 5.6

Problems 1–7 and 8–13 are about the incidence matrices for these two graphs.





1 Write down the 3 by 3 incidence matrix A for the triangle graph. The first row has -1 in column 1 and $+1$ in column 2. What vectors (v_1, v_2, v_3) are in its nullspace? How do you know that $(1, 0, 0)$ is not in its row space?

2 Write down A^{T} for the triangle graph. Find a vector $\boldsymbol{y}$ in its nullspace. The components of $\boldsymbol{y}$ are currents on the edges—how much current is going around the triangle?

3 By elimination on A find the echelon matrix R. What tree corresponds to the two nonzero rows of R?

$$A\boldsymbol{v} = \boldsymbol{b} \qquad \begin{aligned} -v_1 + v_2 &= b_1 \\ -v_1 + v_3 &= b_2 \\ -v_2 + v_3 &= b_3. \end{aligned}$$

4 Choose a vector (b_1, b_2, b_3) for which $A\boldsymbol{v} = \boldsymbol{b}$ can be solved, and another vector $\boldsymbol{b}$ that allows no solution. What are the dot products $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ for $\boldsymbol{y} = (1, -1, 1)$?

5 Choose a vector (f_1, f_2, f_3) for which $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ can be solved, and a vector $\boldsymbol{f}$ that allows no solution. How are those $\boldsymbol{f}$'s related to $\boldsymbol{v} = (1, 1, 1)$? The equation $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is Kirchhoff's _____ law.

6 Multiply matrices to find $A^{\mathrm{T}}A$. Choose a vector $\boldsymbol{f}$ for which $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{f}$ can be solved, and solve for $\boldsymbol{v}$. Put those voltages $\boldsymbol{v}$ and currents $\boldsymbol{y} = -A\boldsymbol{v}$ onto the triangle graph. The vector f represents "current sources."

7 Multiply $A^{\mathrm{T}}A$ (still for the first graph) and find its nullspace—it should be the same as $\boldsymbol{N}(A)$. Which vectors $\boldsymbol{f}$ are in its column space?

8 Write down the 5 by 4 incidence matrix A for the square graph with two loops. Find one solution to $A\boldsymbol{v} = \boldsymbol{0}$ and two solutions to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$. The rank is _____.

9 Find two requirements on the b's for the five differences $v_2 - v_1, v_3 - v_1, v_3 - v_2, v_4 - v_2, v_4 - v_3$ to equal b_1, b_2, b_3, b_4, b_5. You have found Kirchhoff's _____ Law around the two _____ in the graph.

10 By elimination, reduce A to U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

11 Multiply $A^{\mathrm{T}}A$ and explain how its entries come from columns of A (and the graph).

(a) The diagonal of the Laplacian matrix $A^{\mathrm{T}}A$ counts edges into each node (the degree). Why is this the dot product of a column with itself?

(b) The off-diagonals -1 or 0 tell which nodes i and j are connected. Why is -1 or 0 the dot product of column i with another column j?

12 Find the rank and the nullspace of $A^{\mathrm{T}}A$. Why does $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{f}$ have a solution only if $f_1 + f_2 + f_3 + f_4 = 0$?

13 Write down the 4 by 4 adjacency matrix W for the square graph. Its entries 1 or 0 count paths of length 1 between nodes (those are just edges).

Important. Compute W^2 and check that its entries count the paths of length 2 between nodes. Why does $(W^2)_{ii}$ = degree of node i? Those paths go out and back.

14 A connected graph with 7 nodes and 7 edges has how many loops?

15 For the graph with 4 nodes, 6 edges, and 3 loops, add a new node. If you connect it to one old node, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$. If you connect it to two old nodes, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$.

16 Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?

(b) What condition on $\boldsymbol{f}$ makes it possible to solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$?

(c) The diagonal entries of $A^{\mathrm{T}}A$ give the number of edges into each node. What is the sum of those diagonal entries?

17 Why does a complete graph with $n = 6$ nodes have $m = 15$ edges? A tree that connects 6 nodes has only _____ edges and _____ loops.

18 How do you know that *any* $n - 1$ *columns* of the incidence matrix A are independent? If they were dependent, the nullspace would contain a vector with a zero component. But the nullspace of A actually contains _____.

19 (a) Find the Laplacian $A^{\mathrm{T}}A$ for a complete graph with n nodes.

(b) If the edge from node 1 to node 3 is removed, what is the change in $A^{\mathrm{T}}A$?

20 Suppose batteries of strength $b_1, \ldots, b_m$ are inserted into the m edges. Then the voltage differences across edges become $A\boldsymbol{v} - \boldsymbol{b}$. Unit resistances give currents $A\boldsymbol{v} - \boldsymbol{b}$ and Kirchhoff's Current Law is $A^{\mathrm{T}}(A\boldsymbol{v} - \boldsymbol{b}) = \boldsymbol{0}$. Solve this system for the square graph above when $\boldsymbol{b} = (1, 1, \ldots, 1)$.

■ CHAPTER 5 NOTES ■

Vectors are not necessarily column vectors. In the definition of a *vector space*, addition $x + y$ and scalar multiplication cx must obey the following eight rules:

(1) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$

(2) $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$

(3) There is a unique "zero vector" such that $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$

(4) For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$

(5) 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$

(6) $(c_1c_2)\boldsymbol{x} = c_1(c_2\boldsymbol{x})$

(7) $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$

(8) $(c_1 + c_2)\boldsymbol{x} = c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

Here are practice questions to bring out the meaning of those eight rules.

1. Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual multiplication $c\boldsymbol{x} = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

2. Suppose the multiplication $c\boldsymbol{x}$ is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2). With the usual addition in $\mathbf{R}^2$, are the eight conditions satisfied?

3. (a) Which rules are broken if we keep only the positive numbers $x > 0$ in $\mathbf{R}^1$? Every c must be allowed. The half-line is not a subspace.

(b) The positive numbers with $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ redefined to equal the usual xy and x^c *do* satisfy the eight rules. Test rule 7 when $c = 3, x = 2, y = 1$. (Then $\boldsymbol{x}+\boldsymbol{y} = 2$ and $c\boldsymbol{x} = 8$.) Which number acts as the "zero vector"?

4. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write down the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

5. The functions $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$ are "vectors in function space." Which rule is broken if multiplying $\boldsymbol{f}(x)$ by c gives $\boldsymbol{f}(cx)$ instead of $c\boldsymbol{f}(x)$? Keep the usual addition $\boldsymbol{f}(x) + \boldsymbol{g}(x)$.

6. If the sum of the "vectors" $\boldsymbol{f}(x)$ and $\boldsymbol{g}(x)$ is defined to be the function $f(g(x))$, then the "zero vector" is $\boldsymbol{g}(x) = x$. Keep the usual scalar multiplication $c\boldsymbol{f}(x)$ and find two rules that are broken.

Row rank equals column rank: The first big theorem

The dimension of the row space $C(A^{\mathrm{T}})$ equals the dimension of the column space $C(A)$. Here I can outline four proofs (the fourth is neat). Proofs **2, 3, 4** do not use elimination.

Proof 1 Reduce A to R without changing the dimensions of the row and column spaces. The row space actually stays the same. The column space changes, going from A to R, but its dimension stays the same. The theorem is clear for R:

$$\begin{array}{lcl} r \text{ nonzero rows in } R & \leftrightarrow & r = \text{dimension of row space} \\ r \text{ pivot columns in } R & \leftrightarrow & r = \text{dimension of column space} \end{array}$$

Proof 2 (G. Mackiw, *Mathematics Magazine* **68** 1996). Suppose $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ is a basis for the row space of A. The next paragraph will show that $A\boldsymbol{x}_1, \ldots, A\boldsymbol{x}_r$ are independent vectors in the column space. Then dim (row space) = $r \leq$ dim (column space). The same reasoning applies to A^{T}, reversing that inequality. So the two dimensions must be equal.

$$\text{Suppose} \quad c_1 A\boldsymbol{x}_1 + \cdots + c_r A\boldsymbol{x}_r = A(c_1\boldsymbol{x}_1 + \cdots + c_r\boldsymbol{x}_r) = A\boldsymbol{v} = \mathbf{0}.$$

Then $\boldsymbol{v}$ is in the nullspace of A and also in the row space (it is a combination of the $\boldsymbol{x}$'s). So $\boldsymbol{v}$ is orthogonal to itself and $\boldsymbol{v} = \mathbf{0}$. All the c's must be zero since the $\boldsymbol{x}$'s are a basis.

This shows that $c_1 A\boldsymbol{x}_1 + \cdots + c_r A\boldsymbol{x}_r = 0$ requires that all $c_i = 0$. Therefore $A\boldsymbol{x}_1, \ldots, A\boldsymbol{x}_r$ are independent vectors in the column space: dimension of $C(A) \geq r$.

Proof 3 If A has r independent rows and s independent columns, we can move those rows to the top of A and those columns to the left. They meet in an r by s submatrix B:

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \; r \text{ rows} \qquad \begin{bmatrix} B & C \\ D & E \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Suppose $s > r$. Since $B\boldsymbol{v} = \mathbf{0}$ has r equations in s unknowns, it has a solution $\boldsymbol{v} \neq \mathbf{0}$. The upper part of the matrix has $B\boldsymbol{v} + C\mathbf{0} = \mathbf{0}$ as shown. The lower rows of A are combinations of the upper rows, so they also have $D\boldsymbol{v} + E\mathbf{0} = \mathbf{0}$. But now a combination of the first s independent columns $\begin{bmatrix} B \\ D \end{bmatrix}$ of A, with coefficients from $\boldsymbol{v}$, is producing zero.

Conclusion: $s > r$ cannot happen. Thinking similarly for A^{T}, $r > s$ cannot happen.

Proof 4 Suppose r column vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ are a basis for the column space $C(A)$. Then each column of A is a combination of $\boldsymbol{u}$'s. Column 1 of A is $w_{11}\boldsymbol{u}_1 + \cdots + w_{r1}\boldsymbol{u}_r$, with some coefficients w. The whole matrix A equals $UW = (m \text{ by } r)(r \text{ by } n)$.

$$A = \begin{bmatrix} \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_r \\ \\ \end{bmatrix} \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \vdots \\ w_{r1} & \cdots & w_{rn} \end{bmatrix} = UW.$$

Now look differently at $A = UW$. Each row of A is a combination of the r rows of W! Therefore the row space of A has dimension $\leq r$.

This proves that (dimension of row space) $\leq$ (dimension of column space) for any A. Apply this reasoning to A^{T}, and the two dimensions must be equal.

To my way of thinking, that is a really cool proof.

The Transpose and Row Space of d/dt

This book is constantly emphasizing the parallels between linear differential equations and matrix equations. In both cases we have *null solutions* and *particular solutions*. The nullspace for a differential equation $Dy = 0$ contains the null solutions y_n :

$$\textbf{Matrices } A \quad Av_n = \mathbf{0} \qquad \textbf{Derivatives } D \quad Dy_n = y_n'' + By_n' + Cy_n = 0$$

The nullspace of this D has dimension 2. This is the reason that y needs two initial conditions. We look for solutions $y_n = e^{st}$ and usually we find $e^{s_1 t}$ and $e^{s_2 t}$. These functions are a basis for the nullspace. In case $s_2 = s_1$, the second function is $te^{s_1 t}$. All is completely parallel to matrix equations, until we ask this question:

What is the "row space" of D when a differential operator has no rows ?

I want to propose two answers to this question. They come from faithfully imitating the Fundamental Theorem of Linear Algebra. That theorem applies to D, because D is linear.

Answer 1 The row space of D contains all functions $y_r(t)$ orthogonal to $e^{s_1 t}$ and $e^{s_2 t}$.

Answer 2 The row space of D contains all outputs $y_r(t) = D^{\mathrm{T}} q(t)$ from inputs $q(t)$.

This looks good, but when are functions "orthogonal" ? What is the "transpose" of D ?

$$\begin{array}{l}\textbf{Dot product of functions}\\ \textbf{Inner product of } y_n \textbf{ and } y_r\end{array} \qquad (y_n(t), y_r(t)) = \int_{-\infty}^{\infty} y_n(t) y_r(t)\, dt.$$

Do you see this as reasonable ? For vectors, we add the products $v_j w_j$. For functions, we integrate $y_n y_r$. If the vectors or functions are complex, we add $\overline{v}_j w_j$ or integrate $\overline{y}_n y_r$. Then $(\boldsymbol{v}, \boldsymbol{v})$ and (y_r, y_r) give the squared lengths $\|\boldsymbol{v}\|^2$ for vectors and $\|y_r\|^2$ for functions.

The inner product tells us the correct meaning of the transpose. For matrices, A^{T} is the matrix that obeys the inner product law $(Av, w) = (v, A^{\mathrm{T}} w)$. For differential equations,

$$(\boldsymbol{Df}, \boldsymbol{g}) = \int_{-\infty}^{\infty} (f'' + Bf' + Cf) g(t)\, dt = \int_{-\infty}^{\infty} f(t)(g'' - Bg' + Cg)\, dt = (\boldsymbol{f}, \boldsymbol{D}^{\mathbf{T}} \boldsymbol{g}).$$

Integration by parts gave $\int f'g = -\int fg'$. Two integrations gave $\int f''g = \int fg''$ with a plus sign (from two minus signs). Formally, that equation tells us D^{T} :

$$D = \frac{d^2}{dt^2} + B\frac{d}{dt} + C \quad \text{leads to} \quad D^{\mathrm{T}} = \frac{d^2}{dt^2} - B\frac{d}{dt} + C \quad \left(\frac{d}{dt} \textbf{ is antisymmetric}\right)$$

Now the row space of all $D^{\mathrm{T}} q(t)$ makes sense even when D has no rows. Can we just verify that any row space function $D^{\mathrm{T}} q(t)$ is orthogonal to any nullspace function $y_n(t)$?

$$(y_n(t), D^{\mathrm{T}} q(t)) = (Dy_n(t), q(t)) = \int_{-\infty}^{\infty} (0)\, q(t)\, dt = 0.$$

Shakespeare said it best at the end of Hamlet: *The rest is silence*.

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Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

Eigenvalues are the key to a system of n differential equations: $dy/dt = ay$ becomes $d\boldsymbol{y}/dt = A\boldsymbol{y}$. Now A is a matrix and $\boldsymbol{y}$ is a vector $(y_1(t), \ldots, y_n(t))$. The vector $\boldsymbol{y}$ changes with time. Here is a system of two equations with its 2 by 2 matrix A:

$$\begin{aligned} y_1{}' &= 4y_1 + y_2 \\ y_2{}' &= 3y_1 + 2y_2 \end{aligned} \quad \text{is} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{1}$$

How to solve this coupled system, $\boldsymbol{y}' = A\boldsymbol{y}$ with y_1 and y_2 in both equations? The good way is to find solutions that "uncouple" the problem. *We want y_1 and y_2 to grow or decay in exactly the same way* (with the same $e^{\lambda t}$):

Look for $\quad \begin{aligned} y_1(t) &= e^{\lambda t} a \\ y_2(t) &= e^{\lambda t} b \end{aligned} \quad$ **In vector notation this is** $\quad \boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{x} \quad (2)$

That vector $\boldsymbol{x} = (a, b)$ is called an **eigenvector**. The growth rate λ is an **eigenvalue**. This section will show how to find $\boldsymbol{x}$ and λ. Here I will jump to $\boldsymbol{x}$ and λ for the matrix in (1).

First eigenvector $\boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and first eigenvalue $\boldsymbol{\lambda} = \mathbf{5}$ in $\boldsymbol{y} = e^{5t}\boldsymbol{x}$

$$\begin{aligned} y_1 &= \boldsymbol{e^{5t}} \\ y_2 &= \boldsymbol{e^{5t}} \end{aligned} \quad \text{has} \quad \begin{aligned} y_1{}' &= 5\,e^{5t} = 4y_1 + y_2 \\ y_2{}' &= 5\,e^{5t} = 3y_1 + 2y_2 \end{aligned}$$

Second eigenvector $\boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and second eigenvalue $\boldsymbol{\lambda} = \mathbf{1}$ in $\boldsymbol{y} = e^{t}\boldsymbol{x}$

This $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ is a second solution $\quad \begin{aligned} y_1 &= \boldsymbol{e^{t}} \\ y_2 &= \boldsymbol{-3\,e^{t}} \end{aligned} \quad \text{has} \quad \begin{aligned} y_1{}' &= e^{t} = 4y_1 + y_2 \\ y_2{}' &= -3\,e^{t} = 3y_1 + 2y_2 \end{aligned}$

Those two $\boldsymbol{x}$'s and $\boldsymbol{\lambda}$'s combine with any c_1, c_2 to give the complete solution to $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\textbf{Complete solution} \quad \boldsymbol{y}(t) = c_1 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ -3\,e^t \end{bmatrix} = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \qquad (3)$$

This is exactly what we hope to achieve for other equations $\boldsymbol{y}' = A\boldsymbol{y}$ with constant A.

The solutions we want have the special form $\boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{x}$. Substitute that solution into $\boldsymbol{y}' = A\boldsymbol{y}$, to see the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for an eigenvalue λ and its eigenvector $\boldsymbol{x}$:

$$\frac{d}{dt}(e^{\lambda t}\boldsymbol{x}) = A(e^{\lambda t}\boldsymbol{x}) \quad \text{is} \quad \lambda e^{\lambda t}\boldsymbol{x} = Ae^{\lambda t}\boldsymbol{x}. \qquad \text{Divide both sides by } e^{\lambda t}.$$

$$\textbf{Eigenvalue and eigenvector of } \boldsymbol{A} \qquad \boxed{A\boldsymbol{x} = \lambda\boldsymbol{x}} \qquad (4)$$

Those eigenvalues (5 and 1 for this A) are a new way to see into the heart of a matrix. This chapter enters a different part of linear algebra, based on $A\boldsymbol{x} = \lambda\boldsymbol{x}$. **The last page of Chapter 6 has eigenvalue-eigenvector information about many different matrices**.

Finding Eigenvalues from det$(A - \lambda I) = 0$

Almost all vectors change direction, when they are multiplied by A. ***Certain very exceptional vectors $\boldsymbol{x}$ are in the same direction as $A\boldsymbol{x}$. Those are the "eigenvectors."*** The vector $A\boldsymbol{x}$ (in the same direction as $\boldsymbol{x}$) is a number λ times the original $\boldsymbol{x}$.

The eigenvalue λ tells whether the eigenvector $\boldsymbol{x}$ is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! $A\boldsymbol{x} = 0\boldsymbol{x}$ puts this eigenvector $\boldsymbol{x}$ in the nullspace of A.

If A is the identity matrix, every vector has $A\boldsymbol{x} = \boldsymbol{x}$. All vectors are eigenvectors of I. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues λ_1 and λ_2.

To find the eigenvalues, write the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ in the good form $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

If $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, then $A - \lambda I$ is a **singular matrix**. Its determinant must be **zero**.

$$\text{The determinant of } A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \quad \text{is} \quad \boldsymbol{(a - \lambda)(d - \lambda) - bc = 0}.$$

Our goal is to shift A by the right amount λI, so that $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has a solution. Then $\boldsymbol{x}$ is the eigenvector, λ is the eigenvalue, and $A - \lambda I$ is not invertible. So we look for numbers λ that make $\textbf{det}\boldsymbol{(A - \lambda I) = 0}$. I will start with the matrix A in equation (1).

Example 1 For $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, subtract λ from the diagonal and find the determinant:

$$\textbf{det}\boldsymbol{(A - \lambda I)} = \det \begin{bmatrix} 4 - \lambda & 1 \\ 3 & 2 - \lambda \end{bmatrix} = \lambda^2 - 6\lambda + 5 = \boldsymbol{(\lambda - 5)(\lambda - 1)}. \qquad (5)$$

I factored the quadratic, to see the two eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$. The matrices $A - 5I$ and $A - I$ are *singular*. We have found the λ's from det $(A - \lambda I) = 0$.

For each of the eigenvalues 5 and 1, we now find an **eigenvector** x :

$$(A - 5I)\,\boldsymbol{x} = \mathbf{0} \quad \text{is} \quad \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - 1I)\,\boldsymbol{x} = \mathbf{0} \quad \text{is} \quad \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Those were the vectors (a, b) in our special solutions $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$. Both components of $\boldsymbol{y}$ have the growth rate λ, so the differential equation was easily solved: $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$.

Two eigenvectors gave two solutions. Combinations $c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2$ give all solutions.

Example 2 Find the eigenvalues and eigenvectors of the *Markov matrix* $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

$$\det(A - \lambda I) = \det \begin{bmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\boldsymbol{\lambda} - \mathbf{1})\left(\boldsymbol{\lambda} - \tfrac{\mathbf{1}}{\mathbf{2}}\right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda} = \mathbf{1}$ and $\frac{\mathbf{1}}{\mathbf{2}}$. The eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$$\begin{array}{llll} (A - I)\,\boldsymbol{x}_1 = \mathbf{0} & \text{is} \quad A\boldsymbol{x}_1 = \boldsymbol{x}_1 & \text{The first eigenvector is} & \boldsymbol{x}_1 = (.6, .4) \\ (A - \frac{1}{2}I)\,\boldsymbol{x}_2 = \mathbf{0} & \text{is} \quad A\boldsymbol{x}_2 = \frac{1}{2}\boldsymbol{x}_2 & \text{The second eigenvector is} & \boldsymbol{x}_2 = (1, -1) \end{array}$$

$$\boldsymbol{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \boldsymbol{x}_1 \quad (A\boldsymbol{x} = \boldsymbol{x} \text{ means that } \lambda_1 = 1)$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}\,\boldsymbol{x}_2 \text{ so } \lambda_2 = \tfrac{1}{2}).$$

If $\boldsymbol{x}_1$ is multiplied again by A, we still get $\boldsymbol{x}_1$. Every power of A will give $A^n\boldsymbol{x}_1 = \boldsymbol{x}_1$. Multiplying $\boldsymbol{x}_2$ by A gave $\frac{1}{2}\boldsymbol{x}_2$, and if we multiply again we get $(\frac{1}{2})^2$ times $\boldsymbol{x}_2$.

When A is squared, the eigenvectors x stay the same. $\boldsymbol{A^2x = A(\lambda x) = \lambda(Ax) = \lambda^2 x}$.

Notice λ^2. This pattern keeps going, because the eigenvectors stay in their own directions. They never get mixed. The eigenvectors of A^{100} are the same $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100}$ = very small number.

We mention that this particular A is a ***Markov matrix***. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue must be $\lambda = 1$.

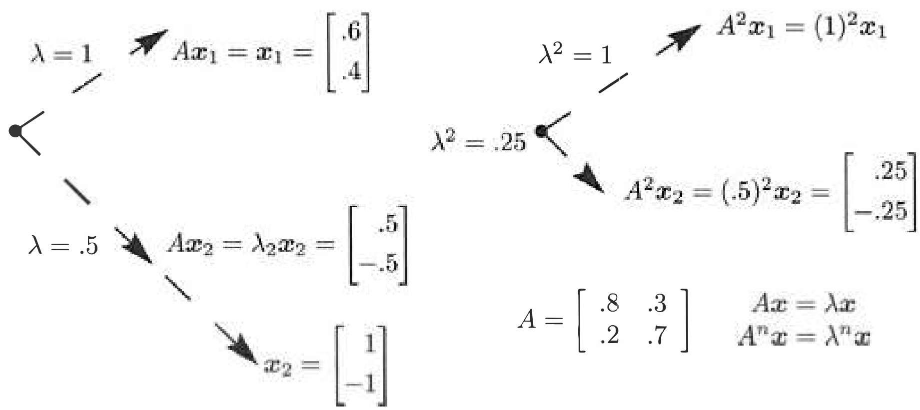


Figure 6.1: The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

The eigenvector $Ax_1 = x_1$ is the *steady state*—which all columns of A^k will approach.

Giant Markov matrices are the key to Google's search algorithm. It ranks web pages. Linear algebra has made Google one of the most valuable companies in the world.

Powers of a Matrix

When the eigenvalues of A are known, we immediately know the eigenvalues of all powers A^k and shifts $A + cI$ and all functions of A. Each eigenvector of A is also an eigenvector of A^k and A^{-1} and $A + cI$:

$$\text{If } Ax = \lambda x \text{ then } A^k x = \lambda^k x \text{ and } A^{-1}x = \frac{1}{\lambda}x \text{ and } (A + cI)x = (\lambda + c)x. \tag{6}$$

Start again with A^2x, which is A times $Ax = \lambda x$. Then $A\lambda x$ is the same as λAx for any number λ, and λAx is $\lambda^2 x$. We have proved that $A^2x = \lambda^2 x$.

For higher powers $A^k x$, continue multiplying $Ax = \lambda x$ by A. Step by step you reach $A^k x = \lambda^k x$. For the eigenvalues of A^{-1}, first multiply by A^{-1} and then divide by λ:

$$\textbf{Eigenvalues of } \boldsymbol{A^{-1}} \textbf{ are } \frac{1}{\lambda} \qquad \boxed{Ax = \lambda x} \qquad x = \lambda A^{-1}x \qquad \boxed{A^{-1}x = \frac{1}{\lambda}x} \tag{7}$$

We are assuming that A^{-1} exists! If A is invertible then λ will never be zero.

Invertible matrices have all $\lambda \neq 0$. Singular matrices have the eigenvalue $\lambda = 0$.

The shift from A to $A + cI$ just adds c to every eigenvalue (*don't change* x):

$$\textbf{Shift of } \boldsymbol{A} \qquad \text{If } Ax = \lambda x \text{ then } (A + cI)x = Ax + cx = (\lambda + c)x. \tag{8}$$

As long as we keep the same eigenvector x, we can allow any function of A:

$$\textbf{Functions of } \boldsymbol{A} \qquad (A^2 + 2A + 5I)x = (\lambda^2 + 2\lambda + 5)x \qquad e^A x = e^\lambda x. \tag{9}$$

I slipped in $e^A = I + A + \frac{1}{2}A^2 + \cdots$ to show that infinite series produce matrices too.

Let me show you the powers of the Markov matrix A in Example 2. That starting matrix is unrecognizable after a few steps.

$$\underset{A}{\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix}} \quad \cdots \quad \underset{A^{100}}{\begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}} \tag{10}$$

A^{100} was found by using $\lambda = 1$ and its eigenvector $[.6, .4]$, not by multiplying 100 matrices. The eigenvalues of A are 1 and $\frac{1}{2}$, so the eigenvalues of A^{100} are 1 and $(\frac{1}{2})^{100}$. That last number is extremely small, and we can't see it in the first 30 digits of A^{100}.

How could you multiply A^{99} times another vector like $\boldsymbol{v} = (.8, .2)$? This is not an eigenvector, but $\boldsymbol{v}$ is a *combination of eigenvectors*. This is a key idea, to express any vector $\boldsymbol{v}$ by using the eigenvectors.

Separate into eigenvectors
$\boldsymbol{v} = \boldsymbol{x}_1 + (.2)\boldsymbol{x}_2$

$$\boldsymbol{v} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \tag{11}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply the vector by A. After 99 steps, $\boldsymbol{x}_1$ is unchanged and $\boldsymbol{x}_2$ is multiplied by $(\frac{1}{2})^{99}$:

$$A^{99}\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is} \quad A^{99}(\boldsymbol{x}_1 + .2\boldsymbol{x}_2) = \boldsymbol{x}_1 + (.2)(\frac{1}{2})^{99}\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100}, because $\boldsymbol{v} = (.8, .2)$ is the first column of A. The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector $\boldsymbol{x}_1 = (.6, .4)$ is a "*steady state*" that doesn't change (because $\lambda_1 = 1$). The eigenvector $\boldsymbol{x}_2$ is a "*decaying mode*" that virtually disappears (because $\lambda_2 = 1/2$). The higher the power of A, the more closely its columns approach the steady state.

Bad News About AB and $A + B$

Normally the eigenvalues of A and B (separately) do not tell us the eigenvalues of AB. We also don't know about $A + B$. When A and B have different eigenvectors, our reasoning fails. The good results for A^2 are wrong for AB and $A + B$, when AB is different from BA. The eigenvalues won't come from A and B separately:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

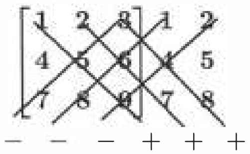
All the eigenvalues of A and B are zero. But AB has an eigenvalue $\lambda = 1$, and $A + B$ has eigenvalues 1 and -1. But one rule holds: ***AB* and *BA* have the same eigenvalues.**

Determinants

The determinant is a single number with amazing properties. It is zero when the matrix has no inverse. That leads to the eigenvalue equation $\det(A - \lambda I) = 0$. When A is invertible, the determinant of A^{-1} is $1/(\det A)$. Every entry in A^{-1} is a ratio of two determinants.

I want to summarize the algebra, leaving the details for my companion textbook *Introduction to Linear Algebra*. The difficulty with $\det(A - \lambda I) = 0$ is that an n by n determinant involves $n!$ terms. For $n = 5$ this is 120 terms––generally impossible to use.

For $n = 3$ there are six terms, three with plus signs and three with minus. Each of those six terms includes **one number from every row and every column**:



Determinant from $n! = 6$ *terms*

Three plus signs, three minus signs

$+(1)(5)(9) \quad +(2)(6)(7) \quad +(3)(4)(8)$

$-(3)(5)(7) \quad -(1)(6)(8) \quad -(2)(4)(9)$

That shows how to find the six terms. For this particular matrix the total must be $\det A = 0$, because the matrix happens to be singular: row 1 + row 3 equals 2(row 2).

Let me start with five useful properties of determinants, for all square matrices.

1. Subtracting a multiple of one row from another row leaves $\det A$ unchanged.

2. The determinant reverses sign when two rows are exchanged.

3. If A is triangular then $\det A =$ product of diagonal entries.

4. The determinant of AB equals $(\det A)$ times $(\det B)$.

5. The determinant of A^{T} equals the determinant of A.

By combining **1**, **2**, **3** you will see how the determinant comes from elimination:

The determinant equals $\pm$ (product of the pivots). (12)

Property **1** says that A and U have the same determinant, unless rows are exchanged.
Property **2** says that an odd number of exchanges would leave $\det A = -\det U$.
Property **3** says that $\det U$ is the product of the pivots on its main diagonal.

When elimination takes A to U, we find $\det A = \pm$ (product of the pivots). This is how all numerical software (like MATLAB or Python or Julia) would compute $\det A$.

Plus and minus signs play a big part in determinants. Half of the $n!$ terms have plus signs, and half come with minus signs. For $n = 3$, one row exchange puts $3 - 5 - 7$ or $1 - 6 - 8$ or $2 - 4 - 9$ on the main diagonal. A minus sign from one row exchange.

Two row exchanges (an even number) take you back to (2) (6) (7) and (3) (4) (8). This indicates how the 24 terms would go for $n = 4$, twelve terms with *plus* and twelve with *minus*.

Even permutation matrices have $\det P = 1$ and odd permutations have $\det P = -1$.

Inverse of A If $\det A \neq 0$, you can solve $A\boldsymbol{v} = \boldsymbol{b}$ and find A^{-1} using determinants:

Cramer's Rule $\quad v_1 = \dfrac{\det B_1}{\det A} \quad v_2 = \dfrac{\det B_2}{\det A} \quad \cdots \quad v_n = \dfrac{\det B_n}{\det A} \qquad (13)$

The matrix B_j replaces the j^{th} column of A by the vector $\boldsymbol{b}$. Cramer's Rule is expensive!

To find the columns of A^{-1}, we solve $AA^{-1} = I$. That is the Gauss-Jordan idea: For each column $\boldsymbol{b}$ in I, solve $A\boldsymbol{v} = \boldsymbol{b}$ to find a column $\boldsymbol{v}$ of A^{-1}.

In this special case, when $\boldsymbol{b}$ is a column of I, the numbers $\det B_j$ in Cramer's Rule are called **cofactors**. They reduce to determinants of size $n-1$, because $\boldsymbol{b}$ has so many zeros. Every entry of A^{-1} is a cofactor of A divided by the determinant of A.

I will close with three examples, to introduce the "trace" of a matrix and to show that real matrices can have imaginary (or complex) eigenvalues and eigenvectors.

Example 3 Find the eigenvalues and eigenvectors of $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution You can see that $\boldsymbol{x} = (1,1)$ will be in the same direction as $S\boldsymbol{x} = (3,3)$. Then $\boldsymbol{x}$ is an eigenvector of S with $\lambda = 3$. We want the matrix $S - \lambda I$ to be singular.

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \det(S - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \boldsymbol{\lambda^2 - 4\lambda + 3} = 0.$$

Notice that **3** is the determinant of S (without λ). And **4** is the sum $2+2$ down the central diagonal of S. **The diagonal sum 4 is the "trace" of A. It equals $\boldsymbol{\lambda_1 + \lambda_2 = 3 + 1}$.**

Now factor $\lambda^2 - 4\lambda + 3$ into $(\lambda - 3)(\lambda - 1)$. The matrix $S - \lambda I$ is singular (zero determinant) for $\lambda = 3$ and $\lambda = 1$. Each eigenvalue has an eigenvector:

$$\lambda_1 = 3 \quad (S - 3I)\,\boldsymbol{x}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1 \quad (S - I)\,\boldsymbol{x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvalues 3 and 1 are *real*. The eigenvectors $(1,1)$ and $(1,-1)$ are *orthogonal*. Those properties always come together for symmetric matrices (Section 6.5).

Here is an *antisymmetric* matrix with $A^{\mathrm{T}} = -A$. It rotates all real vectors by $\theta = 90°$. Real vectors can't be eigenvectors of a rotation matrix because it changes their direction.

Example 4 This real matrix has imaginary eigenvalues i, $-i$ and complex eigenvectors:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A^{\mathrm{T}} \qquad \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \boldsymbol{\lambda^2 + 1 = 0}.$$

That determinant $\lambda^2 + 1$ is zero for $\lambda = i$ and $-i$. The eigenvectors are $(1, -i)$ and $(1, i)$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \boldsymbol{i} \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \boldsymbol{-i} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Somehow those complex vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ don't get rotated (I don't really know how).

Multiplying the eigenvalues $(i)(-i)$ gives $\det A = 1$. Adding the eigenvalues gives $(i) + (-i) = 0$. This equals the sum $0 + 0$ down the diagonal of A.

Product of eigenvalues = determinant **Sum of eigenvalues = "trace"** (14)

Those are true statements for all square matrices. **The trace is the sum** $a_{11} + \cdots + a_{nn}$ **down the main diagonal of A.** This sum and product are is especially valuable for 2 by 2 matrices, when the determinant $\lambda_1 \lambda_2 = \boldsymbol{ad - bc}$ and the trace $\lambda_1 + \lambda_2 = \boldsymbol{a + d}$ completely determine λ_1 and λ_2. Look now at rotation of a plane through any angle θ.

Example 5 Rotation comes from an orthogonal matrix Q. Then $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= \cos\theta + i\sin\theta \\ \lambda_2 &= \cos\theta - i\sin\theta \end{aligned} \qquad \begin{aligned} \lambda_1 + \lambda_2 &= 2\cos\theta = \textbf{trace} \\ \lambda_1 \ \lambda_2 &= 1 = \textbf{determinant} \end{aligned}$$

I multiplied $(\lambda_1)(\lambda_2)$ to get $\cos^2\theta + \sin^2\theta = 1$. In polar form $e^{i\theta}$ times $e^{-i\theta}$ is 1. The eigenvectors of Q are $(1, -i)$ and $(1, i)$ for all rotation angles θ.

Before ending this section, I need to tell you the truth. It is not easy to find eigenvalues and eigenvectors of large matrices. The equation $\det(A - \lambda I) = 0$ is more or less limited to 2 by 2 and 3 by 3. For larger matrices, we can gradually make them triangular without changing the eigenvalues. *For triangular matrices the eigenvalues are on the diagonal.* A good code to compute λ and $\boldsymbol{x}$ is free in LAPACK. The MATLAB command is eig (A).

■ REVIEW OF THE KEY IDEAS ■

1. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ says that eigenvectors $\boldsymbol{x}$ keep the same direction when multiplied by A.
2. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ also says that $\det(A - \lambda I) = 0$. This equation determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors as A.
4. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.
5. The sum down the main diagonal of A (*the trace*) is the sum of the eigenvalues.
6. The determinant is the product of the λ's. It is also $\pm$ (product of the pivots).

Problem Set 6.1

1 Example 2 has powers of this Markov matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

(a) A has eigenvalues 1 and $\frac{1}{2}$. Find the eigenvalues of A^2 and A^∞.

(b) What are the eigenvectors of A^∞ ? One eigenvector is in the nullspace.

(c) Check the determinant of A^2 and A^∞. Compare with $(\det A)^2$ and $(\det A)^\infty$.

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ______ eigenvectors as A. Its eigenvalues are ______ by 1.

3 Compute the eigenvalues and eigenvectors of A and also A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the ______ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues ______. Check that $\lambda_1 + \lambda_2 = \textbf{trace of } A = 0 + 1$.

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same ______ as A. When A has eigenvalues λ_1 and λ_2, the eigenvalues of A^2 are ______. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*) (*might not be equal to*) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A=\begin{bmatrix}1&0\\1&1\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}1&2\\0&1\end{bmatrix} \quad\text{and}\quad AB=\begin{bmatrix}1&2\\1&3\end{bmatrix} \quad\text{and}\quad BA=\begin{bmatrix}3&2\\1&1\end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA? *Yes*!

7 Elimination produces a triangular matrix U. The eigenvalues of U are on its diagonal (*why*?). They are *not the eigenvalues of* A. Give a 2 by 2 example of A and U.

8 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find λ is to ______.

(b) If you know that λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to ______.

9 What do you do to the equation $A\boldsymbol{x}=\lambda\boldsymbol{x}$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

(c) $\lambda+1$ is an eigenvalue of $A+I$, as in Problem 2.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A=\begin{bmatrix}.6&.2\\.4&.8\end{bmatrix} \quad\text{and}\quad A^\infty=\begin{bmatrix}1/3&1/3\\2/3&2/3\end{bmatrix}.$$

11 A 3 by 3 matrix B has eigenvalues $0,1,2$. This information allows you to find:

(a) the rank of B (b) the eigenvalues of B^2 (c) the eigenvalues of $(B^2+I)^{-1}$.

12 Find three eigenvectors for this matrix P. Projection matrices only have $\lambda=1$ and 0. Eigenvectors are *in or orthogonal to* the subspace that P projects onto.

$$\textbf{Projection matrix } \boldsymbol{P^2=P=P^{\mathrm{T}}} \qquad P=\begin{bmatrix}.2&.4&0\\.4&.8&0\\0&0&1\end{bmatrix}.$$

If two eigenvectors $\boldsymbol{x}$ and $\boldsymbol{y}$ share the same repeated eigenvalue λ, so do all their combinations $c\boldsymbol{x}+d\boldsymbol{y}$. Find an eigenvector of P with no zero components.

13 From the unit vector $\boldsymbol{u}=\left(\frac{1}{6},\frac{1}{6},\frac{3}{6},\frac{5}{6}\right)$ construct the rank one projection matrix $P=\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. This matrix has $P^2=P$ because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}=1$.

(a) Explain why $P\boldsymbol{u}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u}$ equals $\boldsymbol{u}$. Then $\boldsymbol{u}$ is an eigenvector with $\lambda=1$.

(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P\boldsymbol{v}=\boldsymbol{0}$. Then $\lambda=0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda=0$.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\boldsymbol{x} = \mathbf{0}$. Use $i^2 = -1$.

15 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. A might not be the zero matrix but check that A^2 is all zeros.

16 This matrix is singular with rank one. Find three λ's and three eigenvectors:

Rank one $$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

17 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues:

Use the trace to find λ_2 $$A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20, so $\lambda = 4$ and 5.

19 Suppose $A\boldsymbol{u} = 0\boldsymbol{u}$ and $A\boldsymbol{v} = 3\boldsymbol{v}$ and $A\boldsymbol{w} = 5\boldsymbol{w}$. The eigenvalues are $0, 3, 5$.

(a) Give a basis for the nullspace of A and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then _____ would be in the column space.

20 Choose the last row of A to produce (a) eigenvalues 4 and 7 (b) any λ_1 and λ_2.

Companion matrix $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}.$$

21 ***The eigenvalues of A equal the eigenvalues of A^{T}.*** This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because _____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

22 Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. Show that $M^{\mathrm{T}}(1,1,1) = (1,1,1)$. By Problem 21, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what λ's?

23 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. **Then $A = B$.** *Reason*: Any vector $\boldsymbol{v}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{v}$? What is $B\boldsymbol{v}$?

24 The block B has eigenvalues $1, 2$ and C has eigenvalues $3, 4$ and D has eigenvalues $5, 7$. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

25 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

26 Subtract I from the previous A. Find the eigenvalues of B and $-B$:

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad -B = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

27 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

28 Every permutation matrix leaves $\boldsymbol{x} = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

29 **The determinant of A equals the product $\lambda_1 \lambda_2 \cdots \lambda_n$.** Start with the polynomial $\det(A - \lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \_\_\_\_\_.$$

30 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{})/2$ and $\lambda =$ _____. Their sum is _____. **If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) =$ _____.**

6.2 Diagonalizing a Matrix

When $\boldsymbol{x}$ is an eigenvector, multiplication by A is just multiplication by a number λ: $A\boldsymbol{x} = \lambda\boldsymbol{x}$. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a *diagonal matrix*, with no off-diagonal interconnections. The 100th power of a diagonal matrix is easy.

The point of this section is very direct. ***The matrix A turns into a diagonal matrix*** $\boldsymbol{\Lambda}$ ***when we use the eigenvectors properly***. This is the matrix form of our key idea. We start right off with that one essential computation.

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Put them into the columns of an ***eigenvector matrix*** V. Then $V^{-1}AV$ is the ***eigenvalue matrix*** Λ, and Λ is diagonal:

Eigenvector matrix $\boldsymbol{V}$
Eigenvalue matrix $\boldsymbol{\Lambda}$

$$\boldsymbol{V^{-1}AV = \Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \tag{1}$$

The matrix A is "diagonalized." We use capital lambda for the eigenvalue matrix, because of the small λ's (the eigenvalues) on its diagonal.

Proof Multiply A times its eigenvectors, which are the columns of V. The first column of AV is $A\boldsymbol{x}_1$. That is $\lambda_1\boldsymbol{x}_1$. Each column of V is multiplied by its eigenvalue λ_i:

A times V
$$AV = A\begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix}.$$

The trick is to split this matrix AV into V times Λ:

V times Λ
$$\begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = V\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column $\boldsymbol{x}_1$, as shown. The diagonalization is complete, and we can write $AV = V\Lambda$ in two good ways:

$$\boxed{AV = V\Lambda \quad \text{is} \quad V^{-1}AV = \Lambda \quad \text{or} \quad A = V\Lambda V^{-1}.} \tag{2}$$

The matrix V has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent. *Without n independent eigenvectors, we can't diagonalize.*

A and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvectors are different. The job of the original eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ was to diagonalize A. Those eigenvectors in V produce $A = V\Lambda V^{-1}$. You will soon see the simplicity and importance and meaning of the kth power $A^k = V\Lambda^k V^{-1}$.

Sections 6.2 and 6.3 solve first order difference and differential equations.

6.2	$\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$	$\boldsymbol{u}_k = A^k\boldsymbol{u}_0$	$= c_1\lambda_1^k\boldsymbol{x}_1 + \cdots + c_n\lambda_n^k\boldsymbol{x}_n$
6.3	$d\boldsymbol{y}/dt = A\boldsymbol{y}$	$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0)$	$= c_1e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_ne^{\lambda_n t}\boldsymbol{x}_n.$

The idea is the same for both problems: $\boldsymbol{n}$ **independent eigenvectors give a basis.** We can write $\boldsymbol{u}_0$ and $\boldsymbol{y}(0)$ as combinations of eigenvectors. Then we follow each eigenvector as k increases and t increases: $\boldsymbol{A^k x}$ **is** $\boldsymbol{\lambda^k x}$ **and** $\boldsymbol{e^{At}x}$ **is** $\boldsymbol{e^{\lambda t}x}$.

Some matrices don't have n independent eigenvectors (because of repeated λ's). Then $A^k\boldsymbol{u}_0$ and $e^{At}\boldsymbol{y}(0)$ are still correct, but they lead to $k\lambda^k\,\boldsymbol{x}$ and $te^{\lambda t}\boldsymbol{x}$: not so good.

Example 1 Here A is triangular so the λ's are on its diagonal: $\lambda = 1$ and $\lambda = 6$.

Eigenvectors in V
$$\underset{V^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}} \underset{V}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \underset{\Lambda}{\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{6} \end{bmatrix}}$$

In other words $A = V\Lambda V^{-1}$. Then watch $A^2 = V\Lambda V^{-1}V\Lambda V^{-1}$. When you remove $V^{-1}V = I$, this becomes $\boldsymbol{A^2 = V\Lambda^2 V^{-1}}$. ***The same eigenvectors for** A **and** A^2 **are in** V**. The squared eigenvalues are in** Λ^2.*

The kth power will be $A^k = V\Lambda^k V^{-1}$. And Λ^k just contains 1^k and 6^k:

Powers A^k
$$\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 6^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}.$$

With $k = 1$ *we get* A. *With* $k = 0$ *we get* $A^0 = I$ (eigenvalues $\lambda^0 = 1$). *With* $k = -1$ *we get the inverse* A^{-1}. You can see how $A^2 = [1\ \ 35;\ \ 0\ \ 36]$ fits the formula when $k = 2$.

Here are four remarks before we use Λ again.

Remark 1 When the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different, the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are independent. ***Any matrix that has no repeated eigenvalues can be diagonalized***.

Remark 2 ***We can multiply eigenvectors by any nonzero constants***. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ will remain true. In Example 1, we can divide the eigenvector $(1, 1)$ by $\sqrt{2}$ to produce a unit vector.

Remark 3 The eigenvectors in V come in the same order as the eigenvalues in Λ. To reverse the order 1, 6 in Λ, put the eigenvector $(1, 1)$ before $(1, 0)$ in V:

New order 6, 1
New order in V
$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda_{\mathbf{new}}$$

To diagonalize A we *must* use an eigenvector matrix. From $V^{-1}AV = \Lambda$ we know that $AV = V\Lambda$. Suppose the first column of V is $\boldsymbol{x}$. Then the first columns of AV and $V\Lambda$ are $A\boldsymbol{x}$ and $\lambda_1\boldsymbol{x}$. For those to be equal, $\boldsymbol{x}$ must be an eigenvector.

Remark 4 (**Warning for repeated eigenvalues**) Some matrices have too few eigenvectors (less than n). *Those matrices cannot be diagonalized.* Here are examples:

$$\begin{array}{l}\textbf{Not diagonalizable}\\ \textbf{Only 1 eigenvector}\end{array} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Their eigenvalues happen to be 0 and 0. The problem is the repetition of λ.

$$\begin{array}{l}\textbf{Only one line}\\ \textbf{of eigenvectors}\end{array} \qquad A\boldsymbol{x} = 0\boldsymbol{x} \quad \text{means} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

There is no second eigenvector, so the unusual matrix A cannot be diagonalized.

Those matrices are the best examples to test any statement about eigenvectors. In many true-false questions, non-diagonalizable matrices lead to *false*.

Remember that there is no connection between invertibility and diagonalizability:

- ***Invertibility*** is concerned with the ***eigenvalues*** ($\lambda = 0$ or $\lambda \neq 0$).
- ***Diagonalizability needs n independent eigenvectors.***

Each eigenvalue has at least one eigenvector! $A - \lambda I$ is singular. If $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ leads you to $\boldsymbol{x} = \mathbf{0}$, λ is *not* an eigenvalue. Look for a mistake in solving $\det(A - \lambda I) = 0$.

Eigenvectors for n different λ's are independent. Then $V^{-1}AV = \Lambda$ will succeed.
Eigenvectors for repeated λ's could be dependent. V might not be invertible.

Example 2 **Powers of A** The Markov matrix A in the last section had $\lambda_1 = 1$ and $\lambda_2 = .5$. Here is $A = V\Lambda V^{-1}$ with those eigenvalues in the matrix Λ:

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = V\Lambda V^{-1}.$$

The eigenvectors $(.6, .4)$ and $(1, -1)$ are in the columns of V. They are also the eigenvectors of A^2. Watch how A^2 has the same V, and ***the eigenvalue matrix of A^2 is Λ^2***:

$$\textbf{Same } V \textbf{ for } A^2 \qquad \boxed{A^2} = V\Lambda V^{-1}V\Lambda V^{-1} = \boxed{V\Lambda^2 V^{-1}.} \tag{3}$$

Just keep going, and you see why the high powers A^k approach a "steady state":

$$\textbf{Powers of } A \qquad A^k = \boldsymbol{V\Lambda^k V^{-1}} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}.$$

As k gets larger, $(.5)^k$ gets smaller. In the limit it disappears completely. That limit is A^∞:

$$\textbf{Limit } k \to \infty \qquad A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}. \tag{4}$$

The limit has the steady state eigenvector $\boldsymbol{x}_1$ in both columns.

Question ***When does $A^k \to$ zero matrix?*** **Answer** ***All* $|\boldsymbol{\lambda}| < 1$.**

Fibonacci Numbers

We present a famous example, where eigenvalues tell how fast the Fibonacci numbers grow. ***Every new Fibonacci number is the sum of the two previous F's*** :

$$\textit{The sequence} \quad 0,1,1,2,3,5,8,13,\ldots \quad \textit{comes from} \quad F_{k+2}=F_{k+1}+F_k.$$

These numbers turn up in a fantastic variety of applications. Plants a grow in spirals, and a pear tree has 8 growths for every 3 turns. The champion is a sunflower that had 233 seeds in 144 loops. Those are the Fibonacci numbers F_{13} and F_{12}. Our problem is more basic.

Problem: Find the Fibonacci number F_{100}. The slow way is to apply the rule $F_{k+2}=F_{k+1}+F_k$ one step at a time. By adding $F_6=8$ to $F_7=13$ we reach $F_8=21$. Eventually we come to F_{100}. Linear algebra gives a better way.

The key is to begin with a matrix equation $\mathbf{u}_{k+1}=A\boldsymbol{u}_k$. That is a *one-step* rule for vectors, while Fibonacci gave a two-step rule for scalars. We match those rules by putting two Fibonacci numbers into a vector $\boldsymbol{u}_k$. Then you will see the matrix A.

$$\boldsymbol{u}_k=\begin{bmatrix}F_{k+1}\\F_k\end{bmatrix}. \qquad \text{The rule} \quad \begin{aligned}F_{k+2}&=F_{k+1}+F_k\\F_{k+1}&=F_{k+1}\end{aligned} \quad \text{is} \quad \boldsymbol{u}_{k+1}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\boldsymbol{u}_k. \tag{5}$$

Every step multiplies by $A=\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$. After 100 steps we reach $\boldsymbol{u}_{100}=A^{100}\boldsymbol{u}_0$:

$$\boldsymbol{u}_0=\begin{bmatrix}1\\0\end{bmatrix},\quad \boldsymbol{u}_1=\begin{bmatrix}1\\1\end{bmatrix},\quad \boldsymbol{u}_2=\begin{bmatrix}2\\1\end{bmatrix},\quad \boldsymbol{u}_3=\begin{bmatrix}3\\2\end{bmatrix},\quad \ldots,\quad \boldsymbol{u}_{100}=\begin{bmatrix}F_{101}\\F_{100}\end{bmatrix}.$$

This problem is just right for eigenvalues. To find them, subtract λI from A:

$$A-\lambda I=\begin{bmatrix}1-\lambda&1\\1&-\lambda\end{bmatrix} \quad \text{leads to} \quad \det(A-\lambda I)=\lambda^2-\lambda-1.$$

The equation $\lambda^2-\lambda-1=0$ is solved by the quadratic formula $\left(-b\pm\sqrt{b^2-4ac}\right)/2a$:

$$\textbf{Eigenvalues} \qquad \lambda_1=\frac{1+\sqrt{5}}{2}\approx 1.618 \quad \text{and} \quad \lambda_2=\frac{1-\sqrt{5}}{2}\approx -.618.$$

These eigenvalues lead to eigenvectors $\boldsymbol{x}_1=(\lambda_1,1)$ and $\boldsymbol{x}_2=(\lambda_2,1)$. Step 2 finds the combination of those eigenvectors that gives $\boldsymbol{u}_0=(1,0)$:

$$\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\lambda_1-\lambda_2}\left(\begin{bmatrix}\lambda_1\\1\end{bmatrix}-\begin{bmatrix}\lambda_2\\1\end{bmatrix}\right) \quad \text{or} \quad \boldsymbol{u}_0=\frac{\boldsymbol{x}_1-\boldsymbol{x}_2}{\lambda_1-\lambda_2}. \tag{6}$$

Step 3 multiplies the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ by $(\lambda_1)^{100}$ and $(\lambda_2)^{100}$:

$$\boldsymbol{A^{100}}\ \textbf{times}\ \boldsymbol{u}_0 \qquad \boldsymbol{u}_{100}=\frac{(\lambda_1)^{100}\boldsymbol{x}_1-(\lambda_2)^{100}\boldsymbol{x}_2}{\lambda_1-\lambda_2}. \tag{7}$$

We want F_{100} = second component of $\boldsymbol{u}_{100}$. The second components of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1. The difference between $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ is $\lambda_1 - \lambda_2 = \sqrt{5}$. We have F_{100} :

$$F_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right] \approx 3.54 \cdot 10^{20}. \tag{8}$$

Is this a whole number? *Yes.* The fractions and square roots must disappear, because Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ stays with integers. The second term in (8) is less than $\frac{1}{2}$, so it must move the first term to the nearest whole number:

$$k\text{th Fibonacci number} = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k. \tag{9}$$

The ratio of F_6 to F_5 is $8/5 = 1.6$. The ratio F_{101}/F_{100} must be very close to the limiting ratio $(1+\sqrt{5})/2$. The Greeks called this number the *"golden mean"*. For some reason a rectangle with sides 1.618 and 1 looks especially graceful.

Matrix Powers A^k

Fibonacci's example is a typical difference equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. ***Each step multiplies by A.*** The solution is $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. We want to make clear how diagonalizing the matrix gives a quick way to compute A^k and find $\boldsymbol{u}_k$ in three steps.

The eigenvector matrix V produces $A = V\Lambda V^{-1}$. This is perfectly suited to computing powers, because ***every time V^{-1} multiplies V we get I***:

$$\textbf{Powers of } \boldsymbol{A} \qquad A^k\boldsymbol{u}_0 = (V\Lambda V^{-1})\cdots(V\Lambda V^{-1})\boldsymbol{u}_0 = V\Lambda^k V^{-1}\boldsymbol{u}_0$$

I will split $V\Lambda^k V^{-1}\boldsymbol{u}_0$ into three steps. Equation (10) puts those steps together in $\boldsymbol{u}_k$.

1. Write $\boldsymbol{u}_0$ as a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ of the eigenvectors. Then $\boldsymbol{c} = V^{-1}\boldsymbol{u}_0$.

2. Multiply each number c_i by $(\lambda_i)^k$. Now we have $\Lambda^k V^{-1}\boldsymbol{u}_0$.

3. Add up the pieces $c_i(\lambda_i)^k\boldsymbol{x}_i$ to find the solution $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. This is $V\Lambda^k V^{-1}\boldsymbol{u}_0$.

$$\boxed{\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n.} \tag{10}$$

In matrix language $A^k\boldsymbol{u}_0$ equals $(V\Lambda V^{-1})^k\boldsymbol{u}_0$. The 3 steps are V times Λ^k times $V^{-1}\boldsymbol{u}_0$.

I am taking time with the three steps to compute $A^k\boldsymbol{u}_0$, because you will see exactly the same steps for differential equations and e^{At}. The equation will be $d\boldsymbol{y}/dt = A\boldsymbol{y}$. Please compare equation (10) for $A^k\boldsymbol{u}_0$ with this solution $e^{At}\boldsymbol{y}(0)$ from Section 6.3.

$$\textbf{Solve } d\boldsymbol{y}/dt = A\boldsymbol{y} \qquad \boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) = c_1e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_ne^{\lambda_n t}\boldsymbol{x}_n. \tag{11}$$

Those parallel equations (10) and (11) show the point of eigenvalues and eigenvectors. They split the solutions into n simple pieces. By following each eigenvector separately—this is the result of diagonalizing the matrix—we have n scalar equations.

The growth factor λ^k in (10) is like $e^{\lambda t}$ in (11).

Summary I will display the matrices in those steps. Here is $\boldsymbol{u}_0 = V\boldsymbol{c}$:

Step 1 $$\boldsymbol{u}_0 = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$ This says that $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ (12)

The coefficients in Step 1 are $\boldsymbol{c} = V^{-1}\boldsymbol{u}_0$. Then Step 2 multiplies by Λ^k. Then Step 3 adds up all the $c_i(\lambda_i)^k\boldsymbol{x}_i$ to get the product of V and Λ^k and $V^{-1}\boldsymbol{u}_0$:

$$A^k\boldsymbol{u}_0 = V\Lambda^k V^{-1}\boldsymbol{u}_0 = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{13}$$

This result is exactly $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n$. It solves $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$.

Example 3 Start from $\boldsymbol{u}_0 = (1, 0)$. Compute $A^k\boldsymbol{u}_0$ when V and Λ contain these eigenvectors and eigenvalues:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 2 \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This matrix A is like Fibonacci except the rule is changed to $F_{k+2} = F_{k+1} + \mathbf{2}F_k$. The new numbers $0, 1, 1, 3, \ldots$ grow faster because $\lambda = 2$ is larger than $(1 + \sqrt{5})/2$.

Example 3 in three steps Find $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ and $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + c_2(\lambda_2)^k\boldsymbol{x}_2$

Step 1 $$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{so} \quad c_1 = c_2 = \frac{1}{3}$$

Step 2 Multiply the two eigenvectors by $(\lambda_1)^k = 2^k$ and $(\lambda_2)^k = (-1)^k$

Step 3 Combine the pieces into $\boldsymbol{u}_k = \frac{1}{3}2^k\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}(-1)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Behind these examples lies the fundamental idea: ***Follow each eigenvector***.

Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A. We discover that fact in two ways:

1. Eigenvectors (geometric) There are nonzero solutions to $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

2. Eigenvalues (algebraic) The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its ***multiplicity***. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor.

For exceptional matrices, an eigenvalue can be ***repeated***. Then there are two *different* ways to count its multiplicity. Always GM $\leq$ AM for each eigenvalue.

1. (Geometric Multiplicity = GM) Count the **independent eigenvectors** for λ. This is the dimension of the nullspace of $A - \lambda I$.

2. (Algebraic Multiplicity = AM) Count the **repetitions of the same $\boldsymbol{\lambda}$** among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

If A has $\lambda = 4, 4, 4$, that eigenvalue has AM $= 3$ (triple root) and GM $=$ **1 or 2 or 3**.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue (AM $= 2$) with only one eigenvector (GM $= 1$).

AM = 2
GM = 1
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2.$$
$\boldsymbol{\lambda} = 0, 0$ but 1 eigenvector

There "should" be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes AM $= 2$. But there is only one eigenvector $\boldsymbol{x} = (1, 0)$. ***This shortage of eigenvectors when*** GM ***is below*** AM ***means that*** A ***is not diagonalizable***.

These three matrices have $\lambda = 5, 5$. Traces are 10, determinants are 25. They only have one eigenvector:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM $= 2$. But each $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM $= 1$. There is only one line of eigenvectors for $\lambda = 5$, and these matrices are not diagonalizable.

■ REVIEW OF THE KEY IDEAS ■

1. If A has n independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, they go into the columns of V.

   A* is diagonalized by *V $\quad \boldsymbol{V^{-1}AV = \Lambda}$ **and** $\boldsymbol{A = V\Lambda V^{-1}}$.

2. The powers of A are $A^k = V\Lambda^k V^{-1}$. The eigenvectors in V are unchanged.

3. The eigenvalues of A^k are $(\lambda_1)^k, \ldots, (\lambda_n)^k$ in the matrix Λ^k.

4. The solution to $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ starting from $\boldsymbol{u}_0$ is $\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = V\Lambda^k V^{-1}\boldsymbol{u}_0$:

$$\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n \quad \text{provided} \quad \boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n.$$

That shows Steps $1, 2, 3$ (c's from $V^{-1}\boldsymbol{u}_0$, powers λ^k from Λ^k, and $\boldsymbol{x}$'s from V).

■ WORKED EXAMPLES ■

6.2 A Find the inverse and the eigenvalues and the determinant of A:

$$A = 5 * \textbf{eye}(4) - \textbf{ones}(4) = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Describe an eigenvector matrix V that gives $V^{-1}AV = \Lambda$.

Solution What are the eigenvalues of the all-ones matrix $\textbf{ones}(4)$? Its rank is certainly 1, so three eigenvalues are $\lambda = 0, 0, 0$. Its trace is 4, so the other eigenvalue is $\lambda = 4$. Subtract the all-ones matrix from $5I$ to get our matrix $A = 5I - \textbf{ones}(4)$:

Subtract the eigenvalues $\mathbf{4, 0, 0, 0}$ from $\mathbf{5, 5, 5, 5}$. The eigenvalues of A are $\mathbf{1, 5, 5, 5}$.

The $\boldsymbol{\lambda}$'s add to 16. So does $\mathbf{4 + 4 + 4 + 4}$ from diag(A). Multiply $\boldsymbol{\lambda}$'s: det $A = \mathbf{125}$.

The eigenvector for $\lambda = 1$ is $\boldsymbol{x} = (1, 1, 1, 1)$. The other eigenvectors are perpendicular to $\boldsymbol{x}$ (since A is symmetric). The nicest eigenvector matrix V is the symmetric orthogonal Hadamard matrix. Multiply by $1/2$ to have unit vectors in its columns.

$$\textbf{Orthonormal eigenvectors} \quad V = Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = Q^{\mathrm{T}} = Q^{-1}.$$

The eigenvalues of A^{-1} are $1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$. The eigenvectors are the same as for A. This inverse matrix $A^{-1} = Q\Lambda^{-1}Q^{-1}$ is surprisingly neat:

$$A^{-1} = \frac{1}{5} * (\textbf{eye}(4) + \textbf{ones}(4)) = \frac{1}{5}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

To check that $AA^{-1} = I$, use **(ones) (ones)** $= 4$ **(ones)**. **Question: Can you find $\boldsymbol{A^3}$?**

Problem Set 6.2

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and V.

1 (a) Factor these two matrices into $A = V\Lambda V^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = V\Lambda V^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, use $V\Lambda V^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

3 Suppose $A = V\Lambda V^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

4 True or false: If the columns of V (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) V is invertible (d) V is diagonalizable.

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix V is triangular, then V^{-1} is triangular. Prove that A is also triangular.

6 Describe all matrices V that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing V^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $V\Lambda^k V^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find A and its eigenvalues and eigenvectors.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = V\Lambda^n V^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

Questions 11–14 are about diagonalizability.

11 True or false: If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

12 True or false: If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $V\Lambda V^{-1}$.

13 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the trace is 10 so the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $A\boldsymbol{x} = 5\boldsymbol{x}$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

14 The matrix $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is _____. Change one entry to make A diagonalizable. Which entries could you change?

Questions 15–19 are about powers of matrices.

15 $A^k = V\Lambda^k V^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

16 (Recommended) Find Λ and V to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $V\Lambda^k V^{-1}$? In the columns of this limiting matrix you see the _____.

17 Find Λ and V to diagonalize A_2 in Problem 15. What is $(A_2)^{10}\boldsymbol{u}_0$ for these $\boldsymbol{u}_0$?

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

18 Diagonalize A and compute $V\Lambda^k V^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

19 Diagonalize B and compute $V\Lambda^k V^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

20 Suppose $A = V\Lambda V^{-1}$. Take determinants to prove $\det A = \det \Lambda = \lambda_1 \lambda_2 \cdots \lambda_n$. This quick proof only works when A can be _____.

21 Show that trace VT = trace TV, by adding the diagonal entries of VT and TV:

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Choose T as ΛV^{-1}. Then $V\Lambda V^{-1}$ has the same trace as $\Lambda V^{-1}V = \Lambda$. The trace of A equals the trace of Λ, which is certainly the sum of the eigenvalues.

22 $AB - BA = I$ is impossible since the left side has trace = _____. But find an elimination matrix so that $A = E$ and $B = E^{\mathrm{T}}$ give

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which has trace zero.}$$

23 If $A = V\Lambda V^{-1}$, diagonalize the block matrix $B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{2A} \end{bmatrix}$. Find its eigenvalue and eigenvector (block) matrices.

24 Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix V. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same V). What is this subspace when $V = I$? What is its dimension?

25 Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So every matrix with $A^2 = A$ can be diagonalized.

26 (Recommended) Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$. If $\lambda = 0$ then $\boldsymbol{x}$ is in the nullspace. If $\lambda \neq 0$ then $\boldsymbol{x}$ is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = V\sqrt{\Lambda}\, V^{-1}$. Why is there no real matrix square root of B?

28 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700} \qquad B^{1024} = I \qquad C^{1024} = -C \qquad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

29 If A and B have the same λ's with the same full set of independent eigenvectors, their factorizations into ______ are the same. So $A = B$.

30 Suppose the same V diagonalizes both A and B. They have the same eigenvectors in $A = V\Lambda_1V^{-1}$ and $B = V\Lambda_2V^{-1}$. Prove that $AB = BA$.

31 (a) If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) =$ *zero matrix*.

(b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

32 Substitute $A = V\Lambda V^{-1}$ into the product $(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The ***Cayley-Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

Challenge Problems

33 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = V\Lambda V^{-1}$. The eigenvectors (columns of V) are $(1, i)$ and $(i, 1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

34 The transpose of $A = V\Lambda V^{-1}$ is $A^{\mathrm{T}} = (V^{-1})^{\mathrm{T}}\Lambda V^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(V^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***.

How do you multiply three matrices $V\Lambda V^{-1}$ to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = V\Lambda V^{-1} = \lambda_1\boldsymbol{x}_1\boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n\boldsymbol{x}_n\boldsymbol{y}_n^{\mathrm{T}}.$$

35 The inverse of $A = \mathbf{eye}(n) + \mathbf{ones}(n)$ is $A^{-1} = \mathbf{eye}(n) + C * \mathbf{ones}(n)$. Multiply AA^{-1} to find that number C (depending on n).

6.3 Linear Systems $\boldsymbol{y}' = A\boldsymbol{y}$

This section is about first order systems of linear differential equations. The key words are *systems* and *linear*. A system allows n equations for n unknown functions $y_1(t), \ldots, y_n(t)$. A linear system multiplies that unknown vector $\boldsymbol{y}(t)$ by a matrix A. Then a first order linear system can include a source term $\boldsymbol{q}(t)$, or not:

$$\boxed{\textbf{Without source} \quad \frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y}(t) \qquad \textbf{With source} \quad \frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y}(t) + \boldsymbol{q}(t)}$$

Without a source term, the only input is $\boldsymbol{y}(0)$ at the start. With $\boldsymbol{q}(t)$ included, there is also a continuing input $\boldsymbol{q}(t)dt$ between times t and $t + dt$. Forward from time t, this input grows or decays along with the $\boldsymbol{y}(t)$ that just arrived from the past. That is important.

The **transient solution** $\boldsymbol{y}_n(t)$ starts from $\boldsymbol{y}(0)$, when $\boldsymbol{q}(t) = \boldsymbol{0}$. The output coming from the source $\boldsymbol{q}(t)$ is one particular solution $\boldsymbol{y}_p(t)$. Linearity allows superposition! **The complete solution with source included is $\boldsymbol{y}(t) = \boldsymbol{y}_n(t) + \boldsymbol{y}_p(t)$ as always.**

The serious work of this section is to find $\boldsymbol{y}_n(t)$, the null solution to $\boldsymbol{y}_n{}' - A\boldsymbol{y}_n = \boldsymbol{0}$. Then Section 6.4 accounts for the source term $\boldsymbol{q}(t)$ and finds a particular solution.

We want to use the eigenvalues and eigenvectors of A. We don't want those to change with time. So we kept our equation linear time-invariant, with a constant matrix A. Fortunately, many important systems have A = constant in the first place. The system is not changing, it is only the *state* of the system that changes: constant A, evolving state $\boldsymbol{y}(t)$.

We will express $\boldsymbol{y}(t)$ as a combination of eigenvectors of A. Section 6.4 uses e^{At}.

Solution by Eigenvectors and Eigenvalues

Suppose the n by n matrix A has n independent eigenvectors. This is automatic if A has n different eigenvalues λ. Then the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are a basis in which we can express any starting vector $\boldsymbol{y}(0)$:

$$\textbf{Initial condition} \qquad \boldsymbol{y}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n \text{ for some numbers } c_1, \ldots, c_n. \tag{1}$$

Computing the c's is Step 1 in the solution, after finding the λ's and $\boldsymbol{x}$'s.

Step 2 solves the equation $\boldsymbol{y}' = A\boldsymbol{y}$ using $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$. *Start from any eigenvector*:

$$\text{If } A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{then} \quad \boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{x} \quad \text{solves} \quad \frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y}. \tag{2}$$

This solution $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ separates the time-dependent $e^{\lambda t}$ from the constant vector $\boldsymbol{x}$:

$$\frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y} \quad \text{becomes} \quad \frac{d}{dt}(e^{\lambda t}\boldsymbol{x}) = \lambda e^{\lambda t}\boldsymbol{x} = A(e^{\lambda t}\boldsymbol{x}). \tag{3}$$

Step 3 is the final solution step. Add the n separate solutions from the n eigenvectors.

$$\boxed{\textbf{Superposition} \qquad \boldsymbol{y}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n.} \tag{4}$$

At $t = 0$ this matches $\boldsymbol{y}(0)$ in equation (1). That was Step 1, where we chose the c's.

Example 1 Find all solutions to $\boldsymbol{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{y}$. Which solution has $\boldsymbol{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$?

Solution First we find $\lambda = -1$ and -3. Their eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ go into V:

$$\det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 3 \quad \text{factors into} \quad (\lambda+1)(\lambda+3)$$

$$\begin{aligned} A\boldsymbol{x}_1 &= -1\,\boldsymbol{x}_1 \\ A\boldsymbol{x}_2 &= -3\,\boldsymbol{x}_2 \end{aligned} \quad \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Step 1 Solve $\boldsymbol{y}(\mathbf{0}) = V\boldsymbol{c}$. Then $\boldsymbol{y}(0)$ is a mixture $4\boldsymbol{x}_1 + 2\boldsymbol{x}_2$ of the eigenvectors:

$$V\boldsymbol{c} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \text{ gives } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \text{ Then } \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Step 2 finds the separate solutions $ce^{\lambda t}\boldsymbol{x}$ given by $4e^{-t}\boldsymbol{x}_1$ and $2e^{-3t}\boldsymbol{x}_2$. Now add:

$$\textbf{Step 3} \qquad \boldsymbol{y}(t) = 4e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4e^{-t} + 2e^{-3t} \\ 4e^{-t} - 2e^{-3t} \end{bmatrix}. \qquad (5)$$

For a larger matrix the computations are harder. The idea doesn't change.

Now I want to show a matrix with complex eigenvalues and eigenvectors. This will lead us to complex numbers in $\boldsymbol{y}(t)$. But A is real and $\boldsymbol{y}(0)$ is real, so $\boldsymbol{y}(t)$ must be real! Euler's formula $e^{it} = \cos t + i \sin t$ will get us back to real numbers.

Example 2 Find all solutions to $\boldsymbol{y}' = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \boldsymbol{y}$. Which solution has $\boldsymbol{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$?

Solution Again we find the eigenvalues and eigenvectors, now complex:

$$\mathbf{det}\,(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0} \qquad \det \begin{bmatrix} -2-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} = \boldsymbol{\lambda}^2 + \mathbf{4}\boldsymbol{\lambda} + \mathbf{5} \ \text{(no real factors)}$$

We use the quadratic formula to solve $\lambda^2 + 4\lambda + 5 = 0$. The eigenvectors are $x = (1, \pm i)$.

$$\begin{aligned} \boldsymbol{\lambda}_1 &= \mathbf{-2} + \boldsymbol{i} \\ \boldsymbol{\lambda}_2 &= \mathbf{-2} - \boldsymbol{i} \end{aligned} \qquad \lambda = \frac{-4 \pm \sqrt{4^2 - 4(5)}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$\begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (-2+i) \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (-2-i) \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

To solve $\boldsymbol{y}' = A\boldsymbol{y}$, Step 1 expresses $\boldsymbol{y}(0) = (6,2)$ as a combination of those eigenvectors:

$$\boldsymbol{y}(0) = V\boldsymbol{c} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 \qquad \begin{bmatrix} 6 \\ 2 \end{bmatrix} = (3-i)\begin{bmatrix} 1 \\ i \end{bmatrix} + (3+i)\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Step 2 finds the solutions $c_1e^{\lambda_1 t}\boldsymbol{x}_1$ and $c_2e^{\lambda_2 t}\boldsymbol{x}_2$. Step 3 combines them into $\boldsymbol{y}(t)$:

$$\textbf{Solution} \quad \boldsymbol{y}(t) = c_1e^{\lambda_1 t}\boldsymbol{x}_1 + c_2e^{\lambda_2 t}\boldsymbol{x}_2 = (3-i)e^{(-2+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + (3+i)e^{(-2-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

As expected, this looks complex. As promised, it must be real. Factoring out e^{-2t} leaves

$$(3-i)(\cos t + i \sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} + (3+i)(\cos t - i \sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 6 \cos t + 2 \sin t \\ 2 \cos t - 6 \sin t \end{bmatrix}. \tag{6}$$

Put back the factor e^{-2t} to find the (real) $\boldsymbol{y}(t)$. It would be wise to check $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\boxed{\boldsymbol{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{y}(t) = e^{-2t}\begin{bmatrix} 6 \cos t + 2 \sin t \\ 2 \cos t - 6 \sin t \end{bmatrix}} \tag{7}$$

The factor e^{-2t} from the real part of λ means decay. The $\cos t$ and $\sin t$ factors from the imaginary part mean oscillation. The oscillation frequency in $\cos t = \cos \omega t$ is $\omega = 1$.

Note The -2's on the diagonal of A (which is exactly $-2I$) are responsible for the real parts -2 of the λ's. They give the decay factor e^{-2t}. Without the -2's we would only have sines and cosines, which converts into **circular motion in the $y_1 - y_2$ plane.** That is a very important example to see by itself.

Example 3 Pure circular motion and pure imaginary eigenvalues

$$\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} \quad \text{sends } \boldsymbol{y} \text{ around a circle.}$$

Discussion The equations are $y_1' = y_2$ and $y_2' = -y_1$. One solution is $y_1 = \sin t$ and $y_2 = \cos t$. A second solution is $y_1 = \cos t$ and $y_2 = -\sin t$. We need two solutions to match two required values $y_1(0)$ and $y_2(0)$. Those solutions would come in the usual way from the eigenvalues $\lambda = \pm i$ and the eigenvectors.

Figure 6.2a shows the solution to Example 2 spiralling in to zero (because of e^{-2t}). Figure 6.2b shows the solution to Example 3 staying on the circle (because of sine and cosine). These are good examples to see the "*phase plane*" with axes y_1 and ${y_1}' = y_2$.

Without the -2's, the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a rotation by $90°$. At every instant, $\boldsymbol{y}'$ is at a $90°$ angle with $\boldsymbol{y}$. That keeps $\boldsymbol{y}$ moving in a circle. Its length is constant:

$$\begin{matrix}\textbf{Constant length} \\ \textbf{Circular orbit}\end{matrix} \qquad \frac{d}{dt}(y_1^2 + y_2^2) = 2y_1y_1' + 2y_2y_2' = 2y_1y_2 - 2y_2y_1 = 0. \tag{8}$$

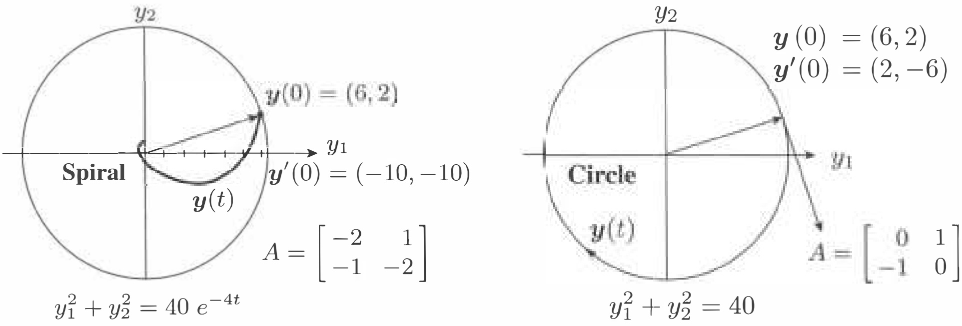


Figure 6.2: (a) The solution (7) including e^{-2t}. (b) The solution (6) without e^{-2t}.

Conservative Motion

Travel around a circle is an example of conservative motion for $n = 2$. The length of $\boldsymbol{y}$ does not change. "Energy is conserved." For $n = 3$ this would become travel on a sphere. For $n > 3$ the vector $\boldsymbol{y}$ would move with constant length around a hypersphere.

Which linear differential equations produce this conservative motion ? We are asking for the squared length $||\boldsymbol{y}||^2 = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}$ to *stay constant*. So its derivative is zero:

$$\frac{d}{dt}(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}) = \left(\frac{d\boldsymbol{y}}{dt}\right)^{\mathrm{T}} \boldsymbol{y} + \boldsymbol{y}^{\mathrm{T}}\frac{d\boldsymbol{y}}{dt} = (A\boldsymbol{y})^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{y}^{\mathrm{T}}(A\boldsymbol{y}) = \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A})\boldsymbol{y} = 0. \qquad (9)$$

The first step was the product rule. Then $d\boldsymbol{y}/dt$ was replaced by $A\boldsymbol{y}$. **Conclusion**:

$$||\boldsymbol{y}||^2 \textbf{ is constant when } A \textbf{ is antisymmetric: } \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A} = \boldsymbol{0} \text{ and } \boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}. \qquad (10)$$

The simplest example is $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\boldsymbol{y}$ goes around the circle in Figure 6.2b. The initial vector $\boldsymbol{y}(0)$ decides the size of the circle: $||\boldsymbol{y}(t)|| = ||\boldsymbol{y}(0)||$ for all time. When A is antisymmetric, its eigenvalues are pure imaginary. This comes in Section 6.5.

Stable Motion

Motion around a circle is only "neutral" stability. **For a truly stable linear system, the solution $\boldsymbol{y}(t)$ always goes to zero.** It is the spiral in Figure 6.2a that shows stability:

$$A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{has eigenvalues } \lambda = -2 \pm i. \text{ This } A \text{ is a } \textbf{stable matrix}.$$

The key is in the eigenvalues of A, which give the simple solutions $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$. When A is diagonalizable (n independent eigenvectors), every solution is a combination of $e^{\lambda_1 t}\boldsymbol{x}_1, \ldots, e^{\lambda_n t}\boldsymbol{x}_n$. So we only have to ask when those simple solutions approach zero:

$$\boxed{\textbf{Stability} \qquad e^{\lambda t}\boldsymbol{x} \to \boldsymbol{0} \textbf{ when the real part of } \lambda \textbf{ is negative: } \textbf{Re}\,\boldsymbol{\lambda} < 0.}$$

The real parts -2 give the exponential decay factor e^{-2t} in the solution $\boldsymbol{y}$. That factor produces the inward spiral in Figure 6.2 a and the stability of the equation $y' = Ay$. The imaginary parts of $\lambda = -2 \pm i$ give oscillations: sines and cosines that stay bounded.

Test for Stability When $n = 2$

For a 2 by 2 matrix, the trace and determinant tell us both eigenvalues. So the trace and determinant must decide stability. A real matrix A has two possibilities **R** and **C**:

R Real eigenvalues λ_1 and λ_2
C Complex conjugate pair $\lambda_1 = s + i\omega$ and $\lambda_2 = s - i\omega$

Adding the eigenvalues gives the trace of A. Multiplying the eigenvalues gives the determinant of A. We check the two possibilities **R** and **C**, to see when Re $(\lambda) < 0$.

R If $\lambda_1 < 0$ and $\lambda_2 < 0$, then **trace** $= \lambda_1 + \lambda_2 < 0$ and **determinant** $= \lambda_1\lambda_2 > 0$
C If $s < 0$ in $\lambda = s \pm i\omega$, then **trace** $= 2s < 0$ and **determinant** $= s^2 + \omega^2 > 0$

Both cases give the same stability requirement: *Negative trace and positive determinant.*

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \textbf{ is stable } \quad \textbf{exactly when} \quad \begin{array}{ll} \textbf{trace} = a + d & < 0 \\ \textbf{det} \quad = ad - bc & > 0 \end{array} \tag{11}$$

It was the quadratic formula that led us to the possibilities **R** and **C**, real or complex. Remember the equation det $(A - \lambda I) = 0$ for the eigenvalues:

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \boldsymbol{\lambda}^2 - (a + d)\,\lambda + (ad - bc) = \boldsymbol{\lambda}^2 - (\textbf{trace})\,\boldsymbol{\lambda} + (\textbf{det}) = 0.$$

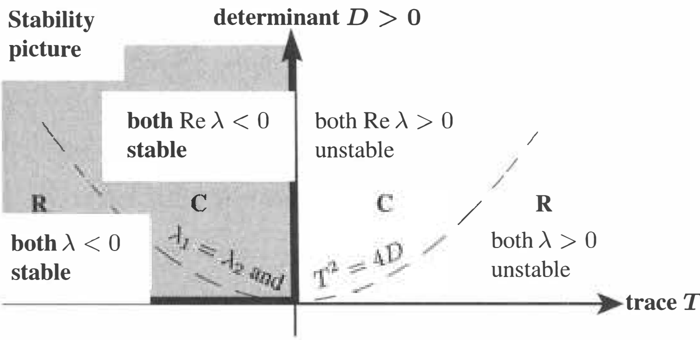
The quadratic formula for the two eigenvalues includes an all–important square root:

$$\textbf{Real or complex } \lambda \qquad \lambda = \frac{1}{2}\left[\text{trace} \pm \sqrt{(\text{trace})^2 - 4(\det)}\right]. \tag{12}$$

The roots are real (case **R**) when $(\text{trace})^2 \geq 4\,(\det)$. The roots are complex (case **C**) when $(\text{trace})^2 < 4\,(\det)$. The line between **R** and **C** is the parabola in the stability picture:

$$(\textbf{Trace})^2 = 4\,(\textbf{det}) \qquad \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \text{ is stable} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ is unstable}$$

Stable matrices only fill one quadrant of the trace-determinant plane: trace < 0, det > 0.



det < 0 means $\lambda_1 < 0$ and $\lambda_2 > 0$: unstable

Examples

$\begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$	**stable**
$\begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}$	**unstable**
$\begin{bmatrix} 0 & 4 \\ 5 & -6 \end{bmatrix}$	**unstable**
$\begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix}$	**neutral**

Second Order Equation to First Order System

Chapter 2 of this book studied the second order equation $y'' + By' + Cy = 0$. Often this is oscillation with underdamping. The solutions $y = e^{(a+i\omega)t}$ and $e^{(a-i\omega)t}$ come from the quadratic equation $\boldsymbol{s^2 + Bs + C = 0}$, when we search for solutions $y = e^{st}$. If B^2 is larger than $4C$, then the roots are real and the solutions are $e^{s_1 t}$ and $e^{s_2 t}$. In that overdamped case, the oscillations are gone.

I want to show you exactly the same solutions in the language of $\boldsymbol{y}' = A\boldsymbol{y}$. Instead of one equation with y'' we will reach **two equations with $\boldsymbol{y}' = (y_1', y_2')$**. You have seen the key idea before: *The original y and y' become y_1 and y_2.* Then the matrix A is a **companion matrix**.

$$\boldsymbol{y'' + By' + Cy = 0} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}. \tag{13}$$

It is important to see why the roots s_1 and s_2 are also the eigenvalues λ_1 and λ_2. The reason is, these are still the roots of the same equation $s^2 + Bs + C = 0$. Only the letter s is changed to λ.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -C & -B - \lambda \end{bmatrix} = \lambda^2 + B\lambda + C = 0. \tag{14}$$

This was foreshadowed when we drew the six solution paths in Section 3.2: Sources, Sinks, Spirals, and Saddles. Those pictures were in the y, y' plane (the phase plane). Now the same pictures are in the y_1, y_2 plane. I specially want to show you again the trace and determinant of A and the whole new-old understanding of stability.

$$\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \text{ has trace} = -B \text{ and determinant} = C.$$

First the test for real roots of $s^2 + Bs + C = 0$ and for real eigenvalues of A:

R	Real roots and real eigenvalues	$\boldsymbol{B^2 \geq 4C}$	**(trace)$^2 \geq$ 4(det)**
C	Complex roots and eigenvalues $\lambda = a \pm i\omega$	$\boldsymbol{B^2 < 4C}$	**(trace)$^2 <$ 4(det)**

In the picture, the dashed parabola $T^2 = 4D$ separates real from complex: **R** from **C**.

More than that, the highlighted quadrant displays the three possibilities for damping. These are all stable: $B > 0$ and $C > 0$.

Underdamping	**Complex roots**	$B^2 < 4AC$	**above the parabola**
Critical damping	**Equal roots**	$B^2 = 4AC$	**on the parabola**
Overdamping	**Real roots**	$B^2 > 4AC$	**below the parabola**

The undamped case $B = 0$ is on the vertical axis: eigenvalues $\pm i\omega$ with $\omega^2 = C$. Everything comes together for 2 by 2 companion matrices. The eigenvectors are attractive too:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \quad \text{agree with} \quad \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \quad \text{at } t = 0. \tag{15}$$

The same method applies to systems with n oscillators. B and C become matrices. The vectors $\boldsymbol{y}$ and $\boldsymbol{y}'$ have n components and the joint vector $\boldsymbol{z} = (\boldsymbol{y}, \boldsymbol{y}')$ has $2n$ components. The network leads to n second order equations for $\boldsymbol{y}$, or $2n$ first order equations for $\boldsymbol{z}$:

$$\boldsymbol{y}'' + B\boldsymbol{y}' + C\boldsymbol{y} = \boldsymbol{0} \qquad \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{y}' \\ \boldsymbol{y}'' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{y}' \end{bmatrix} = A\boldsymbol{z}. \tag{16}$$

Eigenvectors give the null solutions $\boldsymbol{y}_n$. Real problems come with forcing terms $\boldsymbol{q} = \boldsymbol{F}e^{st}$.

Here I make just one point about repeated roots and repeated eigenvalues: **If $\boldsymbol{\lambda_1 = \lambda_2}$ there is no second eigenvector of the companion matrix $\boldsymbol{A}$.** That matrix can't be diagonalized and the eigenvector method fails. The next section will succeed with e^{At}, even without a full set of eigenvectors.

Higher Order Equations Give First Order Systems

A third order (or higher order) equation reduces to first order in the same way. **Introduce derivatives of $\boldsymbol{y}$ as new unknowns.** This is easy to see for a single third order equation with constant coefficients:

$$y''' + By'' + Cy' + Dy = 0 \tag{17}$$

The idea is to create a vector unknown $\boldsymbol{z} = (y, y', y'')$. The first component y satisfies a very simple equation: its derivative is the second component y'. Then the matrix below has $0, 1, 0$ in its first row. Similarly the derivative of y' is y''. The second row of the companion matrix is $0, 0, 1$. The third row contains the original differential equation (17):

$$\boldsymbol{z}' = A\boldsymbol{z} \qquad \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -D & -C & -B \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}. \tag{18}$$

Companion matrices have 1's on their superdiagonal. We want to know their eigenvalues.

Eigenvalues of the Companion Matrix = Roots of the Polynomial

Start with the eigenvalues of the 2 by 2 companion matrix:

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -C & -B-\lambda \end{bmatrix} = \lambda^2 + B\lambda + C = 0. \tag{19}$$

Compare that with substituting $y = e^{\lambda t}$ in the single equation $y'' + By' + Cy = 0$:

$$\lambda^2 e^{\lambda t} + B\lambda e^{\lambda t} + Ce^{\lambda t} \quad \text{gives} \quad \lambda^2 + B\lambda + C = 0. \tag{20}$$

The equations are the same. The λ's in special solutions $y = e^{\lambda t}$ are the same as the eigenvalues in special solutions $\boldsymbol{z} = e^{\lambda t}\boldsymbol{x}$. This is our main point and it is true again for 3 by 3. The eigenvalue equation $\det(A - \lambda I) = 0$ is exactly the polynomial equation from substituting $y = e^{\lambda t}$ in $y''' + By'' + Cy' + Dy = 0$:

$$\det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -D & -C & -B-\lambda \end{bmatrix} = -(\lambda^3 + B\lambda^2 + C\lambda + D) = 0. \tag{21}$$

The eigenvectors of this companion matrix have the special form $\boldsymbol{x} = (1, \lambda, \lambda^2)$. Fourth order equations become $\boldsymbol{z}' = A\boldsymbol{z}$ with $\boldsymbol{z} = (y, y', y'', y''')$. 4 by 4 companion matrix, eigenvalues from $\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$.

Example 4 $(\lambda - 2)^2 = \lambda^2 - 4\lambda + 4 = 0$ comes from $y'' - 4y' + 4y = 0$:

Companion matrix A
Repeated root $\lambda = 2, 2$
$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \qquad \det(A - \lambda I) = \lambda^2 - 4\lambda + 4.$$

$\lambda = 2$ must have one eigenvector, and it is $\boldsymbol{x} = (1, 2)$. *There is no second eigenvector.* The first order system $\boldsymbol{z}' = A\boldsymbol{z}$ and the second order equation $y'' - 4y' + 4y = 0$ are in (*the same*) trouble. **The only pure exponential solution is $y = e^{2t}$.**

The way out for y is the solution te^{2t}. It needs that new form (including t). The way out for $\boldsymbol{z}$ is a "generalized eigenvector" but we are not going there.

■ REVIEW OF THE KEY IDEAS ■

1. The system $\boldsymbol{y}' = A\boldsymbol{y}$ is linear with constant coefficients, starting from $\boldsymbol{y}(0)$.

2. Its solution is usually a combination of exponentials $e^{\lambda t}$ times eigenvectors $\boldsymbol{x}$:

   n independent eigenvectors $\quad \boldsymbol{y}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n.$

3. The constants $c_1, \ldots, c_n$ are determined by $\boldsymbol{y}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. *This is $V\boldsymbol{c}$!*

4. $\boldsymbol{y}(t)$ approaches zero (stability) if every λ has negative real part: Re $\lambda < 0$.

5. 2 by 2 systems are stable if **trace $T = a + d < 0$** and **det $D = ad - bc > 0$**.

6. $y'' + By' + Cy = 0$ leads to a companion matrix with trace $= -B$ and det $= C$.

Problem Set 6.3

1 Find all solutions $\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ to $\boldsymbol{y}' = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \boldsymbol{y}$. Which solution starts from $\boldsymbol{y}(0) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = (2, 2)$?

2 Find two solutions of the form $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ to $\boldsymbol{y}' = \begin{bmatrix} 3 & 10 \\ 2 & 4 \end{bmatrix} \boldsymbol{y}$.

3 If $a \neq d$, find the eigenvalues and eigenvectors and the complete solution to $\boldsymbol{y}' = A\boldsymbol{y}$. This equation is stable when a and d are _____.

$$\boldsymbol{y}' = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \boldsymbol{y}.$$

4 If $a \neq -b$, find the solutions $e^{\lambda_1 t}\boldsymbol{x}_1$ and $e^{\lambda_2 t}\boldsymbol{x}_2$ to $y' = A\boldsymbol{y}$:

$$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}. \quad \text{Why is } \boldsymbol{y}' = A\boldsymbol{y} \text{ not stable?}$$

5 Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and the eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ of A. Write $\boldsymbol{y}(0) = (0, 1, 0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = V\boldsymbol{c}$ and solve $\boldsymbol{y}' = A\boldsymbol{y}$. What is the limit of $\boldsymbol{y}(t)$ as $t \to \infty$ (the steady state) ? *Steady states come from* $\lambda = 0$.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

6 The simplest 2 by 2 matrix without two independent eigenvectors has $\lambda = 0, 0$:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{has a first solution} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{0t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find a second solution to these equations $y_1{}' = y_2$ and $y_2{}' = 0$. That second solution starts with t times the first solution to give $y_1 = t$. What is y_2 ?

Note A complete discussion of $\boldsymbol{y}' = A\boldsymbol{y}$ for all cases of repeated λ's would involve the *Jordan form* of A: too technical. Section 6.4 shows that a triangular form is sufficient, as Problems 6 and 8 confirm. We can solve for y_2 and then y_1.

7 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{y}.$$

What combination $\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{y}(0) = (5, -2)$?

8 Solve Problem 7 for $\boldsymbol{y} = (y, z)$ by back substitution, z before y:

$$\text{Solve } \frac{dz}{dt} = z \text{ from } z(0) = -2. \text{ Then solve } \frac{dy}{dt} = 4y + 3z \text{ from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^t. The λ's are 4 and 1.

9 (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{y}' = A\boldsymbol{y}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$:

$$\text{Solve } \frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{y} \quad \text{with} \quad \boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad \text{What is } \boldsymbol{y}(\infty)\,?$$

10 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{y}/dt = A\boldsymbol{y}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

11 Reverse the diffusion of people in Problem 10 to $d\boldsymbol{z}/dt = -A\boldsymbol{z}$:

$$\frac{dv}{dt} = v - w \quad \text{and} \quad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

12 A has real eigenvalues but B has complex eigenvalues:

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \qquad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \qquad (a \text{ and } b \text{ are real})$$

Find the stability conditions on a and b so that all solutions of $d\boldsymbol{y}/dt = A\boldsymbol{y}$ and $d\boldsymbol{z}/dt = B\boldsymbol{z}$ approach zero as $t \to \infty$.

13 Suppose P is the projection matrix onto the 45° line $y = x$ in $\mathbf{R}^2$. Its eigenvalues are 1 and 0 with eigenvectors $(1, 1)$ and $(1, -1)$. If $d\boldsymbol{y}/dt = -P\boldsymbol{y}$ (notice minus sign) can you find the limit of $\boldsymbol{y}(t)$ at $t = \infty$ starting from $\boldsymbol{y}(0) = (3, 1)$?

14 The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

15 (a) Write $(4, 0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ of these two eigenvectors of A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = -i\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The solution to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ starting from $(4, 0)$ is $c_1e^{it}\boldsymbol{x}_1 + c_2e^{-it}\boldsymbol{x}_2$. Substitute $e^{it} = \cos t + i\sin t$ and $e^{-it} = \cos t - i\sin t$ to find $\boldsymbol{y}(t)$.

Questions 16–19 reduce second-order equations to first-order systems for (y, y').

16 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{y} = (y, y')$:

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

17 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too.* Show that the second solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

18 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}.$$

Find $\boldsymbol{y}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{y}(0) = (3, 0)$.

19 If c is not an eigenvalue of A, substitute $\boldsymbol{y} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{y}/dt = A\boldsymbol{y} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A?

20 A particular solution to $d\boldsymbol{y}/dt = A\boldsymbol{y} - \boldsymbol{b}$ is $\boldsymbol{y}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ give $\boldsymbol{y}_n$. Find the complete solution $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_n$:

(a) $\dfrac{dy}{dt} = y - 4$ (b) $\dfrac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{y} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

21 Find a matrix A to illustrate each of the unstable regions in the stability picture:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$.

22 Which of these matrices are stable ? Then Re $\lambda < 0$, trace < 0, and det > 0.

$$A_1 = \begin{bmatrix} -2 & -3 \\ -4 & -5 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix} \quad A_3 = \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix}.$$

23 For an n by n matrix with trace $(A) = T$ and det $(A) = D$, find the trace and determinant of $-A$. Why is $\boldsymbol{z}' = -A\boldsymbol{z}$ unstable whenever $\boldsymbol{y}' = A\boldsymbol{y}$ is stable ?

24 (a) For a real 3 by 3 matrix with stable eigenvalues (Re $\lambda < 0$), show that trace < 0 and det < 0. Either three real negative λ or else $\lambda_2 = \overline{\lambda}_1$ and λ_3 is real.

(b) The trace and determinant of a 3 by 3 matrix do not determine all three eigenvalues ! Show that A is unstable even with trace < 0 and determinant < 0:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -5 \end{bmatrix}.$$

25 You might think that $\boldsymbol{y}' = -A^2\boldsymbol{y}$ would always be stable because you are squaring the eigenvalues of A. But why is that equation unstable for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

26 Find the three eigenvalues of A and the three roots of $s^3 - s^2 + s - 1 = 0$ (including $s = 1$). The equation $y''' - y'' + y' - y = 0$ becomes

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \quad \text{or} \quad \boldsymbol{z}' = A\boldsymbol{z}.$$

Each eigenvalue λ has an eigenvector $\boldsymbol{x} = (1, \lambda, \lambda^2)$.

27 Find the two eigenvalues of A and the double root of $s^2 + 6s + 9 = 0$:

$$y'' + 6y' + 9y = 0 \text{ becomes } \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \quad \text{or} \quad \boldsymbol{z}' = A\boldsymbol{z}.$$

The repeated eigenvalue gives only one solution $\boldsymbol{z} = e^{\lambda t}\boldsymbol{x}$. Find a second solution $\boldsymbol{z}$ from the second solution $y = te^{\lambda t}$.

28 Explain why a 3 by 3 companion matrix has eigenvectors $\boldsymbol{x} = (\mathbf{1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}^\mathbf{2})$.

First Way: If the first component is $x_1 = 1$, the first row of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ gives the second component $x_2 =$ ______. Then the second row of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ gives the third component $x_3 = \lambda^2$.

Second Way: $\boldsymbol{y}' = A\boldsymbol{y}$ starts with $y_1' = y_2$ and $y_2' = y_3$. $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solves those equations. At $t = 0$ the equations become $\lambda x_1 = x_2$ and ______.

29 Find A to change the scalar equation $y'' = 5y' - 4y$ into a vector equation for $\boldsymbol{z} = (y, y')$:

$$\frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

What are the eigenvalues of the companion matrix A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' - 4y$.

30 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{z}' = A\boldsymbol{z}$:

$$\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

Find $\boldsymbol{z}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{z}(0) = (3, 0)$.

31 (a) Change the third order equation $y''' - 2y'' - y' + 2y = 0$ to a first order system $\boldsymbol{z}' = A\boldsymbol{z}$ for the unknown $\boldsymbol{z} = (y, y', y'')$. The companion matrix A is 3 by 3.

(b) Substitute $y = e^{\lambda t}$ and also find det $(A - \lambda I)$. Those lead to the same λ's.

(c) One root is $\lambda = 1$. Find the other roots and these complete solutions:

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \qquad \boldsymbol{z} = C_1 e^{\lambda_1 t} \boldsymbol{x}_1 + C_2 e^{\lambda_2 t} \boldsymbol{x}_2 + C_3 e^{\lambda_3 t} \boldsymbol{x}_3.$$

32 These companion matrices have $\lambda = 2, 1$ and $\lambda = 4, 1$. Find their eigenvectors:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix} \qquad \text{Notice trace and determinant!}$$

6.4 The Exponential of a Matrix

This section expresses the solution to a system $dy/dt = Ay$ in a different way. Instead of combining eigenvector solutions $e^{\lambda t}\boldsymbol{x}$, the new form uses the **matrix exponential** $e^{\boldsymbol{A}t}$:

Solution to $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$	$\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}(0)$	(1)

This matrix e^{At} matches e^{at} when $n = 1$: the scalar case. For matrices, we can still write the exponential as an infinite series. In one way this is better than depending on eigenvectors—but maybe not in practice:

Advantage We don't need n independent eigenvectors for e^{At}.
Disadvantage An infinite series is usually not so practical.

The new way produces one short symbol e^{At} for the "solution matrix." Still we often compute in the old way with eigenvectors. This is like a linear system $A\boldsymbol{v} = \boldsymbol{b}$, where A^{-1} is the solution matrix but we compute $\boldsymbol{v}$ by elimination.

For large matrices, $\boldsymbol{y}' = A\boldsymbol{y}$ uses completely different ways — often finite differences.

The Exponential Series

The most direct way to define the matrix e^{At} is by an infinite series of powers of A:

Matrix exponential	$e^{\boldsymbol{A}t} = I + At + \frac{1}{2}(At)^2 + \cdots = \sum_{n=0}^{\infty}(At)^n/n!$	(2)

This series always converges, like the scalar case e^{at} in Chapter 1. e^{At} is the great function of matrix calculus. The quickly growing factors $n!$ still assure convergence. The two key properties of e^{at} continue to hold when a becomes a matrix A:

1. **The derivative of** $e^{\boldsymbol{A}t}$ **is** $\boldsymbol{A}e^{\boldsymbol{A}t}$ 2. $\left(e^{\boldsymbol{A}t}\right)\left(e^{\boldsymbol{A}T}\right) = e^{\boldsymbol{A}(t+T)}$

Property **1** says that $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0)$ has derivative $\boldsymbol{y}' = A\boldsymbol{y}$. And $\boldsymbol{y}(t)$ starts correctly from $\boldsymbol{y}(0)$ at $t = 0$, since $e^{A0} = I$ from equation (2). **So** $e^{\boldsymbol{A}t}\boldsymbol{y}(0)$ **solves** $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$.

Suppose we set $T = -t$ in Property **2**. Then $t + T = 0$:

$$\textbf{The inverse of } e^{\boldsymbol{A}t} \textbf{ is } e^{-\boldsymbol{A}t} \qquad e^{At}e^{AT} = e^0 = I \text{ when } T \text{ is } -t. \tag{3}$$

e^{At} has properties **1** and **2** even if A cannot be diagonalized. When A does have n independent eigenvectors, the same eigenvector matrix V diagonalizes A and e^{At}. The next page shows that $e^{\boldsymbol{A}t} = \boldsymbol{V}e^{\boldsymbol{\Lambda}t}\boldsymbol{V}^{-1}$: this is the good way to find e^{At}.

Assume A has n independent eigenvectors, so it is diagonalizable. Substitute $A = V\Lambda V^{-1}$ into the series for e^{At}. Whenever $V\Lambda V^{-1}V\Lambda V^{-1}$ appears, take out $V^{-1}V = I$.

$$\begin{array}{lll} \textbf{Use the series} & e^{At} & = I + V\Lambda V^{-1}t + \frac{1}{2}(V\Lambda V^{-1}t)(V\Lambda V^{-1}t) + \cdots \\ \textbf{Factor out } V \textbf{ and } V^{-1} & & = V\,[I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \cdots]\,V^{-1} \\ \textbf{Diagonalize } \boldsymbol{e^{At}} & e^{At} & = \boldsymbol{V e^{\Lambda t} V^{-1}}. \end{array} \tag{4}$$

The numbers $e^{\lambda_i t}$ are on the diagonal of $e^{\Lambda t}$. Multiply $Ve^{\Lambda t}V^{-1}\boldsymbol{y}(0)$ to see $\boldsymbol{y}(t)$.
Second Proof e^{At} has the same eigenvectors $\boldsymbol{x}$ as A. **The eigenvalues of $\boldsymbol{e^{At}}$ are $\boldsymbol{e^{\lambda t}}$**:

$$A^n\boldsymbol{x} = \lambda^n\boldsymbol{x} \quad \text{leads to} \quad e^{At}\boldsymbol{x} = \left(1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \cdots\right)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}. \tag{5}$$

So the same eigenvector matrix V diagonalizes both A and e^{At}. The eigenvalue matrix for e^{At} is diag $(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$. This is exactly $e^{\Lambda t}$. Again $\boldsymbol{e^{At} = Ve^{\Lambda t}V^{-1}}$.

The eigenvalues of the inverse matrix e^{-At} are $e^{-\lambda t}$. This is $1/e^{\lambda t}$ as expected.

Example 1 The rotation matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$:

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{6}$$

This produces e^{At} without adding up an infinite series. We could also begin the series:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}t^2 & t - \frac{1}{6}t^3 \\ -t + \frac{1}{6}t^3 & 1 - \frac{1}{2}t^2 \end{bmatrix}.$$

The cosine series starts with $1 - \frac{1}{2}t^2$. The sine series starts with $t - \frac{1}{6}t^3$. The full series for e^{At} gives the full series for $\cos t$ and $\sin t$: very exceptional.

Example 1 continued What is the solution to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ with $\boldsymbol{y}(0) = (1, 0)$?
Answer We know that $\boldsymbol{y}(t) = (y_1, y_2)$ is $e^{At}\boldsymbol{y}(0)$, and equation (6) gives e^{At}:

$$\begin{array}{l} y_1{}' = y_2 \\ y_2{}' = -y_1 \end{array} \qquad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}. \tag{7}$$

Right! The derivative of $\cos t$ is $-\sin t$. The derivative of $y_2 = -\sin t$ is $-\cos t$. The equations $\boldsymbol{y}' = A\boldsymbol{y}$ are satisfied. When $t = 0$, we start correctly at $\boldsymbol{y}(0) = (1, 0)$.

This solution is important in physics and engineering. The point $\boldsymbol{y}(t)$ is on the unit circle $y_1^2 + y_2^2 = \cos^2 t + \sin^2 t = 1$. It goes around the circle with constant speed. The second derivative (acceleration) is $\boldsymbol{y}'' = (-\sin t, -\cos t)$ because $A^2 = -I$. This vector $\boldsymbol{y}''$ points in to the center $(0, 0)$. We have a planet going in a circle around the sun.

Example 2 Suppose A is triangular but we can't diagonalize it (only one eigenvector):

$$\boldsymbol{y}' = A\boldsymbol{y} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 0 + y_2 \end{aligned} \tag{8}$$

A has no invertible eigenvector matrix V. How to find $\boldsymbol{y}(t)$ without two eigenvectors?

Solution Since A is triangular, back substitution will solve $\boldsymbol{y}' = A\boldsymbol{y}$. Begin by solving the last equation $y_2{}' = y_2$. Then solve for y_1:

$$y_2(t) = e^t y_2(0) \qquad \text{Then } y_1{}' = y_1 + y_2 = y_1 + e^t y_2(0)$$

That equation for y_1 has a source term $q(t) = e^t y_2(0)$. Chapter 1 found the solution $y_1(t)$:

$$e^t y_1(0) + \int_0^t e^{t-s} q(s)\, ds = e^t y_1(0) + e^t y_2(0) \int_0^t ds = e^t y_1(0) + \boldsymbol{t}\boldsymbol{e}^{\boldsymbol{t}} y_2(0). \tag{9}$$

At last we have a reason for the extra factor $\boldsymbol{t}$. The natural growth rate of y_1 is also the growth rate of y_2. This leads to "resonance" in $y_1{}' = y_1 + y_2$, and the growth of te^t is extra fast. We saw resonance with te^{st} in Chapter 2. Now we are seeing the t in e^{At}.

$$\begin{aligned} y_1(t) &= \boldsymbol{e^t} y_1(0) + \boldsymbol{te^t} y_2(0) \\ y_2(t) &= \boldsymbol{e^t} y_2(0) \end{aligned} \qquad \text{means that} \quad \boldsymbol{e^{At}} = \begin{bmatrix} \boldsymbol{e^t} & \boldsymbol{te^t} \\ 0 & \boldsymbol{e^t} \end{bmatrix}. \tag{10}$$

Example 2 (using $\boldsymbol{e^{At}}$) For this triangular matrix A, we can also add the series for e^{At}:

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & \boldsymbol{t} \\ 0 & t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} t^2 & \boldsymbol{2t^2} \\ 0 & t^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} t^3 & \boldsymbol{3t^3} \\ 0 & t^3 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} e^t & \boldsymbol{te^t} \\ 0 & e^t \end{bmatrix} \qquad \text{because} \quad te^t = t + t^2 + \frac{1}{2}t^3 + \cdots \end{aligned} \tag{11}$$

All the powers of a triangular matrix are triangular. So the diagonal entries of A give the diagonal entries of e^{At}. Those are the eigenvalues of e^{At} and here they are both e^t.

Source Term in $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}$

We can solve $y' = ay + q$ for a single equation (1 by 1). Now allow a matrix A:

$$\textit{Old} \quad y(t) = e^{at} y(0) + \frac{e^{at} - 1}{a} q \qquad \textit{New} \quad \frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y} + \boldsymbol{q} \tag{12}$$

Change a to A! For constant $\boldsymbol{q}$, that is the only change in the formula for $\boldsymbol{y}$:

$$\boxed{\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q} \qquad \text{is solved by} \qquad \boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) + (e^{At} - I)A^{-1}\boldsymbol{q}.} \tag{13}$$

The derivative of $\boldsymbol{y}$ produces $A\boldsymbol{y}$, except for the constant $A^{-1}\boldsymbol{q}$ with derivative $=$ zero. But this term $A^{-1}\boldsymbol{q}$ disappears safely in $A\boldsymbol{y}+\boldsymbol{q}$, because $-AA^{-1}\boldsymbol{q}+\boldsymbol{q}=\boldsymbol{0}$.

Chapter 1 was built on the growth factor e^{at} in the integral for y_p. Now it is e^{At} !

Principle *Each input $\boldsymbol{q}(s)$ has growth factor $e^{A(t-s)}$ from time s to time t.* For constant A, the growth (or decay) over time $t-s$ is just multiplication by $e^{A(t-s)}$:

$$\boxed{\boldsymbol{y}'=A\boldsymbol{y}+\boldsymbol{q}(t)\quad\text{is solved by}\quad \boldsymbol{y}(t)=e^{At}\boldsymbol{y}(0)+\int_0^t e^{A(t-s)}\boldsymbol{q}(s)\,ds.} \tag{14}$$

Similar Matrices A and B

To end this section, I will solve $\boldsymbol{y}'=A\boldsymbol{y}$ in one more way. Same result, new approach.

Change of variables. Write $\boldsymbol{y}=V\boldsymbol{z}$ to change from $\boldsymbol{y}(t)$ to the new variable $\boldsymbol{z}\,(t)$.

$$\frac{d\boldsymbol{y}}{dt}=A\boldsymbol{y}\qquad\text{becomes}\qquad V\frac{d\boldsymbol{z}}{dt}=AV\boldsymbol{z}\qquad\text{which is}\qquad \boldsymbol{\frac{dz}{dt}=V^{-1}AVz}. \tag{15}$$

The matrix A has changed to $B=V^{-1}AV$. Then the solution for $\boldsymbol{z}$ involves e^{Bt} :

$$\boldsymbol{B=V^{-1}AV}\qquad\qquad \boldsymbol{z}'=B\boldsymbol{z}\ \text{ produces }\ \boldsymbol{z}(t)=e^{Bt}\boldsymbol{z}(0) \tag{16}$$

Changing back to $\boldsymbol{y}=V\boldsymbol{z}$, that solution becomes $\boldsymbol{y}(t)=Ve^{Bt}\boldsymbol{z}(0)=\boldsymbol{Ve^{Bt}V^{-1}y(0)}$.

$$\textbf{The exponential of}\quad A=VBV^{-1}\quad\textbf{is}\quad e^{At}=Ve^{Bt}V^{-1}. \tag{17}$$

Special case : When V is the eigenvector matrix, B is the eigenvalue matrix Λ.

Here is my point. Equation (17) is true for any invertible matrix V. Choosing the eigenvector matrix of A makes B diagonal; in fact $B=V^{-1}AV=\Lambda$. This is the outstanding choice for V, to produce $B=\Lambda$ when A has n independent eigenvectors. But *any invertible V is now allowed*, and we have a name for B : **similar matrix**.

$$\boxed{\textbf{Every matrix}\ \ B=V^{-1}AV\ \ \textbf{is “similar” to}\ \ A.\ \ \textbf{They have the same eigenvalues.}}$$

I can quickly prove that eigenvalues stay unchanged. **Eigenvectors change to $\boldsymbol{u}=\boldsymbol{V^{-1}x}$** :

$$\text{If}\ \ A\boldsymbol{x}=\lambda\boldsymbol{x}\quad\text{then}\quad V^{-1}A\boldsymbol{x}=\lambda V^{-1}\boldsymbol{x}\quad\text{which is}\quad V^{-1}AV\boldsymbol{u}=B\boldsymbol{u}=\lambda\boldsymbol{u}. \tag{18}$$

By allowing all invertible V, we have a whole family of matrices $B=V^{-1}AV$. All are similar to A, all have the same eigenvalues as A, only the eigenvectors change with V.

In case A cannot be diagonalized, a good choice of V makes B upper triangular. V is not easy to compute, but it greatly simplifies the problem. Example 2 showed how $\boldsymbol{z}(t)$ comes from back substitution in $\boldsymbol{z}'=\boldsymbol{Bz}$. Then $\boldsymbol{y}(t)=V\boldsymbol{z}(t)$ solves $y'=Ay$ without n independent eigenvectors of A.

Fundamental Matrices (Optional Topic)

A linear system $d\boldsymbol{y}/dt = A(t)\boldsymbol{y}$ is completely solved when you have n independent solutions $\boldsymbol{y}_1(t)$ to $\boldsymbol{y}_n(t)$. Put those solutions into the columns of an n by n matrix $M(t)$:

Fundamental matrix $$M(t) = \left[\boldsymbol{y}_1(t) \ldots \boldsymbol{y}_n(t)\right] \quad \text{has} \quad \frac{d\boldsymbol{M}}{d\boldsymbol{t}} = \boldsymbol{AM(t)}. \tag{19}$$

Every column of dM/dt has $d\boldsymbol{y}/dt = A\boldsymbol{y}$. All columns together give $dM/dt = AM$. "Linear independence" means that M is invertible. The determinant of M is not zero. This determinant $W(t)$ is called the "*Wronskian*" of the n solutions in the columns of M:

$$\boxed{W(t) = \text{Wronskian of } \boldsymbol{y}_1(t), \ldots, \boldsymbol{y}_n(t) = \text{Determinant of } M(t).} \tag{20}$$

The beautiful fact is this: **If the Wronskian starts from $\boldsymbol{W \neq 0}$ at time $\boldsymbol{t = 0}$, then $\boldsymbol{W(t) \neq 0}$ for all $\boldsymbol{t}$.** Independence at the start means independence forever. A combination $\boldsymbol{y}(t) = c_1\boldsymbol{y}_1(t) + \cdots + c_n\boldsymbol{y}_n(t)$ can only be zero at time t if it started from $\boldsymbol{y}(0) = 0$. Solutions to $\boldsymbol{y}' = A\boldsymbol{y}$ don't hit zero! So $W(t) = 0$ requires $W(0) = 0$, as in this neat formula discussed in the Chapter 6 Notes (exponentials are never zero).

$$\frac{dW}{dt} = (\text{trace } A(t))W \quad \text{and then} \quad W(t) = e^{\int \text{trace } A(t)\, dt}\, W(0). \tag{21}$$

What are $M(t)$ and $W(t)$ for a second order equation $y'' + B(t)y' + C(t)y = 0$? We know how to convert this to a first order system $\boldsymbol{y}' = A(t)\boldsymbol{y}$. The vector unknown is $\boldsymbol{y} = (y, y')$ and $A(t)$ is a companion matrix containing $-B(t)$ and $-C(t)$. The two independent solutions in the columns of $M(t)$ are $(y_1, y_1{}')$ and $(y_2, y_2{}')$:

$$\textbf{Matrix } M(t) = \begin{bmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{bmatrix} \quad \textbf{Wronskian } W(t) = \det M = y_1 y_2{}' - y_2 y_1{}'. \tag{22}$$

Again $W(t) \neq 0$ is the test for y_1 and y_2 to be independent. The test is passed for all t if $W(0) \neq 0$. In the mysterious formula (21), the trace of $A(t)$ is $-B(t)$.

You will naturally ask: What is this fundamental matrix $M(t)$? Why are we only seeing it now? One answer is that you already know the *growth factor* G from Chapter 1: $M = G(0,t) = \exp\left(\int a(t)dt\right)$. For systems, you also know $M = e^{At}$. That is the perfect answer when A is constant. e^{At} is the best possible $M(t)$ because it starts from $M(0) = I$.

It is often hard to find $M(t)$ when the matrix A depends on t (then nothing is easy). We know that $\boldsymbol{y}' = A(t)\boldsymbol{y}$ has n independent solutions $\boldsymbol{y}(t)$. But in most cases we don't know what those solutions are. The point of fundamental matrices is that the solution $y(t)$ comes directly from $M(t)$, when and if we know M:

$$\boxed{\boldsymbol{y}(t) = M(t)M(0)^{-1}\boldsymbol{y}(0) \quad \text{for any } M(t)} \tag{23}$$

Let me say a little more about constant A and varying $A(t)$, and then stop.

Constant A with n independent eigenvectors in V We know n solutions $y = e^{\lambda t}x$:

$$\text{Put those } y\text{'s into} \quad M(t) = \begin{bmatrix} e^{\lambda_1 t}x_1 & e^{\lambda_2 t}x_2 & \dots & e^{\lambda_n t}x_n \end{bmatrix} = Ve^{\Lambda t}.$$

How does this differ from e^{At}? You can see everything at $t = 0$, when this $M(t)$ is V. If you want the fundamental matrix that equals I at $t=0$, just multiply by $M(0)^{-1} = V^{-1}$:

$$\text{When } A = V\Lambda V^{-1}, \text{ the best fundamental matrix is } M = Ve^{\Lambda t}V^{-1} \text{ which is } e^{At}.$$

Time-varying $A(t)$ with time-varying eigenvectors The equation $y' = A(t)y$ is more difficult. The next page shows how the expected solution formula fails. The chain rule goes wrong. Finding even one solution $y_1(t)$ is a big challenge. The optimistic point is that if we can find $y_1(t)$, then "variation of parameters" will lead us to $y_2 = C(t)y_1$.

Let me focus on a famous equation that has been studied by great mathematicians:

$$\boxed{\textbf{Bessel's equation} \qquad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0.} \tag{24}$$

The solutions are *Bessel functions of order p*. When the order is $p = \frac{1}{2}$, these solutions y_1 and y_2 are quite special (the variable t is usually changed to x).

$$y_1(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad y_2(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad \text{go into} \quad M = \begin{bmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{bmatrix}$$

Those are independent solutions and the Wronskian $W = y_1y_2' - y_2y_1'$ is never zero.

The most important Bessel functions have $p = 0, 1, 2, \dots$ and whole books are written about these functions. They are not simple! The first and most famous Bessel function is $y = J_0(x)$, with order $p = 0$:

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^24^2} - \frac{x^6}{2^24^26^2} + \cdots \qquad \text{resembles a damped cosine.}$$

The second solution Y_0, independent of J_0, blows up at $x = 0$. When you divide Bessel's equation (24) by x^2, so as to start the equation with y'', you see that its coefficients are singular: $1/x$ and $1 - p^2/x^2$ also blow up at $x = 0$: A singular point.

Failure of a Formula

A single equation $dy/dt = a(t)y$ has a neat solution $y = e^{P(t)}y(0)$. We choose $P(t)$ as the integral of $a(t)$. By the chain rule, dy/dt has the desired factor $a(t) = dP/dt$. I am very sorry to say that $y = e^{P(t)}y(0)$ fails for matrices $A(t)$ and systems $y' = A(t)y$.

There is no doubt that the derivative of the integral of time-varying $A(t)$ is $A(t)$. Even for matrices, this part is true:

$$\textbf{Fundamental Theorem of Calculus} \qquad \frac{d}{dt}\int_0^t A(s)\,ds = \frac{dP}{dt} = A(t). \tag{25}$$

When A is a constant matrix, that integral is $P = At$ and its derivative is A. Then the derivative of e^{At} is Ae^{At}. This whole section is built on that true statement. We hope that the same chain rule will give the answer when $A(t)$ is varying and not constant:

$$\text{The derivative of } G = \exp\left(\int_0^t A(s)\,ds\right) \text{ "should be" } A(t)G. \textit{ Not always!} \tag{26}$$

When the matrix $A(t)$ is changing with time, the chain rule in (26) can let us down. This leaves no simple formula for $\boldsymbol{y}(t)$. How can things go wrong?

The difficulty is that e^A times e^B may not be the same as e^{A+B}. Problem 7 gives an example of A and B. Those matrices do not satisfy $AB = BA$ and this destroys the rule for exponents. It is true that $e^Ae^B = e^{A+B}$ when $AB = BA$, but not here.

Let me use those matrices in Problem 7 to construct a two-part example:

$$\boldsymbol{y}' = B\boldsymbol{y} \quad \text{for } t \le 1 \qquad \text{and then} \qquad \boldsymbol{y}' = A\boldsymbol{y} \quad \text{for } t > 1 \tag{27}$$

Our time-varying matrix $A(t)$ jumps from B to A at $t = 1$. The integral of $A(t)$ is $P(t)$:

$$P(t) = \int_0^t A(s)\,ds = Bt \text{ (for } t \le 1) \quad \text{and} \quad A(t-1) + B \text{ (for } t > 1). \tag{28}$$

But the exponential of $P(t)$ does not solve our differential equation (27) at $t = 2$:

$$P(2) = \int_0^2 A(s)\,ds = A + B \quad \text{is correct but} \quad \boldsymbol{y}(2) = e^{A+B}\boldsymbol{y}(0) \quad \text{is } \textbf{wrong}.$$

The correct answer is $\boldsymbol{y}(2) = e^Ae^B\boldsymbol{y}(0)$. First B then A. The solution is $e^{Bt}\boldsymbol{y}(0)$ up to time $t = 1$, when B changes to A. After $t = 1$ the solution is $e^{A(t-1)}e^B\boldsymbol{y}(0)$.

The chain rule in (26) is wrong, because e^Ae^B is different from e^{A+B}.

■ REVIEW OF THE KEY IDEAS ■

1. The exponential of At is $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$
2. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0)$. This is $Ve^{\Lambda t}V^{-1}\boldsymbol{y}(0)$ if V^{-1} exists.
3. That solution is the same as $c_1e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_ne^{\lambda_n t}\boldsymbol{x}_n$ with $\boldsymbol{c} = V^{-1}\boldsymbol{y}(0)$.
4. The solution to $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}$ (constant source) is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) + (e^{At} - I)A^{-1}\boldsymbol{q}$.
5. All similar matrices $B = VAV^{-1}$ (*with any* V) have the same eigenvalues as A.
6. If $A(t)$ is time-varying, easy formulas for the fundamental matrix $M(t)$ will fail.

WORKED EXAMPLE

Show that $y(t) = e^{At}y(0)$ is exactly $c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n$ if $y(0) = Vc$.

Step 1 Write $\boldsymbol{y}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. This is $\begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = Vc$.

Step 2 Starting from an eigenvector $\boldsymbol{x}$, the solution is $\boldsymbol{y} = ce^{\lambda t}\boldsymbol{x}$.

Step 3 Add those n solutions to get $Ve^{\Lambda t}\boldsymbol{c} = Ve^{\Lambda t}V^{-1}\boldsymbol{y}(0) = \boldsymbol{e^{At}y(0)}$.

Here are those steps for a triangular matrix A. Suppose $\boldsymbol{y}(0) = (5, 3)$. First Λ and V:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ has } \lambda_1 = 1 \text{ and } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \lambda_2 = 2 \text{ and } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step 1 $\boldsymbol{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \mathbf{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix} = V\boldsymbol{c}$.

Step 2 The separate solutions $ce^{\lambda t}\boldsymbol{x}$ from eigenvectors are $\mathbf{2e^t}\boldsymbol{x_1}$ and $\mathbf{3e^{2t}}\boldsymbol{x_2}$.

Step 3 The final $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) = Ve^{\Lambda t}V^{-1}\boldsymbol{y}(0)$ is the sum $2e^t\boldsymbol{x}_1 + 3e^{2t}\boldsymbol{x}_2$.

Challenge Find e^{At} for the companion matrices $\begin{bmatrix} 0 & 1 \\ -C & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}$.
Their eigenvectors in $Ve^{\Lambda t}V^{-1}$ are always $(1, \lambda)$.

Problem Set 6.4

1 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, find an eigenvalue and an eigenvector of e^{At} and also of $-e^{-At}$.

2 (a) From the infinite series $e^{At} = I + At + \cdots$ show that its derivative is Ae^{At}.

(b) The series for e^{At} ends quickly if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Find e^{At} and take its derivative (which should agree with Ae^{At}).

3 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ with eigenvectors in $V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, compute $e^{At} = Ve^{\Lambda t}V^{-1}$.

4 Why is $e^{(A+3I)t}$ equal to e^{At} multiplied by e^{3t} ?

5 Why is $e^{A^{-1}}$ *not* the inverse of e^A ? What is the correct inverse of e^A ?

6 Compute $A^n = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^n$. Add the series to find $e^{At} = \begin{bmatrix} e^t & c(e^t - 1) \\ 0 & 1 \end{bmatrix}$.

7 Find e^A and e^B by using Problem 6 for $c = 4$ and $c = -4$. Multiply to show that the matrices $e^A e^B$ and $e^B e^A$ and e^{A+B} are all different.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

8 Multiply the first terms $I + A + \frac{1}{2}A^2$ of e^A by the first terms $I + B + \frac{1}{2}B^2$ of e^B. Do you get the correct first three terms of e^{A+B} ? *Conclusion*: e^{A+B} is not always equal to $(e^A)(e^B)$. The exponent rule only applies when $AB = BA$.

9 Write $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in the form $V\Lambda V^{-1}$. Find e^{At} from $Ve^{\Lambda t}V^{-1}$.

10 Starting from $\boldsymbol{y}(0)$ the solution at time t is $e^{At}\boldsymbol{y}(0)$. Go an additional time t to reach $e^{At}\, e^{At}\boldsymbol{y}(0)$. Conclusion: e^{At} times e^{At} equals _____.

11 Diagonalize A by V and confirm this formula for e^{At} by using $Ve^{\Lambda t}V^{-1}$:

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{2t} & 4(e^{3t} - e^{2t}) \\ 0 & e^{3t} \end{bmatrix} \quad \text{At } t = 0 \text{ this matrix is \_\_\_\_\_.}$$

12 (a) Find A^2 and A^3 and A^n for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with repeated eigenvalues $\lambda = 1, 1$.

(b) Add the infinite series to find e^{At}. (The $Ve^{\Lambda t}V^{-1}$ method won't work.)

13 (a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$ as a combination of eigenvectors of this matrix A:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y} \qquad \text{with } \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(b) Write the equations as $y_1' = y_2$ and $y_2' = y_1$. Find an equation for y_1'' with y_2 eliminated. Solve for $y_1(t)$ and compare with part (a).

14 Similar matrices A and $B = V^{-1}AV$ have the *same eigenvalues* if V is invertible.

Second proof $\qquad \det(V^{-1}AV - \lambda I) = (\det V^{-1})\,(\det(A - \lambda I))\,(\det V)$.

Why is this equation true ? Then both sides are zero when $\det(A - \lambda I) = 0$.

15 If B is *similar* to A, the growth rates for $\boldsymbol{z}' = B\boldsymbol{z}$ are the same as for $\boldsymbol{y}' = A\boldsymbol{y}$. That equation converts to the equation for $\boldsymbol{z}$ when $B = V^{-1}AV$ and $\boldsymbol{z} =$ _____.

16 If $A\boldsymbol{x} = \lambda\boldsymbol{x} \neq \boldsymbol{0}$, what is an eigenvalue and eigenvector of $(e^{At} - I)A^{-1}$?

17 The matrix $B = \left[\begin{smallmatrix} 0 & -4 \\ 0 & 0 \end{smallmatrix}\right]$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

18 Starting from $\boldsymbol{y}(0) = \mathbf{0}$, solve $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}$ as a combination of the eigenvectors. Suppose the source is $\boldsymbol{q} = q_1\boldsymbol{x}_1 + \cdots + q_n\boldsymbol{x}_n$. Solve for one eigenvector at a time, using the solution $y(t) = (e^{at} - 1)q/a$ to the scalar equation $y' = ay + q$.

Then $\boldsymbol{y}(t) = (e^{At} - I)A^{-1}\boldsymbol{q}$ is a combination of eigenvectors when all $\lambda_i \neq 0$.

19 Solve for $\boldsymbol{y}(t)$ as a combination of the eigenvectors $\boldsymbol{x}_1 = (1, 0)$ and $\boldsymbol{x}_2 = (1, 1)$:

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q} \qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{with} \quad \begin{matrix} y_1(0) = 0 \\ y_2(0) = 0 \end{matrix}$$

20 Solve $\boldsymbol{y}' = A\boldsymbol{y} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$ in three steps. First find the λ's and $\boldsymbol{x}$'s.

(1) Write $\boldsymbol{y}(0) = (3, 1)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$

(2) Multiply c_1 and c_2 by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

(3) Add the solutions $c_1e^{\lambda_1 t}\boldsymbol{x}_1 + c_2e^{\lambda_2 t}\boldsymbol{x}_2$.

21 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}\boldsymbol{y}(0)$ solves $d\boldsymbol{y}/dt = A\boldsymbol{y}$.

Problems 22-25 are about time-varying systems $\boldsymbol{y}' = A(t)\boldsymbol{y}$. Success then failure.

22 Suppose the constant matrix C has $C\boldsymbol{x} = \lambda\boldsymbol{x}$, and $p(t)$ is the integral of $a(t)$. Substitute $\boldsymbol{y} = e^{\lambda p(t)}\boldsymbol{x}$ to show that $d\boldsymbol{y}/dt = a(t)C\boldsymbol{y}$. Eigenvectors still solve this special time-varying system: constant matrix C multiplied by the scalar $a(t)$.

23 Continuing Problem 22, show from the series for $M(t) = e^{p(t)C}$ that $dM/dt = a(t)CM$. Then M is the fundamental matrix for the special system $\boldsymbol{y}' = a(t)C\boldsymbol{y}$. If $a(t) = 1$ then its integral is $p(t) = t$ and we recover $M = e^{Ct}$.

24 The integral of $A = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix}$ is $P = \begin{bmatrix} t & t^2 \\ 0 & 0 \end{bmatrix}$. The exponential of P is $e^P = \begin{bmatrix} e^t & t(e^t - 1) \\ 0 & 1 \end{bmatrix}$. From the chain rule we might hope that the derivative of $e^{P(t)}$ is $P'e^{P(t)} = Ae^{P(t)}$. Compute the derivative of $e^{P(t)}$ and compare with the wrong answer $Ae^{P(t)}$. (One reason this feels wrong: Writing the chain rule as $(d/dt)e^P = e^P dP/dt$ would give $e^P A$ instead of $A\,e^P$. That is wrong too.)

25 Find the solution to $\boldsymbol{y}' = A(t)\boldsymbol{y}$ in Problem 24 by solving for y_2 and then y_1:

$$\text{Solve} \quad \begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{starting from} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}.$$

Certainly $y_2(t)$ stays at $y_2(0)$. Find $y_1(t)$ by "undetermined coefficients" A, B, C: $y_1' = y_1 + 2ty_2(0)$ is solved by $y_1 = y_p + y_n = At + B + Ce^t$.

Choose A, B, C to satisfy the equation and match the initial condition $y_1(0)$.

The wrong answer in Problem 24 included the incorrect factor te^t in $e^{P(t)}$.

6.5 Second Order Systems and Symmetric Matrices

This section solves a differential equation that is crucial in engineering and physics:

Oscillation equation $$\frac{d^2y}{dt^2} + Sy = \mathbf{0}. \tag{1}$$

Since this is second order in time, we need two vectors as initial conditions at $t = 0$:

Starting position and starting velocity $\boldsymbol{y}(0)$ and $\boldsymbol{v}(0) = \frac{d\boldsymbol{y}}{dt}(0)$ are given.

If $\boldsymbol{y}$ has n components, we have n second order equations and $2n$ initial conditions. This is the right number to find $\boldsymbol{y}(t)$. Allow me to say this early: The oscillation equation (1) is the most basic form of the **Fundamental Equation of Engineering**.

The more general equation includes a damping term $B\,d\boldsymbol{y}/dt$ and a forcing term $\boldsymbol{F}\cos\Omega t$. Those give *damped forced oscillations*, where equation (1) is about "free" oscillations. For one mass and one equation, Chapter 2 took that step to damping and forcing. Now we have n masses and n equations and three n by n matrices M, B, K.

Fundamental Equation $$M\frac{d^2\boldsymbol{y}}{dt^2} + B\frac{d\boldsymbol{y}}{dt} + K\boldsymbol{y} = \boldsymbol{F}\cos\Omega t. \tag{2}$$

The *mass matrix* is M, the *stiffness matrix* is K. Those are the pieces we always see and always need. When the damping matrix B and the forcing vector F are removed, that takes us to the heart of the fundamental equation: *free oscillations*.

Mass and stiffness matrices $$\boldsymbol{M}\boldsymbol{y}'' + \boldsymbol{K}\boldsymbol{y} = \mathbf{0}\ . \tag{3}$$

The matrix S in equation (1) is $M^{-1}K$. Its symmetric form is $M^{-1/2}KM^{-1/2}$. In many applications the mass matrix M is diagonal.

If we look for eigenvector solutions $\boldsymbol{y} = e^{i\omega t}\boldsymbol{x}$, the differential equation produces $K\boldsymbol{x} = \omega^2 M\boldsymbol{x}$. This "generalized" eigenvalue problem has an extra matrix M, but it is not more difficult than $S\boldsymbol{x} = \lambda\boldsymbol{x}$. The MATLAB command is $\mathsf{eig}(K, M)$. An essential point is that the eigenvalues are still real and positive, when both M and K are *positive definite*. Positive eigenvalues and positive energy are the key to Chapter 7.

When the forcing term is a constant $\boldsymbol{F}$, the damping brings us to a steady state $\boldsymbol{y}_\infty$. Then the time dependence is gone; those derivatives $d\boldsymbol{y}/dt$ and $d^2\boldsymbol{y}/dt^2$ are zero. The external force $\boldsymbol{F}$ is balanced by the internal force $K\boldsymbol{y}_\infty$. The system is in equilibrium:

Steady state equation $$K\boldsymbol{y}_\infty = \boldsymbol{F} = \text{constant}. \tag{4}$$

The central problem of computational mechanics is to create the stiffness matrix K and force vector $\boldsymbol{F}$. Then the computer solves $M\boldsymbol{y}'' + K\boldsymbol{y} = 0$ and $K\boldsymbol{y}_\infty = \boldsymbol{F}$. For large problems, the *finite element method* is now the favorite way to take those steps. This is a sensational achievement by the collective efforts of thousands of engineers. [1]

Solution by Eigenvalues

We want to solve $\boldsymbol{y}'' + S\boldsymbol{y} = \mathbf{0}$. This is a linear system with constant coefficients. Our solution method will be the same as for $\boldsymbol{y}' = A\boldsymbol{y}$. We use the eigenvectors and eigenvalues of S to find special solutions, and we combine those to find the complete solution.

Each eigenvector of S leads to two special solutions to $\boldsymbol{y}'' + S\boldsymbol{y} = \mathbf{0}$:

Two solutions If $S\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{y}(t) = (\cos\omega t)\boldsymbol{x}$ and $\boldsymbol{y}(t) = (\sin\omega t)\boldsymbol{x}$. (5)

The "frequency" ω is $\sqrt{\lambda}$. Substitute $\boldsymbol{y} = (\cos\omega t)\boldsymbol{x}$ into the differential equation:

$$\boldsymbol{\lambda = \omega^2} \textbf{ and } \boldsymbol{S\boldsymbol{x} = \omega^2\boldsymbol{x}} \qquad \boldsymbol{y}'' + S\boldsymbol{y} = -\omega^2(\cos\omega t)\boldsymbol{x} + S(\cos\omega t)\boldsymbol{x} = \mathbf{0}. \tag{6}$$

When $\cos\omega t$ is factored out, we see the requirement on $\boldsymbol{x}$. It must be an eigenvector of S. We expect n eigenvectors (*normal modes of oscillation*). The eigenvectors don't interact. That is their beauty, each one goes its own way. And each eigenvector gives us two solutions from $(\cos\omega t)\boldsymbol{x}$ and $(\sin\omega t)\boldsymbol{x}$, so we have $2n$ special solutions.

A combination of those $2n$ solutions will match the $2n$ initial conditions (n positions and n velocities at $t = 0$). This determines the $2n$ constants A_i and B_i in the complete solution to $\boldsymbol{y}'' + S\boldsymbol{y} = 0$:

Complete solution
$$\boldsymbol{y(t) = \sum_{i=1}^{n} (A_i \cos\sqrt{\lambda_i}\, t + B_i \sin\sqrt{\lambda_i}\, t)\, x_i}. \tag{7}$$

Since $\sin 0 = 0$, it is the A_i that match the vector $\boldsymbol{y}(0)$ of initial positions. It is the B_i that match the vector $\boldsymbol{v}(0) = \boldsymbol{y}'(0)$ of initial velocities.

Example 1 Two masses are connected by three identical springs in Figure 6.3. Find the stiffness matrix S and its positive eigenvalues $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$. If the system starts from rest, with the top spring unstretched ($y_1(0) = 0$) and the lower mass moved down ($y_2(0) = 2$), find the positions $\boldsymbol{y} = (y_1, y_2)$ at all later times:

$$m\,\frac{d^2\boldsymbol{y}}{dt^2} + S\boldsymbol{y} = \mathbf{0} \text{ with } \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{y}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\boldsymbol{y}(t)$ has eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ times cosine and sine. Four conditions for A_1, A_2, B_1, B_2.

Solution Construct the matrix S that expresses Newton's Law $m\boldsymbol{y}'' + S\boldsymbol{y} = \mathbf{0}$. The acceleration is $\boldsymbol{y}''$, and the force is $-S\boldsymbol{y}$.

[1] The finite element method is a key part of my textbook on *Computational Science and Engineering*. The foundations of the method and the reasons for its success are developed in *An Analysis of the Finite Element Method* (also published by Wellesley-Cambridge Press).

What force F is acting on the upper mass ? The stretched top spring is pulling that mass up. The force is proportional to the stretch y_1. *This is Hooke's Law* $F = -ky_1$.

The middle spring is connected to both masses. It is stretched a distance $y_2 - y_1$. (No stretching if $y_2 = y_1$, the spring would just be shifted up or down.) **The difference $\boldsymbol{y_2 - y_1}$ produces spring forces $\boldsymbol{k(y_2 - y_1)}$,** pulling mass 1 down and mass 2 up.

The bottom spring with fixed end is stretched by $0 - y_2$, so the force is $-ky_2$.

$$\begin{array}{ll} \boldsymbol{F = ma} \textbf{ at the upper mass} & -ky_1 + k(y_2 - y_1) = my_1'' \\ \boldsymbol{F = ma} \textbf{ at the lower mass} & -k(y_2 - y_1) - ky_2 = my_2'' \end{array}$$

These equations $-S\boldsymbol{y} = m\boldsymbol{y}''$ or $m\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ have a symmetric matrix S. Take $k = m = 1$:

$$\boldsymbol{y}'' + S\boldsymbol{y} = \frac{d^2}{dt^2}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8}$$

The modeling part is complete, now for the solution part. The eigenvalues of that matrix are $\lambda_1 = 1$ and $\lambda_2 = 3$. The trace is $1 + 3 = 4$, the determinant is $(1)(3) = 3$. The first eigenvector $\boldsymbol{x}_1 = (1, 1)$ has the springs moving in the same direction in Figure 6.3. The second eigenvector $\boldsymbol{x}_2 = (1, -1)$ has the springs moving oppositely, with higher frequency because $\omega_2^2 = \lambda_2 = 3$.

Formula (7) for $\boldsymbol{y}(t)$ becomes a combination of eigenvectors times cosines:

$$\boxed{\textbf{Solution} \quad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = A_1 \left(\cos\sqrt{1}\,t\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A_2 \left(\cos\sqrt{3}\,t\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.} \tag{9}$$

I removed $B_1 \sin t$ and $B_2 \sin\sqrt{3}t$ because the example started from rest (zero velocity). **At time $\boldsymbol{t = 0}$, cosines give position $\boldsymbol{y(0)}$ and sines give velocity $\boldsymbol{v(0)}$.**

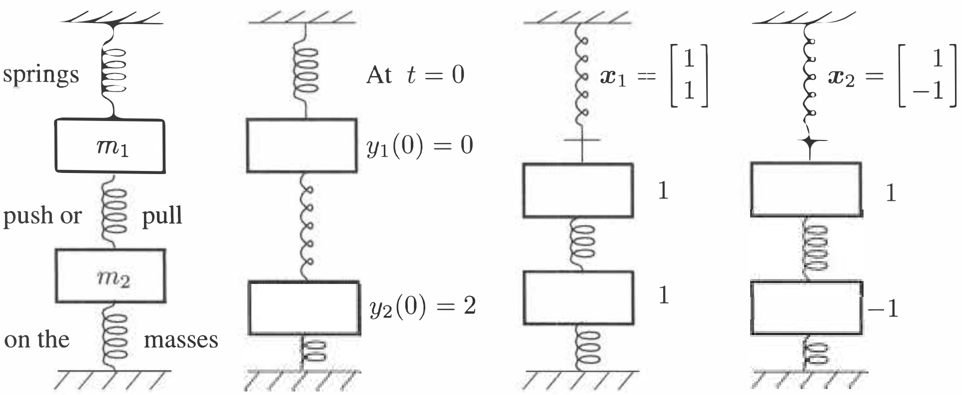


Figure 6.3: The masses oscillate up and down, $\boldsymbol{y}(t)$ combines $(\cos t)\,\boldsymbol{x}_1$ and $(\cos\sqrt{3}t)\,\boldsymbol{x}_2$.

The final step is to find A_1 and A_2 from the initial position $\boldsymbol{y}(0) = (0, 2)$:

$$\textbf{Initial condition} \quad A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \text{gives} \quad A_1 = 1 \text{ and } A_2 = -1.$$

Final answer: $\boldsymbol{y_1(t) = (\cos t - \cos\sqrt{3}t)}$ **and** $\boldsymbol{y_2(t) = (\cos t + \cos\sqrt{3}t)}$. The two masses oscillate forever. The solution part was easier than the modeling part. This is very typical.

Symmetric Matrices

Example 1 led to a symmetric matrix S. *Many many examples* lead to symmetric matrices.

Perhaps this is an extension of Newton's third law, that every action produces an equal and opposite reaction. We really must focus on the special properties of symmetric matrices, because those properties are so useful and the matrices appear so often.

Eigenvalues and eigenvectors—this is the information we need from the matrix. For every class of matrices, we ask about λ and $\boldsymbol{x}$. Are the eigenvalues *real*? Are they *positive*, so we can take square roots in $\lambda = \omega^2$? Are there n *independent* eigenvectors? Are the $\boldsymbol{x}$'s *orthogonal*? The example with $\lambda_1 = 1$ and $\lambda_2 = 3$ was perfect in all respects:

$$S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ is } \textbf{symmetric positive definite} \qquad \begin{array}{l} \textbf{Positive real } \boldsymbol{\lambda = 1 \text{ and } 3} \\ \textbf{Orthogonal } \boldsymbol{x = (1,1), (1,-1)} \end{array}$$

$\boxed{\textbf{Real eigenvalues}}$ **All the eigenvalues of a real symmetric matrix are real.**

Proof Suppose that $S\boldsymbol{x} = \lambda\boldsymbol{x}$. Until we know otherwise, λ might be a complex number and $\boldsymbol{x}$ might be a complex vector. If that did happen, the rules for complex conjugates would give $\overline{S\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$. The key idea is to look at $\overline{\boldsymbol{x}}^{\mathrm{T}} S\boldsymbol{x}$:

$$S \textit{ is symmetric and real} \qquad \overline{\boldsymbol{x}}^{\mathrm{T}} S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}} S^{\mathrm{T}} \boldsymbol{x} = (\overline{S\boldsymbol{x}})^{\mathrm{T}} \boldsymbol{x}. \tag{10}$$

The left side is $\overline{\boldsymbol{x}}^{\mathrm{T}} \lambda\boldsymbol{x}$. The right side is $\overline{\boldsymbol{x}}^{\mathrm{T}} \overline{\lambda}\boldsymbol{x}$. One side has λ, the other side has $\overline{\lambda}$. They multiply $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{x}$ which is not zero—it is the squared length $|x_1|^2 + \cdots + |x_n|^2$. *Therefore* $\lambda = \overline{\lambda}$.

When $\lambda = a + ib$ equals $\overline{\lambda} = a - ib$, we know that $b = 0$ and λ *is real*. Then the vector $\boldsymbol{x}$ in the nullspace of the real matrix $S - \lambda I$ can also be kept real.

$\boxed{\textbf{Orthogonal eigenvectors}}$ If $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$ and $S\boldsymbol{y} = \lambda_2\boldsymbol{y}$ and $\lambda_1 \neq \lambda_2$. **Then** $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

Proof Take the dot product of the first equation with $\boldsymbol{y}$ and the second equation with $\boldsymbol{x}$:

$$\text{Use } S^{\mathrm{T}} = S \qquad (S\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}} S\boldsymbol{y} \text{ is } \lambda_1 \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \lambda_2 \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}. \tag{11}$$

Since $\lambda_1 \neq \lambda_2$, this proves that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$. The eigenvectors are perpendicular.

Remember: The main goal of eigenvectors is to *diagonalize a matrix*, $A = V\Lambda V^{-1}$. Here the matrix is S and its eigenvectors are orthogonal. We can certainly make them unit vectors, so $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1$ and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$. The matrix V with the eigenvectors in its columns has become an **orthogonal matrix**: $V^{\mathrm{T}}V = I$. The right letter for this orthogonal matrix V is Q. **The eigenvector matrix V in $V\Lambda V^{-1}$ can be orthogonal**: $\boldsymbol{Q^{\mathrm{T}}Q = I}$.

Spectral theorem / Principal axis theorem $\quad \boldsymbol{S = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}}$ (12)

In algebra, the eigenvectors are orthogonal. In geometry, the principal axes of an ellipse are orthogonal. If the ellipse equation is $2x^2 - 2xy + 2y^2 = 1$, this corresponds to the example matrix S. Its principal axes $(1, 1)$ and $(1, -1)$ (eigenvectors) are at $+45°$ and $-45°$ from the x axis. The ellipse is turned by $+45°$ from horizontal and vertical axes.

With repeated eigenvalues, $S = Q\Lambda Q^{\mathrm{T}}$ is still correct. Every symmetric S has a full set of n independent eigenvectors (Chapter 6 Notes) even if eigenvalues are repeated.

To summarize, $Q\Lambda Q^{\mathrm{T}}$ is a perfect description of symmetric matrices S. Every S has those factors and every matrix of this form is sure to be symmetric: $(Q\Lambda Q^{\mathrm{T}})^{\mathrm{T}}$ equals $Q^{\mathrm{TT}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}}$ which is $Q\Lambda Q^{\mathrm{T}}$. If we multiply columns of Q times rows of ΛQ^{T}, we see S in a new way (a sum of rank one matrices):

Matrices $\boldsymbol{\lambda x x^{\mathrm{T}}}$
with rank 1
add to S

$$S = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \boldsymbol{x}_1^{\mathrm{T}} \\ \vdots \\ \lambda_n \boldsymbol{x}_n^{\mathrm{T}} \end{bmatrix} = \lambda_1 \boldsymbol{x}_1 \boldsymbol{x}_1^{\mathrm{T}} + \cdots + \lambda_n \boldsymbol{x}_n \boldsymbol{x}_n^{\mathrm{T}}. \tag{13}$$

This is the great factorization $S = Q\Lambda Q^{\mathrm{T}}$, in terms of eigenvalues and eigenvectors.

Example 2 The eigenvectors $(1, 1)$ and $(-1, 1)$ with $\lambda = 16$ and 4 give *unit* eigenvectors $\boldsymbol{x}_1 = (1, 1)/\sqrt{2}$ and $\boldsymbol{x}_2 = (-1, 1)/\sqrt{2}$:

$$S = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} \qquad Q\Lambda Q^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & \\ & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Those eigenvectors still point in the $45°$ direction and the $135°$ direction ($90°$ apart). They are the same as in Example 1, because this new S is 6 times the original S, minus $2I$. Then the new eigenvalues 16 and 4 of S must be 6 times the original 3 and 1, minus 2.

The eigenvectors in Q are the principal axes of an ellipse $10x^2 - 12xy + 10y^2 = 1$.

If I change -6 and -6 off the diagonal to $6i$ and $-6i$, the determinant is still 64. The trace is still 20 and the eigenvalues are still 16 and 4 (**real**!). For complex matrices, we want a symmetric real part and an *antisymmetric* imaginary part. Let me explain why.

Complex Matrices

Important: The squared length is $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{x}$ and not $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ when $\boldsymbol{x}$ has complex components. We want $|x_1|^2 + \cdots + |x_n|^2$ because this is a positive number or zero. We don't want $x_1^2 + \cdots + x_n^2$ because that could be any complex number, and we are looking for $||\boldsymbol{x}||^2$ = length squared ≥ 0. When a component of $\boldsymbol{x}$ is $a + bi$, we want $a^2 + b^2$ and not $(a + bi)^2$. *The length squared of $\boldsymbol{x} = (1, i)$ is $||\boldsymbol{x}||^2 = 1^2 + 1^2 = 2$ and not $1^2 + i^2 = 0$.*

This changes all inner products (dot products) from $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ to $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y}$. Complex vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are perpendicular when $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y} = 0$. This complex inner product forces us to replace the usual transpose by the **conjugate transpose** $(\overline{\boldsymbol{A}})^{\mathbf{T}} = \boldsymbol{A}^*$, when A is complex:

$$\boldsymbol{A}^*_{ij} \text{ is } \overline{\boldsymbol{A}_{ji}} \qquad \text{Then } A\boldsymbol{x} \cdot y = (\overline{A\boldsymbol{x}})^{\mathrm{T}}\boldsymbol{y} = \overline{\boldsymbol{x}}^{\mathrm{T}}\,\overline{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x} \cdot A^*\boldsymbol{y}. \tag{14}$$

MATLAB automatically takes the conjugate transpose to give A^*, when you type $\boldsymbol{x}'$ or A'.

To keep the row space of A perpendicular to the nullspace, we must use $\boldsymbol{C}(A^*)$ for the row space. This is the column space of A^*, not just the column space of A^{T}. Replace every i by $-i$. And an important name: the complex version of a *symmetric matrix* $A^{\mathrm{T}} = A$ is a "*Hermitian matrix*" $\boldsymbol{A}^* = \boldsymbol{A}$.

Hermitian matrix $\boldsymbol{A}_{ij} = \overline{\boldsymbol{A}_{ji}}$ Then $A\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x} \cdot A^*\boldsymbol{y}$ becomes $A\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x} \cdot A\boldsymbol{y}$.

Example 3 This 2 by 2 complex matrix is Hermitian (notice i and $-i$):

$$A = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix} = A^*$$

The determinant is 8 (real). The trace is 6 (the main diagonal of a Hermitian matrix is real). The eigenvalues of this matrix are 2 and 4 (*both real*!).

Hermitian matrices $\boldsymbol{A} = \boldsymbol{A}^*$ have real eigenvalues and perpendicular eigenvectors.

The eigenvectors of A are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. They are perpendicular: $\boldsymbol{x}_1{}^*\boldsymbol{x}_2 = 1^2 + (-i)^2 = 0$. Divide by $\sqrt{2}$ to make them unit vectors. Then they are the columns of a complex orthogonal matrix Q. The right meaning of "complex orthogonal" is $Q^* = Q^{-1}$, and the right name when Q is complex is *unitary*:

Unitary matrix $\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{I}$ The columns of Q are perpendicular unit vectors.

The great factorization $A = Q\Lambda Q^{\mathrm{T}}$ of real symmetric matrices becomes $\boldsymbol{A} = \boldsymbol{Q\Lambda Q}^*$.

Orthogonal Matrices and Unitary Matrices

We have seen the big theorem: If S is symmetric or Hermitian, its eigenvector matrix is orthogonal or unitary. The real case is $S = Q\Lambda Q^{\mathrm{T}} = S^{\mathrm{T}}$ and the complex case is $S = Q\Lambda Q^* = S^*$. The eigenvalues in Λ are real.

What if our matrix is *anti*-symmetric or *anti*-Hermitian? Then $A^{\mathrm{T}} = -A$ or $\boldsymbol{A}^* = -\boldsymbol{A}$. The matrix A could even be i times S. (In that case A^* will be $-i$ times S^* which is exactly $-iS = -A$.) Multiplying by i changes Hermitian to *anti*-Hermitian. The real eigenvalues λ of S change to the imaginary eigenvalues $i\lambda$ of A. The eigenvectors do *not* change: still orthogonal, still going into Q.

Anti-Hermitian matrices have imaginary eigenvalues and orthogonal eigenvectors.

Our standard examples are $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A^{\mathrm{T}}$ and $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -A^*$. $\boldsymbol{\lambda} = \pm \boldsymbol{i}$

Finally, what if our matrix is *orthogonal* or *unitary*? Then $Q^{\mathrm{T}}Q = I$ or $Q^*Q = I$. The eigenvalues of Q are **complex numbers $\boldsymbol{\lambda = e^{i\theta}}$ on the unit circle.**

$$\textbf{If } \boldsymbol{Q^*Q = I} \textbf{ then all eigenvalues of } \boldsymbol{Q} \textbf{ have magnitude } \boldsymbol{|\lambda| = 1}.$$

The proof starts with $Q\boldsymbol{x} = \lambda\boldsymbol{x}$. The conjugate transpose is $\boldsymbol{x}^*Q^* = \overline{\lambda}\boldsymbol{x}^*$. Multiply the left hand sides using $Q^*Q = I$, and multiply the right hand sides using $\overline{\lambda}\lambda = |\lambda|^2$:

$$\boldsymbol{x}^*Q^*Q\boldsymbol{x} = \overline{\lambda}\boldsymbol{x}^*\lambda\boldsymbol{x} \qquad \text{is the same as} \qquad \boldsymbol{x}^*\boldsymbol{x} = |\lambda|^2\boldsymbol{x}^*\boldsymbol{x}. \qquad \text{Then } |\lambda|^2 = 1 \text{ and } |\lambda| = 1.$$

The eigenvectors of Q, like the eigenvectors of S and A, can be chosen orthogonal. *These are the essential facts about the best matrices.* The eigenvalues of S and A and Q are on the *real axis*, the *imaginary axis*, and the *unit circle* in the complex plane.

In the eigenvalue-eigenvector world, a triangular matrix is not really one of the best. Its eigenvalues are easy (on the main diagonal). But its eigenvectors are not orthogonal. It may even fail to be diagonalizable. Matrices without n eigenvectors are the worst.

Symmetric and Orthogonal

At the end of Chapter 4, we looked at symmetric matrices that are also orthogonal: $\boldsymbol{A^{\mathrm{T}} = A}$ **and** $A^{\mathrm{T}} = \boldsymbol{A^{-1}}$. Every diagonal matrix D of 1's and -1's has both properties. Then every $A = QDQ^{\mathrm{T}}$ also has both properties. Symmetry is clear, and a product of orthogonal matrices Q and D and Q^{T} is sure to stay orthogonal.

The question we could not answer was: *Does QDQ^{T} give all possible examples*? The answer is yes, and now we can see why A has this form—based on eigenvalues.

When A is symmetric, its eigenvalues are real. When A is orthogonal, its eigenvalues have $|\lambda| = 1$. The only possibilities for both are $\boldsymbol{\lambda = 1}$ **and** $\boldsymbol{\lambda = -1}$. The eigenvalue matrix $\Lambda = D$ is a diagonal matrix of 1's and -1's. Then the great fact about symmetric matrices (the Spectral Theorem) guarantees that A has the form $Q\Lambda Q^{\mathrm{T}}$ which is QDQ^{T}.

■ REVIEW OF THE KEY IDEAS ■

1. A real symmetric matrix S has *real eigenvalues* and *perpendicular eigenvectors*.

2. Diagonalization $S = V\Lambda V^{-1}$ becomes $S = Q\Lambda Q^{\mathrm{T}}$ with an orthogonal matrix Q.

3. A complex matrix is *Hermitian* if $\overline{S}^{\mathrm{T}} = S$ (often written $S^* = S$): *real* λ's.

4. Every Hermitian matrix is $S = Q\Lambda\overline{Q}^{\mathrm{T}} = Q\Lambda Q^*$. Dot products are $\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}^*\boldsymbol{y}$.

5. All three matrices S and $A = iS = -A^*$ and Q have orthogonal eigenvectors.

6. Symmetric matrices in $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ and $M\boldsymbol{y}'' + K\boldsymbol{y} = \boldsymbol{0}$ give oscillation.

Problem Set 6.5

Problems 1–14 are about eigenvalues. Then come differential equations.

1 Which of A, B, C have two real λ's ? Which have two independent eigenvectors ?

$$A = \begin{bmatrix} 7 & -11 \\ -11 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 7 & -11 \\ 11 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 7 & -11 \\ 0 & 7 \end{bmatrix}$$

2 Show that A has real eigenvalues if $b \geq 0$ and nonreal eigenvalues if $b < 0$:

$$A = \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}.$$

3 Find the eigenvalues and the unit eigenvectors of the symmetric matrices

$$\text{(a) } S = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{(b) } S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

4 Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ?

5 Show that this A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable. Its eigenvalues are 0 and 0.}$$

$A^{\mathrm{T}} = A$ is not so special for complex matrices. *The good property is* $\overline{A}^{\mathrm{T}} = A$.

6 Find *all* orthogonal matrices from all $\boldsymbol{x}_1, \boldsymbol{x}_2$ to diagonalize $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

7 (a) Find a symmetric matrix $S = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know that S must have a negative pivot?

(c) How do you know that S can't have two negative eigenvalues?

8 If $A^2 = 0$ then the eigenvalues of A must be _____. Give an example with $A \neq 0$. But if A is symmetric, diagonalize it to prove that the matrix is $A = 0$.

9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue. (If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then also $A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$.) Prove that every real 3 by 3 matrix has at least one real eigenvalue.

10 Here is a quick "proof" that the eigenvalues of *all* real matrices are real:

$$\textbf{False proof}\quad A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{gives} \quad \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \quad \text{so} \quad \lambda = \frac{\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[\,0\ \ -1;\ \ 1\ \ 0\,]$ with $\lambda = i$ and $\boldsymbol{x} = (i, 1)$.

11 Write A and B in the form $\lambda_1\boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \lambda_2\boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1).$$

12 What number b in $\left[\begin{smallmatrix} 2 & b \\ 1 & 0 \end{smallmatrix}\right]$ makes $A = Q\Lambda Q^{\mathrm{T}}$ possible? What number makes $A = V\Lambda V^{-1}$ impossible? What number makes A^{-1} impossible?

13 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [\ ?\ ]$$

What is the dot product of the two unit eigenvectors ? A small angle !

14 (Recommended) This matrix M is skew-symmetric and also orthogonal. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. They can only be i or $-i$. Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

15 The complete solution to equation (8) for two oscillating springs (Figure 6.3) is

$$\boldsymbol{y}(t) = (A_1 \cos t + B_1 \sin t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (A_2 \cos \sqrt{3}t + B_2 \sin \sqrt{3}t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the numbers A_1, A_2, B_1, B_2 if $\boldsymbol{y}(0) = (3, 5)$ and $\boldsymbol{y}'(0) = (2, 0)$.

16 If the springs in Figure 6.3 have different constants k_1, k_2, k_3 then $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is

$$\begin{array}{ll} \text{Upper mass} & y_1'' + k_1 y_1 - k_2(y_2 - y_1) = 0 \\ \text{Lower mass} & y_2'' + k_2(y_2 - y_1) + k_3 y_2 = 0 \end{array} \qquad S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

For $k_1 = 1, k_2 = 4, k_3 = 1$ find the eigenvalues $\lambda = \omega^2$ of S and the complete sine/cosine solution $\boldsymbol{y}(t)$ in equation (7).

17 Suppose the third spring is removed ($k_3 = 0$ and nothing is below mass 2). With $k_1 = 3, k_2 = 2$ in Problem 16, find S and its real eigenvalues and orthogonal eigenvectors. What is the sine/cosine solution $\boldsymbol{y}(t)$ if $\boldsymbol{y}(0) = (1, 2)$ gives the cosines and $\boldsymbol{y}'(0) = (2, -1)$ gives the sines ?

18 Suppose the top spring is also removed ($k_1 = 0$ and also $k_3 = 0$). S is singular ! Find its eigenvalues and eigenvectors. If $\boldsymbol{y}(0) = (1, -1)$ and $\boldsymbol{y}' = (0, 0)$ find $\boldsymbol{y}(t)$. If $\boldsymbol{y}(0)$ changes from $(1, -1)$ to $(1, 1)$ what is $\boldsymbol{y}(t)$?

19 The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$). Energy is conserved.

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \boldsymbol{y} \quad \text{or} \quad \begin{aligned} y_1' &= cy_2 - by_3 \\ y_2' &= ay_3 - cy_1 \\ y_3' &= by_1 - ay_2. \end{aligned}$$

The derivative of $\|\boldsymbol{y}(t)\|^2 = y_1^2 + y_2^2 + y_3^2$ is $2y_1y_1' + 2y_2y_2' + 2y_3y_3'$. Substitute y_1', y_2', y_3' to get *zero*. The energy $\|\boldsymbol{y}(t)\|^2$ stays equal to $\|\boldsymbol{y}(0)\|^2$.

20 *When $A = -A^{\mathrm{T}}$ is skew-symmetric, e^{At}* is ***orthogonal***. Prove $(e^{At})^{\mathrm{T}} = e^{-At}$ from the series $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$.

21 The mass matrix M can have masses $m_1 = 1$ and $m_2 = 2$. Show that the eigenvalues for $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ are $\lambda = 2 \pm \sqrt{2}$, starting from $\det(K - \lambda M) = 0$:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \quad \text{are positive definite.}$$

Find the two eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Show that $\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 \neq 0$ but $\boldsymbol{x}_1^{\mathrm{T}}M\boldsymbol{x}_2 = 0$.

22 What difference equation would you use to solve $\boldsymbol{y}'' = -S\boldsymbol{y}$?

23 The second order equation $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ reduces to a first order system $\boldsymbol{y}_1{}' = \boldsymbol{y}_2$ and $\boldsymbol{y}_2{}' = -S\boldsymbol{y}_1$. If $S\boldsymbol{x} = \omega^2\boldsymbol{x}$ show that the companion matrix $A = [0\ I\ ;\ -S\ 0]$ has eigenvalues $i\omega$ and $-i\omega$ with eigenvectors $(\boldsymbol{x}, i\omega\boldsymbol{x})$ and $(\boldsymbol{x}, -i\omega\boldsymbol{x})$.

24 Find the eigenvalues λ and eigenfunctions $y(x)$ for the differential equation $y'' = \lambda y$ with $y(0) = y(\pi) = 0$. There are infinitely many !

Table of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. Here are the special properties of the eigenvalues λ_i and the eigenvectors $\boldsymbol{x}_i$.

Symmetric: $S^{\mathrm{T}} = S$	real λ's	orthogonal $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Complex Hermitian: $\overline{S}^{\mathrm{T}} = S$	real λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Positive Definite: $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$	all $\lambda > 0$	orthogonal since $S^{\mathrm{T}} = S$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $\boldsymbol{x} > 0$
Similar: $B = V^{-1}AV$	$\lambda(B) = \lambda(A)$	$\boldsymbol{x}(B) = V^{-1}\boldsymbol{x}(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation: $\cos\theta, \sin\theta$	$e^{i\theta}$ and $e^{-i\theta}$	$\boldsymbol{x} = (1, i)$ and $(1, -i)$
Reflection: $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	$\boldsymbol{u}$; whole plane $\boldsymbol{u}^{\perp}$
Rank One: $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$	$\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u};\ 0, .., 0$	$\boldsymbol{u}$; whole plane $\boldsymbol{v}^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Function: any $f(A)$	$f(\lambda_1), \ldots, f(\lambda_n)$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Tridiagonal: diagonals $-1, 2, -1$	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$\boldsymbol{x}_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$

Factorizations Based on Eigenvalues (Singular Values in Σ)

Diagonalizable: $A = V\Lambda V^{-1}$	diagonal of Λ has λ_i	eigenvectors in V
Symmetric: $S = Q\Lambda Q^{\mathrm{T}}$	diagonal of Λ (real λ_i)	orthonormal eigenvectors in Q
Jordan form: $J = V^{-1}AV$	diagonal of J is Λ	each block gives $\boldsymbol{x} = (0, .., 1, .., 0)$
SVD for any $\boldsymbol{A}$: $A = U\Sigma V^{\mathrm{T}}$	rank(A) = rank(Σ)	eigenvectors of $A^{\mathrm{T}}A, AA^{\mathrm{T}}$ in V, U

■ CHAPTER 6 NOTES ■

A symmetric matrix S has perpendicular eigenvectors. Suppose $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$ and $S\boldsymbol{y} = \lambda_2\boldsymbol{y}$ and $\lambda_1 \neq \lambda_2$. Subtract $\lambda_1 I$ from both equations:

$$(S - \lambda_1 I)\boldsymbol{x} = \boldsymbol{0} \qquad \text{and} \qquad (S - \lambda_1 I)\boldsymbol{y} = (\lambda_2 - \lambda_1)\boldsymbol{y}.$$

This puts $\boldsymbol{x}$ in the nullspace and $\boldsymbol{y}$ in the column space of $S - \lambda_1 I$. That matrix is real symmetric, so its column space is also its row space. Then $\boldsymbol{x}$ in the nullspace is sure to be perpendicular to $\boldsymbol{y}$ in the row space. A new proof that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$.

Several proofs that S has a full set of n independent (and orthogonal) eigenvectors—even in the case of repeated eigenvalues—are on the course website for linear algebra: **web.mit.edu/18.06** (Proofs of the Spectral Theorem).

Similar Matrices and the Jordan Form

For every A, we want to choose V so that $V^{-1}AV$ is as ***nearly diagonal as possible***. When A has a full set of n eigenvectors, they go into the columns of V. Then the matrix $V^{-1}AV$ is diagonal, period. This matrix Λ is the Jordan form of A—when A can be diagonalized. But if eigenvectors are missing, Λ can't be reached.

Suppose A has s independent eigenvectors. Then it is similar to a matrix with s blocks. *Each block has the eigenvalue λ on the diagonal with* 1's *just above it*. This block accounts for one eigenvector. When there are n eigenvectors and n blocks, J is Λ.

(Jordan form) If A has s independent eigenvectors, it is similar to a matrix J that has Jordan blocks J_1 to J_s on its diagonal. Some matrix V puts A into its Jordan form J:

Jordan form $$V^{-1}AV = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J.$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

Jordan block $$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}.$$

A is similar to B if they share the same Jordan form J—not otherwise.

The Jordan form J has an off-diagonal 1 for each missing eigenvector (and the 1's are next to the eigenvalues). This is the big theorem about matrix similarity. In every family of similar matrices, we are picking one outstanding member called J. It is nearly diagonal (or if possible completely diagonal). We can solve $dz/dt = Jz$ by back substitution. Then we have solved $d\boldsymbol{y}/dt = A\boldsymbol{y}$ with $\boldsymbol{y} = V\boldsymbol{z}$.

Jordan's Theorem is proved in my textbook *Linear Algebra and Its Applications*. The reasoning is rather intricate and the Jordan form is not at all popular in computations. A slight change in A will separate the repeated eigenvalues and bring a diagonal Λ.

Time-varying systems $\boldsymbol{y}' = A(t)\boldsymbol{y}$: Wrong formula and correct formula for $\boldsymbol{y}(t)$

Section 6.4 recognized that linear systems are more difficult when the matrix depends on t. The formula $\boldsymbol{y}(t) = \exp(\int A(t)dt)\boldsymbol{y}(0)$ is not correct. The underlying reason is that e^{A+B} (the wrong matrix) is generally different from $e^A e^B$ (the correct matrix at $t = 2$, when the system jumps from $\boldsymbol{y}' = B\boldsymbol{y}$ to $\boldsymbol{y}' = A\boldsymbol{y}$ at $t = 1$.) Go forward in time : e^B *and then* e^A.

It is not usual for a basic textbook to attempt a correct formula. But this is a chance to emphasize that Euler's difference equation goes forward in the right order. It steps from $\boldsymbol{Y}_n$ at time $n\Delta t$ to $\boldsymbol{Y}_{n+1}$ at time $(n+1)\Delta t$, using the current matrix A at time $n\Delta t$.

$$\textbf{Euler's method} \quad \Delta \boldsymbol{Y}/\Delta t = A\boldsymbol{Y} \text{ or } \boldsymbol{Y_{n+1} = E_n Y_n} \quad \text{with } E_n = I + \Delta t A(n\Delta t).$$

When we reach $\boldsymbol{Y}_N$, we have multiplied $\boldsymbol{Y}_0$ by N matrices E_0 to E_{N-1} *in the right order*:

$$\boldsymbol{Y}_N = E_{N-1}E_{N-2}\dots E_1 E_0 \boldsymbol{Y}_0.$$

Basic theory says that Euler's $\boldsymbol{Y}_N$ approaches the correct $\boldsymbol{y}(t)$, when $\Delta t = t/N$ and $N \to \infty$. That product of E's approaches the correct replacement for e^{At}. When A is a constant matrix, not changing with time, all E's are the same and we reach e^{At} from E^N :

$$\textbf{Constant matrix } \boldsymbol{A} \qquad e^{\boldsymbol{A}t} = \text{ limit of } (I + \Delta t A)^N = \text{limit of } \left(I + \frac{At}{N}\right)^N.$$

This came from compound interest in Section 1.3, when A was a number (1 by 1 matrix).

The limit of $E_{N-1}E_{N-2}\dots E_1 E_0$ is called a **product integral**. An ordinary "*sum integral*" $\int A(t)dt$ is the limit of a sum of N terms $\Delta t A$ (each term going to zero). Now we are multiplying N terms $I + \Delta t A$ (each term going to I). Term by term, $I + \Delta t A$ is close to $e^{\Delta t A}$. But matrices don't always commute, and $\exp \int A(t)dt$ is *wrong*. Matrix products $E_{N-1}\dots E_1 E_0$ approach a *product integral* and the correct $\boldsymbol{y}(t)$.

Product integral $M(t) =$ limit of $E_{N-1}E_{N-2}\dots E_1 E_0$. Then $\boldsymbol{y}(t) = M(t)\boldsymbol{y}(0)$.

One final good note. The *determinant* $W(t)$ of the matrix $M(t)$ has a nice formula. This succeeds because numbers $\det A$ (but not matrices A) can be multiplied in any order. Here is the beautiful fact that gives the equation for the Wronskian determinant $W(t)$:

$$\boxed{\text{If } \frac{dM}{dt} = AM \text{ then } \frac{dW}{dt} = (\text{trace}\,(A))W. \text{ Therefore } W(t) = e^{\int \text{trace}\,(A(t))dt} W(0).}$$

This is equation (21) in Section 6.4. We see again that the Wronskian $W(t)$ is never zero, because exponentials are never zero. For $y'' + B(t)y' + C(t)y = 0$, the companion matrix has trace $-B(t)$. The Wronskian is $W(t) = e^{-\int B(t)dt} W(0)$ as Abel discovered.

Chapter 7

Applied Mathematics and $A^{\mathrm{T}}A$

A chapter title that includes the symbols $A^{\mathrm{T}}A$ is not usual. Most textbooks deal with A and its eigenvalues, and stop. When the original problem involves a rectangular matrix, as so many problems do, the steps to reach a square matrix are omitted. In reality, rectangular matrices are everywhere—they connect current and voltage, displacement and force, position and momentum, prices and income, *pairs of unknowns*.

It is true that the eventual equation contains a square matrix (very often symmetric). We start from A and we reach $A^{\mathrm{T}}A$. Those two matrices have the same nullspace. We want $A^{\mathrm{T}}A$ to be invertible so we can solve the problem. Then A must have independent columns (no nullspace except the zero vector) as we now assume: A must be "tall and thin" with $m \geq n$ and full column rank $r = n$.

$S = A^{\mathrm{T}}A$ has positive eigenvalues. It is a **positive definite symmetric matrix**. Its eigenvectors lead us to the **Singular Value Decomposition** of A. The SVD in Section 7.2 is the best way to discover what is important, when a large matrix is filled with data. The singular vectors are like eigenvectors for a square matrix, with the extra guarantee of orthogonality.

The chapter starts with m equations in n unknowns—too many equations, too few unknowns, and *no solution to* $A\boldsymbol{v} = \boldsymbol{b}$. This is a major application of linear algebra (and geometry and calculus). A sensor or a scanner or a counter makes thousands of measurements. Often we are overwhelmed with data. If it lies close to a straight line, that line $v_1 + v_2 t$ or $C + Dt$ has only $n = 2$ parameters. Those are the two numbers we want, coming from $m = 1000$ or 1000000 measurements.

Our first applications, are *least squares* and *weighted least squares*. The 2 by 2 matrix $A^{\mathrm{T}}A$ or $A^{\mathrm{T}}CA$ will appear (C contains the weights). This is the symmetric matrix S of Section 6.5 and Section 7.1, and the stiffness matrix K of Section 7.4, and the conductance matrix of Section 7.5, and the second derivative $A^{\mathrm{T}}A = -d^2/dx^2$ in 7.3. (A minus sign is included, because if $A = d/dx$ is the first derivative then $-d/dx$ is its transpose.)

"Symmetric positive definite"—those are three important words in linear algebra. And they are key ideas in applied mathematics, to be presented in this chapter.

7.1 Least Squares and Projections

Start with $A\boldsymbol{v} = \boldsymbol{b}$. The matrix A has n independent columns; its rank is n. But A has m rows, and m is greater than n. We have m measurements in $\boldsymbol{b}$, and we want to choose $n < m$ parameters $\boldsymbol{v}$ that fit those measurements. An exact fit $A\boldsymbol{v} = \boldsymbol{b}$ is generally impossible. We look for the closest fit to the data—the best solution $\widehat{\boldsymbol{v}}$.

The *error vector* $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{v}}$ tells how close we are to solving $A\boldsymbol{v} = \boldsymbol{b}$. The errors in the m equations are $e_1, \ldots, e_m$. Make the *sum of squares* as small as possible.

Least squares solution $\widehat{\boldsymbol{v}}$ **Minimize** $||\boldsymbol{e}||^2 = e_1^2 + \cdots + e_m^2 = ||\boldsymbol{b} - A\boldsymbol{v}||^2.$

This is our goal, to reduce $\boldsymbol{e}$. If $A\boldsymbol{v} = \boldsymbol{b}$ has a solution (and possibly it could), then the best $\widehat{\boldsymbol{v}}$ is certainly that solution vector $\boldsymbol{v}$. In this case the error is $\boldsymbol{e} = \boldsymbol{0}$, certainly a minimum. But normally there is no exact solution to the m equations $A\boldsymbol{v} = \boldsymbol{b}$. The column space of A is only an n-dimensional subspace of $\mathbf{R}^m$. Almost all vectors $\boldsymbol{b}$ are outside that subspace—they are not combinations of the columns of A. We reduce the error $E = ||\boldsymbol{e}||^2$ as far as possible, but we cannot reach zero error.

Example 1 Find the straight line $b = C + Dt$ that goes through 4 points: $b = 1, 9, 9, 21$ at $t = 0, 1, 3, 4$. Those are four equations for C and D, and they have *no solution*. The four crosses in Figure 7.1 are not on a straight line:

$A\boldsymbol{v} = \boldsymbol{b}$ has no solution

$$\begin{aligned} C + 0D &= 1 \\ C + 1D &= 9 \\ C + 3D &= 9 \\ C + 4D &= 21 \end{aligned} \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 21 \end{bmatrix}. \tag{1}$$

$C = 1$ solves the first equation, then $D = 8$ solves the second equation. Then the other equations fail by a lot. We want a better balance, where no equation is exact but the total squared error $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$ from all four equations is as small as possible.

The best C and D are 2 and 4. The best $\boldsymbol{v}$ is $\widehat{\boldsymbol{v}} = (2, 4)$. The best line is $2 + 4t$. At the four measurement times $t = 0, 1, 3, 4$, this best line has heights $2, 6, 14, 18$. In other words, $A\widehat{\boldsymbol{v}}$ is $\boldsymbol{p} = (2, 6, 14, 18)$ which is as close as possible to $\boldsymbol{b} = (1, 9, 9, 21)$.

For that vector $\boldsymbol{p} = (2, 6, 14, 18)$, the four bullets in Figure 7.1 fall on the line $2 + 4t$. How do we find that best solution $\widehat{\boldsymbol{v}} = (C, D) = (2, 4)$? *It has the smallest error E*:

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = (1 - C - 0D)^2 + (9 - C - 1D)^2 + (9 - C - 3D)^2 + (21 - C - 4D)^2.$$

We can use pure linear algebra to find $C = 2$ and $D = 4$, or pure calculus. To use calculus, set two partial derivatives to zero: $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Solve for C and D.

Linear algebra gives the right triangle in Figure 7.1. The vector $\boldsymbol{b}$ is split into $\boldsymbol{p} + \boldsymbol{e}$. The heights $\boldsymbol{p}$ lie on a line and the errors $\boldsymbol{e}$ are as small as possible. I will use calculus first, and then the linear algebra that I prefer—because it produces a right triangle $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$.

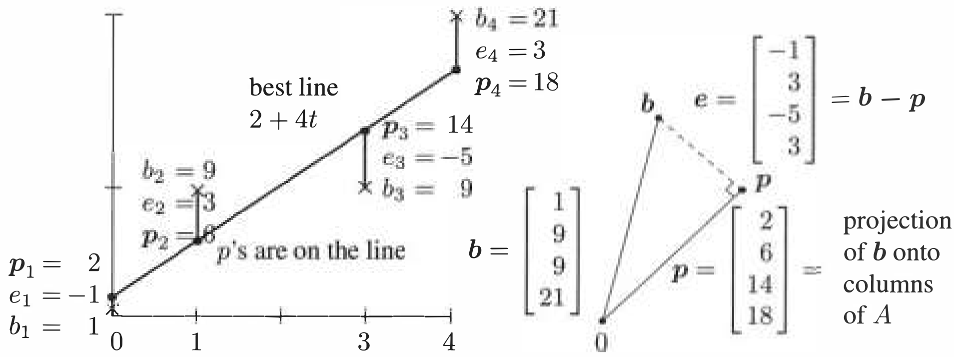


Figure 7.1: Two pictures ! The best line has $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} = 1 + 9 + 25 + 9 = 44 = ||\boldsymbol{b} - \boldsymbol{p}||^2$.

Let me give away the answer immediately (the equation for C and D). Then you can compute the best solution $\widehat{\boldsymbol{v}}$ and the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ and the error $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{v}}$. **The best least squares estimate $\widehat{\boldsymbol{v}} = (C, D)$ solves the "normal equations" using the square symmetric invertible matrix $A^{\mathrm{T}}A$:**

Normal equations to find $\widehat{\boldsymbol{v}}$ $\quad \boxed{A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}.}$ (2)

In short, multiply the unsolvable equations $A\boldsymbol{v} = \boldsymbol{b}$ by A^{T} to get $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Example 1 (completed) The normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$ are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 9 \\ 21 \end{bmatrix}. \tag{3}$$

After multiplication this matrix $A^{\mathrm{T}}A$ is square and symmetric and positive definite:

$$\boldsymbol{A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}b} \qquad \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 40 \\ 120 \end{bmatrix} \text{ gives } \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \tag{4}$$

At $t = 0, 1, 3, 4$ this best line $2 + 4t$ in Figure 7.1 has heights $\boldsymbol{p} = 2, 6, 14, 18$. The minimum error $\boldsymbol{b} - \boldsymbol{p}$ is $\boldsymbol{e} = (-1, 3, -5, 3)$. The picture on the right is the "linear algebra way" to see least squares. We project $\boldsymbol{b}$ to $\boldsymbol{p}$ in the column space of A (you see how $\boldsymbol{p}$ is perpendicular to the error vector $\boldsymbol{e}$). Then $A\widehat{\boldsymbol{v}} = \boldsymbol{p}$ has the best possible right side $\boldsymbol{p}$.

The solution $\widehat{\boldsymbol{v}} = (\widehat{C}, \widehat{D}) = (2, 4)$ is the least squares choice of C and D.

Normal equations using calculus The two equations are $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$.

The first column shows the four terms $e_1^2 + e_2^2 + e_3^2 + e_4^2$ that add to E. Next to them are the derivatives that add to $\partial E/\partial C$ and $\partial E/\partial D$. Notice how the chain rule brings factors $0, 1, 3, 4$ in the third column for $\partial E/\partial D$.

$$\begin{array}{l} \textbf{Add} \\ \textbf{each} \\ \textbf{column} \end{array} \quad \boldsymbol{E} = \begin{array}{l} (C+0D-1)^2 \\ (C+1D-9)^2 \\ (C+3D-9)^2 \\ (C+4D-21)^2 \end{array} \quad \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{C}} = \begin{array}{l} 2(C+0D-1) \\ 2(C+1D-9) \\ 2(C+3D-9) \\ 2(C+4D-21) \end{array} \quad \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{D}} = \begin{array}{l} 2(C+0D-1)(0) \\ 2(C+1D-9)(1) \\ 2(C+3D-9)(3) \\ 2(C+4D-21)(4) \end{array}$$

No problem to divide all derivatives by 2, when $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. The last two columns are added by matrix multiplication (notice the numbers $0, 1, 3, 4$ in $\partial E/\partial D$).

$$\frac{1}{2}\begin{bmatrix} \partial E/\partial C \\ \partial E/\partial D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} C+0D & - & 1 \\ C+1D & - & 9 \\ C+3D & - & 9 \\ C+4D & - & 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5}$$

The 2 by 4 matrix is A^T. The 4 by 1 vector is $A\widehat{\boldsymbol{v}} - \boldsymbol{b}$. Calculus has found $A^TA\boldsymbol{v} = A^T\boldsymbol{b}$.

Example 2 Suppose we have two equations for one unknown $\boldsymbol{v}$. Thus $n = 1$ but $m = 2$ (probably there is no solution). One unknown means only one column in A :

$$A\boldsymbol{v} = \boldsymbol{b} \quad \text{is} \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \text{For example} \quad \begin{array}{l} 2\boldsymbol{v} = 1 \\ 3\boldsymbol{v} = 8 \end{array}. \tag{6}$$

The matrix A is 2 by 1. The squared error is $E = e_1^2 + e_2^2 = (1 - 2\boldsymbol{v})^2 + (8 - 3\boldsymbol{v})^2$.

$$\textbf{Sum of squares} \qquad E(\boldsymbol{v}) = (b_1 - a_1\boldsymbol{v})^2 + (b_2 - a_2\boldsymbol{v})^2.$$

The graph of $E(\boldsymbol{v})$ is a parabola. Its bottom point is at the least squares solution $\widehat{\boldsymbol{v}}$. The minimum error occurs when $dE/d\boldsymbol{v} = 0$:

$$\textbf{Equation for } \widehat{\boldsymbol{v}} \qquad \frac{dE}{d\boldsymbol{v}} = 2a_1\,(a_1\widehat{\boldsymbol{v}} - b_1) + 2a_2\,(a_2\widehat{\boldsymbol{v}} - b_2) = 0. \tag{7}$$

Cancel the 2's, so $(a_1^2 + a_2^2)\widehat{\boldsymbol{v}} = (a_1b_1 + a_2b_2)$. The left side has $a_1^2 + a_2^2 = A^TA$. The right side is $a_1b_1 + a_2b_2 = A^T\boldsymbol{b}$. Calculus has again found $A^TA\widehat{\boldsymbol{v}} = A^T\boldsymbol{b}$:

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \widehat{\boldsymbol{v}} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{produces} \quad \widehat{\boldsymbol{v}} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}. \tag{8}$$

The numerical example has $\boldsymbol{a} = (2, 3)$ and $\boldsymbol{b} = (1, 8)$ and $\widehat{\boldsymbol{v}} = \boldsymbol{a}^T\boldsymbol{b}/\boldsymbol{a}^T\boldsymbol{a} = 26/13 = 2$.

Example 3 The special case $a_1 = a_2 = 1$ has two measurements $v = b_1$ and $v = b_2$ of the same quantity (like pulse rate or blood pressure). The matrix has $A^{\mathrm{T}} = [1 \quad 1]$. To minimize $(v - b_1)^2 + (v - b_2)^2$, the best $\widehat{v}$ is just the average measurement:

$$\text{If} \quad a_1 = a_2 = 1 \quad \text{then} \quad A^{\mathrm{T}}A = 2 \quad \text{and} \quad A^{\mathrm{T}}\boldsymbol{b} = b_1 + b_2 \quad \text{and} \quad \widehat{v} = (b_1 + b_2)/2.$$

The linear algebra picture in Figure 7.2 shows the projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. The projection is $\boldsymbol{p}$, the angle is $90°$, and the other side of the right triangle is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. **The normal equations are saying that $\boldsymbol{e}$ is perpendicular to the line through $\boldsymbol{a}$.**

Least Squares by Linear Algebra

Here is the linear algebra approach to $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$. It takes one wonderful line :

$\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{v}}$ is perpendicular to the column space of A. So $\boldsymbol{e}$ is in the nullspace of $\boldsymbol{A}^{\mathrm{T}}$.

Then $A^{\mathrm{T}}\boldsymbol{b} = A^{\mathrm{T}}A\widehat{\boldsymbol{v}}$. That fourth subspace $\boldsymbol{N}(A^{\mathrm{T}})$ is exactly what least squares needs: $\boldsymbol{e}$ is perpendicular to the whole column space of A and not just to $\boldsymbol{p} = A\widehat{\boldsymbol{v}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

Figure 7.2 shows the projection $\boldsymbol{p}$ as an m by m matrix P multiplying $\boldsymbol{b}$. To project any vector onto the column space of A, multiply by the *projection matrix* P.

Projection matrix gives $\boldsymbol{p} = \boldsymbol{P}\boldsymbol{b}$ $\qquad \boldsymbol{P} = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$ **or** $\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}.$ (9)

The first form of P gives the projection on the line through $\boldsymbol{a}$. Here A has only one column and $A^{\mathrm{T}}A = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. We can divide by that number, but for $n > 1$ the right notation is $(A^{\mathrm{T}}A)^{-1}$. The second form gives P in all cases, provided only that $A^{\mathrm{T}}A$ is invertible:

Two key properties of projection matrices $\qquad P^{\mathrm{T}} = P$ and $P^2 = P.$ (10)

The projection of $\boldsymbol{p}$ is $\boldsymbol{p}$ itself (because $\boldsymbol{p} = P\boldsymbol{b}$ is already in the column space). Then two projections give the same result as one projection : $P(P\boldsymbol{b}) = P\boldsymbol{b}$ and $P^2 = P$.

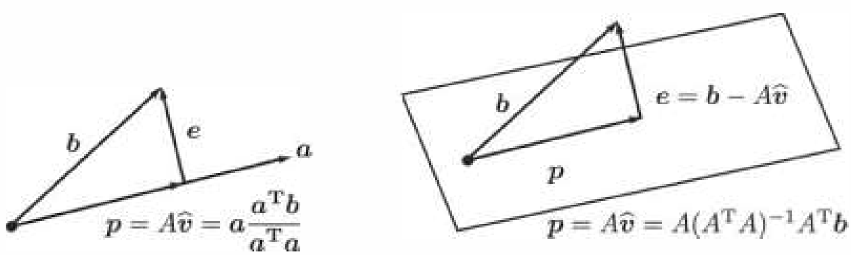


Figure 7.2: The projection $\boldsymbol{p}$ is the nearest point to $\boldsymbol{b}$ in the column space of A. Left $(n = 1)$: column space = line through $\boldsymbol{a}$. Right $(n = 2)$: Column space = plane.

Let me review the four essential equations of (unweighted) least squares :

1.	$A\boldsymbol{v} = \boldsymbol{b}$	m equations, n unknowns, probably **no solution**
2.	$A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$	**normal equations,** $\widehat{\boldsymbol{v}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} =$ best $\boldsymbol{v}$
3.	$\boldsymbol{p} = A\widehat{\boldsymbol{v}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$	**projection** $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space of A
4.	$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$	**projection matrix** $\boldsymbol{P}$ produces $\boldsymbol{p} = P\boldsymbol{b}$ for any $\boldsymbol{b}$

Example 4 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ find $\widehat{\boldsymbol{v}}$ and $\boldsymbol{p}$ and the matrix P.

Solution Compute the square matrix $A^{\mathrm{T}}A$ and also the vector $A^{\mathrm{T}}\boldsymbol{b}$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

Now solve the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$ to find $\widehat{\boldsymbol{v}}$:

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \widehat{v}_1 \\ \widehat{v}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad \text{gives} \quad \widehat{\boldsymbol{v}} = \begin{bmatrix} \widehat{v}_1 \\ \widehat{v}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}. \tag{11}$$

The combination $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ is the projection of $\boldsymbol{b}$ onto the column space of A :

$$\boldsymbol{p} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}. \text{ The error is } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{12}$$

Two checks on the calculation. First, the error $\boldsymbol{e} = (1, -2, 1)$ is perpendicular to both columns $(1, 1, 1)$ and $(0, 1, 2)$. Second, the projection matrix P times $\boldsymbol{b} = (6, 0, 0)$ correctly gives $\boldsymbol{p} = (5, 2, -1)$. That solves the problem for one particular $\boldsymbol{b}$.

To find $\boldsymbol{p} = P\boldsymbol{b}$ for every $\boldsymbol{b}$, compute $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The determinant of $A^{\mathrm{T}}A$ is $15 - 9 = 6$; then $(A^{\mathrm{T}}A)^{-1}$ is easy. Multiply A times $(A^{\mathrm{T}}A)^{-1}$ times A^{T} to reach P:

$$(A^{\mathrm{T}}A)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \quad \text{and} \quad P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{13}$$

We must have $P^2 = P$, because a second projection doesn't change the first projection.

Warning The matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is deceptive. You might try to split $(A^{\mathrm{T}}A)^{-1}$ into A^{-1} times $(A^{\mathrm{T}})^{-1}$. If you make that mistake, and substitute it into P, you will find $P = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}}$. Apparently everything cancels. This looks like $P = I$, the identity matrix. The next two lines explain why this is wrong.

The matrix A is rectangular. It has no inverse matrix. We cannot split $(A^{\mathrm{T}}A)^{-1}$ into A^{-1} times $(A^{\mathrm{T}})^{-1}$ because there is no A^{-1} in the first place.

In our experience, a problem that involves a rectangular matrix almost always leads to $A^{\mathrm{T}}A$. When A has independent columns, $A^{\mathrm{T}}A$ is invertible. This fact is so crucial that we state it clearly and give a proof.

$A^{\mathrm{T}}A$ is invertible if and only if A has linearly independent columns.

Proof $A^{\mathrm{T}}A$ is a square matrix (n by n). For every matrix A, we will now show that $A^{\mathrm{T}}A$ ***has the same nullspace as*** A. When A has independent columns, its nullspace contains only the zero vector. Then $A^{\mathrm{T}}A$, with this same nullspace, is invertible.

Let A be any matrix. If $\boldsymbol{x}$ is in its nullspace, then $A\boldsymbol{x} = \mathbf{0}$. Multiplying by A^{T} gives $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{x}$ is also in the nullspace of $A^{\mathrm{T}}A$.

Now start with the nullspace of $A^{\mathrm{T}}A$. **From** $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ **we must prove** $A\boldsymbol{x} = \mathbf{0}$. We can't multiply by $(A^{\mathrm{T}})^{-1}$, which generally doesn't exist. Just multiply by $\boldsymbol{x}^{\mathrm{T}}$:

$$(\boldsymbol{x}^{\mathrm{T}})A^{\mathrm{T}}A\boldsymbol{x} = 0 \quad \text{or} \quad (A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) = 0 \quad \text{or} \quad \|A\boldsymbol{x}\|^2 = 0.$$

This says: If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ has length zero. Therefore $A\boldsymbol{x} = \mathbf{0}$.

Every vector $\boldsymbol{x}$ in one nullspace is in the other nullspace. If $A^{\mathrm{T}}A$ has dependent columns, so has A. If $A^{\mathrm{T}}A$ has independent columns, so has A. This is the good case:

When A has independent columns, $A^{\mathrm{T}}A$ is square, symmetric, and invertible.

To repeat for emphasis: $A^{\mathrm{T}}A$ is (n by m) times (m by n). Then $A^{\mathrm{T}}A$ is square (n by n). It is symmetric, because its transpose is $(A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}}$ which equals $A^{\mathrm{T}}A$. We just proved that $A^{\mathrm{T}}A$ is invertible—provided A has independent columns. Watch the difference between dependent columns and independent columns:

$$\underset{A^{\mathrm{T}}}{\begin{bmatrix}1&1&0\\2&2&0\end{bmatrix}} \underset{\text{dependent}}{\overset{A}{\begin{bmatrix}1&2\\1&2\\0&0\end{bmatrix}}} = \underset{\text{singular}}{\overset{A^{\mathrm{T}}A}{\begin{bmatrix}2&4\\4&8\end{bmatrix}}} \qquad \underset{A^{\mathrm{T}}}{\begin{bmatrix}1&1&0\\2&2&1\end{bmatrix}} \underset{\text{independent}}{\overset{A}{\begin{bmatrix}1&2\\1&2\\0&1\end{bmatrix}}} = \underset{\text{invertible}}{\overset{A^{\mathrm{T}}A}{\begin{bmatrix}2&4\\4&9\end{bmatrix}}}$$

Very brief summary To find the projection $\boldsymbol{p} = \widehat{v}_1\boldsymbol{a}_1 + \cdots + \widehat{v}_n\boldsymbol{a}_n$, solve $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$. This gives $\widehat{\boldsymbol{v}}$. The projection is $A\widehat{\boldsymbol{v}}$ and the error is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - A\widehat{\boldsymbol{v}}$. The projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ multiplies $\boldsymbol{b}$ to give the projection $\boldsymbol{p} = P\boldsymbol{b}$.

This matrix satisfies $P^2 = P$. ***The distance from b to the subspace is*** $\|e\|$.

Weighted Least Squares

There is normally error in the measurements $\boldsymbol{b}$. That produces error in the output $\widehat{\boldsymbol{v}}$. Some measurements b_i may be more reliable than others (from less accurate sensors). We should give heavier weight to those reliable b_i.

We assume that the expected error in each b_i is zero. Then negative errors balance positive errors in the long run, and *the mean error is zero.* **The expected squared error in the measurement b_i (the "mean squared error") is its variance ${\sigma_i}^2$:**

$$\textbf{Mean } m_i = E[e_i] = 0 \qquad \textbf{Variance } {\sigma_i}^2 = \text{expected squared error } \underline{E}[e_i^2] \tag{14}$$

We should give equation i more weight when σ_i is small. Then b_i is more reliable.

Statistically, the right weight is $w_i = 1/\sigma_i$. We multiply $A\boldsymbol{v} = \boldsymbol{b}$ by the diagonal matrix W with those weights $w_1, \ldots, w_m$. Then solve $WA\boldsymbol{v} = W\boldsymbol{b}$ by ordinary least squares, using WA and $W\boldsymbol{b}$ instead of A and $\boldsymbol{b}$:

$$\textbf{Weighted least squares} \quad \boldsymbol{(WA)^T(WA)\widehat{v} = (WA)^TWb \text{ is } A^TCA\widehat{v} = A^TCb.} \tag{15}$$

$C = W^TW$ goes between A^T and A, to produce the weighted matrix $K = A^TCA$.

Example 5 Your pulse rate v is measured twice. Using unweighted least squares ($w_1 = w_2 = 1$), the best estimate is $\widehat{v} = \frac{1}{2}(b_1 + b_2)$. Example 3 finds that least square solution $\widehat{v}$ to two equations $v = b_1$ and $v = b_2$. But if you were more nervous the first time, then σ_1 is larger than σ_2. The first measurement b_1 has a larger variance than b_2.

We should weight the two measurements by $w_1 = 1/\sigma_1$ and $w_2 = 1/\sigma_2$:

$$\textbf{With weights} \qquad \begin{aligned} w_1 v &= w_1 b_1 \\ w_2 v &= w_2 b_2 \end{aligned} \qquad \widehat{v} = \frac{w_1 b_1 + w_2 b_2}{w_1^2 + w_2^2} \tag{16}$$

When $w_1 = w_2 = 1$, that answer $\widehat{v}$ reduces to the unweighted estimate $\frac{1}{2}(b_1 + b_2)$.

The weighted $K = A^TCA$ has the same good properties as the unweighted A^TA: square, symmetric, and invertible when A has independent columns (as in the example). *Then all eigenvalues of A^TA and A^TCA have $\lambda > 0$*: positive definite matrices!

■ REVIEW OF THE KEY IDEAS ■

1. The least squares solution $\widehat{\boldsymbol{v}}$ minimizes $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$. Then $A^TA\widehat{\boldsymbol{v}} = A^T\boldsymbol{b}$.

2. To fit m points by a line $C + Dt$, A is m by 2 and $\widehat{\boldsymbol{v}} = (\widehat{C}, \widehat{D})$ gives the best line.

3. The projection of $\boldsymbol{b}$ on the column space of A is $\boldsymbol{p} = A\widehat{\boldsymbol{v}} = P\boldsymbol{b}$: closest point to $\boldsymbol{b}$.

4. The error is $e = \boldsymbol{b} - \boldsymbol{p}$. The projection matrix is $P = A(A^TA)^{-1}A^T$ with $P^2 = P$.

5. Weighted least squares has $A^TCA\widehat{\boldsymbol{v}} = A^TC\boldsymbol{b}$. Good weights c_i are 1/variance of b_i.

Problem Set 7.1

1 Suppose your pulse is measured at $b_1 = 70$ beats per minute, then $b_2 = 120$, then $b_3 = 80$. The least squares solution to three equations $v = b_1, v = b_2, v = b_3$ with $A^{\mathrm{T}} = [1\ \ 1\ \ 1]$ is $\widehat{v} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} =$ _____. Use calculus and projections:

(a) Minimize $E = (v-70)^2 + (v-120)^2 + (v-80)^2$ by solving $dE/dv = 0$.

(b) Project $\boldsymbol{b} = (70, 120, 80)$ onto $\boldsymbol{a} = (1,1,1)$ to find $\widehat{v} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

2 Suppose $Av = \boldsymbol{b}$ has m equations $a_i v = b_i$ in *one unknown* v. For the sum of squares $E = (a_1 v - b_1)^2 + \cdots + (a_m v - b_m)^2$, find the minimizing $\widehat{v}$ by calculus. Then form $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ with one column in A, and reach the same $\widehat{v}$.

3 With $\boldsymbol{b} = (4,1,0,1)$ at the points $x = (0,1,2,3)$ set up and solve the normal equation for the coefficients $\widehat{\boldsymbol{v}} = (C, D)$ in the nearest line $C + Dx$. Start with the four equations $A\boldsymbol{v} = \boldsymbol{b}$ that would be solvable if the points fell on a line.

4 In Problem 3, find the projection $\boldsymbol{p} = A\boldsymbol{v}$. Check that those four values lie on the line $C + Dx$. Compute the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and verify that $A^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{0}$.

5 (Problem 3 by calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares: the last one is $(1 - C - 3D)^2$. Find the derivative equations $\partial E/\partial C = \partial E/\partial D = 0$. Divide by 2 to obtain $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

6 For the closest parabola $C + Dt + Et^2$ to the same four points, write down 4 unsolvable equations $A\boldsymbol{v} = \boldsymbol{b}$ for $\boldsymbol{v} = (C, D, E)$. Set up the normal equations for $\widehat{\boldsymbol{v}}$. If you fit the best cubic $C + Dt + Et^2 + Ft^3$ to those four points (thought experiment), what is the error vector $\boldsymbol{e}$?

7 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

8 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 7. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$.

9 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 8. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

10 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is _____.

11 Find the best line $C + Dt$ to fit $\boldsymbol{b} = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

12 Find the *plane* that gives the best fit to the 4 values $\boldsymbol{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. At those 4 points, the equations $C + Dx + Ey = b$ are $A\boldsymbol{v} = \boldsymbol{b}$ with 3 unknowns $\boldsymbol{v} = (C, D, E)$.

13 With $\boldsymbol{b} = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ set up and solve the normal equations $A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b}$. For the best straight line $C + Dt$, find its four heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

14 (By calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Divide by 2 to obtain the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

15 Which of the four subspaces contains the error vector $\boldsymbol{e}$? Which contains p ? Which contains $\widehat{\boldsymbol{v}}$?

16 Find the height C of the best *horizontal line* to fit $\boldsymbol{b} = (0, 8, 8, 20)$. An exact fit would solve the four unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

18 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$. Why is $P\boldsymbol{e} = \boldsymbol{0}$?

19 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 18. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to ______ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

20 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line and $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is ______ ?

Questions 21–26 ask for projections onto lines. Also errors $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and matrices P.

21 Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

$$\text{(a)}\ \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(b)}\ \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}.$$

22 Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it from $\boldsymbol{p} = \widehat{v}\boldsymbol{a}$:

$$\text{(a)}\ \boldsymbol{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(b)}\ \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

23 In Problem 22 find the projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto each vector $\boldsymbol{a}$. Verify in both cases that $P^2 = P$. Multiply $P\boldsymbol{b}$ in each case to find the projection $\boldsymbol{p}$.

24 Construct the projection matrices P_1 and P_2 onto the lines through the $\boldsymbol{a}$'s in Problem 22. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

25 Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those two matrices P_1P_2 and explain the answer.

26 Continuing Problem 25, find the projection matrix P_3 onto $\boldsymbol{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal !

27 Project the vector $\boldsymbol{b} = (1, 1)$ onto the lines through $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (1, 2)$. Draw the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and add $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because the $\boldsymbol{a}$'s are not orthogonal.

28 (Quick and recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

29 If A is doubled, then $P = 2A(4A^{\mathrm{T}}A)^{-1}2A^{\mathrm{T}}$. This is the same as $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The column space of $2A$ is the same as ______. Is $\widehat{\boldsymbol{v}}$ the same for A and $2A$?

30 What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\boldsymbol{b} = (2, 1, 1)$?

31 (*Important*) If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto which fundamental subspace ?

32 If P is the 3 by 3 projection matrix onto the line through $(1, 1, 1)$, then $I - P$ is the projection matrix onto ______.

33 Multiply the matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space so its projection is ______.

34 If A is square and invertible, the warning against splitting $(A^{\mathrm{T}}A)^{-1}$ does not apply. Then $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$ is true. *When A is invertible, why is $P = I$* and $\boldsymbol{e} = \boldsymbol{0}$?

35 An important fact about $A^{\mathrm{T}}A$ is this: ***If*** $\boldsymbol{A^{\mathrm{T}}Ax = 0}$ ***then*** $\boldsymbol{Ax = 0}$. *New proof* : The vector $A\boldsymbol{x}$ is in the nullspace of ______. $A\boldsymbol{x}$ is always in the column space of ______. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero.

Notes on mean and variance and test grades

If all grades on a test are 90, the mean is $m = 90$ and the variance is $\sigma^2 = 0$. Suppose the expected grades are $g_1, \ldots, g_N$. Then σ^2 comes from *squaring distances to the mean* :

$$\textbf{Mean}\ \ m = \frac{g_1 + \cdots + g_N}{N} \qquad \textbf{Variance}\ \ \sigma^2 = \frac{(g_1 - m)^2 + \cdots + (g_N - m)^2}{N}$$

After every test my class wants to know m and σ. My expectations are usually way off.

36 Show that σ^2 also equals $\frac{1}{N}(g_1^2 + \cdots + g_N^2) - m^2$.

37 If you flip a fair coin N times (1 for heads, 0 for tails) what is the expected number m of heads ? What is the variance σ^2 ?

7.2 Positive Definite Matrices and the SVD

This chapter about applications of $A^{\mathrm{T}}A$ depends on two important ideas in linear algebra. These ideas have big parts to play, we focus on them now.

1. Positive definite symmetric matrices (both $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}CA$ are positive definite)

2. Singular Value Decomposition ($A = U\Sigma V^{\mathrm{T}}$ gives perfect bases for the 4 subspaces)

Those are orthogonal matrices U and V in the SVD. Their columns are orthonormal eigenvectors of AA^{T} and $A^{\mathrm{T}}A$. The entries in the diagonal matrix Σ are the *square roots* of the eigenvalues. The matrices AA^{T} and $A^{\mathrm{T}}A$ have the same nonzero eigenvalues.

Section 6.5 showed that the eigenvectors of these symmetric matrices are orthogonal. I will show now that *the eigenvalues of $A^{\mathrm{T}}A$ are positive*, if A has independent columns.

Start with $A^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}$. Then $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. Therefore $\lambda = ||A\boldsymbol{x}||^2/||\boldsymbol{x}||^2 > 0$

I separated $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x}$ into $(A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) = ||A\boldsymbol{x}||^2$. We don't have $\lambda = 0$ because $A^{\mathrm{T}}A$ is invertible (since A has independent columns). The eigenvalues must be positive.

Those are the key steps to understanding positive definite matrices. They give us three tests on S—three ways to recognize when a symmetric matrix S is positive definite :

Positive definite symmetric	1. All the eigenvalues of S are positive. 2. The "energy" $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is positive for all nonzero vectors $\boldsymbol{x}$. 3. S has the form $S = A^{\mathrm{T}}A$ with independent columns in A.

There is also a test on the pivots (all > 0) and a test on n determinants (all > 0).

Example 1 Are these matrices positive definite ? When their eigenvalues are positive, construct matrices A with $S = A^{\mathrm{T}}A$ and find the positive energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$.

$$\text{(a)} \quad S = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(b)} \quad S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \qquad \text{(c)} \quad S = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

Solution The answers are *yes*, *yes*, and *no*. The eigenvalues of those matrices S are

(a) 4 and 1 : *positive* (b) 9 and 1 : *positive* (c) 9 and -1 : *not positive*.

A quicker test than eigenvalues uses **two determinants** : the 1 by 1 determinant S_{11} and the 2 by 2 determinant of S. Example (b) has $S_{11} = \mathbf{5}$ and det $S = 25 - 16 = \mathbf{9}$ (*pass*). Example (c) has $S_{11} = \mathbf{4}$ but det $S = 16 - 25 = \mathbf{-9}$ (*fail the test*).

Positive energy is equivalent to positive eigenvalues, when S is symmetric. Let me test the energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ in all three examples. Two examples pass and the third fails :

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4x_1^2 + x_2^2 > 0 \qquad \text{Positive energy when } \boldsymbol{x} \neq \boldsymbol{0}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5x_1^2 + 8x_1x_2 + 5x_2^2 \qquad \text{Positive energy when } \boldsymbol{x} \neq \boldsymbol{0}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4x_1^2 + 10x_1x_2 + 4x_2^2 \qquad \text{Energy } \mathbf{-2} \text{ when } \boldsymbol{x} = (1, -1)$$

Positive energy is a fundamental property. This is the best definition of *positive definiteness*.

When the eigenvalues are positive, there will be many matrices A that give $A^{\mathrm{T}}A = S$. One choice of A is symmetric and positive definite ! Then $A^{\mathrm{T}}A$ is A^2, and this choice $A = \sqrt{S}$ is a true square root of S. The successful examples (a) and (b) have $\boldsymbol{S = A^2}$:

$$\boxed{\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$

We know that all symmetric matrices have the form $S = V\Lambda V^{\mathrm{T}}$ with orthonormal eigenvectors in V. The diagonal matrix Λ has a square root $\sqrt{\Lambda}$, when all eigenvalues are positive. In this case $A = \sqrt{S} = V\sqrt{\Lambda}V^{\mathrm{T}}$ is the symmetric positive definite square root:

$$\boldsymbol{A^{\mathrm{T}}A} = \sqrt{S}\sqrt{S} = (V\sqrt{\Lambda}V^{\mathrm{T}})(V\sqrt{\Lambda}V^{\mathrm{T}}) = V\sqrt{\Lambda}\sqrt{\Lambda}V^{\mathrm{T}} = \boldsymbol{S} \text{ because } V^{\mathrm{T}}V = I.$$

Starting from this unique square root $\sqrt{S}$, other choices of A come easily. Multiply $\sqrt{S}$ by any matrix Q that has orthonormal columns (so that $Q^{\mathrm{T}}Q = I$). Then $Q\sqrt{S}$ is another choice for A (not a symmetric choice). In fact all choices come this way :

$$A^{\mathrm{T}}A = (Q\sqrt{S})^{\mathrm{T}}\,(Q\sqrt{S}) = \sqrt{S}Q^{\mathrm{T}}Q\sqrt{S} = S. \tag{1}$$

I will choose a particular Q in Example 1, to get particular choices of A.

Example 1 (*continued*) Choose $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to multiply $\sqrt{S}$. Then $A = Q\sqrt{S}$.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \qquad \text{has} \quad S = A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} \qquad \text{has} \quad S = A^{\mathrm{T}}A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

Positive Semidefinite Matrices

Positive *semidefinite* matrices include positive definite matrices, and more. Eigenvalues of S can be zero. Columns of A can be dependent. The energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ can be zero—*but not negative*. This gives new equivalent conditions on a (possibly singular) matrix $S = S^{\mathrm{T}}$.

1′ All eigenvalues of S satisfy $\lambda \geq 0$ (semidefinite allows zero eigenvalues).

2′ The energy is nonnegative for every $\boldsymbol{x}$: $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} \geq 0$ (zero energy is allowed).

3′ S has the form $A^{\mathrm{T}}A$ (every A is allowed; its columns can be dependent).

Example 2 The first two matrices are singular and positive semidefinite—but not the third :

$$\text{(d)}\quad S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(e)}\quad S = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \qquad \text{(f)}\quad S = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}.$$

The eigenvalues are $1, 0$ and $8, 0$ and $-8, 0$. The energies $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ are x_2^2 and $4(x_1 + x_2)^2$ and $-4(x_1 - x_2)^2$. So the third matrix is actually *negative* semidefinite.

Singular Value Decomposition

Now we start with A, square or rectangular. Applications also start this way—the matrix comes from the model. The SVD splits any matrix into *orthogonal* U times *diagonal* Σ times *orthogonal* V^{T}. Those orthogonal factors will give orthogonal bases for the four fundamental subspaces associated with A.

Let me describe the goal for any m by n matrix, and then how to achieve that goal.

Find orthonormal bases $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ **for** $\mathbf{R}^n$ **and** $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ **for** $\mathbf{R}^m$ **so that**

$$\boldsymbol{A}\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1 \quad \ldots \quad \boldsymbol{A}\boldsymbol{v}_r = \sigma_r \boldsymbol{u}_r \qquad \boldsymbol{A}\boldsymbol{v}_{r+1} = \mathbf{0} \quad \ldots \quad \boldsymbol{A}\boldsymbol{v}_n = \mathbf{0} \qquad (2)$$

The rank of A is r. Those requirements in (4) are expressed by a multiplication $\boldsymbol{AV} = \boldsymbol{U\Sigma}$. The r nonzero singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ are on the diagonal of Σ :

$$\boldsymbol{AV} = \boldsymbol{U\Sigma} \qquad A\begin{bmatrix} \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_r & \ldots & \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_r & \ldots & \boldsymbol{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix} \qquad (3)$$

The last $n - r$ vectors in V are a basis for the nullspace of A. The last $m - r$ vectors in U are a basis for the nullspace of A^{T}. The diagonal matrix Σ is m by n, with r nonzeros.

Remember that $V^{-1} = V^{\mathrm{T}}$, because the columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthonormal in $\mathbf{R}^n$:

$$\textbf{Singular Value Decomposition} \qquad \boldsymbol{AV} = \boldsymbol{U\Sigma} \quad \textbf{becomes} \quad \boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathrm{T}} \qquad (4)$$

The SVD has orthogonal matrices U and V, containing eigenvectors of AA^{T} and $A^{\mathrm{T}}A$.

Comment. A square matrix is diagonalized by its eigenvectors : $A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$ is like $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$. But even if A has n eigenvectors, they may not be orthogonal. We need *two bases*—an input basis of $\boldsymbol{v}$'s in $\mathbf{R}^n$ and an output basis of $\boldsymbol{u}$'s in $\mathbf{R}^m$. With two bases, any m by n matrix can be diagonalized. The beauty of those bases is that they can be chosen orthonormal. Then $U^{\mathrm{T}}U = I$ and $V^{\mathrm{T}}V = I$.

The $\boldsymbol{v}$'s are eigenvectors of the symmetric matrix $S = A^{\mathrm{T}}A$. We can guarantee their orthogonality, so that $\boldsymbol{v}_j^{\mathrm{T}}\boldsymbol{v}_i = 0$ for $j \neq i$. That matrix S is positive semidefinite, so its eigenvalues are $\sigma_i^2 \geq 0$. **The key to the SVD is that $A\boldsymbol{v}_j$ is orthogonal to $A\boldsymbol{v}_i$** :

$$\textbf{Orthogonal } \boldsymbol{u}\textbf{'s} \quad (A\boldsymbol{v}_j)^{\mathrm{T}}(A\boldsymbol{v}_i) = \boldsymbol{v}_j^{\mathrm{T}}(A^{\mathrm{T}}A\boldsymbol{v}_i) = \boldsymbol{v}_j^{\mathrm{T}}(\sigma_i^2\boldsymbol{v}_i) = \begin{cases} \sigma_i^2 & \text{if } j = i \\ \mathbf{0} & \text{if } j \neq i \end{cases} \tag{5}$$

This says that the vectors $\boldsymbol{u}_i = A\boldsymbol{v}_i/\sigma_i$ are orthonormal for $i = 1, \ldots, r$. They are a basis for the column space of A. And the $\boldsymbol{u}$'s are eigenvectors of the symmetric matrix AA^{T}, which is usually different from $S = A^{\mathrm{T}}A$ (but the eigenvalues $\sigma_1^2, \ldots, \sigma_r^2$ are the same).

Example 3 Find the input and output eigenvectors $\boldsymbol{v}$ and $\boldsymbol{u}$ for the rectangular matrix A:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} = U\Sigma V^{\mathrm{T}}.$$

Solution Compute $S = A^{\mathrm{T}}A$ and its unit eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. The eigenvalues σ^2 are $8, 2, 0$ so the positive singular values are $\sigma_1 = \sqrt{8}$ and $\sigma_2 = \sqrt{2}$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad \boldsymbol{v}_1 = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The outputs $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sigma_1$ and $\boldsymbol{u}_2 = A\boldsymbol{v}_2/\sigma_2$ are also orthonormal, with $\sigma_1 = \sqrt{8}$ and $\sigma_2 = \sqrt{2}$. Those vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are in the column space of A :

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}\frac{\boldsymbol{v}_1}{\sqrt{8}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}\frac{\boldsymbol{v}_2}{\sqrt{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then $U = I$ and the Singular Value Decomposition for this 2 by 3 matrix is $U\Sigma V^{\mathrm{T}}$:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}^{\mathrm{T}}.$$

The Fundamental Theorem of Linear Algebra

I think of the SVD as the final step in the Fundamental Theorem. First come the *dimensions* of the four subspaces in Figure 7.3. Then come the *orthogonality* of those pairs of subspaces. Now come the *orthonormal bases of $\boldsymbol{v}$'s and $\boldsymbol{u}$'s that diagonalize A*:

SVD

$$\boxed{\begin{aligned} A\boldsymbol{v}_j &= \sigma_j \boldsymbol{u}_j \quad \text{for } j \le r \\ A\boldsymbol{v}_j &= \mathbf{0} \quad\quad\ \text{for } j > r \end{aligned}} \qquad \boxed{\begin{aligned} A^{\mathrm{T}}\boldsymbol{u}_j &= \sigma_j \boldsymbol{v}_j \quad \text{for } j \le r \\ A^{\mathrm{T}}\boldsymbol{u}_j &= \mathbf{0} \quad\quad\ \text{for } j > r \end{aligned}}$$

Multiplying $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$ by A^{T} and dividing by σ_j gives that equation $\boldsymbol{A}^{\mathbf{T}}\boldsymbol{u}_j = \boldsymbol{\sigma}_j \boldsymbol{v}_j$.

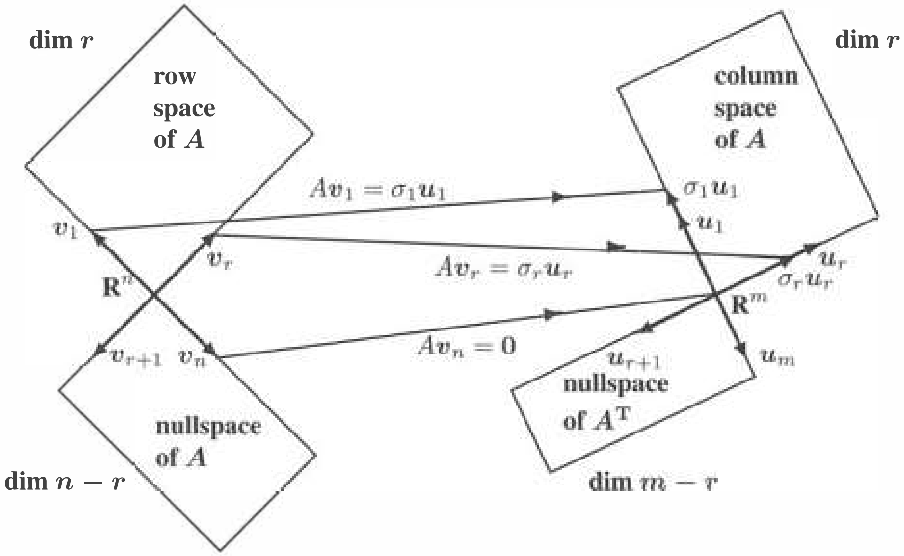


Figure 7.3: Orthonormal bases of $\boldsymbol{v}$'s and $\boldsymbol{u}$'s that diagonalize A: m by n with rank r.

The "**norm**" of A is its largest singular value : $||\boldsymbol{A}|| = \boldsymbol{\sigma}_1$. This measures the largest possible ratio of $||A\boldsymbol{v}||$ to $||\boldsymbol{v}||$. That ratio of lengths is a maximum when $\boldsymbol{v} = \boldsymbol{v}_1$ and $A\boldsymbol{v} = \sigma_1 \boldsymbol{u}_1$. This singular value σ_1 is a much better measure for the size of a matrix than the largest eigenvalue. An extreme case can have zero eigenvalues and just one eigenvector $(1, 1)$ for A. But $A^{\mathrm{T}}A$ can still be large : if $\boldsymbol{v} = (1, -1)$ then $A\boldsymbol{v}$ is 200 times larger.

$$A = \begin{bmatrix} 100 & -100 \\ 100 & -100 \end{bmatrix} \quad \text{has } \lambda_{\max} = \mathbf{0}. \quad \text{But } \boldsymbol{\sigma}_{\mathbf{max}} = \textbf{norm of } \boldsymbol{A} = \mathbf{200}. \tag{6}$$

The Condition Number

A valuable property of $A = U\Sigma V^{\mathrm{T}}$ is that it puts the pieces of A *in order of importance*. Multiplying a column $\boldsymbol{u}_i$ times a row $\sigma_i \boldsymbol{v}_i^{\mathrm{T}}$ produces one piece of the matrix. There will be r nonzero pieces from r nonzero σ's, when A has rank r. The pieces add up to A, when we multiply columns of U times rows of ΣV^{T} :

The pieces have rank 1

$$A = \begin{bmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 \boldsymbol{v}_1^{\mathrm{T}} \\ \dots\dots \\ \sigma_r \boldsymbol{v}_r^{\mathrm{T}} \end{bmatrix} = \boldsymbol{u}_1(\sigma_1 \boldsymbol{v}_1^{\mathrm{T}}) + \cdots + \boldsymbol{u}_r(\sigma_r \boldsymbol{v}_r^{\mathrm{T}}). \qquad (7)$$

The first piece gives the norm of A which is σ_1. The last piece gives the norm of A^{-1}, which is $1/\sigma_n$ when A is invertible. The **condition number** is σ_1 times $1/\sigma_n$:

$$\boxed{\textbf{Condition number of } \boldsymbol{A} \qquad \boldsymbol{c(A)} = ||A||\ ||A^{-1}|| = \frac{\boldsymbol{\sigma_1}}{\boldsymbol{\sigma_n}}.} \qquad (8)$$

This number $c(A)$ is the key to numerical stability in solving $A\boldsymbol{v} = \boldsymbol{b}$. When A is an orthogonal matrix, the symmetric $S = A^{\mathrm{T}}A$ is the identity matrix. So all singular values of an orthogonal matrix are $\sigma = 1$. At the other extreme, a singular matrix has $\sigma_n = 0$. In that case $c = \infty$. Orthogonal matrices have the best condition number $c = 1$.

Data Matrices : Application of the SVD

"*Big data*" is the linear algebra problem of this century (and we won't solve it here). Sensors and scanners and imaging devices produce enormous volumes of information. Making decisive sense of that data is *the* problem for a world of analysts (mathematicians and statisticians of a new type). Most often the data comes in the form of a matrix.

The usual approach is by PCA—*Principal Component Analysis*. That is essentially the SVD. The first piece $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$ holds the most information (in statistics this piece has the greatest variance). It tells us the most. The Chapter 7 Notes include references.

■ REVIEW OF THE KEY IDEAS ■

1. Positive definite symmetric matrices have positive eigenvalues and pivots and energy.
2. $S = A^{\mathrm{T}}A$ is positive definite if and only if A has independent columns.
3. $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x})$ is zero when $A\boldsymbol{x} = \boldsymbol{0}$. $A^{\mathrm{T}}A$ can be positive *semidefinite*.
4. The SVD is a factorization $A = U\Sigma V^{\mathrm{T}} =$ (*orthogonal*) (*diagonal*) (*orthogonal*).
5. The columns of V and U are eigenvectors of $A^{\mathrm{T}}A$ and AA^{T} (singular vectors of A).
6. Those orthonormal bases achieve $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ and A is diagonalized.
7. The largest piece of $A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r \boldsymbol{u}_r \boldsymbol{v}_r^{\mathrm{T}}$ gives the norm $||A|| = \sigma_1$.

Problem Set 7.2

1 For a 2 by 2 matrix, suppose the 1 by 1 and 2 by 2 determinants a and $ac - b^2$ are positive. Then $c > b^2/a$ is also positive.

(i) λ_1 and λ_2 have the *same sign* because their product $\lambda_1\lambda_2$ equals _____.

(i) That sign is positive because $\lambda_1 + \lambda_2$ equals _____.

Conclusion: The tests $a > 0, ac - b^2 > 0$ guarantee positive eigenvalues λ_1, λ_2.

2 Which of S_1, S_2, S_3, S_4 has two positive eigenvalues? Use a and $ac - b^2$, don't compute the λ's. Find an $\boldsymbol{x}$ with $\boldsymbol{x}^{\mathrm{T}}S_1\boldsymbol{x} < 0$, confirming that A_1 fails the test.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

3 For which numbers b and c are these matrices positive definite?

$$S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \qquad S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

4 What is the energy $q = ax^2 + 2bxy + cy^2 = \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ for each of these matrices? Complete the square to write q as a sum of squares $d_1(\quad)^2 + d_2(\quad)^2$.

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

5 $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = 2x_1x_2$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix S produces this energy? What are its eigenvalues?

6 Test to see if $A^{\mathrm{T}}A$ is positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

7 Which 3 by 3 symmetric matrices S and T produce these quadratic energies?

$\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$. Why is S positive definite?

$\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$. Why is T semidefinite?

8 Compute the three upper left determinants of S to establish positive definiteness. (The first is 2.) Verify that their ratios give the second and third pivots.

$$\textbf{Pivots} = \textbf{ratios of determinants} \qquad S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

9 For what numbers c and d are S and T positive definite? Test the 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

10 *If S is positive definite then S^{-1} is positive definite.* Best proof: The eigenvalues of S^{-1} are positive because _____. *Second proof* (only for 2 by 2):

$$\text{The entries of } S^{-1} = \frac{1}{ac - b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad \text{pass the determinant tests} \quad \_\_\_\_\_ .$$

11 *If S and T are positive definite, their sum $S + T$ is positive definite.* Pivots and eigenvalues are not convenient for $S + T$. Better to prove $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} > 0$.

12 A positive definite matrix *cannot have a zero* (or even worse, a negative number) *on its diagonal.* Show that this matrix fails to have $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & \mathbf{0} & 2 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

13 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have _____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a _____ on the main diagonal.

14 Show that *if all $\lambda > 0$ then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$.* We must do this for *every* nonzero $\boldsymbol{x}$, not just the eigenvectors. So write $\boldsymbol{x}$ as a combination of the eigenvectors and *explain why all "cross terms" are $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$*. Then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is

$$(c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n)^{\mathrm{T}}(c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n) = c_1^2\lambda_1\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_1 + \cdots + c_n^2\lambda_n\boldsymbol{x}_n^{\mathrm{T}}\boldsymbol{x}_n > 0.$$

15 Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P = I$.

(c) A diagonal matrix with positive diagonal entries is positive definite.

(d) A symmetric matrix with a positive determinant might not be positive definite!

16 With positive pivots in D, the factorization $S = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $A = \sqrt{D}L^{\mathrm{T}}$ yields the ***Cholesky factorization*** $S = A^{\mathrm{T}}A$ which is "symmetrized LU":

$$\text{From} \quad A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{find } S. \qquad \text{From} \quad S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \quad \text{find } A = \mathbf{chol}(S).$$

17 Without multiplying $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of S (b) the eigenvalues of S
(c) the eigenvectors of S (d) a reason why S is symmetric positive definite.

18 For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$ find the second derivative matrices H_1 and H_2:

$$\textbf{Test for minimum} \quad H = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix} \text{ is positive definite}$$

H_1 is positive definite so F_1 is concave up ($=$ convex). Find the minimum point of F_1 and the saddle point of F_2 (look only where first derivatives are zero).

19 The graph of $z = x^2 + y^2$ is a bowl opening upward. The graph of $z = x^2 - y^2$ is *a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(0,0)$?

20 Which values of c give a bowl and which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

21 When S and T are symmetric positive definite, ST might not even be symmetric. But its eigenvalues are still positive. Start from $ST\boldsymbol{x} = \lambda\boldsymbol{x}$ and take dot products with $T\boldsymbol{x}$. Then prove $\lambda > 0$.

22 Suppose C is positive definite (so $\boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \boldsymbol{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \boldsymbol{0}$ whenever $\boldsymbol{x} \neq \boldsymbol{0}$). Apply the energy test to $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}$ to show that $A^{\mathrm{T}}CA$ is positive definite: *the crucial matrix in engineering*.

23 Find the eigenvalues and unit eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$. Then find $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \text{ and } AA^{\mathrm{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that $\boldsymbol{u}_1$ is a unit eigenvector of AA^{T}. Complete the matrices U, Σ, V.

$$\textbf{SVD} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}}.$$

24 Write down orthonormal bases for the four fundamental subspaces of this A.

25 (a) Why is the trace of $A^{\mathrm{T}}A$ equal to the sum of all a_{ij}^2?

(b) For every rank-one matrix, why is $\sigma_1^2 =$ sum of all a_{ij}^2?

26 Find the eigenvalues and unit eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. Keep each $A\boldsymbol{v} = \sigma\boldsymbol{u}$:

$$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^{\mathrm{T}}$.

27 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide $\pm$ signs in U). Σ *has the same shape as* A.

28 Construct the matrix with rank one that has $A\boldsymbol{v} = 12\boldsymbol{u}$ for $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ and $\boldsymbol{u} = \frac{1}{3}(2,2,1)$. Its only singular value is $\sigma_1 =$ ______.

29 Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by ***as small a matrix as possible*** to produce a singular matrix A_0. Hint: U and V do not change.

$$\text{From} \qquad A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} \qquad \text{find the nearest } A_0.$$

30 The SVD for $A + I$ doesn't use $\Sigma + I$. *Why is* $\sigma(A + I)$ *not just* $\sigma(A) + I$?

31 Multiply $A^{\mathrm{T}}A\boldsymbol{v} = \sigma^2\boldsymbol{v}$ by A. Put in parentheses to show that $A\boldsymbol{v}$ is an eigenvector of AA^{T}. We divide by its length $||A\boldsymbol{v}|| = \sigma$ to get the unit eigenvector $\boldsymbol{u}$.

32 My favorite example of the SVD is when $Av(x) = dv/dx$, with the endpoint conditions $v(0) = 0$ and $v(1) = 0$. We are looking for orthogonal functions $v(x)$ so that their derivatives $Av = dv/dx$ are also orthogonal. The perfect choice is $v_1 = \sin \pi x$ and $v_2 = \sin 2\pi x$ and $v_k = \sin k\pi x$. Then each u_k is a cosine.

The derivative of v_1 is $Av_1 = \pi \cos \pi x = \pi u_1$. The singular values are $\sigma_1 = \pi$ and $\sigma_k = k\pi$. Orthogonality of the sines (and orthogonality of the cosines) is the foundation for Fourier series.

You may object to $AV = U\Sigma$. The derivative $A = d/dx$ is not a matrix ! The orthogonal factor V has functions $\sin k\pi x$ in its columns, not vectors. The matrix U has cosine functions $\cos k\pi x$. Since when is this allowed ? One answer is to refer you to the **chebfun** package on the web. This extends linear algebra to matrices whose columns are functions—not vectors.

Another answer is to replace d/dx by a first difference matrix A. Its shape will be $N + 1$ by N. A has 1's down the diagonal and -1's on the diagonal below. Then $AV = U\Sigma$ has discrete sines in V and discrete cosines in U. For $N = 2$ those will be sines and cosines of $30°$ and $60°$ in $\boldsymbol{v}_1$ and $\boldsymbol{u}_1$.

$**$ Can you construct the difference matrix A (3 by 2) and $A^{\mathrm{T}}A$ (2 by 2) ? The discrete sines are $\boldsymbol{v}_1 = (\sqrt{3}/2, \sqrt{3}/2)$ and $\boldsymbol{v}_2 = (\sqrt{3}/2, -\sqrt{3}/2)$. Test that $A\boldsymbol{v}_1$ is orthogonal to $A\boldsymbol{v}_2$. What are the singular values σ_1 and σ_2 in Σ ?

7.3 Boundary Conditions Replace Initial Conditions

This section is about steady-state problems, not initial-value problems. The time variable t is replaced by the space variable x. Instead of two initial conditions at $t = 0$, we have one boundary condition at $x = 0$ and *another boundary condition at* $x = 1$.

Here is the simplest two-point boundary value problem for $y(x)$. Start with $f(x) = 1$.

Two boundary conditions $$\boldsymbol{-\frac{d^2y}{dx^2} = f(x) \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 0.} \tag{1}$$

One particular solution $y_p(x)$ will come from integrating $f(x)$ twice. If $f(x) = 1$ then two integrations give $x^2/2$, and the minus sign in (1) leads to $y_p = -x^2/2$.

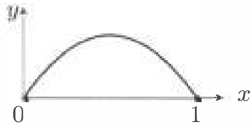
The null solutions $y_n(x)$ solve the equation with zero force: $-y_n'' = 0$. The second derivative is zero for any linear function $y_n = Cx + D$. These are the null solutions.

We can use those two constants C and D to satisfy the two boundary conditions on the complete solution $\boldsymbol{y(x) = y_p + y_n = -x^2/2 + Cx + D}$.

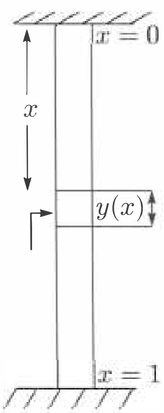
$\boldsymbol{y(0) = 0}$ **and** $\boldsymbol{y(1) = 0}$ Set $x = 0$ and $x = 1$ $D = 0$ and $-\dfrac{1}{2} + C + D = 0$

The boundary conditions give $D = 0$ and $C = \frac{1}{2}$. Then the solution is $y = y_p + y_n$:

Solution to $\boldsymbol{-y'' = 1}$ $$y(x) = -\frac{\boldsymbol{x^2}}{\boldsymbol{2}} + \frac{x}{\boldsymbol{2}} = \frac{x - \boldsymbol{x^2}}{\boldsymbol{2}}$$



The graph of the parabola starts at $y = 0$ and returns (**fixed ends**). The slope $y' = \frac{1}{2} - x$ is decreasing. The second derivative is $y'' = -1$ and the parabola is bending down.



This boundary-value problem describes a bar that has its top and bottom both fixed. The weight of the bar stretches it downward. At point x down the bar, the displacement is $y(x)$. So this fixed-fixed bar has $y(0) = 0$ and $y(1) = 0$. The force of gravity can be $f(x) = 1$. *The bar stretches in the top half where* $dy/dx > 0$. The bottom half is compressed because $dy/dx < 0$. Halfway down at $x = \frac{1}{2}$ is the largest displacement (top of the parabola). That halfway point has $y_{\text{max}} = \frac{1}{2}(x - x^2) = \frac{1}{8}$.

I think of this elastic bar as one long spring. If we pulled it down in the middle, it would start to oscillate. *That is not our problem now*. Our bar is not moving—the oscillation is all damped out. The stretching comes from the bar's own weight.

A Delta Function

This is my chance to introduce again the mysterious but extremely useful function $\boldsymbol{f(x) = \delta(x-a)}$. This **delta function** is zero except at $x = a$. The bar is now so light that we can ignore its weight. All the force on the bar is at *one point* $x = a$. At that point a unit weight is stretching the bar above $x = a$ and compressing the bar below.

Here is an informal definition of the delta function (the symbol ∞ doesn't carry enough information by itself). The good definition is based on *integrating the function across the point* $x = a$. The integral is 1.

Delta function	$\delta(x-a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$	$\int \delta(x-a)\,dx = 1$ $\int \delta(x-a)\,F(x)\,dx = F(a)$

The graph of $\delta(x-a)$ has an *infinite spike* at $x = a$. That spike is at $x = a = 0$ for the standard delta function $\delta(x)$. The function is zero away from the spike and infinite at that one point. **The area under this one-point spike is 1.**

This tells us that $\delta(x)$ cannot be a true function. It is somehow a limit of box functions $B_N(x)$ that have height N over short intervals of width $1/N$. The area of each box is 1 :

$$\textbf{Box functions} \quad B_N(x) = \begin{cases} 0 & |x| > 1/2N \\ N & |x| < 1/2N \end{cases} \qquad \begin{aligned} &\int B_N(x) = \text{ box area } = 1 \\ &\int B_N(x)F(x)\,dx \text{ approaches } F(0) \end{aligned}$$

Mathematically, $\delta(x)$ and its shifts $\delta(x-a)$ are not functions. Physically, they represent action that is concentrated at a single point. In reality that action is probably over a very short interval, like the box functions, but the width of that interval is of no importance. What matters is the total impulse when a bat hits a ball, or the total force when a weight hangs on a bar.

The shifted delta function $\delta(x-a)$ is the derivative of the step function $H(x-a)$. The step function jumps from 0 to 1 at $x = a$. Then δ must integrate to 1.

Response to a Delta Function is a Ramp Function

How to solve the differential equation $-y'' = \delta(x-a)$? One integration of the delta function gives a step function. *A second integration gives a ramp function or corner function.* The solution $y(x)$ must be linear (straight line graph) to the left of $x = a$, because $d^2y/dx^2 = 0$. And $y(x)$ is also linear to the right of $x = a$: constant slope.

The slope of $\boldsymbol{y(x)}$ drops by 1 at the point $\boldsymbol{x = a}$. To see why -1 is the jump in slope (there is no jump in y !), integrate y'' across the point $x = a$ to get the change -1 in y' :

$$\boldsymbol{y'' = -\,\delta\,(x-a)} \qquad \int y''dx = \left[\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right]_{\text{left of } a}^{\text{right of } a} = \int -\,\delta(x-a)dx = \boldsymbol{-1} \qquad (2)$$

The solution $y(x)$ starts with a fixed slope s. At $x = a$ it changes to slope $s - 1$ (the slope drops by 1). At the point $x = 1$, the bottom of the bar is fixed at $y(1) = 0$.

The constant upward slope s over a distance a and the downward slope $s - 1$ over the remaining distance $1 - a$ must bring the function $y(x)$ to zero:

$$sa + (s-1)(1-a) = 0 \text{ gives } sa + s - sa - 1 + a = 0. \textbf{ Then } \boldsymbol{s = 1 - a}. \qquad (3)$$

The graph of $y = sx$ goes up to $sa = (1-a)a$. Then $y(x)$ goes back down to zero.

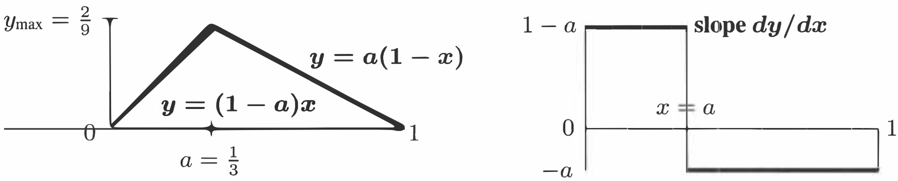


Figure 7.4: $-y'' = \delta(x-a)$ is solved by a **ramp function** that has a corner at $x = a$. At that corner point the slope y' (which is a step function) drops by 1. Then $y'' = -\delta$.

How is the elastic bar stretched and compressed by this point load at $x = a = \frac{1}{3}$? The top third of the bar is stretched, the lower two thirds are compressed. The point $x = a$ shows the highest point on the graph of $y(x)$ and the greatest displacement. That downward displacement is $y(a) = a(1-a) = \frac{2}{9}$.

Uniform stretching above the point load. Uniform compression below the point load.

Eigenvalues and Eigenfunctions

For a square matrix, the eigenvector equation is $A\boldsymbol{x} = \lambda\boldsymbol{x}$. For the second derivative (with a minus sign) and for a boundary condition at both endpoints, the eigenvector $\boldsymbol{x}$ becomes an **eigenfunction $\boldsymbol{y(x)}$**:

Eigenvalues of $-\dfrac{d^2}{dx^2}$

$$-\frac{d^2y}{dx^2} = \lambda y \quad \text{with } y(0) = 0 \text{ and } y(1) = 0. \qquad (4)$$

We can find these eigenfunctions $y(x)$. The solutions to the second order equation $y'' + \lambda y = 0$ are sines and cosines when $\lambda \geq 0$. The boundary conditions choose sines:

$y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$ before applying the boundary conditions
$y(0) = 0$ requires $\boldsymbol{A = 0}$ $\quad y = \sin\sqrt{\lambda} = 0$ at $x = 1$ requires $\boldsymbol{\sqrt{\lambda} = n\pi}$

The eigenfunction is $\boldsymbol{y(x) = \sin n\pi x}$. The eigenvalue is $\boldsymbol{\lambda = n^2\pi^2}$ for $\boldsymbol{n = 1, 2, 3, \ldots}$
Then $-y'' = \lambda y$. We have infinitely many y and λ, not surprising since $S = -d^2/dx^2$ is not a matrix. It is an "operator" and it acts on functions $y(x)$.

The Second Derivative $-d^2/dx^2$ is Symmetric Positive Definite

The derivatives $Ay = dy/dx$ and $Sy = -d^2y/dx^2$ are linear operators. The first derivative A is **antisymmetric**. The second derivative S is **symmetric**. S is also **positive definite**, because of that minus sign. Its eigenvalues $\lambda = n^2\pi^2$ are all positive.

We will use the symbols A^{T} and S^{T}, even though A and S are not matrices. To give meaning to $A^{\mathrm{T}} = -A$ and $S^{\mathrm{T}} = S$, we need *the inner product* (f, g) *of two functions*:

Inner product of f and g $$(\boldsymbol{f(x)}, \boldsymbol{g(x)}) = \int_0^1 \boldsymbol{f(x)\, g(x)\, dx}. \qquad (5)$$

This is the continuous form of the dot product $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}$ of two vectors. For $\boldsymbol{u} \cdot \boldsymbol{v}$ we multiply the components u_i and v_i, and add. For functions we multiply the values of $f(x)$ and $g(x)$, and then integrate as in (5).

A matrix is symmetric if $S\boldsymbol{u} \cdot \boldsymbol{v}$ equals $\boldsymbol{u} \cdot S\boldsymbol{v}$ for all vectors. Then $(S\boldsymbol{u})^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{T}}\left(S^{\mathrm{T}}\boldsymbol{v}\right)$ agrees with $\boldsymbol{u}^{\mathrm{T}}(S\boldsymbol{v})$. An operator is symmetric if (Sf, g) equals (f, Sg) for all functions that satisfy the boundary conditions. Use two integrations by parts to shift the second derivative operator S from f onto g:

Integration by parts twice $$\int_0^1 -\frac{d^2 f}{dx^2}\, g(x)\, dx = \int_0^1 \frac{df}{dx}\frac{dg}{dx}\, dx = \int_0^1 f(x)\left(-\frac{d^2 g}{dx^2}\right) dx. \qquad (6)$$

The integrated terms $[g\, df/dx]_0^1$ and $[f\, dg/dx]_0^1$ in the two integrations by parts are zero because $f = g = 0$ at both endpoints.

The left side and right side of (6) are the inner products (Sf, g) and (f, Sg). Moving S from f onto g always produces S^{T}. Here we have $\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}$ and symmetry is confirmed.

Thus the second derivative $S = -d^2/dx^2$ is symmetric positive definite (this is why we included the minus sign). Section 7.2 gave two other tests, in addition to positive eigenvalues. One test is *positive energy*, and that test is also passed. Choose $g = f$:

Positive energy $\boldsymbol{f}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{f}$ $$(Sf, f) = \int_0^1 -\frac{d^2 f}{dx^2}\, f(x)\, dx = \int_0^1 \left(\frac{df}{dx}\right)^2 dx > \mathbf{0}. \qquad (7)$$

Zero energy requires $df/dx = 0$. Then the boundary conditions ensures $f(x) = 0$.

The third test for a positive definite S looks for A so that $S = A^{\mathrm{T}}A$. Here A is the **first derivative** $(\boldsymbol{Af} = \boldsymbol{df/dx})$. The boundary conditions are still $f(0) = 0$ and $f(1) = 0$. Problem 1 will show that $A^{\mathrm{T}}g$ is $-dg/dx$, with a minus sign from *one* integration by parts. Altogether $\boldsymbol{S} = -\boldsymbol{d^2/dx^2} = (-\boldsymbol{d/dx})(\boldsymbol{d/dx}) = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

Solving the Heat Equation

Differential equations in time give a chance to use all the eigenfunctions $\sin(n\pi x)$. An outstanding example is the **heat equation** $\boldsymbol{\partial u/\partial t = \partial^2 u/\partial x^2 = -Su}$. The eigenvalues of $-S$ are $-n^2\pi^2$, and the negative definite $-S$ leads to decay in time and not growth. Temperatures die out exponentially when there is no fire. Here are the two steps (developed much further in Section 8.3) to solve the heat equation $u_t = u_{xx}$:

1. Write the initial function $u(0,x)$ as a combination of the eigenfunctions $\sin n\pi x$:

Fourier sine series $$u_{\text{start}} = b_1 \sin \pi x + b_2 \sin 2\pi x + \cdots + b_n \sin n\pi x + \cdots \tag{8}$$

2. With $\lambda = -n^2\pi^2$, every eigenfunction decays. Superposition gives u at time t :

$$u(t,x) = b_1 e^{-\pi^2 t} \sin \pi x + b_2 e^{-4\pi^2 t} \sin 2\pi x + \cdots = \sum_1^\infty \boldsymbol{b_n e^{-n^2\pi^2 t}} \sin \boldsymbol{n\pi x} \tag{9}$$

This is the famous **Fourier series solution** to the heat equation. Section 8.1 will show how to compute the Fourier coefficients $b_1, b_2, \ldots$ (a simple formula even when there are infinitely many b's). You see how the solution is exactly analogous to $y(t) = \boldsymbol{c_1 e^{-\lambda_1 t} x_1 + c_2 e^{-\lambda_2 t} x_2}$. That solves an ODE, the heat equation is a PDE.

Second Difference Matrix K

These pages will take a crucial first step in scientific computing. This is where differential equations meet matrix equations. The continuous problem (here continuous in x, previously in t) becomes discrete. Chapter 3 took that step for initial value problems, starting with Euler's forward difference $y(t+\Delta t) - y(t)$. Now we have problems $-y'' = f(x)$ with second derivatives. So we use *second differences* $y(x+\Delta x) - 2y(x) + y(x-\Delta x)$.

The second derivative is the derivative of dy/dx. The second difference is the difference of $\Delta y/\Delta x$. For first differences we have choices—forward or backward or centered differences. To approximate the second derivative $Sy = -y''$ there is *one* outstanding centered choice. This uses the **tridiagonal second difference matrix K** :

$$-\frac{\boldsymbol{d^2y}}{\boldsymbol{dx^2}} \approx \frac{\boldsymbol{KY}}{\boldsymbol{(\Delta x)^2}}$$

$$\boldsymbol{-1 \quad 2 \quad -1} \text{ from}$$

$$\boldsymbol{-Y_{i+1} + 2Y_i - Y_{i-1}}$$

$$\boldsymbol{KY} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ Y_N \end{bmatrix} \tag{10}$$

The numbers Y_1 to Y_N are approximations to the true values $y(\Delta x), \ldots, y(N\Delta x)$ in the continuous problem. The boundary conditions $y(0) = 0$ and $y(1) = 0$ become $Y_0 = 0$ and $Y_{N+1} = 0$. The step Δx has length $1/(N+1)$. The matrix K correctly takes Y_0 and Y_{N+1} to be zero, by working only with Y_1 to Y_N.

The Matrix K is Positive Definite

We know that the operator $S = -d^2/dx^2$ is positive definite. All of its eigenvectors $\sin n\pi x$ have positive eigenvalues $\lambda = n^2\pi^2$. So we hope that the matrix K is also positive definite. That is true—and most unusually for a matrix of any large size N, we can find every eigenvector and eigenvalue of K.

The eigenvectors are the key. It doesn't happen often that *sampling the continuous eigenfunctions at N points produces the discrete eigenvectors.* This is the most important example in all of applied mathematics, of this unprecedented sampling for $y = \sin n\pi x$:

$$\textbf{The } N \textbf{ eigenvectors of } K \textbf{ are } \boldsymbol{y}_n = (\sin n\pi\Delta x, \sin 2n\pi\Delta x, \ldots, \sin Nn\pi\Delta x). \tag{11}$$

$$\textbf{The } N \textbf{ eigenvalues of } K \textbf{ are the positive numbers } \lambda_n = 2 - 2\cos\frac{n\pi}{N+1}. \tag{12}$$

The 2 in every eigenvalue λ comes from the $2'$s along the diagonal of K (that diagonal is $2I$). The cosine in λ and in the equation $K\boldsymbol{y}_n = \lambda_n\boldsymbol{y}_n$ are checked in Problem 12. All eigenvalues are positive because the cosines are below 1. Then K is **positive definite**.

It is natural to try the other positive definite tests too (we don't have to do this, $\lambda > 0$ is enough). With a rectangular first difference matrix A, we have $K = A^{\mathrm{T}}A$:

$$\boldsymbol{A^{\mathrm{T}}A = K} \qquad \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \tag{13}$$

The three columns of that matrix A are certainly independent. Therefore $A^{\mathrm{T}}A$ is a positive definite matrix, now proved twice.

Notice that A^{T} is *minus* the usual forward difference matrix. A is *plus* a backward difference matrix. That sign change reflects the continuous case (for derivatives) where the "transpose" of d/dx is $-\,d/dx$. For every vector $\boldsymbol{f}$, the energy $\boldsymbol{f}^{\mathrm{T}}K\boldsymbol{f}$ is the same as $\boldsymbol{f}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{f} = (A\boldsymbol{f})^{\mathrm{T}}(A\boldsymbol{f}) > 0$:

$$\textbf{The energy} \quad \int_0^1 \left(\frac{d\boldsymbol{f}}{d\boldsymbol{x}}\right)^2 dx \quad \textbf{becomes} \quad \boldsymbol{f}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{f} = (\boldsymbol{A}\boldsymbol{f})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{f}) = \sum_{n=1}^{N+1}(f_n - f_{n-1})^2 > 0.$$

The test of positive energy $\boldsymbol{f}^{\mathrm{T}}K\boldsymbol{f}$ is passed, and K is again proved to be positive definite.

Boundary Conditions on the Slope

The fixed-fixed boundary conditions are $y(0) = 0$ and $y(1) = 0$. One or both of those conditions can change to a *slope condition* on $y' = dy/dx$. If the left condition changes to $y'(0) = 0$, the top of our elastic bar is *free* instead of fixed. This is like a tall building; $x = 0$ is up in the air (**free**) and $x = 1$ is down at the ground (**fixed**).

A fixed-free hanging bar combines $y(0) = 0$ at the top with $y'(1) = 0$ at the bottom. Its matrix is still positive definite. **But a free-free bar has no supports : semidefinite !**

$$\textbf{Free-free } \boldsymbol{Sy = f} \qquad -\frac{d^2y}{dx^2} = f(x) \text{ with } \frac{dy}{dx}(0) = 0 \text{ and } \frac{dy}{dx}(1) = 0. \quad (14)$$

You will see that this problem generally has no solution. **One eigenvalue is now $\boldsymbol{\lambda = 0}$.**

$$\textbf{Free-free } \boldsymbol{Sy = \lambda y} \qquad -\frac{d^2y}{dx^2} = \lambda y(x) \text{ with } \frac{dy}{dx} = 0 \text{ at } x = 0 \text{ and } x = 1. \quad (15)$$

The fixed-fixed problem had eigenfunctions $y(x) = \sin n\pi x$ and eigenvalues $\lambda = n^2\pi^2$. This free-free problem will have $\boldsymbol{y(x) = \cos n\pi x}$ and again $\lambda = n^2\pi^2$. Those cosines start and end with zero slope. Also very important : The free-free problem has an extra eigenfunction $y = \cos 0x$ (which is the constant function $y = 1$). And then $\lambda = 0$:

$$\textbf{Constant } \boldsymbol{y} \textbf{ and zero } \boldsymbol{\lambda} \qquad y = 1 \text{ solves } -\frac{d^2y}{dx^2} = \lambda y \text{ with eigenvalue } \lambda = 0$$

Conclusion : The free-free problem (14) is only positive *semidefinite*. The eigenvalues include $\lambda = 0$. The problem is *singular* and for most loads $f(x)$ there is no solution.

Example with $\boldsymbol{f(x) = x}$ Show that $-y'' = x$ has no solution with $y'(0) = y'(1) = 0$.

Solution Integrate both sides of $-y'' = x$ from $x = 0$ to $x = 1$. The right side gives $\int x\,dx = \frac{1}{2}$. The left side gives $-\int y''\,dx = y'(0) - y'(1)$. But the boundary conditions make this zero and there can be no solution to $0 = \frac{1}{2}$. An operator with a zero eigenvalue is not invertible.

Free-free Difference Matrix B

This problem $-y'' = f(x)$ with free-free conditions $y'(0) = y'(1) = 0$ leads to a **singular matrix** (not invertible). This is still a second difference matrix, to approximate the second derivative. *But row 1 and row N of the matrix are changed by the free-free boundary conditions* :

Free-free matrix B

Change $K_{11} = 2$ to $B_{11} = 1$

Change $K_{NN} = 2$ to $B_{NN} = 1$

$$B = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & \mathbf{1} \end{bmatrix} \text{ is not invertible.}$$

The slope dy/dx is approximated by a first difference in row 1 and row N. All other rows still contain the second difference $-1, 2, -1$. The usual $1, -2, 1$ has signs reversed because the differential equation has $-d^2y/dx^2$.

How to see that B is not invertible ? MATLAB would find pivots $1, 1, \ldots, 1, 0$ from elimination. The zero in the last pivot position means failure. We can see this failure directly by solving $By = \mathbf{0}$. This is the fast way to show that a matrix is singular.

To show that B is not invertible, find the constant solution to $By =$ zero vector.

y = constant vector
B = singular matrix

$$By = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (16)$$

If B^{-1} existed, we could multiply $B\boldsymbol{y} = \mathbf{0}$ by B^{-1} to find $\boldsymbol{y} = \mathbf{0}$. But this $\boldsymbol{y}$ is not zero.

B is positive semidefinite but it is not positive definite. We can still write the matrix B as $A^{\mathrm{T}}A$, but in this free-free case the columns of A will not be independent.

$$\boldsymbol{B = A^{\mathrm{T}}A} \qquad \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{bmatrix}.$$

With only 3 rows, the 4 columns of A must be dependent. They add up to a zero column.

■ REVIEW OF THE KEY IDEAS ■

1. Two initial conditions for $y(0)$ and $y'(0)$ can change to two **boundary conditions**.

2. The fixed-fixed problem $-y'' = \lambda y$ with $y(0) = 0$ and $y(1) = 0$ has $\lambda = n^2\pi^2$.

3. The second difference matrix K has $\lambda_n = 2 - 2\cos\frac{n\pi}{N+1} > 0$. *Positive definite.*

4. Eigenfunctions and eigenvectors are sines, from fixed-fixed boundary conditions.

5. The free-free problem with $y'(0) = y'(1) = 0$ has $y =$ cosines. This allows $\lambda = 0$.

6. The free-free matrix B has $\lambda = 0$ with the eigenvector $y = (1, \ldots, 1)$. Semidefinite.

Problem Set 7.3

1 *Transpose the derivative with integration by parts*: $(dy/dx, g) = -(y, dg/dx)$.

Ay is dy/dx with boundary conditions $y(0) = 0$ and $y(1) = 0$. Why is $\int y'g\,dx$ equal to $-\int yg'\,dx$? Then A^{T} (which is normally written as A^*) is $A^{\mathrm{T}}g = -dg/dx$ with **no** boundary conditions on g. $A^{\mathrm{T}}Ay$ is $-y''$ with $y(0) = 0$ and $y(1) = 0$.

Problems 2-6 have boundary conditions at $x = 0$ and $x = 1$: no initial conditions.

2 Solve this boundary value problem in two steps. Find the complete solution $y_p + y_n$ with two constants in y_n, and find those constants from the boundary conditions:

Solve $-y'' = 12x^2$ with $y(0) = 0$ and $y(1) = 0$ and $y_p = -x^4$.

3 Solve the same equation $-y'' = 12x^2$ with $y(0) = 0$ and $y'(1) = 0$ (zero slope).

4 Solve the same equation $-y'' = 12x^2$ with $y'(0) = 0$ and $y(1) = 0$. Then try for both slopes $y'(0) = 0$ and $y'(1) = 0$: *this has no solution* $y = -x^4 + Ax + B$.

5 Solve $-y'' = 6x$ with $y(0) = 2$ and $y(1) = 4$. Boundary values need not be zero.

6 Solve $-y'' = e^x$ with $y(0) = 5$ and $y(1) = 0$, starting from $y = y_p + y_n$.

Problems 7-11 are about the LU factors and the inverses of second difference matrices.

7 The matrix T with $T_{11} = 1$ factors perfectly into $LU = A^{\mathrm{T}}A$ (all its pivots are 1).

$$\boldsymbol{T} = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = LU.$$

Each elimination step adds the pivot row to the next row (and L subtracts to recover T from U). The inverses of those difference matrices L and U are **sum matrices**. Then the inverse of $T = LU$ is $U^{-1}L^{-1}$:

$$T^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} = U^{-1}L^{-1}.$$

Compute T^{-1} for $N = 4$ (as shown) and for any N.

8 The matrix equation $TY = (0, 1, 0, 0) =$ *delta vector* is like the differential equation $-y'' = \delta(x - a)$ with $a = 2\Delta x = \frac{2}{5}$. The boundary conditions are $y'(0) = 0$ and $y(1) = 0$. Solve for $y(x)$ and graph it from 0 to 1. Also graph $Y =$ second column of T^{-1} at the points $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. The two graphs are ramp functions.

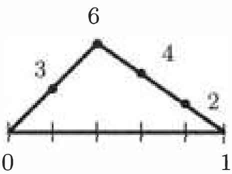
9 The matrix B has $B_{11} = 1$ (like $T_{11} = 1$) and also $B_{NN} = 1$ (where $T_{NN} = 2$). Why does B have the same pivots 1, 1, ... as T, except for zero in the last pivot position? The early pivots don't know $B_{NN} = 1$.

Then B is not invertible: $-y'' = \delta(x - a)$ has no solution with $y'(0) = y'(1) = 0$.

10 When you compute K^{-1}, multiply by $\det K = N+1$ to get nice numbers:

Column 2 of $5K^{-1}$ solves the equation $K\boldsymbol{v} = 5\boldsymbol{\delta}$ when the delta vector is $\boldsymbol{\delta} =$ ____
We know from $KK^{-1} = I$ that K times each column of K^{-1} is a delta vector.

$$5K^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$



graph of column 2

11 K comes with two boundary conditions. T only has $y(1) = 0$. B has no boundary conditions on y. Verify that $K = A^{\mathrm{T}}A$. Then remove the first row of A to get $T = A_1^{\mathrm{T}}A_1$. Then remove the last row to get dependent rows: $B = A_0^{\mathrm{T}}A_0$.

The backward first difference $A = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix}$ gives $K = A^{\mathrm{T}}A$.

12 Multiply K_3 by its eigenvector $\boldsymbol{y}_n = (\sin n\pi h, \sin 2n\pi h, \sin 3n\pi h)$ to verify that the eigenvalues λ_1, λ_2, λ_3 are $\lambda_n = 2 - 2\cos\frac{n\pi}{4}$ in $K\boldsymbol{y}_n = \lambda_n \boldsymbol{y}_n$. This uses the trigonometric identity $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

13 Those eigenvalues of K_3 are $2-\sqrt{2}$ and 2 and $2+\sqrt{2}$. Those add to 6, which is the trace of K_3. Multiply those eigenvalues to get the determinant of K_3.

14 The slope of a ramp function is a step function. The slope of a step function is a delta function. Suppose the ramp function is $r(x) = -x$ for $x \le 0$ and $r(x) = x$ for $x \ge 0$ (so $r(x) = |x|$). *Find dr/dx and d^2r/dx^2.*

15 Find the second differences $y_{n+1} - 2y_n + y_{n-1}$ of these infinitely long vectors $\boldsymbol{y}$:

Constant	$(\ldots, 1, 1, 1, 1, 1, \ldots)$
Linear	$(\ldots, -1, 0, 1, 2, 3, \ldots)$
Quadratic	$(\ldots, 1, 0, 1, 4, 9, \ldots)$
Cubic	$(\ldots, -1, 0, 1, 8, 27, \ldots)$
Ramp	$(\ldots, 0, 0, 0, 1, 2, \ldots)$
Exponential	$(\ldots, e^{-i\omega}, e^0, e^{i\omega}, e^{2i\omega}, \ldots)$.

It is amazing how closely those second differences follow second derivatives for $y(x) = 1, x, x^2, x^3, \max(x, 0)$, and $e^{i\omega x}$. From $e^{i\omega x}$ we also get $\cos\omega x$ and $\sin\omega x$.

7.4 Laplace's Equation and $A^{\mathrm{T}}A$

Section 7.3 solved the differential equation $-d^2y/dx^2 = \delta(x-a)$. Boundary values were given at $x = 0$ and $x = 1$ (our examples began with $y = 0$ at both endpoints). The solutions $y(x)$ went linearly up from zero and linearly back to zero. These boundary value problems correspond to a steady state—with no dependence on time.

Those are "1-dimensional Laplace equations"—certainly the simplest of their kind. This section is more ambitious, in three important ways:

1 We will solve the 2-dimensional Laplace equation—our first PDE. The list of solutions is infinite, and they are particularly beautiful. Amazingly the imaginary number $i = \sqrt{-1}$ enters this real problem.

Laplace's partial differential equation $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

2 The discrete form of (1) is a matrix equation for a vector U. That vector has components $U_1, \ldots, U_n$ at the n nodes of a graph. The graph could be a *line* in 1D or a *grid* in 2D, or any *network of nodes* connected by m edges (Figure 7.5).

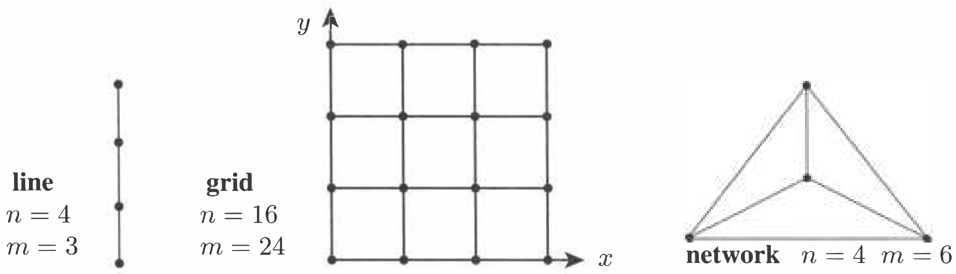


Figure 7.5: A 1D line graph, a 2D grid, and a complete graph: n nodes and m edges.

The natural discrete analog of Laplace's equation (1) is a "5-point scheme" on a grid:

$$\frac{\Delta_x^2 U}{(\Delta x)^2} + \frac{\Delta_y^2 U}{(\Delta y)^2} = \begin{array}{l}\text{2nd difference } \textit{across} \text{ grid}\\ +\text{2nd difference } \textit{down} \text{ grid}\end{array} = 0. \qquad (2)$$

For these equations we are given **boundary values of u and U**. Instead of an interval like $0 \le x \le 1$, there is a region in the plane: u is given along its boundary. U is given at the 12 boundary points of the 4 by 4 grid. Equation (2) holds at each inside point.

3 The continuous and discrete Laplace equations are good examples of $A^{\mathrm{T}}Au$. $A^{\mathrm{T}}A$ is symmetric with eigenvalues $\lambda \ge 0$. And one more matrix will produce $A^{\mathrm{T}}CA$ in Section 7.5. In engineering, C contains the physical properties of the material: stiffness and conductivity and permeability. You will be seeing the structure of applied mathematics.

Laplace's Equation is $A^{\mathrm{T}}Au = 0$

This is our first partial differential equation. It represents *equilibrium*, not change.

Laplace's equation for $u(x,y)$
$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad (3)$$

I have included minus signs to make the left side into $A^{\mathrm{T}}Au$. In one dimension, A was d/dx and A^{T} was $-d/dx$. Now we have two space variables x and y, and two partial derivatives $\partial/\partial x$ and $\partial/\partial y$ will go into A. Then $-\partial/\partial x$ and $-\partial/\partial y$ go into A^{T}.

The vector Au has two components $\partial u/\partial x$ and $\partial u/\partial y$. This is the "*gradient vector*." We are into the 2D world of multivariable calculus and partial derivatives:

Gradient of u
$$Au = \text{grad } u(x,y) = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} u = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}. \qquad (4)$$

I will skip double integrals and the Divergence Theorem (which is the 2D form of the Fundamental Theorem of Calculus). Since A is 2 by 1, you can guess that A^{T} is 1 by 2:

Divergence
$$A^{\mathrm{T}}w = -\text{ div } w = \begin{bmatrix} -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} w_1(x,y) \\ w_2(x,y) \end{bmatrix} = -\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial y}. \qquad (5)$$

Then $A^{\mathrm{T}}Au$ is (minus) the divergence of the gradient of $u(x,y)$. This is the Laplacian:

$$\boxed{\boldsymbol{A^{\mathrm{T}}Au = -\text{ div grad } u} \qquad A^{\mathrm{T}}Au = \begin{bmatrix} -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\[2mm] \dfrac{\partial u}{\partial y} \end{bmatrix} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}.} \qquad (6)$$

You recognize $A^{\mathrm{T}}Au = 0$ as Laplace's equation. With zero on the right hand side, the minus sign can be included or not. We usually give Poisson's name when the equation has a nonzero source (or a sink) $f(x,y)$ on the right hand side:

$$\boldsymbol{u_{xx} + u_{yy} = f(x,y)} \quad \textbf{is Poisson's equation}.$$

The subscripts in u_{xx} and u_{yy} indicate second partial derivatives: $u_{xx} = \partial^2 u/\partial x^2$ and $u_{yy} = \partial^2 u/\partial y^2$. In this notation, u_t indicates $\partial u/\partial t$. Previously that was u', in the ordinary differential equations of earlier chapters. PDEs bring these new notations.

Example 1 $u = xy$ solves Laplace's equation $u_{xx} + u_{yy} = \mathbf{0}$. And $u_p = x^2 + y^2$ solves Poisson's equation $u_{xx} + u_{yy} = 4$ with a constant source. The complete solution for Poisson is this particular solution $x^2 + y^2$ plus any null solution for Laplace.

Solutions to Laplace's Equation

We want a complete set of solutions to $\boldsymbol{u_{xx}+u_{yy}=0}$. The list will be infinitely long. Combinations of those solutions will also be solutions. Laplace's equation is linear, so superposition is allowed. Four solutions are easy to find: $u = \boldsymbol{1, x, y, xy}$. For those four, u_{xx} and u_{yy} are both zero. To find further solutions, we need u_{xx} to cancel u_{yy}.

Start with $u = x^2$, which has $u_{xx} = 2$. Then $u_{yy} = -2$ is achieved by $-y^2$. The combination $u = x^2 - y^2$ solves Laplace's equation. This solution has "*degree* 2" because if x and y are multiplied by C, then u is multiplied by C^2. The same was true of $u = xy$, also degree 2 because $(Cx)(Cy)$ is C^2 times xy.

The real question starts with x^3. *Can this be completed to a solution of degree* 3? From $u = x^3$ we will have $u_{xx} = 6x$. To cancel $6x$, we need a piece that has $u_{yy} = -6x$. *That piece is* $-3xy^2$. The combination $u = x^3 - 3xy^2$ has degree 3 and goes into our list.

The hope is to find two solutions of every degree. Here is the list so far. I will write each pair of solutions in polar coordinates too, starting with $u = x = r\cos\theta$.

degree	**1**	$\boldsymbol{x}$	$\boldsymbol{y}$	$r\cos\theta$	$r\sin\theta$
degree	**2**	$\boldsymbol{x^2-y^2}$	$\boldsymbol{2xy}$	$r^2\cos 2\theta$	$r^2\sin 2\theta$
degree	**3**	$\boldsymbol{x^3-3xy^2}$	**??**	$r^3\cos 3\theta$	$r^3\sin 3\theta$

On the polar coordinate list, the pattern is clear. The pairs of solutions to Laplace's equation are $\boldsymbol{r^n\cos n\theta}$ and $\boldsymbol{r^n\sin n\theta}$. Those will be solutions also for $n = 4, 5, \ldots$

The first list (pairs of x, y polynomials) also has a remarkable pattern. **Those are the real and imaginary parts of** $\boldsymbol{(x+iy)^n}$. Degree $n = 2$ shows the two parts clearly:

$$(x+iy)^2 \text{ is } \boldsymbol{x^2-y^2+i\,2xy} \quad \text{This is } \left(re^{i\theta}\right)^2 = r^2e^{2i\theta} = \boldsymbol{r^2\cos 2\theta + ir^2\sin 2\theta}.$$

The polar pair $r^n\cos n\theta$ and $r^n\sin n\theta$ satisfy Laplace's equation for every n. The x-y pair succeeds because u_{yy} includes $i^2 = -1$, to cancel u_{xx}. We have two solutions for each n:

$$\textbf{Degree } \boldsymbol{n} \quad \boldsymbol{u_n = \mathrm{Re}\,(x+iy)^n = r^n\cos n\theta} \quad \boldsymbol{s_n = \mathrm{Im}(x+iy)^n = r^n\sin n\theta}. \quad (7)$$

All combinations of these solutions will also solve Laplace's equation. For ordinary differential equations (second order with y''), we had two solutions. All null solutions were combinations $c_1y_1 + c_2y_2$. By choosing c_1 and c_2 we matched the two initial conditions $y(0)$ and $y'(0)$. Now we have a partial differential equation with an infinite list of solutions, two of each degree.

By choosing the right coefficients a_n and b_n for every n, including the constant a_0, **we can match any function** $u = u_0(x,y)$ **around the boundary**:

On the boundary $\quad u_0(x,y) = a_0 + a_1x + b_1y + a_2(x^2 - y^2) + b_2(2xy) + \cdots$

Circular boundary $\quad u_0(1,\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta + \cdots$

That last sum is a Fourier series. It enters when we solve Laplace's equation inside a circle. The boundary condition $u = u_0$ is given on the circle $r = 1$. For 1D problems the boundary was the two endpoints $x = 0$ and $x = 1$. We only needed two solutions.

The right choice of all the Fourier coefficients a_n and b_n will come in Chapter 8, and it completes the solution to Laplace's equation inside a circle:

Solution to $\boldsymbol{u_{xx} + u_{yy} = 0}$ $\quad$ $$\boldsymbol{u = a_0 + \sum_{n=1}^{\infty} (a_n\, r^n \cos n\theta + b_n\, r^n \sin n\theta).} \quad (8)$$

Finite Differences and Finite Elements

Laplace's equation is often made discrete. The derivatives u_{xx} and u_{yy} are replaced by finite differences. That produces a large matrix K2D, which is a two-dimensional analog of the tridiagonal $-1, 2, -1$ matrix K. For the square grid in Figure 7.5, there will be entries $-1, 2, -1$ in the x-direction and also in the y-direction. K2D has *five entries*: $2 + 2 = 4$ down its main diagonal and four entries of -1 on a typical inside row.

Suppose the region is not square but curved (like a circle). Then finite differences get complicated. The nodes of a square grid don't fall on circles. The favorite approach changes to the **finite element method**, which can divide the region into triangles of arbitrary shapes. (A triangle can even have a curved edge to fit a boundary.) These finite elements are described in my textbook *Computational Science and Engineering*, with codes that use linear functions $a + bx + cy$ inside each triangle of the mesh. The accuracy is studied in *An Analysis of the Finite Element Method.*

Laplace's Difference Matrix $K2D$

The approach that fits with this book is finite differences. I want to construct the symmetric matrix K2D with rows like $-1, -1, 4, -1, -1$ and show that it is positive definite. K2D comes from second differences in the x and y directions. Each meshpoint needs two indices i and j, to specify its row number and column number on the grid. Go across and up-down:

$$-\frac{\partial^2 u}{\partial x^2} \text{ becomes } \frac{-U_{i+1,j} + 2U_{i,j} - U_{i-1,j}}{(\Delta x)^2} \qquad -\frac{\partial^2 u}{\partial y^2} \text{ becomes } \frac{-U_{i,j+1} + 2U_{i,j} - U_{i,j-1}}{(\Delta y)^2}$$

The square grid has $\Delta x = \Delta y$. Combine $2U_{i,j}$ with $2U_{i,j}$. Then 4 goes on the diagonal of K2D. The difference equation says that *each U_{ij} is the average of its 4 neighbors*:

$\Delta_x^2 U + \Delta_y^2 U = 0$ $\quad$ $$\boldsymbol{4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1} = 0.} \quad (9)$$

If a neighbor of the i, j node falls on the boundary of the square grid, that boundary value of U will be known. Then that term moves to the right side of the difference equation. An entry of -1 disappears from K2D on boundary rows.

If we number the nodes a row at a time, the u_{xx} term puts the 1D matrix K in each block row. The u_{yy} term connects three rows with $-I$ and $2I$ and $-I$.

$$\boldsymbol{K2D} = \begin{bmatrix} \boldsymbol{K} & & & \\ & \boldsymbol{K} & & \\ & & \cdot & \\ & & & \boldsymbol{K} \end{bmatrix} + \begin{bmatrix} \boldsymbol{2I} & \boldsymbol{-I} & & \\ \boldsymbol{-I} & \boldsymbol{2I} & \boldsymbol{-I} & \\ & \boldsymbol{-I} & \cdot & \cdot \\ & & \boldsymbol{-I} & \boldsymbol{2I} \end{bmatrix} = \mathsf{kron}\,(I, K) + \mathsf{kron}\,(K, I).$$

With N interior points in each row, this block matrix K2D is N^2 by N^2. MATLAB's command $\mathsf{kron}(A, B)$ replaces each A_{ij} by the block $A_{ij}B$, so the size grows to N^2.

Here is the matrix for a grid with $3 \times 3 = 9$ squares and $4 \times 4 = 16$ nodes. There are $2 \times 2 = 4$ interior nodes. The other $16 - 4 = 12$ nodes are around the square boundary, where U is given by the boundary condition $u = u_0$. For a large grid, N^2 interior points will far outnumber $4N + 4$ boundary points.

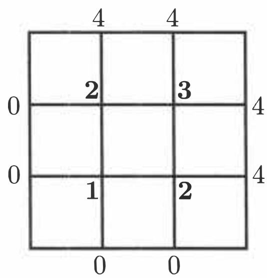
Laplace difference matrix
The interior mesh is 2 by 2

$$K\text{2D} = \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix}.$$

Those rows lost two -1's because each interior gridpoint is next to two boundary points. Normally we see four -1's in almost every row of K2D.

Here is the solution to $K\text{2D}\,\boldsymbol{U} = \boldsymbol{0}$ in the square when boundary values are 0 and 4 :

Each bold value of U is the average of 4 neighbors



The eigenvalues of this matrix K2D are $\lambda = \mathbf{2, 4, 4, 6}$. They add to 16, which is the trace : the sum down the diagonal of $K2D$ above. The eigenvectors are orthogonal :

Eigenvectors of $\boldsymbol{K}$2D $(1, 1, 1, 1)$ and $(1, 1, -1, -1)$, $(1, -1, 1, -1)$ and $(1, -1, -1, 1)$.

Symmetry of K2D guaranteed orthogonal eigenvectors. Positive definiteness produced those positive eigenvalues 2, 4, 4, 6.

Eigenvalues of the Laplacian : Continuous and Discrete

In one dimension, the eigenfunctions for $-u_{xx} = \lambda u$ are $\boldsymbol{u} = \sin \boldsymbol{n\pi x}$ with eigenvalue $\boldsymbol{\lambda} = \boldsymbol{n^2\pi^2}$. These sine functions are zero at the endpoints $x = 0$ and $x = 1$. On a unit square in two dimensions, the eigenfunctions of the Laplacian are just products of sines: $\boldsymbol{u(x,y)} = (\sin \boldsymbol{n\pi x})(\sin \boldsymbol{m\pi y})$ with eigenvalue $\boldsymbol{\lambda} = \boldsymbol{n^2\pi^2 + m^2\pi^2}$. Those functions are zero on the whole boundary of the square, where $x = 0$ or $x = 1$ or $y = 0$ or $y = 1$:

$$\boxed{-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sin n\pi x)(\sin m\pi y) = (\boldsymbol{n^2\pi^2 + m^2\pi^2})(\sin n\pi x)(\sin m\pi y).} \quad (10)$$

The problem on a square allows **separation of variables**. Each of the eigenvectors is a (function of x) times a (function of y). Two 1D problems, just what we hope for.

Equation (6) expressed $-u_{xx} - u_{yy}$ as $-$ div(grad u). **This is $\boldsymbol{A^{\mathrm{T}}A}$** ($A$ = gradient). The test $\lambda \geq 0$ is passed on non-square regions too, when the x, y variables don't separate.

Slope conditions (a derivative of u is zero instead of the function itself) allow the constant eigenfunction $u = 1$. Then $\lambda = 0$ and the Laplacian becomes *semidefinite*.

Turn now to the matrix Laplacian K2D. In one dimension, the eigenvectors of K are discrete sine vectors: Sample the continuous eigenfunction $\sin n\pi x$ at N equally spaced points. The spacing is $\Delta x = 1/(N+1)$ inside the interval from 0 to 1. The eigenvalues of K are $\lambda_n = 2 - 2\cos(n\pi\Delta x)$. We may hope and expect that the eigenvectors of K2D will contain products of sines, and the eigenvalues will be sums of 1D eigenvalues $\lambda(K)$.

The N^2 eigenvalues of K2D are positive. The x and y directions still separate.

$$\lambda_{nm}(K\text{2D}) = \lambda_n(K) + \lambda_m(K) = 4 - 2\cos\frac{n\pi}{N+1} - 2\cos\frac{m\pi}{N+1} > \mathbf{0}. \quad (11)$$

Thus K2D for a square is symmetric positive definite. This formula for the eigenvalues recovers $\lambda = 2, 4, 4, 6$ when $N = 2$, because the cosines of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ are $\frac{1}{2}$ and $-\frac{1}{2}$.

■ REVIEW OF THE KEY IDEAS ■

1. Laplace's equation is solved by the real and the imaginary part of every $(x+iy)^n$.

2. Those are $u = \boldsymbol{r^n} \cos \boldsymbol{n\theta}$ and $s = \boldsymbol{r^n} \sin \boldsymbol{n\theta}$. Their combinations are Fourier series.

3. The discrete equation is $\Delta_x^2 U + \Delta_y^2 U = 0$. The matrix K2D is positive definite.

4. Eigenvectors are (sines in x) (sines in y): $-u_{xx} - u_{yy} = \lambda u$ and $(K\text{2D})\,U = \lambda U$.

Problem Set 7.4

1 What solution to Laplace's equation completes "degree 3" in the table of pairs of solutions? We have one solution $u = x^3 - 3xy^2$, and we need another solution.

2 What are the two solutions of degree 4, the real and imaginary parts of $(x + iy)^4$? Check $u_{xx} + u_{yy} = 0$ for both solutions.

3 What is the second x-derivative of $(x + iy)^n$? What is the second y-derivative? Those cancel in $u_{xx} + u_{yy}$ because $i^2 = -1$.

4 For the solved 2×2 example inside a 4×4 square grid, write the four equations (9) at the four interior nodes. Move the known boundary values 0 and 4 to the right hand sides of the equations. You should see $K2\mathrm{D}$ on the left side multiplying the correct solution $\boldsymbol{U} = (U_{11}, U_{12}, U_{21}, U_{22}) = (1, 2, 2, 3)$.

5 Suppose the boundary values on the 4×4 grid change to $U = 0$ on three sides and $U = 8$ on the fourth side. Find the four inside values so that each one is the average of its neighbors.

6 (MATLAB) Find the inverse $(K2\mathrm{D})^{-1}$ of the 4 by 4 matrix $K2\mathrm{D}$ displayed for the square grid.

7 Solve this Poisson finite difference equation (right side $\neq 0$) for the inside values $U_{11}, U_{12}, U_{21}, U_{22}$. All boundary values like U_{10} and U_{13} are zero. The boundary has i or j equal to 0 or 3, the interior has i *and* j equal to 1 or 2:

$$4U_{ij} - U_{i-1,j} - U_{i+1,j} - U_{i,j-1} - U_{i,j+1} = \mathbf{1} \text{ at four inside points.}$$

8 A 5×5 grid has a 3 by 3 interior grid: 9 unknown values U_{11} to U_{33}. Create the 9×9 difference matrix $K2\mathrm{D}$.

9 Use eig($K2\mathrm{D}$) to find the nine eigenvalues of $K2\mathrm{D}$ in Problem 8. Those eigenvalues will be positive! The matrix $K2\mathrm{D}$ is symmetric positive definite.

10 If $u(x)$ solves $u_{xx} = 0$ and $v(y)$ solves $v_{yy} = 0$, verify that $u(x)v(y)$ solves Laplace's equation. Why is this only a 4-dimensional space of solutions? Separation of variables does not give all solutions—only the solutions with separable boundary conditions.

7.5 Networks and the Graph Laplacian

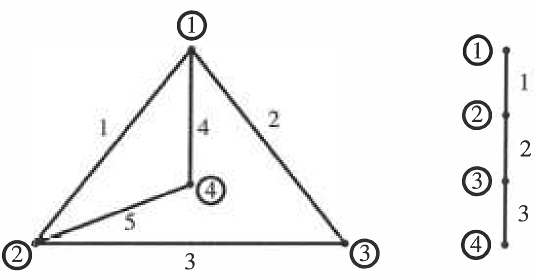
Start with a graph that has n nodes and m edges. Its m by n incidence matrix $\boldsymbol{A}$ was introduced in Section 5.6, with a row in the matrix for every edge in the graph. A single -1 and 1 in the row indicates which two nodes are connected by that edge. Now we take the step to $\boldsymbol{L} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ and $\boldsymbol{K} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{A}$. These are symmetric positive semidefinite matrices that describe the whole network.

Those matrices L and K are the **graph Laplacians**. L is unweighted (with $C = I$) and K is weighted by C. These are the fundamental matrices for *flows in the networks*. They describe electrical networks and their applications go very much further. You see $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}CA$ in descriptions of the brain and the Internet and our nervous system and the power grid.

Social networks and political networks and intellectual networks also use L and K. Graphs have simply become **the most important model in discrete applied mathematics**. This is not a standard topic in teaching linear algebra. But it is today an essential topic in applying linear algebra. It belongs in this book.

Examples of A and $A^{\mathrm{T}}A$

We quickly review incidence matrices, by constructing A for the planar graph and the line graph in Figure 7.6. You will see that every row of A adds to $-1+1=0$. Then the all-ones vector $\boldsymbol{v} = (1,\ldots,1)$ leads to $A\boldsymbol{v} = \boldsymbol{0}$. The columns of A are *dependent*, because their sum is the zero column. $A\boldsymbol{v} = \boldsymbol{0}$ propagates to $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{0}$ and $A^{\mathrm{T}}CA\boldsymbol{v} = \boldsymbol{0}$, so $A^{\mathrm{T}}CA$ for this A will be positive *semidefinite* (but not invertible and not positive definite).



Incidence matrix

$$A_{\textbf{line}} = \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix}$$

Figure 7.6: A planar graph and a line graph: $n = 4$ nodes and $m = 5$ or 3 edges.

$\boldsymbol{A}_{\textbf{line}}$ is a 3 by 4 *difference matrix*. Then $A^{\mathrm{T}}A$ below contains second differences. Notice that the first and last entries of $A^{\mathrm{T}}A$ are 1 and not 2. **The diagonal 1, 2, 2, 1 counts the number of edges that meet at each node** (the "degrees" of the four nodes).

$$\begin{aligned} &\boldsymbol{Av} \;\;\;\; = \textbf{difference of } \boldsymbol{v}\textbf{'s} \\ &\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \textbf{line Laplacian} \end{aligned} \qquad A\boldsymbol{v} = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_2 \\ v_4 - v_3 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} \mathbf{1} & -1 & 0 & 0 \\ -1 & \mathbf{2} & -1 & 0 \\ 0 & -1 & \mathbf{2} & -1 \\ 0 & 0 & -1 & \mathbf{1} \end{bmatrix} \qquad (1)$$

For the planar graph, the incidence matrix A again computes differences $\boldsymbol{v}_{\text{end}} - \boldsymbol{v}_{\text{start}}$ on every edge. The Laplacian matrix $L = A^{\mathrm{T}}A$ again has rows adding to zero. The diagonal of L shows $3, 3, 2, 2$ edges into the four nodes. Everything in A and L can be copied directly from the graph! The missing pair of -1 entries in $L = A^{\mathrm{T}}A$ is because *no edge connects nodes* 3 *and* 4 on the 5-edge graph.

$$\begin{array}{l} \textbf{Incidence matrix} \\ \textbf{Laplacian matrix} \end{array} \quad \boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & \mathbf{0} \\ -1 & -1 & \mathbf{0} & 2 \end{bmatrix} \qquad (2)$$

Note If any arrows change direction on the edges of the graph, this changes A. But $A^{\mathrm{T}}A$ *does not change*. The direction of arrows just multiplies A by a $\pm$ diagonal sign matrix S. Then $(SA)^{\mathrm{T}}(SA)$ is the same as $A^{\mathrm{T}}A$ because $S^{\mathrm{T}}S = I$.

The eigenvalues of $L = A^{\mathrm{T}}A$ always include $\lambda = 0$, from the all-ones eigenvector. The energy $\boldsymbol{v}^{\mathrm{T}}(A^{\mathrm{T}}A)\boldsymbol{v}$ can also be written as $(A\boldsymbol{v})^{\mathrm{T}}(A\boldsymbol{v})$. This just adds up the squares of all the entries of $A\boldsymbol{v}$, which are differences across edges (*not the missing edge from* 3 *to* 4):

$$\textbf{Energy} = (v_2 - v_1)^2 + (v_3 - v_1)^2 + (v_3 - v_2)^2 + (v_4 - v_1)^2 + (v_4 - v_2)^2.$$

We see again that the all-ones vector $\boldsymbol{v} = (1, 1, 1, 1)$ has zero energy.

The Laplacian matrix $L = A^{\mathrm{T}}A$ is *not invertible*! A system of equations $A^{\mathrm{T}}A\boldsymbol{v} = \boldsymbol{f}$ has no solution (or infinitely many). To reach an invertible matrix, **we remove the last column and row of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$**. This corresponds to "grounding a node" by setting the voltage at that node to be zero: $\boldsymbol{v_4} = \mathbf{0}$. It is like fixing one temperature at zero, when the equations only tell us about differences of temperature.

When we know that $v_4 = 0$, column 4 is removed from A. That removes column 4 and also row 4 from $A^{\mathrm{T}}A$. *This reduced* 3 *by* 3 *matrix is positive definite:*

$$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})_{\textbf{reduced}} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} = (A_{\text{reduced}})^{\mathrm{T}}(A_{\text{reduced}}) = (3 \text{ by } 5)\,(5 \text{ by } 3). \qquad (3)$$

The Weighted Laplacian $K = A^{\mathrm{T}}CA$

In many applications the edges come with positive weights $c_1, \ldots, c_m$. Those weights can be *conductances* (through m resistors) or *stiffnesses* (of m springs). In electrical engineering, Ohm's Law connects current w to voltage difference e. In mechanical engineering, Hooke's Law connects spring force w to the stretching e. **Those laws $\boldsymbol{w} = \boldsymbol{ce}$ in every edge give a positive diagonal matrix** C in $\boldsymbol{w} = C\boldsymbol{e} = CA\boldsymbol{v}$. The m currents in $\boldsymbol{w}$ come from the m voltage differences in $A\boldsymbol{v}$.

Kirchhoff's Current Law is $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{w} = \mathbf{0}$. That matrix A^{T} always enters the "balance of currents" and the "balance of forces" between springs. With current sources, or forces applied from outside, the balance equation is $A^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{f}$.

When current sources enter the nodes, the Current Law $A^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{f}$ is "*in equals out*." Then $A^{\mathrm{T}}C\boldsymbol{e} = \boldsymbol{f}$ and $A^{\mathrm{T}}CA\boldsymbol{v} = \boldsymbol{f}$. Thus $K = A^{\mathrm{T}}CA$ is the **conductance matrix for the whole network**. Here is $A^{\mathrm{T}}CA$ for the line of resistors:

$$\begin{aligned} A^{\mathrm{T}}\boldsymbol{w} &= \boldsymbol{f} \quad \textbf{(Kirchhoff)} \\ A^{\mathrm{T}}C\boldsymbol{e} &= \boldsymbol{f} \quad \textbf{(Ohm)} \\ A^{\mathrm{T}}CA\boldsymbol{v} &= \boldsymbol{f} \quad \textbf{(System)} \end{aligned} \qquad (A^{\mathrm{T}}CA)_{\text{line}} = \begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1+c_2 & -c_2 & 0 \\ 0 & -c_2 & c_2+c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix}. \tag{4}$$

The rows of $A^{\mathrm{T}}CA$ still add to zero. The matrix is still positive semidefinite. It becomes positive definite when row and column 4 are removed, which we must do to solve $A^{\mathrm{T}}CA\boldsymbol{v} = \boldsymbol{f}$. *This is a fundamental equation of discrete applied mathematics.*

A network can also have **voltage sources** (like batteries) on the edges. Those go into a vector $\boldsymbol{b}$ with m components. From node to node the voltage drops are $-A\boldsymbol{v}$ (with a minus sign). But Ohm's Law applies to the voltage drops $\boldsymbol{e}$ *across the resistors*. By working with the matrix C and including $\boldsymbol{b}$ in the vector $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{v}$, Ohm's Law is simply $\boldsymbol{w} = C\boldsymbol{e}$. The inputs to the network are $\boldsymbol{f}$ and $\boldsymbol{b}$.

The three equations for $\boldsymbol{e}$, $\boldsymbol{w}$, $\boldsymbol{f}$ use the matrices A, C, A^{T}. Those become *two* equations by eliminating $\boldsymbol{e} = C^{-1}\boldsymbol{w}$. We reach *one* equation by also eliminating $\boldsymbol{w}$.

	3 equations	2 equations	1 equation
Drop	$\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{v}$		
Current	$\boldsymbol{w} = C\boldsymbol{e}$	$\begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}$	$A^{\mathrm{T}}CA\boldsymbol{v} = A^{\mathrm{T}}C\boldsymbol{b} - \boldsymbol{f}$
Balance	$\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{w}$		

I removed $\boldsymbol{e}$ by substituting $\boldsymbol{e} = C^{-1}\boldsymbol{w}$ into the first equation. The step from two equations to one equation substituted $\boldsymbol{w} = C(\boldsymbol{b} - A\boldsymbol{v})$ into $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{w}$. Almost all entries of A and C will be zero. *The weighted graph Laplacian is* $K = A^{\mathrm{T}}CA$.

You see how the sources $\boldsymbol{b}$ and $\boldsymbol{f}$ produce the right side. They make the currents flow.

A Framework for Applied Mathematics

The least squares equation $A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b}$ and the weighted least squares equation $A^{\mathrm{T}}CA\boldsymbol{v} = A^{\mathrm{T}}C\boldsymbol{b}$ are special cases with $\boldsymbol{f} = \boldsymbol{0}$. My experience is that all the symmetric steady state problems of applied mathematics fit into this $A^{\mathrm{T}}CA$ framework.

Voltage Law $\rightarrow$ $\boldsymbol{A}$ | **Ohm's Law** $\rightarrow$ $\boldsymbol{C}$ | **Current Law** $\rightarrow$ $\boldsymbol{A}^{\mathrm{T}}$

I have learned to watch for $A^{\mathrm{T}}CA$ in every lecture about applied mathematics: it is there. Differential equations fit this framework too. Laplace's equation is $A^{\mathrm{T}}Au = 0$ when Au is the gradient of $u(x, y)$. *A typical $A^{\mathrm{T}}CA$ equation is* $-d/dx(c\,du/dx) = f(x)$.

For matrices, those derivatives become differences. The graph analogy with Laplace's equation gave the name *graph Laplacian* to the matrix $A^{\mathrm{T}}A$.

Dynamic problems have time derivatives du/dt. This adds a new step to the $A^{\mathrm{T}}CA$ framework. The equation $du/dt = -A^{\mathrm{T}}Au$ is a matrix analog of the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$. The next chapter will solve the heat equation using the eigenvalues and eigenfunctions (sines and cosines) from $y'' = \lambda y$. **The solutions are Fourier series.**

Example: A Network of Resistors

I will add resistors to the five edges of our four-node graph. The conductances $1/R$ will be the numbers c_1 to c_5. The conductance matrix for the whole network is $A^{\mathrm{T}}CA$. The incidence matrix A in equation (2) above is 5 by 4, and $A^{\mathrm{T}}CA$ is 4 by 4.

Conductance matrix $\boldsymbol{K}$ with five edges

$$A^{\mathrm{T}}CA = \begin{bmatrix} c_1+c_2+c_4 & -c_1 & -c_2 & -c_4 \\ -c_1 & c_1+c_3+c_5 & -c_3 & -c_5 \\ -c_2 & -c_3 & c_2+c_3 & \mathbf{0} \\ -c_4 & -c_5 & \mathbf{0} & c_4+c_5 \end{bmatrix} \tag{5}$$

Please compare this matrix to $A^{\mathrm{T}}A$ in equation (2), where all $c_i = 1$. The new matrix starts with $c_1+c_2+c_4$ because edges $1, 2, 4$ touch node 1. Along that row of K, the entries $-c_1, -c_2, -c_4$ produce *row sum = zero* as we expect. Then $A^{\mathrm{T}}CA$ is singular, not invertible. We must reduce the matrix to 3 by 3 by "grounding a node" and removing column 4 and row 4. The reduced matrix is symmetric positive definite.

Suppose the voltage $v_1 = V$ is fixed, as well as $v_4 = 0$ at the grounded node. Current will flow out of node 1 toward node 4 (with $\boldsymbol{b} = \boldsymbol{f} = \mathbf{0}$). *The terms c_1V and c_2V involving the known $v_1 = V$ move to the right hand side of $A^{\mathrm{T}}CA\boldsymbol{v} = \mathbf{0}$.* There are only two unknown voltages v_2 and v_3, and V is like a boundary value:

Reduced equations $\boldsymbol{v_1 = V}$ and $\boldsymbol{v_4 = 0}$

$$\begin{bmatrix} c_1+c_3+c_5 & -c_3 \\ -c_3 & c_2+c_3 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1V \\ c_2V \end{bmatrix}. \tag{6}$$

When we solve for v_2 and v_3, we know all four voltages $\boldsymbol{v}$ and all five currents $\boldsymbol{w} = CA\boldsymbol{v}$.

Summary

The matrix C changes an "ideal" $A^{\mathrm{T}}A$ problem into an "applied" $A^{\mathrm{T}}CA$ problem. You will see how this three-step framework appears all through applied mathematics. Au is often a derivative of u, or a finite difference. Then CAu comes from Ohm's Law or Hooke's Law. The material constants like conductance and stiffness go into C.

Finally $A^{\mathrm{T}}CA\boldsymbol{v} = \boldsymbol{f}$ is a continuity equation or a balance equation. It represents balance of forces, balance of inputs with outputs, balance of profits with losses. The combined matrix $K = A^{\mathrm{T}}CA$ is symmetric positive definite just like $A^{\mathrm{T}}A$.

To find the forces or the flows inside the network, we solve for $\boldsymbol{v}$ and $\boldsymbol{e}$ and $\boldsymbol{w}$.

The Adjacency Matrix

The Laplacian matrices $L = A^{\mathrm{T}}A$ and $K = A^{\mathrm{T}}CA$ started with the incidence matrix A. The diagonal of L has the degree of each node: the number of edges that touch the node. $A^{\mathrm{T}}A$ also comes directly from the **degree matrix D minus the adjacency matrix W** :

$$A^{\mathrm{T}}A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

The degrees $3, 3, 2, 2$ in D are the row sums in W. Then $D - W$ has zero row sums. When $L = A^{\mathrm{T}}A = D - W$ multiplies $(1, 1, 1, 1)$ the result will be $(0, 0, 0, 0)$.

Question The sum of the degrees is 10. How can this be predicted from the graph?

Answer The graph has five edges. Each edge produces two 1's in the adjacency matrix. There must be ten 1's in W. The degrees in D must add to 10, to balance the 1's in W.

Since the *trace* of L is $3 + 3 + 2 + 2$, the eigenvalues of L must also add to 10.

Question What is the rule for W and D when there are weights $c_1, \ldots, c_m$ on the edges?

Answer Each entry $W_{ij} = 1$ comes from an edge between node i and node j. When this edge k has a weight c_k (the conductance along the edge), the entry W_{ij} changes from 1 to c_k. The weights produce $A^{\mathrm{T}}CA$ in equation (5) and also in equation (8).

$A^{\mathrm{T}}CA = K$ with weights

$$D - W = \begin{bmatrix} c_1 + c_2 + c_4 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & c_4 + c_5 \end{bmatrix} - \begin{bmatrix} 0 & c_1 & c_2 & c_4 \\ c_1 & 0 & c_3 & c_5 \\ c_2 & c_3 & 0 & 0 \\ c_4 & c_5 & 0 & 0 \end{bmatrix}. \tag{8}$$

Problems $1 - 5$ will ask about a **complete graph,** when every pair of nodes is connected by an edge. All off-diagonal entries in the adjacency matrix W are 1. All the degrees in the diagonal D are $n - 1$. The Laplacians L and K have no zeros. Every question about $L = A^{\mathrm{T}}A = D - W$ has a good answer for this graph with all possible edges.

Here is a picture that summarizes this three-step vision of applied mathematics.

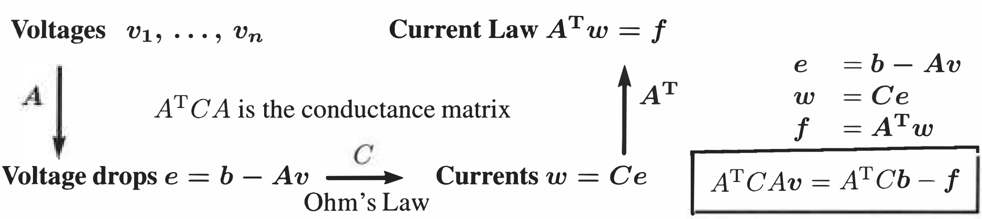


Figure 7.7: The $A^{\mathrm{T}}CA$ framework for steady state problems in science and engineering.

Saddle-Point Matrix

The final matrix is $A^{\mathrm{T}}CA$, after the edge currents $w_1, \ldots, w_m$ are eliminated. Before we took that step, the voltages $\boldsymbol{v}$ and the currents $\boldsymbol{w}$ were the two unknown vectors. With two equations we have a "saddle-point matrix" that contains C^{-1} and A and A^{T}:

Saddle-point problem
Currents and voltages
$$\begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}. \qquad (9)$$

Block matrices of this form appear when there is a constraint like Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{f}$. "Nature minimizes heat loss in the network subject to that constraint." The "KKT matrix" in (9) is symmetric but it is *not at all positive definite*.

A small example will show a positive and also a negative eigenvalue:

$$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \text{ has eigenvalues } 4 \text{ and } -1. \quad \text{The pivots are } 3 \text{ and } -\frac{4}{3}.$$

Eigenvalues and pivots have the same signs! Multiply the eigenvalues or the pivots to reach the determinant -4. The zero on the diagonal rules out positive definiteness.

The saddle-point matrix has m positive and n negative eigenvalues. The energy in $(m+n)$-dimensional space goes upward in m directions and downward in n directions.

An important computational decision has voters on both sides. Is it better to eliminate $\boldsymbol{w}$ and work with one matrix $A^{\mathrm{T}}CA$? Optimizers say no, finite element engineers say yes. Fluids calculations (with pressure dual to velocity) often look for the saddle point.

Computational science and engineering is a highly active subject, a mix of software and hardware and mathematics in solving $A^{\mathrm{T}}CA$ equations with millions of unknowns.

■ REVIEW OF THE KEY IDEAS ■

1. Row k of A (m by n) tells the start node and the end node of edge k in the graph.
2. The Laplacian $L = A^{\mathrm{T}}A$ has $L_{ij} = -1$ when an edge connects nodes i and j.
3. The diagonal of $L = D - W$ shows the degrees of the nodes. Each row adds to zero.
4. With weights c_k on the edges, $K = A^{\mathrm{T}}CA$ is the weighted graph Laplacian.
5. Three steps $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{v}$, $\boldsymbol{w} = C\boldsymbol{e}$, $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{w}$ combine into $A^{\mathrm{T}}CA\boldsymbol{v} = A^{\mathrm{T}}C\boldsymbol{b} - \boldsymbol{f}$.

Problem Set 7.5

Problems 1 — 5 are about complete graphs. Every pair of nodes has an edge.

1 With $n = 5$ nodes and all edges, find the diagonal entries of $A^{\mathrm{T}}A$ (the degrees of the nodes). All the off-diagonal entries of $A^{\mathrm{T}}A$ are -1. Show the reduced matrix R without row 5 and column 5. Node 5 is "grounded" and $v_5 = 0$.

2 Show that the *trace* of $A^{\mathrm{T}}A$ (sum down the diagonal $=$ sum of eigenvalues) is $n^2 - n$. What is the trace of the reduced (and invertible) matrix R of size $n - 1$?

3 For $n = 4$, write the 3 by 3 matrix $R = (A_{\text{reduced}})^{\mathrm{T}}(A_{\text{reduced}})$. Show that $RR^{-1} = I$ when R^{-1} has all entries $\frac{1}{4}$ off the diagonal and $\frac{2}{4}$ on the diagonal.

4 For every n, the reduced matrix R of size $n - 1$ is *invertible*. Show that $RR^{-1} = I$ when R^{-1} has all entries $1/n$ off the diagonal and $2/n$ on the diagonal.

5 Write the 6 by 3 matrix $M = A_{\text{reduced}}$ when $n = 4$. The equation $M\boldsymbol{v} = \boldsymbol{b}$ is to be solved by least squares. The vector $\boldsymbol{b}$ is like scores in 6 games between 4 teams (team 4 always scores zero; it is grounded). Knowing the inverse of $R = M^{\mathrm{T}}M$, what is the least squares ranking $\widehat{v}_1$ for team 1 from solving $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$?

6 For the tree graph with 4 nodes, $A^{\mathrm{T}}A$ is in equation (1). What is the 3 by 3 matrix $R = (A^{\mathrm{T}}A)_{\text{reduced}}$? How do we know it is positive definite?

7 (a) If you are given the matrix A, how could you reconstruct the graph?

(b) If you are given $L = A^{\mathrm{T}}A$, how could you reconstruct the graph (no arrows) ?

(c) If you are given $K = A^{\mathrm{T}}CA$, how could you reconstruct the weighted graph?

8 Find $K = A^{\mathrm{T}}CA$ for a line of 3 resistors with conductances $c_1 = 1$, $c_2 = 4$, $c_3 = 9$. Write K_{reduced} and show that this matrix is positive definite.

9 A 3 by 3 square grid has $n = 9$ nodes and $m = 12$ edges. Number nodes by rows.

(a) How many nonzeros among the 81 entries of $L = A^{\mathrm{T}}A$?

(b) Write down the 9 diagonal entries in the degree matrix D: they are not all 4.

(c) Why does the middle row of $L = D - W$ have four -1's ? Notice $L = K$2D !

10 Suppose all conductances in equation (5) are equal to c. Solve equation (6) for the voltages v_2 and v_3 and find the current I flowing out of node 1 (and into the ground at node 4). What is the "system conductance" I/V from node 1 to node 4 ?

This overall conductance I/V should be larger than the individual conductances c.

11 The multiplication $A^{\mathrm{T}}A$ can be columns of A^{T} times rows of A. For the tree with $m = 3$ edges and $n = 4$ nodes, each (column times row) is $(4 \times 1)(1 \times 4) = 4 \times 4$. Write down those three column-times-row matrices and add to get $L = A^{\mathrm{T}}A$.

12 A graph with two separate 3-node trees is *not connected*. Write its 6 by 4 incidence matrix A. Find *two* solutions to $A\boldsymbol{v} = \boldsymbol{0}$, not just one solution $\boldsymbol{v} = (1, 1, 1, 1, 1, 1)$. To reduce $A^{\mathrm{T}}A$ we must ground *two* nodes and remove two rows and columns.

13 "Element matrices" from column times row appear in the **finite element method**. Include the numbers c_1, c_2, c_3 in the element matrices K_1, K_1, K_3.

$$K_i = (\text{row } i \text{ of } A)^{\mathrm{T}} \ (\boldsymbol{c_i}) \ (\text{row } i \text{ of } A) \qquad K = A^{\mathrm{T}}CA = \boldsymbol{K_1} + \boldsymbol{K_2} + \boldsymbol{K_3}.$$

Write the element matrices that add to $A^{\mathrm{T}}A$ in (1) for the 4-node line graph.

$$A^{\mathrm{T}}A = \left[\begin{array}{cccc} \left[K_1\right] & & & \\ & \left[K_2\right] & & \\ & & \left[K_3\right] & \\ & & & \end{array}\right] = \begin{array}{l}\text{assembly of the nonzero}\\ \text{entries of } K_1 + K_2 + K_3\\ \text{from edges } 1, 2, \text{ and } 3\end{array}$$

14 An n by n grid has n^2 nodes. How many edges in this graph? How many interior nodes? How many nonzeros in A and in $L = A^{\mathrm{T}}A$? *There are no zeros in* L^{-1}!

15 When only $\boldsymbol{e} = C^{-1}\boldsymbol{w}$ is eliminated from the 3-step framework, equation (9) shows

$$\begin{array}{l}\textbf{Saddle-point matrix}\\ \textbf{Not positive definite}\end{array} \qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first block row by $A^{\mathrm{T}}C$ and subtract from the second block row:

$$\textbf{After block elimination} \qquad \begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} - A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

After m positive pivots from C^{-1}, why does this matrix have negative pivots? The two-field problem for $\boldsymbol{w}$ and $\boldsymbol{v}$ is finding a saddle point, not a minimum.

16 The least squares equation $A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b}$ comes from the projection equation $A^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{0}$ for the error $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{v}$. Write those two equations in the symmetric saddle point form of Problem 15 (with $\boldsymbol{f} = \boldsymbol{0}$).

In this case $\boldsymbol{w} = \boldsymbol{e}$ because the weighting matrix is $C = I$.

17 Find the three eigenvalues and three pivots and the determinant of this saddle point matrix with $C = I$. One eigenvalue is negative because A has one column:

$$m = 2, n = 1 \qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

■ CHAPTER 7 NOTES ■

Polar Form of an Invertible Matrix: $A = QS$ = (orthogonal) (positive definite). This is like $re^{i\theta}$ for complex numbers (1 by 1 matrices). $|e^{i\theta}| = 1$ is the orthogonal Q and $r > 0$ is the positive definite S. The matrix factors come directly from the Singular Value Decomposition of A:

$$A = U\Sigma V^{\mathrm{T}} = \left(UV^{\mathrm{T}}\right)\left(V\Sigma V^{\mathrm{T}}\right) = \text{(orthogonal)} \quad \text{times} \quad \text{(positive definite)}.$$

When A is invertible, so is Σ. Then σ_1 to σ_n are the (positive) eigenvalues of $V\Sigma V^{\mathrm{T}}$. In physical language, every motion combines a rotation/reflection Q with a stretching S.

Transpose of $\boldsymbol{A = d/dx}$. It is not enough to say that "the transpose is $-d/dx$." The boundary conditions on the functions f and g in $Af = df/dx$ and $A^{\mathrm{T}}g = -dg/dx$ are important parts of A and A^{T}. In Section 7.3 and especially Problem 1, A comes with *two* conditions $f(0) = 0$ and $f(1) = 0$. Then $A^{\mathrm{T}} = -d/dx$ has no conditions on g. What we want is $(Af, g) = (f, A^{\mathrm{T}}g)$.

Integration by parts is like transposing the operator $\boldsymbol{d/dx}$. The integrated term fg is safely zero when $f(0) = f(1) = 0$. The *fixed-free* operator d/dx with only one condition $f(0) = 0$ would transpose to the *free-fixed* operator $-d/dx$ with the other condition $g(1) = 0$. Then the integrated term is again $fg = 0$ at both ends. In each case, *boundary conditions on* $\boldsymbol{g}$ *make up for missing boundary conditions on* f.

Principal Component Analysis (PCA): Find the most significant (least random) data.

Data often comes in rectangular matrices: A grade for each student in each course. Activity of each gene in each disease. Sales of each product in each store. Income in each age group in each city. An entry goes into each column and each row of the data matrix.

By subtracting off the means, we study the *variances*: measures of useful information as opposed to randomness. The SVD of the data matrix A (showing the eigenvectors and eigenvalues of the correlation $A^{\mathrm{T}}A$) displays the **principal component**: the largest piece $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$ of the matrix. The orthogonal pieces $\sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}$ are in order of importance. The largest σ is the most significant. From a large matrix of partly random data, PCA and the SVD extract its most significant information.

Wikipedia lists many methods that are identical or closely related to PCA. The crucial singular vector $\boldsymbol{v}_1$ (which has $A^{\mathrm{T}}A\boldsymbol{v}_1 = \lambda_{\max}\boldsymbol{v}_1$) is also the vector that maximizes the Rayleigh quotient $(\boldsymbol{v}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{v})/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Computing the first few singular vectors does not require the whole SVD!

Chapter 8

Fourier and Laplace Transforms

This book began with linear differential equations. It will end that way. Those are the equations we can understand and solve—especially when the coefficients are constant. Even the heat equation and wave equation (*those are PDE's*) have good solutions.

These are extremely nice problems, no apologies for that. Almost every application starts with a linear response—current proportional to voltage, output proportional to input. For large voltages or large forces, the true law may become nonlinear. Even then, we often use a sequence of linear problems to deal with nonlinearity. The constant coefficient linear equation is the one we can solve.

This chapter introduces Fourier transforms and Laplace transforms. They express every input $f(x)$ and $f(t)$ and every output $y(x)$ and $y(t)$ as a **combination of exponentials**. For each exponential, the output multiplies the input by a constant that depends on the frequency: $y(t) = Y(s)e^{st}$ or $Y(\omega)e^{i\omega t}$. **That transfer function describes the system by its frequency response : the constants Y that multiply exponentials.**

We have used the complex gain $1/(i\omega - a)$ to invert $y' - ay$, along with transfer functions in Chapters 1 and 2. Now we see them for every time-invariant and shift-invariant partial differential equation—with coefficients that are constant in time and space.

Naturally those ideas appear again for discrete problems with matrix equations. The matrices may be approximating derivatives (like the $-1, 2, -1$ second difference matrix). Or they come on their own from convolutions. Their eigenvectors will be discrete sines or cosines or complex exponentials. A combination of those eigenvectors is a *discrete Fourier series* (**DFT**). We find the coefficients in that combination by using the Fast Fourier Transform (**FFT**)— **the most important algorithm in modern applied mathematics**.

A note about sines and cosines versus complex exponentials. For real problems we may like sines and cosines. But they aren't perfect. We keep $\cos 0$ and we don't keep $\sin 0$. We want one of the highest frequency vectors $(1, -1, 1, -1, \ldots)$ and $(-1, 1, -1, 1, \ldots)$ but not both. In the end (and almost always for the FFT) *the complex exponentials win*. After all, they are eigenfunctions of the derivative d/dx. Transforms are based on combinations of those exponentials—and the derivative of $e^{i\omega x}$ is just $i\omega e^{i\omega x}$.

This page describes a specially nice function space. It is called "*Hilbert space*." **The functions have dot products and lengths**. There are angles between functions, so two functions can be orthogonal (perpendicular). The functions in Hilbert space are just like vectors. In fact *they are vectors*—but Hilbert space is infinite-dimensional.

Here are parallels between real vectors $\boldsymbol{f} = (f_1, \ldots, f_N)$ and real functions $f(x)$. Physicists even separate $< f|$ (bra) from $|g >$ (ket). Not here!

Inner product	$\boldsymbol{f}^{\mathrm{T}}\boldsymbol{g} = f_1 g_1 + \cdots + f_N g_N$	$< \boldsymbol{f}, \boldsymbol{g} > = \int_{-\pi}^{\pi} f(x)\, g(x) dx$
Length squared	$\|\|\boldsymbol{f}\|\|^2 = \boldsymbol{f}^{\mathrm{T}}\boldsymbol{f} = \sum \|f_i\|^2$	$\|\|\boldsymbol{f}\|\|^2 = < f, f > = \int_{-\pi}^{\pi} \|f(x)\|^2 dx$
Angle θ	$\cos\, \theta = \boldsymbol{f}^{\mathrm{T}}\boldsymbol{g}/\|\|\boldsymbol{f}\|\|\, \|\|\boldsymbol{g}\|\|$	$\cos \theta = < f, g > / \|\|f\|\|\, \|\|g\|\|$
Orthogonality	$\boldsymbol{f}^{\mathrm{T}}\boldsymbol{g} = \mathbf{0}$	$< \boldsymbol{f}, \boldsymbol{g} > = \int_{-\pi}^{\pi} f(x)\, g(x) dx = \mathbf{0}$

A function is allowed into Hilbert space if it has a finite length: $\int |f(x)|^2 dx < \infty$. Thus $f(x) = 1/x$ and $f(x) = \delta(x)$ do *not* belong to Hilbert space. But a step function is good. And the function can even blow up at a point—just not too fast. For example $f(x) = 1/|x|^{1/4}$ belongs to Hilbert space and its length is $||f|| = 2\pi^{1/4}$:

$$f(0) \text{ is infinite but } ||f||^2 = \int_{-\pi}^{\pi} |x|^{-1/2} dx = 4\, |x|^{1/2}\Big]_0^{\pi} = 4\pi^{1/2}.$$

When $|f(x)| = |f(-x)|$, the integral from $-\pi$ to π is twice the integral from 0 to π.

There is always an adjustment for *complex* vectors and functions:

$$\textbf{Inner product} \quad \overline{\boldsymbol{f}}^{\mathrm{T}}\boldsymbol{g} = \overline{f}_1 g_1 + \cdots + \overline{f}_N g_N \qquad < \boldsymbol{f}, \boldsymbol{g} > = \int \overline{\boldsymbol{f(x)}}\boldsymbol{g(x)dx}$$

Orthogonality is still $< \boldsymbol{f}, \boldsymbol{g} > = \mathbf{0}$. The best examples are the complex exponentials:

$$\boldsymbol{e^{ikx}} \textbf{ and } \boldsymbol{e^{inx}} \textbf{ are orthogonal} \qquad \int_{-\pi}^{\pi} e^{-ikx} e^{inx} dx = \frac{e^{i(n-k)x}}{n-k}\Big|_{-\pi}^{\pi} = \mathbf{0}.$$

Those e^{ikx} are an **orthogonal basis** for Hilbert space. Instead of xyz axes, functions need infinitely many axes. Every $f(x)$ is a combination of the basis vectors e^{ikx}:

$$f(x) = \frac{e^{ix} - e^{-ix}}{1} + \frac{e^{3ix} - e^{-3ix}}{3} + \cdots \quad \text{has} \quad \int_{-\pi}^{\pi} |f(x)|^2 = 2\pi(1^2 + 1^2 + \frac{1}{3^2} + \frac{1}{3^2} + \cdots).$$

This particular $f(x)$ happens to be a step function. To Hilbert, step functions are vectors. Then Fourier "transformed" $f(x)$ into the numbers (like 1 and $\frac{1}{3}$) that multiply each e^{ikx}.

8.1 Fourier Series

This section explains three Fourier series: **sines, cosines, and exponentials e^{ikx}**. Square waves (1 or 0 or -1) are great examples, with delta functions in the derivative. We look at a spike, a step function, and a ramp—and smoother functions too.

Start with $\sin x$. It has period 2π since $\sin(x+2\pi) = \sin x$. It is an odd function since $\sin(-x) = -\sin x$, and it vanishes at $x = 0$ and $x = \pi$. Every function $\sin nx$ has those three properties, and Fourier looked at *infinite combinations of the sines*:

Fourier sine series $$S(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{n=1}^{\infty} \boldsymbol{b_n \sin nx} \tag{1}$$

If the numbers $b_1, b_2, b_3, \ldots$ drop off quickly enough (we are foreshadowing the importance of their decay rate) then the sum $S(x)$ will inherit all three properties:

Periodic $S(x+2\pi) = S(x)$ **Odd** $S(-x) = -S(x)$ $S(0) = S(\pi) = 0$

200 years ago, Fourier startled the mathematicians in France by suggesting that *any odd periodic function* $S(x)$ could be expressed as an infinite series of sines. This idea started an enormous development of Fourier series. Our first step is to **find the number b_k that multiplies $\sin kx$. The function $S(x)$ is "transformed" to a sequence of b's.**

Suppose $S(x) = \sum b_n \sin nx$. *Multiply both sides by* $\sin kx$. *Integrate from* 0 *to* π:

$$\int_0^{\pi} S(x) \sin kx \, dx = \int_0^{\pi} b_1 \sin x \sin kx \, dx + \cdots + \int_0^{\pi} \boldsymbol{b_k \sin kx \ \sin kx \, dx} + \cdots \tag{2}$$

On the right side, all integrals are zero except the highlighted one with $n = k$. This property of "**orthogonality**" will dominate the whole chapter. For sines, integral $= 0$ is a fact of calculus:

Sines are orthogonal $$\int_0^{\pi} \boldsymbol{\sin nx \ \sin kx \, dx = 0} \quad \text{if} \quad n \neq k\,. \tag{3}$$

Zero comes quickly if we integrate $\int \cos mx \, dx = \left[\frac{\sin mx}{m}\right]_0^{\pi} = 0 - 0$. So we use this:

Product of sines $$\sin nx \ \sin kx = \frac{1}{2}\cos(n-k)x - \frac{1}{2}\cos(n+k)x\,. \tag{4}$$

Integrating $\cos\,(n-k)x$ and $\cos\,(n+k)x$ gives zero, proving orthogonality of the sines.

The exception is when $n = k$. Then we are integrating $(\sin kx)^2 = \frac{1}{2} - \frac{1}{2}\cos 2kx$:

$$\int_0^{\pi} \sin kx \ \sin kx \, dx = \int_0^{\pi} \frac{1}{2} \, dx - \int_0^{\pi} \frac{1}{2} \cos 2kx \, dx = \frac{\pi}{2}\,. \tag{5}$$

The highlighted term in equation (2) is $\boldsymbol{(\pi/2) b_k}$. Multiply both sides by $2/\pi$ to find b_k.

Sine coefficients $S(-x) = -S(x)$

$$b_k = \frac{2}{\pi}\int_0^{\pi} S(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \boldsymbol{S(x)\sin kx\,dx}. \tag{6}$$

Notice that $S(x)\sin kx$ is *even* (equal integrals from $-\pi$ to 0 and from 0 to π).

I will go immediately to the most important example of a Fourier sine series. $S(x)$ is an **odd square wave** with $SW(x) = 1$ for $0 < x < \pi$. It is drawn in Figure 8.1 as an odd function (with period 2π) that vanishes at $x = 0$ and $x = \pi$.

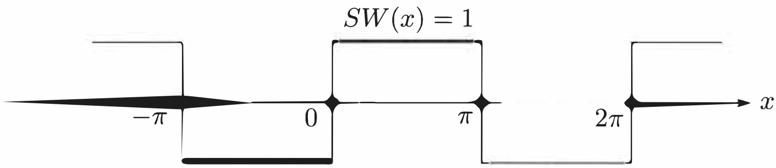


Figure 8.1: The odd square wave with $SW(x + 2\pi) = SW(x) = \{1 \text{ or } 0 \text{ or } -1\}$.

Example 1 Find the Fourier sine coefficients b_k of the odd square wave $SW(x)$.

Solution For $k = 1, 2, \ldots$ use formula (6) with $S(x) = 1$ between 0 and π:

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin kx\,dx = \frac{2}{\pi}\left[\frac{-\cos kx}{k}\right]_{\mathbf{0}}^{\boldsymbol{\pi}} = \frac{2}{\pi}\left\{\frac{\mathbf{2}}{\mathbf{1}}, \frac{0}{2}, \frac{\mathbf{2}}{\mathbf{3}}, \frac{0}{4}, \frac{\mathbf{2}}{\mathbf{5}}, \frac{0}{6}, \ldots\right\} \tag{7}$$

The even-numbered coefficients b_{2k} are all zero because $\cos 2k\pi = \cos 0 = 1$. The odd-numbered coefficients $b_k = \boldsymbol{4/\pi k}$ decrease at the rate $1/k$. We will see that same $1/k$ decay rate for all functions formed from *smooth pieces and jumps*.

Put those coefficients $4/\pi k$ and zero into the Fourier sine series for $SW(x)$:

$$\textbf{Square wave} \qquad \boldsymbol{SW(x)} = \frac{4}{\pi}\left[\frac{\sin x}{\mathbf{1}} + \frac{\sin 3x}{\mathbf{3}} + \frac{\sin 5x}{\mathbf{5}} + \frac{\sin 7x}{\mathbf{7}} + \cdots\right] \tag{8}$$

Figure 8.2 graphs this sum after one term, then two terms, and then five terms. You can see the all-important **Gibbs phenomenon** appearing as these "partial sums" include more terms. Away from the jumps, we safely approach $SW(x) = 1$ or -1. At $x = \pi/2$, the series gives a beautiful alternating formula for the number π :

$$1 = \frac{4}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] \quad \text{so that} \quad \boldsymbol{\pi} = 4\left(\frac{\mathbf{1}}{\mathbf{1}} - \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{7}} + \cdots\right). \tag{9}$$

The Gibbs phenomenon is the overshoot that moves closer and closer to the jumps. Its height approaches $1.18\ldots$ and it does not decrease with more terms of the series. This overshoot is the one greatest obstacle to calculation of all discontinuous functions (like shock waves). We try hard to avoid Gibbs but sometimes we can't.

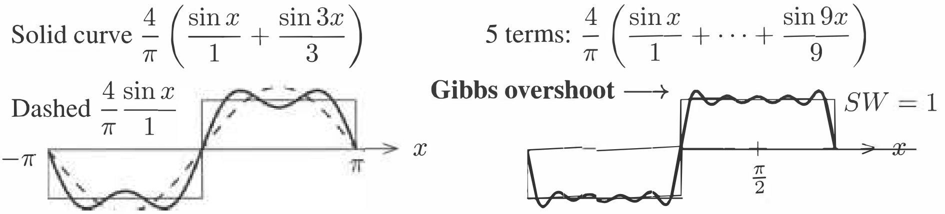


Figure 8.2: The sums $b_1 \sin x + \cdots + b_N \sin Nx$ overshoot the square wave near jumps.

Fourier Cosine Series

The cosine series applies to ***even functions*** $C(x) = C(-x)$. They are symmetric across 0:

Cosine series $$C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots = \boldsymbol{a_0 + \sum_{n=1}^{\infty} a_n \cos nx}. \qquad (10)$$

Every cosine has period 2π. Figure 8.3 shows two even functions, the **repeating ramp** $RR(x)$ and the **up-down train** $UD(x)$ of delta functions. That sawtooth ramp RR is the integral of the square wave. The delta functions in UD give the derivative of the square wave. (For sines, the integral and derivative are cosines.) RR and UD will be valuable examples, one smoother than SW, one less smooth.

First we find formulas for the cosine coefficients a_0 and a_k. *The constant term a_0 is the average value of the function $C(x)$*:

$$\boxed{\boldsymbol{a_0} = \textbf{average} \qquad \boldsymbol{a_0} = \frac{1}{\pi}\int_0^{\pi} C(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(x)\,dx.} \qquad (11)$$

I just integrated every term in the cosine series (10) from 0 to π. On the right side, the integral of a_0 is $a_0\pi$ (divide both sides by π). All other integrals are zero:

$$\int_0^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_0^{\pi} = 0 - 0 = 0. \qquad (12)$$

In words, the constant function 1 is orthogonal to $\cos nx$ over the interval $[0, \pi]$.

The other cosine coefficients a_k come from the *orthogonality of cosines*. As with sines, we multiply both sides of (10) by $\cos kx$ and integrate from 0 to π:

$$\int_0^{\pi} C(x)\cos kx\,dx = \int_0^{\pi} a_0 \cos kx\,dx + \int_0^{\pi} a_1 \cos x \cos kx\,dx + \cdots + \int_0^{\pi} \boldsymbol{a_k(\cos kx)^2\,dx} + \cdots$$

You know what is coming. On the right side, only the highlighted term can be nonzero. For $k > 0$, that bold nonzero term is $\boldsymbol{a_k\pi/2}$. Multiply both sides by $2/\pi$ to find a_k:

Cosine coefficients
$\boldsymbol{C(-x) = C(x)}$
$$a_k = \frac{2}{\pi}\int_0^{\pi} C(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \boldsymbol{C(x)\cos kx\,dx}. \qquad (13)$$

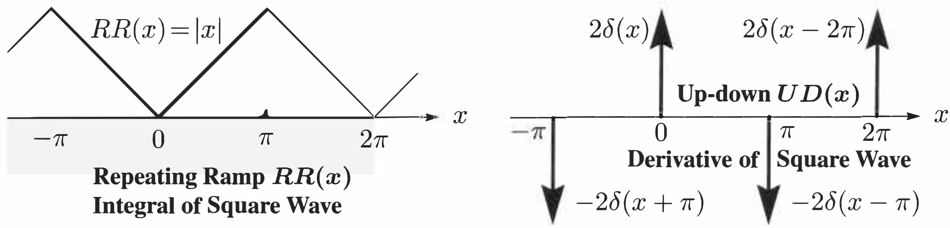


Figure 8.3: The repeating ramp RR and the up-down UD (periodic spikes) are even. The slope of RR is -1 then 1: odd square wave SW. **The next derivative is UD**: $\pm 2\delta$.

Example 2 Find the cosine coefficients of the ramp $RR(x)$ and the up-down $UD(x)$.

Solution The simplest way is to start with the sine series for the square wave:

$$SW(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right] = \text{slope of } RR$$

Take the derivative of every term to produce cosines in the up-down delta function:

Up-down spikes $$\boldsymbol{UD(x)} = \frac{4}{\pi}\left[\cos x + \cos 3x + \cos 5x + \cos 7x + \cdots\right]. \tag{14}$$

Those coefficients don't decay at all. The terms in the series don't approach zero, so officially the series cannot converge. Nevertheless it is correct and important. At $x = 0$, the cosines are all 1 and their sum is $+\infty$. At $x = \pi$, the cosines are all -1. Then their sum is $-\infty$. (The downward spike is $-2\delta(x-\pi)$.) The true way to recognize $\delta(x)$ is by the integral test $\int \delta(x) f(x)\,dx = f(0)$ and Example 3 will do this.

For the repeating ramp, we integrate the square wave series for $SW(x)$ and add a_0. The average ramp height is $a_0 = \pi/2$, halfway from 0 to π:

Ramp series $$\boldsymbol{RR(x)} = \frac{\pi}{2} - \frac{\pi}{4}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right]. \tag{15}$$

The constant of integration is a_0. *Those coefficients a_k drop off like $1/k^2$.* They could be computed directly from formula (13) using $\int x \cos kx\,dx$, and integration by parts (or an appeal to *Mathematica* or *Maple*). It was much easier to integrate every sine separately in $SW(x)$, which makes clear the crucial point: **Each "degree of smoothness" in the function brings a faster decay rate of its Fourier coefficients a_k and b_k.** Every integration divides those numbers by k.

No decay	**Delta** functions (with spikes)
$1/k$ decay	**Step** functions (with jumps)
$1/k^2$ decay	**Ramp** functions (with corners)
$1/k^4$ decay	**Spline** functions (jumps in f''')
r^k decay with $r < 1$	**Analytic** functions like $1/(2-\cos x)$

The Fourier Series for a Delta Function

Example 3 Find the (cosine) coefficients of the ***delta function*** $\boldsymbol{\delta(x)}$, made 2π-periodic.

Solution The spike in $\delta(x)$ occurs at $x = 0$. All the integrals are 1, because the cosine of 0 is 1. We divide by 2π for a_0 and by π for the other cosine coefficients a_k.

$$\textbf{Average}\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)\,dx = \frac{\mathbf{1}}{\mathbf{2\pi}} \qquad \textbf{Cosines}\quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\cos kx\,dx = \frac{\mathbf{1}}{\boldsymbol{\pi}}$$

Then the series for the delta function has *all cosines in equal amounts*: **No decay**.

Delta function
$$\boldsymbol{\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left[\cos x + \cos 2x + \cos 3x + \cdots\right].} \qquad (16)$$

This series cannot truly converge (its terms don't approach zero). But we can graph the sum after $\cos 5x$ and after $\cos 10x$. Figure 8.4 shows how these "partial sums" are doing their best to approach $\delta(x)$. They oscillate faster while going higher.

There is a neat formula for the sum δ_N that stops at $\cos Nx$. Start by writing each term $2\cos x$ as $e^{ix} + e^{-ix}$. We get a geometric progression from e^{-iNx} up to e^{iNx}.

$$\delta_N = \frac{1}{2\pi}\left[1 + e^{ix} + e^{-ix} + \cdots + e^{iNx} + e^{-iNx}\right] = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}. \qquad (17)$$

This is the function graphed in Figure 8.4.

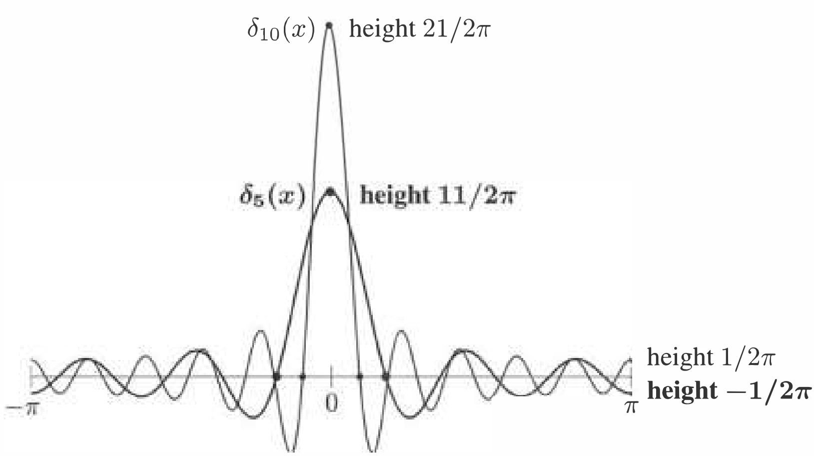


Figure 8.4: The sums $\delta_N(x) = (1 + 2\cos x + \cdots + 2\cos Nx)/2\pi$ try to approach $\delta(x)$.

Complete Series: Sines and Cosines

Over the half-period $[0, \pi]$, the sines are not orthogonal to all the cosines. In fact the integral of $\sin x$ times 1 is not zero. So for functions $F(x)$ that are not odd or even, we must move to the *complete series* (*sines plus cosines*) on the full interval. Since our functions are periodic, that "full interval" can be $[-\pi, \pi]$ or $[0, 2\pi]$. We have both a's and b's.

Complete Fourier series $$F(x) = \boldsymbol{a_0} + \sum_{n=1}^{\infty} \boldsymbol{a_n \cos nx} + \sum_{n=1}^{\infty} \boldsymbol{b_n \sin nx}\,. \tag{18}$$

On every "2π interval" the sines and cosines are orthogonal. We find the Fourier coefficients a_k and b_k in the usual way: **Multiply** (18) **by** 1 **and** $\cos kx$ **and** $\sin kx$. **Then integrate both sides from** $-\pi$ **to** π **to get** a_0 **and** a_k **and** b_k.

$$\boxed{\boldsymbol{a_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx \quad \boldsymbol{a_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\,dx \quad \boldsymbol{b_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\,dx}$$

Orthogonality kills off infinitely many integrals and leaves only the one we want.

Another approach is to split $F(x) = C(x) + S(x)$ into an even part and an odd part. Then we can use the earlier cosine and sine formulas. The two parts are

$$C(x) = F_{\textbf{even}}(x) = \frac{F(x) + F(-x)}{2} \qquad S(x) = F_{\textbf{odd}}(x) = \frac{F(x) - F(-x)}{2}. \tag{19}$$

The even part gives the a's and the odd part gives the b's. Test on a square pulse from $x = 0$ to $x = h$—this one-sided thin box function is not odd or even.

Example 4 Find the a's and b's if $\boldsymbol{F(x)} =$ **tall box** $= \begin{cases} \boldsymbol{1/h} & \text{for } 0 < x < h \\ \boldsymbol{0} & \text{for } h < x < 2\pi \end{cases}$

Solution The integrals for a_0 and a_k and b_k stop at $x = h$ where $F(x)$ drops to zero. The coefficients decay like $1/k$ because of the jump at $x = 0$ and the drop at $x = h$:

Coefficients of square pulse $$\boldsymbol{a_0} = \frac{1}{2\pi}\int_0^h 1/h\,dx = \boldsymbol{\frac{1}{2\pi}} = \textbf{average}$$

$$\boldsymbol{a_k} = \frac{1}{\pi h}\int_0^h \cos kx\,dx = \boldsymbol{\frac{\sin kh}{\pi kh}} \qquad \boldsymbol{b_k} = \frac{1}{\pi h}\int_0^h \sin kx\,dx = \boldsymbol{\frac{1 - \cos kh}{\pi kh}}.$$

Important As h approaches zero, the box gets thinner and taller. Its width is h and its height is $1/h$ and its area is 1. The box approaches a delta function! And its Fourier coefficients approach the coefficients of the delta function as $h \to 0$:

$$a_0 = \boldsymbol{\frac{1}{2\pi}} \qquad a_k = \frac{\sin kh}{\pi kh} \text{ approaches } \boldsymbol{\frac{1}{\pi}} \qquad b_k = \frac{1 - \cos kh}{\pi kh} \text{ approaches } \boldsymbol{0}. \tag{20}$$

Energy in Function = Energy in Coefficients

There is an extremely important equation (***the energy identity***) that comes from integrating $(F(x))^2$. When we square the Fourier series of $F(x)$, and integrate from $-\pi$ to π, all the "cross terms" drop out. The only nonzero integrals come from 1^2 and $\cos^2 kx$ and $\sin^2 kx$. Those integrals give 2π and π and π, multiplied by a_0^2 and a_k^2 and b_k^2 :

Energy $$\int_{-\pi}^{\pi} (F(x))^2 dx = 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots). \quad (21)$$

The energy in $F(x)$ equals the energy in the coefficients. The left side is like the length squared of a vector, except *the vector is a function*. The right side comes from an infinitely long vector of a's and b's. The lengths are equal, which says that the Fourier transform from function to vector is like an orthogonal matrix. Normalized by $\sqrt{2\pi}$ and $\sqrt{\pi}$, **sines and cosines are an orthonormal basis in function space**.

Complex Exponentials $c_k e^{ikx}$

This is a small step and we have to take it. In place of separate formulas for a_0 and a_k and b_k, we will have *one formula* for all the complex coefficients c_k. And the function $F(x)$ might be complex (as in quantum mechanics). The Discrete Fourier Transform will be much simpler when we use N complex exponentials for a vector.

We practice with the complex infinite series for a 2π-periodic function:

Complex Fourier series $$F(x) = c_0 + c_1 e^{ix} + c_{-1}e^{-ix} + \cdots = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (22)$$

If every $c_n = c_{-n}$, we can combine e^{inx} with e^{-inx} into $2\cos nx$. Then (22) is the cosine series for an even function. If every $c_n = -c_{-n}$, we use $e^{inx} - e^{-inx} = 2i \sin nx$. Then (22) is the sine series for an odd function and the c's are pure imaginary.

To find c_k, multiply (22) by e^{-ikx} (not e^{ikx}) **and integrate from $-\pi$ to π:**

$$\int_{-\pi}^{\pi} F(x)e^{-ikx}dx = \int_{-\pi}^{\pi} c_0 e^{-ikx}dx + \int_{-\pi}^{\pi} c_1 e^{ix}e^{-ikx}dx + \cdots + \int_{-\pi}^{\pi} c_k e^{ikx}e^{-ikx}dx + \cdots$$

The complex exponentials are orthogonal. **Every integral on the right side is zero,** except for the highlighted term (when $n = k$ and $e^{ikx}e^{-ikx} = 1$). The integral of 1 is 2π. That surviving term gives the formula for c_k:

Fourier coefficients $$\int_{-\pi}^{\pi} F(x)e^{-ikx}\, dx = 2\pi c_k \qquad \text{for} \quad k = 0, \pm 1, \ldots l \quad (23)$$

Notice that $c_0 = a_0$ is still the average of $F(x)$. The orthogonality of e^{inx} and e^{ikx} is checked by integrating e^{inx} times e^{-ikx}. Remember to use that complex conjugate e^{-ikx}.

Example 5 For a delta function, all integrals are 1 and every c_k is $1/2\pi$. *Flat transform* !

Example 6 Find c_k for the 2π-periodic shifted box $F(x) = \begin{cases} 1 & \text{for } s \le x \le s+h \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$

Solution The integrals (23) have $F = 1$ from s to $s + h$:

$$c_k = \frac{1}{2\pi} \int_s^{s+h} 1 \cdot e^{-ikx}\, dx = \frac{1}{2\pi} \left[\frac{e^{-ikx}}{-ik} \right]_s^{s+h} = e^{-iks} \left(\frac{1 - e^{-ikh}}{2\pi ik} \right). \tag{24}$$

Notice above all the simple effect of the shift by s. *It "modulates" each c_k by e^{-iks}.* The energy is unchanged, the integral of $|F|^2$ just shifts, and $|e^{-iks}| = 1$.

$$\boxed{\textbf{Shift} \quad F(x) \quad \textbf{to} \quad F(x-s) \quad \longleftrightarrow \quad \textbf{Multiply every} \quad c_k \quad \textbf{by} \quad e^{-iks}.} \tag{25}$$

Example 7 A centered box has shift $s = -h/2$. It becomes balanced around $x = 0$. This even function equals 1 on the interval from $-h/2$ to $h/2$:

Centered by $s = -\dfrac{h}{2}$ $$c_k = e^{ikh/2}\, \frac{1 - e^{-ikh}}{2\pi ik} = \frac{1}{2\pi} \frac{\sin(kh/2)}{k/2}.$$

Divide by h for a tall box. The ratio of $\sin(kh/2)$ to $kh/2$ is called the "**sinc**" of $kh/2$.

Tall box $$\frac{F_{\text{centered}}}{h} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{kh}{2}\right) e^{ikx} = \begin{cases} 1/h & \text{for } -h/2 \le x \le h/2 \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

That division by h produces area $= 1$. Every coefficient approaches $\frac{1}{2\pi}$ as $h \to 0$. The Fourier series for the tall thin box again approaches the Fourier series for $\delta(x)$.

The Rules for Derivatives and Integrals

The derivative of e^{ikx} is ike^{ikx}. This great fact puts the Fourier functions e^{ikx} in first place for applications. They are eigenfunctions for d/dx (and the eigenvalues are $\lambda = ik$). Differential equations with constant coefficients are naturally solved by Fourier series.

Multiply by ik **The derivative of** $F(x) = \sum c_k e^{ikx}$ **is** $dF/dx = \sum ik c_k e^{ikx}$

The second derivative has coefficients $(ik)^2 c_k = -k^2 c_k$. High frequencies are growing stronger. And in the opposite direction (when we integrate), we divide by ik and high frequencies get weaker. The solution becomes smoother. Please look at this example:

Response $1/(k^2+1)$
to frequency k $$-\frac{d^2y}{dx^2} + y = e^{ikx} \quad \text{is solved by} \quad y(x) = \frac{e^{ikx}}{k^2+1}$$

This was a typical problem in Chapter 2. The transfer function is $1/(k^2+1)$. There we learned: The forcing function e^{ikx} is exponential so the solution is exponential.

All we are doing now is superposition. Allow all the exponentials at once!

$$-\frac{d^2y}{dx^2} + y = \sum c_k e^{ikx} \quad \text{is solved by} \quad y(x) = \sum \frac{c_k e^{ikx}}{k^2+1}. \tag{26}$$

1. **Derivative rule dF/dx has Fourier coefficients ikc_k** (energy moves to high k).
2. **Shift rule $F(x-s)$ has Fourier coefficients $e^{-iks}c_k$** (no change in energy).

Application: Laplace's Equation in a Circle

Our first application is to Laplace's equation $u_{xx} + u_{yy} = 0$ (Section 7.4). The idea is to construct $u(x,y)$ as an infinite series, choosing its coefficients to match $u_0(x,y)$ along the boundary. The shape of the boundary is crucial, and we take a circle of radius 1.

Begin with the solutions $1,\ r\cos\theta,\ r\sin\theta,\ r^2\cos 2\theta,\ r^2\sin 2\theta,\ \ldots$ to Laplace's equation. Combinations of these special solutions give all solutions in the circle:

$$\boxed{\boldsymbol{u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \cdots}} \tag{27}$$

It remains to choose the constants a_k and b_k to make $u = u_0$ on the boundary. For a circle, θ and $\theta + 2\pi$ give the same point. This means that $u_0(\theta)$ is periodic:

Set $r = 1$ $\qquad u_0(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2 \sin 2\theta + \cdots \qquad (28)$

This is exactly the Fourier series for u_0. **The constants a_k and b_k must be the Fourier coefficients of $u_0(\theta)$.** Thus Laplace's boundary value problem is completely solved, if an infinite series (27) is acceptable as the solution.

Example 8 **Point source** $u_0 = \delta(\theta)$. The boundary is held at $u_0 = 0$, except for the source at $x = 1$, $y = 0$ (where $\theta = 0$). Find the temperature $u(r,\theta)$ inside the circle.

Delta function $\qquad u_0(\theta) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}(\cos\theta + \cos 2\theta + \cos 3\theta + \cdots) = \dfrac{1}{2\pi}\displaystyle\sum_{-\infty}^{\infty} e^{in\theta}$

Inside the circle, each $\cos n\theta$ is multiplied by r^n to solve Laplace's equation:

Inside the circle $\qquad u(r,\theta) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}(r\cos\theta + r^2\cos 2\theta + r^3 \cos 3\theta + \cdots) \qquad (29)$

Poisson managed to sum this infinite series! It involves a series of powers $(re^{i\theta})^n$. His sum gives the response at every (r,θ) to the point source at $r = 1$, $\theta = 0$:

Temperature inside circle $\qquad u(r,\theta) = \dfrac{1}{2\pi}\dfrac{1-r^2}{1+r^2-2r\cos\theta} \qquad (30)$

At the center $r = 0$, this produces the average of $u_0 = \delta(\theta)$ which is $a_0 = 1/2\pi$. On the boundary $r = 1$, this gives $u = 0$ except $u = \infty$ at the point where $\cos 0 = 1$.

Example 9 $u_0(\theta) = 1$ **on the top half of the circle and** $u_0 = -1$ **on the bottom half.**

Solution The boundary values u_0 are a square wave SW. We know its sine series:

$$\textbf{Square wave for } u_0(\theta) \qquad SW(\theta) = \frac{4}{\pi}\left[\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right] \tag{31}$$

Inside the circle, multiplying by $r, r^3, r^5, \ldots$ gives fast decay of high frequencies:

$$\textbf{Rapid decay inside} \qquad u(r,\theta) = \frac{4}{\pi}\left[\frac{r\sin\theta}{1} + \frac{r^3\sin 3\theta}{3} + \frac{r^5\sin 5\theta}{5} + \cdots\right] \tag{32}$$

Laplace's equation has smooth solutions inside, even when $u_0(\theta)$ is not smooth.

Problem Set 8.1

1 (a) To prove that $\cos nx$ is orthogonal to $\cos kx$ when $k \neq n$, use the formula $(\cos nx)\,(\cos kx) = \frac{1}{2}\cos{(n+k)x} + \frac{1}{2}\cos{(n-k)x}$. Integrate from $x = 0$ to $x = \pi$. What is $\int \cos^2 kx\,dx$?

(b) From 0 to π, $\cos x$ is **not** orthogonal to $\sin x$. The period has to be 2π:

$$\text{Find} \int_0^{\pi} (\sin x)\,(\cos x)\,dx \quad \text{and} \quad \int_{-\pi}^{\pi} (\sin x)\,(\cos x)\,dx \quad \text{and} \quad \int_0^{2\pi} (\sin x)\,(\cos x)\,dx.$$

2 Suppose $F(x) = x$ for $0 \le x \le \pi$. Draw graphs for $-2\pi \le x \le 2\pi$ to show three extensions of F: a 2π-periodic even function and a 2π-periodic odd function and a π-periodic function.

3 Find the Fourier series on $-\pi \le x \le \pi$ for

(a) $f_1(x) = \sin^3 x$, an odd function (sine series, only two terms)

(b) $f_2(x) = |\sin x|$, an even function (cosine series)

(c) $f_3(x) = x$ for $-\pi \le x \le \pi$ (sine series with jump at $x = \pi$)

4 Find the complex Fourier series $e^x = \sum c_k e^{ikx}$ on the interval $-\pi \le x \le \pi$. The even part of a function is $\frac{1}{2}(f(x) + f(-x))$, so that $f_{\text{even}}(x) = f_{\text{even}}(-x)$. Find the cosine series for f_{even} and the sine series for f_{odd}. *Notice the jump at* $x = \pi$.

5 From the energy formula (21), the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |SW(x)|^2\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

Substitute the numbers b_k from equation (8) to find that $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

6 If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, \qquad f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients a_k and b_k.

7 Plot the first three partial sums and the function $x(\pi - x)$:

$$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right), 0 < x < \pi.$$

Why is $1/k^3$ the decay rate for this function? What is its second derivative?

8 Sketch the 2π-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

9 Suppose $G(x)$ has period $2L$ instead of 2π. Then $G(x + 2L) = G(x)$. Integrals go from $-L$ to L or from 0 to $2L$. The Fourier formulas change by a factor π/L:

$$\text{The coefficients in } G(x) = \sum_{-\infty}^{\infty} \boldsymbol{C}_k e^{ik\pi x/L} \text{ are } \boldsymbol{C}_k = \frac{1}{2L}\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx.$$

Derive this formula for C_k: Multiply the first equation for $G(x)$ by _____ and integrate both sides. Why is the integral on the right side equal to $2LC_k$?

10 For Geven, use Problem 9 to find the cosine coefficient A_k from $(C_k + C_{-k})/2$:

$$G_{\text{even}}(x) = \sum_{0}^{\infty} A_k \cos\frac{k\pi x}{L} \quad \text{has} \quad A_k = \frac{1}{L}\int_{0}^{L} G_{\text{even}}(x) \cos\frac{k\pi x}{L}\,dx.$$

G_{even} is $\frac{1}{2}(G(x) + G(-x))$. Exception for $A_0 = C_0$: Divide by $2L$ instead of L.

11 Problem 10 tells us that $\boldsymbol{a_k = \frac{1}{2}(c_k + c_{-k})}$ on the usual interval from 0 to π. Find a similar formula for b_k from c_k and c_{-k}. In the reverse direction, find the complex coefficient c_k in $F(x) = \sum c_k e^{ikx}$ from the real coefficients a_k and b_k.

12 Find the solution to Laplace's equation with $u_0 = \theta$ on the boundary. Why is this the imaginary part of $2(z - z^2/2 + z^3/3\cdots) = 2\log(1+z)$? Confirm that on the unit circle $z = e^{i\theta}$, the imaginary part of $2\log(1+z)$ agrees with θ.

13 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin $r = 0$?

14 With boundary values $u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum this geometric series.

15 (a) Verify that the fraction in Poisson's formula (30) satisfies Laplace's equation.

(b) Find the response $u(r, \theta)$ to an impulse at $x = 0, y = 1$ (where $\theta = \frac{\pi}{2}$).

16 With complex exponentials in $F(x) = \sum c_k e^{ikx}$, the energy identity (21) changes to $\int_{-\pi}^{\pi} |F(x)|^2\,dx = 2\pi \sum |c_k|^2$. Derive this by integrating $(\sum c_k e^{ikx})(\sum \bar{c}_k e^{-ikx})$.

17 A centered square wave has $F(x) = 1$ for $|x| \leq \pi/2$.

(a) Find its energy $\int |F(x)|^2\, dx$ by direct integration

(b) Compute its Fourier coefficients c_k as specific numbers

(c) Find the sum in the energy identity (Problem 16).

18 $F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots$ is analytic : infinitely smooth.

(a) If you take 10 derivatives, what is the Fourier series of $d^{10}F/dx^{10}$?

(b) Does that series still converge quickly ? Compare n^{10} with 2^n for $n = 2^{10}$.

19 If $f(x) = 1$ for $|x| \leq \pi/2$ and $f(x) = 0$ for $\pi/2 < |x| < \pi$, find its cosine coefficients. Can you graph and compute the Gibbs overshoot at the jumps ?

20 Find all the coefficients a_k and b_k for F, I, and D on the interval $-\pi \leq x \leq \pi$:

$$F(x) = \delta\left(x - \frac{\pi}{2}\right) \qquad I(x) = \int_0^x \delta\left(x - \frac{\pi}{2}\right) dx \qquad D(x) = \frac{d}{dx}\delta\left(x - \frac{\pi}{2}\right).$$

21 For the one-sided tall box function in Example 4, with $F = 1/h$ for $0 \leq x \leq h$, what is its odd part $\frac{1}{2}(F(x) - F(-x))$? I am surprised that the Fourier coefficients of this odd part disappear as h approaches zero and $F(x)$ approaches $\delta(x)$.

22 Find the series $F(x) = \sum c_k e^{ikx}$ for $F(x) = e^x$ on $-\pi \leq x \leq \pi$. That function e^x looks smooth, but there must be a hidden jump to get coefficients c_k proportional to $1/k$. Where is the jump ?

23 (a) (Old particular solution) Solve $Ay'' + By' + Cy = e^{ikx}$.

(b) (New particular solution) Solve $Ay'' + By' + Cy = \sum c_k e^{ikx}$.

8.2 The Fast Fourier Transform

Fourier series apply to functions. But we compute with vectors. We need to replace the infinite sequence of coefficients c_k (or a_k and b_k) by a **finite sequence** $\boldsymbol{c_0, c_1, \ldots, c_{N-1}}$. We want to preserve and use orthogonality, so the computations will be fast. For the Discrete Fourier Transform, you will see how the FFT makes the computations extra fast.

This section describes two separate ideas. The DFT provides formulas for the c's. *The FFT is an amazing algorithm to compute the c's by rearranging those formulas.*

Discrete Fourier Transform (DFT)

The DFT chooses N orthogonal basis vectors $\boldsymbol{e}_0$ to $\boldsymbol{e}_{N-1}$ for N-dimensional space. The vector $\boldsymbol{e}_k$ comes from e^{ikx}, by sampling that function at N points spaced by $2\pi/N$:

Basis vector $\boldsymbol{e_k}$
Discrete $\boldsymbol{e^{ikx}}$
$$(e^{ik0}, e^{ik2\pi/N}, e^{ik4\pi/N}, \ldots) = (1, \boldsymbol{w}^{\boldsymbol{k}}, \boldsymbol{w}^{\boldsymbol{2k}}, \ldots) \text{ with } \boldsymbol{w} = \boldsymbol{e}^{\boldsymbol{i2\pi/N}}.$$

The continuous Fourier series is $\sum c_k e^{ikx}$. The discrete Fourier series is $\sum c_k \boldsymbol{e}_k$. That sum is a multiplication $\boldsymbol{f} = F\boldsymbol{c}$ with the symmetric N by N **Fourier matrix** F. The basis vectors $\boldsymbol{e}_k$ go into the columns of F.

The matrix F containing powers of w is shown in detail in equation (4).

Fourier matrix
$\boldsymbol{f = Fc}$
$$\boldsymbol{f} = c_0\boldsymbol{e}_0 + c_1\boldsymbol{e}_1 + \cdots = \begin{bmatrix} | & & | \\ \boldsymbol{e}_0 & \cdots & \boldsymbol{e}_{N-1} \\ | & & | \end{bmatrix} \begin{bmatrix} c_0 \\ \cdot \\ \cdot \\ c_{N-1} \end{bmatrix} \tag{1}$$

Inverting $\boldsymbol{f} = F\boldsymbol{c}$ gives $\boldsymbol{c} = F^{-1}\boldsymbol{f}$. The continuous case produced e^{-ikx} in the Fourier coefficient formula $c_k = \int e^{-ikx} f(x)dx/2\pi$. The discrete case produces powers of $\overline{w} = e^{-i2\pi/N}$ in the inverse matrix. Those powers of $\overline{w}$ are displayed in equation (3).

Inverse matrix
$\boldsymbol{c} = F^{-1}\boldsymbol{f}$
$$\boldsymbol{c} = \frac{1}{N}\begin{bmatrix} — & \overline{\boldsymbol{e}}_0^{\mathrm{T}} & — \\ & \cdot & \\ & \cdot & \\ — & \overline{\boldsymbol{e}}_{N-1}^{\mathrm{T}} & — \end{bmatrix} \begin{bmatrix} f_0 \\ \cdot \\ \cdot \\ f_{N-1} \end{bmatrix} = \frac{1}{N}\overline{F}^{\mathrm{T}}\boldsymbol{f}. \tag{2}$$

The constant vector $\boldsymbol{e}_0 = (1, 1, \ldots, 1)$ has $||\boldsymbol{e}_0||^2 = 1 + 1 + \cdots + 1 = N$. Every basis vector has $||\boldsymbol{e}_k||^2 = N$ instead of $\int |e^{ikx}|^2 dx = 2\pi$.

Please notice that F^{-1} produces the coefficients c_k from the vector $\boldsymbol{f}$: the *Fourier transform*. The Fourier matrix F reconstructs $\boldsymbol{f}$ from the c's (the *inverse transform*). The entries of F^{-1} are like e^{-ikx} and the entries of F are like e^{ikx}. Thus $F^{-1} = \overline{F}/N$ contains powers of $\overline{w} = e^{-i2\pi/N}$, while F contains powers of $w = e^{i2\pi/N}$.

The MATLAB command $\boldsymbol{c}$ = fft($\boldsymbol{f}$) uses $\overline{\boldsymbol{w}}$ and the inverse Fourier matrix F^{-1}. The opposite command $\boldsymbol{f}$ = **ifft**($\boldsymbol{c}$) adds up the N-term series $F\boldsymbol{c}$ to reconstruct $\boldsymbol{f}$ in (1).

Example 1 The delta vector $\boldsymbol{f} = (1, 0, 0, \ldots)$ is like a delta function $\delta(x)$. The Fourier coefficients of a delta function are all equal to $c_k = 1/2\pi$. The discrete coefficients of a delta vector are all equal to $c_k = 1/N$. **The transform of $\boldsymbol{f}$ is a constant vector.**

$$\textbf{Fourier transform} \quad \boldsymbol{F^{-1}f = c} \qquad \frac{1}{N}\begin{bmatrix} 1 & 1 & \cdot & \cdot & 1 \\ 1 & \overline{w} & \cdot & \cdot & \overline{w}^{N-1} \\ 1 & \overline{w}^2 & \cdot & \cdot & \overline{w}^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix}. \tag{3}$$

Example 2 The shifted vector $\boldsymbol{f} = (0, 1, 0, \ldots)$ is like a shifted delta function $\delta(x - \frac{2\pi}{N})$. The shifted vector $\boldsymbol{f}$ picks out the next column $(1, \overline{w}, \overline{w}^2, \ldots)$ of F^{-1} in equation (3). The shifted delta function chooses the (same) values of $c_k = \boldsymbol{e}^{-ikx}$ at $x = 2\pi/N$.

The only difference between those discrete and continuous c's is dividing by N or 2π.

Example 3 **The constant vector $\boldsymbol{c} = (1, 1, \ldots)/N$ transforms back to the delta vector!**

$$\textbf{Fourier matrix} \quad \boldsymbol{Fc = f} \qquad \begin{bmatrix} 1 & 1 & \cdot & \cdot & 1 \\ 1 & w & \cdot & \cdot & w^{N-1} \\ 1 & w^2 & \cdot & \cdot & w^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\frac{1}{N}\begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \end{bmatrix} \tag{4}$$

That equation says that $N - 1$ basis vectors starting with $(1, w, w^2, \ldots)$ are orthogonal to the first vector $(1, 1, \ldots, 1)$. ***The basis vectors $\boldsymbol{e}_k$ in the columns of $\boldsymbol{F}$ are orthogonal.***

After a few words about the FFT, equation (7) will confirm this orthogonality.

Fast Fourier Transform (FFT)

The FFT is a brilliant rearrangement of those matrix-vector multiplications $\boldsymbol{f} = F\boldsymbol{c}$ and $\boldsymbol{c} = F^{-1}\boldsymbol{f}$. Normally, multiplying a vector by an N by N matrix takes N^2 separate multiplications. (Each entry in the square matrix is used once. There are N^2 entries.) The FFT computes $\boldsymbol{c}$ and $\boldsymbol{f}$ with only $\frac{1}{2}N \log_2 N$ separate multiplications.

For size $N = 1024 = 2^{10}$, the logarithm is 10. In this case N^2 (a million steps) are reduced to $5N$ (five thousand steps). The transform is speeded up by a factor near 200, which is truly astonishing.

In my opinion, the FFT is the most important algorithm in computational science. It has transformed whole industries. When your instruments measure the response to an input (like the pressure in an oil well), the DFT shows the response to each frequency. The FFT computes N numbers from N numbers, very fast.

The Basis Vectors $\boldsymbol{e_k}$ in the Fourier Matrix $\boldsymbol{F}$

A crucial point is that the basis vectors $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{N-1}$ are **orthogonal**. Those vectors are complex, just as the functions e^{ikx} are complex. So their inner products $\overline{\boldsymbol{e}}_k^{\mathrm{T}}\boldsymbol{e}_n$ require the complex conjugate of one vector, just like $\int e^{inx}\, e^{-ikx}\, dx$.

Here is a typical basis vector e_k, followed by the Fourier matrix that contains $e_0, e_1, \ldots, e_{N-1}$ in its columns:

$$e_k = \begin{bmatrix} 1 \\ e^{2\pi ik/N} \\ e^{4\pi ik/N} \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{w^k} \\ \boldsymbol{w^{2k}} \\ \cdot \\ \cdot \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 1 & \cdot & \cdot & 1 \\ 1 & w & \cdot & \cdot & w^{N-1} \\ 1 & w^2 & \cdot & \cdot & w^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & w^{N-1} & \cdot & \cdot & w^{(N-1)^2} \end{bmatrix} \tag{5}$$

The number w is $e^{2\pi i/N}$. We use the Greek letter ω for its conjugate $\overline{w} = e^{-2\pi i/N} = \omega$. It is the properties of 1, w, w^2, ... that make the basis vectors (columns of F) orthogonal. Our first step is to locate w and $\overline{w}$ in the complex plane. In fact we can locate all the powers of w up to $w^N = (e^{2\pi i/N})^N = e^{2\pi i} = 1$. For $N = 8$, the powers of w produce 8 points evenly spaced around the unit circle . Notice that $w^8 = 1$.

For $N = 4$, the four powers will be $\boldsymbol{i}$, $i^2 = \mathbf{-1}$, $i^3 = \boldsymbol{-i}$, and $i^4 = \mathbf{1}$.

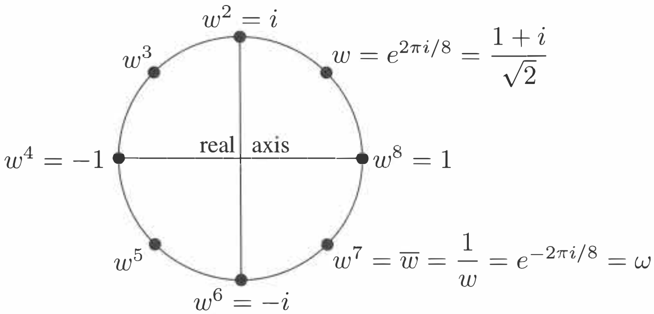


Figure 8.5: The eight powers of $w = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$. The polar form $w = e^{2\pi i/8}$ is best.

Orthogonality of the Discrete Fourier Basis

The key to good formulas for the Fourier coefficients c_k is orthogonality. That property removes every term except term k, when we take a dot product with the basis vector e_k:

$$\boldsymbol{f} = c_0 e_0 + \cdots + c_{N-1} e_{N-1} \quad \text{and} \quad \overline{\boldsymbol{e}}_{\boldsymbol{k}}^{\mathrm{T}} \boldsymbol{f} = \boldsymbol{c_k}\, \overline{\boldsymbol{e}}_{\boldsymbol{k}}^{\mathrm{T}}\, \boldsymbol{e_k} = \boldsymbol{N c_k}. \tag{6}$$

Since $e_0 = (1, 1, 1, \ldots)$ and $e_1 = (1, w, w^2, \ldots)$, the crucial step is their zero dot product: $1 + w + w^2 + \cdots = 0$. **The eight numbers around the circle in Figure 8.5 add to zero.**

Here is the statement and proof that every pair of e's is orthogonal:

If $z^N = 1$ and $z \neq 1$, then the sum $S = 1 + z + z^2 + \cdots + z^{N-1}$ is zero. (7)

Proof. Multiply S times z. This gives $Sz = z + z^2 + z^3 + \cdots + z^N$. Since $z^N = 1$, S times z has all the same terms as the original sum S. Then $Sz = S$. Therefore $S = 0$.

Every dot product $\overline{e}_k^{\mathrm{T}} e_n$ is exactly our sum S. **The number z is $\overline{w}^k w^n$.**

$$(1, \overline{w}^k, \overline{w}^{2k}, \ldots)^{\mathrm{T}}(1, w^n, w^{2n}, \ldots) = 1 + z + z^2 + \cdots = S \tag{8}$$

The Nth power of $z = \overline{w}^k w^n$ is $z^N = \left(\overline{w}^N\right)^k \left(w^N\right)^n = (1)(1)$. Therefore $S = 0$.

Conclusion When we multiply $\overline{F}^{\mathrm{T}}$ times F, the diagonal entries are $\overline{e}_k^{\mathrm{T}} e_k = N$ (because this is a sum of N ones). Off the diagonal we have $k \neq n$ and $\overline{e}_k^{\mathrm{T}} e_n = 0$. Therefore $\overline{F}^{\mathrm{T}} F = NI$. This confirms that **the inverse of the Fourier matrix is $F^{-1} = \frac{1}{N}\overline{F}^{\mathrm{T}}$.**

Note 1. Your eye sees right away that *the* 8 *numbers around the circle add to zero.* Each number cancels its opposite number: $1 + w^4$ is zero, $w + w^5$ is zero, $w^2 + w^6$ is zero, $w^3 + w^7$ is zero. But this proof won't work for $N = 7$ or 5 or 3. We can't pair off the points when N is odd. They still add to zero by equation (8).

Note 2. A cool proof of orthogonality is to see the vectors $e_0, \ldots, e_{N-1}$ as *eigenvectors of a symmetric matrix.* Every symmetric matrix has orthogonal eigenvectors. Problem 14 will choose a suitable matrix (it is a **circulant matrix**) and pursue this idea.

Here are the components of $f = Fc$ and $c = F^{-1}f$: **Discrete Fourier Transform**

$$f_j = e_j^{\mathrm{T}} c = \sum_{j=0}^{N-1} w^{jk} c_k \qquad c_k = \frac{1}{N}\overline{e}_k^{\mathrm{T}} f = \frac{1}{N}\sum_{k=0}^{N-1} \overline{w}^{jk} f_j \tag{9}$$

The symmetry of transform and inverse transform is beautiful. We didn't see this so clearly for Fourier series, where c was a vector but f was a periodic function. The elegant symmetry reappears when the transform is between *function* $f(x)$ *and function* $c(k)$:

Fourier Integral Transform
$$c(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\, e^{ikx}\, dk. \tag{10}$$

Everybody notices e^{-ikx} and e^{ikx}. Be sure to notice dx and dk. The functions $f(x)$ and $c(k)$ are defined for $-\infty < x < \infty$ and $-\infty < k < \infty$. The transform connects $f(x)$ in the space domain to $c(k)$ in the frequency domain. $f(x) = \delta(x)$ transforms to $c(k) = 1$. Section 8.6 will solve $-y'' + y = f(x)$ (no boundaries !) using this integral transform.

Two more examples of the discrete transform are **cos** and **sin**.

Example 4 Sample $\cos x$ and $\sin x$ at $0, \pi/2, \pi, 3\pi/2$ to get discrete vectors **cos** and **sin**. Transform those vectors by F^{-1}. Invert their transforms by F.

Discrete cosine and sine $\quad$ **cos** $= (1, 0, -1, 0)$ and **sin** $= (0, 1, 0, -1)$.

To transform x-space to k-space, we multiply $\boldsymbol{f}$ by F^{-1}. For $N = 4$, this matrix contains powers of $\overline{w} = -i$. We remember to divide by $N = 4$:

$$F^{-1}\ \mathbf{cos} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{1/2} \\ 0 \\ \mathbf{1/2} \end{bmatrix} \qquad F^{-1}\ \mathbf{sin} = \begin{bmatrix} 0 \\ -\boldsymbol{i/2} \\ 0 \\ \boldsymbol{i/2} \end{bmatrix}$$

Multiplication by F transforms back to **cos** and **sin**. This is exactly consistent with the famous formulas of Euler: $\mathbf{cos}\, x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\mathbf{sin}\, x = \frac{-i}{2}(e^{ix} - e^{-ix})$.

Let me also write **exp** for the samples $(1, w, w^2, w^3)$ of e^{ix} at $x = 0, \pi/2, \pi, 3\pi/2$. Then we have Euler's great formulas for vectors:

$$\mathbf{exp} = \mathbf{cos} + i\,\mathbf{sin} \qquad \mathbf{cos} = \frac{1}{2}(\,\mathbf{exp} + \overline{\mathbf{exp}})$$

$$\overline{\mathbf{exp}} = \mathbf{cos} - i\,\mathbf{sin} \qquad \mathbf{sin} = \frac{-i}{2}(\,\mathbf{exp} - \overline{\mathbf{exp}})$$

One Step of the Fast Fourier Transform

Multiplication by an N by N matrix takes N^2 multiplications and additions. Since the Fourier matrix has no zero entries, you might think it is impossible to do better. But the entries w^{jk} are very special. **The FFT idea is to factor F into sparse matrices**.

If you prefer to think of the summation formulas $\sum w^{jk} c_k$ and $\sum \overline{w}^{jk} f_j$, each sum has N terms and a vector needs N sums. In summation language, the FFT idea is to rewrite and regroup the sums to have many fewer terms. I will try to use both languages.

The key idea is to connect F_N with the half-size Fourier matrix $F_{N/2}$. Assume that N is a power of 2 (say $N = 1024$). We will connect F_{1024} to ***two copies*** of F_{512}. When $N = 4$, we connect F_4 to two F_2's:

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}.$$

On the left is F_4, with no zeros. On the right is a matrix that is half zero. The work is cut in half. But wait, those matrices are not the same. The block matrix with F_2's is only one piece of the factorization of F_4. The other pieces also have many zeros:

$$\textbf{Key idea} \qquad F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix}\begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \tag{11}$$

The permutation matrix on the right puts c_0 and c_2 (evens) ahead of c_1 and c_3 (odds). The middle matrix performs separate half-size transforms on those evens and odds. The matrix at the left combines the two half-size outputs, and it produces the correct full-size output $\boldsymbol{f} = F_4\boldsymbol{c}$. You could multiply those three matrices to see F_4.

The same idea applies when $N = 1024$ and $M = \frac{1}{2}N = 512$. The number w is $e^{2\pi i/1024}$. It is at the angle $\theta = 2\pi/1024$ on the unit circle. The Fourier matrix F_{1024} is full of powers of w. The first stage of the FFT is the great factorization discovered by Cooley and Tukey (and foreshadowed in 1805 by Gauss):

FFT (Step 1)
$$F_{1024} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix} \tag{12}$$

I_{512} is the identity matrix. D_{512} is the diagonal matrix with entries $(1, w, \ldots, w^{511})$ using w_{1024}. The two copies of F_{512} are what we expected. They use the 512th root of unity, which is nothing but $w_{512} = (w_{1024})^2$. The even-odd permutation matrix separates the incoming vector $\boldsymbol{c}$ into $\boldsymbol{c}' = (c_0, c_2, \ldots, c_{1022})$ and $\boldsymbol{c}'' = (c_1, c_3, \ldots, c_{1023})$.

Here are the algebra formulas which express this neat FFT factorization of F_N:

(FFT) Set $M = \frac{1}{2}N$. The components of $\boldsymbol{f} = F_N\boldsymbol{c}$ are combinations of the half-size transforms $\boldsymbol{f}' = F_M\boldsymbol{c}'$ and $\boldsymbol{f}'' = F_M\boldsymbol{c}''$. Equation (13) shows $I\boldsymbol{f}' + D\boldsymbol{f}''$ and $I\boldsymbol{f}' - D\boldsymbol{f}''$ with numbers $(w_N)^j$ on the main diagonal of D:

$$\begin{array}{llcll} \textbf{First half} & \boldsymbol{f}_j & = & \boldsymbol{f}'_j + (w_N)^j \boldsymbol{f}''_j, & j = 0, \ldots, M-1 \\ \textbf{Second half} & \boldsymbol{f}_{j+M} & = & \boldsymbol{f}'_j - (w_N)^j \boldsymbol{f}''_j, & j = 0, \ldots, M-1 \end{array} \tag{13}$$

Thus each FFT step has three parts: split $\boldsymbol{c}$ into $\boldsymbol{c}'$ and $\boldsymbol{c}''$, transform them separately by F_M into $\boldsymbol{f}'$ and $\boldsymbol{f}''$, and reconstruct $\boldsymbol{f}$ from equation (13). N must be even!

The algebra of (13) is a splitting into even numbers $2k$ and odd $2k+1$, with $w = w_N$:

Even/Odd
$$f_j = \sum_0^{N-1} w^{jk} c_k = \sum_0^{M-1} w^{2jk} c_{2k} + \sum_0^{M-1} w^{j(2k+1)} c_{2k+1} \text{ with } M = \frac{N}{2}. \tag{14}$$

The even c's go into $\boldsymbol{c}' = (c_0, c_2, \ldots)$ and the odd c's go into $\boldsymbol{c}'' = (c_1, c_3, \ldots)$. Then come the transforms $F_M\boldsymbol{c}'$ and $F_M\boldsymbol{c}''$. The key is $w_N^2 = w_M$. This gives $w_N^{2jk} = w_M^{jk}$.

Rewrite
$$f_j = \sum w_M^{jk} c'_k + (w_N)^j \sum w_M^{jk} c''_k = f'_j + (w_N)^j f''_j. \tag{15}$$

For $j \geq M$, the minus sign in (13) comes from factoring out $(w_N)^M = -1$.

MATLAB easily separates even c's from odd c's. Then two half-size inverse transforms use ifft. The last step produces f from the half-size f' and f''.

Problem 2 shows that F and F^{-1} have the same rows, in different orders.

FFT Step from N to $N/2$ in MATLAB

```
f' = ifft (c(0 : 2 : N − 2)) * N/2; % evens
f'' = ifft (c(1 : 2 : N − 1)) * N/2; % odds
D = w.^(0 : N/2 − 1)'; % diagonal of matrix D
f = [f' + D.*f''; f' − D.*f''];
```

The flow graph shows $\boldsymbol{c}'$ and $\boldsymbol{c}''$ going through the half-size F_2. Those steps are called "*butterflies*," from their shape. Then the outputs $\boldsymbol{f}'$ and $\boldsymbol{f}''$ are combined (multiplying $\boldsymbol{f}''$ by $1, i$ and also by $-1, -i$) to produce $\boldsymbol{f} = F_4\boldsymbol{c}$. The indices $0, 1, 2, 3$ are in binary.

Flow Graph
$\boldsymbol{c}$ to $\boldsymbol{f}$
$N = 4$
$M = 2$

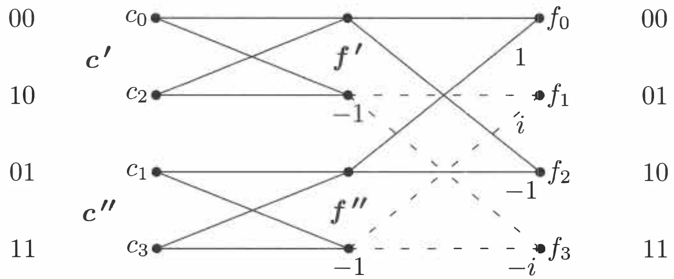


Figure 8.6: Flow graph from $\boldsymbol{c}$ to $\boldsymbol{f}$ for the Fast Fourier Transform with $N = 4$.

This reduction from F_N to two F_M's almost cuts the work in half—you see the zeros in the matrix factorization (12). That reduction is good but not great. The full idea of the FFT is much more powerful. It saves much more time than 50%.

The Full FFT by Recursion

If you have read this far, you may have guessed what comes next. We reduced F_N to $F_{N/2}$. **Keep going to $F_{N/4}$.** The two copies of F_{512} lead to four copies of F_{256}. Then 256 leads to 128. *That is recursion.* It is a basic principle of many fast algorithms. Here is the second stage with $F = F_{256}$ and $D = \text{diag}\,(1, w_{512}, \ldots, (w_{512})^{255})$:

$$\begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{pick} & 0, 4, 8, \ldots \\ \text{pick} & 2, 6, 10, \ldots \\ \text{pick} & 1, 5, 9, \ldots \\ \text{pick} & 3, 7, 11, \ldots \end{bmatrix}.$$

Before the FFT was invented, the operation count was $N^2 = (1024)^2$. This is about a million multiplications. I am not saying that they take a long time. The cost becomes large when we have many transforms to do—which is typical. Then the saving is also large:

$$\textbf{The final count for size } \boldsymbol{N = 2^L} \textbf{ is reduced from } \boldsymbol{N^2} \textbf{ to } \boldsymbol{\frac{1}{2}NL}\,.$$

Here is the reasoning behind $\frac{1}{2}NL$. There are L levels, going from $N = 2^L$ down to $N = 1$. Each level has $\frac{1}{2}N$ multiplications from diagonal matrices D, to reassemble the half-size outputs. This yields the final count $\frac{1}{2}NL$, which is $\frac{1}{2}N \log_2 N$.

Exactly the same idea gives a fast inverse transform. The matrix F_N^{-1} contains powers of the conjugate $\overline{w}$. We just replace w by $\overline{w}$ in the diagonal matrix D, and in formula (13). The fastest FFT will be adapted to the processor and cache capacity of each computer. For free software that automatically adjusts, we highly recommend the website fftw.org. This gives the "fastest Fourier transform in the west."

■ REVIEW OF THE KEY IDEAS ■

1. Multiplying coefficients $\boldsymbol{c}$ by the Fourier matrix F adds the series $f_j = \sum w^{jk} c_k$.
2. The inverse matrix $F^{-1} = \overline{F}/N$ computes the coefficients $c_k = \sum \overline{w}^{jk} f_j/N$.
3. The **FFT** splits those sums in half: $\frac{N}{2}$ terms with powers of w^2. Then recombine.
4. By recursion the FFT has $\log_2 N$ steps with diagonal matrices: $N \log_2 N$ operations.
5. The columns $\boldsymbol{e}_k = (1, w^k, w^{2k}, \ldots)$ are orthogonal, when $w = e^{2\pi i/N}$ and $w^N = 1$.

Problem Set 8.2

1 Multiply the three matrices in equation (11) and compare with F. In which six entries do you need to know that $i^2 = -1$? This is $(w_4)^2 = w_2$. If $M = N/2$, why is $(w_N)^M = -1$?

2 *Why is row i of $\overline{F}$ the same as row $N - i$ of F* (numbered from 0 to $N - 1$)?

3 From Problem 2, find the 4 by 4 permutation matrix P so that $F = P\overline{F}$. Check that $P^2 = I$ so that $P = P^{-1}$. Then from $\overline{F}F = 4I$ show that $F^2 = 4P$.

It is amazing that $F^4 = 16P^2 = 16I$. *Four transforms of any* $\boldsymbol{c}$ *bring back* $\mathbf{16}\,\boldsymbol{c}$. For all N, F^2/N is a permutation matrix P and $\boldsymbol{F^4 = N^2 I}$.

4 Invert the three factors in equation (11) to find a fast factorization of F^{-1}.

5 F is symmetric. Transpose equation (11) to find a new Fast Fourier Transform.

6 All entries in the factorization of F_6 involve powers of w = sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write down these factors with $1, w, w^2$ in D and powers of w^2 in F_3. Multiply!

7 Put the vector $\boldsymbol{c} = (1, 0, 1, 0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0, 1, 0, 1)$.

8 Compute $\boldsymbol{y} = F_8\boldsymbol{c}$ by the three FFT steps for $\boldsymbol{c} = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $\boldsymbol{c} = (0, 1, 0, 1, 0, 1, 0, 1)$.

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

10 F is a symmetric matrix. Its eigenvalues aren't real. How is this possible?

The three great symmetric tridiagonal matrices of applied mathematics are K, B, C. The eigenvectors of K, B, and C are discrete **sines**, **cosines**, and **exponentials**. The eigenvector matrices give the **DST**, **DCT**, and **DFT** — discrete transforms for signal processing. Notice that diagonals of the circulant matrix C loop around to the far corners.

$$K = \begin{bmatrix} \mathbf{2} & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & \mathbf{2} \end{bmatrix} \qquad B = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & \mathbf{1} \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 & \cdot & \mathbf{-1} \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ \mathbf{-1} & \cdot & -1 & 2 \end{bmatrix} \qquad \begin{array}{l} K_{11} = K_{NN} = \mathbf{2} \\ B_{11} = B_{NN} = \mathbf{1} \\ C_{1N} = C_{N1} = \mathbf{-1} \end{array}$$

11 The eigenvectors of K_N and B_N are the discrete sines $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N$ and the discrete cosines $\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{N-1}$. Notice the eigenvector $\boldsymbol{c}_0 = (1, 1, \ldots, 1)$. Here are $\boldsymbol{s}_k$ and $\boldsymbol{c}_k$—these vectors are samples of $\sin kx$ and $\cos kx$ from 0 to π.

$$\left(\sin\frac{\pi k}{N+1}, \sin\frac{2\pi k}{N+1}, \ldots, \sin\frac{N\pi k}{N+1}\right) \text{ and } \left(\cos\frac{\pi k}{2N}, \cos\frac{3\pi k}{2N}, \ldots, \cos\frac{(2N-1)\pi k}{2N}\right)$$

For 2 by 2 matrices K_2 and B_2, verify that $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ are eigenvectors.

12 Show that C_3 has eigenvalues $\lambda = 0, 3, 3$ with eigenvectors $\boldsymbol{e}_0 = (1, 1, 1)$, $\boldsymbol{e}_1 = (1, w, w^2)$, $\boldsymbol{e}_2 = (1, w^2, w^4)$. You may prefer the real eigenvectors $(1, 1, 1)$ and $(1, 0, -1)$ and $(1, -2, 1)$.

13 Multiply to see the eigenvectors $\boldsymbol{e}_k$ and eigenvalues λ_k of C_N. Simplify to $\lambda_k = 2 - 2\cos(2\pi k/N)$. Explain why C_N is only semidefinite. It is not positive definite.

$$C\boldsymbol{e}_k = \begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ -1 & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ w^{(N-1)k} \end{bmatrix} = (2 - w^k - w^{-k}) \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ w^{(N-1)k} \end{bmatrix}.$$

14 The eigenvectors $\boldsymbol{e}_k$ of C are automatically perpendicular because C is a _____ matrix. (To tell the truth, C has repeated eigenvalues as in Problem 12. There was a plane of eigenvectors for $\lambda = 3$ and we chose orthogonal $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in that plane.)

15 Write the 2 eigenvalues for K_2 and the 3 eigenvalues for B_3. Always K_N and B_{N+1} have the same N eigenvalues, with the extra eigenvalue _____ for B_{N+1}. (This is because $K = A^{\mathrm{T}}A$ and $B = AA^{\mathrm{T}}$.)

8.3 The Heat Equation

The first partial differential equation in this book was $u_{xx} + u_{yy} = 0$ (Laplace's equation). This describes a steady state—time is not involved. There is no growth or oscillation or decay. The problem includes boundary conditions on $u(x, y)$, but not initial conditions. This is like a matrix equation $A\boldsymbol{u} = \boldsymbol{b}$ (where $\boldsymbol{b}$ comes from boundary conditions).

Now we move to the **heat equation** $\boldsymbol{u_t = u_{xx}}$. Time is very much involved. We think of u as the temperature along a bar at time t. We are given the initial temperature $u(0, x)$ at time $t = 0$ and at each position x. Then heat begins to flow (from positions with higher temperature to neighbors at lower temperature). This is like a matrix equation $\boldsymbol{u}' = \boldsymbol{Au}$ with an initial condition $u(0)$. Au is now the second derivative u_{xx}.

We have a PDE and not an ODE, a partial and not an ordinary differential equation, because the temperature u is a function of both x and t.

Example 1 (*Infinite bar*) Suppose the bar goes from $x = -\infty$ to $x = \infty$. At time $t = 0$, the temperature is $u = -1$ on the left side $x < 0$ and $u = 1$ on the right side $x > 0$. *Heat will flow from the right side to the left side.* The temperature along the left half will go up from $u = -1$. The right half will go down from $u = 1$. *Solved in Example* 6.

Example 2 (*Finite bar*) Suppose the bar goes from $x = 0$ to $x = 1$. The initial condition $u(0, x) = 1$ tells us the (constant) temperature along the bar at time $t = 0$. We also need boundary conditions like $u(t, 0) = 0$ and $u(t, 1) = 0$ at the ends of the bar. Then the ends stay at zero temperature for all time $t > 0$.

Heat will flow out the ends. Imagine a bar in a freezer, with the sides coated. Heat escapes only at $x = 0$ and $x = 1$. We solve the heat equation to find the temperature $u(t, x)$ at every position $0 < x < 1$ and every time $t > 0$.

$$\boxed{\textbf{Heat equation} \quad \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} = \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2} \quad \text{with } u(0, x) = 1 \text{ and } u(t, 0) = u(t, 1) = 0.} \tag{1}$$

A good form for the solution is a Fourier series. It is natural to choose a sine series, since every basis function $\sin k\pi x$ is zero at $x = 0$ and $x = 1$—exactly what the boundary conditions require: zero temperature at the ends of the bar.

The initial value $u(0, x)$ and the differential equation $u_t = u_{xx}$ will have to tell us the coefficients $b_1(t), b_2(t), \ldots$ in the Fourier sine series. Heat escapes and $b_k(t) \to 0$.

Solution plan The equation $u_t = u_{xx}$ looks different from $du/dt = Au$, but it's not. The solution still combines the eigenvectors. The pieces for the ODE were $ce^{\lambda t}\boldsymbol{x}$. The pieces for the PDE are $be^{\lambda t} \sin k\pi x$.

1. Eigenvectors of A change to eigenfunctions of the second derivative: $(\sin k\pi x)'' = -k^2\pi^2 \sin k\pi x$.
2. $\boldsymbol{u}(0) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots$ changes to $u(0, x) = b_1 \sin \pi x + b_2 \sin 2\pi x + \cdots$ (*with infinitely many* b's)
3. The solution (7) adds up $b_k e^{\lambda_k t} \sin k\pi x$. It is an infinite Fourier series.

Infinity could make the problem difficult, but the $\sin k\pi x$ are orthogonal. Problem solved.

Solution by Fourier Series

Everything comes from choosing the right form for the solution $u(t,x)$. Here it is:

$$\textbf{Sine series} \quad u(t,x) = b_1(t)\sin\pi x + b_2(t)\sin 2\pi x + \cdots = \sum_{k=1}^{\infty} \boldsymbol{b_k(t)\sin k\pi x}. \qquad (2)$$

This form shows **separation of variables**. Functions $\boldsymbol{b_k(t)}$ depending on t multiply functions $\boldsymbol{\sin k\pi x}$ depending on x. When we substitute that product $b_k(t)\sin k\pi x$ into the heat equation, we get a differential equation for each of the coefficients b_k:

$$\frac{\partial}{\partial t}(b_k \sin k\pi x) = \frac{\partial^2}{\partial x^2}(b_k \sin k\pi x) \quad \text{gives} \quad \frac{\partial b_k}{\partial t}\sin k\pi x = -k^2\pi^2 b_k \sin k\pi x. \qquad (3)$$

Then $\boldsymbol{b_k{}' = -k^2\pi^2 b_k}$. Solving this equation will produce every $b_k(t)$ from $b_k(0)$:

$$\textbf{Decay comes from } \boldsymbol{e^{\lambda t}} \qquad \boldsymbol{b_k(t) = e^{-k^2\pi^2 t} b_k(0)}. \qquad (4)$$

Final step: The starting values $b_k(0)$ are decided by the initial condition $u(0,x) = 1$:

$$\textbf{At } \boldsymbol{t = 0} \qquad u(0,x) = \sum_{k=1}^{\infty} b_k(0)\sin k\pi x = 1 \quad \text{for } 0 < x < 1. \qquad (5)$$

This is an ordinary Fourier series question: What are the coefficients of a square wave $\boldsymbol{SW}(x)$? Sines are odd functions, $\sin(-x) = -\sin x$. *The series in* (5) *must add to* -1 *for x between* -1 *and* 0. So the square wave jumps from -1 to 1. It is negative on half of the interval and positive on the other half:

$$\boldsymbol{SW}(x) = \left\{\begin{array}{rcc} -1 & \text{for} & -1 < x < 0 \\ 1 & \text{for} & 0 < x < 1 \end{array}\right\} = \frac{4}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \cdots\right). \qquad (6)$$

The even coefficients $b_2, b_4, \ldots$ are all zero. The odd coefficients are $b_k = 4/\pi k$. Those b's were computed in Section 8.1, as the first example of a Fourier series. Now these numbers are giving the coefficients $b_k(0)$ at $t = 0$. Then the equation $b_k{}' = -k^2\pi^2 b_k$ tells us the coefficients $\boldsymbol{e^{-k^2\pi^2 t} b_k(0)}$ at all future times $t > 0$:

$$\textbf{Solution} \qquad u(t,x) = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} b_k(0)\sin k\pi x = \frac{4}{\pi}\left(\boldsymbol{e^{-\pi^2 t}\sin\pi x} + \cdots\right) \qquad (7)$$

This completes the solution of the heat equation. The heat drops off quickly! Those are powerful exponentials $e^{-\pi^2 t}$ and $e^{-9\pi^2 t}$. The bar will feel extremely cold when $t = 1$.

Note The correct heat equation should be $\boldsymbol{u_t = cu_{xx}}$ with a **diffusion constant** $\boldsymbol{c}$. Otherwise the equation is dimensionally wrong. The units of c are $(\text{distance})^2/\text{time}$, in order to balance u_t with u_{xx}. Then c is large for metals—heat flows easily—compared to its value for water or air. The factor c enters the eigenvalues $-ck^2\pi^2$.

The heat equation is also the **diffusion equation**. A smokestack is almost a point source (a delta function). The smoke spreads out (diffuses into the air). This would involve two space dimensions x and y, or even x, y, z. The PDE could become $\boldsymbol{u_t = c(u_{xx} + u_{yy})}$.

Summary *We had a boundary value problem in x, and an initial value problem in t:*

1. The basis functions $S_k = \sin k\pi x$ **depend on $\boldsymbol{x}$**. They solve $u_{xx} = \lambda u$.

2. The coefficients b_k **depend on $\boldsymbol{t}$**. They solve $b' = \lambda b$ with $b(0)$ coming from $u(0)$.

The basis functions $S_k(x)$ satisfy the *boundary conditions.*
Their coefficients $b_k(t)$ satisfy the *initial conditions*:

$$\textbf{Separation at } \boldsymbol{t = 0} \qquad u(0,x) = \sum b_k(0)\, S_k(x) \tag{8}$$

The PDE for $u(t,x)$ gives an ODE for each coefficient $b_k(t)$. Here are three more bars.

Example 3 (*Insulated bar*) No heat escapes from the ends of the bar. The boundary conditions change to $\partial u/\partial x = 0$ at those ends. *The basis functions change to cosines.* The series (8) becomes a Fourier cosine series.

$$\begin{array}{lll} \text{Initial condition} & u(0,x) & = \sum a_k(0) \cos k\pi x \\ \text{Equation for the } a_k & da_k/dt & = -k^2\pi^2 a_k \text{ for } k = 0, 1, 2, \ldots \end{array}$$

Notice that $k = 0$ is included. The first basis function is $\cos 0\pi x = 1$. Its coefficient is controlled by $da_0/dt = 0$. Thus $k = 0$ contributes a constant a_0 to the solution $u(t,x)$. The temperature approaches this constant everywhere along the bar, since $a_1, a_2, a_3, \ldots$ all die out exponentially fast.

Example 4 (*Circular bar*) Now sines and cosines are both included. The basis functions can also be complex exponentials e^{ikx}. Again u goes to a constant steady state c_0:

$$u(t,x) = \sum_{-\infty}^{\infty} c_k(t) e^{ik\pi x} \quad \text{and} \quad \frac{dc_k}{dt} = -k^2\pi^2 c_k. \tag{9}$$

When you have a separated form for the pieces of u, your problem is nearly solved.

Example 5 (*Infinite bar*) This problem leads to something new and important. There are no boundaries. All exponentials e^{ikx} (not just whole numbers k) are needed. By combining the solutions for $-\infty < k < \infty$ we can solve the heat equation starting from a delta function $\delta(x)$. This "**heat kernel**" is the key to chemical engineering. By a totally unexpected development it is also central to mathematical finance. The prices of stock options are modelled by the Black-Scholes partial differential equation.

To solve for each separate e^{ikx}, look for the right multiplier $e^{i\omega t}$:

$$\boldsymbol{u = e^{i\omega t} e^{ikx}} \ \textbf{ solves } \ \boldsymbol{u_t = u_{xx}} \ \textbf{ when } \ \boldsymbol{i\omega = (ik)^2.} \tag{10}$$

Then $i\omega t = (ik)^2 t = -k^2 t$. The solution $u(t,x)$ has a separated form, with these pieces:

$$\boldsymbol{u(t,x) = e^{-k^2 t} e^{ikx}} \ \textbf{ solves the heat equation.} \text{ It starts from } u(0,x) = e^{ikx}. \tag{11}$$

The Heat Kernel $U(t, x)$

The delta function $\delta(x)$ contains all exponentials e^{ikx} in equal amounts. By superposition, the solution U to the heat equation starting from $\delta(x)$ will contain the solutions $e^{-k^2t}e^{ikx}$ in equal amounts. Integrate $e^{-k^2t}e^{ikx}$ over all k to find the heat kernel U.

$$\text{The solution with } \boldsymbol{U(0,x) = \delta(x)} \quad \text{is} \quad U(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2t}e^{ikx}\,dk. \tag{12}$$

Computing this integral is possible, but unexpected. No simple function of k has the derivative e^{-k^2t}, or close. The neat way is to start with $\partial U/\partial x$. The derivative of e^{ikx} brings the extra factor ik. Then integration by parts connects dU/dx to U:

$$\frac{\boldsymbol{dU}}{\boldsymbol{dx}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (e^{-k^2t}\boldsymbol{k})(\boldsymbol{i}e^{ikx})\,dk = \frac{1}{4\pi t}\int_{-\infty}^{\infty} (e^{-k^2t})(\boldsymbol{x}e^{ikx})\,dk = -\frac{\boldsymbol{xU}}{\boldsymbol{2t}}. \tag{13}$$

Now dU/U equals $-x\,dx/2t$. Integration gives $-x^2/4t$ and then $U = ce^{-x^2/4t}$.

The total heat $\int u\,dx$ starts at $\int \delta(x)\,dx = 1$. To stay at 1, we choose $c = 1/\sqrt{4\pi t}$. Then we have the "fundamental solution" for a point source.

$$\textbf{Heat kernel} \quad U_t = U_{xx} \ \textbf{with} \ U(0,x) = \delta(x) \qquad \boxed{U = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}} \tag{14}$$

Example 6 On an infinite bar, the heat kernel (14) solves $u_t = u_{xx}$ starting from $\delta(x)$ at $t = 0$. Now solve Example 1, which started from $u = -1$ for negative x and $u = 1$ for positive x. Then solve for any initial function $u(0, x)$.

Here is the key idea for Example 1. The derivative of the jump from -1 to 1 at $x = 0$ is $du/dx = \boldsymbol{2\delta(x)}$. The solution starting from $2\delta(x)$ has $du/dx = 2U$, which cancels $\sqrt{4}$ in (14). *Then integrate* $2U$ *to undo the derivative and solve Example* 1 *for* u:

$$\begin{array}{l}\boldsymbol{u = \textbf{Error function}}\\ \textbf{Integral of } \boldsymbol{2U}\end{array} \qquad u(t,x) = \frac{1}{\sqrt{\pi t}}\int_0^x e^{-X^2/4t}\,dX. \tag{15}$$

For $x > 0$ this solution is positive. For $x < 0$ it is negative (the integral in (15) goes backward). At $x = 0$ the solution stays at zero, which we expect by symmetry. I wrote the words "error function" because this important integral has been computed and tabulated to high accuracy (no simple function has the derivative e^{-x^2}). We just change the variable of integration from X to $Y = X/\sqrt{4t}$, to see the standard error function:

$$u = \frac{1}{\sqrt{\pi t}}\int_0^x e^{-X^2/4t}\,dX = \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}} e^{-Y^2}\,dY = \textbf{erf}\left(\frac{x}{\sqrt{4t}}\right). \tag{16}$$

The integral is a cumulative probability for a normal distribution (this is the area under a bell-shaped curve). Statisticians need these integrals erf (x) all the time. At $x = \infty$ we have the total probability $=$ total area under the curve $= 1$.

Finally, we can solve $u_t = u_{xx}$ from *any starting function* $u(0,x)$. The key is to realize that every function of x is **an integral of shifted delta functions $\delta(x-a)$**:

$$\text{Every function } u_0(x) \text{ has } \int_{-\infty}^{\infty} u_0(a)\, \delta(x-a)\, da = u_0(x). \tag{17}$$

By superposition, the solution to $u_t = u_{xx}$ must be an integral of shifted heat kernels.

$$\textbf{Temperature at time } t \qquad \boldsymbol{u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(a) e^{-(x-a)^2/4t}\, da.} \tag{18}$$

I have used the crucial fact that when the point source shifts by a to become $\delta(x-a)$, *the solution also shifts by a*. So I just shifted the heat kernel U, by changing x to $x-a$. The heat equation on the whole line $-\infty < x < \infty$ is **linear shift-invariant**.

The solution (18) is reduced to one infinite integral—still not simple. And for a more realistic finite bar, with boundary conditions at $x = 0$ and $x = 1$, we have to think again. There will also be changes when the diffusion coefficient c in $u_t = (cu_x)_x$ is changing with x or t or u. This thinking probably leads us to finite differences.

Separation of Variables

The basis functions $\sin k\pi x$ are eigenfunctions. The same is true for $\cos k\pi x$ and $e^{ik\pi x}$. Let me show this by substituting $u = B(t)\,A(x)$ into the equation $u_t = u_{xx}$. Right away $\boldsymbol{u_t}$ **gives $\boldsymbol{B'}$ and $\boldsymbol{u_{xx}}$ gives $\boldsymbol{A''}$**. The separated variables are connected by $u_t = u_{xx}$:

$$\boldsymbol{B'(t)\,A(x) = B(t)\,A''(x)} \quad \textbf{leads to} \quad \boxed{\frac{A''(x)}{A(x)} = \frac{B'(t)}{B(t)} = \textbf{constant}} \tag{19}$$

Why a constant? Because A''/A depends only on x and B'/B depends only on t. They are equal, so neither one can move. Call that constant $-\lambda$:

$$\frac{A''}{A} = -\lambda \text{ gives } A = \sin\sqrt{\lambda}\,x \text{ and } \cos\sqrt{\lambda}\,x \qquad \frac{B'}{B} = -\lambda \text{ gives } B = e^{-\lambda t} \tag{20}$$

The products $BA = e^{-\lambda t}\sin\sqrt{\lambda}\,x$ and $BA = e^{-\lambda t}\cos\sqrt{\lambda}\,x$ solve the heat equation for any number λ. But the boundary condition $u(t,0) = 0$ eliminates the cosines. Then $u = 0$ at $x = 1$ requires $\sin\sqrt{\lambda} = 0$ and $\lambda = k^2\pi^2$. Separation of variables has recovered the correct basis functions $\sin k\pi x$ as eigenfunctions for $A'' = -\lambda A$.

Example 7 (Smokestack problem) We backed away from the heat equation in $2+1$ dimensions. The solution to $\boldsymbol{u_t = u_{xx} + u_{yy}}$ involves three variables t, x, y. Put a smokestack at the center point $x = y = 0$, and suppose there is no wind. Then nothing depends on the direction angle θ. Smoke will diffuse out from the center. The concentration depends only on the radial distance r, and we solve the radially symmetric heat equation. Our final solution is $u(t,r)$.

The heat equation is not quite $u_t = u_{rr}$ because r = constant is *curved* (a circle). The correct radial equation is perfect for separation of variables $u = \boldsymbol{B(t)\,A(r)}$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \quad \text{leads to} \quad B'(t)\,A(r) = B(t)\,(A'' + \frac{1}{r}A'). \tag{21}$$

Again B'/B = constant $= -\lambda$ and $B = e^{-\lambda t}$ as before. But instead of $A''/A = -\lambda$, we have **Bessel's equation for the radial eigenfunction $\boldsymbol{A(r)}$:**

$$\textbf{Basis functions } \boldsymbol{A(r)} \qquad \frac{d^2 A}{dr^2} + \frac{1}{r}\frac{dA}{dr} = -\lambda A \ \text{ has a variable coefficient } \frac{1}{r}. \tag{22}$$

The solutions are among the special functions that have been studied for centuries. They are not complex exponentials because the coefficient $1/r$ is not constant. *Bessel replaces Fourier.* This book can't go all the way to solve Bessel's equation, but see Section 6.5. A heat equation with symmetry led Bessel to new eigenfunctions.

■ REVIEW OF THE KEY IDEAS ■

1. The heat equation $u_t = u_{xx}$ is solved by $e^{-k^2\pi^2 t}\sin k\pi x$ for every $k = 1, 2, \ldots$
2. A combination of those solutions matches the initial $u(0, x)$ to its Fourier sine series.
3. With $u_x = 0$ at $x = 0$ and 1, use cosines. With an infinite bar, use all $e^{-k^2 t}e^{ikx}$.
4. The heat kernel $U = e^{-x^2/4t}/\sqrt{4\pi t}$ solves $U_t = U_{xx}$ starting from $U_0 = \delta(x)$.
5. Separation into $B(t)A(x)$ shows that $A(x)$ is an eigenfunction of the "x part" u_{xx}.

Problem Set 8.3

1 Solve the heat equation $u_t = cu_{xx}$ on an infinite bar with coefficient c, starting from $u = e^{ikx}$ at $t = 0$. As in (10) the solution has the product form $u = e^{i\omega t}e^{ikx}$. With c in the equation, *find ω for each k.*

2 Solve the same equation $u_t = cu_{xx}$ starting from the point source $u = \delta(x) = \int e^{ikx}\,dk/2\pi$ at $t = 0$. By superposition, you integrate over all k the solutions u in Problem 1. The result is the heat kernel as in equation (14) but adjusted for c.

3 To solve $u_t = cu_{xx}$ for a bar between $x = 0$ and $x = 1$, the basis functions are still $\sin k\pi x$ (with $u = 0$ at the ends). What are the eigenvalues λ_k that go into the solution $\sum b_k(0)\,e^{-\lambda_k t}\sin k\pi x$?

4 Following Problem 3, solve $u_t = cu_{xx}$ when the initial temperature is $u_0 = 1$ for $\frac{1}{4} \le x \le \frac{3}{4}$ (and $u_0 = 0$ on the first and last quarters of the bar). The problem is to find the coefficients $b_k(0)$ for that initial temperature.

5 Solve the heat equation $u_t = u_{xx}$ from a point source $u(x,0) = \delta(x)$ with free boundary conditions $u'(\pi,t) = u'(-\pi,t) = 0$. Use the infinite cosine series $\delta(x) = (1 + 2\cos x + 2\cos 2x + \cdots)/2\pi$ multiplied by time decay factors $b_k(t)$.

6 (Bar from $x = 0$ to $x = \infty$) Solve $u_t = u_{xx}$ on the positive half of an infinite bar, starting from the shifted delta function $u_0 = \delta(x-a)$ at a point $x = a > 0$. Here is a way to use the full-bar heat kernel U in (14), and still keep $u = 0$ at $x = 0$.

Imagine a negative point source at $x = -a$. Solve the heat equation on the fully infinite bar, including both sources in $u_0 = \delta(x-a) - \delta(x+a)$ at $t = 0$. Your solution (a difference of heat kernels) will stay zero at the boundary $x = 0$ (*Why*?). Then it must be the correct solution on the half-bar, since it started correctly.

7 Check that the basis functions $s_k = \sin\left(k+\frac{1}{2}\right)\pi x$ are orthogonal over $0 \le x \le 1$. Find a formula for the coefficient B_4 in the Fourier series $F(x) = \sum B_k s_k$. (Multiply by $s_4(x)$ and integrate, to isolate B_4.)

8 The basis functions $\sin(k+\frac{1}{2})\pi x$ are for **fixed-free** boundaries ($u = 0$ at $x = 0$ and $u' = 0$ at $x = 1$). What are the basis functions for **free-fixed boundaries** ($u' = 0$ at $x = 0$ and $u = 0$ at $x = 1$)?

9 Suppose $\boldsymbol{u_t = u_{xx} - u}$ with boundary condition $u = 0$ at $x = 0$ and $x = 1$. Find the new numbers λ_k in the general solution $u = \sum b_k(0)\,e^{-\lambda_k t}\sin k\pi x$. (Previously $\lambda_k = -k^2\pi^2$, now there is a new term in λ because of $-u$.)

10 Explain each step in equation (13). Solve $dU/dx = -xU/2t$ to reach $U = e^{-x^2/4t}$. How do the known infinite integrals $\int e^{-x^2}dx = \sqrt{\pi}$ and $\int u\,dx = 1$ lead to the factor $1/\sqrt{4\pi t}$?

11 (**Shift invariance**) What is the solution to $u_t = u_{xx}$ starting from $\boldsymbol{\delta(x-a)}$ at $t = 0$?

12 What are basis functions $A(x,y)$ for heat flow in a square plate, when $u = 0$ along the four sides $x = 0, x = 1, y = 0, y = 1$? The heat equation is $u_t = u_{xx} + u_{yy}$. Find eigenfunctions for $A_{xx} + A_{yy} = \lambda A$ that satisfy the boundary conditions.

The first eigenfunction is $A_{11} = (\sin\pi x)(\sin\pi y)$. Find the eigenvalues λ.

13 Substitute $U = e^{-x^2/4t}/\sqrt{4\,\pi\,t}$ to show that this heat kernel solves $U_t = U_{xx}$.

Notes on a heat bath (This is the opposite problem to a hot bar in a freezer.) The bar is initially at $\boldsymbol{U = 0}$. It is placed into a heat bath at the fixed temperature $U_B = 1$. The boundary conditions are no longer zero and the bar will get hot.

The difference $V = U - U_B$ has zero boundary values, and its initial values are $V = -1$. Now the eigenfunction method (separation of variables) solves for V. The series in (7) is multiplied by -1 to account for $V(x,0) = -1$. Adding back U_B solves the heat bath problem: $U = U_B + V = 1 - u(x,t)$.

Here $U_B \equiv 1$ is the *steady state* solution at $t = \infty$, and V is the *transient* solution. The transient starts at $V = -1$ and decays quickly to $V = 0$.

Heat bath at one end This problem is different in another way too. The fixed "Dirichlet" boundary condition is replaced by the free "Neumann" condition on the slope: $\boldsymbol{u'(1,t) = 0}$. Only the left end is in the heat bath. Heat flows down the metal bar and out at the far end, now located at $x = 1$. How does the solution change for fixed-free?

Again $U_B = 1$ is a steady state. The boundary conditions apply to $V = 1 - U_B$:

Fixed-free eigenfunctions $\quad V(0) = 0$ and $V'(1) = 0$ **lead to** $\quad A(x) = \sin\left(k + \frac{1}{2}\right)\pi x.$

Those new eigenfunctions (adjusted to $A'(1) = 0$) give a new product form $B_k(t)\,A_k(x)$:

Fixed-free solution
$$V(x,t) = \sum_{\text{odd } k} B_k(0)\, e^{-(k+\frac{1}{2})^2\pi^2 t} \sin\left(k + \tfrac{1}{2}\right)\pi x.$$

All frequencies shift by $\frac{1}{2}$ and multiply by π, because $A'' = -\lambda A$ has a free end at $x = 1$. The crucial question is: **Does orthogonality still hold for these new eigenfunctions $\sin\left(k + \frac{1}{2}\right)\pi x$?** The answer to Problem 7 is *yes* because $A'' = -\lambda A$ is symmetric.

Notes on stochastic equations and models for stock prices with Brownian motion. A "stochastic differential equation" has a random term on the right hand side. Instead of a smooth forcing term $q(t)$, or even a delta function $\delta(t)$, the models for stock prices include Brownian motion dW. The idea is subtle and important, and I will just write it down. *A random step has $dW = Z\sqrt{dt}$.* Here Z has a normal Gaussian distribution with mean zero and variance $\sigma^2 = 1$. But a new Z is chosen randomly *at every instant.*

The step size $\sqrt{\Delta t}$ produces a random walk $W(t)$ with wild oscillations. You could see a discrete random walk from $W(t + \Delta t) = W(t) + Z\sqrt{\Delta t}$, and then let Δt approach zero. The true random walk is *nowhere continuous.*

A steady return $S(t)$ on an investment has $S' = aS$. The growth is $S(t) = e^{at}S(0)$ exactly as in Chapter 1. But stock prices also respond to a stochastic part $\sigma\, dW$, where the number σ measures the **volatility of the market**. This mixes ups and downs from Brownian motion $\sigma\, dW$ with steady growth (drift) from $dS = aS\, dt$:

"Diffusion" and "drift"
$$\frac{dS}{S} = \sigma\, dW + a\, dt.$$

Then the basic model for the value of a call option leads to the Black-Scholes equation. The solution comes by a change of variables to reach the heat equation. When they are buying and selling options, traders would have that solution available at all times.

8.4 The Wave Equation

Heat travels with *infinite speed*. Waves travel with *finite speed*. Start both of them from a point source $u_0(x) = \delta(x)$. Compare the solutions at time t:

Heat equation $u_t = u_{xx}$	$u(t,x) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$	is a *smooth function*
Wave equation $u_{tt} = c^2 u_{xx}$	$u(t,x) = \frac{1}{2}\delta(x-ct) + \frac{1}{2}\delta(x+ct)$	*has spikes*

We are starting from a big bang $u = \delta(x)$ at $x = 0$. At a later time t, the bang reaches the two points $x = ct$ and $x = -ct$. That represents travel to the right and to the left with velocities $dx/dt = c$ and $-c$. The speed of sound in air is $c = 342$ meters/second.

Notice another difference from the heat equation. After the bang passes point $x = c$ at time $t = 1$, *silence returns*: $\delta(x - ct) = 0$ when $ct > x$. For the heat equation, temperatures like $e^{-x^2/4t}$ never return to zero. A wavefront passes by and we hear it only once. There is no echo or our ears would be full of sound.

In reality the heat equation is often mixed in with the wave equation. The sound diffuses as it travels. Then we do hear noise forever, but not much: the intensity decays fast.

The One-Way Wave Equation

We begin with a problem that will be particularly clear. It is first order in time $(t \geq 0)$ and first order in space $(-\infty < x < \infty)$. The velocity is still c:

$$\textbf{One-way wave} \qquad \boxed{\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}} \qquad \textbf{with } u = u_0(x) \textbf{ at } t = 0. \tag{1}$$

One solution is $u = e^{x+ct}$. Its time derivative $\partial u/\partial t$ brings a factor c. The same will be true for $\sin(x+ct)$ and $\cos(x+ct)$ and *any function of* $x+ct$. The right function is $u_0(x+ct)$ because this gives the correct start $u_0(x)$ at time $t = 0$:

$$\boxed{\textbf{Solution to } u_t = cu_x \qquad u(t,x) = u_0(x+ct).} \tag{2}$$

Suppose $u_0(x)$ is a step function (a wall of water). We have $u_0(x) = 0$ for negative x and $u_0(x) = 1$ for positive x. Then the dam breaks. A wall of water moves to the left with velocity c. At time t, the water reaches the point $x = -ct$ where $x + ct = 0$.

$$\textbf{Wall at } x = -ct \qquad \begin{array}{ll} u = u_0(x+ct) = \mathbf{0} & \text{for} \quad x + ct < 0 \\ u = u_0(x+ct) = \mathbf{1} & \text{for} \quad x + ct > 0 \end{array} \tag{3}$$

The line $x + ct = 0$ is called a "characteristic." The signal travels (with signal speed c) along that line in space-time, to tell about the jump from $u = 0$ to $u = 1$.

For any initial function $u_0(x)$, the solution $u = u_0(x + ct)$ is a shift of the graph. It is a one-way wave, no change in shape. The waves from $u_{tt} = c^2 u_{xx}$ go both ways.

Waves in Space

Now we solve the wave equation $\partial^2 u/\partial t^2 = c^2\, \partial^2 u/\partial x^2$. The three-dimensional form would be $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$. This is the equation satisfied by light as it travels in empty space: a vacuum. The speed of light c is about 300 million meters per second ($186,000$ miles/second). This is the fastest possible speed in Einstein's relativity theory.

The atmosphere slows down light. Positioning by GPS uses the speed c and the travel time to find the distance from satellite to receiver. (It includes many other extremely small effects.) In fact GPS is the only everyday technology I know that requires both special relativity and general relativity. Amazing that your cell phone can include GPS.

The wave equation is second order in time because of $\partial^2 u/\partial t^2$. We are given the initial velocity $v_0(x)$ as well as the initial position $u_0(x)$.

$$\textbf{At } t = 0 \textbf{ and all } x \qquad u = u_0(x) \quad \textbf{and} \quad \partial u/\partial t = v_0(x). \tag{4}$$

Look for functions that have u_{tt} equal to $c^2 u_{xx}$. Now e^{x+ct} and e^{x-ct} will both succeed. Two time derivatives produce a factor c twice (or a factor $-c$ twice, both cases give c^2). *All functions $f(x + ct)$ and all functions $g(x - ct)$ satisfy the wave equation.* The wave equation is linear, so we can combine those solutions.

$$\textbf{Complete solution to } u_{tt} = c^2 u_{xx} \qquad u(t, x) = f(x + ct) + g(x - ct) \tag{5}$$

Two functions $f(x + ct)$ and $g(x - ct)$ are exactly what we need to match two conditions u_0 and v_0 at $t = 0$:

$$\begin{array}{lrcl} \textbf{Position} & u_0(x) &=& f(x) + g(x) \\ \textbf{Velocity} & v_0(x) &=& cf'(x) - cg'(x) \end{array} \quad \text{and then} \quad \frac{1}{c}\int_0^x v_0\, dx = f(x) - g(x).$$

Add those equations to find $2f(x)$. Subtract those equations to find $2g(x)$. Divide by 2:

$$f(x) = \frac{1}{2}u_0(x) + \frac{1}{2c}\int_0^x v_0\, dx \qquad g(x) = \frac{1}{2}u_0(x) - \frac{1}{2c}\int_0^x v_0\, dx \tag{6}$$

Then d'Alembert's solution u to the wave equation has a wave traveling to the left with shape f and a wave traveling to the right with shape g:

$$\boxed{u = f(x + ct) + g(x - ct) = \frac{u_0(x + ct) + u_0(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x)\, dx} \tag{7}$$

Example 1 Start from rest (velocity $v_0 = 0$) with a sine wave $u_0(x) = \sin \omega x$. That wave splits into two waves:

$$u(t,x) = \frac{u_0(x+ct) + u_0(x-ct)}{2} = \frac{1}{2}\sin(\omega x + c\omega t) + \frac{1}{2}\sin(\omega x - c\omega t). \tag{8}$$

The trigonometry formula $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$ produces a short answer:

$$\boldsymbol{u(t,x) = (\sin \omega x)(\cos c\omega t)} \quad \textbf{Two traveling waves produce one standing wave}.$$

You sometimes see standing waves in the ocean. Not what a surfer wants to find.

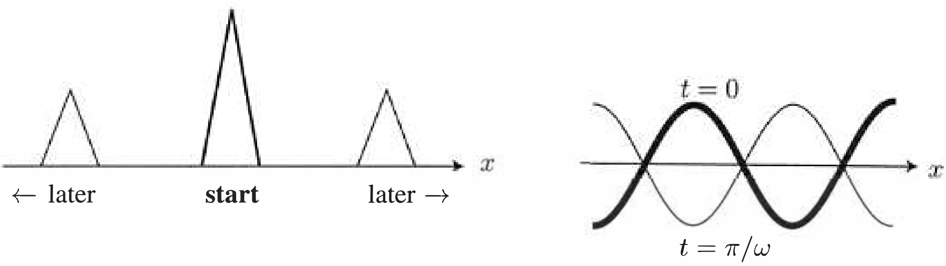


Figure 8.7: Always two traveling waves. Sometimes their sum is a standing wave.

The Wave Equation from $x = 0$ to $x = 1$

Now we leave infinite space-time. The waves we know best are on a finite Earth. They may come from a violin string, fixed at both ends. They could also be water waves (even a tsunami). They may be electromagnetic waves: light or X-rays or TV signals. Or they may be sound waves that our ears convert into words. All these waves are bringing information to our brains, and they are essential to life as we know it.

Start with a violin string of length 1. The velocity c depends on the tension in the string. The ends at $x = 0$ and 1 are assumed to remain fixed:

$$\textbf{Boundary conditions at the ends} \qquad u(t,0) = 0 \text{ and } u(t,1) = 0. \tag{9}$$

If we pluck the string with our finger at time $t = 0$, we give a vertical displacement u_0 and a vertical velocity v_0 (this might be zero):

$$\textbf{Initial conditions at the start} \qquad u(0,x) = u_0(x) \text{ and } \frac{\partial u}{\partial t}(0,x) = v_0(x). \tag{10}$$

If we remove our finger after time zero, waves move along the string. They are reflected back at the ends of the string. The sound is not a single beautiful note (it is a mixture of waves with many frequencies). Still a composer can include this plucking sound in a symphony and a guitarist uses it all the time.

The usual sound from violins comes from a *continuous source*—which is the bow. Now we are solving $u_{tt} = u_{xx} + f(t, x)$. When the violinist puts a finger on the string, *that changes the length and it changes the frequencies.* Instead of waves of length 1 we will have waves of length L and higher notes.

With several strings the violinist or cellist or guitarist is producing several waves of different frequencies to form chords. Let me stay with one string of length 1.

Separation of Variables

We will use the most important method of solving partial differential equations by hand. The wave equation $u_{tt} = c^2 u_{xx}$ has two variables t and x. **The simplest solutions are functions of x multiplied by functions of t.**

$$\text{If } \boldsymbol{u = X(x)T(t)} \quad \text{then} \quad u_{tt} = c^2 u_{xx} \quad \text{is} \quad \boldsymbol{X(x)T''(t) = c^2 X''(x)T(t)}. \tag{11}$$

T'' and X'' are ordinary second derivatives. We can divide equation (11) by c^2XT:

$$\boxed{\textbf{Separation of variables} \qquad \frac{T''}{c^2T} = \frac{X''}{X} = -\omega^2.} \tag{12}$$

The function T''/T depends only on t. The function X''/X depends only on x. So both functions are constant and they are equal. By writing $-\omega^2$ for the constant, the two separated equations have the right form:

$$X'' = -\omega^2 X \qquad \boldsymbol{X = A\cos\omega x + B\sin\omega x} \tag{13}$$

$$T'' = -\omega^2 c^2 T \qquad \boldsymbol{T = C\cos\omega ct + D\sin\omega ct} \tag{14}$$

Key question: Which frequencies ω are allowed? The boundary values at $x = 0$ and $x = 1$ decide this perfectly. We want sines and not cosines, in order to have $X(0) = 0$. We want frequencies that are multiples of π in order to have $X(1) = B\sin\omega = 0$. This gives very specific frequencies $\boldsymbol{\omega = \pi, 2\pi, 3\pi, \ldots}$ and no others.

The base frequency of the violin string is π and the harmonics are multiples $\omega = n\pi$. If we touch the string and reduce its length to L, we want $\sin\omega L = 0$. Then the permitted frequencies increase to $\boldsymbol{\omega = n\pi/L}$. The notes go up the scale, separated by an octave.

Those frequencies ω also go into the time function $T(t)$. The initial condition is $T' = 0$ if the initial velocity is $v_0 = 0$. Only the cosine survives in the time direction:

$$X = B\sin n\pi x \qquad T = C\cos n\pi ct \qquad \boldsymbol{u = XT = b(\sin n\pi x)(\cos n\pi ct)}. \tag{15}$$

With length L, the *natural frequencies* in time are $\omega = n\pi c/L$. The *wavelengths* in space are $2L/n$. The displacement of the string is a combination of solutions $X(x)T(t)$:

$$\boxed{\boldsymbol{u(t,x) = \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi x}{L}\right)\left(\cos\frac{n\pi ct}{L}\right)}.} \tag{16}$$

You see immediately that $u_{tt} = c^2 u_{xx}$ for every one of those terms, and any combination.

Final question: *What are the numbers b_n?* Those are decided by the remaining condition:

$$\textbf{Initial condition} \qquad \boldsymbol{u(0,x)} = u_0(x) = \sum_{n=1}^{\infty} \boldsymbol{b_n \sin \frac{n\pi x}{L}}. \tag{17}$$

This is a Fourier sine series! The formula for b_k comes from multiplying both sides by $\sin k\pi x/L$ and integrating from 0 to L along the string. Only one term $n = k$ survives:

$$\int_0^L u_0(x) \sin k\pi x \, dx = \int_0^L b_k (\sin k\pi x)^2 dx = \frac{L}{2} b_k. \tag{18}$$

Inserting each b_k into (16) completes the solution of the wave equation on $0 \le x \le L$.

Example 2 Suppose the length is $L = 3$ and the initial displacement is a *hat function*:

$$u_0(x) = x \ \text{ for } \ 0 \le x \le 1 \ \text{ and } \ u_0(x) = \frac{1}{2}(3-x) \ \text{ for } \ 1 \le x \le 3.$$

The integrals in (18) lead in *Mathematica* to $b_k = 3/2k^2\pi^2$. The decay rate is $1/k^2$ for this function $u_0(x)$ with a corner. The slope drops from 1 to $-\frac{1}{2}$ at $x = 1$. The infinite series (16) will converge at every point in space-time to the correct solution $u(t,x)$.

Notice also that every piece of u splits into $f + g$, by the formula for $\sin A \cos B$:

$$\sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} = 2 \sin \frac{n\pi(x+ct)}{2L} + 2 \sin \frac{n\pi(x-ct)}{2L} = \boldsymbol{f(x+ct) + g(x-ct)}.$$

We get two wave functions as always, specially chosen to fit the string length L. If the initial velocity v_0 is not zero, then the solution $u(t,x)$ also contains sine functions of t.

Our functions $X(x) = \sin n\pi x/L$ are actually eigenfunctions of the string:

$\boldsymbol{Ax = \lambda x}$ becomes $\boldsymbol{X'' = -\omega^2 X}$ The matrix A changes to a second derivative.

Again linear algebra and differential equations go hand in hand. For *linear* equations.

■ REVIEW OF THE KEY IDEAS ■

1. The one-way wave equation $u_t = cu_x$ is solved by $u(t,x) = u_0(x+ct)$.
2. The two-way equation $u_{tt} = c^2 u_{xx}$ allows two waves $f(x+ct)$ and $g(x-ct)$.
3. At $t = 0$, the d'Alembert solution (7) matches $u_0(x)$ and $v_0(x)$ on the whole line.
4. The Fourier solution (16) chooses b_k so that $u(0,x) = u_0(x)$ for $0 \le x \le L$.
5. **Separation of variables** into $u = X(x)T(t)$ gives $\boldsymbol{X'' = -\omega^2 X}$ and $\boldsymbol{T'' = -\omega^2 c^2 T}$.
6. Zero boundary conditions give $\omega = n\pi/L$ and eigenfunctions $\boldsymbol{X(x) = \sin n\pi x/L}$.

Problem Set 8.4

Problems 1 – 4 are about the one-way wave equation $\partial u/\partial t = c\partial u/\partial x$.

1 Suppose $u(0, x) = \sin 2x$. What is the solution to $u_t = cu_x$? At which times $t_1, t_2, \ldots$ will the solution return to the initial condition $\sin 2x$?

2 Suppose $u_0(x) = \delta(x)$, a big bang at the origin of the one-dimensional universe. At time t the bang is heard at the point $x =$ _____. For $u_{tt} = c^2 u_{xx}$ the bang will reach the two points $x =$ _____ and $x =$ _____ at time t.

3 (a) Integrate both sides of $u_t = cu_x$ from $x = -\infty$ to ∞ to prove that the total mass $M = \int u\,dx$ is constant: $dM/dt = 0$.

(b) Multiply by u and integrate both sides of $uu_t = cuu_x$ to prove that $E = \int u^2\,dx$ is constant.

4 Is the wave $u(t, x) = u_0(x + ct)$ traveling left or right if $c > 0$? To solve $u_t = cu_x$ on the halfline $0 \le x \le \infty$, why is a boundary condition $u(t, 0) = 0$ *not wanted*? With $c < 0$ and waves in the opposite direction, that condition is appropriate.

Problems 5 – 9 are about the one-dimensional wave equation $\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2$.

5 A "box of water" has $u_0(x) = 1$ for $-1 \le x \le 1$. Starting with zero velocity $v_0(x)$, the wave equation $u_{tt} = c^2 u_{xx}$ is solved by $u(t, x) = \frac{1}{2}u_0(x + ct) + \frac{1}{2}u_0(x - ct)$. Graph this solution for small $t = \frac{1}{2}c$ and large $t = 3c$.

6 Under a flat ocean with $u_0(x) = 1$, an earthquake produces $v_0(x) = \delta(x)$. A one-dimensional tsunami starts moving with speed c. What is the solution (7) at time t?

7 Separation of variables gives $u(t, x) = (\sin nx)(\sin nct)$ and three other similar solutions to $u_{tt} = c^2 u_{xx}$. *What are those three*? Which complex functions $e^{ikx}e^{i\omega t}$ solve the wave equation?

8 The 3D wave equation $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ becomes 1D when u has spherical symmetry: u depends only on r and t.

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}.$$

(a) *Multiply by r to find* $(ru)_{tt} = (ru)_{rr}$! Then ru is a function of $r + t$ and $r - t$.

(b) Describe the solution $ru = \delta(r - t - 1)$. This spherical sound wave has the radius $r =$ _____ at $t = 8$.

9 The wave equation along a bar with density ρ and stiffness k is $(\rho u_t)_t = (ku_x)_x$. What is the velocity c in $u_{tt} = c^2 u_{xx}$? What is ω in $u = \sin(\pi x/L)\cos \omega t$?

10 The small vibrations of a beam satisfy the fourth order equation $u_{tt} = -c^2 u_{xxxx}$. Look for solutions $u = X(x)T(t)$ and find separate equations for the functions X and T. Then find four solutions $X(x)$ when $T(t) = \cos \omega t$.

11 If that beam is clamped ($u = 0$ and $\partial u/\partial x = 0$ at both ends $x = 0$ and $x = L$), show that the frequencies ω in Problem 10 must have $(\cos \omega L)(\cosh \omega L) = 1$.

Problems 12 – 16 solve the wave equation with boundary conditions at $x = 0$ and $x = L$.

12 A string plucked halfway along has $u_0(x) = \delta(x - \frac{L}{2})$ and $v_0(x) = 0$. Find the Fourier coefficients b_k from equation (18). Write the first three terms of the Fourier series solution in (16).

13 Suppose the string starts with zero velocity $v_0(x)$ from a *hat function*: $u_0(x) = 2x/L$ for $x < L/2$ and $u_0(x) = 2(L - x)/L$ for $x > L/2$. Find the Fourier coefficients b_k from (18) and the first two nonzero terms of $u(t, x)$ in (16).

14 Suppose the string starts with zero velocity $v_0(x)$ from a *box function*: $u_0(x) = 1$ for $x < L/2$. Find all the b_k in the solution $u = \sum b_k \sin(n\pi x/L) \cos(n\pi ct/L)$.

15 The boundary condition at a *free end* $x = L$ is $\partial u/\partial x = 0$ instead of $u = 0$.

Solve $X'' + \omega^2 X = 0$ to find $X(x)$ and all allowable ω's with this new condition.

Then solve $T'' + \omega^2 c^2 T = 0$ to complete the solution $u = \sum a_n X(x) T(t)$.

16 What is the solution $u(t, x)$ on a string of length $L = 2$ if $u(0, x) = \delta(x - 1)$? The end $x = 0$ is fixed by $u(t, 0) = 0$ and the end $x = 2$ is free: $\partial u/\partial x(t, 0) = 0$.

8.5 The Laplace Transform

When it succeeds, the Laplace transform can turn a linear differential equation into an algebra problem. Laplace transforms are applied to initial value problems ($t > 0$). Fourier transforms are for boundary value problems. Laplace has e^{-st} instead of e^{ikx}.

When does this transform method succeed ? I see two desirable situations :

1. The linear equation should have constant coefficients, as in $Ay'' + By' + Cy = f(t)$.

2. The driving function $f(t)$ should have a "convenient" transform.

Our list of good functions includes $f(t) = \boldsymbol{e^{at}}$ and its transform $F(s) = \boldsymbol{1/(s-a)}$. Then the differential equation will tell us the transform $Y(s)$ of the solution. The final step is to discover which function $y(t)$ has this transform $Y(s)$. Using our list of transforms and especially the rules for finding new transforms, this becomes a problem in algebra: *Invert the transform* $\boldsymbol{Y(s)}$ *to find the solution* $\boldsymbol{y(t)}$. These pages complete Section 2.6.

Particular solutions are easy with $f(t) = e^{at}$. The method of undetermined coefficients taught us to look for $y_p(t) = Ye^{at}$. The Laplace transform is not strictly needed when $f(t) = e^{at}$ or t^n or $\sin \omega t$ or $\cos \omega t$. But for driving functions that turn on and off, and functions that jump or explode (step functions and delta functions and worse), the algebra becomes more systematic and better organized by the Laplace transform. Examples **1**, **2**, **3** with real, imaginary, and complex poles show you the key ideas.

The Transform $F(s)$

Start with a function $f(t)$ defined for $t \geq 0$. Multiply by e^{-st} and integrate from $t = 0$ to $t = \infty$. The result is the Laplace transform $F(s)$ and it depends on the exponent s:

Laplace transform
$$\mathcal{L}\,[f(t)] = F(s) = \int_{t=0}^{\infty} f(t)\, e^{-st}\, dt. \tag{1}$$

The number s can be real or complex. The one key requirement on s is that the infinite integral in (1) must give a finite answer. Here are examples needing $s > 0$ and $s > a$.

$$f(t) = 1 \qquad F(s) = \int_0^{\infty} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty} = \frac{1}{s}. \tag{2}$$

$$f(t) = e^{at} \qquad F(s) = \int_0^{\infty} e^{at}\, e^{-st}\, dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{\infty} = \frac{1}{s-a}. \tag{3}$$

The integral of e^{-st} is finite when s is positive. More than that, it is finite when *the real part of* s *is positive*. A factor $e^{-i\omega t}$ from the imaginary part $i\omega$ has absolute value 1. Laplace transforms are defined when the real part of s exceeds some value s_0. Here $s_0 = a$.

Important All functions in this section have $f(t) = 0$ for $t < 0$. They start at $t = 0$.

So the constant function $f(t) = 1$ is actually the unit step function. It jumps from 0 to 1 at $t = 0$. Its derivative is the delta function $\delta(t)$; this includes the spike at $t = 0$. In this way, the initial value problem $y' + y = 1$ ignores all $t < 0$ and starts from $y(0)$.

You will see that the Laplace transform of that equation is $sY(s) - y(0) + Y(s) = 1/s$. Then algebra gives $Y(s)$ and the inverse Laplace transform gives $y(t)$.

The second example $f = e^{at}$ includes the first example $f = 1$, which has $a = 0$. Then $1/(s-a)$ becomes $1/s$. We need $\operatorname{Re} s > a$ to drive $e^{at}e^{-st}$ to zero at $t = \infty$. There are decreasing functions like $f(t) = e^{-t^2}$ that allow every complex number s. There are also rapidly increasing functions like $f(t) = e^{t^2}$ that allow no s at all.

For a delta function located at $t = T \geq 0$, the integral picks out the transform e^{-sT}:

$$\boldsymbol{f(t) = \delta(t-T)} \qquad \boldsymbol{F(s)} = \int_0^\infty \delta(t-T)\, e^{-st}\, dt = \boldsymbol{e^{-sT}}. \tag{4}$$

To complete this group of examples (the all-star functions), a simple trick gives the transforms of $\cos \omega t$ and $\sin \omega t$. Write Euler's formula $e^{i\omega t} = \cos \omega t + i \sin \omega t$. Take the Laplace transform of every term:

$$\textbf{Linearity} \qquad \mathcal{L}\left[e^{i\omega t}\right] = \mathcal{L}\left[\cos \omega t\right] + i\,\mathcal{L}\left[\sin \omega t\right]$$

The left side is $1/(s - i\omega)$. Multiply by $(s + i\omega)/(s + i\omega)$ to see real and imaginary parts:

$$\frac{1}{s - i\omega}\,\frac{s + i\omega}{s + i\omega} = \frac{s + i\omega}{s^2 + \omega^2} \qquad \boldsymbol{\mathcal{L}\left[\cos \omega t\right] = \frac{s}{s^2 + \omega^2}} \text{ and } \boldsymbol{\mathcal{L}\left[\sin \omega t\right] = \frac{\omega}{s^2 + \omega^2}} \tag{5}$$

Exponents in $f(t)$ are Poles in $F(s)$

Let me pause one minute, before using Laplace transforms to solve differential equations. We can already see the key connection between a function $f(t)$ and its transform $F(s)$. Look at this *Table of Transforms*:

$f(t)$	1	e^{at}	$\delta(t-T)$	$\cos \omega t$	$\sin \omega t$	$t^n e^{ct}$
$F(s)$	$\dfrac{1}{s}$	$\dfrac{1}{s-a}$	e^{-sT}	$\dfrac{s}{s^2+\omega^2}$	$\dfrac{\omega}{s^2+\omega^2}$	$\dfrac{n!}{(s-c)^{n+1}}$

Here is the important message. **If $f(t)$ includes e^{at} then $F(s)$ has a "pole" at $s = a$.** A pole is an isolated point a, real or complex, where the function $F(s)$ blows up. Some integer power $(s-a)^m$ will cancel the pole and leave an "analytic" function $(s-a)^m F(s)$.

An example shows this matchup of exponents in $f(t)$ to poles in the transform $F(s)$:

$$\boldsymbol{f(t)} = e^{0t} + e^{at} + e^{i\omega t} + e^{-i\omega t} + te^{ct} \text{ has exponents } \mathbf{0},\ \boldsymbol{a},\ \boldsymbol{i\omega},\ \boldsymbol{-i\omega},\ \boldsymbol{c}$$

$$\boldsymbol{F(s)} = \frac{1}{s} + \frac{1}{s-a} + \frac{2s}{(s-i\omega)(s+i\omega)} + \frac{1}{(s-c)^2} = \frac{\text{something}}{\boldsymbol{s(s-a)(s-i\omega)(s+i\omega)(s-c)^2}}.$$

The first term $1/s$ has exponent 0 in $f(t)$ and blowup at the pole $s = 0$. The last term $1/(s-c)^2$ has exponent c and double blowup (*double pole*) at $s = c$. In the middle, $2\cos\omega t$ contains two exponents $i\omega$ and $-i\omega$, so the transform $F(s)$ has those two poles.

At the very end you see all the pieces of $F(s)$ tangled together in one big fraction. This is how $F(s)$ comes to us from a differential equation. Normally we must factor the denominator to see five separate poles at $s = 0, a, i\omega, -i\omega, c$. Then $F(s)$ splits into its simple pieces (called partial fractions). The inverse Laplace transform of each piece of $F(s)$ gives a piece of $f(t)$. **PF2** and **PF3** in Section 2.6 allowed two or three pieces.

An engineer moves poles by changing the design. Then the exponents move. The system becomes more stable if their real parts become more negative. A quick accurate picture of stability comes from the poles of $F(s)$. If all those poles are in the left half of the complex plane, where Re $s < 0$, the function will decay to zero (asymptotic stability).

The new function in this example is $\boldsymbol{te^{ct}}$. We remember that the extra factor t appears in the solution $y(t)$ when the exponent c is repeated (c is a *double root* of the polynomial $s^2 - 2cs + c^2$ that comes from $y'' - 2cy' + c^2y$). The double root becomes a *double pole* in the transform, when $(s-c)^2$ shows up in the denominator of $F(s)$. Here is the required step, to confirm that **the transform of $\boldsymbol{f(t) = te^{ct}}$ is $\boldsymbol{F(s) = 1/(s-c)^2}$.**

$$\textbf{The derivative of } \boldsymbol{F(s)} = \int_0^\infty \boldsymbol{f(t)e^{-st}dt} \textbf{ is } \boldsymbol{\frac{dF}{ds}} = \int_0^\infty \boldsymbol{-tf(t)e^{-st}dt}.$$

Rule: If the function $\boldsymbol{f(t)}$ transforms to $\boldsymbol{F(s)}$, then $\boldsymbol{tf(t)}$ transforms to $\boldsymbol{-dF/ds}$.

When this rule is applied to $f(t) = e^{ct}$ with $F(s) = 1/(s-c)$, we learn that te^{ct} transforms to $dF/ds = 1/(s-c)^2$.

This rule extends directly to higher powers of t in $t^n f(t)$. Each time you multiply by t, take the derivative of $F(s)$. Remember to multiply by -1:

$$\boldsymbol{t^2 f(t) \longrightarrow (-1)^2 \frac{d^2F}{ds^2}} \qquad t^2e^{ct} \longrightarrow \frac{d^2}{ds^2}\left(\frac{1}{s-c}\right) = \frac{d}{ds}\frac{-1}{(s-c)^2} = \boldsymbol{\frac{2}{(s-c)^3}}.$$

Continuing this way, the transform of $t^n e^{ct}$ is $n!/(s-c)^{n+1}$. This was the last entry in our Table of Transforms. In the special case $c = 0$, **the transform of $\boldsymbol{t^n}$ is $\boldsymbol{n!/s^{n+1}}$.**

Now we can work with any real poles c or imaginary poles $i\omega$ in $F(s)$. Example 3 will allow complex poles $c + i\omega$. This solves all equations $Ay'' + By' + Cy = 0$.

Transforms of Derivatives

Differential equations involve dy/dt. We must connect the transform $\mathcal{L}[dy/dt]$ to $\mathcal{L}[y]$. This step was especially easy for Fourier transforms—just multiply by ik. *For Laplace transforms we expect to multiply $Y(s)$ by s to get $\mathcal{L}[dy/dt]$, but another term appears.*

The reason this happens is that Laplace completely ignores $t < 0$. The integral starts at $t = 0$ and the number $y(0)$ is important. A good thing that $y(0)$ enters the Laplace transform, because we certainly expect it to enter the solution to a differential equation.

It is integration by parts that connects $\mathcal{L}[dy/dt]$ to $\mathcal{L}[y]$. Two minus signs cancel:

$$\boldsymbol{\mathcal{L}}\left[\frac{\boldsymbol{dy}}{\boldsymbol{dt}}\right] = \int_0^\infty \frac{dy}{dt}e^{-st}dt = \int_0^\infty y(t)(se^{-st})dt + \left[y(t)e^{-st}\right]_0^\infty = \boldsymbol{s\,\mathcal{L}\,[y] - y(0)}. \tag{6}$$

This is the key fact that turns a differential equation for $y(t)$ into an algebra problem for $Y(s)$. If we repeat this step (apply it now to dy/dt), you will see the transform of the second derivative. **Use equations (6) and (7) to transform differential equations**.

$$\boldsymbol{\mathcal{L}}\left[\frac{\boldsymbol{d^2y}}{\boldsymbol{dt^2}}\right] = s\mathcal{L}\left[\frac{dy}{dt}\right] - \frac{dy}{dt}(0) = \boldsymbol{s^2\mathcal{L}\,[y] - sy(0) - \frac{dy}{dt}(0)}. \tag{7}$$

Let me use this rule right away to solve three differential equations. The first has **real poles**. The second has **imaginary poles**. The third has **complex poles** $s = -1 \pm i$.

Example 1 Solve $\boldsymbol{y' - y = 2e^{-t}}$ starting from $y(0) = 1$.

Solution Take the Laplace transform of both sides. We know $\mathcal{L}[2e^{-t}] = 2/(s+1)$:

$$s\mathcal{L}[y] - y(0) - \mathcal{L}[y] = \mathcal{L}[2e^{-t}] \quad \text{is the same as} \quad \boldsymbol{(s-1)Y(s) = 1 + \tfrac{2}{s+1}}.$$

Then algebra gives $Y(s)$ and we split into "partial fractions" to recognize $y(t)$.

$$Y(s) = \frac{1}{s-1} + \frac{2}{(s-1)(s+1)} = \frac{1}{s-1} + \left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \boldsymbol{\frac{2}{s-1} - \frac{1}{s+1}}$$

The inverse transform of $\boldsymbol{Y(s)}$ is $\boldsymbol{y(t) = 2e^t - e^{-t}}$

I always check that $y(0) = 2 - 1 = 1$ and $y'(t) = 2e^t + e^{-t}$ agrees with $y + 2e^{-t}$. And don't forget our usual method. A particular solution is $y_p = -e^{-t}$. It has the same form as the driving function $f(t) = e^{-t}$. The null solution is $y_n = Ce^t$.

From Chapter 2 $\quad \boldsymbol{y = y_p + y_n = -e^{-t} + Ce^t} \quad y(0) = 1$ gives $C = 2$

Maybe the earlier method is simpler for this example? The next examples give practice with second order equations. The complex poles of $Y(s)$ give oscillations $e^{i\omega t}$ in $y(t)$.

Example 2 Solve the equation $y'' + y = \frac{1}{2}\sin 2t$ starting from rest: $y(0) = y'(0) = 0$. The transform of y'' is $s^2 Y(s)$ from (7):

$$s^2 Y(s) + Y(s) = \frac{1}{s^2 + 2^2} \quad \text{and then} \quad Y(s) = \frac{1}{(s^2+1)(s^2+4)}$$

Partial fractions will rewrite that transform $Y(s)$ as

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)} = \frac{\mathbf{1/3}}{\boldsymbol{s^2+1}} - \frac{\mathbf{1/3}}{\boldsymbol{s^2+4}}. \tag{8}$$

We recognize those fractions as transforms of sine functions with $\omega = 1$ and $\omega = 2$:

Solution $\boldsymbol{y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t}$ has initial values $y(0) = 0$ and $y'(0) = 0$.

The transform of sin $2t$ is $2/(s^2+4)$, which explains why $1/3$ becomes $1/6$.

In Chapter 2 we would have found $y_p(t)$ and $y_n(t)$ to reach the same $y(t)$:

$$\boldsymbol{y = y_p + y_n} = -\tfrac{1}{6}\sin 2t + c_1 \cos t + c_2 \sin t.$$

Then $c_1 = 0$ because $y(0) = 0$, and $c_2 = \frac{1}{3}$ because $y'(0) = 0$. Both ways are good.

Example 3 $y'' + 2y' + 2y = 0$ with $y(0) = y'(0) = 1$ has $Y(s) = \dfrac{\boldsymbol{s-1}}{\boldsymbol{s^2+2s+2}}$.

Then the roots of $s^2 + 2s + 2$ are the complex poles $\boldsymbol{s = -1 \pm i}$.

This $Y(s)$ is not yet in our table. But we know the complex solutions $e^{(-1+i)t}$ and $e^{(-1-i)t}$. Their real and imaginary parts are $e^{-t}\cos t$ and $e^{-t}\sin t$. The combination that has $y(0) = y'(0) = 1$ is $y = \boldsymbol{e^{-t}\cos t + 2e^{-t}\sin t}$. This must be the function $y(t)$ that transforms to $Y(s)$.

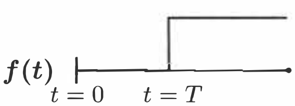
The real and imaginary parts of $e^{ct}e^{i\omega t}$ transform to the real and imaginary parts of $1/(s - c - i\omega)$. Those two new transforms solve Example 3 when $c = -1$ and $\omega = 1$. We can now solve every equation $Ay'' + By' + Cy = 0$.

$$\boxed{\boldsymbol{e^{ct}\cos\omega t} \text{ transforms to } \frac{\boldsymbol{s-c}}{\boldsymbol{(s-c)^2+\omega^2}} \qquad \boldsymbol{e^{ct}\sin\omega t} \text{ transforms to } \frac{\boldsymbol{\omega}}{\boldsymbol{(s-c)^2+\omega^2}}.}$$

Shifts and Step Functions and Cutoffs

Suppose the driving function $f(t)$ in a differential equation turns on at time T. Or suppose it turns off. Or it jumps to a different function. All these jumps in $f(t)$ are realistic in practical problems, and they are automatically handled by the Laplace transform.

Essentially, *we need the transform of a step function*. The basic example is a unit step that jumps from $f = 0$ for $t < T$ to $f = 1$ for $t \geq T$. The transform is an easy integral:



$$\boldsymbol{F(s)} = \int_T^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{\boldsymbol{e^{-sT}}}{\boldsymbol{s}}. \tag{9}$$

A step function at T transforming to e^{-sT}/s is an example of a new rule.

The step at T is a time shift of the step at $t = 0$. Multiply the transform by e^{-sT}.

The original $f(t)$ has the transform $F(s)$. The shifted function is zero until $t = T$, and then it is $f(t-T)$. For the example of a unit step, the shifted step is zero for $t < T$.

Here is the proof of the transform rule for the shifted function: **multiply by e^{-sT}.**

$f(t)$ shifts to $f(t-T)$
$F(s)$ becomes $e^{-sT}F(s)$

$$\int_T^\infty f(t-T)\, e^{-st}\, dt = \int_0^\infty f(\tau)\, e^{-s(\tau+T)}\, d\tau = \boldsymbol{e^{-sT}F(s)}.$$

The first integral has $T \le t < \infty$. The second integral has $0 \le \tau < \infty$. The new variable $\tau = t - T$ shifts the lower limit on the integral back to $\tau = 0$, and it produces the all-important factor e^{-sT}. We end with two examples that need this shift rule.

Example 4 **(Unit step function)** Solve $y' - ay = H(t-T) = \left\{ \begin{array}{ll} \mathbf{0} & t < T \\ \mathbf{1} & t \ge T \end{array} \right\}$.

The transform of every term (with $y(0) = 1$) will give the transform $Y(s)$ of the solution:

$$s\,Y(s) - 1 - a\,Y(s) = \frac{e^{-sT}}{s} \qquad \boldsymbol{Y(s) = \frac{1}{s-a} + \frac{e^{-sT}}{(s-a)s}}. \tag{10}$$

The inverse transform of $1/(s-a)$ is $\boldsymbol{e^{at}}$. Split the other fraction into two parts:

$$\frac{1}{(s-a)s} = \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) \text{ has inverse transform } \boldsymbol{\frac{1}{a}\left(e^{at} - 1\right)}. \tag{11}$$

The factor e^{-sT} in (10) will shift that function in (11). The final solution is

Jump in y'
Corner in y

$$\boldsymbol{y(t)} = \left\{ \begin{array}{ll} \boldsymbol{e^{at}} & \text{for } t \le T \\ \boldsymbol{e^{at} + \frac{1}{a}\left(e^{a(t-T)} - 1\right)} & \text{for } t \ge T \end{array} \right. \tag{12}$$

The first part $y = e^{at}$ has $y' = ay$ as required. This meets the second part correctly at $t = T$ (*no jump in* y). Then the second part of $y(t)$ continues with $y' = ay + 1$:

$$\textit{Check} \quad y' = ae^{at} + e^{a(t-T)} = a\left[e^{at} + \frac{1}{a}e^{a(t-T)} - \frac{1}{a} + \frac{1}{a}\right] = ay + 1.$$

Question Could we have solved this problem without Laplace transforms? Certainly $y = e^{at}$ solves the first part starting from $y(0) = 1$. This is y_n since $f = 0$, and it reaches e^{aT} at time T. Starting from there, we need to add on a particular solution y_p. This y_p will match the driving function $f = 1$ that begins to act at $t = T$:

$${y_p}' - ay_p = 1 \text{ starting from } y_p(T) = 0.$$

Eventually, and somehow, we would find the particular solution $y_p = \left(e^{a(t-T)} - 1\right)/a$. Combined with $y_n = e^{at}$, the complete solution $y_n + y_p$ agrees with equation (12).

Example 5 **Suppose the driving function $f(t) = 1$ turns off instead of on at time T:**

$$\text{Solve } \boldsymbol{y' - ay} = \begin{cases} \mathbf{1} & t \le T \\ \mathbf{0} & t > T \end{cases} \quad \text{with } y(0) = 1.$$

Solution Instead of the previous $H(t-T)$, this new driving function is $1 - H(t-T)$. The step function drops from 1 to 0. We still take the Laplace transform of every term in the differential equation:

$$s\,Y(s) - 1 - a\,Y(s) = \text{ transform of } [1 - H(t-T)] = \frac{1}{s} - \frac{e^{-sT}}{s}.$$

Solve this equation for $Y(s)$ and begin to recognize the inverse transform:

$$Y(s) = \frac{1}{s-a} + \frac{1}{(s-a)s} - \frac{e^{-sT}}{(s-a)s} \quad \text{has the new term } \frac{1}{(s-a)s} \quad \text{compared to (10).}$$

The inverse transform of this new term is $(e^{at} - 1)/a$, according to (11). Since the last term in $Y(s)$ now has a minus sign, the final solution has two pieces meeting at $t = T$:

$$y(t) = \begin{cases} e^{at} + \frac{1}{a}(e^{at} - 1) & \text{for } t \le T \\ e^{at} + \frac{1}{a}(e^{at} - 1) - \frac{1}{a}(e^{a(t-T)} - 1) & \text{for } t \ge T. \end{cases}$$

That first part for $t \le T$ would be our standard $y_n + y_p$, starting from $y(0) = 1$. The second part matches the first part at $t = T$ (*no jump in* y). That second part simplifies to

$$y(t) = e^{at} + \frac{e^{at} - e^{a(t-T)}}{a} \quad \text{and we verify that } y' = ay.$$

Rules for the Laplace Transform

Part of this section is about specific functions $f(t)$. We made a Table of Transforms $F(s)$. The other part of the section is about rules. (This is like calculus. You learn the derivatives of t^n and $\sin t$ and $\cos t$ and e^t. Then you learn the product rule and quotient rule and chain rule.) We need a Table of Rules for the Laplace transform, when we know that $F(s)$ and $G(s)$ are the transforms of $f(t)$ and $g(t)$.

Addition Rule	The transform of $\boldsymbol{f(t) + g(t)}$ is	$\boldsymbol{F(s) + G(s)}$
Shifting Rule	The transform of $\boldsymbol{f(t-T)}$ is	$\boldsymbol{e^{-sT} F(s)}$
Derivative of f	The transform of $\boldsymbol{df/dt}$ is	$\boldsymbol{sF(s) - f(0)}$
Derivative of F	The transform of $\boldsymbol{tf(t)}$ is	$\boldsymbol{-dF/ds}$
Convolution Rule	Section 8.6 will transform $\boldsymbol{f(t)g(t)}$ and invert $\boldsymbol{F(s)G(s)}$	

Problem Set 8.5

1 When the driving function is $f(t) = \delta(t)$, the solution starting from rest is the **impulse response**. The impulse is $\delta(t)$, the response is $y(t)$. Transform this equation to find the **transfer function** $Y(s)$. Invert to find the impulse response $y(t)$.

$$y'' + y = \delta(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

2 (Important) Find the first derivative and second derivative of $f(t) = \sin t$ for $t \geq 0$. Watch for a jump at $t = 0$ which produces a spike (delta function) in the derivative.

3 Find the Laplace transform of the unit box function $b(t) = \{1 \text{ for } 0 \leq t < 1\} = H(t) - H(t-1)$. The unit step function is $H(t)$ in honor of Oliver Heaviside.

4 If the Fourier transform of $f(t)$ is defined by $\widehat{f}(k) = \int f(t)e^{-ikt}dt$ and $f(t) = 0$ for $t < 0$, what is the connection between $\widehat{f}(k)$ and the Laplace transform $F(s)$?

5 What is the Laplace transform $R(s)$ of the standard **ramp function** $r(t) = t$? For $t < 0$ all functions are zero. The derivative of $r(t)$ is the unit step $H(t)$. Then multiplying $R(s)$ by s gives _____ .

6 Find the Laplace transform $F(s)$ of each $f(t)$, and the poles of $F(s)$:

(a) $f = 1 + t$ (b) $f = t\cos\omega t$ (c) $f = \cos(\omega t - \theta)$
(d) $f = \cos^2 t$ (e) $f = e^{-2t}\cos t$ (f) $f = te^{-t}\sin\omega t$

7 Find the Laplace transform s of $f(t) =$ next integer above t and $f(t) = t\,\delta(t)$.

8 *Inverse Laplace Transform*: Find the function $f(t)$ from its transform $F(s)$:

(a) $\dfrac{1}{s - 2\pi i}$ (b) $\dfrac{s+1}{s^2+1}$ (c) $\dfrac{1}{(s-1)(s-2)}$
(d) $1/(s^2 + 2s + 10)$ (e) $e^{-s}/(s-a)$ (f) $2s$

9 Solve $y'' + y = 0$ from $y(0)$ and $y'(0)$ by expressing $Y(s)$ as a combination of $s/(s^2+1)$ and $1/(s^2+1)$. Find the inverse transform $y(t)$ from the table.

10 Solve $y'' + 3y' + 2y = \delta$ starting from $y(0) = 0$ and $y'(0) = 1$ by Laplace transform. Find the poles and partial fractions for $Y(s)$ and invert to find $y(t)$.

11 Solve these initial-value problems by Laplace transform:

(a) $y' + y = e^{i\omega t}, y(0) = 8$ (b) $y'' - y = e^t,\ y(0) = 0,\ y'(0) = 0$
(c) $y' + y = e^{-t}, y(0) = 2$ (d) $y'' + y = 6t,\ y(0) = 0,\ y'(0) = 0$
(e) $y' - i\omega y = \delta(t), y(0) = 0$ (f) $my'' + cy' + ky = 0,\ y(0) = 1, y'(0) = 0$

12 The transform of e^{At} is $(sI - A)^{-1}$. Compute that matrix (the transfer function) when $A = [1\ \ 1;\ 1\ \ 1]$. Compare the poles of the transform to the eigenvalues of A.

13 If dy/dt decays exponentially, show that $sY(s) \to y(0)$ as $s \to \infty$.

14 Transform Bessel's time-varying equation $ty'' + y' + ty = 0$ using $\mathcal{L}[ty] = -dY/ds$ to find a first-order equation for Y. By separating variables or by substituting $Y(s) = C/\sqrt{1+s^2}$, find the Laplace transform of the Bessel function $y = J_0$.

15 Find the Laplace transform of a single arch of $f(t) = \sin \pi t$.

16 Your acceleration $v' = c(v^* - v)$ depends on the velocity v^* of the car ahead:

(a) Find the ratio of Laplace transforms $V^*(s)/V(s)$.

(b) If that car has $v^* = t$ find your velocity $v(t)$ starting from $v(0) = 0$.

17 A line of cars has $v_n' = c[v_{n-1}(t-T) - v_n(t-T)]$ with $v_0(t) = \cos \omega t$ in front.

(a) Find the growth factor $A = 1/(1 + i\omega e^{i\omega T}/c)$ in oscillation $v_n = A^n e^{i\omega t}$.

(b) Show that $|A| < 1$ and the amplitudes are safely decreasing if $cT < \frac{1}{2}$.

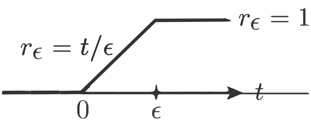
(c) If $cT > \frac{1}{2}$ show that $|A| > 1$ (dangerous) for small ω. (Use $\sin\theta < \theta$.) Human reaction time is $T \geq 1$ sec and human aggressiveness is $c = 0.4/\text{sec}$. Danger is pretty close. Probably drivers adjust to be barely safe.

18 For $f(t) = \delta(t)$, the transform $F(s) = 1$ is the limit of transforms of tall thin box functions $b(t)$. The boxes have width $\epsilon \to 0$ and height $1/\epsilon$ and area 1.

$$\text{Inside integrals,} \quad b(t) = \left\{ \begin{array}{ll} 1/\epsilon & \text{for } 0 \leq t < \epsilon \\ 0 & \text{otherwise} \end{array} \right\} \text{ approaches } \delta(t).$$

Find the transform $B(s)$, depending on ϵ. Compute the limit of $B(s)$ as $\epsilon \to 0$.

19 The transform $1/s$ of the unit step function $H(t)$ comes from the limit of the transforms of short steep ramp functions $r_\epsilon(t)$. These ramps have slope $1/\epsilon$:



Compute $R_\epsilon(s) = \displaystyle\int_0^\epsilon \frac{t}{\epsilon} e^{-st}\,dt + \int_\epsilon^\infty e^{-st}\,dt$. Let $\epsilon \to 0$.

20 In Problems 18 and 19, show that the derivative of the ramp function $r_\epsilon(t)$ is the box function $b(t)$. The "generalized derivative" of a step is the ______ function.

21 What is the Laplace transform of $y'''(t)$ when you are given $Y(s)$ and $y(0), y'(0), y''(0)$?

22 The *Pontryagin maximum principle* says that the optimal control is "bang-bang"— it only takes on the extreme values permitted by the constraints. To go from rest at $x = 0$ to rest at $x = 1$ in minimum time, use maximum acceleration A and deceleration $-B$. At what time t do you change from the accelerator to the brake ? (This is the fastest driving between two red lights.)

8.6 Convolution (Fourier and Laplace)

This section is about multiplication. **Convolution is a different way to multiply functions.** It is also a way to multiply vectors. The rule for vectors may look new, but actually you learned it in third grade. Let me start with ordinary multiplication of numbers, and build up to convolution of vectors and convolution of functions.

When 112 is multiplied by 2 1 3, watch how we collect nine small multiplications:

$$\begin{array}{ccccc} & & 1 & 1 & 2 \\ & & 2 & 1 & 3 \\ \hline & & 3 & 3 & 6 \\ & 1 & 1 & 2 & \\ 2 & 2 & 4 & & \\ \hline \mathbf{2} & \mathbf{3} & \mathbf{8} & \mathbf{5} & \mathbf{6} \end{array} \qquad \begin{array}{ccccc} & & a & b & c \\ & & 2 & 1 & 3 \\ \hline & & 3a & 3b & 3c \\ & a & b & c & \\ 2a & 2b & 2c & & \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

We don't think about this pattern—it is so familiar. In our minds we are just multiplying 112 by 213 in small steps. The new idea is to think of $(1,1,2)$ as a vector and $(2,1,3)$ as another vector. **The convolution of those vectors is the vector** $(\mathbf{2},\mathbf{3},\mathbf{8},\mathbf{5},\mathbf{6})$.

I need a new symbol $*$ for the convolution of two vectors $\boldsymbol{c}$ and $\boldsymbol{d}$:

Convolution of vectors $\quad \boldsymbol{c} * \boldsymbol{d} = (c_0, c_1, \ldots) * (d_0, d_1, \ldots) = (\boldsymbol{c_0 d_0}, \boldsymbol{c_0 d_1} + \boldsymbol{c_1 d_0}, \ldots)$

That line ends with an important hint about $\boldsymbol{c} * \boldsymbol{d}$, if we can see it. First, every c_i multiplies every d_j. (Those are the nine small multiplications.) Then the nine products are collected in a special way. We put $c_0 d_1$ with $c_1 d_0$. **The next component of $\boldsymbol{c} * \boldsymbol{d}$ will be $\boldsymbol{c_0 d_2} + \boldsymbol{c_1 d_1} + \boldsymbol{c_2 d_0}$.**

In the third grade multiplication, we are collecting together all the products $c_i d_j$ that go in the 100s column. Those were $300 + 100 + 400$. To express this with algebra, the n^{th} component of $\boldsymbol{c} * \boldsymbol{d}$ will be $\boldsymbol{c_0 d_n} + \boldsymbol{c_1 d_{n-1}} + \cdots + \boldsymbol{c_n d_0}$. **These are all the products $\boldsymbol{c_i d_j}$ with $\boldsymbol{i + j = n}$.**

Convolution $\boldsymbol{c} * \boldsymbol{d} = \boldsymbol{d} * \boldsymbol{c}$

$$(\boldsymbol{c} * \boldsymbol{d})_n = \sum_{\boldsymbol{i+j=n}} \boldsymbol{c_i d_j} = \sum_{\boldsymbol{i}} \boldsymbol{c_i d_{n-i}}. \tag{1}$$

The summation symbol allows the vectors to be infinitely long. The key point is that small multiplications $c_i d_j$ go together when $\boldsymbol{i + j = n}$, which is the same as $\boldsymbol{j = n - i}$. Let me show that rule again, this time for $2 + x + 3x^2$ times $1 + x + 2x^2$. **We are collecting all the pieces that multiply each power $\boldsymbol{x^n}$.**

$$\begin{array}{rrrrr} & & 1\,+ & x\;+ & 2x^2 \\ & & 2\,+ & x\;+ & 3x^2 \\ \hline & & 3x^2 + & 3x^3 + & 6x^4 \\ & x\;+ & x^2 + & 2x^3 & \\ 2\,+ & 2x\;+ & 4x^2 & & \\ \hline \mathbf{2} + & \mathbf{3}x + & \mathbf{8}x^2 + & \mathbf{5}x^3 + & \mathbf{6}x^4 \end{array}$$

When we multiply polynomials, we take the convolution of the vectors of coefficients.

$(2,1,3) * (1,1,2) = (\mathbf{2},\mathbf{3},\mathbf{8},\mathbf{5},\mathbf{6})$

We will connect convolution of coefficients to multiplication of Fourier series. First, allow me to show one more example that collects the small multiplications $c_i d_j$ in the same "convolution way." That example is a matrix-vector multiplication $C\boldsymbol{d}$. The matrix C has the numbers $c_0, c_1, \ldots$ along its diagonals and C **times $\boldsymbol{d}$ is exactly the convolution $\boldsymbol{c} * \boldsymbol{d}$.**

$C\boldsymbol{d} = \boldsymbol{c} * \boldsymbol{d}$
Constant diagonals
Toeplitz matrix
Shift invariant

$$\begin{bmatrix} c_0 & & \\ c_1 & c_0 & \\ c_2 & c_1 & c_0 \\ & c_2 & c_1 \\ & & c_2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} c_0\,d_0 \\ c_1\,d_0 + c_0\,d_1 \\ c_2\,d_0 + c_1\,d_1 + c_2\,d_0 \\ c_2\,d_1 + c_1\,d_2 \\ c_2\,d_2 \end{bmatrix} \tag{2}$$

These "convolution matrices" are the key to signal processing. In that highly active world, the matrix C is a *filter*. The way to understand this filter is through its frequency response $c_0 + c_1 e^{-i\theta} + c_2 e^{-2i\theta}$.

We are ready to connect convolution with Fourier series and Laplace transforms.

Multiplying $f(x)g(x)$ is Convolution of Coefficients

Convolution answers a question that we unavoidably ask. When $\sum c_k e^{ikx}$ multiplies $\sum d_l e^{ilx}$ (call those functions $f(x)$ and $g(x)$), *what are the Fourier coefficients of the function* $h(x) = f(x)g(x)$? The answer is certainly not $c_k d_k$. We have to multiply every coefficient c_k times every coefficient d_l. All those small multiplications $c_k d_l$ produce the coefficients of $(\sum c_k e^{ikx})(\sum d_l e^{ilx})$. The logic of the convolution rule has two steps:

1. $c_k e^{ikx}$ times $d_l e^{ilx}$ equals $c_k d_l e^{inx}$ when $\boldsymbol{k + l = n}$.
2. The e^{inx} term in $f(x)g(x)$ contains every product $c_k d_l$ in which $\boldsymbol{l = n - k}$.

The n^{th} Fourier coefficient of $(\sum c_k e^{ikx})(\sum d_l e^{ilx})$ is the n^{th} component of $\boldsymbol{c} * \boldsymbol{d}$:

Multiply functions f, g
Convolve coefficients c, d

$$\textbf{Coefficient of } fg = (\boldsymbol{c} * \boldsymbol{d})_n = \sum_{k=-\infty}^{\infty} c_k\, d_{n-k}. \tag{3}$$

Example 1 The "identity vector" in convolution is $\boldsymbol{\delta} = (\ldots, 0, 0, 1, 0, 0, \ldots)$. Then $\boldsymbol{\delta} * \boldsymbol{d} = \boldsymbol{d}$ for every vector $\boldsymbol{d}$. The "identity function" is $i(x) = 1$. Then $i(x)g(x) = g(x)$ for every function g. The Fourier coefficients of $i(x) = 1$ are exactly $\boldsymbol{\delta}$.

You see how **convolution in frequency space (k - space)** leads to **multiplication in function space (x - space)**. This is the central idea of the convolution rule.

Example 2 The **autocorrelation** of a vector $\boldsymbol{c}$ is the convolution $\boldsymbol{c} * \boldsymbol{c'}$. That vector $\boldsymbol{c'}$ is the reverse of $\boldsymbol{c}$. The components of $\boldsymbol{c'}$ are the Fourier coefficients $\overline{c}_{-k}$ of $\overline{f(x)}$. So autocorrelation $\boldsymbol{c} * \boldsymbol{c'}$ gives the Fourier coefficients of the product $f(x)\overline{f(x)} = |f(x)|^2$:

$$f\overline{f} = (1 + e^{ix})(1 + e^{-ix}) = \mathbf{1}\,e^{-ix} + \mathbf{2} + \mathbf{1}\,e^{ix} \qquad \boldsymbol{c} * \boldsymbol{c'} = (0,1,1) * (1,1,0) = (\mathbf{1}, \mathbf{2}, \mathbf{1}).$$

The autocorrelation of the box vector (0,1,1) is the hat vector **(1,2,1)**. *Box* $*$ *box* $=$ *hat.*

Convolution of Functions

The reverse question is equally important and has to be answered. If $f(x)$ and $g(x)$ have Fourier coefficients c_k and d_k, **what function has the Fourier coefficients $\boldsymbol{c_k d_k}$** ? We are multiplying vectors in k-space. Then we have convolution $f * g$ of functions in x-space !

$$\textbf{Periodic Convolution} \quad (f * g)(x) = \int_0^{2\pi} f(y)g(x-y)dy = \int_0^{2\pi} g(y)f(x-y)dy. \tag{4}$$

Vector convolution is $(\boldsymbol{c} * \boldsymbol{d})_n = \sum c_i d_{n-i}$. The key is $\boldsymbol{i} + (\boldsymbol{n} - \boldsymbol{i}) = \boldsymbol{n}$. Convolution of functions has an integral instead of a sum (of course). Above all we notice that $\boldsymbol{y} + (\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{x}$. The pattern stays exactly the same when the functions are not periodic and the integrals go from $-\infty$ to ∞:

$$\textbf{Infinite Convolution} \quad (f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = \int_{-\infty}^{\infty} g(y)f(x-y)\,dy. \tag{5}$$

For the Laplace transform, all functions are zero for $t < 0$. Change x and y to t and T.

$$\begin{matrix}\textbf{One-sided} \\ \textbf{Laplace}\end{matrix} \quad (f * g)(t) = \int_{\boldsymbol{0}}^{\boldsymbol{t}} f(T)g(t-T)\,dT \quad \text{because} \quad \begin{matrix} f(T) = 0 \text{ for } T < 0 \\ g(t-T) = 0 \text{ for } T > t \end{matrix}$$

Solving Differential Equations by Convolution

I want to apply convolution to the main problem of this book—the solution of equations like $y' - ay = f(t)$ and $y'' + y = f(x)$. Those are easy problems and we know the answers. Simplicity is good, it keeps the main point clear. Convolution will offer us a new way to write the solutions $y(t)$ from Laplace and $y(x)$ from Fourier.

I will recall the old ways to solve the same equations. The next page has a summary of the outstanding examples in this book—*linear equations with constant coefficients*.

Example 3 Solve the equation $y' - ay = f(t)$ by convolution, starting from $y(0) = 0$.

Solution Take the Laplace transform of both sides, and divide to find $Y(s)$:

$$sY(s) - aY(s) = F(s) \quad \text{gives} \quad Y(s) = \frac{F(s)}{s-a} = \boldsymbol{G(s)\,F(s)}. \tag{6}$$

The transform $F(s)$ of the driving function is multiplied by the "transfer function" $G(s)$. In this problem $G(s) = 1/(s-a)$. Then $y(t)$ is the inverse transform of $Y(s) = G(s)F(s)$.

The key is convolution. **Multiplication in $\boldsymbol{s}$ - space becomes convolution in $\boldsymbol{t}$ - space.** This rule gives the solution $\boldsymbol{y} = \boldsymbol{g} * \boldsymbol{f}$ from $Y = GF$. Then we prove the rule.

The inverse transform of the transfer function $G(s)$ is the impulse response $g(t)$. For the equation $y' - ay = f(t)$, the transfer function is $G(s) = 1/(s-a)$ and its inverse transform is $g(t) = e^{at}$. Then the multiplication $Y(s) = G(s)F(s)$ becomes a convolution of the impulse response e^{at} with the driving function $f(t)$:

Solution by convolution

$$y(t) = g(t) * f(t) = \int_{T=0}^{t} e^{a(t-T)} f(T)\,dT \tag{7}$$

Please recognize this solution. We are integrating $e^{-at}f(t)$ for the fourth time! The central problem of Chapter 1 was $y' - ay = f(t)$ (*or* $q(t)$). There we proposed three methods.

1. The **integrating factor** e^{-at} multiplies $y' - ay = f(t)$. Integrate $(e^{-at}y)' = e^{-at}f$.
2. **Variation of parameters** in the null solution $y_n = Ce^{at}$ gives $y_p(t) = C(t)\,e^{at}$.
3. Every input $f(T)$ is multiplied by its **growth factor** $e^{a(t-T)}$. Combine the outputs.
4. (New) The solution $y(t)$ is the **convolution** of $f(t)$ with the impulse response e^{at}.

The impulse response is $g(t) = g * \delta$, when the input is the impulse $f(t) = \delta(t)$. The forced response is $y = g * f$, when the force is $f(t)$. **Always the convolution of the driving force $f(t)$ with the Green's function $g(t)$ produces the output $y(t)$.**

Confession I used Green's name partly because the letter g appeared so conveniently. My deeper reason is to express a central idea that connects differential equations and matrix equations—the two themes of this book. *Convolution with the impulse response (the Green's function) is just like multiplication by the inverse matrix* A^{-1}.

Here is the message that comes from $AA^{-1} = I$. The vector $\boldsymbol{g}_j$ in column j of A^{-1} is the response to the delta vector $\boldsymbol{\delta}_j = (\cdot, 0, \mathbf{1}, 0, \cdot)$ in column j of the identity matrix.

$$A\boldsymbol{g}_j = \boldsymbol{\delta}_j \quad \textbf{in linear algebra} \qquad \boldsymbol{g' - ag = \delta(t)} \quad \textbf{in differential equations}$$

I hope you find this helpful. The Green's function $g(t-T)$ gives the response at time t to a unit impulse at time T. The total response at t is the integral of impulses $f(T)$ times responses $g(t-T)$. Compare with the solution $\boldsymbol{v} = A^{-1}\boldsymbol{b}$ to a matrix equation $A\boldsymbol{v} = \boldsymbol{b}$.

The inverse matrix A^{-1} gives the response at position i to a unit impulse at position j. The solution $\boldsymbol{v} = A^{-1}\boldsymbol{b}$ is the sum over all j of impulses b_j times those responses.

For shift-invariant equations, the response at t to an impulse at T depends only on the elapsed time $t - T$. For shift-invariant matrices, the responses $(A^{-1})_{ij}$ depend only on $i - j$. The differential equation has **constant coefficients**. The Toeplitz matrix has **constant diagonals**. Here A is a difference matrix and A^{-1} is a sum matrix.

$$A\boldsymbol{v} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \boldsymbol{v} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{8}$$

Example 4 (**Fourier**) Solve the equation $-y'' + y = f(x)$ for $-\infty < x < \infty$.

Solution This is a boundary value problem, with $y = 0$ at the endpoints $x = -\infty$ and $x = \infty$. Take the Fourier transform of every term, so the two derivatives in y'' become multiplications by ik:

$$-\boldsymbol{y'' + y = f(x)} \qquad -(ik)^2\hat{y} + \hat{y} = \hat{f}(k) \qquad \boldsymbol{\hat{y}(k) = \frac{\hat{f}(k)}{k^2+1} = \hat{g}(k)\,\hat{f}(k)}. \tag{9}$$

In k-space, the transform $\hat{f}(k)$ is multiplied by $\hat{g}(k) = 1/(k^2+1)$. In x-space, the right side $f(x)$ is convolved with the Green's function $g(x)$. *That Green's function $g(x)$ is the solution when the right side $f(x)$ is a delta function $\delta(x)$.*

To complete the solution we need $g(x)$. The transform approach would invert $\hat{g}(k) = 1/(k^2+1)$. The direct approach is to solve $-g'' + g = \delta(x)$. Remember that $\delta(x) = 0$ for $x > 0$ and $x < 0$:

$$\begin{array}{llll} \boldsymbol{x > 0} & -g'' + g = 0 \ \text{ gives } \ g = c_1e^{x} + c_2e^{-x} & & \text{Then } g(\infty) = 0 \text{ requires } c_1 = 0 \\ \boldsymbol{x < 0} & -g'' + g = 0 \ \text{ gives } \ g = C_1e^{x} + C_2e^{-x} & & \text{Then } g(-\infty) = 0 \text{ requires } C_2 = 0 \end{array}$$

The action is all at $x = 0$. There is no jump in the function $g(x)$, so that $C_1 = c_2$. The minus sign in $-g'' + g = \delta(x)$ produces a **drop of 1** in the slope $g'(x)$ at $x = 0$. Comparing the slopes $-c_2e^{-x}$ and C_1e^{x} at $x = 0$ gives $C_1 + c_2 = 1$. The coefficients are $C_1 = c_2 = \frac{1}{2}$ and the Green's function $g(x)$ is found:

$$\boldsymbol{g(x)} = \begin{cases} \boldsymbol{\frac{1}{2}e^{-x}} & \text{for } \ x > 0 \\ \boldsymbol{\frac{1}{2}e^{x}} & \text{for } \ x < 0 \end{cases} \qquad \text{and convolution gives } \ y(x) = \int_{-\infty}^{\infty} f(X)g(x-X)\,dX.$$

Compare with this second order equation in time, when Fourier changes to Laplace. Now we have initial values at $t = 0$ instead of boundary values at $x = \pm\infty$.

Example 5 Solve the equation $y'' + y = f(t)$ starting from $y(0) = y'(0) = 0$.

Solution Take the Laplace transform of both sides, and divide by $s^2 + 1$ to find $Y(s)$:

$$s^2Y(s) + Y(s) = F(s) \ \text{ gives } \ Y(s) = \boldsymbol{\frac{F(s)}{s^2+1} = F(s)G(s)}. \tag{10}$$

The transfer function is $G(s) = 1/(s^2+1)$. That is the Laplace transform of the impulse response (the growth factor) $\boldsymbol{g(t) = \sin t}$. (Problem 8.5.2 confirms that $(\sin t)''$ does surprisingly produce $\delta(t)$. The slope is zero for $t < 0$, and $(\sin t)'$ jumps to $\cos 0 = 1$ at $t = 0$.) Multiplication $F(s)G(s)$ corresponds to convolution $f * g$:

$$\textbf{Laplace convolution} \qquad y(t) = f(t) * g(t) = \int_0^t f(T)\sin(t-T)\,dT. \tag{11}$$

This solves Example 5 quickly—the crucial step is to be able to invert $G(s)$ to find $g(t)$.

Proof of the Convolution Rule

We need to prove that the Laplace transform of $f(t) * g(t)$ is $F(s)G(s)$. Convolution becomes multiplication. Similarly the Fourier transform of $f(x) * g(x)$ is $\widehat{f}(k)\widehat{g}(k)$.

An integral over T produces $f * g$, and then an integral over t gives its transform. The key is to reverse the order in that double integral. Integrate first with respect to t.

$$\int_{t=0}^{\infty}\left(\int_{T=0}^{\infty} f(T)g(t-T)dT\right)e^{-st}dt = \int_{T=0}^{\infty}\left(\int_{t=0}^{\infty} g(t-T)e^{-s(t-T)}dt\right)f(T)e^{-sT}dT.$$

It was safe to extend the integration to $T = \infty$, since $g(t-T) = 0$ for $T > t$. Also safe to insert e^{sT} and e^{-sT}; their product is 1. The inner integral on the right is exactly the Laplace transform $G(s)$, when $t - T$ is replaced by τ:

$$\int_{t=0}^{\infty} g(t-T)e^{-s(t-T)}dt = \int_{\tau=-T}^{\infty} g(\tau)e^{-s\tau}d\tau = \int_{\tau=0}^{\infty} g(\tau)e^{-s\tau}d\tau = G(s). \tag{12}$$

Since the inner integral is $G(s)$, the double integral is $F(s)\,G(s)$ as desired:

$$\int_{T=0}^{\infty} G(s)f(T)\,e^{-sT}dT = F(s)\,G(s).\ \text{ The convolution rule is proved.}$$

The same rule holds for Fourier transforms, except the integrals have $-\infty < x < \infty$ and $-\infty < k < \infty$. With those limits we don't have or need the one-sided condition that $g(t) = 0$ for $t < 0$. The steps are the same and we reach the same conclusion. *The Fourier transform of* $f(x) * g(x)$ *is* $\widehat{f}(k)\,\widehat{g}(k)$.

Point-Spread Functions and Deconvolution

I must not leave the impression that convolution is only useful in solving differential equations. The truth is, we solved those equations earlier. Our solutions now have the neat form $y = f * g$, but they were already found without convolutions. A better application is a telescope looking at the night sky, or a CT-scanner looking inside you.

A telescope produces a *blurred image*. When the actual star is a point source, we don't see that delta function. **The image of $\boldsymbol{\delta(x, y)}$ is a point-spread function $\boldsymbol{g(x, y)}$**: the response to an impulse, the spreading of a point. With diffraction you see an "Airy disk" at the center. The radius of this disk gives the limit of resolution for a telescope.

When the star is shifted, the image is shifted. The source $\delta(x - x_0, y - y_0)$ produces the image $g(x - x_0, y - y_0)$. It is bright at the location x_0, y_0 of the star, and g gets dark quickly away from that point. The image of the whole sky is an integral of blurred points.

The true brightness of the night sky is given by a function $f(x, y)$. *The image we see is the convolution* $c = f * g$. But if we do know the blurring function $g(x, y)$, *deconvolution will bring back $f(x,y)$ from $f * g$.* In transform space, the scanner multiplies by G and the post-processor divides by G. Here is deconvolution:

$$\boldsymbol{c = f * g} \textbf{ transforms to } \boldsymbol{C = FG}. \textbf{ The inverse transform of } \boldsymbol{F = \frac{C}{G}} \textbf{ gives } \boldsymbol{f}.$$

The manufacturer knows the point-spread function g and its Fourier transform G. The telescope or the CT-scanner comes equipped with a code for deconvolution. Transform the blurred output c to C, divide by G, and invert $F = C/G$ to find the true source function f.

Note that two-dimensional functions $f(x,y)$ have two-dimensional transforms $\widehat{f}(k,l)$. The Fourier basis functions of x and y are $e^{ikx}e^{ily}$ with two frequencies k and l.

Cyclic Convolution and the DFT

The Discrete Fourier Transform connects $\boldsymbol{c} = (c_0,\ldots,c_{N-1})$ to $\boldsymbol{f} = (f_0,\ldots,f_{N-1})$. The Fourier matrix gives $\boldsymbol{Fc} = \boldsymbol{f}$. Computations are fast, because all the vectors are N-dimensional and the FFT is available. A convolution rule will lead directly to fast multiplication and fast algorithms. This is convolution in practice.

The rule has to change from $\boldsymbol{c} * \boldsymbol{d} = (1,1,2) * (2,1,3) = (2,3,8,5,6)$. When the inputs $\boldsymbol{c}$ and $\boldsymbol{d}$ have N components, *their cyclic convolution also has N components. The new symbol in* $(1,1,2) \circledast (2,1,3) = (7,9,8)$ *indicates "cyclic" by a circle in* $\circledast$.

The key is that $\boldsymbol{w^3 = 1}$. Cyclic convolution folds $5w^3 + 6w^4$ back into $5 + 6w$.

$$(\mathbf{1} + \mathbf{1}w + \mathbf{2}w^2)(\mathbf{2} + \mathbf{1}w + \mathbf{3}w^2) = 2 + 3w + 8w^2 + 5w^3 + 6w^4 = \mathbf{7} + \mathbf{9}w + \mathbf{8}w^2.$$

In the same way, $(0,\mathbf{1},0) \circledast (0,0,\mathbf{1}) = (\mathbf{1},0,0)$ because w times w^2 equals $w^3 = 1$. I will use this example to test the cyclic convolution rule.

Cyclic convolution rule for the $\boldsymbol{N}$-point transform

> The kth component of $F(\boldsymbol{c} \circledast \boldsymbol{d})$ is $(\boldsymbol{Fc})_k$ times $(\boldsymbol{Fd})_k$. That word "times" means: Multiply $1, w, w^2$ from $\boldsymbol{Fc}$ and $1, w^2, w^4$ from $\boldsymbol{Fd}$ to get $1, w^3, w^6$, which is $1, 1, 1$.

$$\boldsymbol{F} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{w} & \boldsymbol{w^2} \\ \mathbf{1} & \boldsymbol{w^2} & \boldsymbol{w^4} \end{bmatrix} \quad F\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{w} \\ \boldsymbol{w^2} \end{bmatrix} \text{ times } F\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{w^2} \\ \boldsymbol{w^4} \end{bmatrix} \text{ is } F\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The convolution $\boldsymbol{c} \circledast \boldsymbol{d}$ has N^2 small multiplications. Component by component multiplication of two vectors only needs N. So the convolution rule gives a fast way to multiply two very long N-digit numbers (as in the prime factors that banks use for security). When you multiply the numbers, you are convolving those digits.

Transform the numbers to $\boldsymbol{f}$ and $\boldsymbol{g}$. Multiply transforms by $f_k g_k$. Transform back.

When the cost of these three discrete transforms is included, the FFT saves the day:

Go to k-space, multiply, go back $\quad N^2$ multiplications are reduced to $N + 3N \log N$.
In MATLAB, component-by-component multiplication is indicated by $\boldsymbol{f} .* \boldsymbol{g}$ (point-star).

$$F(\boldsymbol{c} \circledast \boldsymbol{d}) = (F\boldsymbol{c}).{*}(F\boldsymbol{d}) \qquad \text{ifft}\,(\boldsymbol{c} \circledast \boldsymbol{d}) = N * \text{ifft}\,(\boldsymbol{c}).{*}\text{ifft}\,(\boldsymbol{d}) \tag{13}$$

Note that the fft command transforms $\boldsymbol{f}$ to $\boldsymbol{c}$ using $\overline{w} = e^{-2\pi i/N}$ and the matrix $\overline{F}$. The ifft command inverts that transform using $w = e^{2\pi i/N}$ and the Fourier matrix F. The factor N appears in equation (13) because $F\overline{F} = NI$.

Circulant Matrices

Multiplication by an infinite constant-diagonal matrix gives an infinite convolution. When row n of C_∞ multiplies $\boldsymbol{d}$, this adds up the small multiplications $c_i d_j$ with $i + j = n$:

Infinite convolution

$$C_\infty \boldsymbol{d} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ c_0 & c_{-1} & c_{-2} & \bullet \\ c_1 & c_0 & c_{-1} & c_{-2} \\ c_2 & c_1 & c_0 & c_{-1} \\ \bullet & c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} \bullet \\ d_0 \\ d_1 \\ d_2 \\ \bullet \end{bmatrix} = \boldsymbol{c} * \boldsymbol{d}. \tag{14}$$

Similarly, **cyclic convolution comes from an N by N matrix**. The matrix is called a "circulant" because every diagonal wraps around (based on $w^N = 1$). All diagonals have N equal entries. The diagonal with c_1 is highlighted for $N = 4$:

Cyclic convolution Circulant matrix

$$C\boldsymbol{d} = \begin{bmatrix} c_0 & c_3 & c_2 & \boldsymbol{c_1} \\ \boldsymbol{c_1} & c_0 & c_3 & c_2 \\ c_2 & \boldsymbol{c_1} & c_0 & c_3 \\ c_3 & c_2 & \boldsymbol{c_1} & c_0 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \boldsymbol{c} \circledast \boldsymbol{d}. \tag{15}$$

Notice how the top row produces $c_0 d_0 + c_3 d_1 + c_2 d_2 + c_1 d_3$. Those subscripts $0 + 0$ and $3 + 1$ and $2 + 2$ are all zero when $N = 4$. *In this cyclic world*, 2 *and* 2 *add to* 0. That comes from $w^2 w^2 = w^4 = w^0$.

Circulant matrices are remarkable. If you multiply circulants B and C you get another circulant. That product BC gives convolution with the vector $\boldsymbol{b} \circledast \boldsymbol{c}$. The amazing part is the eigenvalues from the DFT and eigenvectors from the Fourier matrix:

The eigenvalues of C are the components of the discrete transform $F\boldsymbol{c}$
The eigenvectors of *every* C are the columns of F (also the columns of $\overline{F}$ and F^{-1})

We can verify two eigenvalues $\lambda = c_0 + c_1 + c_2$ and $c_0 + c_1 w + c_2 w^2$ for this circulant:

$$\begin{bmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ w^2 \\ w \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ w^2 \\ w \end{bmatrix}. \tag{16}$$

The equation $FC = \Lambda F$ is the cyclic convolution rule $F(\boldsymbol{c} \circledast \boldsymbol{d}) = (F\boldsymbol{c}).{*}(F\boldsymbol{d})$.

The End of the Book

The book is ending on a high note. Constant coefficient problems have taken a big step from $Ay'' + By' + Cy = 0$. Now we have transforms (Fourier and Laplace) and convolutions. The discrete problems bring constant diagonal matrices. Cyclic problems bring circulants. Time to stop !

I should really say, *stop and look back.* The book has emphasized linear problems, because these are the equations we can understand. It is true that life is not linear. If the input is multiplied by 10, the output might be multiplied by 8 or 12 and not 10. But in most real problems, the input is multiplied or divided by less than 1.1. Then a linear model replaces a curve by its tangent lines (this is the key to calculus). To understand applied mathematics, we need differential equations and linear algebra.

■ REVIEW OF THE KEY IDEAS ■

1. Convolution $(1,2,3) * (4,5,6)$ is the multiplication 123×456 without carrying.

2. $(\sum c_k e^{ikx})(\sum d_l e^{ilx})$ has $(c * d)_n = \sum c_k d_{n-k}$ as the coefficient of e^{inx}. **Multiply functions $\leftrightarrow$ convolve coefficients** as in $(1 + 2x + 3x^2)(4 + 5x + 6x^2)$.

3. Differential equations transform to $Y(s) = F(s)G(s)$. Then $y(t) = f(t) * g(t) =$ driving force $*$ impulse response. The impulse response $g(t)$ is the Green's function.

4. **Shift invariance** : Constant coefficient equations and constant diagonal matrices.

5. Circulants $C\boldsymbol{d}$ give cyclic convolution $\boldsymbol{c} \circledast \boldsymbol{d}$. Multiply components $(F\boldsymbol{c}).*(F\boldsymbol{d})$.

Problem Set 8.6

1 Find the convolution $\boldsymbol{v} * \boldsymbol{w}$ and also the cyclic convolution $\boldsymbol{v} \circledast \boldsymbol{w}$:

(a) $\boldsymbol{v} = (1,2)$ and $\boldsymbol{w} = (2,1)$ (b) $\boldsymbol{v} = (1,2,3)$ and $\boldsymbol{w} = (4,5,6)$.

2 Compute the convolution $(1,3,1) * (2,2,3) = (a,b,c,d,e)$. To check your answer, add $a+b+c+d+e$. That total should be 35 since $1+3+1 = 5$ and $2+2+3 = 7$ and $5 \times 7 = 35$.

3 Multiply $1 + 3x + x^2$ times $2 + 2x + 3x^2$ to find $a + bx + cx^2 + dx^3 + ex^4$. Your multiplication was the same as the convolution $(1,3,1) * (2,2,3)$ in Problem 2. When $x = 1$, your multiplication shows why $1 + 3 + 1 = 5$ times $2 + 2 + 3 = 7$ agrees with $a + b + c + d + e = 35$.

4 (Deconvolution) Which vector $\boldsymbol{v}$ would you convolve with $\boldsymbol{w} = (1,2,3)$ to get $\boldsymbol{v} * \boldsymbol{w} = (0,1,2,3,0)$? Which $\boldsymbol{v}$ gives $\boldsymbol{v} \circledast \boldsymbol{w} = (3,1,2)$?

5 (a) For the periodic functions $f(x) = 4$ and $g(x) = 2\cos x$, show that $f * g$ is **zero** (the zero function)!

(b) In frequency space (k-space) you are multiplying the Fourier coefficients of 4 and $2\cos x$. Those coefficients are $c_0 = 4$ and $d_1 = d_{-1} = 1$. Therefore every product $c_k d_k$ is ______.

6 For periodic functions $f = \sum c_k e^{ikx}$ and $g = \sum d_k e^{ikx}$, the Fourier coefficients of $f * g$ are $\mathbf{2\pi c_k d_k}$. Test this factor 2π when $f(x) = 1$ and $g(x) = 1$ by computing $f * g$ from its definition (4).

7 Show by integration that the periodic convolution $\int_0^{2\pi} \cos x \cos(t-x)\,dx$ is $\pi \cos t$. In k-space you are squaring Fourier coefficients $c_1 = c_{-1} = \frac{1}{2}$ to get $\frac{1}{4}$ and $\frac{1}{4}$; these are the coefficients of $\frac{1}{2}\cos t$. The 2π in Problem 6 makes $\pi \cos t$ correct.

8 Explain why $f * g$ is the same as $g * f$ (periodic or infinite convolution).

9 What 3 by 3 circulant matrix C produces cyclic convolution with the vector $\boldsymbol{c} = (1,2,3)$? Then $C\boldsymbol{d}$ equals $\boldsymbol{c} \circledast \boldsymbol{d}$ for every vector $\boldsymbol{d}$. Compute $\boldsymbol{c} \circledast \boldsymbol{d}$ for $\boldsymbol{d} = (0,1,0)$.

10 What 2 by 2 circulant matrix C produces cyclic convolution with $\boldsymbol{c} = (1,1)$? Show in four ways that this C is not invertible. Deconvolution is impossible.

(1) Find the determinant of C. (2) Find the eigenvalues of C.

(3) Find $\boldsymbol{d}$ so that $C\boldsymbol{d} = \boldsymbol{c} \circledast \boldsymbol{d}$ is zero. (4) $F\boldsymbol{c}$ has a zero component.

11 (a) Change $b(x) * \delta(x-1)$ to a multiplication $\widehat{b}\,\widehat{d}$. Transform the box function $b(x) = \{1 \text{ for } 0 \le x \le 1\}$ to $\widehat{b}(k) = \int_0^1 e^{-ikx}dx$. The shifted delta transforms to $\widehat{d}(k) = \int \delta(x-1)e^{-ikx}dx$.

(b) Show that your result $\widehat{b}\ \widehat{d}$ is the transform of a shifted box function. Then convolution with $\delta(x-1)$ shifts the box.

12 Take the Laplace transform of these equations to find the transfer function $G(s)$:

(a) $Ay'' + By' + Cy = \delta(t)$ (b) $y' - 5y = \delta(t)$ (c) $2y(t) - y(t-1) = \delta(t)$

13 Take the Laplace transform of $y'''' = \delta(t)$ to find $Y(s)$. From the Transform Table in Section 8.5 find $y(t)$. You will see $y''' = 1$ and $y'''' = 0$. But $y(t) = 0$ for negative t, so your y''' is actually a unit step function and your y'''' is actually $\delta(t)$.

14 Solve these equations by Laplace transform to find $Y(s)$. Invert that transform with the Table in Section 8.5 to recognize $y(t)$.

(a) $y' - 6y = e^{-t}$, $y(0) = 2$ (b) $y'' + 9y = 1$, $y(0) = y'(0) = 0$.

15 Find the Laplace transform of the shifted step $H(t-3)$ that jumps from 0 to 1 at $t=3$. Solve $y' - ay = H(t-3)$ with $y(0) = 0$ by finding the Laplace transform $Y(s)$ and then its inverse transform $y(t)$: one part for $t < 3$, second part for $t \geq 3$.

16 Solve $y' = 1$ with $y(0) = 4$—a trivial question. Then solve this problem the slow way by finding $Y(s)$ and inverting that transform.

17 The solution $y(t)$ is the convolution of the input $f(t)$ with what function $g(t)$?

(a) $y' - ay = f(t)$ with $y(0) = 3$ (b) $y' -$ (integral of y) $= f(t)$.

18 For $y' - ay = f(t)$ with $y(0) = 3$, we could replace that initial value by adding $3\delta(t)$ to the forcing function $f(t)$. Explain that sentence.

19 What is $\delta(t) * \delta(t)$? What is $\delta(t-1) * \delta(t-2)$? What is $\delta(t-1)$ *times* $\delta(t-2)$?

20 By Laplace transform, solve $y' = y$ with $y(0) = 1$ to find a very familiar $y(t)$.

21 By Fourier transform as in (9), solve $-y'' + y =$ box function $b(x)$ on $0 \leq x \leq 1$.

22 There is a big difference in the solutions to $y'' + By' + Cy = f(x)$, between the cases $B^2 < 4C$ and $B^2 > 4C$. Solve $y'' + y = \delta$ and $y'' - y = \delta$ with $y(\pm\infty) = 0$.

23 *(Review)* Why do the constant $f(t) = 1$ and the unit step $H(t)$ have the same Laplace transform $1/s$? Answer: Because the transform does not notice _____ .

MATRIX FACTORIZATIONS

1. $\boldsymbol{A = LU} = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $\boldsymbol{A = LDU} = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1's on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges. The pivots in D are divided out to leave 1's on the diagonal of U. If A is symmetric then U is L^{T} and $\boldsymbol{A = LDL^{\mathrm{T}}}$.

3. $\boldsymbol{PA = LU}$ (permutation matrix P to avoid zeros in the pivot positions).

   Requirements: A is invertible. Then P, L, U are invertible. P does all of the row exchanges in advance, to allow normal LU. Alternative: $A = L_1 P_1 U_1$.

4. $\boldsymbol{EA = R}$ (m by m invertible E) (any matrix A) = $\mathsf{rref}(A)$.

   Requirements: None ! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The last $m - r$ rows of E are a basis for the left nullspace of A; they multiply A to give zero rows in R. The first r columns of E^{-1} are a basis for the column space of A.

5. $\boldsymbol{S = C^{\mathrm{T}}C}$ = (lower triangular) (upper triangular) with $\sqrt{D}$ on both diagonals

   Requirements: S is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* $C = \mathsf{chol}(S)$ has $C^{\mathrm{T}} = L\sqrt{D}$, so $C^{\mathrm{T}}C = LDL^{\mathrm{T}}$.

6. $\boldsymbol{A = QR}$ = (orthonormal columns in Q) (upper triangular R).

   Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt or Householder process. If A is square then $Q^{-1} = Q^{\mathrm{T}}$.

7. $\boldsymbol{A = V\Lambda V^{-1}}$ = (eigenvectors in V) (eigenvalues in Λ) (left eigenvectors in V^{-1}).

   Requirements: A must have n linearly independent eigenvectors.

8. $\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$ = (orthogonal matrix Q) (real eigenvalue matrix Λ) (Q^{T} is Q^{-1}).

   Requirements: S is *real and symmetric*. This is the Spectral Theorem.

9. $\boldsymbol{A = MJM^{-1}}$ = (generalized eigenvectors in M) (Jordan blocks in J) (M^{-1}).

Requirements: A is any square matrix. This *Jordan form* J has a block for each independent eigenvector of A. Every block has only one eigenvalue.

10. $\boldsymbol{A = U\Sigma V^{\mathrm{T}}} = \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m \times n \end{pmatrix} \begin{pmatrix} m \times n \text{ singular value matrix} \\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ V \text{ is } n \times n \end{pmatrix}.$

Requirements: None. This *singular value decomposition* (**SVD**) has the eigenvectors of AA^{T} in U and eigenvectors of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$.

11. $\boldsymbol{A^{+} = V\Sigma^{+}U^{\mathrm{T}}} = \begin{pmatrix} \text{orthogonal} \\ n \times n \end{pmatrix} \begin{pmatrix} n \times m \text{ pseudoinverse of } \Sigma \\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ m \times m \end{pmatrix}.$

Requirements: None. The *pseudoinverse* A^{+} has $A^{+}A =$ projection onto row space of A and $AA^{+} =$ projection onto column space. The shortest least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = A^{+}\boldsymbol{b}$. This solves $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. When A is invertible: $A^{+} = A^{-1}$.

12. $\boldsymbol{A = QH}$ = (orthogonal matrix Q) (symmetric positive definite matrix H).

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD.

13. $\boldsymbol{A = U\Lambda U^{-1}}$ = (unitary U) (eigenvalue matrix Λ) (U^{-1} which is $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}}$).

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$: Hermitian case.

14. $\boldsymbol{A = UTU^{-1}}$ = (unitary U) (triangular T with λ's on diagonal) ($U^{-1} = U^{\mathrm{H}}$).

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular:

15. $\boldsymbol{F_n} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} & \\ & F_{n/2} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$ = one step of the (recursive) **FFT**.

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$: $F_n\overline{F}_n = nI$. D has $1, w, \ldots, w^{n/2-1}$ on its diagonal. For $n = 2^{\ell}$ the *Fast Fourier Transform* will compute $F_n\boldsymbol{x}$ with only $\frac{1}{2}n\ell = \frac{1}{2}n\log_2 n$ multiplications from ℓ stages of D's.

Properties of Determinants

1 *The determinant of the n by n identity matrix is* 1.

2 *The determinant changes sign when two rows are exchanged* (sign reversal):

3 *The determinant is a linear function of each row separately* (all other rows stay fixed).

multiply row 1 by any number t	$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
add row 1 of A to row 1 of A'	$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$

Pay special attention to rules 1–3. They completely determine the number $\det A$.

4 ***If two rows of A are equal, then* $\det A = 0$.**

5 ***Subtracting a multiple of one row from another row leaves* $\det A$ *unchanged.***

ℓ times row 1
from row 2

$$\begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

6 ***A matrix with a row of zeros has* $\det A = 0$.**

7 ***If A is triangular then* $\det A = a_{11}a_{22}\cdots a_{nn}$ *=product of diagonal entries.***

8 ***If A is singular then* $\det A = 0$. *If A is invertible then* $\det A \neq 0$.**

Proof Elimination goes from A to U. If A is singular then U has a zero row. The rules give $\det A = \det U = 0$. If A is invertible then U has the pivots along its diagonal. The product of nonzero pivots (using rule 7) gives a nonzero determinant:

Multiply pivots $\quad$ $\det A = \pm \det U = \pm$ **(product of the pivots).**

9 ***The determinant of AB is* $\det A$ *times* $\det B$: $|AB| = |A|\,|B|$.**

A times A^{-1} $\quad$ $AA^{-1} = I$ so $(\det A)(\det A^{-1}) = \det I = 1.$

10 ***The transpose A^{T} has the same determinant as A.***

Index

A

B

C

D

***The single heading* "Matrix" *indexes the active life of linear algebra*.**

Matrix

T

U

Index of Symbols

LINEAR ALGEBRA IN A NUTSHELL

((*The matrix* A *is* n *by* n))

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$A\boldsymbol{x}=\mathbf{0}$ has one solution $\boldsymbol{x}=\mathbf{0}$	$A\boldsymbol{x}=\mathbf{0}$ has infinitely many solutions
$A\boldsymbol{x}=\boldsymbol{b}$ has one solution $\boldsymbol{x}=A^{-1}\boldsymbol{b}$	$A\boldsymbol{x}=\boldsymbol{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r=n$	A has rank $r < n$
The reduced row echelon form is $R=I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values