FOURIER ANALYSIS ON GROUPS

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PREFACE

In classical Fourier analysis the action takes place on the unit circle, on the integers and on the real line. During the last 25 or 30 years, however, an increasing number of mathematicians have adopted the point of view that the most appropriate setting for the development of the theory of Fourier analysis is furnished by the class of all locally compact abelian groups. The relative ease with which the basic concepts and theorems can be transferred to this general context may be one of the factors which contributes to the feeling of some that this extension is a dilution of the classical theory, that it is merely generalization for the sake of generalization.

However, group-theoretic considerations seem to be inherent in the subject. They are implicit in much of the classical work, and their explicit introduction has led to many interesting new analytic problems (it is one of the aims of this book to prove this point) as well as to conceptual clarifications. To cite a very rudimentary example: In discussing Fourier transforms on the line it helps to have *two* lines in mind, one for the functions and one for their transforms, and to realize that each is the dual group of the other.

Also, there are classical subjects which lead almost inevitably to this extension of the theory. For instance, Bohr (1) noticed almost 50 years ago that the unique factorization theorem for positive integers allows us to regard every ordinary Dirichlet series as a power series in infinitely many variables. The boundary values yield a function of infinitely many variables, periodic in each, that is to say, a function on the infinite-dimensional torus T^{ω}. It then becomes of interest to know the closed subgroups of T^{ω}, and it turns out that these comprise all compact metric abelian groups. Once we agree to admit these groups we have to admit their duals, i.e., the countable discrete abelian groups, and since the class of all locally compact abelian groups can be built up from the compact ones, the discrete ones, and the euclidean spaces, it would seem artificial to restrict ourselves to a smaller subclass.

The principal objects of study in the present book are the group algebras $L^1(G)$ and $M(G)$; $L^1(G)$ consists of all complex functions on the group G which are integrable with respect to the Haar measure of G, $M(G)$ consists of all bounded regular Borel measures on G, and multiplication is defined in both cases by convolution. Although certain aspects of these algebras have been studied forn oncommutative groups G, I restrict myself to the abelian case. Other L^p - spaces appear occasionally, but are not treated systematically.

The development of the general theory, given in Chapter 1, is based on some simple facts concerning Banach algebras; these, as well as other background material, are collected in the Appendices at the end of the book. It seems appropriate to develop the material in this way, since much of the early work on Banach algebras was stimulated by Fourier analysis. Chapter 2 contains the structure theory of locally compact abelian groups. These two chapters are introductory, and most of their content is well known.

The material of Chapters 3 to 9, on the other hand, has not previously appeared in book form. Most of it is of very recent vintage, many of the results were obtained only within the last two or three years, and although the solutions of some of the problems under consideration are fairly complete by now, many open questions remain.

My own work in this field has been greatly stimulated by conversations and correspondence with Paul J. Cohen, Edwin Hewitt, Raphael Salem, and Antoni Zygmund, and by my collaboration with Henry Helson, Jean-Pierre Kahane, and YitzhakKatznelson. It is also a pleasure to thank the Alfred P. Sloan Foundation for its generous financial support.

Madison, Wisconsin WALTER RUDIN
November 1960

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CHAPTER 1

The Basic Theorems of Fourier Analysis

The material contained in this chapter forms the core of our subject and is used throughout the later part of this book. Various approaches are possible; the same subject matter is treated, from different points of view, in Cartan and Godement [1], Loomis [1], and Weil [1].

Unless the contrary is explicitly stated, any group mentioned in this book will be abelian and locally compact, with addition as group operation and 0 as identity element (see Appendix B). The abbreviation LCA will be used for "locally compact abelian."

1.1. Haar Measure and Convolution

1.1.1. On every *LCA* group G there exists a non-negative regular measure m (see Appendix E), the so-called *Haar measure* of G, which is not identically 0 and which is *translation-invariant*. That is to say,

$$m(E + x) = m(E) \tag{1}$$

for every $x \in G$ and every Borel set E in G.

For the construction of such a measure, we refer to any of the following standard treatises: Halmos [1], Loomis [1], Montgomery and Zippin [1], and Weil [1]. The idea of the proof is to construct a positive translation-invariant linear functional T on $C_c(G)$, the space of all continuous complex functions on G with compact support. This means that $Tf \geqq 0$ if $f \geqq 0$ and that $T(f_x) = Tf$, where f_x is the translate of f defined by

$$f_x(y) = f(y - x) \qquad (y \in G). \tag{2}$$

As soon as this is done, the Riesz representation theorem shows that there is a measure m with the required properties, such that

$$Tf = \int_G f\,dm \qquad (f \in C_c(G)). \tag{3}$$

1.1.2. *If V is a non-empty open subset of G, then $m(V) > 0$.* For if $m(V) = 0$ and K is compact, finitely many translates of V cover K, and hence $m(K) = 0$. The regularity of m then implies that $m(E) = 0$ for all Borel sets E in G, a contradiction.

1.1.3. We have spoken of *the* Haar measure of G. This is justified by the following uniqueness theorem:

If m and m' are two Haar measures on G, then $m' = \lambda m$, where λ is a positive constant.

Proof: Fix $g \in C_c(G)$ so that $\int_G g\,dm = 1$. Define λ by

$$\int_G g(-x)dm'(x) = \lambda.$$

For any $f \in C_c(G)$ we then have

$$\begin{aligned}
\int_G f\,dm' &= \int_G g(y)dm(y) \int_G f(x)dm'(x) \\
&= \int_G g(y)dm(y) \int_G f(x+y)dm'(x) \\
&= \int_G dm'(x) \int_G g(y)f(x+y)dm(y) \\
&= \int_G dm'(x) \int_G g(y-x)f(y)dm(y) \\
&= \int_G f(y)dm(y) \int_G g(y-x)dm'(x) = \lambda \int_G f\,dm.
\end{aligned}$$

Hence $m' = \lambda m$. Note that the use of Fubini's theorem was legitimate in the preceding calculation, since the integrands $g(y)f(x+y)$ and $g(y-x)f(y)$ are in $C_c(G \times G)$.

Thus Haar measure is unique, up to a multiplicative positive constant. If G is compact, it is customary to normalize m so that $m(G) = 1$. If G is discrete, any set consisting of a single point is assigned the measure 1. These requirements are of course contradictory if G is a finite group, but this will cause us no difficulty.

Having established the uniqueness of m, we shall now change our notation, and write $\int_G f(x)dx$ in place of $\int_G f\,dm$. Thus dx, dy, ... will always denote integration with respect to Haar measure.

1.1.4. *For any Borel set E in G, $m(-E) = m(E)$.* For if we set $m'(E) = m(-E)$, m' is a Haar measure on G, and so there is a constant λ such that $m(-E) = \lambda m(E)$ for all Borel sets E. Taking E so that $-E = E$, we see that $\lambda = 1$.

1.1.5. Translation in $L^p(G)$. If G is a LCA group and $1 \leqq p \leqq \infty$, we shall write $L^p(G)$ in place of $L^p(m)$ (see Appendix E7). It is clear that the L^p-norms are translation invariant, i.e., that

$$(1) \qquad \|f_x\|_p = \|f\|_p \qquad (x \in G),$$

where, we recall, f_x is the translate of f defined by

$$(2) \qquad f_x(y) = f(y - x) \qquad (y \in G).$$

THEOREM. *Suppose $1 \leqq p < \infty$ and $f \in L^p(G)$. The map*

$$(3) \qquad x \to f_x$$

is a uniformly continuous map of G into $L^p(G)$.

Proof: Let $\varepsilon > 0$ be given. Since $C_c(G)$ is dense in $L^p(G)$ (Appendix E8) there exists $g \in C_c(G)$, with compact support K, such that $\|g - f\|_p < \varepsilon/3$, and the uniform continuity of g (Appendix B9) implies that there is a neighborhood V of 0 in G such that

$$(4) \qquad \|g - g_x\|_\infty < \frac{\varepsilon}{3}[m(K)]^{-1/p}$$

for all $x \in V$. Hence $\|g - g_x\|_p < \varepsilon/3$, and so

$$\|f - f_x\|_p \leqq \|f - g\|_p + \|g - g_x\|_p + \|g_x - f_x\|_p < \varepsilon$$

if $x \in V$. Finally, $f_x - f_y = (f - f_{y-x})_x$, so that $\|f_x - f_y\|_p < \varepsilon$ if $y - x \in V$, and the proof is complete.

Note that the same theorem (with the same proof) is true with $C_0(G)$ in place of $L^p(G)$, but that it is false for $L^\infty(G)$, unless G is discrete.

1.1.6. Convolutions. For any pair of Borel functions f and g on the LCA group G we define their convolution $f * g$ by the formula

$$(1) \qquad (f * g)(x) = \int_G f(x - y) g(y) dy$$

provided that

$$\int_G |f(x-y)g(y)|dy < \infty. \tag{2}$$

Note that the integral (1) can also be written in the form

$$\int_G f_y(x)g(y)dy \tag{3}$$

so that $f * g$ may be regarded as a limit of linear combinations of translates of f; this statement may be made precise, but we assign it only heuristic value at present. (See Theorem 7.1.2.)

THEOREM. (a) *If* (2) *holds for some* $x \in G$, *then* $(f * g)(x) = (g * f)(x)$.

(b) *If* $f \in L^1(G)$ *and* $g \in L^\infty(G)$, *then* $f * g$ *is bounded and uniformly continuous.*

(c) *If* f *and* g *are in* $C_c(G)$, *with compact supports* A *and* B, *then the support of* $f * g$ *lies in* $A + B$, *so that* $f * g \in C_c(G)$.

(d) *If* $1 < p < \infty$, $1/p + 1/q = 1$, $f \in L^p(G)$, *and* $g \in L^q(G)$, *then* $f * g \in C_0(G)$.

(e) *If* f *and* g *are in* $L^1(G)$, *then* (2) *holds for almost all* $x \in G$, $f * g \in L^1(G)$, *and the inequality*

$$\|f * g\|_1 \leqq \|f\|_1 \|g\|_1$$

holds.

(f) *If* f, g, h *are in* $L^1(G)$, *then* $(f * g) * h = f * (g * h)$.

Proof: Replacing y by $y + x$ in (1) and applying 1.1.4, we obtain

$$(f * g)(x) = \int_G f(-y)g(y+x)dy = \int_G f(y)g(-y+x)dy = (g * f)(x),$$

and (a) is proved.

Under the hypotheses of (b), it is clear that

$$|(f * g)(x)| \leqq \|f\|_1 \|g\|_\infty \qquad (x \in G)$$

so that $f * g$ is bounded. For $x \in G$, $z \in G$, we have

$$\begin{aligned} |(f * g)(x) - (f * g)(z)| &\leqq \int_G |f(x-y) - f(z-y)||g(y)|dy \\ &\leqq \|f_{-x} - f_{-z}\|_1 \|g\|_\infty. \end{aligned}$$

Theorem 1.1.5 shows that the last expression can be made arbitrarily small by restricting $x - z$ to lie in a suitably chosen neighborhood of 0 and (b) follows.

If f vanishes outside A and g vanishes outside B, then $f(x-y)g(y) = 0$ unless $y \in B$ and $x - y \in A$, i.e., unless $x \in A + B$. Thus $f * g$ vanishes outside $A + B$, and (c) is proved.

To prove (d), choose sequences $\{f_n\}$ and $\{g_n\}$ in $C_c(G)$ such that $\|f_n - f\|_p \to 0$ and $\|g_n - g\|_q \to 0$ as $n \to \infty$. Hölder's inequality shows that $f_n * g_n \to f * g$ uniformly. By (c), $f_n * g_n \in C_c(G)$. Hence $f * g \in C_0(G)$, and (d) follows.

The proof of (e) will depend on Fubini's theorem, and we first have to show that the integrand in (1) is a Borel function on $G \times G$. Fix an open set V in the plane, put $E = f^{-1}(V)$, $E' = E \times G$, and let $E'' = \{(x, y) : x - y \in E\}$. Then E' is a Borel set in $G \times G$, and since the homeomorphism of $G \times G$ onto itself which carries (x, y) to $(x + y, y)$ maps E' onto E'', E'' is also a Borel set. Since $f(x - y) \in V$ if and only if $(x, y) \in E''$, we see that $f(x - y)$ is a Borel function on $G \times G$, and so is the product $f(x - y)g(y)$.

By Fubini's theorem,

$$\int_G \int_G |f(x - y)g(y)|dx\,dy = \|f\|_1 \|g\|_1.$$

Setting $\phi(x) = \int_G |f(x - y)g(y)|dy$, it follows that $\phi \in L^1(G)$. In particular, $\phi(x) < \infty$ for almost all x, and so $(f * g)(x)$ exists for almost all x. Finally, $|(f * g)(x)| \leq \phi(x)$, and the proof of (e) is complete.

The proof of (f) is also an application of Fubini's theorem, justified by (e) for almost all x:

$$\begin{aligned}
(f * (g * h))(x) &= \int_G f(x - z)(g * h)(z)dz \\
&= \int_G \int_G f(x - z)g(z - y)h(y)dy\,dz \\
&= \int_G \int_G f(x - z - y)g(z)h(y)dy\,dz \\
&= \int_G (f * g)(x - y)h(y)dy = ((f * g) * h)(x).
\end{aligned}$$

1.1.7. THEOREM. *For any LCA group G, $L^1(G)$ is a commutative Banach algebra, if multiplication is defined by convolution. If G is discrete, $L^1(G)$ has a unit.*

Proof: The first statement follows from parts (e), (f), and (a) of Theorem 1.1.6, since the distributive law holds: $f * (g + h) = f * g + f * h$.

If G is discrete and the Haar measure is normalized as indicated in Section 1.1.3, then

$$(f * g)(x) = \sum_{y \in G} f(x - y)g(y),$$

and if $e(0) = 1$ but $e(x) = 0$ for all $x \neq 0$, then $e \in L^1(G)$ and $f * e = f$. Thus e is the unit of $L^1(G)$.

1.1.8. If G is not discrete, then $L^1(G)$ has no unit (see Section 1.7.3), but *approximate units* are always available.

THEOREM. *Given $f \in L^1(G)$ and $\varepsilon > 0$, there exists a neighborhood V of 0 in G with the following property: if u is a non-negative Borel function which vanishes outside V, and if $\int_G u(x)dx = 1$, then*

$$\|f - f * u\|_1 < \varepsilon.$$

Proof: By Theorem 1.1.5, we can choose V so that $\|f - f_y\|_1 < \varepsilon$ for all $y \in V$. If u satisfies the hypotheses, we have

$$(f * u)(x) - f(x) = \int_G [f(x - y) - f(x)]u(y)dy$$

so that

$$\|f * u - f\|_1 \leqq \int_G |u(y)|dy \int_G |f(x - y) - f(x)|dx$$
$$= \int_V \|f - f_y\|_1 u(y)dy < \varepsilon.$$

1.2. The Dual Group and the Fourier Transform

1.2.1. Characters. A complex function γ on a LCA group G is called a *character* of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the functional equation

$$\gamma(x + y) = \gamma(x)\gamma(y) \qquad (x, y \in G) \tag{1}$$

is satisfied. The set of all *continuous* characters of G forms a group Γ, the *dual group* of G, if *addition* is defined by

$$(2) \qquad (\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \qquad (x \in G;\ \gamma_1, \gamma_2 \in \Gamma).$$

Throughout this book, the letter Γ will denote the dual group of the LCA group G.

In view of the duality between G and Γ which will be established in Section 1.7, it is customary to write

$$(3) \qquad (x, \gamma)$$

in place of $\gamma(x)$. With this notation, (1) and (2) become

$$(4) \quad (x + y, \gamma) = (x, \gamma)(y, \gamma) \text{ and } (x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2).$$

It follows immediately that

$$(5) \qquad (0, \gamma) = (x, 0) = 1 \qquad (x \in G, \gamma \in \Gamma)$$

and

$$(6) \qquad (-x, \gamma) = (x, -\gamma) = (x, \gamma)^{-1} = \overline{(x, \gamma)}.$$

We shall presently endow Γ with a topology with respect to which Γ will itself be a LCA group. But first we identify Γ with the maximal ideal space of $L^1(G)$ (Appendix D).

1.2.2. THEOREM. *If $\gamma \in \Gamma$ and if*

$$(1) \qquad \hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (f \in L^1(G)),$$

then the map $f \to \hat{f}(\gamma)$ is a complex homomorphism of $L^1(G)$, and is not identically 0. Conversely, every non-zero complex homomorphism of $L^1(G)$ is obtained in this way, and distinct characters induce distinct homomorphisms.

Proof: Suppose $f, g \in L^1(G)$, and $k = f * g$. Then

$$\begin{aligned} \hat{k}(\gamma) &= \int_G (f * g)(x)(-x, \gamma)dx = \int_G (-x, \gamma)dx \int_G f(x - y)g(y)dy \\ &= \int_G g(y)(-y, \gamma)dy \int_G f(x - y)(-x + y, \gamma)dx = \hat{g}(\gamma)\hat{f}(\gamma). \end{aligned}$$

Thus the map $f \to \hat{f}(\gamma)$ is multiplicative on the Banach algebra

$L^1(G)$, and since it is clearly linear, it is a homomorphism. Since $|(-x, \gamma)| = 1$, $\hat{f}(\gamma) \neq 0$ for some $f \in L^1(G)$.

For the converse, suppose h is a complex homomorphism of $L^1(G)$, $h \neq 0$. Then h is a bounded linear functional of norm 1 (Appendix D4), so that

$$h(f) = \int_G f(x)\phi(x)dx \qquad (f \in L^1(G)) \tag{2}$$

for some $\phi \in L^\infty(G)$ with $\|\phi\|_\infty = 1$ (Appendix E10). If f and g are in $L^1(G)$, we have

$$\begin{aligned}\int_G h(f)g(y)\phi(y)dy &= h(f)h(g) = h(f * g) = \int_G (f * g)(x)\phi(x)dx \\ &= \int_G g(y)dy \int_G f(x-y)\phi(x)dx = \int_G g(y)h(f_y)dy,\end{aligned}$$

so that

$$h(f)\phi(y) = h(f_y) \tag{3}$$

for almost all $y \in G$. By Theorem 1.1.5 and the continuity of h, the right side of (3) is a continuous function on G, for each $f \in L^1(G)$. Choosing f so that $h(f) \neq 0$, (3) shows that $\phi(y)$ coincides with a continuous function almost everywhere, and hence we may assume that ϕ is continuous, without affecting (2). Then (3) holds for all $y \in G$.

If we replace y by $x + y$ and then f by f_x in (3), we obtain $h(f)\phi(x + y) = h(f_{x+y}) = h((f_x)_y) = h(f_x)\phi(y) = h(f)\phi(x)\phi(y)$, so that

$$\phi(x + y) = \phi(x)\phi(y) \qquad (x, y \in G). \tag{4}$$

Since $|\phi(x)| \leq 1$ for all x and since (4) implies that $\phi(-x) = \phi(x)^{-1}$, it follows that $|\phi(x)| = 1$. Hence $\phi \in \Gamma$.

Finally, if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in L^1(G)$, (1) implies that $(-x, \gamma_1) = (-x, \gamma_2)$ for almost all $x \in G$, and since γ_1 and γ_2 are continuous, 1.1.2 shows that the equality holds for all $x \in G$, so that $\gamma_1 = \gamma_2$.

1.2.3. The Fourier transform. For all $f \in L^1(G)$, the function $\hat{f}$ defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (\gamma \in \Gamma)$$

is called the *Fourier transform* of f. The set of all functions $\hat{f}$ so obtained will be denoted throughout by $A(\Gamma)$.

By Theorem 1.2.2, $\hat{f}$ is precisely the Gelfand transform of f. If we give Γ the weak topology induced by $A(\Gamma)$ (Appendix A10), the basic facts of the Gelfand theory (Appendix D4) show that $A(\Gamma)$ is a separating subalgebra of $C_0(\Gamma)$. We summarize some of the properties of $A(\Gamma)$.

1.2.4. THEOREM. (a) *$A(\Gamma)$ is a separating self-adjoint subalgebra of $C_0(\Gamma)$, so that $A(\Gamma)$ is dense in $C_0(\Gamma)$, by the Stone-Weierstrass theorem.*

(b) *The Fourier transform of $f * g$ is $\hat{f}\hat{g}$.*

(c) *$A(\Gamma)$ is invariant under translation and under multiplication by (x, γ), for any $x \in G$.*

(d) *The Fourier transform, considered as a map of $L^1(G)$ into $C_0(\Gamma)$, is norm-decreasing and therefore continuous: $\|\hat{f}\|_\infty \leqq \|f\|_1$.*

(e) *For $f \in L^1(G)$ and $\gamma \in \Gamma$, $(f * \gamma)(x) = (x, \gamma)\hat{f}(\gamma)$.*

Proof: For $f \in L^1(G)$, define $\tilde{f}$ by

$$\tilde{f}(x) = \overline{f(-x)}.$$

The Fourier transform of $\tilde{f}$ is the complex conjugate of $\hat{f}$, and (a) follows; (b) is implicit in Theorem 1.2.2. If $\gamma_0 \in \Gamma$ and $g(x) = (x, \gamma_0)f(x)$, then $\hat{g}(\gamma) = \hat{f}(\gamma - \gamma_0)$, so that $A(\Gamma)$ is translation invariant. If $g = f_x$, then

$$\begin{aligned}\hat{g}(\gamma) &= \int_G f(y - x)(-y, \gamma)dy \\ &= (-x, \gamma)\int_G f(y - x)(x - y, \gamma)dy = (-x, \gamma)\hat{f}(\gamma).\end{aligned}$$

This proves (c); (d) and (e) are trivial; (e) allows us to interpret the Fourier transform as a convolution:

$$\hat{f}(\gamma) = (f * \gamma)(0) \qquad (f \in L^1(G), \gamma \in \Gamma).$$

1.2.5. THEOREM. *If G is discrete, Γ is compact. If G is compact, Γ is discrete.*

Proof: If G is discrete, then $L^1(G)$ has a unit (Theorem 1.1.7) and its maximal ideal space Γ is therefore compact (Appendix D4).

If G is compact and its Haar measure is normalized so that $m(G) = 1$, the orthogonality relations

$$\int_G (x, \gamma)dx = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases} \tag{1}$$

hold. The case $\gamma = 0$ is clear. If $\gamma \neq 0$, then $(x_0, \gamma) \neq 1$ for some $x_0 \in G$, and

$$\int_G (x, \gamma)dx = (x_0, \gamma) \int_G (x - x_0, \gamma)dx = (x_0, \gamma) \int_G (x, \gamma)dx,$$

so that (1) is proved. If $f(x) = 1$ for all $x \in G$, then $f \in L^1(G)$ since G is compact, and $\hat{f}(0) = 1$, $\hat{f}(\gamma) = 0$, if $\gamma \neq 0$, by (1). Since $\hat{f}$ is continuous, the set consisting of 0 alone is open in Γ, and so Γ is discrete.

1.2.6. The topology of Γ. So far, Γ is a group and a locally compact Hausdorff space. We shall now prove that these two structures fit together so as to make Γ a LCA group. Our proof depends on an alternative description of the topology of Γ:

THEOREM. (a) *(x, γ) is a continuous function on $G \times \Gamma$.*

(b) *Let K and C be compact subsets of G and Γ, respectively, let U_r be the set of all complex numbers z with $|1 - z| < r$, and put*

$$N(K, r) = \{\gamma\colon (x, \gamma) \in U_r \text{ for all } x \in K\},$$
$$N(C, r) = \{x\colon (x, \gamma) \in U_r \text{ for all } \gamma \in C\}.$$

Then $N(K, r)$ and $N(C, r)$ are open subsets of Γ and G, respectively.

(c) *The family of all sets $N(K, r)$ and their translates is a base for the topology of Γ.*

(d) *Γ is a LCA group.*

Proof: Equation (3) of Section 1.2.2, rewritten in the form

$$\hat{f}(\gamma)(x, \gamma) = \hat{f}_x(\gamma) \qquad (x \in G,\ \gamma \in \Gamma) \tag{1}$$

implies (a), as soon as it is proved that $\hat{f}_x(\gamma)$ is a continuous function on $G \times \Gamma$, for every $f \in L^1(G)$.

Fix x_0, γ_0, and $\varepsilon > 0$. There are neighborhoods V of x_0 and W of γ_0 such that

$$(2) \qquad \|f_x - f_{x_0}\|_1 < \varepsilon \text{ and } |\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| < \varepsilon$$

for all $x \in V$, $\gamma \in W$, by Theorem 1.1.5 and the continuity of $\hat{f}_{x_0}$. Since $|\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma)| \leqq \|f_x - f_{x_0}\|_1$, it follows that $|\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma_0)| < 2\varepsilon$ if $x \in V$ and $\gamma \in W$, and (a) is proved.

Choose a compact set K in G, choose $r > 0$, and fix $\gamma_0 \in N(K, r)$. To every $x_0 \in K$ there correspond neighborhoods V of x_0 and W of γ_0 such that $(x, \gamma) \in U_r$, if $x \in V$ and $\gamma \in W$; this follows from (a). Since K is compact, finitely many of these sets V cover K, and if W^* is the intersection of the corresponding sets W, then $W^* \subset N(K, r)$. Since W^* is a neighborhood of γ_0, $N(K, r)$ is open.

The same proof applies to $N(C, r)$.

To prove (c), assume that V is a neighborhood of γ_0. We have to show that $\gamma_0 + N(K, r) \subset V$ for some choice of K and r. Take $\gamma_0 = 0$, without loss of generality. The definition of the Gelfand topology on Γ shows that there exist functions $f_1, \ldots, f_n \in L^1(G)$ and $\varepsilon > 0$ so that

$$(3) \qquad \bigcap_{i=1}^{n} \{\gamma\colon |\hat{f}_i(\gamma) - \hat{f}_i(0)| < \varepsilon\} \subset V.$$

Since $C_c(G)$ is dense in $L^1(G)$, we may assume that $f_1, \ldots, f_n$ vanish outside a compact set K in G. If

$$(4) \qquad r < \varepsilon / \max_i \|f_i\|_1$$

and if $\gamma \in N(K, r)$, then

$$(5) \qquad |\hat{f}_i(\gamma) - \hat{f}_i(0)| \leqq \int_K |(-x, \gamma) - 1| |f_i(x)| dx \leqq r \|f_i\|_1 < \varepsilon.$$

Hence $N(K, r) \subset V$, and (c) follows.

Given $\gamma', \gamma'' \in \Gamma$ and $N(K, r)$, the obvious relation

$$(6) \qquad [\gamma' + N(K, r/2)] - [\gamma'' + N(K, r/2)] \subset \gamma' - \gamma'' + N(K, r)$$

shows, by (b) and (c), that the map $(\gamma', \gamma'') \to \gamma' - \gamma''$ of $\Gamma \times \Gamma$ onto Γ is continuous. This completes the theorem.

1.2.7. EXAMPLES. The "classical groups" of Fourier analysis are:

(a) the additive group R of the real numbers, with the natural topology of the real line;

(b) the additive group of the reals modulo 2π, or, equivalently, the circle group T, the multiplicative group of all complex numbers of absolute value 1;

(c) the additive group Z of the integers.

The circle group is of particular importance to us, since characters are nothing but homomorphisms into T.

Suppose $G = R$ and fix $\gamma \in \Gamma$. Write $\gamma(x)$ instead of (x, γ), for the moment; there exists $\delta > 0$ such that

$$\int_0^\delta \gamma(t)dt = \alpha \neq 0. \tag{1}$$

The functional equation

$$\gamma(x + t) = \gamma(x)\gamma(t) \qquad (x, t \in R) \tag{2}$$

then implies that

$$\alpha \cdot \gamma(x) = \gamma(x) \int_0^\delta \gamma(t)dt = \int_0^\delta \gamma(x + t)dt = \int_x^{x+\delta} \gamma(t)dt. \tag{3}$$

Since γ is continuous, the last expression is differentiable, and so γ has a continuous derivative γ'. Differentiate (2) with respect to t and then set $t = 0$. The result is the differential equation

$$\gamma'(x) = A\gamma(x), \qquad A = \gamma'(0). \tag{4}$$

Since $\gamma(0) = 1$ and since γ is bounded, (4) implies that

$$\gamma(x) = e^{iyx} \tag{5}$$

for some $y \in R$. The correspondence $\gamma \leftrightarrow y$ is an isomorphism between Γ and R. Thus: *The dual group of R is R.*

We still have to check that the natural topology of R is the same as the Gelfand topology of the dual group. For $r > 0$ and $n = 1, 2, 3, \ldots$, let $V(n, r)$ be the set of all y such that $|1 - e^{ixy}| < r$ if $|x| \leq n$. By Theorem 1.2.6, the sets $V(n, r)$ form a neighborhood base at 0 with respect to the Gelfand topology. But $y \in V(n, r)$ if and only if $|y| < (2/n) \arcsin (r/2)$. Thus the two topologies coincide.

If $G = T$, the same computation as above shows that every character of T must be of the form (5), but now we also must have $\gamma(x + 2\pi) = \gamma(x)$. Hence y must be an integer, and Γ is identified as the discrete group Z (compare Theorem 1.2.5).

If $G = Z$ and $\gamma \in \Gamma$, then $(1, \gamma) = e^{i\alpha}$ for some real α, and it follows that $(n, \gamma) = e^{in\alpha}$. The correspondence $\gamma \leftrightarrow e^{i\alpha}$ is an isomorphism between Γ and T, and we conclude that T is the dual group of Z (the two topologies coincide, as in the case $G = R$).

The Fourier transforms, in these three cases, have the following forms:

$$G = R: \qquad \hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-iyx} dx \qquad (y \in R),$$

$$G = T: \qquad \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \qquad (n \in Z),$$

$$G = Z: \qquad \hat{f}(e^{i\alpha}) = \sum_{n=-\infty}^{\infty} f(n) e^{-in\alpha} \qquad (e^{i\alpha} \in T).$$

1.3. Fourier-Stieltjes Transforms

1.3.1. Convolutions of measures. Suppose G is a LCA group, and μ, λ are members of $M(G)$ (Appendix E1), i.e., bounded regular complex valued measures on G. Let $\mu \times \lambda$ be their product measure on the product space $G^2 = G \times G$, and associate with each Borel set E in G the set

$$(1) \qquad E_{(2)} = \{(x, y) \in G^2 : x + y \in E\}.$$

Then $E_{(2)}$ is a Borel set in G^2 (see the proof of Theorem 1.1.6(d)) and we define $\mu * \lambda$ by

$$(2) \qquad (\mu * \lambda)(E) = (\mu \times \lambda)(E_{(2)}).$$

The set function $\mu * \lambda$ so defined is called the *convolution* of μ and λ.

1.3.2. THEOREM. (a) *If $\mu \in M(G)$ and $\lambda \in M(G)$, then $\mu * \lambda \in M(G)$.*

(b) *Convolution is commutative and associative.*
(c) $\|\mu * \lambda\| \leqq \|\mu\| \cdot \|\lambda\|$.

COROLLARY. *$M(G)$ is a commutative Banach algebra with unit, if multiplication is defined by convolution.*

Proof: The Jordan decomposition theorem shows that in the proof of (a) it is enough to consider non-negative measures only. Since $\mu \times \lambda$ is a measure on G^2, it is clear that $(\mu * \lambda)(E) = \sum (\mu * \lambda)(E_i)$ if E is the union of the disjoint Borel sets E_i ($i = 1, 2, 3, \ldots$). If E is a Borel set in G and if $\varepsilon > 0$, the regularity of $\mu \times \lambda$ shows that there is a compact set $K \subset E_{(2)}$ such that

$$(\mu \times \lambda)(K) > (\mu \times \lambda)(E_{(2)}) - \varepsilon.$$

If C is the image of K under the map $(x, y) \to x + y$, then C is a compact subset of E, $K \subset C_{(2)}$, and hence

$$(\mu * \lambda)(C) = (\mu \times \lambda)(C_{(2)}) \geqq (\mu \times \lambda)(K) > (\mu * \lambda)(E) - \varepsilon.$$

This establishes one half of the requirement that $\mu * \lambda$ be regular. The other half follows by complementation, and (a) is proved. (This argument applies to more general situations; see Stromberg [1].)

Since G is commutative, the condition $x + y \in E$ is the same as the condition $y + x \in E$, and hence $\mu * \lambda = \lambda * \mu$.

The simplest way to prove associativity is to extend the definition of convolution to the case of n measures $\mu_1, \ldots, \mu_n \in M(G)$: with each Borel set E in G associate the set

$$E_n = \{(x_1, \ldots, x_n) \in G^n : x_1 + \ldots + x_n \in E\}, \tag{1}$$

and put

$$(\mu_1 * \mu_2 * \ldots * \mu_n)(E) = (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(E_{(n)}), \tag{2}$$

where the measure on the right is the ordinary product measure on the product space G^n. Associativity now follows from Fubini's theorem, and (b) is proved.

Let χ_E be the characteristic function of the Borel set E in G. The definition of $(\mu * \lambda)(E)$ is equivalent to the equation

$$\int_G \chi_E \, d(\mu * \lambda) = \int_G \int_G \chi_E(x + y) d\mu(x) d\lambda(y). \tag{3}$$

Hence if f is a *simple function* (a finite linear combination of characteristic functions of Borel sets), we have

$$\int_G f d(\mu * \lambda) = \int_G \int_G f(x + y) d\mu(x) d\lambda(y), \tag{4}$$

and since every bounded Borel function is the uniform limit of a sequence of simple functions, (4) holds for every bounded Borel function f. (One could use (4) as the definition of $\mu * \lambda$.) If $|f(x)| \leqq 1$ for all $x \in G$, then $|\int_G f(x + y) d\mu(x)| \leqq ||\mu||$ for all $y \in G$, and hence the right side of (4) does not exceed $||\mu|| \cdot ||\lambda||$. This proves part (c) of the theorem.

As to the Corollary, it only remains to be shown that $M(G)$ has a unit. Let δ_0 be the unit mass concentrated at the point $x = 0$; i.e., $\delta_0(E) = 1$ if $0 \in E$ and $\delta_0(E) = 0$ otherwise. Then $\mu * \delta_0 = \mu$ for all $\mu \in M(G)$, and the proof is complete.

1.3.3. Fourier-Stieltjes transforms. If $\mu \in M(G)$, the function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \qquad (\gamma \in \Gamma) \tag{1}$$

is called the Fourier-Stieltjes transform of μ. The set of all such functions $\hat{\mu}$ will be denoted by $B(\Gamma)$.

THEOREM. (a) *Each $\hat{\mu} \in B(\Gamma)$ is bounded and uniformly continuous.*

(b) *If $\sigma = \mu * \lambda$, then $\hat{\sigma} = \hat{\mu} \cdot \hat{\lambda}$. Hence the map $\mu \to \hat{\mu}(\gamma)$ is, for each $\gamma \in \Gamma$, a complex homomorphism of $M(G)$.*

(c) *$B(\Gamma)$ is invariant under translation, under multiplication by (x, γ) for any $x \in G$, and under complex conjugation.*

Proof: The definition of $\hat{\mu}$ shows immediately that $|\hat{\mu}(\gamma)| \leqq ||\mu||$ for all $\gamma \in \Gamma$. Given $\delta > 0$, the regularity of $|\mu|$ shows that there is a compact set K in G such that $|\mu|(K') < \delta$, where K' is the complement of K. For any $\gamma_1, \gamma_2 \in \Gamma$ we have

$$|\hat{\mu}(\gamma_1) - \hat{\mu}(\gamma_2)| \leqq \int_G |1 - (x, \gamma_1 - \gamma_2)| d|\mu|(x) = \int_K + \int_{K'}.$$

If $\gamma_1 - \gamma_2 \in N(K, \delta)$, as defined in Theorem 1.2.6, the above integrand is less than δ for $x \in K$, hence $\int_K$ does not exceed $\delta\|\mu\|$. The second integral is less than $2|\mu|(K') < 2\delta$. Hence $\hat{\mu}$ is uniformly continuous.

Suppose $\sigma = \mu * \lambda$. Formula (4) in the proof of Theorem 1.3.2 then implies that

$$\begin{aligned}\hat{\sigma}(\gamma) &= \int_G (-z, \gamma) d(\mu * \lambda)(z) = \int_G \int_G (-x-y, \gamma) d\mu(x) d\lambda(y) \\ &= \int_G (-x, \gamma) d\mu(x) \int_G (-y, \gamma) d\lambda(y) = \hat{\mu}(\gamma)\hat{\lambda}(\gamma),\end{aligned}$$

and (b) is proved.

The proof of (c) is quite similar to that of the analogous part of Theorem 1.2.4. If $d\lambda(x) = (x, \gamma_0) d\mu(x)$, then $\hat{\lambda}(\gamma) = \hat{\mu}(\gamma - \gamma_0)$. If $\lambda(E) = \mu(E - x)$, then $\hat{\lambda}(\gamma) = (x, \gamma)\hat{\mu}(\gamma)$. If $\tilde{\mu}(E) = \overline{\mu(-E)}$, then the Fourier-Stieltjes transform of $\tilde{\mu}$ is the complex conjugate of $\hat{\mu}$.

1.3.4. $L^1(G)$ **as a subalgebra of** $M(G)$. Every $f \in L^1(G)$ generates a measure $\mu_f \in M(G)$, defined by

$$\mu_f(E) = \int_E f(x) dx, \tag{1}$$

and which is absolutely continuous with respect to the Haar measure of G. Conversely, the Radon-Nikodym theorem (Appendix E9) shows that every absolutely continuous $\mu \in M(G)$ is μ_f for some $f \in L^1(G)$. Since we identify functions in $L^1(G)$ which differ only on a set of Haar measure 0, the correspondence between f and μ_f is one-to-one, and we may therefore regard $L^1(G)$ as a subset of $M(G)$. It is easily seen that $\hat{f}(\gamma) = \hat{\mu}_f(\gamma)$ for all $\gamma \in \Gamma$ and that $\|f\|_1 = \|\mu_f\|$. Hence we may use f in place of μ_f without causing confusion. For instance, we may write $f * \sigma$ if $f \in L^1(G)$ and $\sigma \in M(G)$, instead of $\mu_f * \sigma$.

1.3.5. Let $M_c(G)$ and $M_d(G)$ denote the sets of all continuous and discrete members of $M(G)$, respectively (Appendix E6).

THEOREM. (a) *$L^1(G)$ and $M_c(G)$ are closed ideals in $M(G)$.*
(b) *$M_d(G)$ is a closed subalgebra of $M(G)$.*

Proof: If we apply the Fubini theorem to the definition of $\mu * \lambda$, we obtain, for any Borel set E in G,

$$(\mu * \lambda)(E) = \int_G \mu(E - y)d\lambda(y). \tag{1}$$

If μ is absolutely continuous and $m(E) = 0$, then $m(E - y) = 0$ for all y, hence $\mu(E - y) = 0$, and so $(\mu * \lambda)(E) = 0$ for every $\lambda \in M(G)$. This says that $\mu * \lambda$ is absolutely continuous, and hence $L^1(G)$ is an ideal in $M(G)$. Since $||f||_1 = ||\mu_f||$ and since $L^1(G)$ is complete, $L^1(G)$ is closed in $M(G)$. If E is countable, $\mu_n \in M_c(G)$, and $||\mu - \mu_n|| \to 0$, then

$$|\mu(E)| = |(\mu - \mu_n)(E)| \leqq |\mu - \mu_n|(E) \leqq ||\mu - \mu_n||,$$

so that $\mu(E) = 0$ and $\mu \in M_c(G)$. Thus $M_c(G)$ is closed, and part (a) is proved. Part (b) follows from the observation that the convolution of two point-measures is a point-measure.

1.3.6. A uniqueness theorem. We shall see later that $\hat{\mu}$ determines μ, i.e. if $\mu \in M(G)$ and $\hat{\mu} = 0$, then $\mu = 0$. At present, we can prove this for the inverse transform:

THEOREM. *If $\mu \in M(\Gamma)$ and if*

$$\int_\Gamma (x, \gamma)d\mu(\gamma) = 0$$

for every $x \in G$, then $\mu = 0$.

Proof: For every $f \in L^1(G)$,

$$\begin{aligned}\int_\Gamma \hat{f}(\gamma)d\mu(\gamma) &= \int_\Gamma \int_G f(x)(-x, \gamma)dx d\mu(\gamma) \\ &= \int_G f(x)dx \int_\Gamma (-x, \gamma)d\mu(\gamma) = 0.\end{aligned}$$

Since $A(\Gamma)$ is dense in $C_0(\Gamma)$ (Theorem 1.2.4), it follows that $\int_\Gamma \phi \, d\mu = 0$ for every $\phi \in C_0(\Gamma)$, and hence $\mu = 0$.

1.4. Positive-Definite Functions

1.4.1. A function ϕ, defined on G, is said to be *positive-definite* if the inequality

$$\sum_{n,m=1}^{N} c_n \overline{c_m} \phi(x_n - x_m) \geqq 0 \tag{1}$$

holds for every choice of $x_1, \ldots, x_N$ in G and for every choice of complex numbers $c_1, \ldots, c_N$.

If ϕ is positive-definite, the following three relations hold:

$$\phi(-x) = \overline{\phi(x)}; \tag{2}$$

$$|\phi(x)| \leqq \phi(0); \tag{3}$$

$$|\phi(x) - \phi(y)|^2 \leqq 2\phi(0) \operatorname{Re} [\phi(0) - \phi(x-y)]. \tag{4}$$

We conclude from (3) that $\phi(0) \geqq 0$ and that ϕ is bounded; (4) implies that ϕ is uniformly continuous if ϕ is continuous at 0.

To prove these relations, take $N = 2$ in (1); $x_1 = 0$, $x_2 = x$; $c_1 = 1$, $c_2 = c$. This gives

$$\{1 + |c|^2\}\phi(0) + c\phi(x) + \bar{c}\phi(-x) \geqq 0. \tag{5}$$

Taking $c = 1$, we see that $\phi(x) + \phi(-x)$ is real; $c = i$ shows that $i(\phi(x) - \phi(-x))$ is real. Hence (2) holds.

If c is chosen so that $c\phi(x) = -|\phi(x)|$, (5) implies (3). To prove (4), take $N = 3$ in (1); $x_1 = 0$, $x_2 = x$, $x_3 = y$; $c_1 = 1$, λ real,

$$c_2 = \frac{\lambda|\phi(x) - \phi(y)|}{\phi(x) - \phi(y)},$$

and $c_3 = -c_2$. Then (1) simplifies to

$$\phi(0)(1 + 2\lambda^2) + 2\lambda|\phi(x) - \phi(y)| - 2\lambda^2 \operatorname{Re} \phi(x-y) \geqq 0. \tag{6}$$

The discriminant of the quadratic polynomial (6) in λ can therefore not be positive, and this gives (4).

1.4.2. Examples of positive-definite functions. (a) *Suppose $f \in L^2(G)$ and $\phi = f * \tilde{f}$. Then ϕ is positive-definite and continuous on G.*

The convolution of any two functions in $L^2(G)$ is continuous (Theorem 1.1.6(d)) and

$$\begin{aligned}\sum c_n \overline{c_m} \phi(x_n - x_m) &= \sum c_n \overline{c_m} \int_G f(x_n - x_m - y) \overline{f(-y)}\, dy \\ &= \sum c_n \overline{c_m} \int_G f(x_n - y) \overline{f(x_m - y)} dy = \int_G |\sum c_n f(x_n - y)|^2 dy \geqq 0.\end{aligned}$$

(b) Every character is positive-definite, hence so is every finite linear combination of characters if the coefficients are positive. More generally, *if* $\mu \in M(\Gamma)$, *if* $\mu \geqq 0$, *and if*

$$\phi(x) = \int_\Gamma (x, \gamma) d\mu(\gamma) \qquad (x \in G), \tag{1}$$

then ϕ *is continuous and positive definite.*

Indeed, (1) shows that

$$\begin{aligned} \sum c_n \overline{c_m} \phi(x_n - x_m) &= \int_\Gamma \sum_{n,m} c_n \overline{c_m} (x_n - x_m, \gamma) d\mu(\gamma) \\ &= \int_\Gamma \Big| \sum_n c_n (x_n, \gamma) \Big|^2 d\mu(\gamma) \geqq 0, \end{aligned}$$

so that ϕ is positive-definite. Since the sets $N(C, r)$ of Theorem 1.2.6 are open in G, our proof of the continuity of $\hat{\mu}$ (Theorem 1.3.3) shows equally well that ϕ is continuous if ϕ is defined by (1).

1.4.3. The previous example (1.4.2(b)) establishes the trivial half of the following important characterization of positive-definite functions:

BOCHNER'S THEOREM. *A continuous function* ϕ *on* G *is positive-definite if and only if there is a non-negative measure* $\mu \in M(\Gamma)$ *such that*

$$\phi(x) = \int_\Gamma (x, \gamma) d\mu(\gamma) \qquad (x \in G). \tag{1}$$

For $G = Z$, this is due to Herglotz [1]; for $G = R$, to Bochner [1]; for the general case, to Weil [1]. Bochner was the first to recognize the key role which this result plays in harmonic analysis. By 1.3.6, the above representation (1) is unique.

Proof: Suppose ϕ is continuous and positive-definite. By 1.4.1(3) we may assume, without loss of generality, that $\phi(0) = 1$.

If $f \in C_c(G)$ and has support K, then $f(x)\overline{f(y)}\phi(x - y)$ is uniformly continuous on $K \times K$, and K can be partitioned into disjoint sets $E_1, \ldots, E_n$ such that the sum

$$\sum_{i,j=1}^{n} f(x_i)\overline{f(x_j)}\phi(x_i - x_j) m(E_i) m(E_j) \qquad (x_i \in E_i) \tag{2}$$

differs from the integral

$$\int_G \int_G f(x)\overline{f(y)}\phi(x-y)dx\,dy \tag{3}$$

by as little as we please. Since ϕ is positive-definite, (2) is always non-negative, and hence so is (3). Since $C_c(G)$ is dense in $L^1(G)$, it follows that (3) *is non-negative for every* $f \in L^1(G)$.

Define a functional T_ϕ by

$$T_\phi(f) = \int_G f(x)\phi(x)dx \qquad (f \in L^1(G)) \tag{4}$$

and put

$$[f, g] = T_\phi(f * \tilde{g}) \qquad (f, g \in L^1(G)). \tag{5}$$

We recall that $\tilde{g}(x) = \overline{g(-x)}$, so that

$$[f, g] = \int_G \int_G f(x)\overline{g(y)}\phi(x-y)dx\,dy. \tag{6}$$

Hence $[f, g]$ is linear in f, $[g, f]$ is the complex conjugate of $[f, g]$, and $[f, f] \geqq 0$. These are just the properties of the Hilbert space inner product which are needed for the standard proof of the Schwarz inequality. In our case, the inequality is

$$|[f, g]|^2 \leqq [f, f][g, g]. \tag{7}$$

Take for g the characteristic function of a symmetric neighborhood V of 0, divided by $m(V)$. By (6),

$$[f, g] - T_\phi(f) = \int_G f(x)\frac{1}{m(V)}\int_V [\phi(x-y) - \phi(x)]dy\,dx$$

and

$$[g, g] - 1 = \frac{1}{m(V)^2}\int_V \int_V [\phi(x-y) - 1]dx\,dy.$$

Since ϕ is uniformly continuous, these expressions can be made arbitrarily small by taking V small enough, and then (7) yields the inequality

$$|T_\phi(f)|^2 \leqq [f, f] = T_\phi(f * \tilde{f}) \qquad (f \in L^1(G)). \tag{8}$$

Put $h = f * \tilde{f}$ and $h^n = h^{n-1} * h$ $(n = 2, 3, 4, \ldots)$. Since $\|\phi\|_\infty = 1$, we have $\|T_\phi\| = 1$, and if we apply (8) with $h, h^2, h^4, \ldots$ in place of f, we obtain

$$|T_\phi(f)|^2 \leqq T_\phi(h) \leqq \{T_\phi(h^2)\}^{\frac{1}{2}} \leqq \ldots \leqq \{T_\phi(h^{2^n})\}^{2^{-n}} \leqq \|h^{2^n}\|_1^{2^{-n}}.$$

As $n \to \infty$, the last expression converges to the spectral radius of h, i.e. to $\|\hat{h}\|_\infty$. (See Appendix D 6 and Theorem 1.2.2.) Hence

$$(9) \quad |T_\phi(f)|^2 \leqq \|\hat{h}\|_\infty = \|\hat{f}\|_\infty^2, \text{ or } |T_\phi(f)| \leqq \|\hat{f}\|_\infty \qquad (f \in L^1(G)).$$

This means that T_ϕ may be regarded as a bounded linear functional on $A(\Gamma)$, with respect to the supremum norm. (We have not yet proved that $\hat{f}_1 = \hat{f}_2$ implies $f_1 = f_2$, but (9) shows that $\hat{f}_1 = \hat{f}_2$ implies $T_\phi(f_1) = T_\phi(f_2)$, and this is sufficient.) We can extend T_ϕ to a bounded linear functional on $C_0(\Gamma)$, preserving its norm, and the Riesz representation theorem then implies that there is a $\mu \in M(\Gamma)$, with $\|\mu\| \leqq 1$, such that

$$(10) \qquad T_\phi(f) = \int_\Gamma \hat{f}(-\gamma)d\mu(\gamma) = \int_G f(x)dx \int_\Gamma (x, \gamma)d\mu(\gamma).$$

Comparison of (10) and (4) shows that (1) holds for almost all $x \in G$, hence for all x, since both sides of (1) are continuous. Finally, taking $x = 0$ in (1), we have

$$1 = \phi(0) = \int_\Gamma d\mu(\gamma) = \mu(\Gamma) \leqq \|\mu\| = 1;$$

hence $\mu(\Gamma) = \|\mu\|$, and this implies that $\mu \geqq 0$.

1.5. The Inversion Theorem

1.5.1. We let $B(G)$ be the set of all functions f on G which are representable in the form

$$(1) \qquad f(x) = \int_\Gamma (x, \gamma)d\mu(\gamma) \qquad (x \in G).$$

Bochner's theorem implies, in combination with the Jordan decomposition theorem, that $B(G)$ is exactly the set of all finite linear combinations of continuous positive-definite functions on G.

THEOREM. (a) *If* $f \in L^1(G) \cap B(G)$, *then* $\hat{f} \in L^1(\Gamma)$.

(b) *If the Haar measure of G is fixed, the Haar measure of* Γ *can be so normalized that the inversion formula*

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma)d\gamma \qquad (x \in G) \tag{2}$$

is valid for every $f \in L^1(G) \cap B(G)$.

Proof: Let us write B^1 in place of $L^1(G) \cap B(G)$, and if μ is associated with f as in (1) above, let us write $\mu = \mu_f$. (This notation has nothing to do with our earlier use of the symbol μ_f in Section 1.3.4.) If $f \in B^1$ and $h \in L^1(G)$, we then have

$$(h * f)(0) = \int_G h(-x)f(x)dx = \int_\Gamma \hat{h}(\gamma)d\mu_f(\gamma), \tag{3}$$

and if g is also in B^1, (3) implies that

$$\int_\Gamma \hat{h}\hat{g}d\mu_f = ((h * g) * f)(0) = ((h * f) * g)(0) = \int_\Gamma \hat{h}\hat{f}d\mu_g.$$

Since $A(\Gamma)$ is dense in $C_0(\Gamma)$, it follows that

$$\hat{g}d\mu_f = \hat{f}d\mu_g. \qquad (f, g \in B^1). \tag{4}$$

We shall now define a positive linear functional T on $C_c(\Gamma)$. Suppose K is the support of some $\psi \in C_c(\Gamma)$. To every $\gamma_0 \in K$ there corresponds a function $u \in C_c(G)$ with $\hat{u}(\gamma_0) \neq 0$, since $C_c(G)$ is dense in $L^1(G)$. The Fourier transform of $u * \tilde{u}$ is positive at γ_0, and is nowhere negative. Since K is compact, there is a finite number of such functions, say $u_1, \ldots, u_n$ such that the function $g = u_1 * \tilde{u}_1 + \ldots + u_n * \tilde{u}_n$ has $\hat{g} > 0$ on K. Since $g \in C_c(G)$, 1.4.2(a) shows that $g \in B^1$. Put

$$T\psi = \int_\Gamma (\psi/\hat{g})d\mu_g. \tag{5}$$

Note that $T\psi$ is well defined: if g were replaced by another function f in B^1 whose Fourier transform has no zero on K, the value of $T\psi$ would not be changed, since (4) implies that

$$\int \frac{\psi}{\hat{f}\hat{g}} \hat{f}d\mu_g = \int \frac{\psi}{\hat{f}\hat{g}} \hat{g}d\mu_f. \tag{6}$$

It is clear that T is linear. The function g in (5) is positive-definite, hence $\mu_g \geqq 0$, and it follows that $T\psi \geqq 0$ if $\psi \geqq 0$. There exists ψ and μ_f such that $\int \psi d\mu_f \neq 0$, and if g is as in (5), we have

$$(7) \qquad T(\psi\hat{f}) = \int_\Gamma (\psi\hat{f}/\hat{g})d\mu_g = \int_\Gamma \psi d\mu_f \neq 0.$$

Thus $T \neq 0$.

Fix $\psi \in C_c(\Gamma)$ and $\gamma_0 \in \Gamma$. Construct g as above, so that $\hat{g} > 0$ on K and also on $K + \gamma_0$. Setting $f(x) = (-x, \gamma_0)g(x)$, we have $\hat{f}(\gamma) = \hat{g}(\gamma + \gamma_0)$ and $\mu_f(E) = \mu_g(E - \gamma_0)$. If $\psi_0(\gamma) = \psi(\gamma - \gamma_0)$, then

$$T\psi_0 = \int_\Gamma [\psi(\gamma - \gamma_0)/\hat{g}(\gamma)]d\mu_g(\gamma) = \int_\Gamma [\psi(\gamma)/\hat{f}(\gamma)]d\mu_f(\gamma) = T\psi.$$

Thus T is translation-invariant, and it follows that

$$(8) \qquad T\psi = \int_\Gamma \psi(\gamma)d\gamma \qquad (\gamma \in C_c(\Gamma)),$$

where $d\gamma$ denotes a Haar measure on Γ.

If now $f \in B^1$ and $\psi \in C_c(\Gamma)$, (7) and (8) show that

$$(9) \qquad \int_\Gamma \psi d\mu_f = T(\psi\hat{f}) = \int_\Gamma \psi\hat{f}d\gamma,$$

and since (9) holds for every $\psi \in C_c(\Gamma)$, we conclude that

$$(10) \qquad \hat{f}d\gamma = d\mu_f \qquad (f \in B^1).$$

Since μ_f is a finite measure, it follows that $\hat{f} \in L^1(\Gamma)$, and substitution of (10) into (1) gives the inversion formula (2).

This completes the proof.

1.5.2. Consequences of the inversion theorem. Let V be a neighborhood of 0 in G, choose a compact neighborhood W of 0 such that $W - W \subset V$, let f be the characteristic function of W, divided by $m(W)^{\frac{1}{2}}$, and put $g = f * \tilde{f}$. Then g is continuous, positive-definite (by 1.4.2(a)), and 0 outside $W - W$. The inversion theorem therefore applies to g. Hence $\hat{g} = |\hat{f}|^2 \geqq 0$,

$$(1) \qquad \int_\Gamma \hat{g}(\gamma)d\gamma = g(0) = 1,$$

and it follows that there is a compact set C in Γ such that

$$\int_C \hat{g}(\gamma)d\gamma > \frac{2}{3}. \tag{2}$$

If $x \in N(C, 1/3)$ (in the notation of Theorem 1.2.6), we write

$$g(x) = \left(\int_C + \int_{C'}\right)\hat{g}(\gamma)(x, \gamma)d\gamma; \tag{3}$$

for $\gamma \in C$, $|1 - (x, \gamma)| < 1/3$, hence $\operatorname{Re}(x, \gamma) > 2/3$, and the integral over C is at least $2/3 \int_C \hat{g} > 4/9$. Since $|\int_{C'}| < 1/3$, we see that $g(x) > 1/9$ if $x \in N(C, 1/3)$, and our conclusion is: $N(C, 1/3) \subset V$.

Since the sets $N(C, r)$ are open in G (Theorem 1.2.6(b)), we now have the following analogue of 1.2.6(c):

The family of all sets $N(C, r)$ and their translates is a base for the topology of G.

If $x_0 \in G$, $x_0 \neq 0$, we can choose V in the preceding paragraph so that $x_0 \notin V$, and we conclude that $(x_0, \gamma) \neq 1$ for some $\gamma \in \Gamma$. Hence Γ *separates points* on G: If $x_1 \neq x_2$, then $(x_1 - x_2, \gamma) \neq 1$ for some γ, and so $(x_1, \gamma) \neq (x_2, \gamma)$.

Any function of the form

$$f(x) = \sum_{j=1}^{n} a_j(x, \gamma_j) \qquad (x \in G)$$

is called a *trigonometric polynomial* on G. The set of all trigonometric polynomials on G is an algebra over the complex field, with respect to pointwise multiplication, and is closed under complex conjugation. Since Γ separates points on G, the Stone-Weierstrass theorem yields the following result:

If G is compact, the trigonometric polynomials on G form a dense subalgebra of $C(G)$.

It follows that the trigonometric polynomials are also dense in $L^p(G)$, $1 \leq p < \infty$, if G is compact (see Appendix E8).

1.5.3. Normalization of Haar measure. If the Haar measure of G is given, the inversion theorem singles out a specific Haar measure of Γ, adjusted so that the inversion formula holds. In

Section 1.1.3 we introduced standard normalizations for the Haar measures of compact and discrete groups. Since Γ is compact [discrete] if G is discrete [compact] (Theorem 1.2.5) the question arises whether these normalizations are "correct," i.e., whether the inversion formula holds for them.

To prove that this is so, it suffices to consider just one function (not identically 0) and its Fourier transform.

If G is compact and $m(G) = 1$, take $f(x) = 1$. Then (see 1.2.5) $\hat{f}(0) = 1$ and $\hat{f}(\gamma) = 0$ if $\gamma \neq 0$. If m_Γ is the Haar measure of Γ, adjusted in accordance with the inversion theorem, then

$$(1)\qquad 1 = f(0) = \int_\Gamma \hat{f}(\gamma)d\gamma = m_\Gamma(\{0\}),$$

and so m_Γ assigns measure 1 to each point of Γ.

If G is discrete and each point has measure 1, take $f(0) = 1$, $f(x) = 0$ if $x \neq 0$. Then $\hat{f}(\gamma) = 1$, and

$$(2)\qquad m(\Gamma) = \int_\Gamma \hat{f}(\gamma)d\gamma = f(0) = 1$$

if the inversion theorem holds.

To consider a non-trivial case, take $G = R$ (see 1.2.7) so that $\Gamma = R$, and let $\alpha\, dx$, $\beta\, dt$ be Haar measures on G and Γ; here dx and dt denote ordinary Lebesgue measure on the real line. Since $e^{-|t|} > 0$, the easily verified formula

$$(3)\qquad \frac{2\beta}{1+x^2} = \int_{-\infty}^{\infty} e^{-|t|}\, e^{ixt}\, \beta\, dt$$

shows that $(1 + x^2)^{-1}$ is positive-definite, and the uniqueness of the inverse transform, combined with the inversion theorem, shows that

$$(4)\qquad e^{-|t|} = 2\alpha\beta \int_{-\infty}^{\infty} \frac{e^{-ixt}}{1+x^2}\, dx.$$

With $t = 0$, (4) becomes

$$(5)\qquad 1 = 2\alpha\beta \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi\alpha\beta,$$

and this is the normalization condition which α and β must satisfy.

Two of the possible choices are frequently used: $\alpha = 1/2\pi$, $\beta = 1$ or $\alpha = \beta = (2\pi)^{-1/2}$.

From now on, it will allways be tacitly assumed that the Haar measures of G and Γ are so adjusted that the inversion theorem holds.

1.6. The Plancherel Theorem

1.6.1. THEOREM. *The Fourier transform, restricted to $(L^1 \cap L^2)(G)$, is an isometry (with respect to the L^2-norms) onto a dense linear subspace of $L^2(\Gamma)$. Hence it may be extended, in a unique manner, to an isometry of $L^2(G)$ onto $L^2(\Gamma)$.*

Proof: If $f \in (L^1 \cap L^2)(G)$ and $g = f * \tilde{f}$, then $g \in L^1(G)$, g is continuous and positive definite, $|\hat{g}| = |\hat{f}|^2$, and the inversion theorem gives

$$\int_G |f(x)|^2 dx = \int_G f(x)\bar{f}(-x)dx = g(0) = \int_\Gamma \hat{g}(\gamma)d\gamma = \int_\Gamma |\hat{f}(\gamma)|^2 d\gamma,$$

or $||\hat{f}||_2 = ||f||_2$.

Let Φ be the set of all $\hat{f} \in A(\Gamma)$ with $f \in (L^1 \cap L^2)(G)$. Since $(L^1 \cap L^2)(G)$ is translation invariant, Φ is invariant under multiplication by (x, γ), for any $x \in G$. Thus if $\psi \in L^2(\Gamma)$ and $\int_\Gamma \phi\bar{\psi} d\gamma = 0$ for all $\phi \in \Phi$, then also

$$\int_\Gamma \phi(\gamma)\overline{\psi(\gamma)}(x, \gamma)d\gamma = 0 \qquad (\phi \in \Phi,\ x \in G).$$

Since $\phi\bar{\psi} \in L^1(\Gamma)$, the uniqueness theorem 1.3.6 implies that $\phi\bar{\psi} = 0$ almost everywhere, for every $\phi \in \Phi$. But $(L^1 \cap L^2)(G)$ is invariant under multiplication by (x, γ), for any $\gamma \in \Gamma$, and so Φ is translation invariant. Hence to every γ_0 there corresponds a $\phi \in \Phi$ which is different from 0 in a neighborhood of γ_0. It follows that $\psi = 0$ almost everywhere. Thus 0 is the only element of $L^2(\Gamma)$ which is orthogonal to Φ, and hence Φ is dense in $L^2(\Gamma)$ (see Appendix C12).

1.6.2. The above extension of the Fourier transform to $L^2(G)$ is sometimes referred to as the *Plancherel transform*; the symbol $\hat{f}$ will be used in this context as well. An important part of the theorem is the assertion that each function in $L^2(\Gamma)$ is the Plancherel transform of some $f \in L^2(G)$. For compact G this is a special case of the Riesz-Fischer theorem about orthogonal systems of functions (Zygmund [1], vol. I, p. 127).

If f and g are in $L^2(G)$, the identity

$$4f\bar{g} = |f+g|^2 - |f-g|^2 + i|f+ig|^2 - i|f-ig|^2,$$

combined with the isometric character of the Plancherel transform, yields the *Parseval formula*

$$\int_G f(x)\overline{g(x)}dx = \int_\Gamma \hat{f}(\gamma)\overline{\hat{g}(\gamma)}d\gamma.$$

1.6.3. THEOREM. *$A(\Gamma)$ consists precisely of the convolutions $F_1 * F_2$, with F_1 and F_2 in $L^2(\Gamma)$.*

Proof: Suppose $f, g \in L^2(G)$. Replacing g by $\bar{g}$, the Parseval formula assumes the form

$$\int_G f(x)g(x)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(-\gamma)d\gamma, \tag{1}$$

and if we replace $g(x)$ by $(-x, \gamma_0)g(x)$ in (1), we obtain

$$\int_G f(x)g(x)(-x,\gamma_0)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(\gamma_0-\gamma)d\gamma = (\hat{f}*\hat{g})(\gamma_0). \tag{2}$$

On the one hand, every $h \in L^1(G)$ is a product $h = fg$, with $f, g \in L^2(G)$, and (2) shows that $\hat{h} = \hat{f} * \hat{g}$, with $\hat{f}, \hat{g} \in L^2(\Gamma)$, by the Plancherel theorem. On the other hand, we can start with $\hat{f}, \hat{g} \in L^2(\Gamma)$, and see from (2) that $\hat{f} * \hat{g} \in A(\Gamma)$.

1.6.4. THEOREM. *If E is a non-empty open set in Γ, there exists $\hat{f} \in A(\Gamma)$, $\hat{f} \neq 0$, such that $\hat{f}(\gamma) = 0$ outside E.*

Proof: Let K be a compact subset of E, with $m(K) > 0$, let V be a compact neighborhood of 0 such that $K + V \subset E$, and set $\hat{f} = \hat{g} * \hat{h}$, where $\hat{g}$ and $\hat{h}$ are the characteristic functions of K and V, respectively. Then $\hat{f}(\gamma) = 0$ outside $K + V$, $\hat{f} \in A(\Gamma)$ by Theorem 1.6.3, and $\int_\Gamma \hat{f}(\gamma)d\gamma = m(K)m(V) > 0$, so that $\hat{f}$ is not identically 0.

1.7. The Pontryagin Duality Theorem

1.7.1. If G is a LCA group, we have seen (Theorem 1.2.6) that its dual Γ is also a LCA group. Hence Γ has a dual group, say $\hat{\Gamma}$, and everything we have proved so far for the ordered pair (G, Γ) holds equally well for the pair $(\Gamma, \hat{\Gamma})$. The value of a character $\hat{\gamma} \in \hat{\Gamma}$ at the point $\gamma \in \Gamma$ will be written $(\gamma, \hat{\gamma})$. (This notation is temporary, and will be abandoned as soon as we prove that $\hat{\Gamma} = G$.)

By Theorem 1.2.6(a) every $x \in G$ may be regarded as a continuous character on Γ, and thus there is a natural map α of G into $\hat{\Gamma}$, defined by

$$(1) \qquad (x, \gamma) = (\gamma, \alpha(x)) \qquad (x \in G, \ \gamma \in \Gamma).$$

1.7.2. THEOREM. *The above map α is an isomorphism and a homeomorphism of G onto $\hat{\Gamma}$.*

Thus $\hat{\Gamma}$ may be identified with G, and a more informal statement of the result would be:

Every LCA *group is the dual group of its dual group.*

This is the Pontryagin duality theorem.

Proof: For $x, y \in G$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}(\gamma, \alpha(x + y)) &= (x + y, \gamma) = (x, \gamma)(y, \gamma) \\ &= (\gamma, \alpha(x))(\gamma, \alpha(y)) = (\gamma, \alpha(x) + \alpha(y)).\end{aligned}$$

Hence $\alpha(x + y) = \alpha(x) + \alpha(y)$, and α is a homomorphism. Since Γ separates points on G (Section 1.5.2), α is one-to-one, and so α is an ismorphism of G into $\hat{\Gamma}$.

The rest of the proof may be broken into three steps:

(a) α is a homeomorphism of G into $\hat{\Gamma}$.

(b) $\alpha(G)$ is closed in $\hat{\Gamma}$.

(c) $\alpha(G)$ is dense in $\hat{\Gamma}$.

Choose a compact set C in Γ, choose $r > 0$, and put

$$(1) \qquad \begin{aligned} V &= \{x \in G: |1 - (x, \gamma)| < r \text{ for all } \gamma \in C\}, \\ W &= \{\hat{\gamma} \in \hat{\Gamma}: |1 - (\gamma, \hat{\gamma})| < r \text{ for all } \gamma \in C\}. \end{aligned}$$

By 1.5.2 and 1.2.6(c), these sets V form a neignborhood base at 0 in G, and the sets W form a neighborhood base at 0 in $\hat{\Gamma}$. The definition of α shows that

$$\alpha(V) = W \cap \alpha(G). \tag{2}$$

It follows that both α and its inverse are continuous at 0, and since α is an isomorphism, the same result holds, by translation, at any other point of G or of $\alpha(G)$.

This proves step (a), and so $\alpha(G)$ is locally compact, in the relative topology which $\alpha(G)$ has as a subset of $\hat{\Gamma}$. Suppose $\hat{\gamma}_0$ is in the closure of $\alpha(G)$, and let U be a neighborhood of $\hat{\gamma}_0$ whose closure $\overline{U}$ is compact. Since $\alpha(G)$ is locally compact, $\alpha(G) \cap \overline{U}$ is compact,* and hence closed in $\hat{\Gamma}$. But $\hat{\gamma}_0$ is in the closure of $\alpha(G) \cap \overline{U}$, and it follows that $\hat{\gamma}_0 \in \alpha(G)$. Thus $\alpha(G)$ is closed, and step (b) is proved.

If $\alpha(G)$ is not dense in $\hat{\Gamma}$, there is a function $F \in A(\hat{\Gamma})$ which is 0 at every point of $\alpha(G)$ but is not identically 0 (see Theorem 1.6.4). For some $\phi \in L^1(\Gamma)$, we have

$$F(\hat{\gamma}) = \int_\Gamma \phi(\gamma)(-\gamma, \hat{\gamma})d\gamma \qquad (\hat{\gamma} \in \hat{\Gamma}). \tag{3}$$

Since $F(\alpha(x)) = 0$ for all $x \in G$, it follows that

$$\int_\Gamma \phi(\gamma)(-x, \gamma)d\gamma = \int_\Gamma \phi(\gamma)(-\gamma, \alpha(x))d\gamma = 0 \qquad (x \in G,) \tag{4}$$

and so $\phi = 0$, by the uniqueness theorem 1.3.6. Hence $F = 0$, by (3), and this contradiction proves step (c) and completes the proof.

1.7.3. Some consequences of the duality theorem. The symmetry between G and Γ which is now established shows that every theorem proved for the ordered pair (G, Γ) also holds for (Γ, G), and this enables us to complete some of the results which were previously established in provisional form only.

(a) *Every compact abelian group is the dual of a discrete abelian group, and every discrete abelian group is the dual of a compact abelian group.* This follows from Theorem 1.2.5.

(b) *If $\mu \in M(G)$ and $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma$, then $\mu = 0$.* This is the dual of Theorem 1.3.6.

(c) *$M(G)$ and $L^1(G)$ are semi-simple Banach algebras.* (See Appendix D5). Since the map $\mu \to \hat{\mu}(\gamma)$ is a complex homomorphism

* This statement is corrected on page 285.

of $M(G)$, for each $\gamma \in \Gamma$, the semi-simplicity of $M(G)$ follows from the uniqueness theorem (b). The same uniqueness theorem evidently holds for $L^1(G)$, and so $L^1(G)$ is semi-simple.

(d) *If G is not discrete, then $L^1(G)$ has no unit. Hence $L^1(G) = M(G)$ if and only if G is discrete.*

For if G is not discrete, then Γ is not compact, by (a), and since $A(\Gamma) \subset C_0(\Gamma)$, $A(\Gamma)$ contains no non-zero constants, hence has no unit. Since $A(\Gamma)$ is isomorphic, as an algebra, to $L^1(G)$, the proof is complete.

(e) *If $\mu \in M(G)$ and $\hat{\mu} \in L^1(\Gamma)$, there exists $f \in L^1(G)$ such that $d\mu(x) = f(x)dx$, and*

$$f(x) = \int_\Gamma \hat{\mu}(\gamma)(x, \gamma)d\gamma \qquad (x \in G). \tag{1}$$

By hypothesis, $\hat{\mu} \in L^1(\Gamma) \cap B(\Gamma)$; hence if f is defined by (1), the inversion theorem (applied to the pair (Γ, G) instead of (G, Γ)), shows that $f \in L^1(G)$ and

$$\hat{\mu}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (\gamma \in \Gamma). \tag{2}$$

Since $\hat{\mu}(\gamma) = \int_G (-x, \gamma)d\mu(x)$, the uniqueness theorem now implies that $d\mu = f dx$, and the proof is complete.

1.8. The Bohr Compactification

1.8.1. Suppose Γ is the dual of the LCA group G, Γ_d is the group Γ with the discrete topology, and $\bar{G}$ is the dual of Γ_d. Then $\bar{G}$ is a compact abelian group which we call the *Bohr compactification* of G (Anzai and Kakutani [1]). Let β be the map of G into $\bar{G}$ defined by

$$(x, \gamma) = (\gamma, \beta(x)) \qquad (x \in G, \gamma \in \Gamma). \tag{1}$$

1.8.2. Theorem. *β is a continuous isomorphism of G onto a dense subgroup $\beta(G)$ of $\bar{G}$.*

This theorem allows us to regard G as a dense subgroup of $\bar{G}$, so that $\bar{G}$ is indeed a compactification of G. Note, however, that $\beta(G)$ is not a locally compact subset of $\bar{G}$ and that β is not a homeomorphism, unless G is compact, in which case $G = \bar{G}$ and $\Gamma = \Gamma_d$.

Proof: Since Γ separates points on G, β is one-to-one, and it is easy to verify, as in the beginning of the proof of the Pontryagin duality theorem, that β is an isomorphism.

Let W be a neighborhood of 0 in $\bar{G}$. Since a subset of Γ_d is compact if and only if it is finite, Theorem 1.2.6 shows that there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $r > 0$, such that W contains the set

$$\{\bar{x} \in \bar{G}: |1 - (\gamma_i, \bar{x})| < r;\ i = 1, \ldots, n\}$$

which is a neighborhood of 0 in $\bar{G}$. Let

$$V = \{x \in G: |1 - (x, \gamma_i)| < r;\ i = 1, \ldots, n\}.$$

Then V is a neighborhood of 0 in G, and $x \in V$ implies $\beta(x) \in W$. Thus β is continuous at 0, and hence at all points of G, by translation.

Finally, let H be the closure in $\bar{G}$ of $\beta(G)$. If $H \neq \bar{G}$, then $\bar{G}/H$ is a non-trivial compact group, and hence there is a character ϕ on $\bar{G}/H$ which is not identically 1. The map $\bar{x} \to \phi(\bar{x} + H)$ is then a continuous character on $\bar{G}$, not identically 1, which is 1 if $\bar{x} \in H$. Consequently there exists $\gamma_0 \in \Gamma$, $\gamma_0 \neq 0$, such that $(x, \gamma_0) = (\gamma_0, \beta(x)) = 1$ for all $x \in G$. This last equation implies that $\gamma_0 = 0$, and this contradiction completes the proof.

1.8.3. We may interpret the theorem in the following way: G and Γ are given, G is the group of all continuous characters on Γ, $\bar{G}$ is the group of *all* characters on Γ, and the fact that G (or $\beta(G)$) is dense in $\bar{G}$ leads to an approximation theorem (Hewitt and Zuckerman [1]):

THEOREM. *Given $\gamma_1, \ldots, \gamma_n \in \Gamma$, given $\varepsilon > 0$, and given any character ϕ on Γ, there is a continuous character ψ on Γ such that*

$$|\psi(\gamma_i) - \phi(\gamma_i)| < \varepsilon \qquad (i = 1, \ldots, n). \tag{1}$$

Proof: $\phi \in \bar{G}$, and the set of all $\psi \in \bar{G}$ satisfying (1) is open in $\bar{G}$, hence intersects $\beta(G)$.

1.8.4. A function f on a LCA group G is *almost periodic* if and only if it is a uniform limit of trigonometric polynomials on G. (This is not the usual definition, but is equivalent to it). The almost periodic functions on G are precisely those which have continuous extensions to $\bar{G}$; in other words, they are the restrictions to G of the continuous functions on $\bar{G}$, and so $\bar{G}$ may also be obtained as the maximal ideal space of the Banach algebra whose members are the almost periodic functions on G (Loomis [1]). These relations between almost periodicity and $\bar{G}$ are the reason for associating Bohr's name with $\bar{G}$. We shall not use these relations and omit their proof.

1.9. A Characterization of $B(\Gamma)$

1.9.1. We recall that $B(\Gamma)$ is the set of all functions $\hat{\mu}$ on Γ which are Fourier-Stieltjes transforms of measures $\mu \in M(G)$. We norm $B(\Gamma)$ by defining $\|\hat{\mu}\| = \|\mu\|$.

We already know one characterization of $B(\Gamma)$: $\phi \in B(\Gamma)$ if and only if ϕ is a finite linear combination of continuous positive-definite functions. It seems difficult to apply this, however, whereas the following criterion will be very useful to us. It was proved by Bochner [2] on the real line; an integral analogue is due to Schoenberg [1]; for the general case, see Eberlein [1].

THEOREM. *Each of the following two statements about a function ϕ, defined on Γ, implies the other*:

(a) $\phi \in B(\Gamma)$ *and* $\|\phi\| \leqq A$.

(b) ϕ *is continuous, and*

$$|\sum_{i=1}^{n} c_i \phi(\gamma_i)| \leqq A\|f\|_\infty \tag{1}$$

for every trigonometric polynomial f on G, of the form

$$f(x) = \sum_{i=1}^{n} c_i(x, \gamma_i). \tag{2}$$

Proof: If (a) holds, then $\phi = \hat{\mu}$, $\|\mu\| \leqq A$, and

$$\sum c_i \phi(\gamma_i) = \sum c_i \int_G (-x, \gamma_i) d\mu(x) = \int_G f(-x) d\mu(x). \tag{3}$$

Hence (a) implies (b).

To prove the converse, we pass to the Bohr compactification $\bar{G}$ of G. In the notation of Section 1.8, the formula

$$f(\bar{x}) = \sum_{k=1}^{n} c_k (\gamma_k, \bar{x}) \qquad (\bar{x} \in \bar{G}) \tag{4}$$

extends each trigonometric polynomial f on G to a trigonometric polynomial on $\bar{G}$, and since G is dense in $\bar{G}$, the norm $\|f\|_\infty$ is not altered by this extension. The linear functional T defined on the space of all trigonometric polynomials f of the form (4) by

$$Tf = \sum c_k \phi(\gamma_k) \tag{5}$$

thus satisfies the inequality

$$|Tf| \leqq A \|f\|_\infty; \tag{6}$$

thus T can be extended to a bounded linear functional on $C(\bar{G})$, of norm not exceeding A, and the Riesz representation theorem implies that there is a measure $\mu \in M(\bar{G})$ such that $\|\mu\| \leqq A$ and

$$\sum c_k \phi(\gamma_k) = \int_{\bar{G}} f(-\bar{x}) d\mu(\bar{x}) \tag{7}$$

for all f of the form (4). Taking $f(\bar{x}) = (\gamma, \bar{x})$, for some $\gamma \in \Gamma$, we obtain

$$\phi(\gamma) = \int_{\bar{G}} (-\gamma, \bar{x}) d\mu(\bar{x}) \qquad (\gamma \in \Gamma). \tag{8}$$

To complete the proof, we have to show that μ is concentrated on G (more precisely, on $\beta(G)$, in the notation of 1.8).

It follows from the Radon-Nikodym theorem (Appendix E9) that there is a Borel function g on $\bar{G}$, of absolute value 1, such that $g d\mu = d|\mu|$, and since $C(\bar{G})$ is dense in $L^1(|\mu|)$ (Appendix E8), there is a sequence of trigonometric polynomials f_n on $\bar{G}$ such that

$$\lim_{n\to\infty} \int_{\bar{G}} |f_n - g| d|\mu| = 0. \tag{9}$$

By (8), the transforms

$$\phi_n(\gamma) = \int_{\bar{G}} (-\gamma, \bar{x}) f_n(\bar{x}) d\mu(\bar{x}) \qquad (\gamma \in \Gamma) \tag{10}$$

are finite linear combinations of translates of ϕ and hence are continuous on Γ (not merely on Γ_d!). By (9), $\{\phi_n\}$ converges uniformly to

$$\Phi(\gamma) = \int_{\bar{G}} (-\gamma, \bar{x}) d|\mu|(\bar{x}) \qquad (\gamma \in \Gamma), \tag{11}$$

and Φ is continuous on Γ. The representation (11) shows that Φ is positive-definite, and so, by Bochner's theorem, Φ is the Fourier-Stieltjes transform of a measure σ on the dual group G of Γ. The uniqueness theorem for Fourier-Stieltjes transforms now implies that $\sigma = |\mu|$, hence $|\mu|$ is concentrated on G, and so is μ.

1.9.2. Theorem. *If $\phi_n \in B(\Gamma)$ and $||\phi_n|| \leqq A$ for $n = 1, 2, 3, \ldots$, if $\phi \in C(\Gamma)$ and if*

$$\phi(\gamma) = \lim_{n \to \infty} \phi_n(\gamma) \qquad (\gamma \in \Gamma), \tag{1}$$

then $\phi \in B(\Gamma)$ and $||\phi|| \leqq A$.

This is a corollary of Theorem 1.9.1.

CHAPTER 2

The Structure of Locally Compact Abelian Groups

This chapter contains those structure theorems which will be useful later. The proofs make strong use of the duality theorem. For results which go beyond what is presented here, the books by Montgomery and Zippin [1], Pontryagin [1], and Weil [1] may be consulted. Some material on local identities is also included. Theorems 2.6.1 to 2.6.6 use a device introduced by Helson [1] and Reiter [1].

2.1. The Duality between Subgroups and Quotient Groups

2.1.1. Suppose H is a closed subgroup of the LCA group G, and Λ is the set of all $\gamma \in \Gamma$ (the dual group of G) such that $(x, \gamma) = 1$ for all $x \in H$. We call Λ the *annihilator* of H.

For any fixed $x \in H$, the continuity of (x, γ) shows that the set of all γ with $(x, \gamma) = 1$ is closed, so that Λ is an intersection of closed sets. Since Λ is evidently a group, we conclude that *Λ is a closed subgroup of Γ.*

2.1.2. Theorem. *With the above notation Λ and Γ/Λ are (isomorphically homeomorphic to) the dual groups of G/H and H, respectively.*

Proof: Let h be the natural homomorphism of G onto G/H (Appendix B2). The equation

$$(1) \qquad (x, \gamma) = (h(x), \phi) \qquad (x \in G)$$

defines a one-to-one correspondence between the elements $\gamma \in \Lambda$ and the continuous characters ϕ on G/H, and if γ_i corresponds to ϕ_i $(i = 1, 2)$, then $\gamma_1 + \gamma_2$ corresponds to $\phi_1 + \phi_2$, since
$(x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2) = (h(x), \phi_1)(h(x), \phi_2) = (h(x), {}_1 + \phi_2)$.
Hence (1) defines an isomorphism τ between Λ and the dual

group of G/H. To every compact set C_1 in G/H there corresponds a compact set C_2 in G such that $C_1 = h(C_2)$, since h is a continuous open map. If

$$N(C_1, r) = \{\phi: |1 - (h(x), \phi)| < r \text{ for all } h(x) \in C_1\}$$

and

$$N(C_2, r) = \{\gamma: |1 - (x, \gamma)| < r \text{ for all } x \in C_2\} \cap \Lambda,$$

then it is clear that τ maps $N(C_2, r)$ onto $N(C_1, r)$, and Theorem 1.2.6 shows that τ is a homeomorphism.

The second part of the theorem (that Γ/Λ is the dual group of H) follows from the first part by the Pontryagin duality theorem, as soon as the following lemma is proved:

2.1.3. LEMMA. *If Λ is the annihilator of H, then H is the annihilator of Λ.*

Proof: If $x_0 \in H$, the definition of Λ shows that $(x_0, \gamma) = 1$ for all $\gamma \in \Lambda$. If $x_0 \notin H$, the argument used at the end of the proof of Theorem 1.8.2 shows that there exists $\gamma \in \Lambda$ such that $(x_0, \gamma) \neq 1$.

2.1.4. THEOREM. *If H is a closed subgroup of G, every continuous character on H can be extended to a continuous character on G.*

Proof: If ϕ is a continuous character on H, then $\phi \in \Gamma/\Lambda$, in the notation used above, and if h is the natural homomorphism of Γ onto Γ/Λ and $h(\gamma) = \phi$, then $(x, \gamma) = (x, \phi)$ for all $x \in H$. Hence γ is an extension of ϕ.

2.2. Direct Sums

2.2.1. The notions of *direct sum* and *complete direct sum* of LCA groups are defined in Appendix B7. The direct sum of G_1 and G_2 will be written $G_1 \oplus G_2$, and the direct sum of n copies of G will be denoted by G^n. In particular, T^n is the n-dimensional torus, R^n is n-dimensional euclidean space, and Z^n is the group of all lattice points in R^n, i.e., the group of all points in R^n with integral coordinates. (Compare Section 1.2.7).

2.2.2. THEOREM. *If $G = G_1 \oplus G_2 \oplus \ldots \oplus G_n$ and if Γ_i is the dual group of G_i $(1 \leq i \leq n)$, then $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$.*

Proof: It is clearly enough to consider the case $n = 2$. If $x = x_1 + x_2$ is the unique representation of $x \in G$ as a sum of elements of G_1 and G_2, if $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, the pair γ_1, γ_2 determines a character $\gamma \in \Gamma$ by the formula

$$(1) \qquad (x, \gamma) = (x_1, \gamma_1)(x_2, \gamma_2).$$

Since every $\gamma \in \Gamma$ is determined by its action on the subgroups G_1 and G_2, (1) shows that Γ is algebraically the direct sum of Γ_1 and Γ_2. Since Γ_1 and Γ_2 are the annihilators of G_2 and G_1, they are closed subgroups of Γ, and since Γ is thus algebraically the direct sum of two of its closed subgroups, the topology of Γ is identical with the product topology of $\Gamma_1 \times \Gamma_2$.

COROLLARY. *R^n is its own dual; T^n and Z^n are the duals of each other.*

2.2.3. THEOREM. *If G is the complete direct sum of a family $\{G_\alpha\}$ of compact abelian groups, then Γ is the direct sum of the corresponding dual groups Γ_α.*

Proof: Each $x \in G$ may be thought of as a string

$$(1) \qquad x = (\ldots, x_\alpha, \ldots),$$

the group operating being componentwise addition. If

$$(2) \qquad \gamma = (\ldots, \gamma_\alpha, \ldots),$$

with only finitely many $\gamma_\alpha \neq 0$, then γ is a continuous character on G, defined by

$$(3) \qquad (x, \gamma) = \prod_\alpha (x_\alpha, \gamma_\alpha),$$

since each factor in this product is a continuous character on G and only finitely many factors are different from 1.

Conversely, for each index α, Γ_α is the dual of the subgroup G_α of G which consists of all elements of the form $(\ldots, 0, 0, x_\alpha, 0, 0, \ldots)$. It follows that every $\gamma \in \Gamma$ is of the form (2), and that (x, γ) is given by (3). It remains to be proved that only finitely many γ_α can be different from 0 for any γ.

Suppose infinitely many γ_α are different from 0 in (2), and let V be a neighborhood of 0 in G. The definition of the product topology shows that V restricts only finitely many of the coordinates x_α. Hence there exists α such that $\gamma_\alpha \neq 0$ and $V \supset G_\alpha$. Then

$$(V, \gamma) \supset (G_\alpha, \gamma) = (G_\alpha, \gamma_\alpha),$$

which is a non-trivial subgroup of the circle T. It follows that (V, γ) is not contained in $\{z: |1 - z| < 1\}$, and since V was chosen arbitrarily, the continuity of γ is contradicted.

2.2.4. Let q be an integer, $q \geqq 2$, and let Γ be the direct sum of countably many copies of the cyclic group Z_q of order q. Its dual G is compact, is the complete sum of countably many copies of Z_q, by Theorem 2.2.3 (since Z_q is its own dual), and is homeomorphic to the Cantor set. We shall denote this group G by D_q.

2.2.5. Another interesting example is the infinite-dimensional torus T^ω, the complete direct sum of countably many copies of T. Its dual is the direct sum Z^∞ of countably many copies of Z. Functions on T^ω may be regarded as periodic functions in countably many variables. If $f \in L^1(T^\omega)$, then

$$\hat{f}(n_1, n_2, \ldots) = \int_{T^\omega} f(x_1, x_2, \ldots) \exp\{-i \sum n_k x_k\}\, dx,$$

where only finitely many of the integers n_k are different from 0, and the x_k are real numbers modulo 2π. The inversion formula has the form

$$f(x_1, x_2, \ldots) = \sum \hat{f}(n_1, n_2, \ldots) \exp\{i \sum n_k x_k\}.$$

T^ω is metrizable, and is, in fact, a universal compact metric abelian group (we use *metric* synonymously with *metrizable*):

2.2.6. THEOREM. *In the class of all compact abelian groups G, the following three properties are equivalent*:

(a) *G is metric.*

(b) *Γ is countable.*

(c) *G is a closed subgroup of T^ω.*

Proof: If G is metric, then $C(G)$ is separable. (Appendix A16). If $\gamma_1 \neq \gamma_2$ $(\gamma_i \in \Gamma)$, then

$$\|\gamma_1 - \gamma_2\|_\infty^2 \geqq \int_G |(x, \gamma_1) - (x, \gamma_2)|^2 dx = 2,$$

and so the presence of uncountably many γ would contradict the separability of $C(G)$. Hence (a) implies (b).

Every countable Γ is a quotient group of Z^{∞}, and so the implication (b) $\rightarrow$ (c) is a consequence of Theorem 2.1.2.

Finally, the dual group of T^{ω} is countable, the trigonometric polynomials on T^{ω} are dense in $C(T^{\omega})$, and hence $C(T^{\omega})$ is separable and T^{ω} is metric (Appendix A16). Thus (c) implies (a).

2.3. *Monothetic Groups*

2.3.1. A topological group G is called *monothetic* if it has a dense subgroup which is a homomorphic image of Z. In other words, G is to contain a dense set of points x_n $(n \in Z)$ such that $x_n + x_m = x_{n+m}$ $(n, m \in Z)$.

2.3.2. THEOREM. *Suppose G is a monothetic* LCA *group. If G is not compact, then $G = Z$.*

Proof: If G is discrete, then either $G = Z$ or G is a finite cyclic group, hence is compact. Thus we have to prove that G is compact if G is not discrete.

Let V be an open symmetric neighborhood of 0 in G, with compact closure $\bar{V}$. If $y \in G$, then $y \in x_k + V$ for some k, where $\{x_n\}$ is the dense subset of G described in 2.3.1, and there is a symmetric neighborhood W of 0 in G such that $y - x_k + W \subset V$. Since G is not discrete, W contains infinitely many of the points x_n, and since W is symmetric, $x_{-n} \in W$ if $x_n \in W$. Hence there exists $j < k$ so that $x_j \in W$. Putting $i = k - j$, we have $i > 0$, and

$$y - x_i = y - x_k + x_j \in y - x_k + W \subset V.$$

This proves that

$$G = \bigcup_{i=1}^{\infty} (x_i + V); \tag{1}$$

the point is that it suffices to take positive subscripts in (1).

Since $\bar{V}$ is compact, (1) shows that

$$\bar{V} \subset \bigcup_{i=1}^{N} (x_i + V) \tag{2}$$

for some integer N. For $y \in G$, let $n = n(y)$ be the smallest positive integer such that $y \in x_n + \bar{V}$. Then $x_n - y \in x_i + V$, for some i $(1 \leqq i \leqq N)$, by (2), so that $y \in x_{n-i} + V$. Since $i > 0, n - i < n$, and so $n - i \leqq 0$, by our choice of n. Thus $n \leqq i \leqq N$ for all $y \in G$, and so

$$G = \bigcup_{i=1}^{N} (x_i + \bar{V}). \tag{3}$$

Being a finite union of compact sets, G is compact, and the proof is complete.

2.3.3. The compact monothetic groups have a simple characterization in terms of their duals (Halmos and Samelson [1], Anzai and Kakutani [1]):

THEOREM. *A compact abelian group G is monothetic if and only if its dual Γ is a subgroup of T_d, the circle group with the discrete topology.*

Proof: If G is monothetic, the continuous characters of G are evidently determined by their values on the dense homomorphic image of Z in G. Hence Γ is a subgroup of the dual T of Z. Since G is compact, Γ must be discrete.

Conversely, if Γ is a subgroup of T_d, then G is a quotient group of the dual of T_d (by Theorem 2.1.2), i.e., of the Bohr compactification $\bar{Z}$ of Z. Since $\bar{Z}$ is obviously monothetic, so is its continuous homomorphic image G.

2.4. *The Principal Structure Theorem*

2.4.1. THEOREM. *Every* LCA *group G has an open subgroup G_1 which is the direct sum of a compact group H and a euclidean space R^n $(n \geqq 0)$.*

Note that G_1 is also closed (Appendix B5), and that G/G_1 is discrete, since the natural homomorphism of G onto G/G_1 maps the open set G_1 onto the 0 of G/G_1.

We shall begin with some lemmas which are of independent interest.

2.4.2. LEMMA. *If G is generated by a compact neighborhood V of 0, then G contains a closed subgroup (isomorphic to) Z^n, for some $n \geqq 0$, such that G/Z^n is compact, and such that $V \cap Z^n = \{0\}$.*

Proof: Without loss of generality, assume V is symmetric. Putting $V_1 = V$, $V_{n+1} = V_n + V$, we have $G = \bigcup V_n$ $(n \geqq 1)$. Since V_2 is compact, there are points $x_1, \ldots, x_p \in G$ such that $V_2 \subset \bigcup (x_i + V)$ $(1 \leqq i \leqq p)$. Let H be the group generated by $x_1, \ldots, x_p$. Assuming that $V_n \subset V + H$ (which is trivial for $n = 1$ and true for $n = 2$, by our choice of $x_1, \ldots, x_p$), we have

$$V_{n+1} \subset V + V + H = V_2 + H \subset V + H + H = V + H;$$

by induction, $V_n \subset V + H$ for all $n \geqq 1$, and so $G = V + H$.

Let $\bar{H}_i$ be the closure in G of the group H_i generated by x_i $(1 \leqq i \leqq p)$. If each $\bar{H}_i$ is compact, then $\bar{H}$ is compact, hence $G = V + \bar{H}$ is compact, and the lemma is true with $n = 0$. If G is not compact, it follows that one of the monothetic LCA groups $\bar{H}_i$ is isomorphic to Z (Theorem 2.3.2). In this case $\bar{H}_i = H_i$, and we conclude:

If $G = V + H$, where H is a finitely generated group, and if G is not compact, then H contains a closed infinite cyclic subgroup of G.

Since H is finitely generated, there is a *largest* integer n such that H contains a closed subgroup of G, say H', which is isomorphic to Z^n. Since $H' \cap V$ is a finite set, we may also assume (replacing H' by one of its subgroups of finite index, if necessary) that $H' \cap V = \{0\}$. Let ϕ be the natural homomorphism of G onto $G' = G/H'$. Then $G' = \phi(V) + \phi(H)$, our choice of n shows that $\phi(H)$ contains no closed infinite cyclic subgroup of G', and the preceding italicized statement, applied to G' instead of G, implies that G' is compact.

2.4.3. LEMMA. *Suppose E is a compact open set in G.*

(a) *There is a symmetric neighborhood W of 0 in G such that $E + W = E$.*

(b) *If $0 \in E$, then E contains a compact open subgroup of G.*

(c) *E is a finite union of cosets of open subgroups of G.*

Proof: Since E is open, to every $x \in E$ is associated a symmetric neighborhood V_x of 0 such that $x + V_x + V_x \subset E$. Since E is compact, there are points $x_1, \ldots, x_n \in E$ such that $E = \bigcup (x_i + V_{x_i})$ $(1 \leqq i \leqq n)$. Put $W = \bigcap V_{x_i}$. If $x \in E$ and $w \in W$, then $x \in x_i + V_{x_i}$ for some i, and so

$$x + w \in x_i + V_{x_i} + W \subset x_i + V_{x_i} + V_{x_i} \subset E.$$

This proves (a).

To prove (b), choose W as in (a) and let H be the group generated by W. Then $H \subset E$, H is open, hence H is closed, and since E is compact, H is compact. Finally, (b) shows that E is a union of cosets of open subgroups of G, and since E is compact, (c) is proved.

2.4.4. Corollary. *If G is totally disconnected, then every neighborhood of 0 contains a compact open subgroup of G.*

This follows from 2.4.3(b), since the compact open sets form a base for the topology of G (Appendix A4).

2.4.5. Lemma. *Suppose G is connected, locally isomorphic to R^k, for some $k \geqq 0$, and G contains no infinite compact subgroup. Then G is R^k.*

Proof: To say that G is locally isomorphic to R^k means that there is a spherical neighborhood Q of 0 in R^k, a neighborhood V of 0 in G, and a homeomorphism ϕ of Q onto V such that $\phi(x + y) = \phi(x) + \phi(y)$ whenever x, y, and $x + y$ are in Q.

For each $x \in R^k$, $x/n \in Q$ for all sufficiently large positive integers n. Define $\phi(x) = n\phi(x/n)$. Since

$$n\phi(x/n) = nm\phi(x/nm) = m\phi(x/m)$$

provided x/n and x/m are in Q, ϕ is well defined; ϕ is clearly a continuous homomorphism of R^k into G; and ϕ is one-to-one, for otherwise G would contain a compact subgroup isomorphic to T. Finally, ϕ is an open map, hence $\phi(R^k)$ is an open subgroup of G. Since G is connected, $G = \phi(R^k)$, and the proof is complete.

2.4.6. Proof of theorem 2.4.1. Let G_0 be the component of 0 in G, i.e., G_0 is the largest connected subset of G which contains 0. Then G_0 is closed, and if $x \in G_0$, then $x - G_0$ is connected and intersects G_0, so that $x - G_0 \subset G_0$. Thus G_0 is a closed subgroup of G;

the quotient group G/G_0 is LCA and totally disconnected and so has a compact open subgroup K, by 2.4.4. Let ϕ be the natural homomorphism of G onto G/G_0 and put $G_1 = \phi^{-1}(K)$. Since K is open, G_1 is an open subgroup of G.

Since K is compact, K has no open subgroup of infinite index (otherwise K would be the union of infinitely many disjoint open sets), and since every open subgroup of G_1 contains G_0, it follows that G_1 has no open subgroup of infinite index.

There is a compact neighborhood V in G_1 such that $\phi(V) = K$ (compare Appendix A7). The group H generated by V is an open subgroup of G_1 which intersects every coset of G_0 in G_1. Since these cosets are connected, $H = G_1$. Thus Lemma 2.4.2 implies that G_1 contains a closed subgroup Z^n, for some $n \geqq 0$, such that G_1/Z^n is compact.

If Γ_1 is the dual of G_1, Theorems 2.1.2 and 2.2.2 now show that $\Gamma_1/D = T^n$, where D is the discrete dual of G_1/Z^n. Thus Γ_1 is locally isomorphic to R^n and hence Γ_0, the component of 0 in Γ_1, is open in Γ_1. Since G_1 has no open subgroup of infinite index, Theorem 2.1.2 implies that Γ_1 has no infinite compact subgroup. Thus Lemma 2.4.5 applies to Γ_0 and shows that $\Gamma_0 = R^n$.

So far, then, we see that Γ_1 has R^n as an open subgroup. If we can show that Γ_1 is the direct sum of R^n and a discrete group Λ, then G_1 is the direct sum of R^n and the compact dual of Λ (by Theorem 2.2.2), and the proof is complete.

Let Λ be a subgroup of Γ_1, maximal with respect to the property: $\Lambda \cap R^n = \{0\}$. Since Λ has at most one point in each coset of R^n, Λ is discrete. Also, the sum $R^n + \Lambda$ is direct. Suppose, to reach a contradiction, that $R^n + \Lambda \neq \Gamma_1$. Then there exists $\gamma \in \Gamma_1$, $\gamma \notin R^n + \Lambda$, and the maximality of Λ shows that there exists $\gamma_0 \in \Lambda$ such that $\gamma_0 + k\gamma = x$, for some integer $k \neq 0$ and some $x \in R^n, x \neq 0$. If $y = x/k$ and $\gamma_1 = \gamma - y$, then $k\gamma_1 \in \Lambda, \gamma_1 \notin R^n + \Lambda$, and hence there exists $\gamma_2 \in \Lambda$ such that $\gamma_2 + m\gamma_1 = z$ for some integer $m \neq 0$ and some $z \in R^n$, $z \neq 0$. This last relation may be rewritten in the form

$$k\gamma_2 + km\gamma_1 = kz \neq 0.$$

Since $k\gamma_1$ and γ_2 are in Λ, $k\gamma_2 + km\gamma_1 \in \Lambda$; but $kz \in R^n$, which contradicts the fact that $R^n + \Lambda$ was a direct sum.

Thus $\Gamma_1 = R^n \oplus \Lambda$, and the proof is complete.

2.4.7. Example. As will be apparent from the proof of Theorem 2.4.1, several choices may be possible for G_1, and the question arises whether G_1 can always be chosen so that G is the direct sum of G_1 and a discrete group. The following example (communicated to the author orally by Kaplansky) shows that the answer is negative.

Let G be the set of all sequences $x = \{\xi_n\}$, $n = 1, 2, 3, \ldots$, where $\xi_n = 0, 1, 2, 3$, only finitely many ξ_n are 1 or 3 for any x, and the group operation is componentwise addition modulo 4. Let K be the set of all $x \in G$ with $2x = 0$ (i.e., $\xi_n = 0$ or 2); K is the complete direct sum of countably many groups of order 2. Give K the corresponding product topology, and declare K to be an open subgroup of G. Then G is a LCA group, and since G is totally disconnected, G_1 must be compact. If G_1 were a direct summand of G, then G would contain an infinite closed discrete subgroup, but this is impossible since every infinite subgroup of G has infinitely many elements in K: if $x \in G$, then $2x \in K$.

2.5. The Duality between Compact and Discrete Groups

Since the compact abelian groups are precisely those whose duals are discrete (Section 1.7.3), purely algebraic properties of abelian groups give information about topological properties of compact ones. We begin with some algebraic preliminaries.

2.5.1. An abelian group D is called *divisible* if to every $x \in D$ and to every integer $n \neq 0$ there corresponds at least one $y \in D$ such that $ny = x$.

Theorem (Kaplansky [2]). (a) *Every abelian group G can be embedded in a divisible group D; if G is countable, D may be chosen countable.*

(b) *If ϕ is a homomorphism of a subgroup H of G into a divisible group D, then ϕ can be extended to a homomorphism of G into D.*

Proof: Every G can be defined by specifying generators and relations. Thus $G = F/H$, where F is the direct sum of a certain number of copies of Z, and H is a subgroup of F which corresponds to the relations. F can be embedded in a direct sum E of copies of the additive group of the rational numbers. Since E is divisible, so is E/H, and it is clear that G is a subgroup of E/H. If G is countable, then F (hence E) may be chosen countable and (a) follows.

To prove (b), choose $x_0 \in G$ so that $x_0 \notin H$, and let H' be the group generated by H and x_0. If $nx_0 \notin H$ for $n = 1, 2, 3, \ldots$, let $\phi(x_0)$ be an arbitrary element of D. In the contrary case, let k be the smallest positive integer such that $kx_0 \in H$, and choose $\phi(x) \in D$ so that $k\phi(x_0) = \phi(kx_0)$; since D is divisible, this choice is possible. In either case, extend ϕ to H' by defining

$$\phi(x + nx_0) = \phi(x) + n\phi(x_0) \qquad (x \in H,\ n = 0, \pm 1, \pm 2, \ldots).$$

It is easily verified that ϕ is a homomorphism of H' into D. The proof is completed by transfinite induction (or Zorn's lemma), exactly as in the standard proof of the Hahn-Banach theorem.

2.5.2. THEOREM. *Every infinite compact abelian group G contains an infinite compact metric subgroup.*

Proof: A compact subgroup H of G is metric if and only if its dual is countable (Theorem 2.2.6). By Theorem 2.1.2, the result to be proved is therefore equivalent to the following algebraic proposition:

Every infinite abelian group Γ can be mapped homomorphically onto a countably infinite group.

If Γ is infinite, then Γ contains a countably infinite subgroup Λ which may be embedded in a countable divisible group D (Theorem 2.5.1); this embedding is an isomorphism of Λ into D and can therefore be extended to a homomorphism ϕ of Γ into D. Since $\Lambda = \phi(\Lambda) \subset \phi(\Gamma) \subset D$, $\phi(\Gamma)$ is countable and infinite, and the proof is complete.

(If the word „abelian" is omitted from the above proposition, a false statement results: Schreier and Ulam [1] have shown that the group P of all permutations of a countable set has only two normal subgroups, the finite permutations and the even finite ones; hence every non-trivial homomorphic image of P has the power of the continuum.)

2.5.3. THEOREM. *If G is compact and not of bounded order, then G contains a dense set of elements of infinite order.*

Proof: For $n = 1, 2, 3, \ldots$, let E_n be the set of all $x \in G$ such that $nx = 0$, and assume that one of these sets E_n contains a non-empty open set V. If $W = V - V$, then $nx = 0$ for all $x \in W$. The group H generated by W is compact and open, hence G/H is finite. If G/H has q elements, it follows that $qx \in H$ and so $nqx = 0$, for every $x \in G$. Hence G is of bounded order.

This contradiction implies that none of the compact sets E_n has an interior, and the Baire theorem implies that the complement of $\bigcup E_n$ is dense in G.

2.5.4. THEOREM. *A* LCA *group G is of bounded order if and only if its dual Γ is of bounded order.*

Proof: If $nx = 0$ for all $x \in G$, then

$$(x, n\gamma) = (x, \gamma)^n = (nx, \gamma) = (0, \gamma) = 1$$

for all $\gamma \in \Gamma$, so that $n\gamma = 0$.

2.5.5. We call a LCA group G an I-group if every neighborhood of 0 in G contains an element of infinite order.

THEOREM. (a) *Every I-group contains a closed subgroup which is a metric I-group.*

(b) *If G is not discrete and is not an I-group, then G contains D_q as a closed subgroup, for some $q > 1$.*

Proof: (a) Let G be an I-group. If $n > 0$ in the structure theorem 2.4.1, then G contains R^n, a metric I-group. If $n = 0$, then the open subgroup G_1 of G is a compact I-group, and we may as well assume that G is compact. By Theorem 2.5.3, it is enough to show that G contains a compact subgroup H which is not of bounded order, and Theorems 2.1.2 and 2.5.4 show that this is equivalent to the following algebraic proposition:

Every abelian group Γ which is not of bounded order can be mapped homomorphically onto a countable group which is not of bounded order.

To prove this, note that Γ contains a countable group Λ which is not of bounded order, and proceed as in the proof of Theorem 2.5.2.

(b) If G is not an I-group and is not discrete, Theorems 2.4.1 and 2.5.3 show that G contains an infinite compact subgroup G_1 of bounded order, whose dual Γ_1 is also of bounded order, and hence (Appendix B8) is the direct sum of infinitely many finite cyclic groups. Some countable subset of these has the same order, say q; their direct sum is a direct summand of Γ_1, hence is a quotient group of Γ_1, hence is the dual of a compact subgroup of G, isomorphic to D_q.

2.5.6. THEOREM. *Suppose G is compact.*

(a) *If every element of Γ has finite order, then G is totally disconnected.*

(b) *If Γ contains an element of infinite order, then G contains a one-parameter subgroup.*

(c) *G is connected if and only if Γ contains no element of finite order (except $\gamma = 0$).*

A *one-parameter subgroup* of G is, by definition, a non-trivial subgroup H of G which is the image of R under a continuous homomorphism ϕ. For instance, for any real λ, the set of all points $(e^{ix}, e^{i\lambda x})$ $(-\infty < x < \infty)$ is a one-parameter subgroup H of the torus T^2; if λ is rational, H is compact; if λ is irrational, H is dense in T^2, and hence is not locally compact.

Note that (b) asserts more than just the converse of (a).

Proof: Let G_0 be the component of 0 in G; G_0 is a closed subgroup of G, and if G_0 consists of more than one point, then G_0 has a non-constant character, which may be extended to a continuous character γ on G, by Theorem 2.1.4. Since γ maps G_0 onto a connected subgroup of T, we see that γ maps G_0 onto T.

If γ had order n, then $(x, \gamma)^n = (x, n\gamma) = (x, 0) = 1$ for each $x \in G$, so that γ would map G onto a finite subgroup of T. This contradiction shows that γ has infinite order, and proves (a).

If Γ contains an element of infinite order, then Γ contains Z as a subgroup. The identity map of Z into R_d (the real numbers with the discrete topology) is an isomorphism which, by Theorem 2.5.1, can be extended to a homomorphism of Γ into R_d. Theorem 2.1.2 now shows that G contains a compact subgroup H whose dual is a non-trivial subgroup of R_d, and that H is therefore the continuous image of the Bohr compactification $\bar{R}$ of R, under a homomorphism ϕ. Since R is a dense one-parameter subgroup of $\bar{R}$, $\phi(R)$ is a dense one-parameter subgroup of H, and (b) is proved.

To prove (c), assume first that G is not connected. By Lemma 2.4.3, G then contains a proper open subgroup H. The quotient G/H is finite and its dual is a subgroup of Γ, by Theorem 2.1.2. Hence Γ contains a non-trivial finite subgroup.

Conversely, if $\gamma \in \Gamma$ has finite order and $\gamma \neq 0$, then γ maps G onto a non-trivial finite subgroup of T, and since γ is continuous, G cannot be connected.

2.5.7. Examples. To illustrate the preceding theorem, let $\bar{G}$ be the Bohr compactification of the discrete group G. The knowledge that G is discrete tells us nothing about the topology of $\bar{G}$; the algebraic structure of G is decisive:

(i) If G is the discrete additive group of the rational numbers, then G has no subgroup of finite index, hence Γ has no element of finite order, hence $\bar{G}$ is connected and contains one-parameter subgroups.

(ii) If $G = Z$, then $\Gamma = T$; T has elements of finite and infinite orders; hence $\bar{Z}$ is neither connected nor totally disconnected and contains one-parameter subgroups.

(iii) If G is a discrete group of bounded order, then Γ is of bounded order, hence $\bar{G}$ is totally disconnected.

2.6. Local Units in $A(\Gamma)$

In this section we gather some technical results which should be regarded as tools for our later work.

2.6.1. Theorem. *Suppose C is a compact subset of Γ, $V \subset \Gamma$, and $0 < m(V) < \infty$, where m is the Haar measure of Γ. Then there exists $k \in L^1(G)$ such that*

(a) $\hat{k}(\gamma) = 1$ *on* C, $\hat{k}(\gamma) = 0$ *outside* $C + V - V$, *and* $0 \leqq \hat{k}(\gamma) \leqq 1$ *for all* $\gamma \in \Gamma$.

(b) $\|k\|_1 \leqq \{m(C - V)/m(V)\}^{\frac{1}{2}}$.

Proof: Let g and h be the functions in $L^2(G)$ whose Plancherel transforms are the characteristic functions of V and $C - V$, respectively, and define

$$k(x) = \frac{g(x)h(x)}{m(V)} \qquad (x \in G). \tag{1}$$

Then (see Section 1.6.3) $\hat{k} = m(V)^{-1}(\hat{g} * \hat{h})$, or

$$\hat{k}(\gamma) = \frac{1}{m(V)} \int_V \hat{h}(\gamma - \gamma')d\gamma' \qquad (\gamma \in \Gamma). \tag{2}$$

If $\gamma \in C$, then $\hat{h}(\gamma - \gamma') = 1$ for all $\gamma' \in V$, hence $\hat{k}(\gamma) = 1$. If $\gamma \notin C + V - V$, then $\hat{h}(\gamma - \gamma') = 0$ for all $\gamma' \in V$. Since $0 \leqq \hat{h} \leqq 1$, (a) follows.

By the Plancherel theorem, $\|g\|_2 = m(V)^{\frac{1}{2}}$, $\|h\|_2 = m(C - V)^{\frac{1}{2}}$, and the Schwarz inequality, applied to (1), shows that $\|k\|_1 \leqq m(V)^{-1}\|g\|_2\|h\|_2$. This implies (b).

2.6.2. THEOREM. *If W is an open set in Γ which contains a compact set C, then there exists $f \in L^1(G)$ such that $\hat{f} = 1$ on C and $\hat{f} = 0$ outside W.*

Proof: Choose an eighborhood V of 0 in Γ such that $C+V-V \subset W$, and apply Theorem 2.6.1.

2.6.3. THEOREM. *Suppose $f \in L^1(G)$, $\gamma_0 \in \Gamma$, $\hat{f}(\gamma_0) = 0$, W is a neighborhood of γ_0, and $\varepsilon > 0$. There exists $k \in L^1(G)$, such that*

(a) $\|k\|_1 < 2$,

(b) $\hat{k} = 1$ *in a neighborhood of* γ_0 *and* $\hat{k} = 0$ *outside* W,

(c) $\|f * k\|_1 < \varepsilon$.

Proof: Without loss of generality, we assume $\gamma_0 = 0$. Put

$$\delta = \frac{\varepsilon}{4(1 + \|f\|_1)}. \tag{1}$$

There exists a compact set E in G such that the integral of $|f|$ over the complement E' of E is less than δ. We can find C and V, as in Theorem 2.6.1, subject to these further conditions: (i) 0 is an interior point of C, (ii) $m(C - V) < 2m(V)$, (iii) $C + V - V \subset W$, and (iv) $|1 - (x, \gamma)| < \delta$ whenever $x \in E$ and $\gamma \in C + V - V$. Define k as in the proof of Theorem 2.6.1. Then (a) and (b) hold, and since $\hat{f}(0) = 0$, we have

$$(2) \qquad (f * k)(x) = \int_G f(y)\{k(x-y) - k(x)\}dy \qquad (x \in G),$$

so that

$$(3) \qquad \|f * k\|_1 \leqq \int_G |f(y)| \cdot \|k_y - k\|_1 dy = \int_E + \int_{E'}.$$

The integral over E' is less than

$$(4) \qquad 2\|k\|_1\delta \leqq 2\delta\{m(C-V)/m(V)\}^{\frac{1}{2}} < 4\delta,$$

by (ii), and the integral over E does not exceed

$$(5) \qquad \|f\|_1 \cdot \sup_{y \in E} \|k_y - k\|_1.$$

Hence the inequality

$$(6) \qquad \|k_y - k\|_1 < 4\delta \qquad (y \in E)$$

will complete the proof.

In the notation of Theorem 2.6.1,

$$(7) \qquad m(V)(k_y - k) = g(h_y - h) + (g_y - g)h_y.$$

For $y \in E$ (iv) implies, by the Plancherel theorem, that

$$(8) \qquad \int_G |g_y - g|^2 = \int_V |1 - (y, \gamma)|^2 d\gamma < \delta^2 m(V),$$

so that

$$(9) \qquad \|g_y - g\|_2 < \delta\{m(V)\}^{\frac{1}{2}} \qquad (y \in E).$$

Similarly, $\|h_y - h\|_2 < \delta\{m(C-V)\}^{\frac{1}{2}}$, and since $\|g\|_2 = m(V)^{\frac{1}{2}}$ and $\|h\|_2 = m(C-V)^{\frac{1}{2}}$, we obtain

$$(10) \qquad m(V)\|k_y - k\|_1 < 2\delta\{m(V)m(C-V)\}^{\frac{1}{2}} \qquad (y \in E).$$

By (ii), (10) implies (6), and the proof is complete.

2.6.4. THEOREM. *Suppose $f \in L^1(G)$, $\gamma_0 \in \Gamma$, $\hat{f}(\gamma_0) = 0$, and $\varepsilon > 0$. Then there exists $v \in L^1(G)$ such that $\hat{v} = 0$ in a neighborhood of γ_0, $\|v\|_1 < 3$, and $\|f - f * v\|_1 < \varepsilon$.*

Proof: By Theorem 1.1.8, there exists $u \in L^1(G)$ such that $\|u\|_1 = 1$ and $\|f - f * u\|_1 < \varepsilon/2$. Since $(\hat{f}\hat{u})(\gamma_0) = 0$, Theorem 2.6.3 applies to $f * u$, and so there exists $k \in L^1(G)$ such that $\hat{k} = 1$ in a neighborhood of γ_0, $\|k\|_1 < 2$, and $\|f * u * k\|_1 < \varepsilon/2$. Put $v = u - u * k$. Then $\hat{v} = 0$ when $\hat{k} = 1$, and

$$\|f - f * v\|_1 \leqq \|f - f * u\|_1 + \|f * u * k\|_1 < \varepsilon.$$

2.6.5. THEOREM. *Suppose $f \in L^1(G)$, $\gamma_0 \in \Gamma$, W is a neighborhood of γ_0, and $\varepsilon > 0$. There exists $h \in L^1(G)$ such that $\|h\|_1 < \varepsilon$, $\hat{h} = 0$ outside W, and*

$$\hat{f}(\gamma) - \hat{h}(\gamma) = \hat{f}(\gamma_0)$$

in some neighborhood of γ_0.

Proof: Choose $g \in L^1(G)$ such that $\hat{g}(\gamma) = \hat{f}(\gamma_0)$ in some neighborhood of γ_0. Theorem 2.6.3 applies to $f - g$, and so there exists $k \in L^1(G)$ such that $\hat{k} = 1$ in a neighborhood of γ_0, $\hat{k} = 0$ outside W, and $\|(f - g) * k\|_1 < \varepsilon$. Put $h = (f - g) * k$. Then $\hat{h} = (\hat{f} - \hat{g})\hat{k}$, and so there is a neighborhood of γ_0 in which $\hat{h} = \hat{f} - \hat{g} = \hat{f} - \hat{f}(\gamma_0)$.

2.6.6. THEOREM. *Suppose $f \in L^1(G)$ and $\varepsilon > 0$. There exists $v \in L^1(G)$ such that $\hat{v}$ has compact support and $\|f - f * v\|_1 < \varepsilon$.*

Proof: Let X be the set of all $g \in L^2(G)$ such that $\hat{g}$ has compact support. By the Plancherel theorem, X is dense in $L^2(G)$. If $v = gh$, with $g, h \in X$, then $\hat{v} = \hat{g} * \hat{h}$, hence $\hat{v}$ has compact support. Since X is dense in $L^2(G)$, the set of all such v is dense in $L^1(G)$.

By 1.1.8, there exists $u \in L^1(G)$ such that $\|f - f * u\|_1 < \varepsilon/2$, and we can choose $v \in L^1(G)$ such that $\hat{v}$ has compact support and $\|u - v\|_1 < \varepsilon/(2\|f\|_1)$. Then

$$\|f - f * v\|_1 \leqq \|f - f * u\|_1 + \|f * (u - v)\|_1 < \varepsilon.$$

2.6.7. Theorems 2.6.1 to 2.6.6 did not depend on any structure theorems, but our next result does:

THEOREM. *If C is a compact subset of Γ and if $\varepsilon > 0$, there exists a Borel set V in Γ, with compact closure, such that*

$$m(C - V) < (1 + \varepsilon)m(V).$$

Proof: Let W be a compact neighborhood of 0 in Γ which contains $C - C$ in its interior, and let Γ_1 be the group generated by W. Since Γ_1 is open in Γ, we may assume, without loss of generality, that $\Gamma_1 = \Gamma$.

By Lemma 2.4.2, Γ has a closed subgroup Z^k which has only 0 in common with W, such that Γ/Z^k is a compact group, say H. Let ϕ be the natural homomorphism of Γ onto H. Our choice of W and Z^k shows that there is an open set X_1 in Γ, with compact closure which contains C and on which ϕ is a homeomorphism. Put $Y_1 = \phi(X_1)$.

Since H is compact, finitely many translates of Y_1, say $Y_1, \ldots, Y_r$ will cover H, and there are open sets X_i in Γ, with compact closure, such that ϕ maps X_i homeomorphically onto Y_i. If Y'_i is the part of Y_i not in $Y_1 \cup \ldots \cup Y_{i-1}$, if $X'_i = X_i \cap \phi^{-1}(Y'_i)$, and if $E = \bigcup X'_i$ $(1 \leqq i \leqq n)$, then E is a Borel set in Γ, ϕ is one-to-one on E, and $\phi(E) = H$. In other words, each $x \in \Gamma$ has a unique representation $x = e + n$, with $e \in E$ and $n \in Z^k$; we may visualize Γ as being "paved" by the translates $E + n$ of E.

Note also that $C \subset X_1 = X'_1 \subset E$ and that $\bar{E}$ is compact.

If $n = (n_1, \ldots, n_k) \in Z^k$, set $||n|| = \max_i |n_i|$. Since $\bar{E}$ is compact and Z^k is discrete, $Z^k \cap (E + E - E)$ is a finite set, and so there is an integer s such that $||n|| \leqq s$ for all $n \in Z^k \cap (E + E - E)$.

For $N = 1, 2, 3, \ldots$, let $V_N = \bigcup (E + n)$ $(||n|| \leqq N)$. If $x \in V_N + E$ then $x = n + e_1 + e_2 = n' + e$; since $e_1 + e_2 - e \in Z^k$ we have $||n' - n|| \leqq s$; hence $||n'|| \leqq ||n|| + s \leqq N + s$, and so $V_N + E \subset V_{N+s}$.

Since $m(V_N) = (2N + 1)^k m(E)$, we have

$$\frac{m(C + V_N)}{m(V_N)} \leqq \frac{m(E + V_N)}{m(V_N)} \leqq \frac{m(V_{N+s})}{m(V_N)} = \left\{1 + \frac{2s}{2N+1}\right\}^k,$$

and the last expression tends to 1 as $N \to \infty$.

The theorem follows if we take $V = -V_N$, with N large enough.

Remark. The conclusion can be strengthened: There is an open set V with compact closure for which the desired inequality holds. We shall not need this stronger result.

2.6.8. Theorem. *Suppose C is a compact set in Γ, and $\varepsilon > 0$. Then there exists $k \in L^1(G)$ such that $\hat{k} = 1$ on C, $\hat{k}$ has compact support, and $\|k\|_1 < 1 + \varepsilon$.*

Proof: Combine Theorems 2.6.1 and 2.6.7.

2.7. Fourier Transforms on Subgroups and on Quotient Groups

Throughout this section, H will be a closed subgroup of G, and Λ will be the annihilator of H, as in 2.1.1.

2.7.1. Theorem. *A measure $\mu \in M(G)$ is concentrated on H if and only if $\hat{\mu}$ is constant on the cosets of Λ.*

Proof: If μ is concentrated on H and $\gamma_0 \in \Lambda$, then

$$(1) \qquad (-x, \gamma_0)d\mu(x) = d\mu(x),$$

since $(x, \gamma_0) = 1$ on H, and so

$$(2) \qquad \hat{\mu}(\gamma + \gamma_0) = \hat{\mu}(\gamma) \qquad (\gamma_0 \in \Lambda,\ \gamma \in \Gamma).$$

Conversely, if (2) holds, then (1) holds by the uniqueness theorem for Fourier-Stieltjes transforms, so that $(x, \gamma_0) = 1$ almost everywhere with respect to $|\mu|$, for all $\gamma_0 \in \Lambda$. This implies that the support of μ lies in H.

2.7.2. Theorem. *The functions belonging to $B(\Lambda)$ are presicely the restrictions to Λ of the functions belonging to $B(\Gamma)$.*

Proof: Let ϕ be the natural homomorphism of G onto G/H. If $\mu \in M(G)$, the map

$$(1) \qquad f \to \int_G f(\phi(x))d\mu(x)$$

is a bounded linear functional on $C_0(G/H)$, and hence there is a unique measure $\sigma \in M(G/H)$, with $\|\sigma\| \leqq \|\mu\|$, such that

$$(2) \qquad \int_G f(\phi(x))d\mu(x) = \int_{G/H} f d\sigma \qquad (f \in C_0(G/H)).$$

We write $\sigma = \pi\mu$ if (2) holds. In this case (2) also holds for all bounded Borel functions on G/H, and in particular (2) holds for all continuous characters on G/H, i.e., for all $\gamma \in \Lambda$. (Recall that Λ plays a double role: it is the dual group of G/H, and it is a subgroup of Γ.) Hence $\hat{\mu}(\gamma) = \hat{\sigma}(\gamma)$ if $\sigma = \pi\mu$ and if $\gamma \in \Lambda$, and the proof of the theorem will be complete as soon as we show that π maps $M(G)$ *onto* $M(G/H)$.

If V is a compact neighborhood of 0 in G/H, then, as in the proof of Lemma 2.4.2, the group generated by V is $V + D$, where D is a discrete subgroup of G/H. Hence G/H is covered by a collection $\{V_\alpha\}$ of translates of V, such that every compact subset of G/H is covered by a finite subcollection of $\{V_\alpha\}$. To each V_α there corresponds a compact set $E_\alpha \subset G$ such that $\phi(E_\alpha) = V_\alpha$. Put $X = \bigcup E_\alpha$. Then X is locally compact, $\phi(X) = G/H$, and $\phi^{-1}(K) \cap X$ is compact for every compact subset K of G/H.

Let S be the subspace of all $g \in C_0(X)$ which are constant on the intersections of X with the cosets of H; S is isometrically isomorphic to $C_0(G/H)$. If $\sigma \in M(G/H)$ and $f(x) = g(\phi^{-1}(x))$, the map $g \to \int f d\sigma$ is a bounded linear functional on S; extending it to $C_0(X)$, we find that there is a measure $\mu \in M(X)$, with $\|\mu\| \leqq \|\sigma\|$, such that (2) holds.

This completes the proof.

2.7.3. Suppose m_G, m_H, and $m_{G/H}$ are the Haar measures of the indicated groups, and let $\xi = \xi(x)$ be the coset of H (the element of G/H) which contains x, where $x \in G$. For any $f \in C_c(G)$, the integral

$$(1) \qquad \int_H f(x+y)\,dm_H(y)$$

is not changed if x is replaced by $x + h$, where $h \in H$. Hence (1) is a function of ξ, which belongs to $C_c(G/H)$. The Haar measures can be adjusted so that

$$(2) \qquad \int_G f\,dm_G = \int_{G/H} dm_{G/H}(\xi) \int_H f(x+y)\,dm_H(y)$$

for every $f \in C_c(G)$, since the right side of (2) is a positive translation invariant linear functional on $C_c(G)$.

Suppose that the measures are so adjusted, and denote the integral (1) by $F(\xi)$. The map T defined by $F = Tf$ is a bounded linear transformation of $L^1(G)$ into $L^1(G/H)$; it is easy to see that T is actually onto, and that T is nothing but the restriction of the map π of Theorem 2.7.2 to $L^1(G)$. Hence $\hat{F}(\gamma) = \hat{f}(\gamma)$ if $\gamma \in \Lambda$.

Summarizing, we obtain the following result:

2.7.4. Theorem. *The functions belonging to $A(\Lambda)$ are precisely the restrictions to Λ of the functions belonging to $A(\Gamma)$. For $f \in L^1(G)$, $\hat{f}$ vanishes on Λ if and only if*

$$\int_H f(x+y)dm_H(y) = 0$$

for almost all $x \in G$.

2.7.5. Theorem (Calderon [2]). *If $g \in L^1(G)$, $\eta > 0$, and $\hat{g}$ vanishes on Λ, then there exists $\mu \in M(G)$, concentrated on H, such that $\|\mu\| < 2$, $\|g * \mu\| < \eta$, and $\hat{\mu} = 1$ on an open set containing Λ.*

Proof: By Theorem 1.1.8, there exists $u \in C_c(G)$, such that $\|f - g\|_1 < \eta/3$ if $f = g * u$. Since

$$\text{(1)}\quad \int_H |f(x+s) - f(x'+s)|dm_H(s) \leqq \int_G |g(y)|dm_G(y)\int_H |u(x+s-y) - u(x'+s-y)|dm_H(s),$$

the fact that $u \in C_c(G)$ shows that to each $\delta > 0$ there corresponds a neighborhood V_δ of 0 in G such that the left side of (1) is less than δ if $x - x' \in V_\delta$. Put

$$\text{(2)}\quad \alpha(\xi) = \int_H |f(x+s)|dm_H(s),$$

where ξ is the coset of H which contains x. If $k \in L^1(H)$, $\|k\|_1 < 2$, and

$$\text{(3)}\quad \beta_k(\xi) = \int_H \left| \int_H f(x+s)k(t-s)dm_H(s)\right| dm_H(t),$$

then $\beta_k \leqq 2\alpha$, and if $x' - x \in V_\delta$, we have

$$\text{(4)}\quad \begin{aligned} |\beta_k(\xi) - \beta_k(\xi')| &\leqq \int_H\int_H |f(x+s) - f(x'+s)||k(t-s)|dm_H(s)\,dm_H(t) \\ &\leqq 2\int_H |f(x+s) - f(x'+s)|dm_H(s) < 2\delta. \end{aligned}$$

The proof of Theorem 2.6.3 can be modified so that it applies to a finite set of functions f_j with $\hat{f}_j(\gamma_0) = 0$. Considering $f(x + s)$, for fixed $x \in G$, as a function $\phi(s)$ on H, Theorem 2.7.4. shows that $\hat{\phi}(0) = 0$, since $\hat{f}(\gamma) = 0$ for all $\gamma \in \Lambda$. Hence if $\delta > 0$ and $x_1, \ldots, x_n$ are points of G, we can find $k \in L^1(H)$, so that $\|k\|_1 < 2$, $\hat{k} = 1$ in a neighborhood of 0 in Γ/Λ, and

$$\beta_k(\xi_j) < \delta \qquad (1 \leqq j \leqq n). \tag{5}$$

Suppose $0 < \varepsilon < \eta/6$. Since $\alpha \in L^1(G/H)$, there is a compact set C in G/H such that the integral of α over the complement of C is less than ε. Choose $\delta > 0$ so that $3\delta \cdot m_{G/H}(C) + 2\varepsilon < \eta/3$. There are finitely many ξ_j in C and there is a function k of the above type, so that to each ξ in C there corresponds a ξ_j for which $|\beta_k(\xi) - \beta_k(\xi_j)| < 2\delta$, as in (4), and so that $\beta_k(\xi_j) < \delta$ for all j. Then $\beta_k(\xi) < 3\delta$ on C, and our choice of ε and δ shows that

$$\int_{G/H} \beta_k(\xi) dm_{G/H}(\xi) < \eta/3. \tag{6}$$

But the integral (6) is just $\|f * \mu\|_1$, where $d\mu = k\, dm_H$. Finally, $\|g * \mu\|_1 \leqq \|f - g\|_1 \|\mu\| + \|f * \mu\|_1 < 2\eta/3 + \eta/3 = \eta$.

2.7.6. Theorem. *Suppose f is a function on the circle T, $0 < \delta < \pi$, and $f(e^{i\theta}) = 0$ if $\pi - \delta \leqq \theta \leqq \pi + \delta$. Let g be defined on the line by*

$$g(x) = \begin{cases} f(e^{ix}) & \text{if } |x| \leqq \pi, \\ 0 & \text{if } |x| > \pi. \end{cases} \tag{1}$$

Then $f \in A(T)$ if and only if $g \in A(R)$. Moreover, there are positive numbers c_1, c_2 (depending on δ) such that

$$c_1\|g\| \leqq \|f\| \leqq c_2\|g\|, \tag{2}$$

the norms being those of $A(T)$ and $A(R)$, respectively.

Proof: Let h be a function on R with two continuous derivatives, such that $h(x) = 1$ on $[-\pi + \delta, \pi - \delta]$ and $h(x) = 0$ if $|x| \geqq \pi$. The Fourier transform of h'' is $-y^2\hat{h}(y)$; it follows that

$$|\hat{h}(y)| < \frac{a}{1 + y^2} \qquad (-\infty < y < \infty) \tag{3}$$

for some constant a. The inversion theorem implies that $h \in A(R)$.

If $f \in A(T)$ and $F(x) = f(e^{ix})$ for all real x, then $F \in B(R)$, $||F|| = ||f||$, $g = hF$, and hence $g \in A(R)$ and $||g|| \leqq ||h|| \cdot ||f||$.

If $g \in A(R)$, then $g = gh$, and so

$$(4) \qquad \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)h(x)e^{-inx}\,dx.$$

By (3), the inversion formula holds for h; substitution into (4) yields

$$(5) \qquad \hat{f}(n) = \int_{-\infty}^{\infty} \hat{g}(y)\hat{h}(n-y)dy \qquad (n \in Z).$$

By (3), there is a constant b such that $\sum_{-\infty}^{\infty} |\hat{h}(n-y)| < b$ for all $y \in R$. Hence

$$(6) \qquad ||f|| = \sum_{-\infty}^{\infty} |\hat{f}(n)| \leqq b\int_{-\infty}^{\infty} |\hat{g}(y)|dy = b||g||.$$

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CHAPTER 3

Idempotent Measures

A measure $\mu \in M(G)$ is said to be *idempotent* if $\mu * \mu = \mu$. The set of all idempotents in $M(G)$ will be denoted by $J(G)$. The problem considered in this chapter is the determination of all members of $J(G)$.

Apart from its intrinsic interest, the solution of this problem turns out to be the crucial link in the description of the homomorphisms between group algebras (Chapter 4); it yields all bounded linear projections in $L^1(G)$ which commute with translation (Section 3.8); and it determines the class of all simple functions on Γ (i.e., those whose range is a finite set) which belong to $B(\Gamma)$.

3.1. Outline of the Main Result

3.1.1. If $\mu \in J(G)$, then $\hat{\mu}^2 = \hat{\mu}$, so that $\hat{\mu}(\gamma) = 1$ or 0 for all $\gamma \in \Gamma$. Define

$$S(\mu) = \{\gamma \in \Gamma : \hat{\mu}(\gamma) = 1\} \qquad (\mu \in J(G)).$$

The problem of finding all $\mu \in J(G)$ is obviously equivalent to the problem of finding all subsets of Γ whose characteristic function belongs to $B(\Gamma)$.

3.1.2. Suppose Λ is an open subgroup of Γ and H is its annihilator. Since Γ/Λ is discrete, H is compact, and if m_H is the Haar measure of H, normalized so that $m_H(H) = 1$, then m_H may be regarded as a measure on G. The orthogonality relations 1.2.5 show that $\hat{m}_H(\gamma) = 1$ if $\gamma \in \Lambda$ and $\hat{m}_H(\gamma) = 0$ otherwise. Hence $\Lambda = S(m_H)$.

If $E = \Lambda + \gamma_0$, it follows that $E = S(\mu)$, where $d\mu(x) = (x, \gamma_0)dm_H(x)$. Thus every open coset in Γ is $S(\mu)$ for some $\mu \in J(G)$. (We call a subset E of Γ a *coset in Γ* if E is a coset of some subgroup of Γ; it is frequently of no interest to name the subgroup.)

If μ and λ are in $J(G)$, then so are the measures $\mu * \lambda$ and $\mu \vee \lambda = \mu + \lambda - \mu * \lambda$, as well as $\delta_0 - \mu$, where δ_0 is the point measure of norm 1 concentrated at the point 0 in G. Since

$$S(\mu * \lambda) = S(\mu) \cap S(\lambda), \qquad S(\mu \vee \lambda) = S(\mu) \cup S(\lambda),$$

and $S(\delta_0 - \mu)$ is the complement of $S(\mu)$, the family Ω of all sets $S(\mu)$ in Γ is closed under the formation of finite unions, finite intersections, and complements. In other words, Ω is a *ring* of sets, and the preceding remarks show that Ω contains the *coset-ring* of Γ; the latter is defined as the smallest ring of subsets of Γ which contains all *open* cosets in Γ.

The solution of our problem is simply that Ω is equal to the coset ring:

3.1.3. Theorem. *A subset E of Γ is $S(\mu)$ for some $\mu \in J(G)$ if and only if E belongs to the coset-ring of Γ.*

3.1.4. The result may also be stated without reference to Fourier-Stieltjes transforms:

Call μ an *elementary idempotent* if $d\mu(x) = (x, \gamma_0)dm_H(x)$ where $\gamma_0 \in \Gamma$ and H is a compact subgroup of G. Then every measure on G which can be obtained from the elementary idempotents by finitely many applications of the binary operations $*$ and $\vee$ (see Section 3.1.2) and of complementation ($\delta_0 - \mu$ is the "complement" of μ) is idempotent; moreover (and this is the non-trivial part of the theorem) every $\mu \in J(G)$ is obtained in this manner.

3.1.5. If $\mu \in M(G)$, the *support group* of μ is defined to be the smallest closed subgroup of G on which μ is concentrated. A closed subgroup K of a compact group H is called a *singular* subgroup of H if H/K is infinite; this is equivalent to the requirement that $m_H(K) = 0$. If $|\mu|(K) = 0$ for every singular subgroup K of the support group of μ, then we call μ *irreducible*.

The proof of Theorem 3.1.3 proceeds in three major steps:

(A) *If $\mu \in J(G)$, then the support group of μ is compact.*

(B) *If $\mu \in J(G)$, then $\mu = a_1\mu_1 + \ldots + a_n\mu_n$, where the a_i are integers and the μ_i are irreducible idempotents.*

(C) *If G is compact, if $\mu \in J(G)$, and if $|\mu|(K) = 0$ for every singular subgroup K of G, then $S(\mu)$ is a finite subset of Γ.*

Once this is done, Theorem 3.1.3 follows easily:

Suppose G is LCA, $\mu \in J(G)$, and $H_1, \ldots, H_n$ are the support groups of the measures $\mu_1, \ldots, \mu_n$ which appear in (B). Their annihilators $\Lambda_1, \ldots, \Lambda_n$ are open subgroups of Γ, since (A) implies that the groups H_i are compact. By (C), $S(\mu_i)$ is a finite subset of Γ/Λ_i, the dual of H_i, since $\mu_i \in J(H_i)$. Regarding μ_i as an element of $J(G)$, $S(\mu_i)$ is thus a finite union of cosets of Λ_i, hence belongs to the coset-ring of Γ. It follows that the set of all $\gamma \in \Gamma$ at which any finite linear combination $\sum a_i\hat{\mu}_i(\gamma)$ assumes a given value belongs to the coset-ring of Γ, and this completes the proof, by (B).

3.1.6. A subset E of the integer group Z is a coset in Z either if E consists of a single point or if E is an arithmetic progression, infinite in both directions. If S belongs to the coset-ring of Z, if $A_1, \ldots, A_n$ are the arithmetic progressions involved in the formation of S, and if d is the least common multiple of the differences d_i of the progressions A_i, then it is clear that S differs from a set with period d in at most finitely many places. Thus we obtain the following special case of Theorem 3.1.3:

A sequence $\{c_n\}$ $(-\infty < n < \infty)$ of zeros and ones is the sequence of Fourier-Stieltjes coefficients of some measure on the unit circle if and only if $\{c_n\}$ differs from a periodic sequence in at most finitely many places.

This result is due to Helson [4], [7]. The case $G = T^n$ of Theorem 3.1.3 was proved by Rudin [13]; steps (A) and (B) of 3.1.5 are also in that paper. P. J. Cohen [2] proved the general case of Theorem 3.1.3; in particular, the introduction of "pseudo-periods," Lemma 3.5.5, and the combinatorial argument of Section 3.6 are due to him.

3.2. Some Trivial Cases

3.2.1. Since $\hat{\mu}$ is continuous, $S(\mu)$ is open and closed, for every $\mu \in J(G)$. Consequently, if Γ is connected, there are only two possibilities for $S(\mu)$: $S(\mu) = \Gamma$ or $S(\mu)$ is empty. In other words, δ_0 and 0 are the only members of $J(G)$.

3.2.2. *Every compact open subset of Γ belongs to the coset ring of Γ.* This follows from Lemma 2.4.3.

3.2.3. Suppose $\mu \in J(G)$, $\mu \neq 0$ and $\mu \geqq 0$. Then $\hat{\mu}$ is positive definite and $\hat{\mu}(0) = 1$. If γ and γ' are in $S(\mu)$, then $-\gamma' \in S(\mu)$, and the inequality 1.4.1(4) shows that

$$|\hat{\mu}(\gamma - \gamma') - \hat{\mu}(\gamma)| \leqq 2\hat{\mu}(0) \operatorname{Re} [\hat{\mu}(0) - \hat{\mu}(-\gamma')] = 0.$$

Hence $\gamma - \gamma' \in S(\mu)$, and we conclude that *$S(\mu)$ is an open subgroup of Γ.*

3.2.4. If $\mu \in J(G)$ and $\mu \neq 0$, then $\|\mu\| = \|\mu * \mu\| \leqq \|\mu\|^2$, so that $\|\mu\| \geqq 1$. Suppose $\|\mu\| = 1$. Setting $d\sigma(x) = (x, \gamma)d\mu(x)$, proper choice of γ assures that $\hat{\sigma}(0) = 1$. Then

$$1 = \hat{\sigma}(0) = \sigma(G) \leqq \|\sigma\| = 1;$$

hence $\sigma(G) = \|\sigma\|$, $\sigma \geqq 0$, and the preceding result implies: *If $\mu \in J(G)$ and $\|\mu\| = 1$, then $S(\mu)$ is an open coset in Γ.*

3.3. Reduction to Compact Groups

3.3.1. For technical reasons which will become apparent in the proof of Theorem 3.4.3, it is convenient to enlarge the class $J(G)$ somewhat. We let $F(G)$ be the class of all $\mu \in M(G)$ such that $\hat{\mu}$ is an integer-valued function. Since $\hat{\mu}$ is a bounded function, $\hat{\mu}$ has only finitely many distinct values if $\mu \in F(G)$.

3.3.2. THEOREM. *If $\mu \in F(G)$, then the support group of μ is compact.*

Proof: Let H be the support group of μ. Since μ may be regarded as an element of $F(H)$, we may assume that $G = H$; i.e., that μ is not concentrated on any proper closed subgroup of G. By Theorem 2.7.1 this means that $\hat{\mu}$ is not invariant under any non-zero translation of Γ. If we define μ_γ by

$$d\mu_\gamma(x) = (x, \gamma)d\mu(x) \qquad (\gamma \in \Gamma), \tag{1}$$

it follows that $\mu_\gamma \neq \mu$ if $\gamma \neq 0$. Since $\hat{\mu}_\gamma - \hat{\mu}$ is an integer-valued function, we have

$$(2) \qquad \|\mu_\gamma - \mu\| \geqq \|\hat{\mu}_\gamma - \hat{\mu}\|_\infty \geqq 1 \qquad (\gamma \neq 0).$$

There is a compact set C in G, with complement C', such that $|\mu|(C') < 1/4$. If V is the set of all γ such that

$$(3) \qquad |1 - (x, \gamma)| < (3\|\mu\|)^{-1} \qquad (x \in C),$$

then V is open in Γ, and for $\gamma \in V$ we have

$$(4) \quad \|\mu - \mu_\gamma\| \leqq \int_G |1 - (x, \gamma)| d|\mu|(x) = \int_C + \int_{C'} \leqq \frac{1}{3} + \frac{1}{2} < 1.$$

Comparison of (2) and (4) shows that the open set V consists of 0 alone. Hence Γ is discrete, G is compact, and the theorem is proved.

We note that this contains step (A) of Section 3.1.5 as a special case.

3.4. Decomposition into Irreducible Measures

3.4.1. A homomorphism of $M(G)$. Let H be a singular compact subgroup of the compact group G, let $\{H_\alpha\}$ be the collection of all cosets of H in G, let μ_α be the restriction of μ to H_α, for any $\mu \in M(G)$, and define

$$(1) \qquad \pi_H \mu = \sum_\alpha \mu_\alpha.$$

At most countably many terms are different from 0 in this sum, so that $\pi_H \mu$ is well defined. Also, $\sum \|\mu_\alpha\| \leqq \|\mu\|$. If H were not singular, π_H would be the identity operator.

THEOREM. *π_H is a homomorphism of $M(G)$ into $M(G)$.*

Proof: It is clear that π_H is a bounded linear map of $M(G)$ into $M(G)$. Let R_H and N_H be the range and null-space, respectively, of π_H.

If μ and λ are concentrated on H_α and H_β, then $\mu * \lambda$ is concentrated on $H_\alpha + H_\beta$, which is again a coset of H. It follows that R_H is a subalgebra of $M(G)$. Also, $\pi_H \mu = \mu$ if $\mu \in R_H$.

The null-space N_H, on the other hand, is an ideal in $M(G)$. For if $\pi_H\mu = 0$, if $\sigma = \mu * \lambda$, and if $E \subset H_\alpha$, then $\mu(E - x) = \pi_H\mu(E - x) = 0$ for all $x \in G$, so that

$$\sigma(E) = \int_G \mu(E - x) d\lambda(x) = 0, \tag{2}$$

and so $\sigma_\alpha = 0$. Consequently $\pi_H\sigma = 0$, and $\sigma \in N_H$.

The formula $\mu = \pi_H\mu + (\mu - \pi_H\mu)$ represents μ as a sum of two measures, one in R_H, the other in N_H. This representation is unique, for if $\mu = \mu_1 + \mu_2$, with $\mu_1 \in R_H$, $\mu_2 \in N_H$, then

$$\pi_H\mu = \pi_H\mu_1 + \pi_H\mu_2 = \mu_1.$$

Finally, if $\mu, \lambda \in M(G)$, then

$$\mu * \lambda - (\pi_H\mu) * (\pi_H\lambda) = \mu * (\lambda - \pi_H\lambda) + (\mu - \pi_H\mu) * \pi_H\lambda,$$

and this lies in N_H, since N_H is an ideal; since R_H is an algebra, $(\pi_H\mu) * (\pi_H\lambda) \in R_H$. The uniqueness just established implies now that

$$\pi_H(\mu * \lambda) = (\pi_H\mu) * (\pi_H\lambda), \tag{3}$$

and the proof is complete.

3.4.2. THEOREM. *If H and π_H are as above, and if $\mu \in F(G)$, then $\pi_H\mu \in F(G)$, and $\pi_H\mu$ is concentrated on a singular compact subgroup K of G.*

Proof: Define $\mu^1 = \mu$, $\mu^n = \mu^{n-1} * \mu$, and if P is a polynomial, $P(t) = \sum c_n t^n$, define $P(\mu) = \sum c_n \mu^n$, where $\mu^0 = \delta_0$.

Let $a_1, \ldots, a_n$ be the distinct values of $\hat{\mu}$, where μ is our given measure in $F(G)$, and put $P(t) = \Pi(t - a_i)$. Then $P(\hat{\mu}(\gamma)) = 0$ for all $\gamma \in \Gamma$, and so $P(\mu) = 0$. Since π_H is a homomorphism, $P(\pi_H\mu) = \pi_H P(\mu) = 0$, so that the range of the Fourier-Stieltjes transform of $\pi_H\mu$ is a subset of the set $\{a_1, \ldots, a_n\}$. Hence $\pi_H\mu \in F(G)$.

We now change the topology of G by adjoining to the original collection τ of open sets arbitrary unions of sets of the form $(H + x) \cap V$, where $x \in G$ and $V \in \tau$. The result is an LCA group G_H, which differs from G only insofar as H is now a compact *open* subgroup of G_H; within H the topology is unchanged. We may regard $\pi_H\mu$ as a member of $M(G_H)$. Since $P(\pi_H\mu) = 0$,

$\pi_H\mu \in F(G_H)$, and Theorem 3.3.2 implies that $\pi_H\mu$ is concentrated on a compact subgroup K of G_H. Since H is open in G_H, K intersects only finitely many cosets of H. Since H is a compact singular subgroup of G, it follows that K is also a compact singular subgroup of G, and the proof is complete.

3.4.3. THEOREM. *Suppose G is compact and $\mu \in F(G)$. Then there exist integers $a_1, \ldots, a_n$ and irreducible measures $\mu_1, \ldots, \mu_n \in J(G)$ such that*

(1) $$\mu = a_1\mu_1 + \ldots + a_n\mu_n.$$

Proof: The theorem is trivial if $\mu = 0$. Suppose the theorem is true for all G and for all $\mu \in F(G)$ with $\|\mu\| \leqq p - 1$, where p is a positive integer.

Consider a fixed $\mu \in F(G)$ with $\|\mu\| \leqq p$; without loss of generality, we may assume that G is the support group of μ.

If $\pi_H\mu = 0$ for every singular subgroup H of G, let $a_1, \ldots, a_n$ be the distinct non-zero values of $\hat{\mu}$, construct polynomials P_i such that $P_i(0) = P_i(a_j) = 0$ if $i \neq j$ and $P_i(a_i) = 1$, and put $\mu_i = P_i(\mu)$. Then $\hat{\mu}_i(\gamma) = 1$ if $\hat{\mu}(\gamma) = a_i$ and $\hat{\mu}_i(\gamma) = 0$ otherwise, so that $\mu_i \in J(G)$, and (1) holds. Since π_H is a homomorphism, we have

$$\pi_H\mu_i = \pi_H P_i(\mu) = P_i(\pi_H\mu) = P_i(0) = 0 \qquad (1 \leqq i \leqq n)$$

for every singular subgroup H of G, and this proves that the measures μ_i are irreducible.

If $\pi_H\mu \neq 0$ for some singular subgroup H of G, then $\pi_H\mu$ is concentrated on a compact singular subgroup K of G, by Theorem 3.4.2. Since G is the support group of μ, it follows that $\pi_H\mu \neq \mu$, and so $\mu_1 = \mu - \pi_H\mu \neq 0$. By Theorem 3.4.2, $\pi_H\mu \in F(G)$, and hence $\mu_1 \in F(G)$; since both of these measures are different from 0, their norms are at least 1; since they are concentrated on disjoint sets, we have

$$\|\pi_H\mu\| + \|\mu_1\| = \|\mu\| \leqq p;$$

consequently, $\|\pi_H\mu\| \leqq p - 1$ and $\|\mu_1\| \leqq p - 1$. By our induction hypothesis, both $\pi_H\mu$ and μ_1 are of the form (1), and so is their sum μ.

This completes the proof. Step (B) of Section 3.1.5 follows.

3.5. Five Lemmas

In this section we shall assume that G is compact and that Γ is countable. This latter assumption is merely a matter of convenience. It assures that $C(G)$ is separable, so that the weak* topology of the unit ball in $M(G)$ is metrizable (Appendix C8), and hence every infinite subset of this unit ball contains a convergent sequence. We thus avoid the use of directed sets.

3.5.1. LEMMA. *Suppose $\mu \in M(G)$ and $\{\gamma_n\}$ is a sequence of distinct elements of Γ $(n = 1, 2, 3, \ldots)$. Define λ_n by*

$$d\lambda_n(x) = (x, \gamma_n)d\mu(x) \qquad (n = 1, 2, 3, \ldots).$$

If $\{\lambda_n\}$ converges to $\sigma \in M(G)$, in the weak topology of $M(G)$, then σ is singular. In fact, if $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to the Haar measure of G, then $|\sigma|(E) \leqq |\mu_s|(E)$ for every Borel set E in G.*

Since $\{\hat{\lambda}_n\}$ is a sequence of translates of $\hat{\mu}$, we shall refer to this as the *translation lemma* (Helson [5] [7]).

Proof: Since μ_a is absolutely continuous, $\hat{\mu}_a \in A(\Gamma)$, and so $\hat{\mu}_a(\gamma_n) \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int_G f(x)(x, \gamma_n)d\mu_a(x) = 0 \tag{1}$$

for every trigonometric polynomial f on G and since every $f \in C(G)$ is a uniform limit of trigonometric polynomials, (1) holds for every $f \in C(G)$.

It follows that σ is the weak* limit of the measures defined by $(x, \gamma_n)d\mu_s(x)$. If V is open in G, if $f \in C(G)$, $\|f\|_\infty = 1$, and $f = 0$ outside V, then

$$\left|\int_G f d\sigma\right| = \lim_{n\to\infty} \left|\int_G f(x)(x, \gamma_n)d\mu_s(x)\right| \leqq |\mu_s|(V), \tag{2}$$

so that $|\sigma|(V) \leqq |\mu_s|(V)$. This inequality also holds for all Borel sets, by the regularity of the measures $|\sigma|$ and $|\mu_s|$, and the lemma is proved.

3.5.2. Suppose $\mu \in J(G)$. We call a set $P \subset \Gamma$ a *set of pseudo-periods* of $S(\mu)$ if to every $\gamma \in S(\mu)$ there corresponds a $\gamma' \in P$ such that $\gamma + \gamma' \in S(\mu)$.

LEMMA. *If $S(\mu)$ is infinite and if E is a finite subset of Γ, then there exists a finite set P of pseudo-periods of $S(\mu)$ which does not intersect E.*

Proof: Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be an ordering of the elements of Γ which are not in E. If the lemma is false, then for each $n > 0$ there exists $\gamma_n \in S(\mu)$ such that none of the sums $\gamma_n + \alpha_1, \ldots, \gamma_n + \alpha_n$ lies in $S(\mu)$. If only finitely many γ_n were obtained in this way, then for one of them it would be true that $\gamma_n + \alpha_i \notin S(\mu)$ for $i = 1, 2, 3, \ldots$, and hence $S(\mu)$ would be finite. Thus there are infinitely many distinct γ_n, and there is a subsequence $\{\gamma_{n_k}\}$ of distinct elements such that the measures λ_k defined by

$$d\lambda_k(x) = (-x, \gamma_{n_k})d\mu(x) \tag{1}$$

converge to a measure $\sigma \in M(G)$ in the weak* topology.

For $1 \leqq j \leqq n_k$, $\hat{\lambda}_k(\alpha_j) = \mu(\alpha_j + \gamma_{n_k}) = 0$, so that $\hat{\sigma}(\alpha_j) = 0$ for $j = 1, 2, 3, \ldots$. Thus σ has its support in the finite set E, and hence σ is absolutely continuous. By the translation lemma, σ is singular, and so $\sigma = 0$. But $\hat{\lambda}_k(0) = \hat{\mu}(\gamma_{n_k}) = 1$, so that $\hat{\sigma}(0) = 1$, and $\sigma \neq 0$. This contradiction proves the lemma.

3.5.3. LEMMA. *Suppose $\mu \in J(G)$, and suppose that there are finite subgroups Λ_n of Γ $(n = 1, 2, 3, \ldots)$ such that Λ_n is a proper subgroup of Λ_{n+1} and such that $S(\mu)$ contains a coset of each Λ_n. Then $|\mu|(H) > 0$ for some singular subgroup H of G.*

Proof: There is a sequence $\{\gamma_n\}$ such that $\gamma_n + \Lambda_n \subset S(\mu)$ and such that infinitely many of the γ_n are distinct. A subsequence $\{\gamma_{n_k}\}$ of these will be such that the measures λ_k, defined as in the proof of Lemma 3.5.2, converge to a measure σ in the weak* topology. Since $\Lambda_{n_k} \subset S(\lambda_k)$, $S(\sigma)$ contains the infinite group $\Lambda = \bigcup_1^\infty \Lambda_n$, so that $\sigma * m_H = m_H$, where m_H is the Haar measure of H, the annihilator of Λ. Hence

$$\begin{aligned} 1 = m_H(H) = (\sigma * m_H)(H) &= \int_H \sigma(H - x)dm_H(x) \\ &= \int_H \sigma(H)dm_H(x) = \sigma(H). \end{aligned}$$

By the translation lemma, $|\mu|(H) \geqq |\sigma|(H) \geqq \sigma(H) \geqq 1$. Since Λ is infinite its dual G/H is infinite and the lemma is proved.

3.5.4. Lemma. *Suppose $\mu \in J(G)$ and $|\mu|(H) = 0$ for every singular subgroup H of G. Suppose Λ is an infinite cyclic subgroup of Γ, generated by an element $\gamma_0 \in \Gamma$. Then there is an integer N, depending on μ and Λ, such that $S(\mu)$ contains no arithmetic progression of the form*

$$\gamma + \gamma_0, \gamma + 2\gamma_0, \ldots, \gamma + N\gamma_0 \qquad (\gamma \in \Gamma).$$

Proof: If the conclusion is false, then there exist infinitely many distinct $\gamma_n \in \Gamma$ $(n = 1, 2, 3, \ldots)$ such that $\gamma_n + j\gamma_0 \in S(\mu)$ for $-n \leqq j \leqq n$. Setting $d\lambda_n(x) = (-x, \gamma_n) d\mu(x)$, a subsequence of $\{\lambda_n\}$ converges in the weak* topology to a measure $\sigma \in J(G)$ such that $S(\sigma)$ contains the infinite group Λ, and we conclude that $|\mu|(H) \geqq 1$, where H is the annihilator of Λ, as in the proof of Lemma 3.5.3. Since H is a singular subgroup of G, we have a contradiction.

3.5.5. Lemma. *Suppose P and Q are real numbers such that*

$$1 \leqq P < Q \leqq P + \frac{1}{10P}.$$

Then there exist positive numbers a and b, depending only on P and Q, with the following properties: If f, g, h are Borel functions on G, if $\mu \in M(G)$, and if

(a) $|f(x)| = 1, \quad |g(x)| \leqq 1, \quad |h(x)| \leqq 1$ *for all* $x \in G$,

(b) $\int_G h\, d\mu = \int_G fh\, d\mu = P, \quad \int_G g\, d\mu = 1, \quad \int_G fg\, d\mu = 0,$

(c) $\psi = ah + afh + bg - bfg,$

then $|\psi(x)| \leqq 1$ for all $x \in G$ and $\int_G \psi\, d\mu = Q$.

Proof: Put

$$a = \frac{Q}{2P} \cdot \frac{9P^2 - 1}{9P^2 + 2}, \qquad b = \frac{3Q}{9P^2 + 2}.$$

Then $\int_G \psi\, d\mu = 2aP + b = Q$. Setting $f(x) = e^{2i\alpha}$, we have

$$|\psi| \leqq a|1 + f| + b|1 - f| = 2a|\cos \alpha| + 2b|\sin \alpha| \leqq 2(a^2 + b^2)^{1/2}.$$

But

$$4(a^2+b^2) = \left[\frac{Q}{P}\cdot\frac{9P^2+1}{9P^2+2}\right]^2 \leqq \left[\frac{10P^2+1}{10P^2}\cdot\frac{9P^2+1}{9P^2+2}\right]^2 \leqq 1.$$

Hence $|\psi(x)| \leqq 1$, and the proof is complete.

3.6. Characterization of Irreducible Idempotents

3.6.1. We are now ready to complete step (C) of Section 3.1.5:

THEOREM. *If G is compact, if $\mu \in J(G)$, and if $|\mu|(K) = 0$ for every singular subgroup K of G, then $S(\mu)$ is a finite subset of Γ.*

In other words, there are finitely many distinct characters $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$d\mu(x) = [(x, \gamma_1) + \ldots + (x, \gamma_n)]dx.$$

If the theorem is false, there exists a compact group G and an irreducible $\mu \in J(G)$, whose support group is G, such that $S(\mu)$ is infinite. Then Γ has a countable subgroup Λ such that $S(\mu) \cap \Lambda$ is infinite. Let H be the annihilator of Λ, let ϕ be the natural homomorphism of G onto G/H, and define $\sigma(E) = \mu(\phi^{-1}(E))$ for every Borel set E in G/H. Then $\sigma \in M(G/H)$ and $\hat{\sigma}(\gamma) = \hat{\mu}(\gamma)$ if $\gamma \in \Lambda$ (see Theorem 2.7.2). Hence $\sigma \in J(G/H)$, $S(\sigma)$ is infinite, and since $\phi^{-1}(K)$ is a singular subgroup of G for every singular subgroup K of G/H, σ is irreducible. Hence it is sufficient to prove the theorem under the additional assumption that Γ is countable.

3.6.2. *We now assume that the hypotheses of Theorem* 3.6.1 hold, that Γ is countable, and that $S(\mu)$ *is infinite.*

Under these (contradictory) assymptions we will be able to construct, for each non-negative integer j, a finite collection U_j of cosets of *finite* subgroups of Γ and a finite collection Φ_j of trigonometric polynomials ϕ on G, such that $\|\phi\|_\infty \leqq 1$, with the following properties:

(1) $U_j \subset U_{j+1}$ and $\Phi_j \subset \Phi_{j+1}$.

(2) For every coset $K \in U_j$, $\gamma + K \subset S(\mu)$ for some $\gamma \in S(\mu)$.

(3) With every $\gamma \in S(\mu)$ there is associated a coset $K_{j,\gamma} \in U_j$, a function $\phi_{j,\gamma} \in \Phi_j$, and an integer $p_{j,\gamma}$, $0 \leqq p_{j\gamma} \leqq j$, such that

(a) $\gamma + K_{j,\gamma} \subset S(\mu)$;

(b) $\int_G (-x, \gamma)\phi_{j,\gamma}(x) d\mu(x) = 1 + p_{j,\gamma}/10\|\mu\|$;

(c) There is a chain $C_{j,\gamma}$ of cosets $K_p, K_{p+1}, \ldots, K_{j-1}, K_j$, where $p = p_{j,\gamma}$, $K_j = K_{j,\gamma}$, $K_i \in U_i$ $(p \leqq i \leqq j)$ and the subgroup of which K_i is a coset is a *proper* subgroup of the one which has K_{i+1} as a coset.

The existence of U_j and Φ_j for all positive integers j leads to an easy contradiction. By (3)(b), no $p_{j,\gamma}$ exceeds $10\|\mu\|^2$. If q is an integer larger than $10\|\mu\|^2$, then (3)(c) implies that there are arbitrarily long chains which contain a certain $K^*_q \in U_q$, since U_q is finite. Similarly, there exist arbitrarily long chains, as in (3)(c), which contain K^*_q and a certain coset $K^*_{q+1} \in U_{q+1}$. Continuing, we obtain an infinite chain of cosets K^*_n which have translates in $S(\mu)$, such that the corresponding subgroups of Γ form a strictly increasing sequence. By Lemma 3.5.3 this is impossible.

We now have to construct U_j and Φ_j, under the assumptions 3.6.2. We proceed inductively. U_0 contains only the group consisting of 0 alone, Φ_0 contains only the function $\phi = 1$. Taking $p_{0,\gamma} = 0$ in (3), U_0 and Φ_0 satisfy the requirements.

Suppose U_j and Φ_j are constructed. Let α be an integer larger than $10\|\mu\|^2 + 1$ times the number of elements of Φ_j, and construct pairwise disjoint finite sets $P_1, \ldots, P_\alpha$ of pseudo-periods of $S(\mu)$, with the following additional property: if $n \neq m$, if $\gamma' \in P_n$ and $\gamma'' \in P_m$, then $\gamma'' - \gamma'$ does not belong to any of the groups whose cosets are members of U_j. This is possible, by Lemma 3.5.2.

Now fix $\gamma \in S(\mu)$. If $1 \leqq n \leqq \alpha$, then $\gamma + \gamma_n \in S(\mu)$ for some $\gamma_n \in P_n$. Our choice of α implies that for some $n_1 \neq n_2$ the same $\phi \in \Phi_j$ and the same integer p are associated by (3) with the points $u = \gamma + \gamma_{n_1}$ and $v = \gamma + \gamma_{n_2}$. Thus

$$(4) \qquad \int_G (-x, u)\phi(x) d\mu(x) = \int_G (-x, v)\phi(x) d\mu(x) = 1 + \frac{p}{10\|\mu\|},$$

where $\phi = \phi_{j,u} = \phi_{j,v}$ and $p = p_{j,u} = p_{j,v}$.

Put $w = \gamma_{n_1} - \gamma_{n_2} = u - v$. Our choice of $\{P_n\}$ shows that $w \neq 0$. There are two possibilities:

(i) $\gamma + rw + K_{j,\gamma} \subset S(\mu)$ for all integers r;

(ii) for some r, $\gamma + rw + K_{j,\gamma}$ is not a subset of $S(\mu)$.

In case (i), Lemma 3.5.4 shows that w must generate a finite cyclic subgroup Λ of Γ. Put $K_{j+1,\gamma} = K_{j,\gamma} + \Lambda$, $p_{j+1,\gamma} = p$, $\phi_{j+1,\gamma}(x) = (-x, \gamma_{n_1})\phi(x)$, and let $C_{j+1,\gamma}$ be the chain $C_{j,\gamma}$ followed by $K_{j+1,\gamma}$. Our choice of $\{P_n\}$ shows that w is not in the subgroup K of Γ of which $K_{j,\gamma}$ is a coset, so that K is properly contained in $K + \Lambda$. Hence $C_{j+1,\gamma}$ has the required properties.

In case (ii), there exists $\gamma_0 \in K_{j,\gamma}$ and a smallest non-negative integer r_0 such that

$$(5) \qquad \gamma + r_0 w + \gamma_0 \in S(\mu) \text{ but } \gamma + (r_0 + 1)w + \gamma_0 \notin S(\mu);$$

since $\gamma + \gamma_0 \in S(\mu)$, this follows from Lemma 3.5.4. Put

$$(6) \qquad P = 1 + \frac{p}{10||\mu||}, \qquad Q = 1 + \frac{p+1}{10||\mu||}.$$

By (3)(b), $P \leqq ||\mu||$, and so $Q = P + (10||\mu||)^{-1} \leqq P + (10P)^{-1}$. Choose positive numbers a and b, in accordance with Lemma 3.5.5, put

$$(7) \quad f(x) = (-x, w), \; g(x) = (-x, r_0 w + \gamma_0), \; h(x) = (-x, \gamma_{n_2})\phi(x)$$

and

$$(8) \qquad \phi_{j+1,\gamma} = ah + afh + bg - bfg.$$

Then (4), (5) and Lemma 3.5.5 imply that $||\phi_{j+1,\gamma}||_\infty \leqq 1$ and that

$$(9) \qquad \int_G (-x, \gamma)\phi_{j+1,\gamma}(x)d\mu(x) = Q.$$

If now $K_{j+1,\gamma} = K_{j,u} + \gamma_{n_1}$, if $p_{j+1,\gamma} = p + 1$, and if $C_{j+1,\gamma}$ is the chain obtained from $C_{j,u}$ by replacing $K_{j,u}$ by $K_{j+1,\gamma}$, it is easily verified that the conditions (3) hold, with $j + 1$ in place of j.

Define U_{j+1} as the union of U_j and the collection of the cosets $K_{j+1,\gamma}$, let Φ_{j+1} be the union of Φ_j and the trigonometric polynomials $\phi_{j+1,\gamma}$, for all $\gamma \in S(\mu)$. To complete the induction, we must prove that U_{j+1} and Φ_{j+1} are finite collections.

We had finitely many sets P_n, hence only finitely many possibilities for γ_{n_1}, γ_{n_2}, and w, and hence we defined only finitely many cosets $K_{j+1,\gamma}$. It is clear that only finitely many $\phi_{j+1,\gamma}$ arose in case (i). In case (ii), Lemma 3.5.4 shows that for each w and for each γ_0 there were only finitely many choices of r_0, and since U_j was a finite collection of finite cosets, there were only finitely many possibilities for γ_0. Hence only finitely many $\phi_{j+1,\gamma}$ were constructed by (8).

This completes the proof.

3.7. *Norms of Idempotent Measures*

As we saw in Section 3.2.4, an idempotent measure μ has norm 1 if and only if $S(\mu)$ is a coset. A simple combinatorial characterization of cosets leads to the curious result that $||\mu|| \geqq \sqrt{5}/2 \sim 1.118$ if $||\mu|| > 1$.

3.7.1. LEMMA. *A set $E \subset \Gamma$ is a coset in Γ if and only if $E+E-E \subset E$.*

Proof: If E is a coset of a group Λ, then $E - E = \Lambda$, and hence $E = E + \Lambda = E + E - E$.

Conversely, if $E + E - E \subset E$, put $\Lambda = E - E$ and fix $\gamma_0 \in E$. If $\gamma \in E$, then $\gamma - \gamma_0 \in \Lambda$, and so $E = \gamma_0 + \Lambda$, since

$$E \subset \gamma_0 + \Lambda = \gamma_0 + E - E \subset E.$$

Consequently, $\Lambda - \Lambda = (E - \gamma_0) - (E - \gamma_0) = E - E = \Lambda$, so that Λ is a group.

3.7.2. THEOREM. *If $\mu \in J(G)$ and $||\mu|| > 1$, then $||\mu|| \geqq \sqrt{5}/2$.*

Proof: Since $||\mu|| > 1$, $S(\mu)$ is not a coset, and Lemma 3.7.1 shows that there exist $\gamma_1, \gamma_2, \gamma_3 \in S(\mu)$ so that $\gamma_1 + \gamma_2 - \gamma_3 \notin S(\mu)$. Put

$$\begin{aligned} f(x) &= 2(-x, \gamma_1)[1 + (x, \gamma_1 - \gamma_3)] + (-x, \gamma_2)[1 - (x, \gamma_3 - \gamma_1)] \\ &= 2(-x, \gamma_1) + 2(-x, \gamma_3) + (-x, \gamma_2) - (-x, \gamma_1 + \gamma_2 - \gamma_3). \end{aligned}$$

The second expression for f shows that $\int_G f\,d\mu = 5$. Setting $(x, \gamma_1 - \gamma_3) = e^{2i\alpha}$, the first expression for f shows that

$$|f(x)| \leqq 2|1 + e^{2i\alpha}| + |1 - e^{-2i\alpha}| = 4|\cos \alpha| + 2|\sin \alpha| \leqq 2\sqrt{5}.$$

Hence

$$5 = \int_G f d\mu \leqq ||f||_\infty ||\mu|| \leqq 2\sqrt{5}||\mu||.$$

3.7.3. It is not known whether the constant $\sqrt{5}/2$ is the best possible one in the preceding theorem. However, an example shows that it cannot be increased beyond $(1 + \sqrt{2})/2 \sim 1.207\ldots$.

Let γ be a character on a compact group G, $\gamma \neq 0$. If γ has infinite order, then $\|1 + \gamma\|_1 = 4/\pi$. If γ has finite order q, then

$$\|1+\gamma\|_1 = \frac{1}{q}\sum_{j=1}^{q} |1 + e^{2\pi i j/q}| = \begin{cases} \dfrac{2}{q \sin(\pi/2q)} & (q \text{ odd}) \\[2ex] \dfrac{2}{q \tan(\pi/2q)} & (q \text{ even}) \end{cases}$$

For odd q, $\|1 + \gamma\|_1$ decreases therefore to $4/\pi$, as $q \to \infty$; for even q, $\|1 + \gamma\|_1$ increases to $4/\pi$. The smallest value larger than 1 is obtained when $q = 4$, and is $(1 + \sqrt{2})/2$.

3.7.4. If $n_1, n_2, \ldots, n_k$ are distinct integers and

$$d\mu(x) = \sum_{j=1}^{k} e^{in_jx}\,dx, \tag{1}$$

then μ is an idempotent measure on the circle group T, and

$$\|\mu\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{in_1x} + \ldots + e^{in_kx}|\,dx. \tag{2}$$

It is an interesting problem to determine the order of magnitude of $m(k)$, the greatest lower bound of the numbers (2) for all possible choices of $n_1, \ldots, n_k$. The best result in this direction so far is that

$$m(k) > A\left\{\frac{\log k}{\log\log k}\right\}^{1/4}, \tag{3}$$

where A is an absolute constant (Cohen [2]). If the integers $n_1, \ldots, n_k$ are in arithmetic progression then (2) is asymptotic to $A \log k$, and it is conceivable that this is the true order of magnitude of $m(k)$.

3.8. A Multiplier Problem

3.8.1. Theorem (Helson [3], Edwards [2]). *Suppose ϕ is a function defined on Γ such that $\phi\hat{f} \in B(\Gamma)$ for every $\hat{f} \in A(\Gamma)$. Then $\phi \in B(\Gamma)$.*

Proof: First we show that the linear map T which takes $\hat{f}$ to $\phi\hat{f}$ is a continuous map of $A(\Gamma)$ into $B(\Gamma)$. Suppose $\hat{f}_n \to \hat{f}$ in the norm of $A(\Gamma)$ and $\phi\hat{f}_n \to \hat{\mu}$ in the norm of $B(\Gamma)$. (We define $\|\hat{f}\| = \|f\|_1$ if $\hat{f} \in A(\Gamma)$, and $\|\hat{\mu}\| = \|\mu\|$ if $\hat{\mu} \in B(\Gamma)$.) For any $\gamma \in \Gamma$ we then have

$$\hat{\mu}(\gamma) = \lim_{n\to\infty} \phi(\gamma)\hat{f}_n(\gamma) = \phi(\gamma)\hat{f}(\gamma) = (T\hat{f})(\gamma).$$

Thus $T\hat{f} = \hat{\mu}$, and the continuity of T follows from the closed graph theorem (Appendix C6). Hence there is a constant K such that

$$(1) \qquad \|\phi\hat{f}\| \leqq K\|\hat{f}\| \qquad (\hat{f} \in A(\Gamma)).$$

Given $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\varepsilon > 0$, Theorem 2.6.8 shows that there exists $\hat{f} \in A(\Gamma)$ with $\|\hat{f}\| < 1 + \varepsilon$, such that $\hat{f}(\gamma_i) = 1$ for $1 \leqq i \leqq n$. If $c_1, \ldots, c_n$ are complex numbers and if $\hat{\mu} = \phi\hat{f}$,

$$|\sum_i c_i\phi(\gamma_i)| =$$
$$|\sum_i c_i\hat{\mu}(\gamma_i)| \leqq \|\mu\| \cdot \sup_{x\in G} |\sum_i c_i(x, \gamma_i)| \leqq K(1+\varepsilon) \sup_{x\in G} |\sum_i c_i(x, \gamma_i)|.$$

Taking $\hat{f}$ constant on an open set V, it follows that ϕ is continuous on V, hence ϕ is continuous. Thus Theorem 1.9.1 applies and shows that $\phi \in B(\Gamma)$, with $\|\phi\| \leqq K$.

3.8.2. COROLLARY. *If $\mu \in M(G)$, the transformation $f \to f * \mu$ of $L^1(G)$ into $L^1(G)$ has norm $\|\mu\|$.*

3.8.3. For $a \in G$, let τ_a be the translation operator defined by $\tau_a f = f_a$, where $f_a(x) = f(x - a)$.

THEOREM. *Suppose Ψ is a bounded linear transformation of $L^1(G)$ into $L^1(G)$ which commutes with all translations*; i.e., *$\Psi\tau_a = \tau_a\Psi$ for all $a \in G$. Then there is a function ϕ on Γ such that*

$$(1) \qquad \hat{\Psi f}(\gamma) = \phi(\gamma)\hat{f}(\gamma) \qquad (f \in L^1(G),\ \gamma \in \Gamma),$$

where $\hat{\Psi f}$ is the Fourier transform of Ψf. Conversely, if Ψ satisfies (1), *then $\Psi\tau_a = \tau_a\Psi$.*

Proof: Fix $b \in L^\infty(G)$. The map $f \to \int (\Psi f)b$ is a bounded linear functional on $L^1(G)$, and hence there exists $\beta \in L^\infty(G)$ such that

$$(2) \qquad \int_G (\Psi f)(x)b(x)dx = \int_G f(x)\beta(x)dx \qquad (f \in L^1(G)).$$

If now $f, g \in L^1(G)$, we obtain

$$\int_G ((\Psi f) * g)(x)b(x)dx = \int_G \int_G (\tau_y \Psi f)(x)g(y)b(x)dydx$$
$$= \int_G g(y)dy \int_G (\Psi \tau_y f)(x)b(x)dx = \int_G g(y)dy \int_G (\tau_y f)(x)\beta(x)dx$$
$$= \int_G (f * g)(x)\beta(x)dx.$$

Since the last expression is unaltered if f and g are interchanged and since b was an arbitrary member of $L^\infty(G)$, Ψ satisfies the identity

$$(3) \qquad (\Psi f) * g = f * (\Psi g) \qquad (f, g \in L^1(G)).$$

Hence $(\widehat{\Psi f}) \cdot \hat{g} = \hat{f} \cdot (\widehat{\Psi g})$, and this implies the existence of a function ϕ on Γ for which (1) holds.

The converse is trivial, since $\hat{f}_a(\gamma) = (-a, \gamma)\hat{f}(\gamma)$.

3.8.4. Combining Theorems 3.8.1 and 3.8.3, we see that *the bounded linear transformations of $L^1(G)$ into $L^1(G)$ which commute with all translations of G are precisely the transformations of the form*

$$\Psi f = f * \mu$$

where $\mu \in M(G)$.

Moreover, $\|\Psi\| = \|\mu\|$, and if Ψ is a projection, i.e., if $\Psi^2 = \Psi$, then $\mu * \mu = \mu$.

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CHAPTER 4

Homomorphisms of Group Algebras

4.1. Outline of the Main Result

4.1.1. Consider two LCA groups G_1, G_2 and their duals Γ_1, Γ_2, and let Ψ be a homomorphism of $L^1(G_1)$ into $M(G_2)$; that is to say, Ψ is a linear transformation of $L^1(G_1)$ into $M(G_2)$ which is also multiplicative with respect to convolution:

$$\Psi(f * g) = (\Psi f) * (\Psi g) \qquad (f, g \in L^1(G_1)). \tag{1}$$

Since $M(G_2)$ is semi-simple, Ψ is bounded (Appendix D5). To avoid trivialities, we assume that Ψ is not identically zero.

Associated with Ψ there is a homomorphism $\hat{\Psi}$ of $A(\Gamma_1)$ into $B(\Gamma_2)$, defined by the requirement that $\hat{\Psi}\hat{f}$ be the Fourier-Stieltjes transform of Ψf, for all $\hat{f} \in A(\Gamma_1)$.

For every $\gamma \in \Gamma_2$, the map $f \to (\hat{\Psi}\hat{f})(\gamma)$ is a complex homomorphism of $L^1(G_1)$. Let Y be the set of all $\gamma \in \Gamma_2$ for which this homomorphism is not identically 0. If $\gamma \in Y$, Theorem 1.2.2 shows that there is a character $\alpha(\gamma) \in \Gamma_1$ such that $(\hat{\Psi}\hat{f})(\gamma) = \hat{f}(\alpha(\gamma))$.

Thus each homomorphism Ψ of $L^1(G_1)$ into $M(G_2)$ induces a map α of a subset Y of Γ_2 into Γ_1, such that

$$(\hat{\Psi}\hat{f})(\gamma) = \begin{cases} \hat{f}(\alpha(\gamma)) & \text{if } \gamma \in Y, \\ 0 & \text{if } \gamma \notin Y. \end{cases} \qquad (f \in L^1(G_1), \qquad \gamma \in \Gamma_2). \tag{2}$$

We shall abbreviate (2) by using the notation

$$\hat{\Psi}\hat{f} = \hat{f} \circ \alpha. \tag{3}$$

The problem considered and solved in this chapter is the characterization of all homomorphisms of $L^1(G_1)$ into $M(G_2)$. The above remarks show that this is equivalent to the problem of finding all maps α of subsets Y of Γ_2 into Γ_1 such that the transformation $\hat{f} \to \hat{f} \circ \alpha$ carries $A(\Gamma_1)$ into $B(\Gamma_2)$.

4.1.2. Affine and piecewise affine maps. If E is a coset in Γ_2 and if α is a continuous map of E into Γ_1 which satisfies the identity

$$(1) \quad \alpha(\gamma + \gamma' - \gamma'') = \alpha(\gamma) + \alpha(\gamma') - \alpha(\gamma'') \qquad (\gamma, \gamma', \gamma'' \in E),$$

then α is said to be *affine*; to justify the definition we refer to Lemma 3.7.1.

Suppose that

(a) $S_1, \ldots, S_n$ are pairwise disjoint sets belonging to the coset-ring of Γ_2;

(b) each S_i is contained in an open coset K_i in Γ_2;

(c) for each i, α_i is an affine map of K_i into Γ_1;

(d) α is the map of $Y = S_1 \cup \ldots \cup S_n$ into Γ_1 which coincides on S_i with α_i.

Then α is said to be a *piecewise affine map of* Y *into* Γ_1.

4.1.3. We can now state the main result of this chapter, in the terminology developed in Section 4.1.1:

THEOREM. *If Ψ is a homomorphism of $L^1(G_1)$ into $M(G_2)$, then $\hat{\Psi f} = \hat{f} \circ \alpha$, where α is a piecewise affine map of Y into Γ_1 and Y belongs to the coset-ring of Γ_2.*

Conversely, if Y belongs to the coset-ring of Γ_2 and if α is a piecewise affine map of Y into Γ_1, then $\hat{\mu} \circ \alpha \in B(\Gamma_2)$ for every $\hat{\mu} \in B(\Gamma_1)$.

We add that there are homomorphisms of $M(G_1)$ into $M(G_2)$ which are not of the above type. An example is the homomorphism π_H described in Section 3.4.1 which maps $M(G)$ into $M(G)$, is not identically 0, but maps $L^1(G)$ into 0.

The general case of this theorem was proved by Cohen [2]; the proof depends on knowing the idempotent measures on $G_1 \oplus G_2$. Special cases, obtained earlier by Helson [3], Beurling and Helson [1], Leibenson [1], Kahane [1], [2], and Rudin [3], [10] are described in Section 4.7.

We first prove the second part of the theorem (Theorem 4.2.3). Then, after some combinatorial preparation, we prove the first part for compact G_1 and G_2 (Section 4.4); since the dual groups are now discrete, no topological considerations are involved in the characterization of α. The general case is deduced by passing to the Bohr compactifications of G_1 and G_2 (Section 4.5).

4.2. The Action of Piecewise Affine Maps

4.2.1. LEMMA. *Suppose Λ is an open subgroup of Γ_2, α is a continuous homomorphism of Λ into Γ_1, $\hat{\mu} \in B(\Gamma_1)$,* and $\phi = \hat{\mu} \circ \alpha$. Then $\phi \in B(\Gamma_2)$, *and* $\|\phi\| \leqq \|\mu\|$.

Proof: The annihilator H of Λ is a compact subgroup of G_2. Let $P(y) = \sum_1^n c_i(y, \gamma_i)$ be a trigonometric polynomial on G_2 and put $Q = P * m_H$. Then $\|Q\|_\infty \leqq \|P\|_\infty \|m_H\| = \|P\|_\infty$, and $Q(y) = \sum c_i \chi(\gamma_i)(y, \gamma_i)$, where χ is the characteristic function of Λ.

Since α is a homomorphism, the map $\gamma \to (x, \alpha(\gamma))$ is a character on Λ, for each $x \in G_1$, and so α induces a continuous homomorphism β of G_1 into G_2/H for which

$$(x, \alpha(\gamma)) = (\beta(x), \gamma) \qquad (x \in G_1,\ \gamma \in \Lambda). \tag{1}$$

Then

$$\sum_{\gamma_i \in \Lambda} c_i(x, \alpha(\gamma_i)) = Q(\beta(x)) \qquad (x \in G_1), \tag{2}$$

and so Theorem 1.9.1 implies that

$$\Big|\sum_{i=1}^{n} c_i \phi(\gamma_i)\Big| = \Big|\sum_{\gamma_i \in \Lambda} c_i \hat{\mu}(\alpha(\gamma_i))\Big| \leqq \|\mu\| \cdot \|Q\|_\infty \leqq \|\mu\| \cdot \|P\|_\infty \tag{3}$$

and hence that $\phi \in B(\Gamma_2)$ and $\|\phi\| \leqq \|\mu\|$.

4.2.2. LEMMA. *The conclusion of Lemma* 4.2.1 *holds also if Λ is an open coset in Γ_2 and if α is an affine map of Λ into Γ_1.*

Proof: Fix $\gamma_0 \in \Lambda$ and define $\alpha_1(\gamma) = \alpha(\gamma + \gamma_0) - \alpha(\gamma_0)$, for $\gamma \in \Lambda - \gamma_0$. Then α_1 is a continuous homomorphism of the open subgroup $\Lambda - \gamma_0$ into Γ_1. Since both $B(\Gamma_1)$ and $B(\Gamma_2)$ are invariant under translation and since translations leave their norms invariant, the result follows from 4.2.1.

4.2.3. THEOREM. *If Y belongs to the coset-ring of Γ_2, if α is a piecewise affine map of Y into Γ_1, if $\hat{\mu} \in B(\Gamma_1)$ and if $\phi = \hat{\mu} \circ \alpha$, then $\phi \in B(\Gamma_2)$.*

Proof: We use the symbols S_i, K_i, α_i as in Section 4.1.2. If $\phi_i = \hat{\mu} \circ \alpha_i$ and if χ_i is the characteristic function of S_i, then $\phi = \sum \chi_i \phi_i$. Since S_i belongs to the coset-ring of Γ_2, $\chi_i \in B(\Gamma_2)$ (see Section 3.1.2); since α_i is affine on the open coset K_i, $\phi_i \in B(\Gamma_2)$, by Lemma 4.2.2. Since $B(\Gamma_2)$ is an algebra, it follows that $\phi \in B(\Gamma_2)$.

4.2.4. LEMMA. *Suppose E is a coset in Γ_2 and α is an affine map of E into Γ_1.* Then α can be extended to an affine map of the closure $\bar{E}$ *of E, and $\alpha(\bar{E})$ is a closed coset in Γ_1.*

Proof: Fix $\gamma^* \in E$. Since

(1) $\quad \alpha(\gamma') - \alpha(\gamma'') = \alpha(\gamma^* + \gamma' - \gamma'') - \alpha(\gamma^*) \qquad (\gamma', \gamma'' \in E),$

the continuity of α implies that α is *uniformly continuous*; i.e., to each neighborhood W of 0 in Γ_1, there corresponds a neighborhood V of 0 in Γ_2 such that $\alpha(\gamma') - \alpha(\gamma'') \in W$ whenever $\gamma' \in E$, $\gamma'' \in E$, and $\gamma' - \gamma'' \in V$.

Fix $\gamma_0 \in \bar{E}$, and let A be a subset of E, with $\bar{A}$ compact. If W is a compact neighborhood of 0 in Γ_1, the uniform continuity of α shows that there is an open neighborhood V of 0 in Γ_2 such that $\alpha((\gamma + V) \cap E) \subset \alpha(\gamma) + W$, for all $\gamma \in E$. Since $\bar{A} \subset A + V$, there exist $\gamma_1, \ldots, \gamma_n \in A$ such that $\bar{A} \subset \bigcup (\gamma_i + V)$. Hence $\alpha(A) \subset \bigcup \alpha((\gamma_i) + W)$, and so $\alpha(A)$ has compact closure in Γ_1.

It follows that the closures F_N of the sets $\alpha(E \cap N)$ are compact, where N runs through the compact neighborhoods of γ_0. Hence $\bigcap F_N$ is not empty. The uniform continuity of α implies then that the sets F_N have exactly one point in common, and we define $\alpha(\gamma_0)$ to be that point.

It is now a routine matter to verify that α, so extended to $\bar{E}$, is continuous on $\bar{E}$. Once this is done, the continuity of the group operations shows that the extension is affine. Since $\bar{E}$ is a coset, Lemma 3.7.1 shows that $\alpha(\bar{E})$ is a coset and the uniform continuity of α implies that $\alpha(\bar{E})$ is closed.

4.3. Graphs in the Coset Ring

4.3.1. Assume that Γ_1 and Γ_2 are discrete abelian groups and that $\Gamma = \Gamma_1 \oplus \Gamma_2$ is their direct sum. A set $E \subset \Gamma$ will be called a *graph* if to every $\gamma_2 \in \Gamma_2$ there is at most *one* $\gamma_1 \in \Gamma_1$ such that $(\gamma_1, \gamma_2) \in E$.

THEOREM. *Suppose $Y \subset \Gamma_2$, α is a map of Y into Γ_1, and E is the graph of α; i.e., E is the set of all points $(\alpha(\gamma_2), \gamma_2) \in \Gamma$, with $\gamma_2 \in Y$. If E belongs to the coset-ring of Γ, then Y belongs to the coset-ring of Γ_2 and α is piecewise affine.*

This will be proved in Section 4.3.4.

4.3.2. If Λ is an abelian group and E is a coset of a subgroup Λ_1 of Λ, we define the *index of E* to be the index of Λ_1 in Λ, i.e., the number of elements in Λ/Λ_1. If E_1 and E_2 are cosets in Λ, of the subgroups Λ_1 and Λ_2, *the index of E_1 in E_2* is defined to be the index of $\Lambda_1 \cap \Lambda_2$ in Λ_2.

4.3.3. LEMMA. *An abelian group is not a finite union of cosets of infinite index.*

Proof: This can be proved by quite elementary means, but it may be of interest to use the analytical apparatus which is at our disposal.

Suppose $E_1, \ldots, E_n$ are cosets of infinite index in the discrete group Λ whose dual is H. The characteristic function of E_i is the Fourier-Stieltjes transform of a measure μ_i which is the Haar measure of an infinite compact subgroup of H, multiplied by a character of H (see Section 3.1.2) so that μ_i is continuous. Defining $\mu \vee \sigma = \mu + \sigma - \mu * \sigma$, the characteristic function of $E = \bigcup E_i$ is the Fourier-Stieltjes transform of the measure $\mu = \mu_1 \vee \mu_2 \vee \ldots \vee \mu_n$, which is continuous. Hence $\mu \neq \delta_0$, $\hat{\mu}$ is not identically 1 on Λ, and so $E \neq \Lambda$.

4.3.4. Proof of Theorem 4.3.1. If Σ is a finite collection of subgroups of Γ, let $R(\Sigma)$ be the ring generated by the cosets of the groups belonging to Σ; i.e., $R(\Sigma)$ is the smallest family of subsets of Γ which contains all cosets of the groups in Σ and which is closed under the formation of finite unions, finite intersections, and complements.

Suppose now that the graph E of α belongs to the coset-ring of Γ. Then $E \in R(\Sigma)$, for some finite collection Σ. We may enlarge

Σ so that $\Gamma \in \Sigma$ and so that the intersection of any two members of Σ belongs to Σ.

We call H a *minimal element* of Σ if $H \in \Sigma$ and if Σ contains no proper subgroup of H which has finite index in H. Remove all non-minimal elements of Σ and let Σ' be the remaining collection of groups. Since Σ is finite, each $H \in \Sigma$ belongs to $R(\Sigma')$. Hence $E \in R(\Sigma')$. If $H_i \in \Sigma'$ $(i = 1, 2)$, the index of H_1 in H_2 is either 1 or ∞.

It follows that E is a finite disjoint union of sets of the form

$$(1) \qquad E_i = L_i \cap \bigcap_j M'_{ij} \qquad (1 \leqq i \leqq n),$$

where L_i and M_{ij} are cosets in Γ, M'_{ij} is the complement of M_{ij}, each M_{ij} has infinite index in L_i, and there are only finitely many M_{ij} for each i.

We claim that each L_i is a graph. Without loss of generality, we may assume that L_i is a subgroup of $\Gamma = \Gamma_1 \oplus \Gamma_2$. If $(\gamma_0, 0) \in L_i$ and $\gamma_0 \neq 0$, and if $(\gamma_1, \gamma_2) \in E_i$, then the element

$$(2) \qquad (\gamma_0, 0) + (\gamma_1, \gamma_2) = (\gamma_0 + \gamma_1, \gamma_2)$$

belongs to at least one M_{ij}, since E_i is a graph. It follows that E_i is covered by the union of the cosets $M_{ij} - (\gamma_0, 0)$, and so L_i is covered by a finite union of cosets of infinite index, in contradiction to Lemma 4.3.3.

Let π be the homomorphism $(\gamma_1, \gamma_2) \to \gamma_2$ of Γ onto Γ_2 and put $K_i = \pi(L_i)$, $S_i = \pi(E_i)$. Since L_i is a graph, there is a uniquely defined map α_i of K_i into Γ_1 such that

$$(3) \qquad (\alpha_i(\gamma), \gamma) \in L_i \qquad (\gamma \in K_i).$$

This α_i is affine. Since $S_i = K_i \cap \bigcap_j N'_{ij}$ where $N_{ij} = \pi(L_i \cap M_{ij})$, S_i is in the coset-ring of Γ_2.

Since $Y = \bigcup S_i$ and since α_i coincides with α on S_i $(1 \leqq i \leqq n)$, the proof is complete.

4.4. Compact Groups

We now assume that G_1 and G_2 are compact, that Ψ is a homomorphism of $L^1(G_1)$ into $M(G_2)$, and that Y and α are associated with Ψ as in 4.1.1(2).

4.4.1. LEMMA. *Ψ has a norm preserving extension to a homomorphism of $M(G_1)$ into $M(G_2)$, given by*

$$(1)\qquad (\hat{\Psi}\hat{\mu})(\gamma) = (\hat{\mu} \circ \alpha)(\gamma) \qquad (\mu \in M(G_1),\ \gamma \in \Gamma_2).$$

Proof: Fix $\mu \in M(G_1)$, put $\phi = \hat{\mu} \circ \alpha$, and let $P(y) = \sum_1^n c_i(y, \gamma_i)$ be a trigonometric polynomial on G_2. Given $\varepsilon > 0$, Theorem 2.6.8 shows that there exists $k \in L^1(G_1)$ such that $||k||_1 < 1 + \varepsilon$ and $\hat{k}(\alpha(\gamma_i)) = 1$ for those γ_i which lie in Y and which occur in the definition of $P(y)$. If $f = k * \mu$, then $f \in L^1(G_1)$,

$$(2)\qquad (\hat{\Psi}\hat{f})(\gamma) = (\hat{f} \circ \alpha)(\gamma) = (\hat{k} \circ \alpha)(\gamma)\phi(\gamma) \qquad (\gamma \in \Gamma_2)$$

so that

$$(3)\qquad \sum_{i=1}^{n} c_i\phi(\gamma_i) = \sum_{\gamma_i \in Y} c_i\hat{k}(\alpha(\gamma_i))\phi(\gamma_i) = \sum_{i=1}^{n} c_i(\hat{\Psi}\hat{f})(\gamma_i).$$

Theorem 1.9.1 now implies that

$$(4)\qquad |\sum_{i=1}^{n} c_i\phi(\gamma_i)| \leqq ||\Psi f|| \cdot ||P||_\infty$$

and hence that $\phi \in B(\Gamma_2)$, with

$$(5)\quad ||\phi|| \leqq ||\Psi f|| \leqq ||\Psi|| \cdot ||f||_1 \leqq ||\Psi||\ ||\mu|| \cdot ||k||_1 \leqq (1+\varepsilon)||\Psi|| \cdot ||\mu||.$$

Since ε was arbitrary, $||\phi|| \leqq ||\Psi||\ ||\mu||$, and the proof is complete.

4.4.2. If δ_x is the unit mass concentrated at the point $x \in G_1$ and if Ψ is extended as in Lemma 4.4.1, put $\mu_x = \Psi\delta_x$. Then $\mu_x \in M(G_2)$, $||\mu_x|| \leqq ||\Psi||$, and

$$(1)\qquad \hat{\mu}_x(\gamma) = \begin{cases} (-x, \alpha(\gamma)) & (\gamma \in Y), \\ 0 & (\gamma \notin Y). \end{cases}$$

These are the only properties of the extended homomorphism that will be used in the proof of the next theorem. It may be of interest to note that the map $x \to \mu_x$ is a bounded homomorphism of G_1 into $M(G_2)$ since $\mu_{x+y} = \mu_x * \mu_y$, by (1).

4.4.3. THEOREM. *The graph E of α is a member of the coset-ring of $\Gamma_1 \oplus \Gamma_2$.*

Proof: Let $G = G_1 \oplus G_2$, $\Gamma = \Gamma_1 \oplus \Gamma_2$. The graph of α was defined in Theorem 4.3.1. By Theorem 3.1.3, we have to show that the characteristic function χ of E belongs to $B(\Gamma)$.

The letters $x, y, z, \gamma', \gamma'', \gamma$ will denote points of $G_1, G_2, G, \Gamma_1, \Gamma_2$, and Γ, respectively. If k is a trigonometric polynomial on G_2,

$$k(y) = \sum_{\Gamma_2} a(\gamma'')(y, \gamma''), \tag{1}$$

define $\hat{k}$ as a function on Γ by

$$\hat{k}(\gamma) = \hat{k}(\gamma', \gamma'') = a(\gamma''), \tag{2}$$

and put

$$\phi(x, y) = \sum_{\gamma'' \in Y} a(\gamma'')(x, \alpha(\gamma''))(y, \gamma''). \tag{3}$$

Then ϕ is a trigonometric polynomial on G, whose Fourier transform is $\hat{k}\chi$. Defining μ_x as in 4.4.2, we have

$$\begin{aligned} \phi(x, y) &= \sum_{\gamma'' \in \Gamma_2} a(\gamma'')(y, \gamma'') \int_{G_2} (-t, \gamma'') d\mu_{-x}(t) \\ &= \int_{G_2} \sum_{\Gamma_2} a(\gamma'')(y - t, \gamma'') d\mu_{-x}(t) = \int_{G_2} k(y - t) d\mu_{-x}(t), \end{aligned} \tag{4}$$

so that

$$\int_{G_2} |\phi(x, y)| dy \leqq \|k\|_1 \|\mu_{-x}\| \leqq \|k\|_1 \|\Psi\| \qquad (x \in G_1). \tag{5}$$

Integrating (5) with respect to x, we thus have

$$\|\phi\|_1 = \int_{G_1}\int_{G_2} |\phi(x, y)| dx dy \leqq \|k\|_1 \|\Psi\|. \tag{6}$$

Now choose $\gamma_1, \ldots, \gamma_n \in \Gamma$, complex numbers $c_1, \ldots, c_n$, and $\varepsilon > 0$. By Theorem 2.6.8 there exists k of the form (1) such that $\|k\|_1 < 1 + \varepsilon$ and $\hat{k}(\gamma_i) = 1$ $(1 \leqq i \leqq n)$. Setting $P(x) = \sum c_i(x, \gamma_i)$, we then have

$$\begin{aligned} \left|\sum_1^n c_i \chi(\gamma_i)\right| &= \left|\sum_1^n c_i \hat{k}(\gamma_i)\chi(\gamma_i)\right| = \left|\sum_1^n c_i \hat{\phi}(\gamma_i)\right| \\ &\leqq \|\phi\|_1 \|P\|_\infty \leqq (1 + \varepsilon)\|\Psi\| \cdot \|P\|_\infty. \end{aligned} \tag{7}$$

Hence $\chi \in B(\Gamma)$, by Theorem 1.9.1, and since ε was arbitrary, we also see that $\|\chi\| \leqq \|\Psi\|$.

4.4.4. Theorems 4.4.3 and 4.3.1 establish Theorem 4.1.3 for compact G_1 and G_2.

4.5. The General Case

4.5.1. We now turn to the proof of Theorem 4.1.3 for arbitrary *LCA* groups G_1 and G_2; Ψ is a homomorphism of $L^1(G_1)$ into $M(G_2)$ and α is the induced map of Y into Γ_1. We have to prove that Y belongs to the coset-ring of Γ_2 and that α is piecewise affine.

Let $\bar{G}_1$ and $\bar{G}_2$ be the Bohr compactifications of G_1 and G_2. Their duals are the discrete groups $\Gamma_{1,d}$ and $\Gamma_{2,d}$. Choose $\mu \in M(\bar{G}_1)$, $\varepsilon > 0$, and $\gamma_1, \ldots, \gamma_n \in \Gamma_2$. Since G_1 is dense in $\bar{G}_1$ and since $\{\gamma_1, \ldots, \gamma_n\}$ is a finite set, there is a measure $\mu_1 \in M(G_1)$ with $\|\mu_1\| \leqq \|\mu\|$, such that

$$(1) \qquad |(\hat{\mu}_1 \circ \alpha)(\gamma_i) - (\hat{\mu} \circ \alpha)(\gamma_i)| < \varepsilon \qquad (1 \leqq i \leqq n).$$

Replacing μ_1 by $k * \mu_1$, where $k \in L^1(G_1)$ is as in Theorem 2.6.8, we see that there exists $f \in L^1(G_1)$ with $\|f\|_1 \leqq (1+\varepsilon)\|\mu\|$, such that (1) holds with $\hat{f}$ in place of $\hat{\mu}_1$.

Setting $\phi = \hat{f} \circ \alpha$, it follows that $\phi \in B(\Gamma_2)$,

$$\|\phi\| \leqq (1+\varepsilon)\|\mu\| \cdot \|\Psi\|,$$

and

$$(2) \qquad |\phi(\gamma_i) - (\hat{\mu} \circ \alpha)(\gamma_i)| < \varepsilon \qquad (1 \leqq i \leqq n).$$

Since $\gamma_1, \ldots, \gamma_n$ and ε were arbitrary, these conditions imply, by Theorem 1.9.1, that $\hat{\mu} \circ \alpha \in B(\Gamma_{2,d})$ and that

$$(3) \qquad \|\hat{\mu} \circ \alpha\| \leqq \|\Psi\| \cdot \|\mu\|.$$

Thus α, regarded as a map of Y_d (the set Y in the discrete topology) into $\Gamma_{1,d}$ carries $B(\Gamma_{1,d})$ into $B(\Gamma_{2,d})$. Hence the result of Section 4.4 applies, and we conclude:

The set Y_d belongs to the coset-ring of $\Gamma_{2,d}$ and α is a piecewise affine map of Y_d into $\Gamma_{1,d}$.

4.5.2. To complete the proof of the theorem, we have to show that Y and α satisfy the topological requirements. In other words, we have to show that the conclusion of Section 4.5.1 remains true if the subscripts d are removed.

If $\gamma_0 \in Y$ and if V is an open neighborhood of $\alpha(\gamma_0)$ in Γ_1, there exists $\hat{f} \in A(\Gamma_1)$ whose support lies in V, such that $\hat{f}(\alpha(\gamma_0)) \neq 0$. If $W = \{\gamma \in \Gamma_2 \colon (\hat{f} \circ \alpha)(\gamma) \neq 0\}$, then W is open in Γ_2, $\gamma_0 \in W$, and $\alpha(W) \subset V$. *Hence Y is open, and α is continuous.*

Gathering up the information obtained so far, we now have the following situation:

There are finitely many cosets K_i and N_{ij} in Γ_2, and there are finitely many disjoint sets S_i of the form

$$S_i = K_i \cap \bigcap_j N'_{ij}, \tag{1}$$

where N_{ij} has infinite index in K_i and N'_{ij} is the complement of N_{ij}. The set $Y = \bigcup S_i$ is open. To each K_i corresponds an affine map α_i of K_i into Γ_i, and α_i coincides with α on S_i.

By Lemma 4.2.4, each α_i can be extended to an affine map of the closure $\bar{K}_i$ of K_i. Suppose this is done. Fix $\gamma_0 \in \bar{Y}$. Then $\gamma_0 \in \bar{S}_i$ for some i, and $\alpha_i(\gamma_0) \in \Gamma_1$. Choose $f \in L^1(G_1)$ so that $\hat{f}(\alpha_i(\gamma_0)) \neq 0$. On S_i, $(\hat{f} \circ \alpha)(\gamma) = \hat{f}(\alpha_i(\gamma))$. Since α_i and $\hat{f} \circ \alpha$ are continuous, $(\hat{f} \circ \alpha)(\gamma_0) = \hat{f}(\alpha_i(\gamma_0)) \neq 0$. Hence $\gamma_0 \in Y$, and so Y is closed in Γ_2.

Let I be the set of all i such that $\bar{S}_i$ has non-empty interior. Then $\bigcup \bar{S}_i$ $(i \in I)$ covers Y, since Y is open, and thus is equal to Y, since Y is closed. If N_{ij} has non-empty interior, then N_{ij} is open and hence does not intersect $\bar{S}_i$. Hence, for $i \in I$,

$$\bar{S}_i = \bar{K}_i \cap \bigcap_j N'_{ij}, \tag{2}$$

the intersection being taken over those N_{ij} which are open. Since $\bar{K}_i$ is open and closed, it follows that $\bar{S}_i$ is open and closed and belongs to the coset ring of Γ_2. Let $\bar{S}_1, \ldots, \bar{S}_n$ be an enumeration of these open and closed sets, and let E_i be the set of all points in $\bar{S}_i$ which are not in $\bar{S}_1 \cup \ldots \cup \bar{S}_{i-1}$ $(1 \leqq i \leqq n)$.

This completes the proof: Y is the disjoint union of the sets E_i which belong to the coset-ring of Γ_2, each E_i lies in an open coset

$\bar{K}_i$, and there are affine maps α_i of $\bar{K}_i$ into Γ_1 such that α_i coincides on E_i with α.

4.6. *Complements to the Main Result*

The map α and the set Y will be associated with the homomorphism Ψ as in Section 4.1.1.

4.6.1. THEOREM. *Let Ψ be a homomorphism of $L^1(G_1)$ into $M(G_2)$.*

(a) *There is a norm-preserving extension of Ψ to a homomorphism of $M(G_1)$ into $M(G_2)$.*

(b) *Suppose G_1 is not discrete. Then Ψ has a unique extension to a homomorphism of $M(G_1)$ into $M(G_2)$ if and only if $Y = \Gamma_2$.*

If G_1 is discrete, then $M(G_1) = L^1(G_1)$, and the extension problem does not arise.

Proof: For compact G_1 and G_2, (a) was proved in Lemma 4.4.1. The same proof applies in the general situation, provided we know that the function ϕ used in that proof is continuous, so that Theorem 1.9.1 applies. But we know now that α is continuous and that Y is open and closed, and the continuity of ϕ is then trivial.

To prove (b), suppose Ψ has been extended to $M(G_1)$, choose $f \in L^1(G_1)$, $\mu \in M(G_1)$, and put $\sigma = \Psi f$, $\lambda = \Psi\mu$. If $g = f * \mu$, then $g \in L^1(G_1)$, and hence the Fourier transform of Ψg is 0 outside Y and is

$$(1) \qquad (\hat{f}\hat{\mu})(\alpha(\gamma)) = \hat{f}(\alpha(\gamma))\hat{\mu}(\alpha(\gamma))$$

on Y. But since $\Psi g = (\Psi f) * (\Psi\mu) = \sigma * \lambda$, this transform is also equal on Y to

$$(2) \qquad \hat{\sigma}(\gamma)\hat{\lambda}(\gamma) = \hat{f}(\alpha(\gamma))\hat{\lambda}(\gamma).$$

Since (2) holds for all $f \in L^1(G_1)$, we have $\hat{\lambda}(\gamma) = \hat{\mu}(\alpha(\gamma))$ for $\gamma \in Y$.

Thus $\hat{\lambda}$ is uniquely determined on Y by α, i.e., by the action of Ψ on $L^1(G_1)$, and hence the extension of Ψ is unique if $Y = \Gamma_2$.

If $Y \neq \Gamma_2$, let h be a complex homomorphism of $M(G_1)$ which is identically 0 on $L^1(G_1)$, and for any $\mu \in M(G_1)$, let $\Psi\mu$ be the measure whose transform is

$$\phi(\gamma) = \begin{cases} \hat{\mu}(\alpha(\gamma)) & (\gamma \in Y), \\ h(\mu) & (\gamma \notin Y). \end{cases} \tag{3}$$

Since $L^1(G_1)$ is a proper closed ideal in $M(G_1)$, there exists such an h which is not identically 0 on $M(G_1)$, and the extension of Ψ so obtained is different from the one with $h = 0$.

It is also possible to define such an h explicitly: let π_H be the homomorphism of $M(G_1)$ into itself defined in Section 3.4.1, fix $\gamma \in \Gamma_1$, and put $h(\mu) = \hat{\sigma}(\gamma)$, where $\sigma = \pi_H \mu$.

4.6.2. THEOREM. *The homomorphism Ψ of $L^1(G_1)$ into $M(G_2)$ maps $L^1(G_1)$ into $L^1(G_2)$ if and only if $\alpha^{-1}(C)$ is compact for every compact subset C of Γ_1.*

Proof: If there is a compact set C in Γ_1 such that $\alpha^{-1}(C)$ is not compact, choose $f \in L^1(G_1)$ such that $\hat{f} = 1$ on C. Then the set of all $\gamma \in Y$ at which $\hat{f}(\alpha(\gamma)) = 1$ contains the closed set $\alpha^{-1}(C)$ and hence is not compact, so that $\Psi f \notin L^1(G_2)$.

Conversely, if $\alpha^{-1}(C)$ is compact for every compact C and if $f \in L^1(G_1)$, we can find $f_n \in L^1(G_1)$ such that $\hat{f}_n$ has compact support and such that $f_n \to f$ in the norm of $L^1(G_1)$. Each $\hat{f}_n \circ \alpha$ then has compact support, so that $\Psi f_n \in L^1(G_2)$. Since Ψ is continuous, $\Psi f_n \to \Psi f$ in the norm of $M(G_2)$, and since $L^1(G_2)$ is closed in $M(G_2)$, $\Psi f \in L^1(G_2)$.

4.6.3. If we examine the proof of Theorem 4.1.3 and pay attention to the norms, we find:

(a) If $||\Psi|| < 1$, then $\Psi = 0$, by Appendix D5.

(b) If $||\Psi|| = 1$, then $||\chi|| = 1$ in Lemma 4.4.3, hence the graph of α is a coset in $\Gamma_1 \oplus \Gamma_2$ (Section 3.2.4), and this means that Y is an open coset in Γ_2 and that α is affine on Y. By Lemma 4.2.2, this last condition in turn implies that $||\Psi|| = 1$.

(c) It follows from (b) and Theorem 3.7.2 that $||\Psi|| \geq \sqrt{5}/2$ if $||\Psi|| > 1$.

4.6.4. We recall that an isomorphism is a homomorphism which is one-to-one.

THEOREM. *If Ψ is an isomorphism of $M(G_1)$ onto $M(G_2)$, then Ψ maps $L^1(G_1)$ onto $L^1(G_2)$. Conversely, every isomorphism of $L^1(G_1)$*

onto $L^1(G_2)$ has a unique extension to an isomorphism of $M(G_1)$ onto $M(G_2)$.

Proof: If Ψ is an isomorphism of $M(G_1)$ onto $M(G_2)$, then the restriction of Ψ to $L^1(G_1)$ is an isomorphism of $L^1(G_1)$ into $M(G_2)$ and hence determines α and Y, as before. The proof of Theorem 4.6.1(b) shows that if $\mu \in M(G_1)$ and $\sigma = \Psi\mu$, then

$$\hat{\sigma}(\gamma) = \hat{\mu}(\alpha(\gamma)) \qquad (\gamma \in Y). \tag{1}$$

Since the range of Ψ covers $M(G_2)$, α must be one-to-one on Y. Since α is piecewise affine and Y is closed, $\alpha(Y)$ is closed in Γ_1. If $\alpha(Y) \neq \Gamma_1$ it follows that there exists $f \in L^1(G_1)$ such that $\hat{f} = 0$ on $\alpha(Y)$ but $\hat{f} \neq 0$ at some point of Γ_1; since $\hat{f} \circ \alpha = 0$, we have $\Psi f = 0$, and this contradicts the assumption that Ψ is one-to-one. If $Y \neq \Gamma_2$, there exists $\sigma \in M(G_2)$, $\sigma \neq 0$, such that $\hat{\sigma} = 0$ on Y, and there exists $\mu \in M(G_1)$ such that $\sigma = \Psi\mu$; by (1), $\hat{\mu} = 0$ on Γ_1, thus $\mu = 0$; but $\Psi\mu = \sigma \neq 0$, a contradiction.

Summing up, α is a piecewise affine homeomorphism of Γ_2 onto Γ_1, and by Theorem 4.6.2 the first part of 4.6.4 is proved.

If Ψ is an isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ then again $Y = \Gamma_2$, $\alpha(Y) = \Gamma_1$, and α is one-to-one. The extension of Theorem 4.6.1 is thus an isomorphism of $M(G_1)$ onto $M(G_2)$.

COROLLARY. *If Ψ is an automorphism of $M(G)$,* (i.e., an isomorphism of $M(G)$ onto $M(G)$) *then $\Psi(L^1(G)) = L^1(G)$.*

This means that the ideal of all absolutely continuous measures in $M(G)$ is algebraically distinguishable from all other ideals in $M(G)$.

4.6.5. The simplest isomorphisms of $L^1(G_1)$ onto $L^1(G_2)$ are obtained by taking for α an affine homeomorphism (not merely a piecewise affine one) of Γ_2 onto Γ_1. Then there exists $\gamma_0 \in \Gamma_1$ and an isomorphism τ of Γ_2 onto Γ_1 such that $\alpha(\gamma) = \tau(\gamma - \gamma_0)$, and τ induces an isomorphism β of G_1 onto G_2:

$$(x, \tau(\gamma)) = (\beta(x), \gamma) \qquad (x \in G_1, \gamma \in \Gamma_2). \tag{1}$$

The map

$$f \to \int_{G_2} f(\beta^{-1}(y))dy \tag{2}$$

is a translation invariant linear functional on $C_c(G_1)$; hence there is a positive constant $k = k(\beta)$, such that

$$(3) \qquad k\int_{G_2} f(\beta^{-1}(y))dy = \int_{G_1} f(x)dx \qquad (f \in L^1(G_1)).$$

If $f \in L^1(G_1)$, the Fourier transform of $k \cdot (y, \gamma_0) \cdot f(\beta^{-1}(y))$ is

$$(4) \quad k\int_{G_2} (y, \gamma_0) f(\beta^{-1}(y))(-y, \gamma)dy = \int_{G_1} f(x)(\beta(x), \gamma_0 - \gamma)dx$$
$$= \int_{G_1} f(x)(x, \tau(\gamma_0 - \gamma))dx = \hat{f}(\alpha(\gamma)).$$

Thus

$$(5) \qquad (\Psi f)(y) = k(y, \gamma_0) f(\beta^{-1}(y)) \qquad (f \in L^1(G_1), y \in G_2).$$

We note that this Ψ is an isometry.

4.6.6. Let us say that Ψ *preserves positivity* if $\Psi f \geqq 0$ whenever $f \geqq 0$. Our next theorem characterizes the maps α such that $\phi \circ \alpha$ is positive-definite on Γ_2 whenever ϕ is positive-definite on Γ_1.

THEOREM. *If Ψ preserves positivity, then Y is an open subgroup of Γ_2 and α is a continuous homomorphism of Y into Γ_1.*

Proof: Considering first the case of compact G_1 and G_2, the extension 4.4.1 of Ψ carries $\mu \geqq 0$ to $\Psi\mu \geqq 0$, so that the measures μ_x of Section 4.4.2 are non-negative. If $\Psi \neq 0$ then Y is not empty, hence $\mu_x \neq 0$ for all $x \in G_1$, and so $\hat{\mu}_x(0) = 1$. This shows that $0 \in Y$ and that $\|\mu_x\| = 1$, so that $\|\chi\| = 1$ in the proof of Theorem 4.4.3. Hence the graph of α is a coset, Y is a coset, and α is affine. Since $0 \in Y$, Y is a subgroup of Γ_2; since $\hat{\mu}_x(0) = 1$ for all $x \in G_1$, $\alpha(0) = 0$, and hence α is a homomorphism.

The general case follows; for if we pass to the Bohr compactifications, the induced homomorphism of $M(\bar{G}_1)$ into $M(\bar{G}_2)$ also preserves positivity.

4.6.7. COROLLARY. *If Ψ is an isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ which preserves positivity, then Ψ has the form described in Section 4.6.5, with $\gamma_0 = 0$.*

4.6.8. THEOREM. *If Γ is an infinite LCA group, then $A(\Gamma)$ is a proper subset of $C_0(\Gamma)$.*

Proof: There is a homeomorphism α of Γ onto Γ which is not piecewise affine, and so there exists $\hat{f} \in A(\Gamma)$ such that $\hat{f} \circ \alpha \notin A(\Gamma)$. But $\hat{f} \circ \alpha \in C_0(\Gamma)$.

A different proof of this theorem was given by Segal [3]; see also Hewitt [2] and Edwards [1].

4.6.9. This chapter is primarily devoted to a study of the effect of a map $\hat{f} \to \hat{f} \circ \alpha$ on L^1-norms. Similar problems can of course be posed for other norms. We will prove the analogue of Theorem 4.1.3 for the L^∞-norm. Section 5.7.8 contains a comment on other L^p-spaces.

THEOREM. *Suppose G_1 and G_2 are compact, $Y \subset \Gamma_2$, and α maps Y into Γ_1. The following two conditions are equivalent:*

(a) *Y belongs to the coset-ring of Γ_2, α is piecewise affine, and $\alpha^{-1}(\gamma_1)$ is a finite set, for each $\gamma_1 \in \Gamma_1$;*

(b) *to every $f \in C(G_1)$ there corresponds a function $g \in L^\infty(G_2)$ such that $\hat{g} = \hat{f} \circ \alpha$.*

Proof: If α is a homomorphism of Γ_2 onto Γ_1 with finite kernel Λ, then $\Gamma_1 = \Gamma_2/\Lambda$, and so G_1 may be regarded as the annihilator of Λ, i.e., as a compact open subgroup of G_2, of index n, where n is the number of elements of Λ. If $f \in C(G_1)$ and if $\hat{g} = \hat{f} \circ \alpha$, it is easily found (by first considering trigonometric polynomials) that $g(y) = nf(y)$ if $y \in G_1$ and that $g(y) = 0$ at all other points of G_2. Arguing as in Section 4.2, it follows from this special case that α carries $C(G_1)$ into $C(G_2)$ if (a) holds; we omit the details. Thus (a) implies (b).

The proof that (b) implies (a) is more interesting. Let Ψ be the linear transformation of $C(G_1)$ into $L^\infty(G_2)$ defined by setting $g = \Psi f$ if $\hat{g} = \hat{f} \circ \alpha$. If $\gamma_1 \in \Gamma_1$ and $f(x) = (x, \gamma_1)$, then $\hat{g}(\gamma) = 1$ for all $\gamma \in \alpha^{-1}(\gamma_1)$, and since $\hat{g} \in C_0(\Gamma_2)$, it follows that $\alpha^{-1}(\gamma_1)$ is finite. Also

$$(1) \qquad (\Psi f)(y) = \sum_{\alpha(\gamma)=\gamma_1} (y, \gamma) \qquad (y \in G_2).$$

Suppose $f_n \in C(G_1)$ for $n = 1, 2, 3, \ldots$, $\|f_n - f\|_\infty \to 0$, and $\|\Psi f_n - g\|_\infty \to 0$ for some $g \in L^\infty(G_2)$. Then $\hat{f}_n(\gamma) \to \hat{f}(\gamma)$ for all $\gamma \in \Gamma_1$ and $(\hat{f}_n \circ \alpha)(\gamma) \to \hat{g}(\gamma)$ for all $\gamma \in \Gamma_2$. Hence $\hat{g} = \hat{f} \circ \alpha$, or

$g = \Psi f$. We conclude from the closed graph theorem that Ψ is a *bounded* linear transformation.

If F is a trigonometric polynomial on $G_1 \oplus G_2$ and if we write $F^y(x)$ for $F(x, y)$, then F^y is a trigonometric polynomial on G_1, for every $y \in G_2$, and since $\alpha^{-1}(\gamma_1)$ is finite for each $\gamma_1 \in \Gamma_1$, (1) shows that ΨF^y is a trigonometric polynomial on G_2. Setting $\phi(y) = (\Psi F^y)(y)$, it follows that ϕ is a trigonometric polynomial on G_2. Also,

$$||\phi||_1 \leqq ||\phi||_\infty \leqq \sup_{y \in G_2} ||\Psi F^y||_\infty \leqq ||\Psi|| \cdot \sup_{y \in G_2} ||F^y||_\infty = ||\Psi|| \cdot ||F||_\infty.$$

We conclude that there exists $\sigma \in M(G_1 \oplus G_2)$ such that

$$\int_{G_1}\int_{G_2} F(-x, y)d\sigma(x, y) = \int_{G_2} (\Psi F^y)(y)dy \tag{2}$$

for every trigonometric polynomial F on $G_1 \oplus G_2$.

Fix $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, put $F(x, y) = (x, \gamma_1)(-y, \gamma_2)$. The left side of (2) is then $\hat{\sigma}(\gamma_1, \gamma_2)$, and the integrand on the right side is $\sum (y, \gamma - \gamma_2)$, the sum being extended over all $\gamma \in \Gamma_2$ for which $\alpha(\gamma) = \gamma_1$. Thus $\hat{\sigma}(\gamma_1, \gamma_2) = 1$ if $\alpha(\gamma_2) = \gamma_1$ and is 0 otherwise. This means that $\hat{\sigma}$ is the characteristic function of the graph of α. Theorem 3.1.3 shows that this graph belongs to the coset ring of $\Gamma_1 \oplus \Gamma_2$, and Theorem 4.3.1 completes the proof.

4.7. *Special Cases*

4.7.1. THEOREM. *If Ψ is an isomorphism of $L^1(G_1)$ onto $L^1(G_2)$, and if $||\Psi|| \leqq 1$, then G_1 and G_2 are isomorphic and Ψ is of the form* 4.6.5 (Helson [3]).

Proof: By 4.6.3, $||\Psi|| = 1$ and α is affine; by 4.6.4, α is an affine homeomorphism of Γ_2 onto Γ_1, and the result follows from 4.6.5.

4.7.2. THEOREM. *Suppose Γ_2 is connected. If Ψ is an isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ then G_1 and G_2 are isomorphic and Ψ is of the form* 4.6.5. (Beurling and Helson [1]).

Proof: Since Γ_2 is connected, Γ_2 is the only non-empty member of the coset ring of Γ_2. Thus $Y = \Gamma_2$, α is affine, and the proof is completed as in 4.7.1.

4.7.3. THEOREM. *Suppose $\phi \in B(\Gamma)$, and suppose there is a constant C such that*

$$(1) \qquad \|\phi^n\| \leqq C \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

the norm being that of $B(\Gamma)$. Then ϕ is a piecewise affine map of Γ into the circle group T.

COROLLARY (Beurling and Helson [1]). *If, in addition, Γ is connected, then there is a complex number a with $|a| = 1$, and an element $x \in G$, such that*

$$(2) \qquad \phi(\gamma) = a(x, \gamma) \qquad (\gamma \in \Gamma).$$

Proof: The spectral radius formula shows that

$$(3) \qquad \|\phi\|_\infty \leqq \lim_{n\to\infty} \|\phi^n\|^{1/n} \leqq \lim_{n\to\infty} C^{1/n} = 1.$$

The same holds for $1/\phi$. Hence $|\phi(\gamma)| = 1$ for all $\gamma \in \Gamma$; i.e., ϕ maps Γ into T.

Let $\mu \in M(G)$ be the measure such that $\phi = \hat{\mu}$. For any $f \in L^1(Z)$, define

$$(4) \qquad \Psi f = \sum_{n=-\infty}^{\infty} f(n)\mu^{-n}.$$

Then Ψ is a homomorphism of $L^1(Z)$ into $M(G)$, since $\|\mu^n\| \leqq C$ for all $n \in Z$. The Fourier-Stieltjes transform of Ψf is

$$(5) \qquad \sum_{n=-\infty}^{\infty} f(n)\phi(\gamma)^{-n} = \hat{f}(\phi(\gamma)) \qquad (\gamma \in \Gamma).$$

By Theorem 4.1.3, it follows that ϕ is piecewise affine.

If Γ is connected, then ϕ is affine, hence is a continuous homomorphism into T (i.e., a character) followed by a translation in T (i.e., by multiplication by a complex number of absolute value 1).

REMARK. If $\Gamma = R$, the real line, the preceding result specializes to

$$(6) \qquad \phi(t) = e^{i(at+b)} \qquad (-\infty < t < \infty),$$

where a and b are real numbers.

4.7.4. Piecewise affine maps of Γ into T can be described quite explicitly: Let $S_1, \ldots, S_n$ be disjoint sets belonging to the coset-ring of Γ, whose union is Γ; choose $x_1, \ldots, x_n \in G$, and choose complex numbers $c_1, \ldots, c_n$ of absolute value 1. Put

$$(1) \qquad \phi(\gamma) = c_j(x_j, \gamma) \qquad (\gamma \in S_j).$$

Then ϕ is piecewise affine, and every piecewise affine map of Γ into T is so obtained.

In particular, the affine maps ϕ of T into T are of the form

$$(2) \qquad \phi(e^{i\theta}) = ce^{in\theta}$$

where $|c| = 1$ and n is an integer.

4.7.5. Theorem (Leibenson [1], Kahane [1], [2]). *Let ϕ be a map of T into T such that the Fourier series of $f(\phi)$ is absolutely convergent whenever the Fourier series of f is absolutely convergent. Then there is an integer n and a real number a such that*

$$\phi(e^{i\theta}) = e^{i(n\theta + a)} \qquad (e^{i\theta} \in T).$$

Proof: The map $f \to f(\phi)$ is a homomorphism Ψ of $A(T)$ into $A(T)$, or of $L^1(Z)$ into $L^1(Z)$, and ϕ is nothing but the map α induced by Ψ. Since T is connected, ϕ is affine.

4.7.6. In the preceding special cases, Γ_2 was either connected, in which case its coset-ring was trivial, or Ψ was an isomorphism of norm 1. In either case, α was affine on Γ_2, and so neither coset-rings nor piecewise affine maps appeared in these results. The role played by these two concepts in the homomorphism problem became apparent for the first time after the case $\Gamma_1 = \Gamma_2 = Z$ had been settled; the problem, and the result, can here be stated quite concretely:

Suppose Y is a subset of the integer group Z and α maps Y into Z. For which Y and α is it true that

$$(1) \qquad \sum_{n \in Y} c(\alpha(n)) e^{in\theta}$$

is a Fourier series (of a function in $L^1(T)$) whenever the series

$$(2) \qquad \sum_{n \in Z} c(n) e^{in\theta}$$

is a Fourier series? Or, in our previous terminology, for which α does the map $\hat{f} \to \hat{f} \circ \alpha$ carry $A(Z)$ into $A(Z)$?

THEOREM (Rudin [3]). *Necessary and sufficient conditions for this are the existence of a positive integer q and of a map β of Z into Z with the following properties:*

(a) *If $A_1, \ldots, A_q$ are the residue classes modulo q, then $Y = S_1 \cup \ldots \cup S_n$ where each S_i is either finite or is contained in some A_j from which it differs by a finite set, and the sets S_i are pairwise disjoint.*

(b) *$\alpha(n) = \beta(n)$ for all $n \in Y$, with possibly finitely many exceptions, $\beta(n + q) \neq \beta(n)$ for all $n \in Z$, and*

$$\beta(n + q) + \beta(n - q) = 2\beta(n) \qquad (n \in Z). \tag{3}$$

Proof: This is just a restatement of Theorem 4.1.3, adapted to the case $\Gamma_2 = Z$. We saw in Section 3.1.3 that Y belongs to the coset-ring of Z if and only if Y has the structure described in (a). A map β on A_j is affine if and only if it is of the form

$$\beta(a + kq) = u + kv \qquad (-\infty < k < \infty), \tag{4}$$

hence if and only if (3) holds.

The condition $\beta(n + q) \neq \beta(n)$ assures that α is not constant on any infinite set (compare Theorem 4.6.2); if this condition is omitted, the remaining conditions are necessary and sufficient for α to carry Fourier series to Fourier-Stieltjes series.

4.7.7. Suppose the algebras $L^1(G_1)$ and $L^1(G_2)$ are isomorphic. What can be said about the relation between G_1 and G_2? In particular, does it follow that G_1 and G_2 are isomorphic?

It follows from Theorem 4.1.3 (see also the proof of Theorem 4.6.4) that *a necessary and sufficient condition for $L^1(G_1)$ and $L^1(G_2)$ to be isomorphic is the existence of a piecewise affine homeomorphism of Γ_2 onto Γ_1.*

The following special case is an application of this remark:

THEOREM (Rudin [10]). *$L^1(G)$ is isomorphic to $L^1(T)$ if and only if $G = T \oplus F$, where F is a finite abelian group.*

Proof: If $G = T \oplus F$ then $\Gamma = Z \oplus F$; if F has q elements $f_1, \ldots, f_q$ and if

$$\alpha(nq + k) = (n, f_k) \qquad (n \in Z, 1 \leqq k \leqq q) \tag{1}$$

then α is a piecewise affine map of Z onto Γ and α is one-to-one. Hence $L^1(G)$ is isomorphic to $L^1(T)$.

Conversely, if α is a piecewise affine map of Z into Γ then α has the structure described in part (b) of Theorem 4.7.6. In particular, if $\alpha(Z) = \Gamma$, then $\beta(Z) = \Gamma$, Γ is a union of finitely many arithmetic progressions, hence Γ is finitely generated and so is a direct sum of cyclic groups. Since Γ is the union of *finitely* many arithmetic progressions, only *one* of these cyclic groups can be infinite. Thus $\Gamma = Z \oplus F$ and hence $G = T \oplus F$, with F finite.

CHAPTER 5

Measures and Fourier Transforms on Thin Sets

Measures concentrated on sets which are "thin" in a certain arithmetic (or group-theoretic) sense havé some unexpected pathological properties. In the present chapter, we study some of these phenomena; as a by-product we obtain an easy proof of the asymmetry of the Banach algebra $M(G)$ for all non-discrete LCA groups G. We discuss the behavior of the Fourier-Stieltjes transforms of measures concentrated on thin sets, as well as the restrictions of Fourier transforms to such sets.

5.1. Independent Sets and Kronecker Sets

5.1.1. A subset E of an abelian group G is said to be *independent* if E has the following property: for every choice of distinct points $x_1, \ldots, x_k$ of E and integers $n_1, \ldots, n_k$, either

$$n_1x_1 = n_2x_2 = \ldots = n_kx_k = 0 \tag{1}$$

or

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k \neq 0. \tag{2}$$

In other words, no linear combination (2) can be 0 unless every summand is 0.

5.1.2. A subset E of a LCA group G will be called a *Kronecker set* if E has the following property: to every continuous function f on E, of absolute value 1, and to every $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\sup_{x \in E} |f(x) - (x, \gamma)| < \varepsilon. \tag{1}$$

This definition is motivated by the classical theorem of Kronecker which asserts, in the present terminolcgy, that every finite independent subset of the real line is a Kronecker set (see Theorem 5.1.3). Since groups of bounded order contain no non-empty Kronecker sets, we state a modified definition which is applicable to that case.

For any integer $q \geqq 2$, let Z_q be the set of all numbers $\exp\{2\pi ij/q\}$, $0 \leqq j \leqq q-1$; Z_q is the cyclic subgroup of T whose order is q. A subset E of D_q (see Section 2.2.4 for the definition) is said to be *of type* K_q if E has the following property: every continuous function on E which maps E into Z_q *coincides* on E with a continuous character of D_q.

5.1.3. For $x \in G$, put $S(x) = T$ if x has infinite order; if x has order q, put $S(x) = Z_q$.

Theorem. *Suppose E is a finite independent set in a LCA group G, f is a function on E such that $f(x) \in S(x)$ for all $x \in E$, and $\varepsilon > 0$. Then there exists $\gamma \in \Gamma$ such that*

$$(1) \qquad |(x, \gamma) - f(x)| < \varepsilon \qquad (x \in E).$$

Proof: Suppose $E = \{x_1, \ldots, x_k\}$. The group H generated by E consists of all linear combinations $\sum n_i x_i$ with integral coefficients n_i, and the independence of E shows that each $x \in H$ has a unique representation $x = \sum n_i x_i$. It follows that the formula

$$(2) \qquad \phi\Big(\sum_{i=1}^{k} n_i x_i\Big) = \prod_{i=1}^{k} [f(x_i)]^{n_i}$$

defines a function ϕ on H; also, $\phi(x_i) = f(x_i)$ $(1 \leqq i \leqq k)$, and ϕ is a character of H, i.e., a homomorphism of H into T. Since T is divisible, Theorem 2.5.1 shows that ϕ can be extended to a character of G, and by Theorem 1.8.3 there exists $\gamma \in \Gamma$ such that $|(x, \gamma) - \phi(x)| < \varepsilon$ for all $x \in E$. This completes the proof.

There are more elaborate approximation theorems which may be proved by this method with equal ease; see Hewitt and Zuckerman [1].

Corollary. *Suppose E is a finite independent set in G.*

(a) *If every $x \in E$ has infinite order; then E is a Kronecker set.*

(b) *If* $G = D_q$ *and every* $x \in E$ *has order* q, *then* E *is of the type* K_q.

Part (a) is evident. Taking $\varepsilon < \sin(\pi/q)$ in the theorem establishes (b), since both (x, γ) and $f(x)$ are constrained to lie in Z_q.

5.1.4. THEOREM. (a) *Kronecker sets are independent, and contain no elements of finite order.*

(b) *Sets of type* K_q *are independent subsets of* D_q, *and contain only elements of order* q.

Proof: Suppose E is a Kronecker set in G, $x_1, \ldots, x_k$ are distinct points of E, $n_1, \ldots, n_k$ are integers and $\sum n_i x_i = 0$. Then

$$\prod_{i=1}^{k} (x_i, \gamma)^{n_i} = \prod_{i=1}^{k} (n_i x_i, \gamma) = (\sum_{1}^{k} n_i x_i, \gamma) = 1 \tag{1}$$

for every $\gamma \in \Gamma$, so that any function f on E which can be uniformly approximated on E by characters must satisfy the condition

$$\prod_{i=1}^{k} [f(x_i)]^{n_i} = 1. \tag{2}$$

Since E is a Kronecker set, (2) must hold for arbitrary complex numbers $f(x_i)$ of absolute value 1. Hence $n_1 = n_2 = \ldots = n_k = 0$, and (a) is proved.

If E is of type K_q in D_q the same proof applies, except that now (2) holds for all choices of $f(x_i) \in Z_q$. Hence $n_i \equiv 0 \pmod q$ $(1 \leqq i \leqq k)$, and this implies (b).

5.2. Existence of Perfect Kronecker Sets

5.2.1. We call a subset E of a topological space *perfect* if E is compact and non-empty and if no point of E is an isolated point of E. We call E a *Cantor set* if E is homeomorphic to Cantor's "middle third" set on the line. A set is a Cantor set if and only if it is metric, perfect, and totally disconnected. Our present objective is the construction of Cantor sets which are Kronecker sets or sets of type K_q, respectively, and to show that this can be done in every non-discrete LCA group. We recall that if a non-discrete LCA group is not an I-group then it contains D_q (for some $q \geqq 2$)

as a closed subgroup (Theorem 2.5.5). It will be convenient to separate these two cases:

5.2.2. THEOREM. (a) *Every I-group contains a Cantor set which is a Kronecker set.*

(b) *The group D_q contains a Cantor set which is of type K_q.*

Our proof will imitate the usual construction of a Cantor set on the line as the intersection of a decreasing sequence of sets E_r, each of which is the union of 2^r disjoint closed intervals.

5.2.3. LEMMA. *Suppose $V_1, \ldots, V_k$ are disjoint non-empty open sets in G, and G is either an I-group or $G = D_q$. Then there are points $x_i \in V_i$ $(1 \leqq i \leqq k)$ such that*

(a) *$\{x_1, \ldots, x_k\}$ is a Kronecker set, if G is an I-group,*

(b) *$\{x_1, \ldots, x_k\}$ is a set of type K_q, if $G = D_q$.*

Proof: Suppose first that G is an I-group. If $y \in G$ and if k is an integer, $k \neq 0$, then the set $E_{k,y}$ of all $x \in G$ such that $kx = y$ is closed and contains no open set; for if it did, then $E_{k,y} - E_{k,y}$ would be a neighborhood W of 0; since $kx = 0$ for all $x \in W$, this contradicts the definition of an I-group.

By Baire's theorem, V_1 is therefore not covered by the union of sets $E_{k,0}$ $(k = 1, 2, 3, \ldots)$, so that V_1 contains an element x_1 of infinite order. Suppose $x_1, \ldots, x_j$ are chosen, $x_i \in V_i$ $(1 \leqq i \leqq j)$, and the set $\{x_1, \ldots, x_j\}$ is independent. Let H be the group generated by $x_1, \ldots, x_j$. Since H is countable, Baire's theorem shows that V_{j+1} is not covered by the union of the sets $E_{k,y}$ $(k = 1, 2, 3, \ldots;$ $y \in H)$; hence there exists $x_{j+1} \in V_{j+1}$ such that none of the multiples kx_{j+1} $(k \neq 0)$ lies in H. In a finite number of steps we thus obtain an independent set $\{x_1, \ldots, x_k\}$ each of whose elements has infinite order, with $x_i \in V_i$.

If $G = D_q$, define $E_{k,y}$ as above. Now $E_{k,y}$ contains no open set if $0 < k < q$, since each neighborhood of 0 in D_q contains elements of order q. Having chosen independent elements $x_1, \ldots, x_j$, with $x_i \in V_i$, of order q, it follows that V_{j+1} contains a point x_{j+1} such that kx_{j+1} is not in the finite group generated by $x_1, \ldots, x_j$, unless q divides k.

The lemma now follows from the corollary to Theorem 5.1.3.

5.2.4. Proof of Theorem 5.2.2. Since every I-group has a closed subgroup which is a metric I-group (Theorem 2.5.5) and since every Kronecker set in a closed subgroup of G is a Kronecker set in G, we may assume in part (a) that G is metric.

Let P_1^0 be any compact neighborhood in G. Suppose $r \geqq 1$, $s = 2^{r-1}$, and suppose that disjoint compact neighborhoods

$$P_1^{r-1}, P_2^{r-1}, \ldots, P_s^{r-1} \tag{1}$$

have been constructed. Let W_{2j-1} and W_{2j} be non-empty disjoint open sets in P_j^{r-1} $(1 \leqq j \leqq s)$. By Lemma 5.2.3, there is a Kronecker set $\{x_1^r, \ldots, x_{2s}^r\}$ with $x_j^r \in W_j$ $(1 \leqq j \leqq 2s)$. It follows that there is a finite set $F_r \subset \Gamma$ with the following property: to each choice of real numbers α_j there exists at least one $\gamma \in F_r$ which satisfies the simultaneous inequalities

$$|e^{i\alpha_j} - (x_j^r, \gamma)| < 1/r \qquad (1 \leqq j \leqq 2^r). \tag{2}$$

There exist disjoint compact neighborhoods P_j^r of x_j^r $(1 \leqq j \leqq 2^r)$, such that $P_j^r \subset W_j$, such that

$$|(x, \gamma) - (x_j^r, \gamma)| < 1/r \qquad (x \in P_j^r,\ \gamma \in F_r), \tag{3}$$

and such that $d(x, x_j^r) < 1/r$ for all $x \in p_j^r$, where d is the metric of G.

This completes the induction. We define

$$P = \bigcap_{r=1}^{\infty} \bigcup_{j=1}^{2^r} P_j^r. \tag{4}$$

It is evident that P is a Cantor set. Suppose $f \in C(P)$, $|f| = 1$, and $\varepsilon > 0$. Since f is uniformly continuous, there exists r_0 such that f maps each of the sets $P \cap P_j^{r_0}$ $(1 \leqq j \leqq 2^{r_0})$ into an arc on the unit circle (not onto the whole circle), and the Tietze extension theorem shows that f may be extended to a continuous map of $\bigcup P_j^{r_0}$ $(1 \leqq j \leqq 2^{r_0})$ into the unit circle; in particular, $f(x_j^r)$ will be defined if $r \geqq r_0$. We can choose $r > r_0$, $r > 3/\varepsilon$, and such that

$$|f(x) - f(x_j^r)| < \varepsilon/3 \qquad (x \in P_j^r,\ 1 \leqq j \leqq 2^r). \tag{5}$$

Our choice of F_r shows that there exists $\gamma \in F_r$ for which

(6) $$|f(x_j^r) - (x_j^r, \gamma)| < 1/r \qquad (1 \leqq j \leqq 2^r).$$

If we add the inequalities (3), (5), and (6), we obtain

(7) $$|f(x) - (x, \gamma)| < 2/r + \varepsilon/3 < \varepsilon$$

for all $x \in \bigcup P_j^{r_0}$, hence for all $x \in P$, and part (a) is proved.

If $G = D_q$, we proceed in almost the same fashion. Having constructed neighborhoods (1), we choose $\{W_j\}$ as before; Lemma 5.2.3 shows that there are points $x_j^r \in W_j$ $(1 \leqq j \leqq 2s)$ such that $\{x_1^r, \ldots, x_{2s}^r\}$ is a set of type K_q. Hence there is a finite set $F_r \subset \Gamma$ with the following property: to each choice of numbers $z_j \in Z_q$ there exists at least one $\gamma \in F_r$ which satisfies the simultaneous equations

(8) $$(x_j^r, \gamma) = z_j \qquad (1 \leqq j \leqq 2^r).$$

Choose disjoint compact neighborhoods P_j^r of x_j^r $(1 \leqq j \leqq 2^r)$ such that $P_j^r \subset W_j$, $d(x, x_j^r) < 1/r$ for all $x \in P_j^r$, and

(9) $$(x, \gamma) = (x_j^r, \gamma) \qquad (1 \leqq j \leqq 2^r);$$

this is possible, since each γ is constant in a neighborhood of x_j^r.

Now define P by (4). If f is a continuous map of P into Z_q, then $P = E_1 \cup \ldots \cup E_q$, where f is constant on each E_i. There are open-closed sets $K_i \supset E_i$ whose union is D_q. Extend f so that it is constant on each K_i. Then f is a continuous map of D_q into Z_q. Choose r so large that f is constant on each of the sets P_j^r $(1 \leqq j \leqq 2^r)$. Our choice of F_r shows that

(10) $$f(x_j^r) = (x_j^r, \gamma) \qquad (1 \leqq j \leqq 2^r)$$

for some $\gamma \in F_r$, and by (9) this implies that $f(x) = (x, \gamma)$ for all $x \in \bigcup P_j^r$, hence for all $x \in P$.

This completes the proof.

5.2.5. Remark. If μ_r is the measure which is concentrated on the set $\{x_1^r, \ldots, x_{2s}^r\}$ such that $\mu_r(\{x_j^r\}) = 2^{-r}$ $(1 \leqq j \leqq 2^r)$, then the measures μ_r have a weak*-limit $\mu \in M(P)$ such that $\|\mu\| = 1$, $\mu \geqq 0$, and μ is continuous.

Hence there exist non-trivial continuous measures on each Cantor set.

5.2.6. THEOREM. *Every non-discrete* LCA *group G contains an independent Cantor set.*

Proof: This is a corollary of Theorem 5.2.2. For if G is not an I-group, then G contains D_q as a closed subgroup (Theorem 2.5.5). Hence G either contains a Cantor set which is a Kronecker set, or one which is of type K_q, and the theorem follows from Theorem 5.1.4.

In some of our applications (for instance, in Section 5.3) this result is all that is needed. However, the proof of Theorem 5.2.2 was no harder than a direct proof of Theorem 5.2.6 would have been.

5.2.7. One might think that compact independent sets have to be totally disconnected. This is trivially true on the line, and is true in finite-dimensional groups (Theorem 5.2.9). However, it is not true in all cases:

THEOREM. *The infinite-dimensional torus T^ω contains an arc which is a Kronecker set.*

Proof: Each $x \in T^\omega$ is of the form

$$x = (\xi_1, \xi_2, \xi_3, \ldots) \qquad (\xi_j \text{ real mod } 2\pi). \tag{1}$$

Suppose $0 < \alpha < \beta < 1$, and let L be the set of all $x(t) \in T^\omega$ of the form

$$x(t) = (2\pi t, 2\pi t^2, 2\pi t^3, \ldots) \qquad (\alpha \leqq t \leqq \beta). \tag{2}$$

Then L is clearly an arc. If $f \in C(L)$ and $|f| = 1$, then there is a real continuous function h on $[\alpha, \beta]$ such that $f(x(t)) = \exp\{2\pi i h(t)\}$. If $n_1, \ldots, n_k$ are integers and if γ is the character on T^ω defined by

$$(x, \gamma) = \exp\{i(n_1\xi_1 + \ldots + n_k\xi_k)\}, \tag{3}$$

then

$$\begin{aligned} |f(x(t)) - (x(t), \gamma)| &= |\exp\{2\pi i h(t)\} - \exp\{2\pi i \sum_1^k n_r t^r\}| \\ &\leqq 2\pi |h(t) - \sum_1^k n_r t^r| \qquad (\alpha \leqq t \leqq \beta). \end{aligned}$$

Hence the theorem is a consequence of the following lemma:

5.2.8. LEMMA. *If $0 < \alpha < \beta < 1$, then every real continuous function on $[\alpha, \beta]$ can be uniformly approximated by polynomials $\sum_1^k n_r t^r$ with integral coefficients n_r.*

Proof: Let R be the set of all functions on $[\alpha, \beta]$ which can be so approximated. It is clear that R is closed under addition and multiplication, and since R contains the identity function, R separates points on $[\alpha, \beta]$. If we can show that R contains all constants, then the lemma follows from the Stone-Weierstrass theorem.

So, choose a constant c, and define

$$S_p(t) = \frac{1}{p}\{1 - t^p - (1 - t)^p\}, \tag{1}$$

where p is a prime. Applying the binomial theorem to $(1 - t)^p$, we see that S_p is a polynomial with integral coefficients. Also $pS_p(t) \to 1$ uniformly on $[\alpha, \beta]$, as $p \to \infty$. For each p, there is an integer q_p such that $|c - q_p/p| < 1/p$, and it is clear that $q_p S_p(t) \to c$ uniformly on $[\alpha, \beta]$. This completes the proof.

A full discussion of approximation by polynomials with integral coefficients is given by Hewitt and Zuckerman [2].

5.2.9. THEOREM. *If G is a metric LCA group, if $\dim G = n < \infty$, and if E is a compact independent subset of G, then E is totally disconnected.*

Proof: If E is not totally disconnected, then E contains $n + 1$ disjoint compact connected sets $X_1, \ldots, X_{n+1}$. If $X = X_1 \times X_2 \times \ldots \times X_{n+1}$, then $\dim X \geqq n + 1$ (Hurewicz [1]). Since E is independent, the map

$$(x_1, \ldots, x_{n+1}) \to x_1 + \ldots + x_{n+1}$$

is a homeomorphism of X into G. Since $\dim G = n$, we have a contradiction.

5.3. The Asymmetry of $M(G)$

5.3.1. A commutative semi-simple Banach algebra A is called *symmetric* or *self-adjoint* if, regarded as a function algebra on its maximal ideal space, it is closed under complex conjugation. Without reference to the maximal ideal space, the condition may be expressed like this: it is required that to each $x \in A$ there corresponds an element $x^* \in A$ such that

$$h(x^*) = \overline{h(x)} \tag{1}$$

for every complex homomorphism h of A.

In this section we prove the remarkable result (Theorem 5.3.4) that $M(G)$ is not symmetric, unless G is discrete. (For discrete G there is no problem: $M(G) = L^1(G)$, Γ is the maximal ideal space of $L^1(G)$, and $A(\Gamma)$ is closed under complex conjugation.) For $G = R$, this was proved by Šreider [1]; Hewitt [3] extended it to I-groups, and Williamson [1] completed the theorem; see also Rudin [11], [14].

The asymmetry of $M(G)$ seems so remarkable for the following reason: Define h_γ, for $\gamma \in \Gamma$, by

$$h_\gamma(\mu) = \hat{\mu}(\gamma) \qquad (\mu \in M(G)). \tag{2}$$

Then h_γ is a complex homomorphism of $M(G)$, and if $\tilde{\mu}$ is the measure defined by $\tilde{\mu}(E) = \overline{\mu(-E)}$, then we *do* have

$$h_\gamma(\tilde{\mu}) = \overline{h_\gamma(\mu)} \qquad (\gamma \in \Gamma). \tag{3}$$

Thus the symmetry requirement is satisfied for the homomorphisms (2); moreover, these homomorphisms determine μ, by the uniqueness theorem for Fourier-Stieltjes transforms, and we are led to the following conclusion:

If $M(G)$ is symmetric, then $\mu^* = \tilde{\mu}$.

5.3.2. THEOREM. *Suppose G is not discrete, P is an independent compact set in G, and $Q = P \cup (-P)$. If $\mu \in M(G)$ is a continuous measure concentrated on Q, then the measures $\delta_0, \mu, \mu^2, \mu^3, \ldots$ are mutually singular.*

We recall that δ_0 is the unit of $M(G)$, and that $\mu^n = \mu^{n-1} * \mu$.

Proof: Replacing μ by $|\mu|$, we may assume, without loss of generality, that $\mu \geqq 0$. Since μ^n is concentrated on Q_n, where $Q_1 = Q$ and $Q_n = Q_{n-1} + Q$, it is enough to show that

(1) $$\mu^n(Q_m) = 0 \qquad (m < n).$$

Fix m and n, $m < n$, and let S be the set of all points $(x_1, \ldots, x_n) \in G^n$ such that

(2) $$x_1 \in Q, \ldots, x_n \in Q; \qquad x_1 + \ldots + x_n \in Q_m.$$

Since μ is concentrated on Q, we have, by 1.3.2(2),

(3) $$\mu^n(Q_m) = \mu_n(S)$$

where $\mu_{(1)} = \mu$, $\mu_{(n)} = \mu \times \mu_{(n-1)}$; note that $\mu_{(n)} \in M(G^n)$.

Suppose $(x_1, \ldots, x_n) \in S$. By (2), there are points $y_1, \ldots, y_m \in Q$ such that

(4) $$x_1 + \ldots + x_n = y_1 + \ldots + y_m.$$

The definition of Q shows that $x_i = \varepsilon_i p_i$, where $p_i \in P$ and $\varepsilon_i = \pm 1$. If $p_i \neq p_j$ whenever $i \neq j$, then since $n > m$, (4) leads to a relation between elements of P which contradicts the independence of P. Hence $x_i \pm x_j = 0$ for some $i \neq j$, and we conclude that S is contained in the union of the sets E'_{ij} and E''_{ij} $(i, j = 1, \ldots, n; i \neq j)$ which are defined by $x_i + x_j = 0$, $x_i - x_j = 0$. Since μ is continuous, the Fubini theorem shows that $\mu_{(n)}(E'_{ij}) = \mu_{(n)}(E''_{ij}) = 0$ if $i \neq j$; hence $\mu_{(n)}(S) = 0$, and the theorem follows from (3).

5.3.3. Corollary. *Suppose μ is a non-negative continuous measure concentrated on $Q = P \cup (-P)$, where P is compact and independent in G. Then*

(a) $$\left\|\sum_{k=0}^{n} a_k \mu^k\right\| = \sum_{k=0}^{n} |a_k| \cdot \|\mu\|^k$$

for arbitrary $n \geqq 0$ and arbitrary complex numbers $a_0, \ldots, a_n$;

(b) *if $e^{i\mu}$ is defined by*

$$e^{i\mu} = \sum_{k=0}^{\infty} \frac{i^k}{k!} \mu^k,$$

then $\|e^{i\mu}\| = e^{\|\mu\|}$.

Proof: Since the μ^k are mutually singular $(\mu^0 = \delta_0)$ we have $\|\sum a_k \mu^k\| = \sum |a_k| \cdot \|\mu\|^k$ (Appendix E 2); since $\mu \geqq 0$, $\|\mu^k\| = \|\mu\|^k$; (a) follows.

The series defining $e^{i\mu}$ converges in the norm of $M(G)$, so that (a) implies

$$||e^{i\mu}|| = \lim_{n\to\infty} ||\sum_0^n \frac{i^k}{k!}\mu^k|| = \lim_{n\to\infty} \sum_0^n \frac{1}{k!}||\mu||^k = e^{||\mu||},$$

and the proof is complete.

We note that results analogous to (b) hold of course if the exponential function is replaced by any entire function.

5.3.4. THEOREM. *Suppose G is not discrete. Then*

(a) *$M(G)$ is not symmetric,*

(b) *there exists a real function $\phi \in B(\Gamma)$ such that $\phi(\gamma) \geqq 1$ for all $\gamma \in \Gamma$, but $1/\phi \notin B(\Gamma)$.*

Proof: By Theorem 5.2.6, G contains an independent Cantor set P and (see 5.2.5) there is a non-negative continuous measure μ_1, concentrated on P, with $||\mu_1|| = 1$. If

$$\mu = \tfrac{1}{2}(\mu_1 + \tilde{\mu}_1) \tag{1}$$

then $\mu = \tilde{\mu}$, μ is concentrated on $Q = P \cup (-P)$, $\mu \geqq 0$, and μ is continuous. Put $\sigma = \delta_0 - \mu^2$. Applying 5.3.3, we obtain

$$||\sigma^n|| = ||\sum_{k=0}^{n} \binom{n}{k} (-1)^k \mu^{2k}|| = \sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad (n = 1, 2, 3, \ldots), \tag{2}$$

so that the spectral norm of σ, $\lim ||\sigma^n||^{1/n}$, is 2 (Appendix D6). Hence there is a complex homomorphism h of $M(G)$ such that $|h(\sigma)| = 2$. Since $||\mu|| = 1$ we have $|h(\mu^2)| \leqq 1$, and so the equation

$$|1 - h(\mu^2)| = |h(\sigma)| = 2 \tag{3}$$

is possible only if $h(\mu^2) = -1$ and $h(\sigma) = 2$.

Since $h(\mu)^2 = h(\mu^2)$, we have $h(\mu) = \pm i$. Since $\mu = \tilde{\mu}$, $h(\tilde{\mu}) = h(\mu)$. Hence $h(\tilde{\mu}) \neq \overline{h(\mu)}$, and the conclusion of Section 5.3.1 shows that $M(G)$ is not symmetric.

To prove (b), put $\tau = \delta_0 + \mu^2$. Since $\tilde{\mu} = \mu$, $\hat{\mu}$ is real on Γ, hence $-1 \leqq \hat{\mu} \leqq 1$, and thus $1 \leqq \hat{\tau}(\gamma) \leqq 2$ for all $\gamma \in \Gamma$. But $h(\tau) = 1 + h(\mu^2) = 0$, so that τ^{-1} does not exist in $M(G)$.

This proof is due to Williamson [1]. For $G = R$, (b) was proved by Wiener and Pitt [1].

5.3.5. COROLLARY. *Γ is not a dense subset of Δ, the maximal ideal space of $M(G)$, unless Γ is compact (in which case $\Gamma = \Delta$).*

5.3.6. The proof of Theorem 5.3.2 can be adapted to show that independent sets are thin in terms of Haar measure:

THEOREM. *If P is a compact independent set in a non-discrete LCA group G, and if H is the group generated by P, then $m(H) = 0$, where m is the Haar mesaure of G.*

Proof: If $Q = P \cup (-P)$ and $Q_k = Q_{k-1} + Q$, then $H = \bigcup Q_k$ $(k = 1, 2, 3, \ldots)$ and it is enough to show that $m(Q_k) = 0$ for all k. Suppose $m(Q_k) > 0$ for some fixed k. If χ is the characteristic function of Q_k and $f = \chi * \chi$, then f is continuous, $f(0) = m(Q_k) > 0$, hence $f(x) > 0$ for all x in a neighborhood V of 0. Hence $V \subset Q_k + Q_k = Q_{2k}$.

Let S be the set of all points $(p_1, \ldots, p_{2k+2}) \in G^{2k+2}$ with $p_i \in P$, such that $\sum \varepsilon_i p_i$ lies in V for some choice of $\{\varepsilon_i\}$, $\varepsilon_i = \pm 1$ $(1 \leqq i \leqq 2k+2)$. For each $\{\varepsilon_i\}$, the map $(p_1 \ldots, p_{2k+2}) \to \sum \varepsilon_i p_i$ is continuous, and thus S is a non-empty open set in G^{2k+2}.

But since $V \subset Q_{2k}$, the argument used in the proof of Theorem 5.3.2 shows that S lies in the union of finitely many sets defined by by $p_i \pm p_j = 0$ $(i \neq j)$. Since G is not discrete, these sets.have no interior, and we have reached a contradiction.

5.4. *Multiplicative Extension of Certain Linear Functionals*

5.4.1. Let $M_c(X)$ denote the set of all continuous measures which belong to $M(X)$, where X is a compact Hausdorff space. It is easy to see that $M_c(X)$ is a closed linear subspace of $M(X)$; if X is perfect, $M_c(X)$ is infinite-dimensional. The following theorem, due to Hewitt and Kakutani [1], is an elaboration of an earlier result of Šreider [2]:

THEOREM. *Every non-discrete* LCA *group G contains a Cantor set P such that every linear functional on $M_c(P)$, of norm $\leqq 1$, can be extended to a complex homomorphism of $M(G)$.*

5.4.2. LEMMA. *Suppose P is a compact independent set in G, $\mu_1, \ldots, \mu_r$ are non-negative continuous measures, concentrated on disjoint subsets $E_1, \ldots, E_r$ of P, and $\|\mu_i\| = 1$ $(1 \leqq i \leqq r)$. If $z_1, \ldots, z_r$ are complex numbers satisfying $|z_i| \leqq 1$, then there exists a complex homomorphism h of $M(G)$ such that*

$$h(\mu_i) = z_i \qquad (1 \leqq i \leqq r). \tag{1}$$

Proof: Let $S(n_1, \ldots, n_r) = n_1E_1 + \ldots + n_rE_r$, where nE denotes the set of all $x_1 + \ldots + x_n$ with $x_j \in E$. Consider two fixed measures

$$\lambda = \mu_1{}^{n_1} * \ldots * \mu_r{}^{n_r}, \quad \nu = \mu_1{}^{m_1} * \ldots * \mu_r{}^{m_r}, \tag{1}$$

with $n_1 + \ldots + n_r \geqq m_1 + \ldots + m_r$. They are concentrated on $S(n_1, \ldots, n_r)$ and $S(m_1, \ldots, m_r)$, respectively. Put

$$S = S(n_1, \ldots, n_r) \cap S(m_1, \ldots, m_r). \tag{2}$$

Every $x \in S$ has two representations

$$\begin{aligned}(x_1^1 + \ldots + x_1^{n_1}) + \ldots + (x_r^1 + \ldots + x_r^{n_r})\\ = (y_1^1 + \ldots + y_1^{m_1}) + \ldots + (y_r^1 + \ldots + y_r^{m_r}),\end{aligned} \tag{3}$$

where $x_i^k \in E_i$ and $y_i^k \in E_i$.

Fix $i, j, k, j \neq k$, and let D be the set of all points represented by the left side of (3), with the restriction that $x_i^k = x_i^j$. Then

$$\lambda(D) \leqq (\mu_1^{n_1} \times \ldots \times \mu_r^{n_r})(D') \tag{4}$$

where D' is the set of all points

$$(x_1^1, \ldots, x_1^{n_1}, \ldots, x_r^1, \ldots, x_r^{n_r}) \tag{5}$$

in $G^{n_1+\cdots+n_r}$ such that $x_i^k = x_i^j$; since the measures μ_i are continuous, Fubini's theorem shows that the right side of (4) is 0. Hence $\lambda(D) = 0$.

Thus, if $\lambda(S) > 0$, it follows that for some $x \in S$ the x_i^k appearing on the left side of (3) are all distinct. Since $\sum n_i \geqq \sum m_i$, the independence of P then implies that $\sum n_i = \sum m_i$, that the y_i^k are just a permutation of the x_i^k, and hence that $m_i = n_i$ for $1 \leqq i \leqq r$.

This proves that λ and ν are mutually singular, unless $m_i = n_i$ for $1 \leqq i \leqq r$. Hence if f is any polynomial in r viarables, say

(6) $$f(t_1, \ldots, t_r) = \sum a(n_1, \ldots, n_r) t^{n_1} \ldots t^{n_r},$$

we have

(7) $$\|f(\mu_1, \ldots, \mu_r)\| = \sum |a(n_1, \ldots, n_r)|.$$

Let us now assume that we have a special case, namely

(8) $$|z_i| = 1 \qquad (1 \leqq i \leqq r).$$

We put $\sigma = \delta_0 + \bar{z}_1\mu_1 + \ldots + \bar{z}_r\mu_r$, express σ^n as a polynomial in $\mu_1, \ldots, \mu_r$, and apply (7); the result is

(9) $$\|\sigma^n\| = (r+1)^n \qquad (n = 1, 2, 3, \ldots).$$

Hence the spectral norm of σ is $r+1$, and it follows that there is a complex homomorphism h of $M(G)$ such that

(10) $$|1 + \bar{z}_1 h(\mu_1) + \ldots + \bar{z}_r h(\mu_r)| = |h(\sigma)| = r + 1.$$

Since $|h(\mu_i)| \leqq 1$ for $1 \leqq i \leqq r$, (10) implies that $\bar{z}_i h(\mu_i) = 1$, or

(11) $$h(\mu_i) = z_i \qquad (1 \leqq i \leqq r).$$

To remove the assumption (8), note that if $|z_i| \leqq 1$ then $z_i = \frac{1}{2}(z_i' + z_i'')$, with $|z_i'| = |z_i''| = 1$. The continuity of μ_i shows that $E_i = E_i' \cup E_i''$, where $\mu_i(E_i') = \mu_i(E_i'') = \frac{1}{2}$ and $E_i' \cap E_i''$ is empty. If μ_i', μ_i'' are the restrictions of μ_i to E_i', E_i'', the special case applies to the measures $2\mu_1'$, $2\mu_1''$, $\ldots$, $2\mu_r'$, $2\mu_r''$ and yields a homomorphism h such that

(12) $$2h(\mu_i') = z_i', \qquad 2h(\mu_i'') = z_i'' \qquad (1 \leqq i \leqq r).$$

This is the h whose existence the lemma asserts.

Remark. We could equally well consider measures concentrated on $P \cup (-P)$. The notation would be more complicated ($\pm x_i^k$ in place of x_i^k, etc.) but the idea of the proof is unchanged.

5.4.3. Proof of Theorem 5.4.1. By Theorem 5.2.6, G contains an independent Cantor set P. Let L be a linear functional on $M_c(P)$, with $\|L\| \leqq 1$.

For $\mu \in M_c(P)$ and $\varepsilon > 0$, let $H(\mu, \varepsilon)$ be the set of all complex homomorphisms h of $M(G)$ such that

$$(1) \qquad |h(\mu) - L(\mu)| \leqq \varepsilon.$$

The definition of the Gelfand topology (Appendix D4) shows that each $H(\mu, \varepsilon)$ is a compact subset of the maximal ideal space of $M(G)$. If we can show that the collection of all sets $H(\mu, \varepsilon)$ has the finite intersection property, it will follow that there is an h_0 which belongs to all $H(\mu, \varepsilon)$; this h_0 must satisfy the equation

$$(2) \qquad h_0(\mu) = L(\mu)$$

for every $\mu \in M_c(P)$ and hence furnishes the desired extension of L.

So, suppose $\mu_1, \ldots, \mu_r \in M_c(P)$, $\varepsilon_1 > 0, \ldots, \varepsilon_r > 0$. Put $\sigma = \sum |\mu_i|$, where, we recall, $|\mu_i|$ is the total variation of μ_i. By the Radon-Nikodym theorem (Appendix E9) there are Borel functions f_i such that $d\mu_i = f_i d\sigma$, and there are simple Borel functions g_i on P such that

$$(3) \qquad \int_P |f_i - g_i| d\sigma < \varepsilon_i/2 \qquad (1 \leqq i \leqq r).$$

If $d\tau_i = g_i d\sigma$, it follows that $\|\tau_i - \mu_i\| < \varepsilon_i/2$.

The set P is the union of finitely many disjoint Borel sets $A_1, \ldots, A_n$ such that g_i is constant (equal to c_{ik}) on A_k, for $1 \leqq i \leqq r$, $1 \leqq k \leqq n$. Let σ_k be the restriction of σ to A_k. Since $\|L\| \leqq 1$, Lemma 5.4.2 assures the existence of an h such that $h(\sigma_k) = L(\sigma_k)$, for $k = 1, \ldots, n$. Since $\tau_i = \sum_k c_{ik}\sigma_k$, we have

$$(4) \quad h(\tau_i) = \sum_k c_{ik} h(\sigma_k) = \sum_k c_{ik} L(\sigma_k) = L(\tau_i) \qquad (1 \leqq i \leqq r),$$

and since $\|L\| \leqq 1$ and $\|h\| = 1$, we conclude that

$$(5) \quad |h(\mu_i) - L(\mu_i)| \leqq |h(\mu_i - \tau_i)| + |L(\tau_i - \mu_i)| \leqq 2\|\mu_i - \tau_i\| < \varepsilon_i.$$

The sets $H(\mu_i, \varepsilon_i)$, $1 \leqq i \leqq r$, thus have non-empty intersection, and the proof is complete.

5.4.4. As an illustration, suppose P is an independent Cantor set in D_2 and μ is a real continuous measure on P, with $\|\mu\| = 1$.

Then the Fourier-Stieltjes transform of μ is real, but the range of the Gelfand transform of μ (i. e., the spectrum of μ; see Appendix D6) *consists of the whole closed unit disc.*

Since each $x \in D_2$ has order 2, the characters of D_2 are homomorphisms into Z_2, hence have only ± 1 as values. Thus $\hat{\mu}(\gamma)$ is real for all γ in the dual of D_2. But if $|z| \leqq 1$, there exists a linear functional L on $M_c(P)$, such that $||L|| \leqq 1$ and $L(\mu) = z$; by Theorem 5.4.1, L extends to a complex homomorphism h of $M(G)$. Thus $h(\mu) = z$, and since z was arbitrary, the proof is complete.

5.4.5. Theorem 5.4.1 is probably the first known example of an infinite-dimensional subspace X of a Banach algebra A such that every linear functional on X, of norm not exceeding 1, coincides on X with a complex homomorphism of A. However, this phenomenon may be fairly common.

For instance, let $A = L^\infty(Z)$, the algebra of all bounded functions on the integers. Let α run through an index set which has the power of the continuum. There exist real numbers a_α, b_α such that the set consisting of all a_α, all b_α, and π, is independent. Let X be the linear space generated by the functions f_α, where

(1) $$f_\alpha(n) = \{\exp\{ina_\alpha\}\cos nb_\alpha \qquad (n \in Z),$$

and let L be a linear functional on X, such that $||L|| \leqq 1$.

By Kronecker's theorem (Section 5.1.2) there corresponds to each finite set $\alpha_1, \ldots, \alpha_k$ and to each $\varepsilon > 0$ an integer n such that

(2) $$|f_{\alpha_r}(n) - L(f_{\alpha_r})| < \varepsilon \qquad (1 \leqq r \leqq k).$$

If $H(\alpha_1, \ldots, \alpha_k, \varepsilon)$ is the set of all complex homomorphisms h of A such that $|h(f_{\alpha_r}) - L(f_{\alpha_r})| < \varepsilon$ for $1 \leqq r \leqq k$, it follows that these sets have the finite intersection property, and the proof is completed just as in 5.4.3.

5.5. Transforms of Measures on Kronecker Sets

5.5.1. LEMMA. *Suppose X is a compact Hausdorff space and U is the set of all $f \in C(X)$ with $|f| = 1$.*

(a) *If $\mu \in M(X)$, then* $||\mu|| = \sup \left|\int_X f\,d\mu\right| \qquad (f \in U).$

(b) *If* $\mu \in M_c(X)$, *the set of all numbers* $\int_X f\,d\mu$, *where* f *ranges over* U, *is dense in the disc* $|z| \leqq ||\mu||$.

Proof: We first prove (b). Suppose $||\mu|| = 1$, without loss of generality, and fix z, $|z| \leqq 1$, and $\varepsilon > 0$. Then $z = z_1 + z_2$, where $|z_1| = |z_2| = \frac{1}{2}$. Since μ is continuous, X is the union of disjoint sets X_1, X_2 such that $|\mu|(X_1) = |\mu|(X_2) = \frac{1}{2}$. There is a Borel function ϕ on X, with $|\phi| = 1$, such that $\phi\,d\mu = d|\mu|$; this follows from the Radon-Nikodym theorem. It follows that there are Borel functions g_k on X_k, $0 \leqq g_k \leqq 2\pi$, which satisfy the equations

$$\int_{X_k} e^{ig_k}\,d\mu = z_k \qquad (k = 1, 2). \tag{1}$$

If $g = g_k$ on X_k, then g is a Borel function on X, $0 \leqq g \leqq 2\pi$, and

$$\int_X e^{ig}\,d\mu = z. \tag{2}$$

By Lusin's theorem (Appendix E8) there exists $h \in C(X)$, $0 \leqq h \leqq 2\pi$, such that $h = g$ in the complement of a set E with $|\mu|(E) < \varepsilon/2$. Putting $f = e^{ih}$, we obtain

$$\left|\int_X f\,d\mu - z\right| = \left|\int_E (e^{ih} - e^{ig})\,d\mu\right| \leqq 2|\mu|(E) < \varepsilon, \tag{3}$$

and (b) follows.

To prove (a), note that $||\mu|| = \int_X e^{ig}\,d\mu$ for some Borel function g, $0 \leqq g \leqq 2\pi$, and apply Lusin's theorem, as above.

5.5.2. THEOREM. *Suppose* P *is a compact Kronecker set in* G.

(a) *If* $\mu \in M(P)$, *then* $||\hat{\mu}||_\infty = ||\mu||$.

(b) *If* $\mu \in M_c(P)$, *then* $\hat{\mu}$ *maps* Γ *onto a dense subset of the disc* $|z| \leqq ||\mu||$.

Proof: By Lemma 5.5.1(a) there exists $f \in C(P)$, with $|f| = 1$, such that $\int f\,d\mu$ differs from $||\mu||$ by as little as we please. Since P is a Kronecker set, $|f(x) - (x, \gamma)|$ can be made arbitrarily small for all $x \in P$ by proper choice of γ. Hence the difference between $|\hat{\mu}(\gamma)|$ and $||\mu||$ can be made arbitrarily small, and this proves (a). Part (b) follows similarly from 5.5.1(b).

It is not known whether property (a) characterizes Kronecker sets.

5.5.3. Theorem. *Suppose P is a compact set of type K_q in D_q, and $\mu \in M(P)$. Then* $\|\hat{\mu}\|_\infty \geqq \frac{1}{2}\|\mu\|$.

Proof: Assume $\|\mu\| = 1$. There exist Borel functions α, g, and $h = e^{i\alpha}g$ on P such that $|g| = 1$, $g\,d\mu = d|\mu|$, $-\pi/q \leqq \alpha(x) \leqq \pi/q$, and h maps P into Z_q. Hence

$$(1) \qquad \left|\int h\,d\mu\right| = \left|\int e^{i\alpha} d|\mu|\right| \geqq \int \cos\alpha\, d|\mu| \geqq \cos(\pi/q).$$

There are continuous functions f_n on P whose range lies in Z_q, such that $\int f_n d\mu \to \int h\,d\mu$. Since P is of type K_q, each f_n is the restriction to P of a continuous character of G, and so

$$(2) \qquad \|\hat{\mu}\|_\infty \geqq \cos(\pi/q).$$

This proves the theorem for $q \geqq 3$.

If $q = 2$, put $\mu = \mu_1 + i\mu_2$ (μ_j real). Since the characters are now real, $\hat{\mu} = \hat{\mu}_1 + i\hat{\mu}_2$ and $\hat{\mu}_1$, $\hat{\mu}_2$ are real. Fix j ($j = 1$ or 2) so that $\|\mu_j\| \geqq 1/2$. There is a Borel function h, with values ± 1, such that $h\,d\mu_j = d|\mu_j|$; hence $\int h\,d\mu_j \geqq 1/2$. It follows, as above, that $\|\hat{\mu}_j\|_\infty \geqq 1/2$. Since $|\hat{\mu}(\gamma)| \geqq |\hat{\mu}_j(\gamma)|$, the proof is complete.

5.6. Helson Sets

5.6.1. We call a compact set P in G a Helson set if every $F \in C(P)$ is the restriction to P of a member of $A(G)$, i.e., if to every $F \in C(P)$ there corresponds a function $f \in L^1(\Gamma)$ such that

$$(1) \qquad F(x) = \hat{f}(x) = \int_\Gamma f(\gamma)(x, \gamma)d\gamma \qquad (x \in P).$$

The reason for this terminology is Theorem 5.6.10 (Helson [6]). Carleson [1] had previously studied sets with a similar property relative to absolutely convergent power series. See Wik [1].

Note that $\hat{f}$, as defined by (1), is really the *inverse* transform of f, since we have written (x, γ) in (1), and not $(-x, \gamma)$. This choice is imposed by the inversion theorem.

5.6.2. For any compact set P, let $I(P)$ be the set of all functions in $L^1(\Gamma)$ such that $\hat{f}(x) = 0$ for all $x \in P$. It is clear that $I(P)$ is a closed ideal in $L^1(\Gamma)$. For $f \in L^1(\Gamma)$, let πf be the restriction of $\hat{f}$

to P. Since $\pi f = 0$ for all $f \in I(P)$, π may be regarded as a linear transformation of the quotient space $L^1(\Gamma)/I(P)$ into $C(P)$. The definition of the quotient norm (Appendix C2) shows that *π is a bounded linear transformation*, with $\|\pi\| \leq 1$; *π is one-to-one on* $L^1(\Gamma)/I(P)$, and the Stone-Weierstrass theorem shows *that the range of π is dense in* $C(P)$. Thus π satisfies the hypotheses of the theorem on adjoints (Appendix C11). Let us compute π^*.

The dual space of $L^1(\Gamma)/I(P)$ is the space $\Phi(P)$ which consists of all $\phi \in L^\infty(\Gamma)$ such that

$$\int_\Gamma f(\gamma)\phi(-\gamma)d\gamma = 0 \qquad (f \in I(P)). \tag{1}$$

If $\mu \in M(P)$ and $\phi = \pi^*\mu$, then $\phi \in \Phi(P)$, and the definition of the adjoint of a linear transformation shows that

$$\int_\Gamma f(\gamma)\phi(-\gamma)d\gamma = \int_P \hat{f}(x)d\mu(x) \qquad (f \in L^1(\Gamma)). \tag{2}$$

The right side of (2) is equal to

$$\begin{aligned}\int_P\int_\Gamma f(\gamma)(x,\gamma)d\gamma d\mu(x) &= \int_\Gamma f(\gamma)\int_P(x,\gamma)d\mu(x)d\gamma \\ &= \int_\Gamma f(\gamma)\hat{\mu}(-\gamma)d\gamma.\end{aligned} \tag{3}$$

Comparison of (2) and (3) shows that $\phi = \hat{\mu}$ almost everywhere. Since we identify functions in $L^\infty(\Gamma)$ which coincide almost everywhere, we have

$$\pi^*\mu = \hat{\mu} \qquad (\mu \in M(P)). \tag{4}$$

By definition, P is a Helson set if and only if the range of π covers $C(P)$. Theorem 11 of Appendix C therefore yields the following equivalences:

5.6.3. Theorem. *The following three properties of a compact set P in a* LCA *group G are equivalent:*

(a) *P is a Helson set.*

(b) *$\|\mu\|$ and $\|\hat{\mu}\|_\infty$ are equivalent norms on $M(P)$.*

(c) *Each $\phi \in \Phi(P)$ is (equal almost everywhere to) the Fourier-Stieltjes transform of a $\mu \in M(P)$.*

Property (c) is of interest in connection with the problem of spectral synthesis (see Theorem 7.8.8).

5.6.4. If P is a Helson set, then the transformation π introduced in 5.6.2 has a continuous inverse (Appendix C6), and the definition of the quotient norm implies that there is a constant K with the following property: if $F \in C(P)$, there exists $f \in L^1(\Gamma)$ such that $F = \hat{f}$ on P and $\|f\|_1 \leqq K\|F\|_\infty$.

It is interesting that an apparently much weaker interpolation property is in fact equivalent to this:

5.6.5. THEOREM. *Suppose P is compact in G, $\delta > 0$, $K < \infty$, and suppose that to every $F \in C(P)$ with $|F| = 1$ there corresponds a function $f \in L^1(\Gamma)$ such that $\|f\|_1 \leqq K$ and such that*

$$\sup_{x \in P} |\hat{f}(x) - F(x)| < 1 - \delta. \tag{1}$$

Then P is a Helson set.

Proof: If F and f satisfy these conditions, and if $\mu \in M(P)$, then

$$\left|\int_P \hat{f}(x)d\mu(x)\right| = \left|\int_\Gamma f(\gamma)\hat{\mu}(-\gamma)d\gamma\right| \leqq \|f\|_1\|\hat{\mu}\|_\infty \leqq K\|\hat{\mu}\|_\infty, \tag{2}$$

so that

$$\left|\int_P F\,d\mu\right| \leqq \int_P |F - \hat{f}|d|\mu| + \left|\int_P \hat{f}d\mu\right| \leqq (1-\delta)\|\mu\| + K\|\hat{\mu}\|_\infty. \tag{3}$$

The supremum of the left side of (3) is $\|\mu\|$, by Lemma 5.5.1. Hence

$$\delta\|\mu\| \leqq K\|\hat{\mu}\|_\infty \qquad (\mu \in M(P)), \tag{4}$$

which shows that P has property (b) of Theorem 5.6.3.

5.6.6. The existence of perfect Helson sets in every non-discrete LCA group G is now easily established:

THEOREM. *Every compact Kronecker set is a Helson set, and so is every compact set of type K_q in D_q.*

Proof: This follows immediately from Theorems 5.5.2(a), 5.5.3, and 5.6.3(b).

5.6.7. THEOREM. *If P is a countable, compact, independent set in G, then P is a Helson set.*

Proof: Let $E = \{x_1, \ldots, x_n\}$ be a finite subset of P, let $x_1, \ldots, x_r$ be those points of E (if any) whose order is 2, and let μ be a measure concentrated on E, with $\|\mu\| = 1$. We will show that there are numbers $a_j \in S(x_j)$, (this notation was introduced in 5.1.3) such that

$$|\sum_1^n a_j\mu(\{x_j\})| \geqq \tfrac{1}{2}. \tag{1}$$

Since neither $\|\mu\|$ nor the left side of (1) are changed if μ is multiplied by a scalar of absolute value 1, we may assume that

$$\sum_1^r |\operatorname{Re} p_j| \geqq \tfrac{1}{2} \sum_1^r |p_j|, \tag{2}$$

where $p_j = \mu(\{x_j\})$. For $1 \leqq j \leqq r$, put $a_j = \pm 1$ so that $a_j \operatorname{Re} p_j \geqq 0$. For $r + 1 \leqq j \leqq n$, choose b_j, $|b_j| = 1$, so that $b_j p_j \geqq 0$, and choose θ_j, $-\pi/3 \leqq \theta_j \leqq \pi/3$, so that $a_j = e^{i\theta_j} b_j \in S(x_j)$. Then

$$\operatorname{Re} \sum_j a_j p_j = \sum_1^r a_j \operatorname{Re} p_j + \sum_{r+1}^n \cos\theta_j |p_j| \geqq \tfrac{1}{2} \sum_1^n |p_j| = \tfrac{1}{2},$$

and (1) is proved.

By Theorem 5.1.3, this means that

$$\|\hat{\mu}\|_\infty \geqq \tfrac{1}{2}\|\mu\| \tag{3}$$

for every measure μ concentrated on E. Since every $\mu \in M(P)$ is a limit, in the norm of $M(P)$, of measures concentrated on finite subsets of P, the inequality (3) persists for all $\mu \in M(P)$. Thus P has property (b) of Theorem 5.6.3, and the proof is complete.

5.6.8. It is not true that every countable compact subset of a LCA group is a Helson set. For instance, a Helson set P on the line cannot contain arbitrarily long arithmetic progressions (Section 6.8); a stronger result (Kahane and Salem [2]) is that no arithmetic progression of N terms ($N \geqq 2$) contains more than $A \log N$ terms of P, where A is a constant depending only on P.

It does not seem to be known whether the union of two Helson sets is a Helson set.

5.6.9. We now insert a theorem which concerns the "mean-value" of a Fourier-Stieltjes transform. Suppose $\{V_\alpha\}$ is a neighborhood base of 0 in G, associate with each V_α a continuous positive-definite function f_α whose compact support lies in V_α, such that $f_\alpha(0) = 1$, and define

$$A_\alpha(\mu) = \int_\Gamma \hat{f}_\alpha(\gamma)|\hat{\mu}(\gamma)|^2\, d\gamma \qquad (\mu \in M(G)). \tag{1}$$

Since $\hat{f}_\alpha \geqq 0$ and $\int_\Gamma \hat{f}_\alpha(\gamma)d\gamma = 1$, $A_\alpha(\mu)$ may be regarded as an average of $|\hat{\mu}|^2$.

We say that $\lim_\alpha A_\alpha(\mu) = A$ if to every $\epsilon > 0$ there exists a neighborhood V of 0 in G such that $|A_\alpha(\mu) - A| < \varepsilon$ for all $V_\alpha \subset V$.

THEOREM. *For any $\mu \in M(G)$, we have*

$$\lim_\alpha A_\alpha(\mu) = \sum_{x \in G} |\mu(\{x\})|^2. \tag{2}$$

Note that at most countably many terms of this sum are different from 0. For $G = T$, the theorem is due to Wiener; see Zygmund [1], vol. I, p. 108.

Proof: Put $\sigma = \mu * \tilde{\mu}$, so that $\sigma = |\hat{\mu}|^2$. If we apply the inversion formula to f_α, (1) becomes

$$\int_G f_\alpha(-x)d\sigma(x) = A_\alpha(\mu), \tag{3}$$

and this shows clearly that $\lim A_\alpha(\mu) = \sigma(\{0\})$. Since $\sigma(E) = \int_G \overline{\mu(x - E)}d\mu(x)$, we have

$$\sigma(\{0\}) = \int_G \overline{\mu(\{x\})}d\mu(x) = \sum_{x \in G} |\mu(\{x\})|^2, \tag{4}$$

and (2) follows.

COROLLARIES. (a) *If $\hat{\mu} \in C_0(\Gamma)$, then μ is continuous.*

(b) *If $|\hat{\mu}| = 1$, then $\sum |\mu(\{x\})|^2 = 1$* (Helson [3], Glicksberg [1]).

To prove (a), choose $\varepsilon > 0$, let K be a compact set in Γ such that $|\hat{\mu}| < \varepsilon$ in the complement K' of K, and write (1) in the form

$\int_K + \int_{K'}$. Since $|\hat{f}_\alpha| \leq m_G(V_\alpha)$, the first integral is no larger than $m_G(V_\alpha) \cdot m_\Gamma(K) \cdot \|\mu\|^2$, and the second is less than ε^2. Hence $\lim A_\alpha(\mu) = 0$ and so $\mu(\{x\}) = 0$ for every $x \in G$.

Part (b) is an immediate consequence of the theorem.

5.6.10. THEOREM. *Suppose P is a Helson set in a* LCA *group G, $\sigma \in M(P)$, and $\sigma \neq 0$. Then $\hat{\sigma}$ is not in $C_0(\Gamma)$. That is to say, $\hat{\sigma}$ does not vanish at infinity.*

Proof: Let M_0 be the set of all $\mu \in M(P)$ such that $\hat{\mu} \in C_0(\Gamma)$. Since $\|\mu\|$ and $\|\hat{\mu}\|_\infty$ are equivalent norms on $M(P)$ (Theorem 5.6.3), the map $\mu \to \hat{\mu}$ carries $M_0(P)$ onto a closed subspace of $C_0(\Gamma)$ and has a continuous inverse. It follows that every $T \in M_0^*$ (the dual space of M_0) is of the form

$$T\mu = \int_\Gamma \hat{\mu}\, d\lambda \qquad (\mu \in M_0) \tag{1}$$

for some $\lambda \in M(\Gamma)$. But $\int_\Gamma \hat{\mu}\, d\lambda = \int_G \hat{\lambda}\, d\mu$, so that every $T \in M_0^*$ is of the form

$$T\mu = \int_P f\, d\mu \qquad (\mu \in M_0), \tag{2}$$

where $f \in C(P)$.

If ϕ is a bounded Borel function on P, then $\int \phi\, d\mu$ is a bounded linear functional on M_0, and it follows that there exists $f \in C(P)$ such that

$$\int_P (\phi - f)\, d\mu = 0 \qquad (\mu \in M_0). \tag{3}$$

If $d\mu_\gamma(x) = (x, \gamma)\, d\mu(x)$, then $\hat{\mu}_\gamma$ is a translate of $\hat{\mu}$, hence $\hat{\mu}_\gamma \in C_0(\Gamma)$, and so

$$\int_P (\phi(x) - f(x))\, (x, \gamma)\, d\mu(x) = 0 \qquad (\mu \in M_0,\ \gamma \in \Gamma). \tag{4}$$

The uniqueness theorem for Fourier-Stieltjes transforms thus implies: *to every bounded Borel function ϕ on P there exists $f \in C(P)$ such that*

$$\phi\, d\mu = f\, d\mu \qquad (\mu \in M_0). \tag{5}$$

Suppose now that $\sigma \in M_0$ and $\sigma \neq 0$. By Theorem 5.6.9, σ is continuous, and its support S is therefore a perfect subset of P. If S contains two disjoint open sets V_1 and V_2 (open relative to S!) which have a common boundary point, and if ϕ is the characteristic function of V_1, then the measure $\phi d\sigma$ is not equal to $f d\sigma$ for any $f \in C(P)$, since $|\sigma|(V) > 0$ for every non-empty relatively open subset of S. This contradicts (5).

The proof will thus be complete if we can show that S contains two disjoint relatively open sets whose closures intersect; i.e., we have to show that S is not "extremally disconnected" (see Kelley [1]). Since $\hat{\sigma} \in C_0(\Gamma)$, $\hat{\sigma}$ vanishes outside an open subgroup Λ of Γ which is generated by a compact neighborhood of 0 in Γ. The annihilator H of Λ is compact, and since $\sigma = \sigma * m_H$, it is clear that $\sigma(E) = \sigma(E - x)$ for all Borel sets E in G and all $x \in H$. Thus S is a union of cosets of H. This implies that H is a Helson set in G, and hence (Theorem 2.7.4) $A(H) = C(H)$. But then H is finite (Theorem 4.6.8), Λ has finite index in Γ, Γ is itself generated by a compact neighborhood of 0, and so every point of G has a countable neighborhood base (Theorem 1.2.6). If $x \in S$, we now see that there is a simple countable sequence $\{x_n\}$ in S, with $x_n \neq x$ and $x_n \neq x_m$ if $n \neq m$, such that $\lim x_n = x$, and there exist disjoint open neighborhoods E_n of x_n. Put $V_1 = S \cap \bigcup E_{2n-1}$, $V_2 = S \cap \bigcup E_{2n}$. Since these sets have x as a common boundary point, the proof is complete.

5.6.11. It is known that there exists an independent Cantor set on the line which carries a positive continuous measure whose Fourier-Stieltjes transform vanishes at infinity (Rudin [19]). *Hence there exist independent Cantor sets which are not Helson sets and, a fortiori, are not Kronecker sets.*

5.7. Sidon Sets

5.7.1. So far we have concentrated on compact sets, with particular emphasis on perfect sets, although the definitions of Kronecker sets and of Helson sets can easily be extended to closed sets in LCA groups. The new phenomena which are caused by the loss of compactness are most conveniently studied by considering closed discrete sets, and we shall actually restrict ourselves to subsets of discrete groups.

For the remainder of this chapter, G will therefore be a compact abelian group and E will be a subset of its dual Γ. A function $f \in L^1(G)$ will be called an *E-function* if $\hat{f}(\gamma) = 0$ for all γ not in E. A trigonometric polynomial on G which is an E-function will be called an *E-polynomial.*

5.7.2. We say that E is a *Sidon set* if there is a constant B (depending on E) such that

$$\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| \leqq B\|f\|_\infty \tag{1}$$

for every E-polynomial f on G. We shall see that Sidon sets are the discrete analogues of Helson sets. The results which follow are generalizations (from the case $G = T$, $\Gamma = Z$) of theorems about lacunary trigonometric series. As references we cite Sidon [1], [2], Zygmund [1] (vol. I, pp. 208, 215, 247; vol. II, p. 131), Kaczmarz and Steinhaus [1], Stečkin [1], Hewitt and Zuckerman [3], and Rudin [17].

5.7.3. Theorem. *Each of the following five properties of a set E in the discrete group Γ implies the others:*

(a) *E is a Sidon set.*

(b) *Every bounded E-function f has $\sum |\hat{f}(\gamma)| < \infty$.*

(c) *Every continuous E-function f has $\sum |\hat{f}(\gamma)| < \infty$*

(d) *To every bounded function ϕ on E there corresponds a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$.*

(e) *To every $\phi \in C_0(E)$ there corresponds a function $f \in L^1(G)$ such that $\hat{f}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$.*

Proof: Suppose E is a Sidon set, with constant B, and f is a bounded E-function. Given $\varepsilon > 0$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, Theorem 2.6.8 shows that there is a trigonometric polynomial k on G such that $\|k\|_1 < 1 + \varepsilon$ and $\hat{k}(\gamma_i) = 1$ $(1 \leqq i \leqq n)$. Since $f * k$ is an E-polynomial, we have

$$\sum_{i=1}^{n} |\hat{f}(\gamma_i)| = \sum_{i=1}^{n} |(\hat{f}\hat{k})(\gamma_i)| \leqq \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)\hat{k}(\gamma)| \leqq B\|f * k\|_\infty. \tag{1}$$

and since $\|f * k\|_\infty \leqq \|f\|_\infty \|k\|_1$, we conclude:

(2) $$\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| \leqq B\|f\|_\infty.$$

Thus (a) implies (b). It is trivial that (b) implies (c).

Let $C_E(G)$ be the set of all continuous E-functions on G; $C_E(G)$ is a closed subspace of $C(G)$, and if E has property (c), then the map $f \to \hat{f}$ is an isomorphism of $C_E(G)$ onto $L^1(E)$. Since $\|f\|_\infty \leqq \|\hat{f}\|_1$, the two norms are equivalent on $C_E(G)$ (Appendix C6) so that E is a Sidon set.

Having proved the equivalence of (a), (b), (c), we now show that these conditions imply (d): If E is a Sidon set and $|\phi(\gamma)| \leqq 1$ for all $\gamma \in E$, then the map

(3) $$f \to \sum_{\gamma \in E} \hat{f}(\gamma)\phi(\gamma)$$

is a bounded linear functional on $C_E(G)$ of norm $\leqq B$ which may be extended to $C(G)$. by the Hahn-Banach theorem. Hence there is a measure $\mu \in M(G)$ such that $\|\mu\| \leqq B$ and

(4) $$\sum \hat{f}(\gamma)\phi(\gamma) = \int_G f(-x)d\mu(x) \qquad (f \in C_E(G)).$$

If $\gamma \in E$ and $f(x) = (x, \gamma)$, (4) shows that $\phi(\gamma) = \hat{\mu}(\gamma)$, and so E has property (d).

Suppose (d) holds and M' is the space of all $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = 0$ for all $\gamma \in E$. Then $M(G)/M'$ is continuously isomorphic to the space of all bounded functions on E, and the definition of the quotient norm shows that to each bounded function ϕ on E there corresponds a measure $\mu \in M(G)$, with $\hat{\mu} = \phi$ on E, such that $\|\mu\| \leqq B\|\phi\|_\infty$, where B is a constant depending only on E. If now $\phi \in C_0(E)$ and $|\phi| \leqq 1$, let E_n be the (finite) subset of E at which $2^{-n} < |\phi(\gamma)| \leqq 2^{1-n}$ $(n = 1, 2, 3, \ldots)$ and define $\phi_n(\gamma) = \phi(\gamma)$ on E_n, $\phi_n(\gamma) = 0$ at all other points of E. There are measures $\mu_n \in M(G)$ such that $\hat{\mu}_n = \phi_n$ on E and $\|\mu_n\| \leqq 2^{1-n}B$, and there are trigonometric polynomials k_n on G such that $\hat{k}_n = 1$ on E_n and $\|k_n\|_1 \leqq 2$. Put

(5) $$f = \sum_{n=1}^{\infty} k_n * \mu_n.$$

Since $k_n * \mu_n$ is a trigonometric polynomial whose L^1-norm is less than $2^{2-n}B$, $f \in L^1(G)$. Also, if $\gamma \in E_n$, then

$$\hat{f}(\gamma) = \hat{k}_n(\gamma)\hat{\mu}_n(\gamma) = \hat{\mu}_n(\gamma) = \phi(\gamma), \tag{6}$$

so that $\hat{f} = \phi$ on $E_1 \cup E_2 \cup \ldots$. At all other points of E, $\hat{f}(\gamma) = 0 = \phi(\gamma)$. Thus (d) implies (e).

If (e) holds we see, as in the preceding paragraph, that to every $\phi \in C_0(E)$ there is an $f \in L^1(G)$ such that $\hat{f} = \phi$ on E and $\|f\|_1 \leqq B\|\phi\|_\infty$. Let g be an E-polynomial, define ϕ so that $\phi(\gamma)\hat{g}(\gamma) = |\hat{g}(\gamma)|$ if $\hat{g}(\gamma) \neq 0$, and $\phi(\gamma) = 0$ at all other points of E, and choose $f \in L^1(G)$ so that $\hat{f} = \phi$ on E and $\|f\|_1 \leqq B$. Then

$$\sum |\hat{g}(\gamma)| = \sum \hat{f}(\gamma)\hat{g}(\gamma) = (f * g)(0) \leqq \|f\|_1\|g\|_\infty \leqq B\|g\|_\infty, \tag{7}$$

and hence (a) holds.

This completes the proof.

5.7.4. THEOREM. *A set E in the discrete group Γ is a Sidon set if there is a constant $\delta > 0$ with the following property: To every function ϕ on E with $\phi(\gamma) = \pm 1$ there corresponds a measure $\mu \in M(G)$ such that*

$$\sup_{\gamma \in E} |\hat{\mu}(\gamma) - \phi(\gamma)| \leqq 1 - \delta. \tag{1}$$

In contrast to Theorem 5.6.5, no bound on $\|\mu\|$ is needed here. Comparison with property (d) of Theorem 5.7.3 leads to the following interesting dichotomy: *Either every bounded function on E coincides on E with a Fourier-Stieltjes transform, or there is a function ϕ on E, with $\phi(\gamma) = \pm 1$, such that*

$$\sup_{\gamma \in E} |\hat{\mu}(\gamma) - \phi(\gamma)| = 1 \tag{2}$$

for every $\mu \in M(G)$.

Proof: Let f be a continuous E-function with $\hat{f}$ real, and define ϕ on Γ, so that $\phi = \pm 1$ and $\phi\hat{f} = |\hat{f}|$. By hypothesis, there is a measure $\mu \in M(G)$ which satisfies (1); if $\sigma = \frac{1}{2}(\mu + \tilde{\mu})$, σ also satisfies (1), since $\hat{\sigma}$ is the real part of $\hat{\mu}$, and we have

$$|\hat{f}\hat{\sigma} - |\hat{f}|| = |\hat{f}||\hat{\sigma} - \phi| \leqq (1 - \delta)|\hat{f}|. \tag{3}$$

Hence, setting $g = f * \sigma$, we have

$$\hat{g} = \hat{f}\hat{\sigma} \geqq \delta|\hat{f}|. \tag{4}$$

Corresponding to any choice of $\gamma_1, \ldots, \gamma_n \in \Gamma$ there is a trigonometric polynomial k on G such that $\|k\|_1 < 2$, $\hat{k} \geqq 0$, and $\hat{k}(\gamma_i) = 1$ $(1 \leqq i \leqq n)$. Then $k * g$ is a trigonometric polynomial, and

$$\begin{aligned}\delta \sum_1^n |\hat{f}(\gamma_i)| \leqq \sum_1^n \hat{k}(\gamma_i)\hat{g}(\gamma_i) &\leqq \sum_{\gamma \in \Gamma} \hat{k}(\gamma)\hat{g}(\gamma) = (k * g)(0) \\ &\leqq \|k\|_1\|g\|_\infty \leqq 2\|\sigma\| \cdot \|f\|_\infty.\end{aligned}$$

Hence $\sum |\hat{f}(\gamma)| < \infty$.

If $\hat{f}$ is not real, put $f_1 = f + \tilde{f}$, $f_2 = i(f - \tilde{f})$. Then f_1 and f_2 are continuous E-functions, $\hat{f}_1$ and $\hat{f}_2$ are real, and $2f = f_1 - if_2$. It follows that E has property (c) of Theorem 5.7.3.

This completes the proof. We note that the introduction of the polynomials k amounts to applying a summability method to the Fourier series of g.

5.7.5. We shall now apply the preceding theorem to exhibit a large class of infinite Sidon sets in any infinite discrete abelian group Γ. It is clear that E is a Sidon set if and only if every countable subset of E is a Sidon set. Hence we may restrict our attention to countable sets E, and therefore also to countable groups Γ.

If $E \subset \Gamma$, let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be an enumeration of the elements of E, and for any $\gamma \in \Gamma$ and any positive integer s, let $R_s(E, \gamma)$ be the number of representations of γ in the form

$$\gamma = \pm\gamma_{n_1} \pm\gamma_{n_2} \pm \cdots \pm \gamma_{n_s} \qquad (n_1 < n_2 < \ldots < n_s). \tag{1}$$

THEOREM. *Suppose $E \subset \Gamma$ and E satisfies the following conditions:*

(a) *If $\gamma \in E$ and $2\gamma \neq 0$, then $-\gamma \notin E$.*

(b) *There is a constant B and a decomposition of E into a union of disjoint sets $E_1, \ldots, E_t$ such that*

$$R_s(E_j, \gamma) \leqq B^s \qquad (1 \leqq j \leqq t;\ s = 1, 2, 3, \ldots) \tag{2}$$

for all $\gamma \in E$ and for $\gamma = 0$.

Then E is a Sidon set.

Proof: Without loss of generality we assume that $0 \notin E$. Put $\beta = (3tB^2)^{-1}$, and let ϕ be an arbitrary function on E such that $\phi(\gamma) = \pm \beta$.

Fix j $(1 \leqq j \leqq t)$, let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be an enumeration of the elements of E_j, put

$$(3) \quad f_i(x) = \begin{cases} 1+\phi(\gamma_i)(x, \gamma_i)+\phi(\gamma_i)(x, -\gamma_i) & \text{if } 2\gamma_i \neq 0, \\ 1+\phi(\gamma_i)(x, \gamma_i) & \text{if } 2\gamma_i = 0, \end{cases}$$

and define

$$(4) \qquad P_N(x) = \prod_{i=1}^{N} f_i(x) \qquad (x \in G;\ N = 1, 2, 3, \ldots).$$

Multiplying out, we see that $P_N(x)$ equals

$$(5) \quad 1 + \sum_{i=1}^{N} \phi(\gamma_i)(x, \gamma_i) + \sum_{\substack{i=1 \\ 2\gamma_i \neq 0}}^{N} \phi(\gamma_i)(x, -\gamma_i) + \sum_{\gamma \in \Gamma} c_N(\gamma)(x, \gamma),$$

where

$$(6) \qquad |c_N(\gamma)| \leqq \sum_{s=2}^{N} \sum |\phi(\gamma_{n_1}) \cdots \phi(\gamma_{n_s})|;$$

the inner sum extends over all $\gamma_{n_1}, \ldots, \gamma_{n_s}$ which satisfy (1) and hence has at most B^s terms if $\gamma \in E$ or if $\gamma = 0$. Hence

$$(7) \quad |c_N(\gamma)| \leqq \sum_{2}^{\infty} B^s\beta^s = \frac{B^2\beta^2}{1 - B\beta} \leqq \frac{1}{6t^2B^2} \qquad (\gamma \in E,\ \gamma = 0).$$

Since $\beta < \frac{1}{2}$, $P_N(x) \geqq 0$, and so

$$(8) \quad ||P_N||_1 = 1 + c_N(0) \leqq 1 + \frac{1}{6t^2B^2} \qquad (N = 1, 2, 3, \ldots)$$

by (7); in particular $\{||P_N||_1\}$ is bounded, and a subsequence of $\{P_N\}$ therefore converges, in the weak*-topology of $M(G)$, to a measure $\mu_j \in M(G)$; (5) and (7) imply that

$$(9) \qquad \begin{cases} |\hat{\mu}_j(\gamma_i) - \phi(\gamma_i)| \leqq (6t^2B^2)^{-1} & (\gamma_i \in E_j), \\ \qquad\qquad |\hat{\mu}_j(\gamma)| \leqq (6t^2B^2)^{-1} & (\gamma \in E,\ \gamma \notin E_j). \end{cases}$$

We now put $\mu = \mu_1 + \ldots + \mu_t$, add the inequalities (9), and obtain

$$(10) \qquad |\hat{\mu}(\gamma) - \phi(\gamma)| \leqq (6tB^2)^{-1} \leqq \beta/2 \qquad (\gamma \in E).$$

Hence E satisfies the hypothesis of Theorem 5.7.4, with $\delta = 1/2$, and the proof is complete.

5.7.6. Examples. (a) Suppose $\gamma_1, \ldots, \gamma_n$ are chosen, and S_n is the set of all $\gamma \in \Gamma$ of the form

$$(1) \quad \gamma = \pm\gamma_{i_1} \cdots \pm\gamma_{i_r} \qquad (i_1 < i_2 < \ldots < i_r;\ 1 \leqq r \leqq n).$$

Then S_n is finite, and we can choose γ_{n+1} outside S_n. Proceeding, we obtain an infinite set $E = \{\gamma_i\}$ which satisfies the hypotheses of Theorem 5.7.5 with $t = 1$, $B = 1$. *Hence every infinite subset of a discrete group Γ contains an infinite Sidon set.*

(b) Similarly, every *independent* subset of Γ is a Sidon set. A case of special interest is obtained by taking $G = D_2$. Every $x \in D_2$ can be expressed in the form

$$(2) \qquad x = (\xi_1, \xi_2, \xi_3, \ldots) \qquad (\xi_n = 0, 1);$$

the group operation is componentwise addition mod 2. Put

$$(3) \qquad r_n(x) = (-1)^{\xi_n} \qquad (x \in D_2,\ n = 1, 2, 3, \ldots).$$

The functions r_n are continuous characters on D_2. They form an independent subset of the dual group Γ (they also generate Γ). If we associate with each x of the form (2), the real number

$$(4) \qquad t = \sum_{n=1}^{\infty} 2^{-n}\xi_n,$$

the map $x \to t$ is a measure-preserving map of D_2 (with its Haar measure) onto $[0, 1]$ (with Lebesgue measure) which is one-to-one except for a countable set, and if we identify x and t, our functions r_n turn out to be the well-known Rademacher functions (Zygmund [1]).

(c) A set $\{n_i\}$ of positive integers is called a *Hadamard set* if there is a constant $\lambda > 1$ such that

(5) $$n_{i+1}/n_i > \lambda \qquad (i = 1, 2, 3, \ldots),$$

in view of Hadamard's classical theorem concerning the natural boundary of power series of the form $\sum a_i z^{n_i}$.

If E is a Hadamard set with $\lambda \geqq 3$, then $R_s(n, E) \leqq 1$ for every $n \in Z$. Since every Hadamard set is a finite union of sets with $\lambda \geqq 3$, Theorem 5.7.5, implies:

Every finite union of Hadamard sets is a Sidon set.

The simplest known example of a Sidon set in Z which is not a finite union of Hadamard sets seems to be the following (Hewitt and Zuckerman [3]):

For $m = 0, 1, 2, \ldots$, set $M = 2^m$, and let E be the set of all numbers

(6) $$3^{4M} + 3^{M+j} \qquad (j = 0, \ldots, M-1;\ m = 0, 1, 2, \ldots).$$

It is not hard to see that $R_s(n, E) \leqq 1$, so that E is a Sidon set. On the other hand, the number of elements which E has on the interval $[x, 2x]$ is an unbounded function of x, and so E is not a finite union of Hadamard sets.

(d) Suppose $\Gamma = Z^2$; consider Z^2 as the set of all points in the plane whose coordinates are integers. Choose integers a, b, c, d, with $ad - bc = 1$ and $a + d > 2$. Let A be the transformation of Z^2 onto Z^2 which carries each $n = (n_1, n_2)$ to the point $An = (an_1 + bn_2, cn_1 + dn_2)$. The *orbit* of n is the set of all points $A^i n$ $(-\infty < i < \infty)$.

The transformation A has two distinct eigenvectors v_1, v_2 in the plane; they do not lie in Z^2; the corresponding eigenvalues λ_1, λ_2 are positive, and $\lambda_1 > 2$. If $n \in Z^2$ $(n \neq 0)$, the orbit of n lies on one branch of the hyperbola

(7) $$cx^2 - (a - d)xy - by^2 = \text{constant}.$$

THEOREM. *Each finite union of such orbits is a Sidon set in Z^2.*

Proof: If $n = \alpha_1 v_1 + \alpha_2 v_2$, then $A^i n = \lambda_1^i \alpha_1 v_1 + \lambda_2^i \alpha_2 v_2$; since $\lambda_1 > 2$, we have $\lambda_1^i \neq \pm \lambda_1^{i_1} \pm \ldots \pm \lambda_k^{i_k}$ if $i_1, \ldots, i_k, i$ are distinct integers. Hence $A^i n \neq \pm A^{i_1} n \pm \ldots \pm A^{i_k} n$, and it follows that each orbit satisfies the hypotheses of Theorem 5.7.5.

(e) It is not known in general whether the union of two Sidon sets is a Sidon set.

5.7.7. If E is a Hadamard set, then every E-function on the circle T belongs to $L^p(T)$ for all $p < \infty$ (Zygmund [1], vol. I, p. 215). Our next theorem implies that this property is shared by all Sidon sets.

THEOREM. *If E is a subset of the discrete group Γ and if*

$$\sum |\hat{f}(\gamma)| \leqq B\|f\|_\infty \tag{1}$$

for every E-polynomial f on G, then we also have

$$\|f\|_p \leqq B\sqrt{p}\|f\|_2 \qquad (2 < p < \infty), \tag{2}$$

$$\|f\|_2 \leqq 2B\|f\|_1 \tag{3}$$

for every E-polynomial f.

Proof: We use the fact that (2) is known for the Rademacher functions r_n: if $g(t) = \sum a_n r_n(t)$, then

$$\int_0^1 |g(t)|^{2m} dt \leqq m^m \{\sum |a_k|^2\}^m \qquad (m = 1, 2, 3 \ldots). \tag{4}$$

This is usually proved for real a_k (Zygmund [1], vol. I, p. 213), but the proof holds equally well for complex a_k. Writing $|g|^2 = |g|^{2/3}|g|^{4/3}$, Hölder's inequality shows that

$$\|g\|_2 \leqq \|g\|_1^{1/3}\|g\|_4^{2/3}. \tag{5}$$

Substituting (4) into (5), with $m = 2$, we obtain

$$\{\sum |a_k|^2\}^{1/2} \leqq 2\int_0^1 |g(t)|dt. \tag{6}$$

Suppose now that f is an E-polynomial on G, and define

$$g_t(x) = g(x, t) = \sum_\gamma \hat{f}(\gamma) r_\gamma(t)(x, \gamma) \qquad (x \in G,\ 0 \leqq t \leqq 1); \tag{7}$$

instead of writing $r_1, r_2, \ldots$, we use the elements of Γ as indices. The proof that every Sidon set has property (d) of Theorem 5.7.3 shows that we can associate to each $t \in [0, 1]$ a measure $\mu_t \in M(G)$ such that $\|\mu_t\| \leqq B$ and $\hat{\mu}_t(\gamma) = r_\gamma(t)$ for all $\gamma \in E$. Hence $f = g_t * \mu_t$, and so

$$(8) \qquad \|f\|_p \leqq \|g_t\|_p \|\mu_t\| \leqq B\|g_t\|_p \qquad (0 \leqq t \leqq 1).$$

But we also have $g_t = f * \mu_t$, so that

$$(9) \qquad \|g_t\|_1 \leqq B\|f\|_1 \qquad (0 \leqq t \leqq 1).$$

With $p = 2m$ (8) becomes

$$(10) \qquad \int_G |f(x)|^{2m} \leqq B^{2m} \int_G |g(x, t)|^{2m} dx \qquad (0 \leqq t \leqq 1).$$

We integrate this over $[0, 1]$ and apply (4), with coefficients $\hat{f}(\gamma)(x, \gamma)$. The result is

$$(11) \qquad \int_G |f(x)|^{2m} dx \leqq B^{2m} m^m \{\sum |\hat{f}(\gamma)|^2\}^m.$$

If now $2m - 2 \leqq p \leqq 2m$, then $m \leqq p$, since $p \geqq 2$, and (11) implies

$$(12) \qquad \|f\|_p \leqq \|f\|_{2m} \leqq B\sqrt{m}\|f\|_2 \leqq B\sqrt{p}\|f\|_2.$$

This proves (2). Similarly, (6) gives

$$(13) \qquad \{\sum |\hat{f}(\gamma)|^2\}^{1/2} \leqq 2 \int_0^1 |g(x, t)| dt \qquad (x \in G).$$

If we integrate this over G and use (9) we obtain (3).

REMARK. The term $\sqrt{p}$ in the inequality (2) cannot be replaced by anything whose order of magnitude is smaller. For if E is any infinite subset of Γ and if k is a positive integer, there is an E-polynomial f on G such that

$$(14) \qquad \|f\|_k \geqq \frac{1}{4}\sqrt{k}\,\|f\|_2.$$

We refer to Rudin [17] for a proof of this statement, as well as for a more detailed discussion of problems concerning L^p-norms and lacunarity.

5.7.8. Suppose G is compact. We saw in Chapter 4 that the only maps of Γ into Γ which carry $L^1(G)$ into $L^1(G)$ are the piecewise affine ones, and likewise for $L^\infty(G)$. On the other hand, if $1 < p < \infty$, there exist permutations of Γ (i.e., one-to-one maps of Γ onto Γ) which carry $L^p(G)$ onto $L^p(G)$ but which are not so intimately related to the group structure of Γ.

To see this, suppose $1 < p \leqq 2$, and let $\{\gamma_k\}$ be an infinite Sidon set in Γ. If $f \in L^p(G)$, if $h(x) = \sum c_k(x, \gamma_k)$ is a trigonometric polynomial on G, and if $1/p + 1/q = 1$, we have

$$|\sum c_k \hat{f}(\gamma_k)| = \left|\int_G f(x)h(-x)dx\right|$$
$$\leqq \|f\|_p \|h\|_q \leqq B\sqrt{q}\,\|f\|_p \{\sum |c_k|^2\}^{\frac{1}{2}},$$

by Theorem 5.7.7 (2). Hence

$$\{\sum_{k=1}^{\infty} |\hat{f}(\gamma_k)|^2\}^{\frac{1}{2}} \leqq B\sqrt{q}\|f\|_p$$

for all $f \in L^p(G)$. If α is an arbitrary permutation of $\{\gamma_k\}$, if $\alpha(\gamma) = \gamma$ for all $\gamma \notin \{\gamma_k\}$, if $f \in L^p(G)$, and if $\hat{g}(\gamma) = \hat{f}(\alpha(\gamma))$, it follows that $f - g \in L^2(G)$, so that $g \in L^p(G)$. Thus α carries $L^p(G)$ onto $L^p(G)$.

Consideration of the adjoint map shows that α also carries $L^q(G)$ onto $L^q(G)$.

CHAPTER 6

Functions of Fourier Transforms

6.1. Introduction

6.1.1. Range transformations. Let Ω be a family of functions, defined on some set S, and let F be a function defined on some set E in the complex plane. If the range of a function $\phi \in \Omega$ lies in E, then $F(\phi)$ denotes the function whose value at a point $x \in S$ is $F(\phi(x))$.

We say that F *operates in* Ω if $F(\phi) \in \Omega$ for every $\phi \in \Omega$ whose range lies in E. Let (F) denote the map $\phi \to F(\phi)$. If F operates in Ω, we call (F) a *range transformation on* Ω. Analogously, the maps $\hat{\mu} \to \hat{\mu} \circ \alpha$ studied in Chapter 4 could be called *domain transformations*.

The topic of the present chapter is the determination of the range transformations of $A(\Gamma)$ and $B(\Gamma)$. The first result in this direction is due to Wiener [1] and Lévy [1]. A simple proof, based on an idea of Calderòn, is contained in Zygmund [1] (vol. I, pp. 245–246). This theorem asserts that *if $f \in A(T)$ where T is the unit circle, and if F is analytic on the range of f, then $F(f) \in A(T)$.* We shall see (Section 6.2) that the word "analytic" can be replaced by "real-analytic", but (and this is one of the main results of this chapter) that the class of all real-analytic functions cannot be replaced by a larger one.

Let F be defined on the interval $[-1, 1]$ of the real axis. The following three theorems are prototypes of the more general results proved in Sections 6.6, 6.5, and 6.3.

6.1.2. THEOREM. *If $F(f) \in A(T)$ whenever $f \in A(T)$ and $-1 \leqq f \leqq 1$, then F is analytic on $[-1, 1]$.* (Katznelson [1].)

6.1.3. THEOREM. *If $\{F(c_n)\}$ is a sequence of Fourier coefficients for every sequence $\{c_n\}$ of Fourier coefficients such that $-1 \leqq c_n \leqq 1$*

$(-\infty < n < \infty)$ (i.e., *if* F *operates in* $A(Z)$), *then* F *is analytic in some neighborhood of the origin and* $F(0) = 0$. (Helson and Kahane [1].)

6.1.4. THEOREM. *If* $\{F(c_n)\}$ *is a sequence of Fourier-Stieltjes coefficients for every sequence* $\{c_n\}$ *of Fourier-Stieltjes coefficients such that* $-1 \leqq c_n \leqq 1$ $(-\infty < n < \infty)$ (i.e., *if* F *operates in* $B(Z)$), *then* F *can be extended to an entire function in the complex plane.* (Kahane and Rudin [1].)

The conclusions may be stated in terms of power series: in 6.1.4, $F(t) = \sum_0^\infty a_n t^n$, and this series converges for all t; in 6.1.3, such a representation of F is valid in some neighborhood of the origin; in 6.1.2, to each t_0 in $[-1, 1]$ there corresponds a series $\sum_0^\infty a_n(t - t_0)$ which converges to $F(t)$ in some neighborhood of t_0.

6.1.5. Throughout this chapter, G and Γ will be *infinite LCA* groups, to avoid trivialities. The symbols $A_R(\Gamma)$ and $B_R(\Gamma)$ will denote the subsets of $A(\Gamma)$ and $B(\Gamma)$, respectively, which consist of real-valued functions.

The principal references for this chapter are the paper by Helson, Kahane, Katznelson and Rudin [1] and the thesis of Katznelson [3]. Earlier results (besides those already cited) were obtained by Kahane [1], [2], [4], [5] and Rudin [2], [4].

6.2. Sufficient Conditions

6.2.1. Real-analytic and real-entire functions. A complex-valued function F, defined on an open set E in the plane, is said to be *real-analytic* in E if to every point (s_0, t_0) in E there corresponds an expansion with complex coefficients

$$F(s, t) = \sum_{m, n=0}^{\infty} a_{mn}(s - s_0)^m (t - t_0)^n, \tag{1}$$

which converges absolutely for all (s, t) in some neighborhood of (s_0, t_0).

A function F, defined on some plane set E, is real-analytic on E, by definition, if F is real-analytic in some open set containing E. If E is a subset of the real axis, then "analytic on E" and "real-analytic on E" mean the same thing.

If F is defined in the whole plane by a series

$$F(s, t) = \sum_{m,n=0}^{\infty} a_{mn} s^m t^n \tag{2}$$

which converges absolutely for every (s, t), then we call F *real-entire*. The example

$$F(s, t) = \frac{1}{(1+s^2)(1+t^2)} \tag{3}$$

shows that a function may be real-analytic in the whole plane without being real-entire.

6.2.2. THEOREM. *Every real-entire function operates in $B(\Gamma)$.*

Proof: If $\phi \in B(\Gamma)$ and $\phi = \phi_1 + i\phi_2$ (ϕ_1, ϕ_2 real), then $\phi_1, \phi_2 \in B(\Gamma)$. If $F(s, t) = \sum a_{mn} s^m t^n$ converges absolutely for all (s, t), then the series $\sum a_{mn} \phi_1^m \phi_2^n$ converges in the norm of $B(\Gamma)$; its sum is $F(\phi_1, \phi_2) = F(\phi)$.

6.2.3. THEOREM. *If F is real-analytic in a neighborhood of the origin and if $F(0) = 0$, then F operates in $A(\Gamma)$ if Γ is discrete.*

Proof: Suppose $F(s, t) = \sum a_{mn} s^m t^n$, $a_{00} = 0$, and the series converges absolutely if $|s| < \delta$, $|t| < \delta$. Suppose also, for simplicity, that F is defined in the rest of the plane, it does not matter how. Given $f \in L^1(G)$, there is a trigonometric polynomial P on G such that $\|f - P\|_1 < \delta$. Setting $g = f - P$, it follows, as in the proof of Theorem 6.2.2, that $F(\hat{g}) \in A(\Gamma)$. Since $F(\hat{g}(\gamma))$ differs from $F(\hat{f}(\gamma))$ for only finitely many $\gamma \in \Gamma$, we also have $F(\hat{f}) \in A(\Gamma)$.

6.2.4. THEOREM. *If F is real-analytic in an open set E in the plane, if $\hat{f} \in A(\Gamma)$ and if the closure of the range of $\hat{f}$ lies in E, then* $F(\hat{f}) \in A(\Gamma)$. (If Γ is not compact, we also require that $F(0) = 0$.)

This could be proved by an appeal to a theorem concerning the action of an analytic function of two complex variables on a Banach algebra (see Arens and Calderòn [1], for instance). However, we shall present a proof which is essentially that of Wiener [1] and Lévy [1], since its technique will be useful to us later.

6.2.5. Let I be an ideal in $A(\Gamma)$ and let ϕ be a function defined on Γ. We say that ϕ *belongs to I locally at a point $\gamma_0 \in \Gamma$* if there is a neighborhood V of γ_0 and a function $\hat{f} \in I$ such that $\phi(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in V$. If Γ is not compact and if there is a compact set $K \subset \Gamma$ and a function $\hat{f} \in I$ such that $\phi(\gamma) = \hat{f}(\gamma)$ in the complement of K, then ϕ is said to *belong to I locally at infinity.*

At present we shall need the following lemma only for the case $I = A(\Gamma)$.

6.2.6. LEMMA. *If ϕ belongs to I locally at every point of Γ (including the point at infinity if Γ is not compact), then $\phi \in I$.*

We note that I is not required to be closed.

Proof: Suppose first that ϕ has compact support C. Then there exist (i) open sets $V_1, \ldots, V_n$ and functions $\hat{f}_1, \ldots, \hat{f}_n \in I$ such that $\phi = \hat{f}_i$ in V_i and $V_1 \cup \ldots \cup V_n$ covers C, (ii) open sets $W_1, \ldots, W_n$ with compact closures $\overline{W}_i \subset V_i$ such that $W_1 \cup \ldots \cup W_n$ covers C, and (iii) functions $\hat{k}_i \in A(\Gamma)$ such that $\hat{k}_i = 1$ on $\overline{W}_i$ and $\hat{k}_i = 0$ outside V_i (Theorem 2.6.2).

Hence $\phi\hat{k}_i = \hat{f}_i\hat{k}_i \in I$, since I is an ideal, and if

$$(1) \qquad \psi = \phi\{1 - (1 - \hat{k}_1)(1 - \hat{k}_2) \ldots (1 - \hat{k}_n)\},$$

it follows that $\psi \in I$. The multiplier of ϕ in (1) is 1 whenever one of the $\hat{k}_i$ is 1, and this happens at every point of C; outside C, $\phi = 0$; hence $\psi = \phi$, and thus $\phi \in I$.

In the general case, ϕ belongs to I locally at infinity, so that there is a function $\hat{g} \in I$ which coincides with ϕ outside some compact subset of Γ. Then $\phi - \hat{g}$ has compact support and belongs to I locally at every point of Γ; by the first case, $\phi - \hat{g} \in I$, and so $\phi \in I$.

6.2.7. Proof of theorem 6.2.4. By the preceding lemma it is enough to show that $F(\hat{f})$ belongs to $A(\Gamma)$ locally at every point of $\Gamma \cup \{\infty\}$ (Appendix A5). Fix $\gamma_0 \in \Gamma \cup \{\infty\}$, put $\hat{f}(\gamma_0) = s_0 + it_0$, and choose $\delta > 0$ such that the series

$$(1) \quad F(s, t) = F(s_0, t_0) + \sum_{m,n=0}^{\infty} a_{mn}(s - s_0)^m(t - t_0)^n \qquad (a_{00} = 0)$$

converges absolutely for $|s - s_0| \leqq \delta$, $|t - t_0| \leqq \delta$.

There exists a function $g \in L^1(G)$ such that $||g||_1 < \delta$ and such that

$$\hat{f}(\gamma) = \hat{f}(\gamma_0) + \hat{g}(\gamma) \tag{2}$$

in some neighborhood V of γ_0. If $\gamma_0 \in \Gamma$, this follows from Theorem 2.6.5; if $\gamma_0 = \infty$, then $\hat{f}(\gamma_0) = 0$, and we put $g = f - f * v$, where v is chosen as in Theorem 2.6.6. Put $\hat{g} = \hat{g}_1 + i\hat{g}_2$ ($\hat{g}_1$, $\hat{g}_2$ real). Then $||g_1||_1 < \delta$ and $||g_2||_1 < \delta$. The series

$$\sum_{m,n=0}^{\infty} a_{mn} g_1^m g_2^n \tag{3}$$

therefore converges, in the norm of $L^1(G)$, to a function $h \in L^1(G)$; we recall that $g_1^m = g_1 * g_1^{m-1}$, etc. But if $\gamma \in V$, we have

$$\begin{aligned} F(\hat{f}(\gamma)) &= F(s_0 + \hat{g}_1(\gamma), t_0 + \hat{g}_2(\gamma)) \\ &= F(s_0, t_0) + \sum_{m,n=0}^{\infty} a_{mn} \hat{g}_1(\gamma)^m \hat{g}_2(\gamma)^n \\ &= F(s_0, t_0) + \hat{h}(\gamma). \end{aligned} \tag{4}$$

Thus $F(\hat{f})$ belongs to $A(\Gamma)$ locally at γ_0 and the proof is complete.

6.3. Range Transformations on $B(\Gamma)$ for Non-Compact Γ

We begin by stating two theorems. The first of these evidently contains the second. We shall show, conversely, that the first follows from the second, and will then prove the second.

6.3.1. THEOREM. *Suppose F is defined on the interval $[-1, 1]$, Γ is a non-compact LCA group, and F operates in $B(\Gamma)$. Then F can be extended to an entire function in the complex plane.*

6.3.2. THEOREM. *Suppose F is defined on the real line, F has period 2π, Γ is discrete and countable, and F operates in $B(\Gamma)$. Then F can be extended to an entire function in the complex plane.*

6.3.3. Reduction of theorem 6.3.1. to theorem 6.3.2. Suppose the hypotheses of Theorem 6.3.1 are satisfied. The structure theorem 2.4.1 asserts that Γ has an open subgroup Γ_0 which is the direct sum of a compact group and a euclidean space R^p, for some

$p \geqq 0$. If $\Gamma_1 = \Gamma/\Gamma_0$ is infinite, then F operates in the algebra of all functions in $B(\Gamma)$ which are constant on the cosets of Γ_0, and this means that F operates in $B(\Gamma_1)$. It follows (Theorem 2.7.2) that F operates in $B(\Lambda)$ where Λ is any countable subgroup of Γ_1.

If Γ_1 is finite, then $p > 0$, since Γ is not compact, and so Γ contains R^p, and hence also Z^p, as a closed subgroup. It follows (again by Theorem 2.7.2) that F operates in $B(Z^p)$.

We have proved that F operates in $B(\Lambda)$, where Λ is a countable discrete group. Consider the functions F_1 and F_2 defined by

$$(1) \quad F_1(s) = F(r_1 \sin s), \qquad F_2(s) = F(r_2 \sin s) \qquad (-\infty < s < \infty)$$

where $0 < r_1 < r_2 < 1$. Then F_1 and F_2 operate in $B(\Lambda)$, and if Theorem 6.3.2 is true, then F_1 and F_2 are entire. The formula

$$(2) \qquad F(s) = F_1(\text{arc sin } (s/r_1)) \qquad (-r_1 < s < r_1)$$

shows that F can be expanded in a power series about the origin, and that this power series can be analytically continued to a (possibly multi-valued) function in the finite plane, except for possible branch points at $s = \pm r_1$. Using F_2 in place of F_1, the same argument shows that $s = \pm r_2$ are the only possible singular points of F in the finite plane. Since $r_1 \neq r_2$, the analytic extension of F is an entire function.

Hence 6.3.1 follows from 6.3.2.

6.3.4. Lemma. *Suppose F is defined on the real line, $\eta > 0$, Γ is discrete and countable, and $F(\hat{f}) \in B(\Gamma)$ whenever $\hat{f} \in A_R(\Gamma)$ and $\|f\|_1 < \eta$. Then F is continuous at the origin.*

Proof: Replacing F by $F - F(0)$, we may assume that $F(0) = 0$. If F is not continuous at 0, there exists a sequence $\{a_n\}$ of real numbers such that $\sum |a_n| < \eta$ but $|F(a_n)| > \delta$ for some $\delta > 0$ $(n = 1, 2, 3, \ldots)$. Choose a sequence $\{\gamma_n\}$ in Γ such that

$$(1) \qquad \gamma_n \neq \gamma_i + \gamma_j - \gamma_k$$

if $i, j, k < n$, and put

$$(2) \qquad f(x) = \sum_{1}^{\infty} a_n (x, \gamma_n) \qquad (x \in G).$$

The hypotheses of the lemma imply that there is a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma_n) = F(a_n)$ $(n = 1, 2, 3, \ldots)$ and $\hat{\mu}(\gamma) = 0$ for all other $\gamma \in \Gamma$. Define $\mu_n \in M(G)$ by

$$(2) \qquad d\mu_n(x) = (-x, \gamma_n)d\mu(x) \qquad (n = 1, 2, 3, \ldots)$$

and let σ be a weak* limit of a subsequence of $\{\mu_n\}$. By the translation lemma 3.5.1, σ is singular.

Since $|\hat{\mu}_n(0)| = |\hat{\mu}(\gamma_n)| = |F(a_n)| > \delta$, we have $|\hat{\sigma}(0)| \geqq \delta$, and hence $\sigma \neq 0$.

Fix $\gamma \neq 0$. Since $\hat{\mu}_n(\gamma) = \hat{\mu}(\gamma + \gamma_n)$, we see that $\hat{\mu}_n(\gamma) \neq 0$ only if $\gamma + \gamma_n = \gamma_m$ for some m. Our choice (1) shows that no $\gamma \neq 0$ has more than one representation of the form $\gamma = \gamma_m - \gamma_n$. Hence $\hat{\mu}_n(\gamma) \neq 0$ for at most one value of n. It follows that $\hat{\sigma}(\gamma) = 0$, and so σ is absolutely continuous.

This contradiction proves the lemma.

6.3.5. COROLLARY. *Under the hypotheses of Theorem 6.3.2, F is continuous on the whole line.*

Proof: Apply the lemma to $F(s + s_0)$ in place of $F(s)$.

6.3.6. If F operates in $B(\Gamma)$, we write $F \circ \mu$ for the measure whose Fourier-Stieltjes transform is $F(\hat{\mu})$.

LEMMA. *Suppose the hypotheses of Theorem 6.3.2 are satisfied.*

(a) *If $\hat{\mu} \in B_R(\Gamma)$, there exists $\delta > 0$ and $C < \infty$ such that the inequality*

$$(1) \qquad \|F \circ (\mu + \sigma)\| \leqq C$$

holds for all $\sigma \in M(G)$ with $\hat{\sigma}$ real and $\|\sigma\| < \delta$.

(b) *The map (F) carries each compact subset of $B_R(\Gamma)$ into a bounded subset of $B(\Gamma)$.*

Proof: We recall that (F) maps $\phi \in B_R(\Gamma)$ to $F(\phi) \in B(\Gamma)$. Part (a) asserts that each $\phi \in B_R(\Gamma)$ has a neighborhood in $B_R(\Gamma)$ on which (F) is bounded, and this immediately implies (b).

To prove (a), it is enough to show that for some $\delta > 0$ and $C < \infty$ the inequality

$$(2) \qquad \|F \circ (\mu + f)\| \leqq C$$

holds whenever f is a trigonometric polynomial on G with $\|f\|_1 < \delta$ and $\hat{f}$ real. For if this is proved and if $\|\sigma\| < \delta$, there exist trigonometric polynomials k_j on G, such that $\hat{k}_j$ is real, $\|k_j * \sigma\| < \delta$, and $\hat{k}_j = 1$ on E_j, where $\{E_j\}$ is an expanding sequence of finite sets whose union is Γ; this follows from Theorem 2.6.8. By (2) we then have

$$\|F \circ (\mu + k_j * \sigma)\| \leqq C \qquad (j = 1, 2, 3, \ldots). \tag{3}$$

But

$$\lim_{j\to\infty} F(\hat{\mu}(\gamma) + \hat{k}_j(\gamma)\hat{\sigma}(\gamma)) = F(\hat{\mu}(\gamma) + \hat{\sigma}(\gamma)) \qquad (\gamma \in \Gamma) \tag{4}$$

since, for each $\gamma \in \Gamma$, $\hat{k}_j(\gamma) = 1$ for all but finitely many values of j. By (3) and (4), Theorem 1.9.2 implies that (1) holds.

Thus, if the lemma is false, there is a sequence $\{f_n\}$ of trigonometric polynomials on G, with $\hat{f}_n$ real, such that $\|f_n\|_1 \to 0$ but $\|F \circ (\mu + f_n)\| \to \infty$.

Take $n_1 = 1$. If integers $n_1, \ldots, n_j$ and trigonometric polynomials $k_1, \ldots, k_{j-1}$ are chosen, put

$$\lambda_j = \mu + f_{n_1} + \ldots + f_{n_j} \tag{5}$$

and let k_j be a trigonometric polynomial on G, with $\hat{k}_j$ real and $\|k_j\|_1 < 2$, such that

$$\|k_j * (F \circ \lambda_j)\| > \tfrac{1}{2}\|F \circ \lambda_j\|; \tag{6}$$

this is possible, by Theorems 2.6.8 and 1.9.2. Then let n_{j+1} be an integer, so large that

$$\|f_{n_{j+1}}\|_1 < 2^{-j} \tag{7}$$

$$\|F \circ (\lambda_j + f_{n_{j+1}})\| \geqq j + 1 \tag{8}$$

$$\|k_i * (F \circ \lambda_j - F \circ (\lambda_j + f_{n_{j+1}}))\| < 2^{-j} \qquad (1 \leqq i \leqq j). \tag{9}$$

Observe that (8) can be achieved since $\|F \circ (\mu + f_n)\| \to \infty$ and since $\hat{\lambda}_j$ differs from $\hat{\mu}$ at only a finite number of points; (9) can be achieved since $\hat{f}_n(\gamma) \to 0$ as $n \to \infty$, and since each $\hat{k}_i$ has finite support; note that F is continuous.

Having defined $\{n_j\}$ and $\{k_j\}$ by induction, we put

$$\lambda = \mu + \sum_{j=1}^{\infty} f_{n_j} \tag{10}$$

and $\tau = F \circ \lambda$.

We cannot yet assert that $\tau = \lim F \circ \lambda_N$ in the norm of $M(G)$. But we do assert that

$$\lim_{N\to\infty} \|k_r * (\tau - F \circ \lambda_N)\| = 0 \qquad (r = 1, 2, 3, \ldots) \tag{11}$$

since $\hat{\tau}(\gamma) = \lim F(\hat{\lambda}_N(\gamma))$ and since $\hat{k}_r$ has finite support. Hence

$$k_r * (\tau - F \circ \lambda_r) = \lim_{N\to\infty} \sum_{j=r}^{N-1} k_r * (F \circ \lambda_{j+1} - F \circ \lambda_j), \tag{12}$$

and combined with (9) this implies

$$\|k_r * (\tau - F \circ \lambda_r)\| \leqq \sum_{j=r}^{\infty} 2^{-j} \leqq 1 \qquad (r = 1, 2, 3, \ldots). \tag{13}$$

Finally, (13), (6) and (8) yield the inequalities

$$\begin{aligned} 2\|\tau\| \geqq \|k_r * \tau\| &\geqq \|k_r * (F \circ \lambda_r)\| - 1 \\ &\geqq \tfrac{1}{2}\|F \circ \lambda_r\| - 1 \geqq \tfrac{1}{2}r - 1 \end{aligned}$$

for every positive integer r, which is absurd.

The lemma follows.

6.3.7. We are now ready to prove Theorem 6.3.2. Since F is continuous and periodic, F has a Fourier series

$$F(s) \sim \sum_{-\infty}^{\infty} c_n e^{ins}. \tag{1}$$

Let P be an independent perfect set in G and let μ be a positive continuous measure concentrated on $P \cup (-P)$ with $\hat{\mu}$ real (i.e., such that $\mu = \tilde{\mu}$), as in the proof of Theorem 5.3.4. The set $\{\mu + a\delta_0\}$, $0 \leqq a \leqq 2\pi$, is a continuous image of $[0, 2\pi]$ and is therefore compact in $M(G)$, and Lemma 6.3.6(b) shows that there is a constant $C < \infty$ such that

$$\|F \circ (\mu + a\delta_0)\| \leqq C \qquad (0 \leqq a \leqq 2\pi); \tag{2}$$

we recall that δ_0 is the unit of $M(G)$. Since F is continuous, (1) shows that

$$(3)\qquad \begin{aligned} c_n \exp\{in\hat{\mu}(\gamma)\} &= \frac{1}{2\pi}\int_0^{2\pi} F(s+\hat{\mu}(\gamma))e^{-ins}ds \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} F\left(\frac{2\pi k}{N}+\hat{\mu}(\gamma)\right)e^{-2\pi ik/N} \end{aligned}$$

for all $n \in Z$ and $\gamma \in \Gamma$. Since $F(\hat{\mu}(\gamma) + 2\pi k/N)$ is the transform of $F \circ (\mu + a\delta_0)$ with $a = 2\pi k/N$, (2), (3) and Theorem 1.9.2 imply that

$$(4)\qquad |c_n| \cdot \|e^{in\mu}\| \leqq C \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The special way in which μ was chosen shows, by 5.3.3(b), that

$$(5)\qquad \|e^{in\mu}\| = e^{|n|\,\|\mu\|} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By (4), the series

$$(6)\qquad \sum_{-\infty}^{\infty} c_n e^{in(s+it)}$$

therefore converges absolutely in the strip $|t| < \|\mu\|$. Since $\|\mu\|$ can be taken arbitrarily large, the sum of (6) is an entire function of $s + it$ which coincides with F on the real axis.

This completes the proof.

6.4. Some Consequences

6.4.1. THEOREM. *Suppose G is not discrete and z_0 is a complex number. Then there exists a measure $\mu \in M(G)$ whose Fourier-Stieltjes transform has its range in the interval* $[-1, 1]$ *and whose spectrum contains the point z_0.*

Proof: If this were not so, then $\mu - z_0\delta_0$ would have an inverse in $M(G)$ provided that $-1 \leqq \hat{\mu}(\gamma) \leqq 1$ for all $\gamma \in \Gamma$. This means that the function

$$(1)\qquad F(s) = \frac{1}{s - z_0} \qquad (-1 \leqq s \leqq 1)$$

would operate in $B(\Gamma)$, in contradiction to Theorem 6.3.1.

6.4.2. We saw earlier (Theorem 5.3.4) that $M(G)$ is not symmetric unless G is discrete, i.e., unless Γ is compact. In other words, the function F defined in the complex plane by $F(z) = \bar{z}$ does not operate in the algebra of all Gelfand transforms of $M(G)$. This result can now be strengthened:

6.4.3. THEOREM. *Suppose G is not discrete. If F is defined in the complex plane and if F operates in the algebra of all Gelfand transforms of $M(G)$, then F is an entire function.*

The hypothesis may be restated without reference to the Gelfand transform: it is assumed that F associates with each $\mu \in M(G)$ a measure $\sigma \in M(G)$ such that $h(\sigma) = F(h(\mu))$ for every complex homomorphism h of $M(G)$.

Proof: Since the members of $B(\Gamma)$ are precisely the restrictions to Γ of the Gelfand transforms of $M(G)$ (see 5.3.1), F operates in $B(\Gamma)$, and hence the restriction of F to the real axis operates in $B(\Gamma)$. By Theorem 6.3.1, there is an entire function F_1 which coincides with F on the real axis. Being entire, F_1 operates in $B(\Gamma)$, and so does $F - F_1$. Since $F - F_1 = 0$ on the real axis, $F - F_1$ associates the zero-measure to each $\mu \in M(G)$ with real $\hat{\mu}$, by the uniqueness theorem for Fourier-Stieltjes transforms. Thus $F(h(\mu)) = F_1(h(\mu))$ for every μ with $\hat{\mu}$ real and for every complex homomorphism h, and Theorem 6.4.1 therefore implies that $F(z) = F_1(z)$ for all z.

6.5. Range Transformations on $A(\Gamma)$ for Discrete Γ

6.5.1. THEOREM. *Suppose F is defined on $[-1, 1]$ and Γ is an infinite discrete abelian group. If F operates in $A(\Gamma)$, then $F(0) = 0$ and F is analytic in some neighborhood of the origin.*

Since $A(\Gamma)$ contains no constant except 0, it is clear that $F(0) = 0$. If $f \in A(\Gamma)$, then $|\hat{f}(\gamma)| > 1$ for only finitely many γ; hence we may extend F from $[-1, 1]$ to the whole real axis in any way whatever, and the extension will operate in $A(\Gamma)$. We will assume that F is so extended. Finally, we may assume without loss of generality that Γ is countable, as in Section 6.3.

6.5.2. LEMMA. *If the hypotheses of Theorem 6.5.1 hold, there exists $\delta > 0$ and $C < \infty$ with the following property: If $\sigma \in M(G)$, $\|\sigma\| < \delta$, and $\hat{\sigma}$ is real, then $F(\hat{\sigma}) \in B(\Gamma)$ and $\|F \circ \sigma\| \leqq C$.*

Proof: If the lemma is false, then, as in the proof of Lemma 6.3.6, there are trigonometric polynomials f_n on G, with $\hat{f}_n$ real, such that $\|f_n\|_1 \to 0$ but $\|F \circ f_n\|_1 \to \infty$. By taking a subsequence, we may also assume that $\|f_n\|_1 < 2^{-n}$ $(n = 1, 2, 3, \ldots)$. Choose trigonometric polynomials k_n on G, with $\hat{k}_n$ real, $\|k_n\|_1 < 2$ and $\hat{k}_n = 1$ on the support of $\hat{f}_n$. Then translate the pairs $\hat{f}_n$, $\hat{k}_n$ so that the supports of $\hat{k}_n$ and $\hat{k}_m$ are disjoint if $n \neq m$. This changes none of the norms. Put

$$g = \sum_{n=1}^{\infty} f_n. \tag{1}$$

Since $F(\hat{g}(\gamma)) = F(\hat{f}_n(\gamma))$ if γ is in the support of $\hat{f}_n$, and $F(\hat{g}(\gamma)) = 0$ otherwise, we have

$$k_n * (F \circ g) = k_n * (F \circ f_n) = F \circ f_n \qquad (n = 1, 2, 3, \ldots). \tag{2}$$

Hence $\|F \circ f_n\|_1 \leqq 2\|F \circ g\|$, which contradicts the assumption that $\|F \circ f_n\|_1 \to \infty$.

6.5.3. Proof of theorem 6.5.1. Suppose $-\delta < s_0 < \delta$, where δ is as in Lemma 6.5.2, and put $F_1(s) = F(s_0 + s) - F(s_0)$. Then $F_1(\hat{f}) \in B(\Gamma)$ for all $\hat{f} \in A_R(\Gamma)$ such that $\|f\|_1 < \delta - |s_0|$, so that F_1 is continuous at the origin (Lemma 6.3.4); hence F is continuous in $(-\delta, \delta)$. Define

$$F_2(s) = F(r \sin s) \qquad (-\infty < s < \infty) \tag{1}$$

where r is fixed, $0 < r < \delta/e$.

If $\hat{\mu} \in B_R(\Gamma)$ and $\|\mu\| \leqq 1$, then

$$\begin{aligned} \|\sin(\hat{\mu} + a)\| &\leqq |\cos a| \cdot \|\sin \mu\| + |\sin a| \cdot \|\cos \mu\| \\ &\leqq \sum \frac{1}{(2n-1)!}\|\mu^{2n-1}\| + \sum \frac{1}{(2n)!}\|\mu^{2n}\| \leqq e^{\|\mu\|} \leqq e \end{aligned} \tag{2}$$

for every real number a, and Lemma 6.5.2 implies that

$$\|F_2 \circ (\mu + a\delta_0)\| \leqq C \qquad (-\infty < a < \infty). \tag{3}$$

The argument of Section 6.3.7 can now be repeated, with F_2 in place of F, and with μ restricted so that $\|\mu\| \leqq 1$, and leads to the conclusion that F_2 can be extended to a function which is analytic in a horizontal strip of width 2, bisected by the real axis.

Hence $F(s) = F_2(\arcsin (s/r))$ is analytic in a neighborhood of the origin and the proof is complete.

6.5.4. REMARK. The preceding proof actually yields a little more than Theorem 6.5.1, namely:

THEOREM. *If F is defined in $[-1, 1]$, if Γ is an infinite discrete abelian group, and if $F(\hat{f}) \in B(\Gamma)$ for all $\hat{f} \in A(\Gamma)$ such that $-1 \leqq \hat{f} \leqq 1$, then F is analytic in a neighborhood of the origin.*

6.6. Range Transformations on $A(\Gamma)$ for Non-Discrete Γ

6.6.1. THEOREM. *Suppose F is defined on $[-1, 1]$ and Γ is a non-discrete LCA group. If F operates in $A(\Gamma)$, then F is analytic on $[-1, 1]$. Moreover, $F(0) = 0$ if Γ is not compact.*

In Sections 6.3 and 6.5 we used the fact that if Γ is not compact then there exists $\hat{\mu} \in B_R(\Gamma)$ such that

$$\|\exp \{in\mu\}\| = \exp \{|n| \, \|\mu\|\} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that $\|\mu\|$ may be taken arbitrarily large. This is not true for all Γ, but the following lemma will suffice:

6.6.2. LEMMA. *Suppose Γ is an infinite LCA group, $r > 0$, and S_r is the set of all $\mu \in M(G)$ with $\hat{\mu}$ real and $\|\mu\| \leqq r$. Then*

$$\sup_{\mu \in S_r} \|e^{i\mu}\| = e^r. \tag{1}$$

Proof: The left side of (1) cannot exceed the right, since

$$\|e^{i\mu}\| = \|\sum_0^\infty (i\mu)^n/n!\| \leqq \sum_0^\infty \|\mu\|^n/n! \leqq \sum_0^\infty r^n/n! = e^r \tag{2}$$

if $\|\mu\| \leqq r$.

To prove the opposite inequality, we pass to the Bohr compactification $\bar{G}$ of G. There is a measure $\sigma \in M(\bar{G})$ such that $\|\sigma\| = r$, $\hat{\sigma}$ is real on Γ_d, and $\|e^{i\sigma}\| = e^r$. Fix $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\varepsilon > 0$.

Since G is dense in $\bar{G}$ there exists $\mu \in M(G)$ with $\hat{\mu}$ real and $||\mu|| \leqq r$ such that

(3) $$|\hat{\mu}(\gamma_j) - \hat{\sigma}(\gamma_j)| < \varepsilon \qquad (1 \leqq j \leqq n).$$

Hence

(4) $$|\sum c_j \exp \{i\hat{\sigma}(\gamma_j)\}| \leqq \sup_{\mu \in S_r} |\sum c_j \exp \{i\hat{\mu}(\gamma_j)\}|,$$

for arbitrary constants $c_1, \ldots, c_n$. If $f(x) = \sum c_j(x, \gamma_j)$, the right side of (4) does not exceed

(5) $$\sup_{\mu \in S_r} ||e^{i\mu}|| \cdot ||f||_\infty$$

and hence Theorem 1.9.1. implies

(6) $$e^r = ||e^{i\sigma}|| \leqq \sup_{\mu \in S_r} ||e^{i\mu}||.$$

6.6.3. We shall now prove Theorem 6.6.1 under the additional assumption that Γ is compact. We shall also assume, without loss of generality, that $F(0) = 0$.

Let us say that (F) is *locally bounded at a point* $\gamma \in \Gamma$ if there exist two positive numbers η, K and a neighborhood V of γ such that $||F(\phi)|| \leqq K$ for all $\phi \in A_R(\Gamma)$ whose support lies in V and which satisfy the inequality $||\phi|| \leqq \eta$.

Since the map $\phi \to F(\phi)$ commutes with all translations of Γ and since the norm of $A(\Gamma)$ is translation-invariant, there are only two possibilities: either (F) is locally bounded at every point of Γ, or at no point of Γ.

Suppose the second alternative occurs. Choose disjoint open sets V_n in Γ which contain non-empty open sets W_n, such that $\overline{W}_n \subset V_n$, and choose $\phi_n \in A(\Gamma)$ such that $0 \leqq \phi_n \leqq 1$, $\phi_n = 1$ on W_n, $\phi_n = 0$ outside V_n $(n = 1, 2, 3, \ldots)$. Since (F) is not locally bounded at any point, there is a sequence $\{f_n\}$ in $A_R(\Gamma)$ such that (i) the support of f_n is in W_n, (ii) $||f_n|| < n^{-2}$, and (iii) $||F(f_n)|| > n||\phi_n||$.

If $f = \sum_1^\infty f_n$, then $f \in A(\Gamma)$, $-1 < f(\gamma) < 1$ for all $\gamma \in \Gamma$, and

(1) $$\phi_n(\gamma)F(f(\gamma)) = \phi_n(\gamma)F(f_n(\gamma)) = F(f_n(\gamma)) \quad (n = 1, 2, 3, \ldots),$$

here we have used the properties of ϕ_n, property (i) of f_n, and the assumption that $F(0) = 0$. Hence

$$(2) \qquad \|\phi_n\| \cdot \|F(f)\| \geqq \|\phi_n \cdot F(f)\| = \|F(f_n)\| > n\|\phi_n\|$$

so that $\|F(f)\| > n$ for every positive integer n, which is absurd.

Thus (F) is locally bounded at every point of Γ, and the translation-invariance of the norms shows that the following statement is true:

There exists a neighborhood V of 0 in Γ and two positive numbers η, K such that $\|F(\phi)\| \leqq K$ for every $\phi \in A_R(\Gamma)$ with $\|\phi\| \leqq \eta$ whose support lies in some translate of V.

Now let U and W be neighborhoods of 0 in Γ such that $\overline{W} \subset U \subset \overline{U} \subset V$, and choose $\alpha, \beta \in A(\Gamma)$ such that $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha = 1$ on W, $\alpha = 0$ outside U, $\beta = 1$ on U, $\beta = 0$ outside V. A finite union of translates of W, say $W_1, \ldots, W_n$, covers Γ. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be the corresponding translates of α and β, and put

$$(3) \qquad \phi_i = \frac{\alpha_i}{\alpha_1 + \ldots + \alpha_n} \qquad (1 \leqq i \leqq n).$$

Since $\alpha_1 + \ldots + \alpha_n > 0$, Theorem 6.2.4 implies that $\phi_i \in A_R(\Gamma)$; also, $\phi_i \geqq 0$, and $\sum \phi_i = 1$. (The functions ϕ_i form a partition of unity.)

Suppose $f \in A_R(\Gamma)$ and $\|f\| \leqq \eta/\|\beta\|$. Then the support of $\beta_i f$ lies in a translate of V and so $\|F(\beta_i f)\| \leqq K$. Since

$$(4) \qquad F(f) = \sum_{i=1}^{n} \phi_i \cdot F(f) = \sum_{i=1}^{n} \phi_i F(\beta_i f),$$

we have

$$(5) \qquad \|F(f)\| \leqq K \sum_{i=1}^{n} \|\phi_i\|.$$

Putting $\delta = \eta/\|\beta\|$ and $C = K \sum \|\phi_i\|$ we have proved: *If $f \in A_R(\Gamma)$, and $\|f\| \leqq \delta$, then $\|F(f)\| \leqq C$.* In other words, (F) maps a certain neighborhood of 0 in $A_R(\Gamma)$ into a bounded subset of $A(\Gamma)$.

Now define

(6) $$F_1(s) = F(r \sin s) \qquad (-\infty < s < \infty)$$

where $0 < r < \delta/e$. It follows, exactly as in Section 6.5.3, that

(7) $$\|F_1(f + a)\| \leqq C \qquad (-\infty < a < \infty)$$

for all $f \in A_R(\Gamma)$ with $\|f\| \leqq 1$.

Since Γ is not discrete, F is continuous on $[-1, 1]$; for if $-1 \leqq t_n \leqq 1$ and $t_n \to t$, there exists $f \in A_R(\Gamma)$ such that $f(\gamma_n) = t_n$ for some sequence $\{\gamma_n\}$ which has a limit point $\gamma \in \Gamma$, and the continuity of f and $F(f)$ implies that

(8) $$\lim F(t_n) = \lim F(f(\gamma_n)) = F(f(\gamma)) = F(t).$$

Hence F_1 is continuous and can be expanded in a Fourier series

(9) $$F_1(s) \sim \sum_{-\infty}^{\infty} c_n e^{ins}.$$

The argument used in Section 6.3.7, combined with the inequality (7), now yields

(10) $$|c_n| \, \|e^{inf}\| \leqq C \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

for all $f \in A_R(\Gamma)$ such that $\|f\| \leqq 1$. Fix n and take the supremum of the left side of (10); Lemma 6.6.2 implies that

(11) $$|c_n| \leqq C \cdot e^{-|n|} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence $\sum c_n e^{in(s+it)}$ converges absolutely if $|t| < 1$, and so F_1 can be extended to a function which is analytic in the strip $|t| < 1$.

By (6), we conclude that F *is analytic in a neighborhood of* 0.

Translation shows that F is analytic at any interior point of $[-1, 1]$. To prove analyticity at the end-points, put

(12) $$F_2(s) = F(1 - s^2) \qquad (-1 \leqq s \leqq 1).$$

Then F_2 operates in $A(\Gamma)$, and since F_2 is an even function, we have, for some $\varepsilon > 0$,

(13) $$F_2(s) = \sum_{0}^{\infty} a_n s^{2n} \qquad (-\varepsilon < s < \varepsilon).$$

Hence

$$F(1-s) = \sum_0^\infty a_n s^n \qquad (0 \leqq s < \varepsilon^2), \tag{14}$$

and so F is analytic at the right end-point.

The other end-point can be treated similarly, and the proof is complete for compact Γ.

6.6.4. The general case of Theorem 6.6.1 follows easily. If Γ contains an infinite compact subgroup Λ, and if F operates in $A(\Gamma)$, then F also operates in $A(\Lambda)$, by Theorem 2.7.4, and so F is analytic on $[-1, 1]$.

If every compact subgroup of Γ is finite, then, since Γ is not discrete, Γ has a closed subgroup which is isomorphic to the real line R, by the structure theorem 2.4.1. Hence F operates in $A(R)$. But this implies that F operates in $A(T)$, and hence the problem is again reduced to the compact case:

Choose $f \in A(T)$, such that $-1 \leqq f(e^{ix}) \leqq 1$, and put $g(x) = f(e^{ix})$ $(-\infty < x < \infty)$. Then $g \in B(R)$, and if $\phi \in A(R)$, $0 \leqq \phi \leqq 1$, $\phi = 1$ on some interval J, then $\phi g \in A(R)$, hence $F(\phi g) \in A(R)$, hence $F(g)$ belongs to $A(R)$ locally at every point of J. Since J was arbitrary, $F(g)$ belongs to $A(R)$ locally at every point of R, hence (Theorem 2.7.6) $F(f)$ belongs to $A(T)$ locally at every point of T, and Lemma 6.2.6 implies that $F(f) \in A(T)$.

This completes the proof.

6.7. Comments on the Preceding Theorems

6.7.1. In Theorems 6.3.1, 6.5.1, and 6.6.1, we strongly used the knowledge that the algebras in question contain real elements f such that $\|e^{if}\| = e^{\|f\|}$, or at least that the equality can be almost attained. The use of this fact is quite natural. For suppose A is a semi-simple self-adjoint Banach algebra, represented as an algebra of functions on its maximal ideal space, and suppose there is a sequence $\{\omega_n\}$ such that

$$\|e^{inf}\| \leqq C_f \omega_n \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{1}$$

for every real $f \in A$, where C_f is a constant depending on f. If now $\sum |a_n|\omega_n < \infty$ and

$$(2) \qquad F(s) = \sum_{-\infty}^{\infty} a_n e^{ins} \qquad (-\infty < s < \infty),$$

then F operates in A; unless ω_n increases exponentially, we thus have operating functions which cannot be analytically extended to any open set containing the real axis.

6.7.2. A major step in the preceding theorems was the proof that the map (F) has some boundedness properties. In the case of $B(\Gamma)$, with Γ not compact, we first proved that (F) is bounded on each compact set. The conclusion was that F is entire, and hence we have the stronger result that (F) is bounded on every bounded subset of $B_R(\Gamma)$.

For $A(\Gamma)$ the situation is different. We proved again that (F) must carry some sphere about the origin into a bounded subset of $A(\Gamma)$. Examination of the proof of Theorem 6.6.1 shows that if (F) is bounded on every bounded set, then F must be entire. But there are functions F on $[-1, 1]$ which operate in $A(\Gamma)$ and which are not entire. Hence it may happen that (F) is unbounded on some sphere in $A_R(\Gamma)$, although (F) must also be bounded on some sphere. This sort of behavior is of course impossible for linear transformations.

Nevertheless, we can show that if F operates in $A(\Gamma)$, then (F) is an *analytic transformation*, in the following sense:

6.7.3. THEOREM. *Suppose Γ is a non-discrete LCA group, F is defined on $[-1, 1]$, and F operates in $A(\Gamma)$. If $f \in A(\Gamma)$, and $-1 \leqq f \leqq 1$, there exists $\delta > 0$ with the following property: if $g \in A(\Gamma)$, $-1 \leqq f + g \leqq 1$, and $\|g\| < \delta$, then*

$$(1) \qquad F(f + g) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(f) g^n.$$

Here $F^{(n)}$ denotes the nth derivative of F, and the "Taylor series" in (1) converges absolutely in the norm of $A(\Gamma)$.

Proof: By Theorem 6.6.1, F can be extended so as to be analytic in a simply connected region D which contains $[-1, 1]$. Since $F^{(n)}$ is analytic on $[-1, 1]$, $F^{(n)}$ operates in $A(\Gamma)$, and the terms of the series (1) are defined. Let C be a simple closed curve in D which surrounds $[-1, 1]$. Then there exist constants $\delta > 0$, $K_1 < 1$, $K_2 < \infty$, such that

$$\left\|\frac{g}{\lambda - f}\right\| \leqq K_1, \quad \left\|\frac{g}{(\lambda - f)^2}\right\| \leqq K_2, \text{ if } \|g\| < \delta, \; \lambda \in C. \tag{2}$$

We have

$$F^{(n)}(f(\gamma))g^n(\gamma) = \frac{n!}{2\pi i}\int_C \frac{g^n(\gamma)}{(\lambda - f(\gamma))^{n+1}} F(\lambda)d\lambda \qquad (\gamma \in \Gamma). \tag{3}$$

Approximating this integral by Riemann sums, (2) and Theorem 1.9.2 show that

$$\left\|\frac{1}{n!} F^{(n)}(f)g^n\right\| \leqq K_2 K_1^{\,n-1} \cdot \sup_{\lambda \in C} |F(\lambda)| \cdot L, \tag{4}$$

where L is the length of C, so that

$$\left\|\frac{1}{n!} F^{(n)}(f)g^n\right\| \leqq \text{const.}\, K_1^n \qquad (n = 1, 2, 3, \ldots). \tag{5}$$

Since $K_1 < 1$, the series (1) converges absolutely in $A(\Gamma)$. Since the ordinary Taylor formula shows that

$$F(f(\gamma) + g(\gamma)) = \sum_0^\infty \frac{1}{n!} F^{(n)}(f(\gamma))g^n(\gamma) \qquad (\gamma \in \Gamma), \tag{6}$$

provided that $\|g\|_\infty$ is less than the distance from $[-1, 1]$ to the boundary of D, we see that the series (1) converges to $F(f + g)$.

6.7.4. The same result holds for discrete Γ, if we replace $[-1, 1]$ by the interval J on which F is analytic. Since F can be a perfectly arbitrary function outside J (F need not be bounded, for instance) it is clear that this restriction is needed.

6.8. Range Transformations on Some Quotient Algebras

6.8.1. Let E be a compact set in Γ, and let $A(E)$ be the set of all functions on E which are restrictions to E of functions belonging to $A(\Gamma)$. If I is the set of all $f \in A(\Gamma)$ such that $f(\gamma) = 0$ on E, then $A(E)$ is the quotient algebra $A(\Gamma)/I$. One may ask which functions operate in $A(E)$. The following result is due to Kahane and Katznelson [1]; see also Katznelson [3].

6.8.2. THEOREM. *Suppose E is a compact set in R which contains arbitrarily long arithmetic progressions. If F is defined on the real axis and if F operates in $A(E)$, then F is analytic on the real axis.*

If $m(E) > 0$, then our hypothesis is satisfied; for if g is the characteristic function of E and if

$$(1) \qquad h(x) = \int_{-\infty}^{\infty} g(t + x)g(t + 2x) \dots g(t + nx)dt,$$

then h is continuous, $h(0) = m(E) > 0$, and so $h(x) > 0$ for some $x > 0$. For this x there exists t such that each of the points $t + x$, $t + 2x, \dots, t + nx$ lies in E. (The same argument shows that E contains an affine image of every finite subset of R.)

Another example, which is perhaps more interesting, is obtained by taking for E the set of all points $1/n$ ($n = 1, 2, 3, \dots$) plus their limit point 0. This example illustrates the arithmetic nature of the theorem: there are arbitrarily small displacements of E which produce an independent compact set E'; since E' is a Helson set (Theorem 5.6.7), $A(E') = C(E')$, and thus *every* continuous function operates in $A(E')$.

No example is known which lies between these two extremes; i.e., no set E is known such that some non-analytic function operaates in $A(E)$ although $A(E) \neq C(E)$. (Compare Katznelson [2], [4].)

6.8.3. To prove the theorem, it is clearly enough to show that F is analytic at 0, and we may assume that $F(0) = 0$. The union of any collection of arbitrarily long arithmetic progressions in E has a limit point, and from this it follows that there are sets S_N in E ($N = 1, 2, 3, \dots$) which consist of the points

$$(1) \qquad y_N + j\varepsilon_N \qquad (-2N \leqq j \leqq 2N;\ \varepsilon_N > 0;\ y_N \in R),$$

such that the intervals $I_N = [y_N - 2N\varepsilon_N, y_N + 2N\varepsilon_N]$ are disjoint.

Replacing E by one of its subsets, we may assume that $E \cap I_N = S_N$, for $N = 1, 2, 3, \ldots$.

The norm in $A(E)$ is the usual quotient norm: If $\phi \in A(E)$, then $\|\phi\| = \inf \|g\|_1$, the infimum being taken over all $g \in L^1(R)$ such that $\hat{g} = \phi$ on E.

6.8.4. LEMMA. *Suppose $\phi \in A(E)$, the support of ϕ lies in S_N, and*

$$\phi(y_N + j\,\varepsilon_N) = a_j \qquad (-2N \leqq j \leqq 2N), \tag{1}$$

where $a_j = 0$ if $|j| > N$. Let P be the trigonometric polynomial defined on T by

$$P(e^{i\theta}) = \sum_{-N}^{N} a_j e^{ij\theta} \qquad (e^{i\theta} \in T). \tag{2}$$

Then

$$\|\phi\| \leqq \|P\|_1 \leqq 3\|\phi\|. \tag{3}$$

Proof: Since affine transformations of R do not affect the norm in $A(R)$, we may assume that $y_N = 0$ and $\varepsilon_N = 1$.

Define $\hat{k}(y) = \max(1 - |y|, 0)$. Since $\hat{k} = u * u$, where u is the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$, $\hat{k}$ is positive-definite, and is therefore the Fourier transform of a non-negative function $k \in L^1(R)$. If $f(e^{ix}) = \sum k(x + 2r\pi)$ $(r \in Z)$, then $f \in L^1(T)$ and

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{ix})e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)e^{-inx}\,dx \tag{4}$$

$$= \hat{k}(n) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0. \end{cases}$$

Thus $f(e^{ix}) = 1$ for all $x \in R$. If now

$$\hat{g}(y) = \sum_{-N}^{N} a_j \hat{k}(y - j) \qquad (y \in R), \tag{5}$$

then $\hat{g} \in A(R)$, $\hat{g} = \phi$ on E, and $g(x) = P(e^{ix})k(x)$ $(x \in R)$. Hence

$$\|g\|_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |P(e^{ix})|k(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(e^{ix})|f(e^{ix})dx = \|P\|_1, \tag{6}$$

so that $\|\phi\| \leqq \|P\|_1$.

To prove the other inequality in (3), suppose $h \in L^1(R)$ and $\hat{h} = \phi$ on E, and define $\hat{v}_N(y) = 2\hat{k}(y/2N) - \hat{k}(y/N)$. Then $\hat{v}_N \in A(R)$, $\hat{v}_N(y) = 1$ if $|y| \leqq N$, $\|v_N\|_1 \leqq 3$, and

$$(7) \qquad P(e^{ix}) = \sum_{j=-\infty}^{\infty} \hat{v}_N(j)\hat{h}(j)e^{ijx} = \sum_{n=-\infty}^{\infty} (v_N * h)(x + 2n\pi),$$

so that

$$(8) \qquad \|P\|_1 \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} |(v_N * h)(x)|dx \leqq 3\|h\|_1.$$

The lemma follows.

6.8.5. We can now complete the proof of Theorem 6.8.2. Let us suppose, to get a contradiction, that there is a sequence $\{P_n\}$ of trigonometric polynomials on T such that $\hat{P}_n$ is real, $\|P_n\|_1 \to 0$, but $\|F \circ P_n\|_1 \to \infty$, where

$$(1) \qquad (F \circ P_n)(e^{i\theta}) = \sum F(a_j)e^{ij\theta}$$

if $P_n(e^{i\theta}) = \sum a_j e^{ij\theta}$.

If we replace $\{P_n\}$ by a suitable subsequence and apply Lemma 6.8.4, we see that there is a sequence $\{\phi_n\}$ in $A(E)$, with the following properties: $\phi_n = 0$ except on S_{N_n}, $\|\phi_n\| \leqq n^{-2}$, and $\|F(\phi_n)\| \to \infty$. If $\phi = \sum_1^\infty \phi_n$, then $\phi \in A(E)$. Setting

$$(2) \qquad \alpha_n(y) = \hat{v}_{N_n}((y - y_{N_n})/\varepsilon_{N_n}) \qquad (y \in E;\ n = 1, 2, 3, \ldots),$$

where $\hat{v}_N$ is the function used in the proof of Lemma 6.8.4, we obtain

$$(3) \qquad \alpha_n F(\phi) = \alpha_n F(\phi_n) = F(\phi_n) \qquad (n = 1, 2, 3, \ldots),$$

so that

$$(4) \qquad \|F(\phi_n)\| \leqq \|\alpha_n\| \cdot \|F(\phi)\| \leqq 3\|F(\phi)\|,$$

contradicting the assumption that $\|F(\phi_n)\| \to \infty$.

Hence $\|F \circ P\|_1 \leqq C$ for all trigonometric polynomials P on T with $\hat{P}$ real and $\|P\|_1 < \delta$, for some $\delta > 0$, $C < \infty$. This implies that F operates in $A(Z)$, and thus F is analytic in a neighborhood of the origin, by Theorem 6.5.1.

6.9. Operating Functions Defined in Plane Regions

6.9.1. Since $A(\Gamma)$ and $B(\Gamma)$ are algebras of complex-valued functions, it may seem unnatural and unduly restrictive to have confined our attention to functions F defined on the real axis, or even on an interval. However, this was done primarily to simplify the exposition, and no difficulty is encountered in extending the results.

Let us suppose that F is defined in an open plane region E which contains the origin. The analogues of Theorems **6.3.1**, **6.5.1**, and 6.6.1 are then the precise converses of the sufficient conditions obtained in Section **6.2**.

6.9.2. THEOREM. *If F operates in $B(\Gamma)$ and Γ is not compact, then F can be extended to a real-entire function in the plane. If F operates in $A(\Gamma)$ and Γ is discrete, then F is real-analytic in some neighborhood of the origin; if Γ is not discrete, then F is real-analytic in E.*

Only one new device is needed in the proof: in place of the periodic functions $F(r \sin s)$ we now use the doubly periodic functions

$$F_1(s, t) = F(r \sin s, r \sin t)$$

which we expand in Fourier series of the form

$$\sum_{n,\, m=-\infty}^{\infty} c_{nm}\, e^{i(ns+mt)},$$

and we estimate the coefficients c_{nm} in the same way in which we previously estimated the coefficients c_n.

6.9.3. If E is a closed convex set in the plane, if F is defined on E, and if F operates in $A(\Gamma)$ for some non-discrete Γ, then the full analogue of Theorem **6.6.1** holds (Helson and Kahane [1]): *F is real-analytic on E* (not just in the interior of E).

To prove this, suppose p is a boundary point of E. If τ is any affine transformation of the plane, then $F \circ \tau$ operates in $A(\Gamma)$, and since the class of all real-analytic functions is invariant under affine transformations, we may assume that p is the origin and that E contains the set of all (s, t) with $s \geqq 0$, $t \geqq 0$, $s^2 + t^2 \leqq 1$.

If now $F_1(s, t) = F(s^2, t^2)$, then F_1 operates in $A(\Gamma)$, F_1 is defined in a full neighborhood of the origin, and since F_1 is an even function of s and t, Theorem 6.9.2 shows that $F_1(s,t) = \sum a_{mn} s^{2m} t^{2n}$ in some neighborhood of the origin. Hence there exists $\delta > 0$ such that

$$(1) \qquad F(s, t) = \sum a_{mn} s^m t^n \qquad (s \geqq 0,\ t \geqq 0,\ s^2 + t^2 < \delta^2).$$

Finally, two real-analytic functions which coincide in an open set are identical. and since F is real-analytic in the interior of E, the equality in (1) holds for all $(s, t) \in E$ such that $s^2 + t^2 < \delta^2$. Thus F is real-analytic at $(0, 0)$.

6.9.4. We conclude this chapter with some open problems.

(a) Which functions F operate in the set of all positive-definite functions on Γ? *If $\Gamma = Z$ and if F is defined on* $[-1, 1]$, *a necessary and sufficient condition is that*

$$(1) \qquad F(t) = \sum_{n=0}^{\infty} a_n t^n \qquad \left(a_n \geqq 0,\ \sum_{0}^{\infty} a_n < \infty\right).$$

The proof (Rudin [15]) extends to any Γ which is not of bounded order. For groups of bounded order the problem is open. Also, the problem is open for every Γ if we assume that F is defined in the closed unit disc. One may conjecture that F must then be of the form

$$(2) \qquad F(z) = \sum_{m,\, n=0}^{\infty} a_{mn} z^m \bar{z}^n \qquad \left(a_{mn} \geqq 0,\ \sum a_{mn} < \infty\right).$$

(b) For discrete Γ (or even for $\Gamma = Z$), which functions F have the property that $F(\hat{f})$ is the Fourier transform of a function in $L^p(G)$ (or in $C(G)$) whenever $f \in L^p(G)$ (or $f \in C(G)$)? The case $p = 2$ is trivial here; a necessary and sufficient condition is that $|F(z)/z|$ be bounded in a neighborhood of the origin, and $F(0) = 0$. In the other cases, only partial results are known (Rudin [16]).

(c) Define $B_0(\Gamma) = B(\Gamma) \cap C_0(\Gamma)$; i.e., $B_0(\Gamma)$ consists of all Fourier-Stieltjes transforms on Γ which vanish at infinity. Let $M_0(G)$ be the set of all $\mu \in M(G)$ with $\hat{\mu} \in B_0(\Gamma)$, and suppose Γ is discrete. If F, defined on $[-1, 1]$, operates in $B_0(\Gamma)$, must F

coincide with an entire function in some neighborhood of 0, or is it enough for F to be analytic in some neighborhood of 0?

The latter condition is necessary since $B_0(\Gamma) \supset A(\Gamma)$. Since there is an independent perfect set P in R which carries a measure $\mu \in M_0(R)$, (see 5.6.11) our proof of the asymmetry of $M(G)$ applies to the algebras $M_0(R)$ and $M_0(T)$. It seems plausible that $M_0(G)$ is asymmetric for all non-discrete G, and this may imply that the entire functions are the only ones which operate in $B_0(\Gamma)$.

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CHAPTER 7

Closed Ideals in $L^1(G)$

7.1. Introduction

7.1.1. In the group context, it is naturally of interest to study function spaces which are invariant under translation; the relevance of the Fourier transform is illustrated by the fact that it converts translation to multiplication by a character. The closed translation-invariant subspaces of $L^1(G)$ can be very neatly characterized in algebraic terms: they are exactly the closed ideals in $L^1(G)$. This may be regarded as one of the "reasons" for the close connection between Fourier analysis and the theory of Banach algebras.

Let I be a translation invariant subspace of $L^1(G)$, and suppose $\phi \in L^\infty(G)$ annihilates I; that is to say,

$$(1) \qquad \int_G f(-y)\phi(y)dy = 0 \qquad (f \in I).$$

Since I contains every translate of f if $f \in I$, we also have

$$(2) \qquad \int_G f(x-y)\phi(y)dy = 0 \qquad (f \in I,\ x \in G).$$

Hence, *to say that ϕ annihilates I is the same as to say that $f * \phi = 0$ for all $f \in I$.*

With the aid of this remark, it is easy to prove the equivalence stated above:

7.1.2. THEOREM. *Every closed translation-invariant subspace of $L^1(G)$ is an ideal; conversely, every closed ideal in $L^1(G)$ is translation invariant.*

Proof: For $f, g \in L^1(G)$ and $\phi \in L^\infty(G)$ we have

$$(1) \qquad \int_G (f * g)(-x)\phi(x)dx = \int_G g(-y)(f * \phi)(y)dy,$$

since each of these expressions is $(f * g * \phi)(0)$.

Suppose I is closed and translation-invariant, ϕ annihilates I, and $f \in I$. Then $f * \phi = 0$, the right side of (1) is 0, hence ϕ annihilates $f * g$, for every $g \in L^1(G)$. Since this is true for every ϕ which annihilates I, the Hahn-Banach theorem implies that $f * g \in I$, and so I is an ideal.

Suppose I is a closed ideal, ϕ annihilates I, and $f \in I$. Then $f * g \in I$, the left side of (1) is 0, hence $f * \phi$ annihilates every $g \in L^1(G)$, and so $f * \phi = 0$. This says that ϕ annihilates every translate of f, and if we apply the Hahn-Banach theorem once more, we see that I contains every translate of f.

7.1.3. For $f \in L^1(G)$, we define $Z(f)$ to be the set of all $\gamma \in \Gamma$ such that $\hat{f}(\gamma) = 0$, and if I is an ideal in $L^1(G)$, we define the *zero-set* of I by

(1) $$Z(I) = \bigcap_{f \in I} Z(f).$$

Thus $\gamma \in Z(I)$ if and only if $\hat{f}(\gamma) = 0$ for all $f \in I$.

Since $\hat{f}$ is continuous on Γ, each $Z(f)$ is closed, hence $Z(I)$ is closed for every I. Conversely, each closed set E in Γ is $Z(I)$ for some closed ideal I of $L^1(G)$: simply take for I the set of *all* $f \in L^1(G)$ such that $E \subset Z(f)$. Since convolution in $L^1(G)$ corresponds to pointwise multiplication in $A(\Gamma)$, I is an ideal; since norm convergence in $L^1(G)$ implies uniform (hence pointwise) convergence in $A(\Gamma)$, I is closed; if $\gamma_0 \notin E$, there exists $f \in L^1(G)$ such that $\hat{f}(\gamma_0) = 1$, but $E \subset Z(f)$, and this shows that $Z(I) = E$.

This ideal is evidently the largest one with the property that $Z(I) = E$. We shall denote it by $I(E)$.

7.1.4. We can now state the question to which the present chapter is devoted:

Can there be two distinct closed ideals I_1 and I_2 in $L^1(G)$ such that $Z(I_1) = Z(I_2)$? Or does $Z(I)$ determine I?

A set $E \subset \Gamma$ such that $E = Z(I)$ for a *unique* closed ideal I in $L^1(G)$ will be called an S-set. The letter S stands for "spectral synthesis"; this will be discussed in Section 7.8. Our question can now be rephrased.

Is every closed set in Γ an S-set?

The answer turns out to be affirmative if Γ is discrete. If Γ is not discrete, then Γ contains certain types of sets which are S-sets and also contains closed sets which are not S-sets. The examples described in the present chapter are of such diverse nature that the problem of finding structural conditions which are necessary and sufficient for a closed set to be an S-set seems hopelessly difficult.

7.1.5. For discrete Γ, the problem is so simple that it is worthwhile to deal with this case separately, although the result is contained in Theorem 7.2.4. The simplification is due to the fact that the continuous characters on G belong to $L^1(G)$ if G is compact.

THEOREM. *Suppose G is compact and I is a closed ideal in $L^1(G)$. If $f \in L^1(G)$ and $Z(I) \subset Z(f)$, then $f \in I$.*

Proof: If $\gamma_0 \notin Z(I)$, there exists $g \in I$ with $\hat{g}(\gamma_0) = 1$, and hence $g * \gamma_0 = \gamma_0$, regarding γ_0 as a member of $L^1(G)$. Since I is an ideal, $g * \gamma_0 \in I$, and so $\gamma_0 \in I$. It follows that I contains every trigonometric polynomial on G of the form $\sum a_\gamma(x, \gamma)$, provided that $a_\gamma = 0$ for all $\gamma \in Z(I)$. If $Z(f) \supset Z(I)$, then $f * k$ satisfies this condition for every trigonometric polynomial k on G. Since $\|f - f * k\|_1$ can be arbitrarily small (Theorem 2.6.6) and since I is closed, we conclude that $f \in I$.

7.2. Wiener's Tauberian Theorem

7.2.1. Wiener's theorem has several equivalent formulations. One of these asserts that the empty set is an S-set; in other words, $L^1(G)$ is the only closed ideal I in $L^1(G)$ for which $Z(I)$ is empty. The proof which follows is, in essence, that of Wiener, in spite of the fact that the terminology and the details are quite different; it has evolved through several stages (Wiener [1], Ditkin [1], Mandelbrojt and Agmon [1], Kaplansky [1], Helson [1], Reiter [1], Loomis [1]) and now yields a considerably stronger result (Theorem 7.2.4).

In what follows, G is an arbitrary LCA group. If I is an ideal in $L^1(G)$, $\hat{I}$ denotes the set of all $\hat{f} \in A(\Gamma)$ with $f \in I$; $\hat{I}$ is then an ideal in $A(\Gamma)$.

7.2.2. Lemma. *Suppose $f \in L^1(G)$, I is an ideal in $L^1(G)$, and $\gamma_0 \in \Gamma$. Then $\hat{f}$ belongs to $\hat{I}$ locally at γ_0 if either of the following conditions is satisfied:*

(a) *γ_0 is not in $Z(I)$;*

(b) *γ_0 is in the interior of $Z(f)$.*

Proof: If (a) holds, there exists $g \in I$ with $\hat{g}(\gamma_0) = 1$, and Theorem 2.6.5 shows that there exists $h \in L^1(G)$ such that $\|h\| < \frac{1}{2}$ and $\hat{h}(\gamma) = 1 - \hat{g}(\gamma)$ in some neighborhood V of γ_0. The series $\sum_0^\infty \hat{f}\hat{h}^n$ converges, in the norm of $A(\Gamma)$, to a function $\hat{\jmath} \in A(\Gamma)$, and $\hat{\jmath}(\gamma) = \{1 - \hat{h}(\gamma)\}^{-1}\hat{f}(\gamma)$ for all $\gamma \in \Gamma$. If $\gamma \in V$, then $\hat{g}(\gamma)\hat{\jmath}(\gamma) = \hat{f}(\gamma)$; since $\hat{g} \in \hat{I}$ and $\hat{I}$ is an ideal, $\hat{g}\hat{\jmath} \in \hat{I}$, and so $\hat{f}$ belongs to $\hat{I}$ locally at γ_0.

If (b) holds, then $\hat{f} = 0$ in a neighborhood of γ_0, and since $\hat{I}$ contains the constant 0, $\hat{f}$ belongs to $\hat{I}$ locally at γ_0.

7.2.3. Lemma. *Suppose $f \in L^1(G)$, I is a closed ideal in $L^1(G)$, $Z(I) \subset Z(f)$, and Q is the set of all points of Γ at which $\hat{f}$ does not belong to $\hat{I}$ locally. Then Q is closed and has no isolated point.*

Proof: It is trivial that the set of all points at which $\hat{f}$ belongs to $\hat{I}$ locally is an open set. Hence Q is closed.

Suppose γ_0 is an isolated point of Q. By Lemma 7.2.2.(a), $\gamma_0 \in Z(I)$, hence $\hat{f}(\gamma_0) = 0$. Let W be a compact neighborhood of γ_0 which contains no other point of Q, and choose $k \in L^1(G)$ such that $\hat{k} = 0$ outside W and $\hat{k} = 1$ in some neighborhood of γ_0. By Theorem 2.6.4 there is a sequence $\{v_n\}$ in $L^1(G)$ such that each $\hat{v}_n$ is 0 in some neighborhood of γ_0 and such that

$$\lim_{n\to\infty} \|f - f * v_n\|_1 = 0.$$

For each n, $\hat{f}\hat{k}\hat{v}_n$ belongs to $\hat{I}$ locally at every point of $\Gamma \cup \{\infty\}$: at γ_0, consider $\hat{v}_n$; at other points of W, consider $\hat{f}$; and in the complement of W, $\hat{k} = 0$. Lemma 6.2.6 now implies that $\hat{f}\hat{k}\hat{v}_n \in \hat{I}$, for $n = 1, 2, 3, \ldots$. Since $\hat{I}$ is closed in $A(\Gamma)$, $\hat{f}\hat{k} \in \hat{I}$; and since $\hat{k} = 1$ in a neighborhood of γ_0, $\hat{f}$ belongs to $\hat{I}$ locally at γ_0. But this means that $\gamma_0 \notin Q$, a contradiction.

7.2.4. Theorem. *Suppose $f \in L^1(G)$, I is a closed ideal in $L^1(G)$,* *and* $Z(I) \subset Z(f)$. *If the intersection of the boundaries of* $Z(I)$ *and* $Z(f)$ *contains no perfect set, then* $f \in I$.

Proof: The symbol ∂E will denote the boundary of a set E in Γ. For $n = 1, 2, 3, \ldots$, put $f_n = f * u_n$, where $u_n \in L^1(G)$ is so selected that $\hat{u}_n$ has compact support and $\|f - f_n\|_1 \to 0$ (Theorem 2.6.6). Since $Z(I) \subset Z(f) \subset Z(f_n)$, we have

$$(1) \qquad Z(I) \cap \partial Z(f_n) \subset Z(I) \cap \partial Z(f) = \partial Z(I) \cap \partial Z(f).$$

If $\gamma_0 \in \Gamma$, Lemma 7.2.2 shows that $\hat{f}_n$ belongs to $\hat{I}$ locally at γ_0 unless $\gamma_0 \in Z(I) \cap \partial Z(f_n)$; by (1), this set contains no perfect subset, and hence Lemma 7.2.3 shows that $\hat{f}_n$ belongs to $\hat{I}$ locally at every $\gamma_0 \in \Gamma$. Since $\hat{f}_n$ has compact support, $\hat{f}_n$ belongs to $\hat{I}$ locally at infinity. Hence Lemma 6.2.6 implies that $f_n \in I$, for $n = 1, 2, 3, \ldots$, and so $f \in I$, since I is closed.

7.2.5. Corollaries of theorem 7.2.4. (a) *If* $Z(I)$ *is in the interior of* $Z(f)$, *then* $f \in I$.

Or, if $\hat{f} = 0$ in an open set containing $Z(I)$, then $f \in I$. This has some interesting consequences:

Let E be a closed set in Γ, and let $I_0(E)$ be the closure of the set of all $f \in L^1(G)$ such that E is in the interior of $Z(f)$. It follows from (a) that $I_0(E)$ is the *smallest* closed ideal I of $L^1(G)$ such that $Z(I) = E$. Thus each closed set E in Γ has a largest ideal $I(E)$ (Section 7.1.3) and a smallest closed ideal $I_0(E)$ associated with it, and E is an S-set if and only it $I(E) = I_0(E)$. The question whether E is an S-set can therefore be restated in two ways:

(i) *If* $\phi \in L^\infty(G)$ *and* $f * \phi = 0$ *for every* $f \in I_0(E)$, *does it follow that* $f * \phi = 0$ *for every* $f \in I(E)$?

(ii) *If* $f \in L^1(G)$ *and* $\hat{f} = 0$ *on* E, *can* f *be approximated, in the norm of* $L^1(G)$, *by functions* $g \in L^1(G)$ *such that* $\hat{g} = 0$ *on an open set containing* E?

(b) *If the boundary of a closed set* E *in* Γ *contains no perfect set, then* E *is an* S-*set.*

Note that the hypothesis involves only the *topological* structure of E as a subset of Γ. No stronger result of this type is known.

(c) *If* I *is a closed ideal in* $L^1(G)$ *and if* $Z(I)$ *is empty, then* $I = L^1(G)$.

This is Wiener's theorem (note that its proof does not require Lemma 7.2.3). If I is a closed ideal in $L^1(G)$ and $I \neq L^1(G)$, it follows that $Z(I)$ is not empty, and hence I is contained in a closed ideal J such that $Z(J)$ consists of just one point $\gamma_0 \in \Gamma$. Since J is the kernel of the homomorphism $f \to \hat{f}(\gamma_0)$, J is a regular maximal ideal (Appendix D3), and Wiener's theorem can be rephrased in the following terms:

Every proper closed ideal in $L^1(G)$ is contained in a regular maximal ideal.

For discrete groups G, this statement is almost trivial, since $L^1(G)$ then has a unit, and every proper ideal in a commutative ring with unit is contained in a maximal one (Appendix D2; the word "regular" is redundant in rings with unit). Keeping Theorem 7.1.5 in mind, we can therefore say that Wiener's theorem is most significant if G is neither compact nor discrete; the importance of the special case $G = R$ thus becomes apparent.

(d) *If $f \in L^1(G)$, the translates of f span $L^1(G)$* (i.e., the set of all finite linear combinations of translates of f is dense in $L^1(G)$) *if and only if $\hat{f}$ has no zero in Γ.*

To see this, let I be the smallest closed ideal of $L^1(G)$ which contains f; by Theorem 7.1.2, I is precisely the space spanned by the translates of f; since $Z(I) = Z(f)$ and since $I = L^1(G)$ if and only if $Z(I)$ is empty (by (c)), the proof is complete.

7.2.6. The tauberian character of Wiener's theorem. A tauberian theorem is, roughly speaking, one which asserts that if certain averages of a function have a limit, then the function itself has a limit. The original form of Wiener's theorem is of this type, although the conclusion is not quite so strong.

If $\phi \in L^\infty(G)$, the statement "$\phi(x) \to a$ as $x \to \infty$" will mean that to every $\varepsilon > 0$ there exists a compact set K in G such that $|\phi(x) - a| < \varepsilon$ in the complement of K. If $f \in L^1(G)$, the convolution $(f * \phi)(x)$ may be regarded as an "average" of ϕ, obtained by assigning a weight factor $f(x - y)$ to the value $\phi(y)$; this terminology is most appropriate if $\hat{f}(0) = 1$. It is easy to prove that $(f * \phi)(x) \to a\hat{f}(0)$ if $\phi(x) \to a$ as $x \to \infty$; we omit the proof, since we will use this fact only for constant ϕ in which case it is quite trivial.

THEOREM. *Suppose* $\phi \in L^\infty(G)$, $f \in L^1(G)$, $\hat{f}(\gamma) \neq 0$ *for all* $\gamma \in \Gamma$, *and*

$$(1) \qquad (f * \phi)(x) \to a\hat{f}(0) \qquad (x \to \infty).$$

Then the limit relation

$$(2) \qquad (g * \phi)(x) \to a\hat{g}(0) \qquad (x \to \infty)$$

holds for every $g \in L^1(G)$.

Proof: Replacing ϕ by $\phi - a$, we may assume, without loss of generality, that $a = 0$. The set I of all $g \in L^1(G)$ such that $(g * \phi)(x) \to 0$ as $x \to \infty$ is a linear subspace of $L^1(G)$ which is clearly translation-invariant; I is closed, for if $\|g_n - g\|_1 \to 0$, then $\|g_n * \phi - g * \phi\|_\infty \to 0$; and $f \in I$. Hence I is a closed ideal in $L^1(G)$ with $Z(I)$ empty, and so $I = L^1(G)$.

7.2.7. If we impose slightly stronger conditions on ϕ, the conclusion of the preceding theorem may be replaced by the stronger assertion that $\phi(x) \to a$ as $x \to \infty$.

Let us call a function $\phi \in L^\infty(G)$ *slowly oscillating* if $\phi(x) - \phi(y) \to 0$ as $x \to \infty$ and $x - y \to 0$. More explicitly, we require that to each $\varepsilon > 0$ there should exist a compact set K in G and a compact neighborhood V of 0 in G such that $|\phi(x) - \phi(y)| < \varepsilon$ if $x - y \in V$ and $x \notin K$. For instance, uniformly continuous bounded functions are slowly oscillating; but slowly oscillating functions need not be continuous.

THEOREM (Pitt [1]). *Suppose* $\phi \in L^\infty(G)$, ϕ *is slowly oscillating*, $f \in L^1(G)$, $\hat{f}(\gamma) \neq 0$ *for all* $\gamma \in \Gamma$, *and*

$$(1) \qquad (f * \phi)(x) \to a\hat{f}(0) \qquad (x \to \infty).$$

Then $\phi(x) \to a$ *as* $x \to \infty$.

Proof: Given $\varepsilon > 0$, choose K and V as above, and let g be the characteristic function of V, divided by $m(V)$. Then

$$(2) \qquad \phi(x) - (g * \phi)(x) = \frac{1}{m(V)} \int_V \{\phi(x) - \phi(x - y)\} dy,$$

so that $|\phi(x) - (g * \phi)(x)| < \varepsilon$ in the complement of K. By 7.2.6, $(g * \phi)(x) \to a$ as $x \to \infty$, and the desired conclusion follows.

7.2.8. If $G = R$, we may consider the behavior of $\phi(x)$ or of $(f * \phi)(x)$ as $x \to +\infty$ or as $x \to -\infty$. Theorems 7.2.6 and 7.2.7 remain true (with the same proofs) if we replace "$x \to \infty$" by "$x \to +\infty$" or "$x \to -\infty$" in the hypotheses as well as in the conclusions. For applications one usually needs the results in this form (Wiener [1], Pitt [1]).

7.2.9. To illustrate how much easier the L^2-theory is, let us consider the analogue of Theorem 7.2.5(d) in $L^2(G)$:

THEOREM. *If $f \in L^2(G)$, the translates of f span $L^2(G)$ if and only if $\hat{f}(\gamma) \neq 0$ for almost all $\gamma \in \Gamma$.*

Proof: Note that $\hat{f}$ now denotes the Plancherel transform of f and is defined as an element of $L^2(\Gamma)$; as a point function, $\hat{f}$ is only defined up to sets of Haar measure zero. Hence it is quite natural to have "almost all" in the statement.

If $g \in L^2(G)$ and $\int_G f_x \bar{g} = 0$ for every translate f_x of f, the Parseval formula shows that

$$\int_\Gamma \hat{f}(\gamma)\overline{\hat{g}(\gamma)}(-x, \gamma)d\gamma = 0 \qquad (x \in G). \tag{1}$$

Since $\hat{f}$ and $\bar{\hat{g}}$ are in $L^2(\Gamma)$, their product is in $L^1(\Gamma)$ and the uniqueness theorem for Fourier transforms implies that $\hat{f}\bar{\hat{g}} = 0$ almost everywhere on Γ.

If $\hat{f} \neq 0$ almost everywhere, it follows that $g = 0$, and hence (by the Hahn-Banach theorem) that the translates of f span $L^2(G)$.

If $\hat{f} = 0$ on a set E in Γ with $m(E) > 0$, there exists $g \neq 0$, $g \in L^2(G)$, such that $\hat{g} = 0$ outside E. Since $\hat{f}\bar{\hat{g}} = 0$, the Parseval formula shows, as above, that $\int_G f_x \bar{g} = 0$ for every $x \in G$. Hence g is orthogonal to the span of the translates of f.

7.2.10. Arithmetic conditions can play a role in the problem of determining whether the translates of a function do or do not span $L^1(G)$.

For example, take $G = R$, let $f(x) = 2$ if $0 < x < 1$, $f(x) = 1$ if $1 < x < \alpha$, where α is a given real number, and $f(x) = 0$ for all other $x \in R$. Then $\hat{f}(y)$ is a constant multiple of $(2-e^{-iy}-e^{-i\alpha y})/y$, and hence $\hat{f}(y) = 0$ if and only if $y = 2n\pi$ and $\alpha y = 2m\pi$, where m, n are non-zero integers.

We conclude that the translates of f span $L^1(R)$ if and only if α is irrational. They span $L^2(R)$ for every $\alpha > 1$.

7.3. The Example of Schwartz

The first and simplest example of a closed set which is not an S-set, as defined in Section 7.1.4, is the unit sphere in the euclidean space R^3. We take $G = R^3$, so that $\Gamma = R^3$ (Theorem 2.2.2), and we let E be the set of all $y \in \Gamma$ whose distance from the origin is 1.

7.3.1. THEOREM. (Schwartz [1]). *E is not an S-set.*

Proof: Let Ω be the set of all infinitely differentiable complex functions on R^3 with compact support. If $\phi \in \Omega$ and f is the inverse transform of ϕ,

$$f(x) = \int_{R^3} \phi(y) e^{ix\cdot y}\, dy \qquad (x \in R^3) \tag{1}$$

where $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$, then the inverse transform of $\partial\phi/\partial y_k$ is $-ix_k f(x)$, etc.; since all derivatives of ϕ are in $L^1(R^3)$, it follows that $|x|^p|f(x)|$ is in $L^\infty(R^3)$, for $p = 0, 1, 2, \ldots$. Hence $f \in L^1(R^3)$, and so $\Omega \subset A(R^3)$, by the inversion theorem 1.7.3 (e).

Let J be the set of all $f \in L^1(R^3)$ such that $\hat{f} \in \Omega$ and $\hat{f}(y) = 0$ for all $y \in E$. Let I be the set of all $f \in J$ such that $\partial\hat{f}/\partial y_1 = 0$ on E. Then I and J are translation-invariant linear spaces, and their L^1-closures $\bar{I}$, $\bar{J}$ have $Z(\bar{I}) = Z(\bar{J}) = E$. We shall show that $\bar{I} \neq \bar{J}$ by constructing a bounded linear functional in $L^1(R^3)$ which annihilates I but not J.

Let μ be the unit mass, uniformly distributed over E. The inverse transform of μ is

$$\hat{\mu}(x) = \int_E e^{ix\cdot y}\, d\mu(y) \qquad (x \in R^3). \tag{2}$$

Fix $x \in R^3$, and introduce spherical coordinates on E, with pole at the point x/r, where r is the distance from 0 to x. Then

$$\hat{\mu}(x) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} e^{ir\cos\theta} \sin\theta\, d\theta = \frac{\sin r}{r}. \tag{3}$$

Hence $|x_1\hat{\mu}(x)| \leqq 1$ for all $x \in R^3$, and the expression

$$(4) \qquad \Psi f = \int_{R^3} f(x) x_1 \hat{\mu}(-x) dx$$

is a bounded linear functional on $L^1(R^3)$. If $\hat{f} \in \Omega$, then $x_1 f(x) \in L^1(R^3)$, and so

$$(5) \qquad \Psi f = \int_{R^3} x_1 f(x) dx \int_E e^{-ix \cdot y} \, d\mu(y) = i(2\pi)^3 \int_E \frac{\partial \hat{f}}{\partial y_1} \, d\mu.$$

Thus $\Psi f = 0$ if $f \in I$. But it is clear that there are functions in J for which the last integral is not 0.

7.3.2. The analogue of Theorem 7.3.1 holds if R^3 is replaced by R^n $(n \geqq 3)$ (Schwartz [1]), but does not hold in R^2 (Herz [2]), and evidently not in R (see 7.2.5(b), for instance).

7.4. *The Examples of Herz*

Cantor's "middle third" set is an S-set on the real line. This was proved by Herz [1] and was the first example of a totally disconnected perfect set in R which is an S-set. The idea of the proof was later extended to other cases (Herz [3]). We shall present the construction for compact Γ. This simplifies some of the technical details, and, as we shall see in Section 7.6, the compact case has the most important features of the general case.

7.4.1. Lemma. *Suppose Λ is a finite subgroup of the compact abelian group Γ. Then there is a Borel set Y in Γ with the following properties:*

(i) *$m(Y) = 1/n$, where n is the number of elements of Λ and m is the Haar measure of Γ;*

(ii) *the set $V = Y - Y$ is open;*

(iii) *$V \cap \Lambda$ contains only one point, namely* 0.

Proof: Every finite abelian group is a direct sum of cyclic groups. Thus Λ is generated by independent elements $\gamma_1, \ldots, \gamma_k$, of order $q_1, \ldots, q_k$, and $n = \Pi q_r$. There exist points $x_1, \ldots, x_k$ in G such that

$$(1)\qquad (x_r, \gamma_s) = \begin{cases} \exp\{2\pi i/q_r\} & \text{if } s = r \\ 1 & \text{if } s \neq r \end{cases} \quad (r, s = 1, \ldots, k).$$

If $\gamma \in \Lambda$, then $\gamma = \sum a_s \gamma_s$, where $a_1, \ldots, a_k$ are integers, and

$$(2)\quad (x_r, \gamma) = \prod_{s=1}^{k} (x_r, \gamma_s)^{a_s} = (x_r, \gamma_r)^{a_r} = \exp\{2\pi i a_r/q_r\} \qquad (1 \leqq r \leqq k).$$

Let Y_r be the set of all $\gamma \in \Gamma$ such that $0 \leqq \arg (x_r, \gamma) < 2\pi/q_r$, and put $Y = Y_1 \cap \ldots \cap Y_k$. Then $m(Y_r) = 1/q_r$. Since (1) shows that $\{x_1, \ldots, x_k\}$ is an independent set, it follows that

$$(3)\qquad m(Y) = \prod_{r=1}^{k} m(Y_r) = 1/n.$$

Since V is the set of all $\gamma \in \Gamma$ such that

$$(4)\qquad -2\pi/q_r < \arg (x_r, \gamma) < 2\pi/q_r \qquad (1 \leqq r \leqq k),$$

V is open, and (2) implies that $V \cap \Lambda = \{0\}$.

7.4.2. We now suppose that $\{\Lambda_i\}$ is a countable family of finite subgroups of Γ and we associate with each Λ_i two sets Y_i, V_i as in Lemma 7.4.1. We also assume that $\{V_i\}$ forms a neighborhood base at 0.

Suppose E is a compact subset of Γ such that

$$(1)\qquad (E + V_i) \cap \Lambda_i \subset E \qquad (i = 1, 2, 3 \ldots);$$

in other words, if $\gamma \in \Lambda_i$, then either $\gamma \in E$ or $\gamma + V_i$ contains no point of E.

THEOREM. *Under these conditions, E is an S-set.*

To give a class of examples on the circle group T (regarded as the reals mod 2π), let $\{p_i\}$ be a sequence of integers greater than 2, put $n_i = p_1 p_2 \ldots p_i$ $(i = 1, 2, 3, \ldots)$, let $\Lambda_i = \{2\pi j/n_i\}$ $(j = 1, \ldots, n_i)$, $Y_i = [0, 2\pi/n_i)$, $V_i = (-2\pi/n_i, 2\pi/n_i)$, and take for E the set of all numbers of the form

$$(2)\qquad x = 2\pi \sum_{i=1}^{\infty} a_i/n_i \qquad (a_i = 0 \text{ or } a_i = p_i - 1).$$

Then E is a Cantor set, and every point of Λ_i which is not in E is at a distance from E which is at least $2\pi/n_i$.

There are compact groups Γ in which this construction cannot be carried out. For instance, if Γ is the dual of the additive group R^* of the rational numbers, then Γ has no non-trivial finite subgroup, since R^* has no proper subgroup of finite index.

7.4.3. Proof of theorem 7.4.2. Let g_i be the characteristic function of Y_i, and define $k_i \in L^1(G)$ by putting

$$\hat{k}_i = n_i^2(g_i * \tilde{g}_i) \qquad (i = 1, 2, 3, \dots) \tag{1}$$

where n_i is the number of elements of Λ_i. Since $\hat{k}_i$ is continuous and positive-definite, we verify easily that

$$k_i \geqq 0, \quad k_i(0) = 1, \quad \hat{k}_i(0) = n_i, \quad \hat{k}_i(\gamma) = 0 \text{ if } \gamma \notin V_i. \tag{2}$$

For $\phi \in L^\infty(G)$, put

$$\Phi_i(\gamma) = \sum_{x \in G} \phi(x) k_i(-x)(-x, \gamma) \qquad (\gamma \in \Gamma) \tag{3}$$

and

$$\phi_i(x) = \frac{1}{n_i} \sum_{\gamma \in \Lambda_i} \Phi_i(\gamma)(x, \gamma) \qquad (x \in G). \tag{4}$$

If H_i is the subgroup of G which annihilates Λ_i, we have

$$\phi_i(x) = \sum_{y \in G} \phi(y) k_i(-y) \frac{1}{n_i} \sum_{\gamma \in \Lambda_i} (x - y, \gamma) = \sum_{y - x \in H_i} \phi(y) k_i(-y),$$

or

$$\phi_i(x) = \sum_{y \in H_i} \phi(x + y) k_i(-x - y). \tag{5}$$

If now $\phi = 1$, then $\Phi_i(\gamma) = \hat{k}_i(-\gamma)$. Since $\hat{k}_i = 0$ outside V_i and since $V_i \cap \Lambda_i = \{0\}$, (4) shows that in this case $\phi_i = 1$. Hence (5) implies

$$\sum_{y \in H_i} k_i(-x - y) = 1 \qquad (x \in G;\ i = 1, 2, 3, \dots). \tag{6}$$

Returning to a general $\phi \in L^\infty(G)$, (5) and (6) imply that

(7) $$\|\phi_i\|_\infty \leqq \|\phi\|_\infty \qquad (i = 1, 2, 3, \ldots),$$

since $k_i \geqq 0$. Since $\{V_i\}$ is a neighborhood base at 0, the definition of k_i shows that $k_i(x) \to 1$ for every $x \in G$, as $i \to \infty$. Hence

(8) $$\lim_{i\to\infty} \sum_{\substack{y \in H_i \\ y \neq 0}} k_i(-x-y) = 0 \qquad (x \in G),$$

by (6), and we conclude from (5) that

(9) $$\lim_{i\to\infty} \phi_i(x) = \phi(x) \qquad (x \in G).$$

Having proved (7) and (9), we choose $\phi \in L^\infty(G)$ so that $g * \phi = 0$ for every $g \in L^1(G)$ which has $\hat{g} = 0$ on an open set containing E. Suppose $f \in L^1(G)$ and $\hat{f} = 0$ on E. We have to prove that $f * \phi = 0$ (see 7.2.5(a) (i)).

Put $h_i(x) = k_i(x)(x, \gamma)$. Then $\hat{h}_i = 0$ outside $V_i + \gamma$, and so $h_i * \phi = 0$ if the closure of $V_i + \gamma$ does not intersect E. By (3), this means that $\Phi_i(\gamma) = 0$ if $\gamma \notin E + \bar{V}_i$. Since Φ_i is continuous, we have

(10) $$\Phi_i(\gamma) = 0 \qquad (\gamma \notin E + V_i).$$

By (4)

(11) $$(f * \phi_i)(x) = \frac{1}{n_i} \sum_{\gamma \in A_i} \Phi_i(\gamma)\hat{f}(\gamma)(x, \gamma) \qquad (i = 1, 2, 3, \ldots).$$

Our hypothesis about the structure of E, combined with (10) and the fact that $\hat{f}(\gamma) = 0$ on E, now shows that each term in the sum (11) is 0. Hence $f * \phi_i = 0$ for $i = 1, 2, \ldots$. It follows, by (7) and (9), that $f * \phi = 0$, and this completes the proof.

7.5. Polyhedral Sets

7.5.1. Suppose E is a closed subset of Γ, with the following property: if $f \in L^1(G)$, if $\hat{f} = 0$ on E, and if $\varepsilon > 0$, there exists $g \in L^1(G)$ such that $\|f - f * g\|_1 < \varepsilon$ and such that $\hat{g}$ has compact support, disjoint from E. Under these conditions, we call E a C-set in Γ.

Since $f * g \in I_0(E)$ (see 7.2.5(a)), it is clear that every C-set is an S-set. It is not known whether the converse is true.

This definition of C-sets is very similar to one introduced by Calderon [2] and the theorem which follows is analogous to his.

7.5.2. THEOREM. (a) *Every point of Γ is a C-set in Γ.*

(b) *Finite unions of C-sets in Γ are C-sets in Γ.*

(c) *If the boundary of E is a C-set in Γ, so is E.*

(d) *Each closed subgroup of Γ is a C-set in Γ.*

(e) *If E is a closed subset of a closed subgroup Λ of Γ, if ∂E is the boundary of E relative to Λ, and if ∂E is a C-set in Γ, then E is also a C-set in Γ.*

Before proving this, let us see what the theorem tells us if $\Gamma = R^n$. Note that (e) holds equally well if Λ is a coset rather than a subgroup, since the family of all C-sets is evidently invariant under translation.

By (a), (b), and (e), each compact straight-line interval in R^n is a C-set, hence the union of any three of these intervals is a C-set, by (b), and if we apply (e) again, we see that each triangle is a C-set. Continuing in this way, we find that every rectilinear simplex, of dimension $\leqq n$, is a C-set in R^n. So are hyperplanes (of all dimensions $\leqq n - 1$), by (d), and half-spaces, by (e), as well as quadrants in the plane (bounded by two closed half-lines), octants in R^3, etc.

We conclude that *every polyhedral set in R^n* (i.e., any set which is a finite union of sets built up in the above manner) *is a C-set in R^n.*

In particular, every polyhedral set is an S-set.

7.5.3. Proof of theorem 7.5.2. (a) If $\hat{f}(\gamma_0) = 0$ and $\varepsilon > 0$, there exists $v \in L^1(G)$ such that $\hat{v} = 0$ in a neighborhood of γ_0 and $\|f - f * v\|_1 < \varepsilon/2$, by Theorem 2.6.4. Also, there exists $k \in L^1(G)$ such that $\hat{k}$ has compact support and $\|v - v * k\|_1 < \varepsilon/2\,\|f\|$. If $g = v * k$, then $\hat{g}$ has compact support, disjoint from γ_0, and

$$\|f - g\|_1 \leqq \|f - f * v\|_1 + \|f * (v - v * k)\| < \varepsilon.$$

(b) Suppose E_1 and E_2 are C-sets in Γ, $E = E_1 \cup E_2, \varepsilon > 0$,

$f \in L^1(G)$, and $\hat{f} = 0$ on E. The definition of C-sets shows that there exist functions $g_i \in L^1(G)$, $(i = 1, 2)$ such that $\hat{g}_i$ has compact support, disjoint from E_i, and such that $\|f - f * g_1\|_1 < \varepsilon/2$ and $\|f * g_1 - f * g_1 * g_2\|_1 < \varepsilon/2$. If $g = g_1 * g_2$, then $\hat{g} = \hat{g}_1 \hat{g}_2$ has compact support disjoint from E, and $\|f - f * g\|_1 < \varepsilon$.

Part (c) is a special case of (e) ($\Lambda = \Gamma$), and so is (d) ($E = \Lambda$), since the empty set is a C-set in Γ (Theorem 2.6.6).

(e) Suppose $f \in L^1(G)$, where $\hat{f} = 0$ on E, $\varepsilon > 0$, and E satisfies the hypotheses of (e). Since ∂E is a C-set in Γ, there exists $g \in L^1(G)$ such that the support K of $\hat{g}$ is compact and disjoint from ∂E, and such that $\|f - f * g\|_1 < \varepsilon/2$. Let E' be the complement of E, relative to Λ. Since $K \cap E$ is compact and disjoint from the closure of E', it follows that there exists $h \in L^1(G)$ such that $\hat{h} = 1$ on an open set containing $K \cap E$ and such that $\hat{h} = 0$ on E'. Hence $\hat{f}\hat{h} = 0$ on Λ.

By Theorem 2.7.5, there exists $\mu \in M(G)$ such that $\hat{\mu} = 1$ on an open set containing Λ and $\|f * g * h * \mu\|_1 < \varepsilon/2$. Since $\hat{g} = 0$ outside K and $\hat{h} = 1$ on an open set containing $K \cap E$, the function $\hat{g} - \hat{g}\hat{h}\hat{\mu}$ has compact support disjoint from E, and

$$\|f - f * (g - g * h * \mu)\|_1 \leqq \|f - f * g\|_1 + \|f * g * h * \mu\|_1 < \varepsilon.$$

This completes the proof.

We conclude Section 7.5 with a few other classes of sets which are easily seen to be S-sets.

7.5.4. Star-shaped bodies. Let E be a closed set in R^n which has an interior point p_0 such that each straight line through p_0 intersects the boundary of E in at most 2 points. Such a set is called *star-shaped*. For example, every convex body is star-shaped.

THEOREM. *Every star-shaped body E in R^n is an S-set.*

Proof: Without loss of generality, we may assume that p_0 is the origin of R^n. Choose $f \in L^1(R^n)$ such that $\hat{f} = 0$ on E, suppose $0 < \alpha < 1$, $\alpha\beta = 1$, and put $g(x) = f(\beta x)$, $x \in R^n$. Then $\hat{g}(y) = \alpha^n \hat{f}(\alpha y)$. Since $\alpha < 1$, $\hat{g}(y) = 0$ on an open set containing E. Hence $g \in I_0(E)$. As $\beta \to 1$, $\|g - f\|_1 \to 0$, hence f is in the closure of $I_0(E)$ and the theorem follows.

7.5.5. In particular the closed unit ball in R^n is an S-set. The same proof shows that the closure of the exterior of the unit sphere is an S-set. The example of Schwartz thus shows that the intersection of two S-sets need not be an S-set.

It is not known whether the union of any two S-sets is an S-set.

7.5.6. A class of semigroups. A subset E of a group Γ is a *semigroup if* $E + E \subset E$.

THEOREM. *If E is a closed semigroup in Γ and if* 0 *is in the closure of the interior of E, then E is an S-set.*

Proof: Choose γ_0 in the interior of E, and let V be a neighborhood of 0 in Γ such that $\gamma_0 + V \subset E$. Then $V \subset E - \gamma_0$, hence $E + V \subset E + E - \gamma_0 = E - \gamma_0$, and so E lies in the interior of $E - \gamma_0$.

Now if $f \in L^1(G)$ and $\hat{f} = 0$ on E, then $\hat{f}(\gamma + \gamma_0) = 0$ if $\gamma \in E - \gamma_0$. Setting $g(x) = f(x)(-x, \gamma_0)$, it follows that $g \in I_0(E)$. Since 0 is in the closure of the interior of E, $\|f - g\|_1$ can be made arbitrarily small by taking γ_0 sufficiently close to 0. Hence $f \in I_0(E)$, and the theorem follows.

7.6. Malliavin's Theorem

7.6.1. Theorem 7.1.5 shows that every subset of a discrete abelian group Γ is an S-set in Γ. That this is false in every other case was proved by Malliavin [1], [2], [3]:

THEOREM. *If Γ is a non-discrete LCA group, then Γ contains a closed set which is not an S-set.*

We divide the proof into two parts (7.6.3 and 7.6.4). The first part contains the main idea, in a form which is a little stronger than Malliavin's statement, although all the necessary ingredients are contained in his work. The second part consists of a construction which, though not simple, is merely a matter of technique, and several possibilities exist. Following Kahane [6] we use a method based on probability considerations; this simplifies the required computations and also shows that, in a certain sense, "randomly selected" compact sets fail to be S-sets.

But first of all we show that it is enough to consider compact groups Γ.

7.6.2. Lemma. (a) *If Λ is a closed subgroup of Γ and if E is a closed subset of Λ which is not an S-set in Λ, then E is not an S-set in Γ.*

(b) *If the circle group T contains a closed set which is not an S-set, then so does the real line R.*

(c) *If the conclusion of Theorem 7.6.1 is true for every infinite compact Γ, then it is true for every non-discrete Γ.*

Proof: (a) follows from Theorem 2.7.4. For if ϕ is the restriction map of $A(\Gamma)$ to Λ, then ϕ is a homomorphism of $A(\Gamma)$ onto $A(\Lambda)$; since $A(\Lambda)$ is semi-simple, ϕ is continuous; hence if I_1, I_2 are distinct closed ideals in $A(\Lambda)$, $\phi^{-1}(I_1)$ and $\phi^{-1}(I_2)$ are distinct closed ideals in $A(\Gamma)$; finally, I and $\phi^{-1}(I)$ have the same zero-set.

To prove (b), let E_1 be a closed subset of the circle T which is disjoint from an arc $E_2 \subset T$; assume $-1 \in E_2$. Let K_1 be the set of all $y \in (-\pi, \pi)$ such that $e^{iy} \in E_1$. Let K_2 consist of all $y \in (-\pi, \pi)$ such that $e^{iy} \in E_2$, and of all y with $|y| \geqq \pi$. Put $E = E_1 \cup E_2$ and $K = K_1 \cup K_2$.

If $\hat{f} \in A(T)$ and if $\hat{f} = 0$ on E, then the set of points at which $\hat{f}$ does not belong locally to $\hat{I}_0(E)$ is a perfect subset of the boundary of E (or is empty), by Lemma 7.2.3. Thus E is an S-set if and only if E_1 is an S-set. Similarly, K is an S-set if and only if K_1 is an S-set.

Suppose $\hat{f} \in A(T)$, $\hat{f} = 0$ on E, and define $\hat{g}(y) = \hat{f}(e^{iy})$ for $|y| \leqq \pi$, $\hat{g}(y) = 0$ for $|y| > \pi$. If K is an S-set, there is a sequence $g_n \in L^1(R)$ such that $\|g - g_n\|_1 \to 0$ and $\hat{g}_n = 0$ on an open set V_n containing K. By Theorem 2.7.6 it follows that E is an S-set. The argument can be reversed, and shows that E is an S-set if and only if K is an S-set.

We conclude that E_1 is an S-set if and only if K_1 is an S-set.

The proof of (c) is now immediate. If Γ is not discrete, the structure theorem 2.4.1 shows that Γ either contains an infinite compact subgroup Λ, in which case we appeal to (a), or that Γ contains a closed subgroup isomorphic to R, in which case we appeal to (b) and (a).

7.6.3. We now assume that Γ is compact and infinite. If $f \in L^1(G)$, then $\exp(iu\hat{f}) \in A(\Gamma)$ for every real number u. We define $a_x(u)$, for $x \in G$, by

$$\exp\{-iu\hat{f}(\gamma)\} = \sum_{x \in G} a_x(u)(-x, \gamma), \tag{1}$$

and put

$$M_n = \sup_{x \in G} \frac{1}{2\pi}\int_{-\infty}^{\infty} |a_x(u)u^n|du \qquad (n = 1, 2, 3, \ldots). \tag{2}$$

We shall see later that there exist real functions $\hat{f} \in A(\Gamma)$ such that

$$M_n < \infty \qquad (n = 1, 2, 3, \ldots). \tag{3}$$

THEOREM (Rudin [18]). *If $\hat{f}$ satisfies the above conditions, there exists a real number ξ such that the closed ideals in $A(\Gamma)$ which are generated by $(\hat{f} - \xi)^n$ $(n = 1, 2, 3, \ldots)$ are all distinct.*

Since all of these ideals have the same zero-set, Malliavin's theorem follows from this as soon as the existence of an appropriate $\hat{f}$ is assured.

Proof: The map

$$\phi \to \int_\Gamma \phi(\hat{f}(\gamma))(x, \gamma)d\gamma \tag{4}$$

is, for each $x \in G$, a bounded linear functional (of norm $\leqq 1$) on the space $C(Y)$, where Y is the range of $\hat{f}$, a compact subset of the line R. The Riesz representation theorem implies that there are measures $\mu_x \in M(R)$, concentrated on Y, such that

$$\int_\Gamma \phi(\hat{f}(\gamma))(x, \gamma)d\gamma = \int_{-\infty}^{\infty} \phi(t)d\mu_x(t) \qquad (x \in G) \tag{5}$$

for every continuous function ϕ on R. If we take $\phi(t) = e^{-iut}$, comparison of (4) and (1) shows that $a_x(u) = \hat{\mu}_x(u)$. We infer from (2), (3), and the inversion theorem 1.7.3(e) that $d\mu_x(t) = m_x(t)dt$, where $m_x \in C^\infty$ (the class of infinitely differentiable functions on R), and since

$$(6)\qquad (D^n m_x)(t) = \frac{i^n}{2\pi}\int_{-\infty}^{\infty} u^n a_x(u)e^{iut}\,du,$$

(2) implies that

$$(7)\qquad |(D^n m_x)(t)| \leqq M_n \qquad (x \in G,\ t \in R,\ n = 1, 2, 3, \ldots).$$

Here D^n denotes the nth derivative. We now rewrite (5) in the form

$$(8)\qquad \int_\Gamma \phi(\hat{f}(\gamma))(x, \gamma)d\gamma = \int_{-\infty}^{\infty} \phi(t)m_x(t)dt.$$

Taking $u = 0$ in (1), we see that $a_0(0) = 1$. But a_0 is the Fourier transform of m_0. It follows that

$$(9)\qquad m_0(\xi) \neq 0,$$

for some real number ξ.

With this ξ, we define bounded linear functionals T_n on $A(\Gamma)$, for $n = 1, 2, 3, \ldots$:

$$(10)\qquad T_n\hat{g} = \sum_{x \in G} g(-x)(D^n m_x)(\xi) \qquad (g \in L^1(G)).$$

If $\hat{I}_n$ is the closed ideal generated by $(\hat{f} - \xi)^n$ we shall see that T_n annihilates $\hat{I}_{n+1}$ but not $\hat{I}_n$, and this will complete the proof. We do this by first obtaining an alternative description of $T_n\hat{g}$ for a certain class of functions $\hat{g}$.

Suppose $x \in G$, $\phi(\hat{f}) \in A(\Gamma)$, and

$$(11)\qquad \hat{g}(\gamma) = \phi(\hat{f}(\gamma))(x, \gamma) \qquad (\gamma \in \Gamma).$$

By (8), (11), and (1) the Fourier transform of ϕm_x is

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)m_x(t)e^{-iut}\,dt &= \int_\Gamma \phi(\hat{f}(\gamma)) \exp\{-iu\hat{f}(\gamma)\}(x, \gamma)d\gamma \\ &= \int_\Gamma \hat{g}(\gamma) \sum_{y \in G} a_y(u)(-y, \gamma)d\gamma \\ &= \sum_{y \in G} g(-y)\hat{m}_y(u)\end{aligned}$$

so that

$$(12)\qquad \phi(t)m_x(t) = \sum_{y \in G} g(-y)m_y(t) \qquad (t \in R).$$

(Observe that the right side of (12) is in C^∞. This means that $\phi \in C^\infty$ on every segment on which one of the functions m_x is different from 0. Hence $\phi(\hat{f}) \in A(\Gamma)$ only for very smooth functions ϕ. This remark establishes a connection with the problems treated in Chapter 6, and has been pursued further by Malliavin [4].)

By (7), the series in (12) may be differentiated term by term any number of times. Putting $t = \xi$, comparison with (10) then shows that

$$(13) \qquad T_n\hat{g} = D^n(\phi m_x)(\xi) \qquad (n = 1, 2, 3, \ldots)$$

if $\hat{g}$ is of the form (11).

Taking $\phi(t) = (t - \xi)^{n+1}$ and $\hat{g}(\gamma) = (\hat{f}(\gamma) - \xi)^{n+1}(x, \gamma)$, it follows that $T_n\hat{g}$ is the nth derivative of $(t - \xi)^{n+1}m_x(t)$, evaluated at $t = \xi$, and this is 0. Hence T_n annihilates $\hat{I}_{n+1}$.

But if $\phi(t) = (t - \xi)^n$ and $\hat{g}(\gamma) = (\hat{f}(\gamma) - \xi)^n$, (13) implies that $T_n\hat{g}$ is the nth derivative of $(t - \xi)^n m_0(t)$, evaluated at $t = \xi$, and this is $n!m_0(\xi)$. Since $m_0(\xi) \neq 0$, $T_n\hat{g} \neq 0$.

Hence $(\hat{f} - \xi)^n$ is not in $\hat{I}_{n+1}$, and the proof is complete.

The set E_ξ which is thus shown not to be an S-set is the set of all $\gamma \in \Gamma$ such that $\hat{f}(\gamma) = \xi$. Since E_α and E_β are disjoint if $\beta \neq \alpha$, *we conclude that there are uncountably many pairwise disjoint compact sets in Γ which are not S-sets and whose union is an open set.*

7.6.4. We now have to prove the existence of a real function $\hat{f} \in A(\Gamma)$ which satisfies the hypotheses made in 7.6.3. Put

$$(1) \qquad \psi(t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{t} e^{-u^2/2}\,du \qquad (-\infty < t < \infty).$$

Let Ω be the Cartesian product of countably many copies of the segment $(0, 1)$; the coordinates of a point $\omega \in \Omega$ are $\omega_1, \omega_2, \omega_3, \ldots$, where $0 < \omega_i < 1$. Define

$$(2) \qquad \phi_k(\omega) = \phi(\omega_k) \qquad (\omega \in \Omega,\ k = 1, 2, 3, \ldots),$$

where ϕ is the inverse of ψ: $\psi(\phi(t)) = t$ for $0 < t < 1$. In the language of probability theory, the functions ϕ_k form an independent sequence of random variables, with the same Gaussian distribution.

THEOREM. *If Γ is compact and infinite, there is a sequence $\{x_k\}$ in G with the following property: for almost all $\omega \in \Omega$, the function $\hat{f}$ defined on Γ by*

$$\hat{f}(\gamma) = \hat{f}(\gamma; \omega) = \sum_{k=1}^{\infty} k^{-2}\phi_k(\omega) \operatorname{Re}(x_k, \gamma) \tag{3}$$

belongs to $A(\Gamma)$ and satisfies the hypotheses of 7.6.3.

The measure on Ω with respect to which the phrase "almost all" is to be understood is the product measure, characterized as follows: if $E_1, \ldots, E_n$ are sets in $(0, 1)$ and $E \subset \Omega$ is the "cylinder set" consisting of all ω such that $\omega_i \in E_i$ $(1 \leqq i \leqq n)$, the measure of E is $E = \prod_{i=1}^{n} m(E_i)$, where m is Lebesgue measure on $(0, 1)$.

Proof: By (1) and (2), we have

$$\int_{\Omega} F(\phi_k(\omega))d\omega = \int_0^1 F(\phi(x))dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)e^{-t^2/2}\, dt \tag{4}$$

for any function F for which these integrals exist as Lebesgue integrals. Taking $F(t) = |t|$, it follows that

$$\int_{\Omega} \sum_{k=1}^{\infty} k^{-2}|\phi_k(\omega)|d\omega < \infty;$$

by (3), *this implies that $\hat{f} \in A(\Gamma)$ for almost all ω.*

If we define $a_x(u) = a_x(u; \omega)$ as in 7.6.3, then

$$\int_{\Gamma} \exp\{iu[\hat{f}(\gamma + \gamma') - \hat{f}(\gamma')]\}d\gamma' = \sum_{x \in G} |a_x(u)|^2(x, \gamma), \tag{5}$$

and if

$$B(u) = B(u; \omega) = \sum_{x \in G} |a_x(u; \omega)|^4, \tag{6}$$

the Parseval formula gives

$$\begin{aligned} B(u) &= \int_{\Gamma}\int_{\Gamma}\int_{\Gamma} \exp\{iu[\hat{f}(\gamma + \gamma') - \hat{f}(\gamma') \\ &\qquad\qquad - \hat{f}(\gamma + \gamma'') + \hat{f}(\gamma'')]\}d\gamma\, d\gamma'\, d\gamma'' \\ &= \int_{\Gamma}\int_{\Gamma}\int_{\Gamma} \prod_{k=1}^{\infty} \exp\left\{\frac{iu}{k^2}\phi_k(\omega)P_k(\gamma, \gamma', \gamma'')\right\} d\gamma\, d\gamma'\, d\gamma'', \end{aligned} \tag{7}$$

where

$$P_k(\gamma, \gamma', \gamma'') = \operatorname{Re}\,[(x_k, \gamma + \gamma') - (x_k, \gamma') - (x_k, \gamma + \gamma'') + (x_k, \gamma'')]. \tag{8}$$

Let E be the expected value of B:

$$E(u) = \int_\Omega B(u; \omega) d\omega. \tag{9}$$

If we integrate (7) over Ω, (2) shows that $\int_\Omega$ may be moved inside the product sign. Taking $F(u) = e^{isu}$ in (4), with s real, the well-known formula

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{isu}\, e^{-u^2/2}\, du = e^{-s^2/2}$$

therefore gives

$$E(u) = \int_\Gamma \int_\Gamma \int_\Gamma \prod_{k=1}^{\infty} \exp\left\{- \frac{u^2}{2k^4} P_k^2(\gamma, \gamma', \gamma'')\right\} d\gamma\, d\gamma'\, d\gamma''. \tag{10}$$

Fix u, $|u| \geqq 1$, and let N be the largest integer such that $N^2 \leqq |u|$. Since no factor in (10) exceeds 1,

$$E(u) \leqq \int_\Gamma \int_\Gamma \int_\Gamma \prod_{k=1}^{N} \exp\{- \tfrac{1}{2} P_k^2(\gamma, \gamma', \gamma'')\} d\gamma\, d\gamma'\, d\gamma''. \tag{11}$$

Two facts will be needed for our next estimate of $E(u)$. First, there is a constant $A < 1$ such that

$$\int_\Gamma \int_\Gamma \int_\Gamma \exp\{- \tfrac{1}{2} P_k^2\} < A \tag{12}$$

whenever $x_k \neq 0$ (compare (8)); secondly, $\{x_k\}$ can be so chosen in G that the right side of (11) is less than

$$2 \prod_{k=1}^{N} \iiint \exp\{- \tfrac{1}{2} P_k^2\} \qquad (N = 1, 2, 3, \ldots). \tag{13}$$

Once these are proved, it follows from (11) that

$$E(u) < 2 \exp\{- \delta |u|^{1/2}\} \qquad (-\infty < u < \infty) \tag{14}$$

for some $\delta > 0$. If $0 < \varepsilon < \delta$, (9) and (14) imply that

$$\int_\Omega d\omega \int_{-\infty}^{\infty} B(u;\omega) \exp\{\varepsilon|u|^{1/2}\}du < 2\int_{-\infty}^{\infty} \exp\{(\varepsilon-\delta)|u|^{1/2}\}du < \infty. \tag{15}$$

The inner integral on the left side of (15) is therefore finite for almost all ω. For any such ω, (6) shows, via Hölder's equality, that

$$\int_{-\infty}^{\infty} |a_x(u)u^n|du \leqq \int_{-\infty}^{\infty} B(u)^{1/4}|u|^n\, du$$

$$\leqq \left\{\int_{-\infty}^{\infty} B(u) \exp\{\varepsilon\sqrt{|u|}\}du\right\}^{1/4} \cdot \left\{\int_{-\infty}^{\infty} \exp\left\{-\frac{\varepsilon}{3}\sqrt{|u|}\right\}|u|^{4n/3}\, du\right\}^{3/4},$$

for $n = 1, 2, 3, \ldots$ and $x \in G$. Thus the hypotheses of Theorem 7.6.3 are satisfied.

We now turn to the proof of (12) and (13). Define P on the torus T^3 by

$$P(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) = \cos(\theta_1+\theta_2) - \cos\theta_2 - \cos(\theta_1+\theta_3) + \cos\theta_3, \tag{17}$$

put $Q = \exp\{-\frac{1}{2}P^2\}$,

$$Q_k(\gamma, \gamma', \gamma'') = Q((x_k, \gamma), (x_k, \gamma'), (x_k, \gamma'')), \tag{18}$$

and

$$J_k = \int_\Gamma \int_\Gamma \int_\Gamma Q_k(\gamma, \gamma', \gamma'')d\gamma\, d\gamma'\, d\gamma''. \tag{19}$$

Since $Q_k = \exp\{-\frac{1}{2}P_k^2\}$, J_k is the integral in (12). If x_k has order n, then

$$J_k = J(n) = \frac{1}{n^3} \sum_{p,q,r=1}^{n} Q(e^{2\pi ip/n}, e^{2\pi iq/n}, e^{2\pi ir/n}), \tag{20}$$

and if x_k is of infinite order, we have

$$J_k = J(\infty) = \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} Q(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})d\theta_1\, d\theta_2\, d\theta_3. \tag{21}$$

Since $P(e^{i\theta}, e^{-i\theta}, 1) > 0$ if $0 < \theta < 2\pi$, it follows that for $n \geqq 2$ at least one of the summands in (20) is less than 1. Hence $J(n) < 1$. Since $J(\infty) < 1$ and since $\lim J(n) = J(\infty)$, we have proved (12).

Choose $\varepsilon_k > 0$ so that $\prod_1^\infty (1 + \varepsilon_k) < 2$. Choose $x_1 \in G$, $x_1 \neq 0$, and suppose $x_1, \ldots, x_{N-1}$ are selected and satisfy the induction hypothesis

$$(22) \qquad \int_\Gamma \int_\Gamma \int_\Gamma \prod_{k=1}^{N-1} Q_k \, d\gamma \, d\gamma' \, d\gamma'' < A^{N-1} \prod_{k=1}^{N-1} (1 + \varepsilon_k),$$

where A is as in (12). Put $\Phi_{N-1} = \prod_1^{N-1} Q_k$. We will choose $x_N \in G$ so that

$$(23) \quad \iiint \Phi_{N-1} Q_N \, d\gamma \, d\gamma' \, d\gamma'' \leqq \iiint \Phi_{N-1} \cdot \iiint Q_N \cdot (1 + \varepsilon_N).$$

Since P is a trigonometric polynomial, $Q \in A(T^3)$, and

$$(24) \qquad Q_N(\gamma, \gamma', \gamma'') = \sum_{p,q,r=-\infty}^{\infty} a_{pqr}(x_N, p\gamma + q\gamma' + r\gamma''),$$

where the numbers a_{pqr} are the Fourier coefficients of Q. Note also that $\Phi_{N-1} \in A(\Gamma^3)$ and that $\hat{\Phi}_{N-1} = 0$ outside $G^3{}_{N-1}$, where G_{N-1} is the subgroup of G generated by $x_1, \ldots, x_{N-1}$. To evaluate the integral on the left side of (23) we replace Φ_{N-1} and Q_N by their Fourier series, multiply the series, and integrate term by term.

Case I. If G is of bounded order, then G is a direct sum of infinitely many cyclic groups, G_{N-1} is finite, and we can find $x_N \neq 0$ so that the group generated by x_N has only 0 in common with G_{N-1}. Then (23) holds with $\varepsilon_N = 0$, and the two sides are actually equal.

Case II. If G is not of bounded order but contains no element of infinite order, then G_{N-1} is again finite, and to every positive integer s there corresponds an element $x_N \in G$ such that $ax_N \notin G_{N-1}$ for $a = 1, 2, \ldots, s$. (Otherwise no $x \in G$ would have order greater than s times the number of elements of G_{N-1}, a contradiction.) Given $\eta > 0$, we can choose s so that $\sum |a_{pqr}| < \eta$, the summation being extended over all (p, q, r) with $|p| > s$ or $|q| > s$ or $|r| > s$. By (24), we then have

$$\left|\iiint Q_N - a_{000}\right| < \eta. \tag{25}$$

Our choice of x_N shows that

$$\begin{aligned}\iiint \Phi_{N-1}\, Q_N &= a_{000}\iiint \Phi_{N-1} \\ &\quad + \textstyle\sum' a_{pqr}\displaystyle\iiint (px_N, \gamma)(qx_N, \gamma')(rx_N, \gamma'')\Phi_{N-1},\end{aligned} \tag{26}$$

where $|p| > s$, $|q| > s$, and $|r| > s$ in $\sum'$. This sum is less than η, since $\Phi_{N-1} \leqq 1$. Comparison of (25) and (26) shows that (23) holds if η is small enough.

Case III. If G has an element x_0 of infinite order, and if $\eta > 0$, then G_{N-1} has a finite subset H such that $\sum |\hat{\Phi}_{N-1}(x, x', x'')| < \eta$, the sum being extended over all points (x, x', x'') with at least one coordinate outside H. Let x_N be a multiple of x_0, so that $ax_N \notin H$ unless $a = 0$. Then $\iiint Q_N = a_{000}$ and

$$\iiint \Phi_{N-1}\, Q_N \leqq a_{000}\iiint \Phi_{N-1} + \eta. \tag{27}$$

Taking η small enough, we again obtain (23).

The proof is complete.

7.7. Closed Ideals Which Are Not Self-Adjoint

7.7.1. The work of Section 7.6 can be modified so as to yield the following result:

THEOREM. *If Γ is not discrete, then $A(\Gamma)$ contains a closed ideal which is not self-adjoint* (i.e., which is not closed under complex conjugation).

Proof: By an argument quite analogous to that used in Lemma 7.6.2 we see that it is enough to prove the theorem for compact infinite Γ.

Pick $\{\phi_k\}$ as in 7.6.4 (2), put

$$\hat{f}_1(\gamma) = \sum_{k\ \mathrm{odd}} k^{-2}\phi_k(\omega)\ \mathrm{Re}\ (x_k, \gamma), \quad \hat{f}_2(\gamma) = \sum_{k\ \mathrm{even}} k^{-2}\phi_k(\omega)\ \mathrm{Re}\ (x_k, \gamma),$$

define $a_x(u, v)$ for u, v real and $x \in G$ by

$$\exp\{-i(u\hat{f}_1(\gamma) + v\hat{f}_2(\gamma))\} = \sum_{x \in G} a_x(u, v)(-x, \gamma),$$

put

$$B(u, v) = B(u, v; \omega) = \sum_{x \in G} |a_x(u, v)|^4$$

and

$$E(u, v) = \int_\Omega B(u, v; \omega) d\omega.$$

As in 7.6.4(10), we obtain

$$E(u, v) = \iiint \prod_{k \text{ odd}} \exp\left\{-\frac{u^2}{2k^4} P_k^2\right\} \prod_{k \text{ even}} \exp\left\{-\frac{v^2}{2k^4} P_k^2\right\} d\gamma\, d\gamma'\, d\gamma'',$$

and the final argument in 7.6.4 shows that $\{x_k\}$ can be so chosen in G that

$$E(u, v) < 2 \exp\{-\delta(|u|^{1/2} + |v|^{1/2})\}.$$

A weak consequence of this is a condition analogous to 7.6.3(2):

$$\sup_{x \in G} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |a_x(u, v)| (u^2 + v^2)^{1/2}\, du dv = M < \infty.$$

We now define $\hat{f} = \hat{f}_1 + i\hat{f}_2$ and find, as in 7.6.3(8) that

$$\int_\Gamma \phi(\hat{f}(\gamma)) (x, \gamma) d\gamma = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(s, t) m_x(s, t) ds\, dt \qquad (x \in G),$$

for every continuous function ϕ in the plane; the functions m_x have compact support, and satisfy the inequalities

$$\left|\frac{\partial m_x}{\partial s}\right| \leqq M, \quad \left|\frac{\partial m_x}{\partial t}\right| \leqq M.$$

Since m_0 is not identically 0, we may, by adding a constant to $\hat{f}$, assume that $m_0(0, 0) \neq 0$. We now define a bounded linear functional T on $A(\Gamma)$:

$$T\hat{g} = \sum_{x \in G} g(-x) \frac{\partial m_x}{\partial z} (0, 0)$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right) \qquad (z = s + it).$$

If $\hat{g}(\gamma) = \phi(\hat{f}(\gamma))(x, \gamma)$, with $\phi(\hat{f}) \in A(\Gamma)$ and $x \in G$, we see, as in 7.6.3 (13), that

$$T\hat{g} = \frac{\partial}{\partial z}(\phi m_x)(0, 0).$$

Taking $\phi(z) = z$, $x = 0$, this gives

$$T\hat{f} = \frac{\partial}{\partial z}(zm_0)(0, 0) = m_0(0, 0) \neq 0.$$

But if $\phi(z) = \bar{z}$ and $x \in G$, we get

$$T\hat{g} = \frac{\partial}{\partial z}(\bar{z}m_x)(0, 0) = \bar{z}\left.\frac{\partial m_x}{\partial z}\right|_{z=0} = 0.$$

Thus T annihilates the ideal generated by $\bar{\hat{f}}$ without annihilating $\hat{f}$. The theorem follows.

7.7.2. One can produce other examples of this sort, based on other differential operators. However, the following result of Helson [2] shows that this does not exhaust all possibilities:

THEOREM. *If I_1 and I_2 are distinct closed ideals in $L^1(G)$, if $I_1 \subset I_2$, and if $Z(I_1) = Z(I_2)$, then there exists a closed ideal I such that $I \neq I_1$, $I \neq I_2$, and $I_1 \subset I \subset I_2$.*

7.8. Spectral Synthesis of Bounded Functions

7.8.1. It often happens that a problem in a Banach space X can be replaced by an equivalent one in the dual space X^*. For example, we shall see that the study of the closed ideals of $L^1(G)$ is equivalent to the study of the translation-invariant subspaces of $L^\infty(G)$ which are closed in the weak*-topology. This approach was suggested by Beurling [2], [3], and various aspects of it have been discussed by Segal [2], Godement [1], Pollard [1], and Herz [3].

If I is a closed ideal in $L^1(G)$ and if Φ is a translation-invariant weak*-closed subspace of $L^\infty(G)$, define

(1) $$\alpha(I) = \{\phi \in L^\infty(G) \colon f * \phi = 0 \text{ for all } f \in I\},$$

(2) $$\beta(\Phi) = \{f \in L^1(G) \colon f * \phi = 0 \text{ for all } \phi \in \Phi\}.$$

The letters I, Φ, α, β will have these meanings throughout the remainder of this chapter.

The spectrum of Φ, written $\sigma(\Phi)$, is defined as the set of all continuous characters of G which belong to Φ; if $\phi \in L^\infty(G)$, its spectrum $\sigma(\phi)$ is defined to be $\sigma(\Phi(\phi))$, where $\Phi(\phi)$ is the smallest Φ which contains ϕ. The problem of *spectral synthesis* in $L^\infty(G)$ is the question whether each $\phi \in \Phi$ can be "synthesized" from $\sigma(\Phi)$. More precisely: *Is every Φ identical with the weak*-closed subspace Φ_1 of $L^\infty(G)$ which is generated by the continuous characters in Φ? Or can distinct Φ's have the same spectrum?* (It is clear that $\Phi_1 \subset \Phi$, that $\sigma(\Phi_1) = \sigma(\Phi)$, and that Φ_1 is the smallest weak*-closed subspace of $L^\infty(G)$ with this spectrum.)

If a set $E \subset \Gamma$ is the spectrum of a unique Φ, we say that *spectral synthesis* holds for E, or that E is an S-set. The equivalence of this definition with the one adopted in Section 7.1.4 is a consequence of part (e) of the theorem which follows.

7.8.2. THEOREM. (a) *Each $\alpha(I)$ is a Φ, each $\beta(\Phi)$ is an I.*

(b) *$\alpha(\beta(\Phi)) = \Phi$ and $\beta(\alpha(I)) = I$.*

(c) *If $\Phi = \alpha(I)$ then $\sigma(\Phi) = Z(I)$.*

(d) *$\sigma(\Phi)$ is always closed, and every closed set $E \subset \Gamma$ is $\sigma(\Phi)$ for some Φ.*

(e) *A closed set $E \subset \Gamma$ is $\sigma(\Phi)$ for a unique Φ if and only if $E = Z(I)$ for a unique I.*

Proof: Part (a) follows from Theorem 7.1.2 and from the definition of the weak*-topology.

The inclusions

(1) $$\alpha(\beta(\Phi)) \supset \Phi \text{ and } \beta(\alpha(I)) \supset I$$

are obvious from the definitions of α and β. The Hahn-Banach theorem implies that $\alpha(I_1) = \alpha(I_2)$ if and only if $I_1 = I_2$; its dual-space analogue (Appendix C9) shows similarly that $\beta(\Phi_1) = \beta(\Phi_2)$ if and only if $\Phi_1 = \Phi_2$. The first of the relations (1) shows that

$$(2) \qquad \beta(\alpha(\beta(\Phi))) \subset \beta(\Phi),$$

and if we put $I = \beta(\Phi)$ in the second, we obtain the reverse of (2). Hence $\beta(\alpha(\beta(\Phi))) = \beta(\Phi)$, and this implies that $\alpha(\beta(\Phi)) = \Phi$. The second part of (b) is proved the same way.

If $\Phi = \alpha(I)$, then $\gamma \in \sigma(\Phi)$ if and only if $f * \gamma = 0$ for all $f \in I$. But $(f * \gamma)(x) = (x, \gamma)\hat{f}(\gamma)$, and this is 0 for all $x \in G$ and all $f \in I$ if and only if $\gamma \in Z(I)$. This proves (c).

By (a) and (b), every Φ is $\alpha(I)$ for some I; hence (d) follows immediately from (c).

Since $\sigma(\Phi) = Z(\beta(\Phi))$, by (c), and since $\beta(\Phi_1) = \beta(\Phi_2)$ if and only if $\Phi_1 = \Phi_2$, we see that (e) is true.

7.8.3. We can now rephrase some of our earlier results.

(a) *If V is an open set containing $\sigma(\Phi)$ and if Φ_1 is the weak*-closed subspace generated by the characters in V (not just those in $\sigma(\Phi)$!) then $\Phi_1 \supset \Phi$.*

For if $I = \beta(\Phi)$ and $I_1 = \beta(\Phi_1)$, then $Z(I)$ is in the interior of $Z(I_1)$, and so $I_1 \subset I$, by 7.2.5(a). Hence $\Phi_1 = \alpha(I_1) \supset \alpha(I) = \Phi$.

(b) *If $f \in L^1(G)$ and $\hat{f} = 0$ on an open set containing $\sigma(\phi)$, then $f * \phi = 0$.*

This is a special case of (a).

(c) *If Φ contains a non-zero function, then $\sigma(\Phi)$ is not empty.*

Since we identify functions $L^\infty(G)$ which differ only on sets of measure zero, the hypothesis means that Φ contains a function which differs from zero on a set of positive measure. Then $\beta(\Phi) \neq L^1(G)$, and so $Z(\beta(\Phi))$ is not empty, by Wiener's theorem. But $Z(\beta(\Phi)) = \sigma(\Phi)$, by Theorem 7.8.2.

(d) *If $\phi \in L^\infty(G)$ and $\phi \neq 0$, then at least one continuous character of G can be approximated, in the weak*-topology, by linear combinations of translates of ϕ.*

This follows from (c).

(e) *If $\sigma(\Phi) = \{\gamma_1, \ldots, \gamma_n\}$, a finite set, then every $\phi \in \Phi$ is a trigonometric polynomial of the form*

(1) $$\phi(x) = \sum_{i=1}^{n} c_i(x, \gamma_i).$$

This is so since every finite set in Γ is an S-set and since the space of all polynomials (1) has $\{\gamma_1, \ldots, \gamma_n\}$ as its spectrum.

7.8.4. *Although a space Φ may not be determined by the continuous characters which it contains, Φ is determined by the uniformly continuous members of Φ.*

For if $\phi \in \Phi$ and $g \in L^1(G)$, then $g * \phi$ is bounded and uniformly continuous (Theorem 1.1.6(b)) and (a) $g * \phi \in \Phi$ (b) ϕ is in the weak*-closure of the set of all $g * \phi$ $(g \in L^1(G))$.

To prove (a), take $f \in \beta(\Phi)$. Since $\beta(\Phi)$ is an ideal, $f * g \in \beta(\Phi)$, hence $f * (g * \phi) = 0$, and so $g * \phi \in \alpha(\beta(\Phi)) = \Phi$. To prove (b), suppose $f \in L^1(G)$ and $f * g * \phi = 0$ for all $g \in L^1(G)$. Taking $\{g_n\}$ so that $\|f - f * g_n\|_1 \to 0$, we see that $f * \phi = 0$, and (b) follows from Appendix C9.

7.8.5. Let us say that a function $\phi \in L^\infty(G)$ *admits spectral synthesis* if ϕ is in the weak*-closed subspace of $L^\infty(G)$ generated by $\sigma(\phi)$; in other words, if ϕ is in the weak*-closure of the set of all trigonometric polynomials f of the form

(1) $$f(x) = \sum_{i=1}^{n} c_i(x, \gamma_i) \qquad (x \in G,\ \gamma_i \in \sigma(\phi)).$$

THEOREM. *If $\mu \in M(\Gamma)$ and if*

(2) $$\phi(x) = \int_\Gamma (x, \gamma) d\mu(\gamma) \qquad (x \in G),$$

then $\sigma(\phi)$ is the support of μ, and ϕ admits spectral synthesis.

(For any $\phi \in L^\infty(G)$, $\sigma(\phi)$ may thus be regarded, heuristically, as the support of the "Fourier transform" of ϕ, although we have not defined any such transform on $L^\infty(G)$.)

Proof: Let Φ be the smallest weak*-closed translation-invariant subspace of $L^\infty(G)$ which contains ϕ, and put $I = \beta(\Phi)$. If $f \in L^1(G)$, (2) implies

(3) $$(f * \phi)(x) = \int_\Gamma (x, \gamma) \hat{f}(\gamma) d\mu(\gamma),$$

and hence $f \in I$ if and only if $\hat{f} = 0$ on the support E of μ. In other words, I is the *largest* ideal in $L^1(G)$ which has $Z(I) = E$. By Theorem 7.8.2, $\sigma(\phi) = \sigma(\Phi) = Z(I)$, and so $\sigma(\phi) = E$.

Since $\Phi = \alpha(I)$, it also follows that Φ is the *smallest* subspace of $L^\infty(G)$ with $\sigma(\Phi) = E$, and the proof is complete.

COROLLARY. *If G is discrete, and if $\phi \in L^2(G)$, then ϕ admits spectral synthesis.* (Note that $L^2(G) \subset L^\infty(G)$.)

For if $\phi \in L^2(G)$, ϕ is the Plancherel transform of a function in $L^2(\Gamma)$, and $L^2(\Gamma) \subset L^1(\Gamma) \subset M(\Gamma)$ since Γ is compact.

7.8.6. With $q > 2$ in place of 2, the preceding corollary is false.

THEOREM. *If G is discrete and infinite and if $q > 2$, there exists $\phi \in L^q(G)$ which does not admit spectral synthesis.*

Proof: We shall use the results, notation, and terminology of Sections 7.6.3 and 7.6.4.

Let I_n be the ideals defined after 7.6.3(10), put $\Phi_n = \alpha(I_n)$, and $\phi_n(x) = (D^n m_x)(\xi)$, for $n = 1, 2, 3, \ldots$ and $x \in G$. We saw in 7.6.3 that ϕ_n is in Φ_{n+1} but not in Φ_n. Since $\phi_n \in \Phi_{n+1}$, and since $\sigma(\Phi_{n+1}) = Z(I_{n+1})$, $\sigma(\phi_n) \subset Z(I_{n+1}) = Z(I_n)$. Hence every ϕ in the weak*-closed space generated by $\sigma(\phi_n)$ belongs to $\alpha(I_n) = \Phi_n$, and it follows that none of the functions ϕ_n admit spectral synthesis.

In the notation of 7.6.4, we had

$$\text{(1)} \qquad \sum_{x \in G} |a_x(u)|^4 = B(u), \qquad \sum_{x \in G} |a_x(u)|^2 = 1,$$

and Hölder's inequality therefore gives

$$\text{(2)} \qquad \sum_{x \in G} |a_x(u)|^q \leqq B(u)^{(q-2)/2} \qquad (2 < q < 4).$$

By 7.6.3(6), we have

$$2\pi|\phi_1(x)| \leqq \int_{-\infty}^{\infty} |a_x(u)u|du \leqq \left\{\int_{-\infty}^{\infty} \left(\frac{|u|}{1+u^2}\right)^p du\right\}^{1/p}$$
$$\left\{\int_{-\infty}^{\infty} (1+u^2)^q |a_x(u)|^q \, du\right\}^{1/q},$$

where $1/p + 1/q = 1$, and it follows from (2) that

$$\sum_{x \in G} |\phi_1(x)|^q \leqq \text{const.} \int_{-\infty}^{\infty} (1 + u^2)^q B(u)^{(q-2)/2}\, du. \tag{3}$$

Since $\int_\Omega B(u; \omega)^{(q-2)/2} d\omega \leqq E(u)^{(q-2)/2}$, 7.6.14 shows that the integral in (3) is finite for almost all ω. For any such ω, $\phi_1 \in L^q(G)$ for every $q > 2$, and the proof is complete.

7.8.7. *Similarly, Theorem* 7.2.9 *becomes false if the exponent* 2 *is replaced by any* $p < 2$ (Segal [1]).

This can also be deduced from Sections 7.6.3 and 7.6.4. The remark at the end of 7.6.3 shows that for some ξ the set $\hat{f}^{-1}(\xi)$ has measure 0 and the various powers $(\hat{f} - \xi)^n$ generate distinct closed ideals in $A(\Gamma)$. Put $\hat{g} = (\hat{f} - \xi)^2$. Then $g \in L^1(G)$, and since G is discrete, $g \in L^p(G)$ for all $p \geqq 1$.

We saw in 7.8.6 that the construction used in the proof of Malliavin's theorem yields a function $\phi_1 (\phi_1(x) = m'_x(\xi))$ which belongs to $L^q(G)$ for every $q > 2$. The conclusion of 7.6.3 shows that ϕ_1 annihilates the ideal in $A(\Gamma)$ generated by $\hat{g}$; in other words

$$\sum_{y \in G} g(x - y)\phi_1(y) = 0 \qquad (x \in G). \tag{1}$$

Since ϕ_1 does not annihilate $\hat{f} - \xi$, $\phi_1 \not\equiv 0$. Hence (1) implies that the set of all finite linear combinations of translates of g is not dense in $L^p(G)$ if $p < 2$, although $Z(g)$ has measure 0.

7.8.8. We end this chapter with the construction of another class of S-sets; for $\Gamma = T$, this class was discovered by Kahane and Salem [1].

THEOREM. *Every infinite compact abelian group Γ contains a perfect set E with the following property: if $\phi \in L^\infty(G)$ and if $\sigma(\phi) \subset E$, then $\phi \in B(G)$.*

By Theorem 7.8.5, these sets are S-sets. They are also Helson sets, as may be seen by restating the theorem: *if $\phi \in L^\infty(G)$ and if $f * \phi = 0$ for all $f \in I_0(E)$, then $\phi \in B(G)$*; in particular, $\phi \in B(G)$ under the stronger hypothesis that $f * \phi = 0$ for all $f \in I(E)$, and hence E is a Helson set, by Theorem 5.6.3(c). It is not known whether every Helson set is an S-set.

It is also not known whether every Kronecker set is an S-set, but it seems quite possible that the Fourier-Stieltjes transforms on G are the only bounded functions whose spectrum can lie in a Kronecker set in Γ. In any case, the construction by which we will prove the theorem shows that the sets obtained have no "arithmetic cohesion" at all, and thus differ radically from the S-sets constructed by Herz whose arithmetic structure is quite rigid.

Proof: Suppose Λ is a closed subgroup of Γ, H is the annihilator of Λ, $\phi \in L^\infty(G)$, and $\sigma(\phi) \subset \Lambda$. If $y \in H$, $f(0) = 1$, $f(y) = -1$, and $f = 0$ at all other points of G, then $\hat{f} = 0$ on Λ, and since Λ is an S-set (Theorem 7.5.2) we have

$$\phi(x) - \phi(x - y) = (f * \phi)(x) = 0$$

for all $x \in G$. Thus ϕ is constant on the cosets of H, and ϕ may therefore be regarded as a member of $L^\infty(G/H)$.

We conclude that if the theorem is true for some closed subgroup of Γ it is also true for Γ. As in Section 5.2, the problem therefore reduces to two cases: (a) compact metric I-groups, and (b) groups D_q, with $q \geqq 2$.

Suppose Γ is a compact metric I-group. Since G is then countable (Theorem 2.2.6) we may arrange the elements of G in a sequence $x_1, x_2, x_3, \ldots$. Fix $r \geqq 1$, and suppose that disjoint compact neighborhoods E_j^{r-1} have been chosen $(1 \leqq j \leqq 2^{r-1})$, whose union is E_{r-1}. By Lemma 5.2.3, there is a Kronecker set K_r, consisting of 2^r points, which has 2 points in the interior of each set E_j^{r-1}, and since K_r is a Kronecker set, there is an integer $N_r > r$ with the following property: If $|\alpha(\gamma)| = 1$ for all $\gamma \in K_r$, the inequality

$$\inf_{\gamma \in K_r} \operatorname{Re} [\alpha(\gamma)(x_n, \gamma)] \geqq \tfrac{1}{2} \tag{1}$$

holds for some x_n with $1 \leqq n \leqq N_r$. If μ is a measure concentrated on K_r, then $d\mu = \alpha d|\mu|$ with $|\alpha| = 1$, and it follows that

$$\sup_{1 \leqq n \leqq N_r} |\hat{\mu}(x_n)| \geqq \tfrac{1}{2}\|\mu\|, \tag{2}$$

where

$$\hat{\mu}(x) = \int_\Gamma (x, \gamma) d\mu(\gamma) \qquad (x \in G). \tag{3}$$

By Theorem 2.6.1, we can associate a function $f_j \in L^1(G)$ with each point γ_j of K_r so that $\|f_j\|_1 < 2$, so that $\hat{f}_j(\gamma) = 1$ in a compact neighborhood V_j of γ_j and so that the supports of the functions $\hat{f}_j$ are disjoint and lie in the interior of E_{r-1}.

By Theorem 2.6.5, there are functions $g_n \in L^1(G)$, for $1 \leqq n \leqq N_r$, such that $\hat{g}_n(\gamma) = (x_n, \gamma_j)$ in a compact neighborhood $W_{j,n}$ of each point $\gamma_j \in K_r$, and such that

$$|g_n(-x_n) - 1| + \sum_{x \neq x_n} |g_n(-x)| < 3^{-r}. \tag{4}$$

Now put

$$E_j^r = V_j \cap \bigcap_{n=1}^{N_r} W_{j,n}, \quad E_r = \bigcup_{j=1}^{2^r} E_j^r. \tag{5}$$

(It should be borne in mind that the functions f_j, g_n and the sets V_j, $W_{j,n}$ also depend on r.) The set E_r is in the interior of E_{r-1}, and $E = \bigcap_1^\infty E_r$ is the desired set.

Suppose $\phi \in L^\infty(G)$, $\|\phi\|_\infty = 1$, and $\sigma(\phi) \subset E$. Fix r, let f_j be as above, define

$$p_j = (f_j * \phi)(0) \qquad (1 \leqq j \leqq 2^r), \tag{6}$$

and let μ_r be the measure, concentrated on K_r, such that $\mu_r(\{\gamma_j\}) = p_j$. Then

$$\|\mu_r\| \leqq 2^r \|f_j\|_1 \|\phi\|_\infty \leqq 2^{r+1} \tag{7}$$

and

$$p_j = \int \hat{f}_j d\mu_r = (f_j * \hat{\mu}_r)(0) \qquad (1 \leqq j \leqq 2^r). \tag{8}$$

Since each of functions $\hat{f}_j$ and $\hat{g}_n$ is constant on each of the sets E_j^r, $\hat{g}_n$ coincides on E_r with a linear combination of the functions $\hat{f}_j$, and since E_r contains E and K_r in its interior, our assumption that $\sigma(\phi) \subset E$, together with (6) and (8), implies that $(g_n * \psi)(0) = 0$, where $\psi = \phi - \hat{\mu}_r$. Thus

$$(9)\qquad -\psi(x_n) = [g_n(-x_n) - 1]\psi(x_n) + \sum_{x \neq x_n} g_n(-x)\psi(x),$$

and (4), (7), and (9) imply

$$(10)\qquad |\psi(x_n)| \leqq 3^{-r}||\psi||_\infty \leqq 3^{-r}(1 + 2^{r+1}) \qquad (1 \leqq n \leqq N_r).$$

It follows that

$$(11)\qquad \phi(x) = \lim_{r\to\infty} \hat{\mu}_r(x) \qquad (x \in G).$$

Since $|\hat{\mu}_r| \leqq 1 + |\psi|$, (10) and (2) imply

$$(12)\qquad ||\mu_r|| \leqq 2\{1 + 3^{-r}(1 + 2^{r+1})\} \qquad (r = 1, 2, 3, \ldots)$$

so that $\{||\mu_r||\}$ is bounded. Hence $\phi \in B(G)$, by (11) and Theorem 1.9.2.

This completes the proof in case (a). In case (b), we use sets of type K_q in place of Kronecker sets. The argument is then so similar that we omit the details; the modifications are like those used in Sections 5.2 and 5.5.

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CHAPTER 8

Fourier Analysis on Ordered Groups

The theory of analytic functions in the unit disc of the complex plane can be extended in several ways. In one type of extension, the unit disc is replaced by other plane domains (the Riemann mapping theorem plays an important role here), by domains on Riemann surfaces, and by domains in spaces of several complex variables. Another type of extension is based on the observation that every power series $\sum_0^\infty a_n z^n$, when restricted to the unit circle, is a trigonometric series $\sum_{-\infty}^{\infty} a_n e^{in\theta}$ whose coefficients are 0 if $n < 0$. This suggests that one might replace the circle T by any LCA group G whose dual Γ is *ordered* (the definition is given below; Z is the simplest example of an ordered group) and study those functions on G whose Fourier transforms vanish on the negative half of Γ.

The present chapter is devoted to extensions of the second type. G will play the role of the boundary T of the unit disc. Although it is possible to define an analogue of the interior of the disc whenever the dual of G is ordered (Arens [1], [2], Arens and Singer [1], Hoffman [1], Hoffman and Singer [1]), we shall restrict ourselves to problems which can be discussed and solved on the group itself. In the classical case ($G = T$) this amounts to proving theorems about functions analytic in the unit disc by studying only the boundary values of these functions; in some cases, the proofs so obtained are distinctly different from the more familiar ones. Sections 8.2 to 8.5 are to a large extent based on the work of Helson and Lowdenslager [1].

8.1. Ordered Groups

8.1.1. Suppose P is a semigroup in a LCA group G (this means simply that $P + P \subset P$) which is closed and has two additional properties:

(1) $$P \cap (-P) = \{0\}, \quad P \cup (-P) = G.$$

Under these conditions, P induces an order in G. For if we define $x \geqq y$ to mean that $x - y \in P$ (it is understood that $x, y, \ldots$ are elements of G), the axioms for a linear order are satisfied: if $x - y \geqq 0$ and $y - z \geqq 0$, then $x - z \geqq 0$ since P is a semigroup, so that the relation $\geqq$ is transitive, and (1) shows that each pair x, y satisfies one and only one of the relations $x > y, x = y, y > x$. Also, if $x > y$, then $x + z > y + z$.

The choice of a semigroup P with the above properties (i.e., the choice of an order in G which is compatible with the group operations) makes G into an *ordered group*. A given group G may have many different orders.

An order is said to be *archimedean* if it has the following property: to every pair of elements x, y of G such that $x > 0$ and $y > 0$, there corresponds a positive integer n such that $nx > y$.

8.1.2. THEOREM. *Let G be a discrete abelian group.*

(a) *G can be ordered if and only if G has no element of finite order.*

(b) *If G has no element of finite order and if the cardinality of G does not exceed the power of the continuum, then G can be given an archimedean order.*

(c) *If G has an archimedean order, then there is an order-preserving isomorphism of G onto a subgroup of R.*

Proof: We begin with the observation that if G is the direct sum of ordered groups G_α, where α runs through an index set A, then G can be ordered. For we can well-order A; then every $x \in G$ has the form

$$x = (x_1, x_2, \ldots, x_\alpha, \ldots) \qquad (x_\alpha \in G_\alpha),$$

and if $x \neq y$ and $y \in G$, there is a *first* $\alpha \in A$ for which $x_\alpha \neq y_\alpha$. Define $x < y$ if and only if $x_\alpha < y_\alpha$ for this α. (This is usually called a *lexicographic* order).

It is clear that no finite cyclic group can be ordered. Since every subgroup of an ordered group is ordered, it follows that an ordered group contains no element of finite order. Conversely, if G has no element of finite order, G can be embedded in a divisible group D of the same cardinality as G (Theorem 2.5.1) which has no element of finite order. Since D is a direct sum of copies of the rationals [Kaplansky [2]), D can be ordered, and the same is true of its subgroup G. This proves (a).

Under the assumptions of (b), the above group D is the direct sum of at most c copies of the rationals (where c is the power of the continuum) and since R is the direct sum of c copies of the rationals, G is isomorphic to a subgroup of R, and hence has an archimedean order.

To prove (c), suppose G has an archimedean order, fix $x_0 \in G$, $x_0 > 0$, and if $x \in G$, $x > 0$, let $E(x)$ be the set of all rational numbers m/n (m, n positive integers) such that $nx > mx_0$. If $\phi(x)$ is the least upper bound of $E(x)$, and if $\phi(-x) = -\phi(x)$, it is a simple exercise to verify that ϕ is an isomorphism of G into R and that ϕ preserves the order of G.

8.1.3. Let us say that P is a *maximal* semigroup in G if G is the only semigroup in G which contains P as a proper subset.

THEOREM. (a) *If P induces an archimedean order in G, then P is a maximal semigroup in G.* (b) *Conversely, if P is a closed maximal semigroup in G,* if *$P \neq \{0\}$ and if $P \cap (-P) = \{0\}$, then P defines an archimedean order in G.*

Proof: Fix $x \in P$, $x \neq 0$, and let S be the semigroup generated by P and $-x$. If the order induced by P is archimedean, then to every $y \in P$ there corresponds $n > 0$ such that $nx - y \in P$. Hence $-y \in S$, and so $S = G$. This proves (a).

To prove that P induces an order in G, under the assumptions of (b), we have to show that $G = P \cup (-P)$. If this is false, there exists $y \in G$ such that $y \notin P$ and $y \notin -P$. Fix $x' \in P$, $x' \neq 0$. Since P is maximal, the semigroup generated by P and y is G. Hence $-x' = x'' + ny$, where $x'' \in P$, $n > 0$, and $ny \neq 0$, so that $ny \in -P$. Since $-P$ is also a maximal semigroup, the same argument shows that there is a positive integer r such that $ry \neq 0$ and $ry \in P$. Then $nry \in P \cap (-P)$, hence $nry = 0$. Now $-ry \in -P$, and $-ry = (n-1)ry \in P$, contradicting the assumption that $P \cap (-P) = \{0\}$.

Suppose now that $a \in P$, $b \in P$, $b \neq 0$. The semigroup generated by P and $-b$ is G, and so $-a = p - nb$ for some $p \in P$ and $n > 0$. This says that $nb - a \in P$, and hence the order induced by P is archimedean.

8.1.4. THEOREM. *No non-trivial compact abelian group G can be ordered.*

Proof: Suppose G is compact and P is a closed semigroup in G which induces an order in G. Let S be the intersection of the sets $P + x$, where x ranges over P. These sets are compact and have the finite intersection property, so that S is not empty. Fix $x_0 \in S$ and $x \in P$. Since $x_0 \in P + x$, $x_0 - x \in P$; since $x_0 \in P$, we also have $x_0 + x \in P$. Thus $x_0 + G \subset P$, or $G = P$. Hence $G \cap (-G) = \{0\}$. But $-G = G$, and so $G = \{0\}$.

8.1.5. THEOREM. *If G is an ordered LCA group, and if G is not discrete, then $G = R \oplus D$, where D is a discrete ordered group.*

Proof: Since every subgroup of an ordered group is ordered, Theorem 8.1.4 shows that G has no non-trivial compact subgroups. Thus G has R^n as an open subgroup, for some $n > 0$, by the structure theorem 2.4.1, and R^n is ordered. In any ordered group, the set of all negative elements is open, and since the map $x \to -x$ is a homeomorphism, so is the set of all positive elements; it follows that removal of 0 disconnects an ordered group; but R^n is disconnected by the removal of a point if and only if $n = 1$. Hence R is an open subgroup of G. The conclusion of the proof of Theorem 2.4.1 shows that R is then a direct summand of G, and this completes the proof.

8.1.6. THEOREM. *If G is an archimedean ordered LCA group, then $G = R$ or G is a discrete subgroup of R.*

Proof: If G is discrete, we refer to Theorem 8.1.2(c). If G is not discrete, then $G = R \oplus D$, as in Theorem 8.1.5. Suppose D is not trivial. Let P be the semigroup which induces the order of G. Since P is closed and since removal of 0 from P leaves an open set, it follows that P contains all cosets of R (except R itself) which it intersects. Hence there are cosets of R which are not intersected by P. It follows that $P \cup R$ is a semigroup between P and G, which contradicts Theorem 8.1.3. Hence D is trivial and $G = R$.

8.1.7. Examples. The group Z^2, the dual of the torus T^2, is simple enough to be easily visualized (regard it as the set of all points in the plane whose coordinates are integers) and yet it offers enough diversity to be interesting.

If α and β are real numbers and if α/β is irrational, let P be the set of all $(m, n) \in Z^2$ such that

$$\alpha m + \beta n \geqq 0. \tag{1}$$

The map $(m, n) \to \alpha m + \beta n$ is an isomorphism of Z^2 into R which preserves the order induces by P. Hence P induces an archimedean order in Z^2.

If α/β is rational, the map $(m, n) \to \alpha m + \beta n$ is no longer one-to-one. Suppose $\alpha \neq 0$, and let $(m, n) \in P$ either if $\alpha m + \beta n > 0$ or if $\alpha m + \beta n = 0$ and $n \geqq 0$. This order is not archimedean, and P is not a maximal semigroup: the set of all (m, n) with $\alpha m + \beta n \geqq 0$ is a larger one.

The case $\alpha = 1$, $\beta = 0$ gives a simple lexicographic order;

$$(m, n) > (m', n') \text{ if } m > m' \text{ or if } m = m' \text{ and } n > n'.$$

8.1.8. Suppose now that G is a compact abelian group. By Theorems 8.1.2(a) and 2.5.6(c), Γ can be ordered if and only if G is connected. Suppose this is so, and suppose that a certain order has been chosen for Γ.

In this situation, we shall say that a function $f \in L^1(G)$ is of *analytic type* if $\hat{f}(\gamma) = 0$ for all $\gamma < 0$; this terminology is suggested by the opening remarks of the present chapter. Similarly, if $\mu \in M(G)$ and if $\hat{\mu}(\gamma) = 0$ for all $\gamma < 0$, we shall say that μ is a *measure of analytic type*. The set of all functions of analytic type which belong to $L^p(G)$ will be denoted by $H^p(G)$ $(1 \leqq p \leqq \infty)$. The continuous functions of analytic type will also be of interest to us, as will the trigonometric polynomials of analytic type.

It should be realized that this nomenclature is not quite complete. The class $H^p(G)$, for instance, does not depend on G alone, but depends also on the particular order which is imposed on Γ. Since we will usually work with one fixed order, this will cause no difficulty.

8.2. The Theorem of F. and M. Riesz

8.2.1. THEOREM. *If μ is a measure on the unit circle such that*

$$(1) \qquad \int_0^{2\pi} e^{in\theta}\, d\mu(\theta) = 0 \qquad (n = 1, 2, 3, \dots)$$

then μ is absolutely continuous with respect to Lebesgue measure.

In other words, if $\mu \in M(T)$ and if μ is of analytic type, then $d\mu(\theta) = g(e^{i\theta})d\theta$, for some $g \in L^1(T)$. Setting $d\lambda(\theta) = e^{i\theta}\, d\mu(\theta)$, the hypothesis of the theorem is equivalent to the assumption that

$$(2) \qquad \int_0^{2\pi} f(e^{i\theta})d\lambda(\theta) = 0$$

for every f which is continuous on the closed unit disc and which is analytic in the interior of the disc. It is thus not surprising that the theorem was originally proved by complex variable methods (F. and M. Riesz [1], Zygmund [1], vol. I, p. 285), particularly since its first applications lay in that direction. If the hypothesis is formulated as in (2), the theorem extends to multiply connected plane regions (Rudin [1]) and to regions on Riemann surfaces (Wermer [3]).

Helson and Lowdenslager [1] discovered a different proof in which the integers can be replaced by any discrete ordered group (although the statement of the theorem must be slightly altered if there is no smallest positive element). This proof is based on a minimizing process in a certain Hilbert space which we now describe and which will be used in other situations as well.

8.2.2. Suppose G is compact and connected, and an order has been selected for Γ. Suppose $\sigma \in M(G)$, $\sigma \geqq 0$, and

$$(1) \qquad d\sigma = d\sigma_s + w\,dx$$

where σ_s is singular with respect to the Haar measure of G and $w \in L^1(G)$ (Appendix E3).

Let Ω be the set of all trigonometric polynomials Q on G, of the form

(2) $$Q(x) = \sum_{\gamma>0} a(\gamma)(x, \gamma),$$

let K be the set of all functions $1 + Q(Q \in \Omega)$, and let $\bar{K}$ be the closure of K in the Hilbert space $L^2(\sigma)$. Since $\bar{K}$ is convex, there is a unique $\phi \in \bar{K}$ such that

(3) $$||\phi|| = \inf_{Q \in \Omega} ||1 + Q||;$$

the norms in (3) are taken in $L^2(\sigma)$. As a point function on G, ϕ is determined almost everywhere with respect to σ.

LEMMA. *In the above situation, ϕ has the following properties:*

(i) $\phi = 0$ *almost everywhere with respect to* σ_s.

(ii) $\phi w \in L^2(G)$ *and* $|\phi|^2 w = c^2$ *almost everywhere (with respect to Haar measure), where* $c = ||\phi||$.

(iii) *If* $||\phi|| > 0$ *and if* $h = 1/\phi$, *then* $h \in H^2(G)$ *and* $\hat{h}(0) = 1$.

Proof: If $g \in L^2(\sigma)$ and if $\phi + \alpha g \in \bar{K}$ for all scalars α, then ϕ and g are orthogonal, by the minimum property of $||\phi||$. This condition is satisfied, for $\gamma > 0$, if $g(x) = (x, \gamma)$ and if $g(x) = \phi(x)(x, \gamma)$. Hence

(5) $$\int_G \overline{\phi(x)}(x, \gamma) d\sigma(x) = 0 \qquad (\gamma > 0)$$

and

(6) $$\int_G |\phi(x)|^2 (x, \gamma) d\sigma(x) = 0 \qquad (\gamma > 0).$$

Since $|\phi|^2$ is real, (6) is also true for all $\gamma < 0$. If $d\lambda = |\phi|^2 d\sigma$, this says that $\hat{\lambda}(\gamma) = 0$ for $\gamma \neq 0$, and $\hat{\lambda}(0) = c^2$. Hence

(7) $$|\phi|^2 d\sigma = c^2 dx.$$

This implies that the singular part of $|\phi|^2 d\sigma$ is 0 and (i) follows. Hence $|\phi|^2 d\sigma = |\phi|^2 w\, dx$, and so $|\phi|^2 w = c^2$ almost everywhere; since $|\phi w|^2 = c^2 w$ and $w \in L^1(G)$, it follows that $\phi w \in L^2(G)$.

If $c > 0$, then $|h|^2 = |\phi|^{-2} = c^{-2} w$, so that $h \in L^2(G)$. For $\gamma \in \Gamma$, we have

$$8)\ \int_G h(x)(x,\gamma)dx = c^{-2}\int_G \overline{\phi(x)}w(x)(x,\gamma)dx = c^{-2}\int_G \overline{\phi(x)}(x,\gamma)d\sigma(x).$$

By (5), the last integral is 0 if $\gamma > 0$. Hence $\hat{h}(\gamma) = 0$ if $\gamma < 0$, and $h \in H^2(G)$. With $\gamma = 0$, (8) and (5) imply

$$(9)\quad c^2\hat{h}(0) = \int_G \bar{\phi}d\sigma = \int_G (1 + Q_n)\bar{\phi}d\sigma \qquad (n = 1, 2, 3, \ldots),$$

where $Q_n \in \Omega$ and $1 + Q_n \to \phi$ in the norm of $L^2(\sigma)$. Letting $n \to \infty$, (9) becomes

$$(10)\qquad c^2\hat{h}(0) = \int_G |\phi|^2 d\sigma = \|\phi\|^2 = c^2,$$

and the proof is complete.

8.2.3. We now come to the generalized version of the F. and M. Riesz theorem; we refer to Section 8.1.8 for the terminology used.

THEOREM. *Suppose G is compact and connected, Γ is ordered, $\mu \in M(G)$, and μ is of analytic type. If*

$$(1)\qquad d\mu = d\mu_s + f\,dx$$

where μ_s is singular and $f \in L^1(G)$, then

(a) *both μ_s and f are of analytic type, and*

(b) $\hat{\mu}_s(0) = 0$.

Proof: We may add any multiple of the Haar measure of G to μ without affecting either the hypothesis or the conclusion of the theorem. Hence we may assume, without loss of generality, that

$$(2)\qquad \inf_{Q\in\Omega} \int_G |1 + Q|^2 d\sigma > 0,$$

where $\sigma = |\mu|$ and Ω is as in 8.2.2. Choose ϕ as in 8.2.2. Since $\phi = \lim (1 + Q_n)$ in the norm of $L^2(\sigma)$, for some sequence $\{Q_n\}$ in Ω, the hypothesis that $\hat{\mu}(\gamma) = 0$ if $\gamma < 0$ implies

$$(3)\qquad \int_G (1 + Q(x))\phi(x)(x,\gamma)d\mu(x) = 0 \qquad (Q \in \Omega, \gamma > 0).$$

Since $\sigma_s = |\mu_s|$, part (i) of Lemma 8.2.2 shows that (3) is the same as

$$(4)\qquad \int_G (1 + Q(x))\phi(x)(x,\gamma)f(x)dx = 0 \qquad (Q \in \Omega, \gamma > 0).$$

By part (iii) of the lemma there is a sequence $\{Q_n\}$ in Ω such that $1 + Q_n \to h$ in the norm of $L^2(G)$. Since $\phi f \in L^2(G)$, by part (ii) of the lemma, since (4) holds with Q_n in place of Q, and since $\phi h = 1$, it follows that $\hat{f}(\gamma) = 0$ if $\gamma < 0$. Hence we also have $\hat{\mu}_s(\gamma) = \hat{\mu}(\gamma) - \hat{f}(\gamma) = 0$ if $\gamma < 0$, and (a) is proved.

To prove (b), we again apply Lemma 8.2.2, but this time with $\sigma = |\mu_s|$. Then $\sigma = \sigma_s$ and part (i) of the lemma implies that

$$\lim_{n\to\infty} \int_G |1 + Q_n)|^2 d\sigma = 0 \tag{5}$$

for a certain sequence $\{Q_n\}$ in Ω. By the Schwarz inequality, it follows that

$$\lim_{n\to\infty} \int_G (1 + Q_n) d\mu_s = 0. \tag{6}$$

But $\int Q d\mu_s = 0$ for every $Q \in \Omega$, since $\hat{\mu}_s(\gamma) = 0$ if $\gamma < 0$. Hence $\int d\mu_s = 0$, and the proof is complete.

8.2.4. We can now prove Theorem 8.2.1. For if $\mu \in M(T)$ and $\hat{\mu}(n) = 0$ for all $n < 0$, Theorem 8.2.3 implies that $\hat{\mu}_s(n) = 0$ if $n \leqq 0$. If $\mu_s \neq 0$, there is a *first* positive integer n_0 such that $\hat{\mu}_s(n_0) \neq 0$. Set $\hat{\lambda}(n) = \hat{\mu}_s(n_0 + n)$. Then λ is singular, $\hat{\lambda}(n) = 0$ for $n < 0$, and $\hat{\lambda}(0) = \hat{\mu}_s(n_0) \neq 0$, in contradiction to Theorem 8.2.3. Hence $\mu_s = 0$, and the proof is complete.

In the general case, the following observation can be extracted from the preceding argument:

If $\mu \in M(G)$, if μ is singular, and if $\hat{\mu}(\gamma) = 0$ for all $\gamma < 0$, then there cannot be a first element $\gamma_0 \in \Gamma$ at which $\hat{\mu}(\gamma_0) \neq 0$.

The word "first" is of course to be interpreted with respect to the given order of Γ.

8.2.5. To give another application of Theorem 8.2.3, take $G = T^2$; the dual group Z^2 is the set of all lattice points in the plane (see Section 8.1.6).

THEOREM (Bochner [4]). *Suppose Y is a closed sector in the plane, whose opening is less than π radians. If $\mu \in M(T^2)$ and if $\hat{\mu} = 0$ outside Y, then μ is absolutely continuous.*

Proof: Y is contained in a sector of the same opening whose vertex is a lattice point, and by translation we may therefore assume that the vertex of Y is at $(0, 0)$.

The proof will involve three distinct orders of Z^2. There are two distinct closed half-planes Π_1, Π_2, bounded by lines of irrational slope, which contain Y. By Theorem 8.2.3, $\hat{\mu}_s$ has its support in Π_1 and also in Π_2, hence in $\Pi_1 \cap \Pi_2$. Let Π be a closed half-plane, bounded by a line of irrational slope through $(0, 0)$, which contains $\Pi_1 \cap \Pi_2$ in its interior (except for $(0, 0)$). It is clear geometrically that $\Pi_1 \cap \Pi_2 \cap Z^2$ is well-ordered with respect to the order induced by Π. Hence the remark made in 8.2.4 shows that $\mu_s = 0$.

8.2.6. In general, however, the conclusion of Theorem 8.2.3 cannot be strengthened to "$\mu_s = 0$". For example, let Γ be a dense subgroup of R, with the natural order, and give Γ the discrete topology. The function $\psi(y) = \max(1 - |y|, 0)$ is positive-definite on R (compare the proof of Lemma 6.8.4); if ϕ is the restriction of ψ to Γ, it follows that ϕ is positive-definite on Γ, and hence $\phi \in B(\Gamma)$. Since $\phi(\gamma) > \frac{1}{2}$ at infinitely many points of Γ, and since Γ is discrete, $\phi \notin A(\Gamma)$. If $\gamma_0 \in \Gamma$ and $\gamma_0 > 1$, the function $\phi(\gamma - \gamma_0)$ is in $B(\Gamma)$, vanishes for all $\gamma \leqq 0$, but is not in $A(\Gamma)$.

8.2.7. Since T is a quotient group of R, we can transfer Theorem 8.2.1 from T to R:

THEOREM. *If $\mu \in M(R)$ and if*

$$\int_{-\infty}^{\infty} e^{-iyx}\, d\mu(x) = 0 \tag{1}$$

for all $y < 0$, then μ is absolutely continuous.

Proof: The change of variable $x \to cx$, where c is a constant, does not affect the absolute continuity of any $\mu \in M(R)$. Hence, if the theorem is false it is false for some measure μ such that

$$|\mu_s|([-\pi, \pi]) > \tfrac{1}{2}\|\mu_s\|. \tag{2}$$

Define $\sigma \in M(T)$ (as in the proof of Theorem 2.7.2) by

$$\sigma(E) = \mu(\{x : e^{ix} \in E\}). \tag{3}$$

Since (2) holds, σ has a non-zero singular component. But $\hat{\sigma}(n) = \hat{\mu}(n)$ for all $n \in Z$. Hence $\hat{\sigma}(n) = 0$ if $n < 0$, and so σ must be absolutely continuous, by Theorem 8.2.1. This contradiction establishes the theorem.

8.3. Geometric Means

8.3.1. Suppose $w \in L^1(G)$ and $w \geqq 0$. The *geometric mean* of w is defined by

$$\Delta(w) = \exp \int_G \log w(x) dx. \tag{1}$$

If the integral in (1) is $-\infty$, $\Delta(w) = 0$.

LEMMA. *Suppose G is a compact abelian group, $w \in L^1(G)$, and $w \geqq 0$. Then*

$$\exp \int_G \log w(x) dx = \inf \int_G e^{f(x)} w(x) dx, \tag{2}$$

the infimum being taken over all real trigonometric polynomials f on G such that $\hat{f}(0) = 0$.

Proof: If $\hat{f}(0) = 0$, then $\Delta(w) = \Delta(e^f w)$, and the familiar inequality between the arithmetic and geometric means shows that

$$\Delta(w) \leqq \int_G e^{f(x)} w(x) dx \tag{3}$$

for all real $f \in L^1(G)$ with $\hat{f}(0) = 0$.

Suppose, temporarily, that $\int \log w\, dx > -\infty$. Division of w by a positive constant does not affect (2). We may therefore assume that

$$\int_G \log w(x) dx = 0 \tag{4}$$

and we decompose $\log w$ into its positive and negative parts: $\log w = u - v$, $u \geqq 0$, $v \geqq 0$, $uv = 0$. Since $\int u = \int v$, there exist monotonically increasing sequences $\{u_n\}$, $\{v_n\}$ of bounded non-negative Borel functions on G such that $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ for all $x \in G$, and $\int u_n = \int v_n$. Put $g_n = v_n - u_n$. Then $\hat{g}_n(0) = 0$, and

$$g_n + \log w = \begin{cases} u - u_n \leqq \log w & \text{if } w \geqq 1, \\ -v + v_n \leqq 0 & \text{if } w < 1. \end{cases}$$

Hence $\exp\{g_n + \log w\} \leqq \max(w, 1)$. Since $g_n + \log w_n \to 0$ almost everywhere, Lebesgue's dominated convergence theorem yields, in conjunction with (4),

$$\lim_{n\to\infty} \int_G w(x) \exp\{g_n(x)\} dx = 1 = \Delta(w). \tag{5}$$

Fix n. By Lusin's theorem (Appendix E8) there is a uniformly bounded sequence of real continuous functions h_i on G such that $\lim h_i(x) = g_n(x)$ for almost all $x \in G$. Since $\hat{g}_n(0) = 0$, we have $\hat{h}_i(0) \to 0$ as $i \to \infty$. If $k_i = h_i - \hat{h}_i(0)$, then k_i is continuous, $\hat{k}_i(0) = 0$, $k_i(x) \to g_n(x)$ for almost all $x \in G$ as $i \to \infty$, and $\{\|k_i\|_\infty\}$ is bounded. The functions k_i can be uniformly approximated by real trigonometric polynomials f_i with $\hat{f}_i(0) = 0$, and we have

$$\lim_{i\to\infty} \int_G w(x) \exp\{f_i(x)\} dx = \int_G w(x) \exp\{g_n(x)\} dx. \tag{6}$$

If we combine (3), (5), and (6), we see that we have proved (2), provided that $\int \log w(x) dx$ is finite.

In the general case, we replace w by $w + \varepsilon$, where $\varepsilon > 0$. What we have just proved shows that

$$\Delta(w + \varepsilon) = \inf \int_G e^f (w + \varepsilon) dx \geqq \inf \int_G e^f w \, dx, \tag{7}$$

where f ranges over all real trigonometric polynomials with $\hat{f}(0) = 0$. Letting $\varepsilon \to 0$ in (7), comparison with (3) gives (2).

8.3.2. We now come to a theorem of Szegö [1], generalized by Helson and Lowdenslager [1]:

THEOREM. *Suppose G is compact and connected, Γ is ordered, $\sigma \in M(G)$, $\sigma \geqq 0$, and*

$$d\sigma = d\sigma_s + w\, dx, \tag{1}$$

where σ_s is singular and $w \in L^1(G)$. Then

$$\exp \int_G \log w(x) dx = \inf_{Q \in \Omega} \int_G |1 + Q(x)|^2 d\sigma(x), \tag{2}$$

where Ω is the set of all trigonometric polynomials Q on G of the form

$$Q(x) = \sum_{\gamma > 0} a(\gamma)(x, \gamma). \tag{3}$$

Proof: Since every real trigonometric polynomial f on G with $\hat{f}(0) = 0$ is of the form $f = Q + \bar{Q} = 2 \operatorname{Re} Q$, for some $Q \in \Omega$, Lemma 8.3.1 asserts that

$$\Delta(w) = \inf_{Q \in \Omega} \int_G |e^Q|^2 w \, dx. \tag{4}$$

Note that $e^Q - 1 = Q + Q^2/2! + \ldots$. This series converges uniformly on G, and each of its partial sums belongs to Ω, if $Q \in \Omega$. Hence (4) implies

$$\Delta(w) \geqq \inf_{Q \in \Omega} \int_G |1 + Q|^2 w \, dx. \tag{5}$$

This inequality holds for all non-negative $w \in L^1(G)$, and the opposite inequality can be deduced from it. Put $w = |1 + P|^2$ for some $P \in \Omega$. Then (5) gives

$$\Delta(|1 + P|^2) \geqq \inf_{Q \in \Omega} \int_G |1 + P + Q + PQ|^2 dx \geqq 1; \tag{6}$$

the last inequality follows from the Parseval formula, since $PQ \in \Omega$ for every $Q \in \Omega$. Hence

$$\Delta(w) \leqq \Delta(w)\Delta(|1 + P|^2) = \Delta(|1 + P|^2 w) \leqq \int_G |1 + P|^2 w \, dx \tag{7}$$

for every $P \in \Omega$, since the arithmetic mean is never less than the geometric mean.

Thus equality holds in (5). Finally, part (i) of Lemma 8.2.2 shows that the right side of (5) is equal to the right side of (2). This completes the proof.

8.4. Factorization Theorems in $H^1(G)$ and in $H^2(G)$

In this section, G is a compact connected abelian group, Γ is an ordered group, and the spaces $H^p(G)$ are defined with respect to this order.

8.4.1. THEOREM. *If $f \in H^1(G)$, then*

$$|\hat{f}(0)| \leqq \exp \int_G \log |f(x)| dx; \tag{1}$$

in particular, $\log |f| \in L^1(G)$ *if* $\hat{f}(0) \neq 0$.

COROLLARY. *Let $E_\alpha = \{x \in G: f(x) = \alpha\}$. If $f \in H^1(G)$, then there is at most one number α for which E_α has positive Haar measure.*

Proof: Put $f_n = f * k_n$, where $\{k_n\}$ is a sequence of trigonometric polynomials on G with $\hat{k}_n(0) = 1$, so that $\|f_n - f\|_1 \to 0$. Since each f_n is a trigonometric polynomial of analytic type, the Parseval formula implies

$$|\hat{f}(0)|^2 = |\hat{f}_n(0)|^2 \leqq \int_G |(1 + Q)f_n|^2 \, dx \tag{2}$$

for every $Q \in \Omega$. By Theorem 8.3.2, the greatest lower bound of the last expression in (2) is $\Delta(|f_n|^2)$. Hence $|\hat{f}(0)| \leqq \Delta(|f_n|)$. For any $\varepsilon > 0$, it follows that

$$\log |\hat{f}(0)| \leqq \int_G \log (|f_n| + \varepsilon) dx. \tag{3}$$

Since

$$|\log (|f_n| + \varepsilon) - \log (|f| + \varepsilon)| \leqq \varepsilon^{-1} |f_n - f|, \tag{4}$$

the integral of the left side of (4) tends to 0 as $n \to \infty$. Hence (3) implies that

$$\log |\hat{f}(0)| \leqq \int_G \log (|f| + \varepsilon) dx, \tag{5}$$

and (1) follows from (5) as $\varepsilon \to 0$, by the monotone convergence theorem.

8.4.2. If $\hat{f}(0) = 0$ but if there is a *first* $\gamma_0 > 0$ at which $\hat{f}(\gamma_0) \neq 0$, then we can still conclude that $\int \log |f| > -\infty$; the argument is quite similar to that used in 8.2.4. In particular, we obtain the classical theorem that

$$\int_0^{2\pi} \log |f(e^{i\theta})| d\theta > -\infty \tag{1}$$

for every $f \in H^1(T)$, except when f is identically 0.

However, the assumption $\hat{f}(0) \neq 0$ cannot be dropped altogether from the second statement in Theorem 8.4.1. This is very easily seen if the order of Γ is not archimedean. In that case Γ contains an element γ_0 and a non-trivial subgroup Λ such that $\gamma < \gamma_0$ for all $\gamma \in \Lambda$, and there is a function $g \in A(G)$, not identically 0, of the form

$$g(x) = \sum_{\gamma \in \Lambda} c_\gamma (x, \gamma)$$

which vanishes on a non-empty open subset V of G. If $f(x) = (x, \gamma_0) g(x)$, then $f = 0$ on V, hence $\int_G \log |f| = -\infty$, although f is of analytic type.

The problem is more delicate if the order of Γ is archimedean. In this situation Arens [2] has proved that $\int_G \log |f| > -\infty$ if f is a continuous function of analytic type on G which does not vanish identically. Helson and Lowdenslager, on the other hand, have recently discovered that the word "continuous" cannot be replaced by "bounded" in Arens' theorem (the example is unpublished at the time of this writing). This difference between bounded functions and continuous functions is curious and quite unexpected.

8.4.3. THEOREM. *Suppose $w \in L^1(G)$ and $w \geqq 0$. Then $w = |f|^2$ for some $f \in H^2(G)$ with $\hat{f}(0) \neq 0$ if and only if*

$$\int_G \log w(x) dx > -\infty. \tag{1}$$

Proof: If $w = |f|^2$ and $f \in H^2(G)$, we obtain, as in the proof of Theorem 8.4.1, that

$$\Delta(w) = \Delta(|f|^2) = \inf_{Q \in \Omega} \int_G |(1 + Q) f|^2 \, dx \geqq |\hat{f}(0)|^2. \tag{2}$$

Hence (1) holds if $\hat{f}(0) \neq 0$.

Conversely, suppose (1) holds, and define c by

$$c^2 = \inf_{Q \in \Omega} \int_G |1 + Q|^2 w \, dx, \qquad c \geqq 0. \tag{3}$$

By Theorem 8.3.2, $c > 0$, and Lemma 8.2.2 (with $d\sigma = w\,dx$)

therefore implies that there is a function $h \in H^2(G)$ with $\hat{h}(0) = 1$ and $|ch|^2 = w$. To complete the proof, put $f = ch$.

8.4.4. THEOREM. *Suppose $f \in H^1(G)$ and $\hat{f}(0) \neq 0$. Then there are functions α and β in $H^2(G)$ such that $f = \alpha\beta$ and $\|\alpha\|_2^2 = \|\beta\|_2^2 = \|f\|_1$.*

Proof: By Theorem 8.4.1, $\log |f| \in L^1(G)$. We put $d\sigma = |f|dx$, apply Lemma 8.2.2, and conclude, as in the proof of Theorem 8.4.3, that there is a function $h \in H^2(G)$ such that $|f| = |ch|^2$, where $c > 0$. Moreover, $h = 1/\phi$, and

$$\lim_{n\to\infty} \int_G |\phi - (1 + Q_n)|^2 |f| dx = 0 \tag{1}$$

for a certain sequence $\{Q_n\}$ in Ω.

Define $\alpha = ch$, $\beta = f/\alpha$. Then $f = \alpha\beta$, $|\alpha|^2 = |\beta|^2 = |f|$, and it remains to be proved that $\beta \in H^2(G)$. By (1), the Schwarz inequality implies that

$$\lim_{n\to\infty} \int_G |\phi f - (1 + Q_n) f| dx = 0. \tag{2}$$

Since $(1 + Q_n)f \in H^1(G)$ and since $H^1(G)$ is a closed subspace of $L^1(G)$, it follows that $\phi f \in H^1(G)$. But $\phi f = f/h = cf/\alpha = c\beta$. Thus $\beta \in H^1(G)$; since $|\beta|^2 = |f|$, $\beta \in L^2(G)$; hence $\beta \in H^2(G)$, and the proof is complete.

This proof may be of interest even in the classical case, i.e., in the case $G = T$. There the theorem is usually proved by first factoring out a suitable Blaschke product; the remaining factor has no zero in the unit disc and hence has an analytic square root (Zygmund [1] vol. I, p. 275).

8.4.5. If $G = T$, Theorem 8.4.3 has an analogue for trigonometric polynomials, due to Fejér [1] and F. Riesz. It is interesting that this analogue, unlike the preceding theorems, does not even extend to the case $G = T^2$:

THEOREM. (a) *If p is a non-negative trigonometric polynomial on T, then $p = |f|^2$, where f is of the form*

$$f(e^{i\theta}) = \sum_{k=0}^{n} a_k e^{ik\theta}. \tag{1}$$

(b) *If* $0 < \delta < 1/4$ *and if* $w = 1 + \delta(e^{ix} + e^{-ix} + e^{iy} + e^{-iy})$, *then* w *is positive on* T^2, *but* w *is not a product of two trigonometric polynomials on* T^2, *unless one is a constant multiple of a character.*

Note that Z^2 can be made into an ordered group in many ways, as was shown in 8.1.7. If $H^2(T^2)$ is defined with respect to any of these orderings, Theorem 8.4.3 implies that $w = |f|^2$ for some $f \in H^2(T^2)$; it follows from (b) that this f cannot be a trigonometric polynomial.

Proof: Write $p(e^{i\theta}) = \sum_{-n}^{n} c_k e^{ik\theta}$. Since p is real, $c_{-k} = \bar{c}_k$, and we may therefore choose n so that $c_n \neq 0$. Put $F(z) = \sum c_k z^k$. Then F is a rational function, $F(z) \geqq 0$ if $|z| = 1$, and the reflection principle implies that $F(1/\bar{a}) = 0$ if $F(a) = 0$. Since $z^n F(z)$ is a polynomial of degree $2n$, it has $2n$ zeros; any zeros on T have even order; hence

$$F(z) = c \prod_{j=1}^{n} (z - z_j)(z^{-1} - \bar{z}_j), \tag{2}$$

where c is a positive constant. Since $z^{-1} = \bar{z}$ when $|z| = 1$, (2) shows that

$$p(e^{i\theta}) = F(e^{i\theta}) = c\left| \prod_{j=1}^{n} (e^{i\theta} - z_j)\right|^2, \tag{3}$$

and (a) follows.

To prove (b), assume that $w = f_1 f_2$, where f_1 and f_2 are trigonometric polynomials on T^2. Let E_i be the set of all $n \in Z^2$ at which $\hat{f}_i(n) \neq 0$. Each E_i is finite and not empty. Similarly, let E be the set of all $n \in Z^2$ at which $\hat{w}(n) \neq 0$; E consists of the points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

Regarding Z^2 as the set of all lattice points in the plane, let $a_i[b_i]$ be the highest [lowest] point among those points of E_i which are farthest to the right. Then the point $a_1 + a_2$ has no other representation as a sum of an element of E_1 and an element of E_2. It follows that $a_1 + a_2 \in E$, and hence $a_1 + a_2 = (1, 0)$. Similarly, $b_1 + b_2 = (1, 0)$. Thus $a_1 = b_1$, $a_2 = b_2$, and each of the sets E_i has only *one* point which is farthest to the right. The same is true of the other three directions.

Let ρ, ρ_1, ρ_2 denote the horizontal widths of E, E_1, E_2. Then $\rho_1 + \rho_2 = \rho = 2$. If $\rho_1 = 2$, then $\rho_2 = 0$, so that E_2 lies on a vertical line; the preceding paragraph then implies that E_2 consists of a single point, hence f_2 is a character on T^2, and this is what (b) asserts. If (b) is false, we must therefore have $\rho_1 = \rho_2 = 1$, and the same must be true of the vertical widths. Combined with the preceding paragraph, this implies that each E_i consists of 2 points. Since $E \subset E_1 + E_2$ and E has 5 points, we have a contradiction.

8.5. *Invariant Subspaces of* $H^2(G)$

8.5.1. We return to the general situation (Γ discrete and ordered). For any $\gamma \in \Gamma$, we define the multiplication operator M_γ on $H^2(G)$ by

$$(M_\gamma f)(x) = (x, \gamma) f(x). \tag{1}$$

A linear subspace of $H^2(G)$ is said to be *invariant* if

$$M_\gamma X \subset X \tag{2}$$

for all $\gamma > 0$.

Adapting a definition of Beurling [4], we call a function $f_0 \in H^2(G)$ an *inner function* if $|f_0| = 1$ (almost everywhere on G); a function $f_1 \in H^2(G)$ is an *outer function* if

$$\int_G \log |f_1(x)| dx = \log \left| \int_G f_1(x) dx \right|. \tag{3}$$

In other words, f_1 is outer if the geometric mean of $|f_1|$ is equal to $|\hat{f}_1(0)|$. We recall that the left side of (3) is never less than the right side, by Theorem 8.4.1.

For any $f \in H^2(G)$, we let X_f denote the smallest closed invariant subspace of $H^2(G)$ which contains f.

8.5.2. Theorem. (Beurling [4], Helson and Lowdenslager [1]). *Suppose* $f \in H^2(G)$ *and*

$$\int_G \log |f(x)| dx > -\infty. \tag{1}$$

Then $f = f_0 f_1$, *where* f_0 *is an inner function and* f_1 *is an outer func-*

tion. This factorization is unique, except for multiplication by constants of absolute value 1. *Furthermore,*

$$X_f = f_0 \cdot H^2(G). \tag{2}$$

The last assertion states, explicitly, that $g \in X_f$ if and only if $g = f_0 h$ for some $h \in H^2(G)$, and also that $X_f = X_{f_0}$.

Proof: Let K_f be the closure (in $H^2(G)$) of the set of all functions $(1 + Q)f$, where Q ranges over Ω. Since K_f is convex, it has a unique element ψ whose norm is a minimum. By Lemma 8.2.2 (taking $d\sigma = |f|^2\, dx$), we see that $|\psi| = c$, where

$$c^2 = \inf_{Q \in \Omega} \int_G |(1 + Q)f|^2\, dx = \exp \int_G \log |f|^2\, dx > 0. \tag{3}$$

Here we used (1) and Theorem 8.3.2. Setting $f_0 = \psi/c$, it is evident that f_0 is inner, and since $K_f \subset X_f$, we have $f_0 \in X_f$. If $f_1 = f/f_0$, part (iii) of Lemma 8.2.2 shows that $f_1 \in H^2(G)$ and $\hat{f}_1(0) = c$. Since $|f_1| = |f|$, we therefore conclude from (3) that $\hat{f}_1(0) = \Delta(|f_1|)$, and so f_1 is outer.

Let us consider the case of an outer function f which satisfies (1). The preceding construction then yields

$$0 < \hat{f}_1(0) = \Delta(|f_1|) = \Delta(|f|) = |\hat{f}(0)|. \tag{4}$$

Approximating f_0 and f_1 by trigonometric polynomials in $H^2(G)$, the equation $f = f_0 f_1$ is seen to imply that $\hat{f}(0) = \hat{f}_0(0)\hat{f}_1(0)$, so that $|\hat{f}_0(0)| = 1$, by (4). Since $|f_0| = 1$, the Parseval equation therefore shows that $\hat{f}_0(\gamma) = 0$ for all $\gamma \neq 0$, and so f_0 is a constant of absolute value 1. Since $f_0 \in X_f$, it follows that $X_f = H^2(G)$.

Returning to the general case, we thus have $X_{f_1} = H^2(G)$. Since $|f_0| = 1$, multiplication by f_0 is an isometry in $H^2(G)$, and so $X_f = f_0 \cdot X_{f_1}$. This proves (2).

Finally, we prove the uniqueness of the factorization. Suppose $f = gh$ with g inner and h outer. Since $|h| = |f|$, $\Delta(|h|) > 0$, and hence $X_h = H^2(G)$, by the preceding argument. As above, $X_f = g \cdot X_h$, since $|g| = 1$. Thus $f_0 \cdot H^2(G) = g \cdot H^2(G)$. This shows that both f_0/g and its complex conjugate g/f_0 belong to $H^2(G)$, and therefore f_0/g must be constant.

This completes the proof.

COROLLARY. *If $f \in H^2(G)$, then $X_f = H^2(G)$ if and only if f is an outer function and the geometric mean of $|f|$ is positive.*

If $\Delta(|f|) > 0$, this follows from the preceding theorem. If $\Delta(|f|) = 0$, then $\hat{f}(0) = 0$, by Theorem 8.4.1, and so $\hat{g}(0) = 0$ for every $g \in X_f$; thus $X_f \neq H^2(G)$.

8.5.3. The proof of the next theorem is due to Helson and Lowdenslager and is unpublished at the time of this writing; it extends a theorem of Beurling [4] (whose proof used Nevanlinna's representation of analytic functions of bounded characteristic by means of Poisson-Stieltjes integrals), and its idea has been used by Wermer [5] in a study of function algebras.

THEOREM. *If X is an invariant closed subspace of $H^2(G)$ which contains a function g with $\hat{g}(0) \neq 0$, then there is an inner function f_0 such that $X = X_{f_0}$.*

Proof: Since $\hat{g}(0) \neq 0$, the constant function 1 is not orthogonal to X in $H^2(G)$. Let ψ be the orthogonal projection of 1 into X. Then $\psi \neq 0$. Since X is invariant and since $1 - \psi$ is orthogonal to X, we have

$$\int_G \{1 - \overline{\psi(x)}\}\psi(x)(x, \gamma)dx = 0 \qquad (\gamma \geqq 0). \tag{1}$$

But $\int \psi(x)(x, \gamma)dx = 0$ for all $\gamma > 0$, since $\psi \in H^2(G)$. Hence

$$\int_G |\psi(x)|^2(x, \gamma)dx = 0 \qquad (\gamma > 0), \tag{2}$$

and so $|\psi| = c$, a constant. Put $f_0 = \psi/c$. Then $X_{f_0} = X_\psi$.

Since $\psi \in X$, it is clear that $X_\psi \subset X$. Let $h \in X$ be orthogonal to X_ψ. Then

$$\int_G \overline{h(x)}\psi(x)(x, \gamma)dx = 0 \qquad (\gamma \geqq 0). \tag{3}$$

Since $1 - \psi$ is orthogonal to X, we also have

$$\int_G \{1 - \overline{\psi(x)}\}h(x)(x, \gamma)dx = 0 \qquad (\gamma \geqq 0) \tag{4}$$

and hence

$$\int_G \overline{\psi(x)}h(x)(x, \gamma)dx = 0 \qquad (\gamma > 0). \tag{5}$$

By (3) and (5), $\bar{\psi}h = 0$, and so $h = 0$, since $|\psi| = c \neq 0$. Thus $X_\psi = X$, and the proof is complete.

8.5.4. The conclusion of Theorem 8.5.3 holds for every invariant closed subspace X of $H^2(T)$ (except for $X = \{0\}$) since there is always a first $n \geqq 0$ such that $\hat{f}(n) \neq 0$ for some $f \in X$ (the argument is as in 8.2.4). However, if Γ contains no smallest positive element and if X is the set of all $f \in H^2(G)$ with $\hat{f}(0) = 0$, then X is an invariant closed subspace for which the conclusion of Theorem 8.5.3 is false.

Other results involving inner and outer functions may be found in Lax [1], de Leeuw and Rudin [1], and Rudin [7]. These papers are based on complex variable methods.

8.6. A Gap Theorem of Paley

We suppose again that G is compact and connected, so that Γ can be given an order which is compatible with its group structure. We fix such an order and define $H^1(G)$ with respect to this order. For each $\gamma \geqq 0$, put

(1) $$L_\gamma = \{\gamma' : \gamma \leqq \gamma' \leqq 2\gamma\},$$

and if E is a set of positive elements of Γ, let $N(E,\gamma)$ be the number of terms of E in L_γ.

THEOREM. *The following properties of E are equivalent:*

(a) *$N(E,\gamma)$ is a bounded function of γ.*

(b) *If $f \in H^1(G)$, then $\sum_{\gamma \in E} |\hat{f}(\gamma)|^2 < \infty$.*

(c) *If $\mu \in M(G)$ and $\hat{\mu}(\gamma) = 0$ for $\gamma < 0$, then $\sum_{\gamma \in E} |\hat{\mu}(\gamma)|^2 < \infty$.*

For $G = T$, Paley [2] proved that (a) implies (b); for the converse, see Rudin [8]. The theorem of F. and M. Riesz shows that (b) and (c) are identical statements if $G = T$, but in general (c) asserts more; this follows from the example in 8.2.6.

Let (b′) be the statement (b) with $L^1(G)$ in place of $H^1(G)$. Then (b′) is *false* for every infinite set E. For if (b′) were true for some infinite E, it would also be true for every subset of E, hence for an infinite Sidon set (see Example 5.7.6(a)) but this contradicts Theorem 5.7.3(e).

Proof: We first show that (a) implies (b). Fix $f \in H^1(G)$ and assume, without loss of generality, that $\hat{f}(0) \neq 0$. By Theorem 8.4.4, we then have $f = \alpha\beta$, with $\alpha, \beta \in H^2(G)$ and $\|\alpha\|_2^2 = \|\beta\|_2^2 = \|f\|_1$. If $\gamma_1, \gamma_2, \gamma_3, \ldots$ is an enumeration of those elements of E at which $\hat{f} \neq 0$, we have

$$\hat{f}(\gamma_i) = \sum_{0 \leqq \gamma \leqq \gamma_i} \hat{\alpha}(\gamma_i - \gamma)\hat{\beta}(\gamma) \qquad (i = 1, 2, 3, \ldots). \tag{2}$$

Let $S_i = \{\gamma : \gamma_i \in L_\gamma\}$. If $0 \leqq \gamma \leqq \gamma_i$ and $\gamma \notin S_i$, then $\gamma_i > 2\gamma$, and so $\gamma_i - \gamma \in S_i$. The sum in (2) can therefore be split in two; in one sum $\gamma \in S_i$, and in the other $\gamma_i - \gamma \in S_i$. The Schwarz inequality, applied to each of these sums, yields

$$|\hat{f}(\gamma_i)| \leqq \|\alpha\|_2 \left\{ \sum_{\gamma \in S_i} |\hat{\beta}(\gamma)|^2 \right\}^{1/2} + \|\beta\|_2 \left\{ \sum_{\gamma \in S_i} |\hat{\alpha}(\gamma)|^2 \right\}^{1/2}, \tag{3}$$

and the inequality $(a + b)^2 \leqq 2a^2 + 2b^2$ transforms (3) into

$$|\hat{f}(\gamma_i)|^2 \leqq 2\|f\|_1 \sum_{\gamma \in S_i} \{|\hat{\beta}(\gamma)|^2 + |\hat{\alpha}(\gamma)|^2\} \qquad (i = 1, 2, 3, \ldots). \tag{4}$$

Since (a) holds and since $\gamma_i \in L_\gamma$ if and only if $\gamma \in S_i$, there is a constant K such that no γ is contained in more than K of the sets S_i. If we add the inequalities (4) we therefore obtain

$$\sum_{\gamma \in E} |\hat{f}(\gamma)|^2 \leqq 4K\|f\|_1^2. \tag{5}$$

Thus (a) implies (b).

Next, let X be the set of all $f \in H^1(G)$ such that $\hat{f}(\gamma) = 0$ for all $\gamma \in E$. If ϕ is a function on E and $\sum |\phi(\gamma)|^2 < \infty$ (i.e., if $\phi \in L^2(E)$), then there is an $f \in H^2(G)$ such that $\hat{f} = \phi$ on E, $\hat{f} = 0$ outside E, and $\|f\|_1 \leqq \|f\|_2 = \|\phi\|_2$. Hence the norm (in $H^1(G)/X$) of the coset Y_ϕ of X which contains f does not exceed $\|\phi\|_2$. This says that the map $\phi \to Y_\phi$ of $L^2(E)$ into $H^1(G)/X$ is continuous; it is clearly one-to-one.

If now (b) holds, this map is onto, and hence has a continuous inverse (Appendix C6). Thus there is a constant K such that (5) holds for every $f \in H^1(G)$. Fix $\mu \in M(G)$ with $\hat{\mu}(\gamma) = 0$ if $\gamma < 0$, and choose $\gamma_1, \ldots, \gamma_n$ in E. Given $\varepsilon > 0$, there exists $g \in L^1(G)$ such that $\hat{g}(\gamma_1) = \ldots = \hat{g}(\gamma_n) = 1$ and $\|g\|_1 < 1 + \varepsilon$ (Theorem 2.6.8). If $f = g * \mu$, then $f \in H^1(G)$, and so

$$(6) \qquad \sum_1^n |\hat{\mu}(\gamma_i)|^2 = \sum_1^n |\hat{f}(\gamma_i)|^2 \leqq 4K\|f\|_1^2 \leqq 4K(1+\varepsilon)^2\|\mu\|^2.$$

Since $\{\gamma_1, \ldots, \gamma_n\}$ was an arbitrary finite subset of E and since ε was arbitrary, (6) implies

$$(7) \qquad \sum_{\gamma \in E} |\hat{\mu}(\gamma)|^2 \leqq 4K\|\mu\|^2.$$

Thus (b) implies (c).

Suppose now that (c) holds. Then (b) holds, and the preceding proof shows that the inequality (7) holds for some K and for all $\mu \in M(G)$ which are of analytic type. Fix an integer $p > 16K$. If E does not satisfy (a), then there exists γ_0 such that L_{γ_0} contains more than p elements of E. Define

$$S_0 = \{\gamma \geqq 0\colon n\gamma < \gamma_0 \text{ for every positive integer } n\},$$
$$S_1 = \{\gamma \geqq 0\colon m\gamma_0 > \gamma \text{ for some positive integer } m\},$$

and let Λ_0, Λ_1 be the groups generated by S_0, S_1. We may think of S_0 as the set of all $\gamma \geqq 0$ which are "infinitely small" relative to γ_0; similarly, S_1 is the set of all $\gamma \geqq 0$ which are not "infinitely large" relative to γ_0. It is easy to see that $\Lambda_0 = S_0 \cup (-S_0)$ and $\Lambda_1 = S_1 \cup (-S_1)$.

If h is the natural homomorphism of Λ_1 onto $\Lambda = \Lambda_1/\Lambda_0$, then h induces an order in Λ ($h(\gamma) > 0$ means: $\gamma > 0$ and $\gamma \notin \Lambda_0$), and this order of Λ is archimedean. We may therefore regard h as an order-preserving homomorphism of Λ_1 into R (i.e., $h(\gamma) \geqq 0$ if and only if $\gamma \geqq 0$) such that $h(\gamma_0) = 1$.

Define $w(t) = \max(2 - |t-2|, 0)$, for $t \in R$. Then w is a translate of a positive-definite function on R, and if $\psi(\gamma) = w(h(\gamma))$, then ψ is a translate of a positive-definite function on Λ_1. Extend ψ to Γ by setting $\psi(\gamma) = 0$ outside Λ_1. Then $\psi = \hat{\mu}$ for some $\mu \in M(G)$, and we see that $\hat{\mu}(\gamma) = 0$ if $\gamma < 0$ and that $\|\mu\| = 2$. Since $w(t) \geqq 1$ on $[1, 2]$, it follows that $\hat{\mu}(\gamma) \geqq 1$ on

L_{γ_0}, and since L_{γ_0} contains more than p elements of E, the sum in (7) exceeds p. Hence $p < 4K||\mu||^2 = 16K$, which contradicts our choice of p.

8.7. *Conjugate Functions*

8.7.1. We again assume that G is compact and connected so that Γ can be ordered. With respect to any fixed order of Γ, one can define a notion of conjugacy. We first do this for trigonometric polynomials. If

$$u(x) = \sum c_\gamma (x, \gamma) \qquad (x \in G) \tag{1}$$

is a trigonometric polynomial on G, the *conjugate function of u* is the trigonometric polynomial

$$v(x) = -i \sum_{\gamma > 0} c_\gamma (x, \gamma) + i \sum_{\gamma < 0} c_\gamma (x, \gamma) \qquad (x \in G). \tag{2}$$

We also define

$$w(x) = u(x) + iv(x) = c_0 + 2 \sum_{\gamma > 0} c_\gamma (x, \gamma) \tag{3}$$

and

$$F(x) = \sum_{\gamma \geqq 0} c_\gamma (x, \gamma). \tag{4}$$

Then w and F are trigonometric polynomials of analytic type; we call F the *analytic contraction* of u, since it is obtained from u by simply suppressing the coefficients c_γ with $\gamma < 0$. If u is real, so is its conjugate v, and w is the unique trigonometric polynomial of analytic type which has u for its real part and which satisfies the condition $\hat{w}(0) = \hat{u}(0)$. The equations

$$\Phi u = F, \qquad \Psi u = w \tag{5}$$

define linear operators on the space of all trigonometric polynomials on G.

In general, if $u \in L^1(G)$, and if $\hat{u}\chi \in A(\Gamma)$, where $\chi(\gamma) = 1$ for $\gamma \geqq 0$, $\chi(\gamma) = 0$ for $\gamma < 0$, then the function F defined by the equation $\hat{F} = \hat{u}\chi$ will be called the analytic contraction of u, and we will write $F = \Phi u$.

We shall prove an extension of a classical theorem of M. Riesz (Zygmund [1], vol. I, p. 253; see Bochner [3], [5] for generalizations) which asserts that Φ and Ψ are bounded linear operators on $L^p(G)$ if $1 < p < \infty$. Since $iv = w - u$, it follows that the map $u \to v$ is also bounded in $L^p(G)$.

8.7.2. THEOREM. *Suppose $1 < p < \infty$. There exist constants A_p and B_p (they do not depend on G) such that the inequalities*

$$(1) \qquad \|\Phi u\|_p \leqq A_p \|u\|_p, \qquad \|\Psi u\|_p \leqq B_p \|u\|_p$$

hold for every trigonometric polynomial u on G. Hence Φ and Ψ can be extended to bounded linear operators on $L^p(G)$.

The operator Φ is a projection (i.e., $\Phi^2 = \Phi$) *which maps $L^p(G)$ onto $H^p(G)$.*

Since $2\Phi u = \hat{u}(0) + \Psi u$, it is enough to prove one of the inequalities (1). In fact, $|B_p - 2A_p| \leqq 1$.

We postpone the proof of the theorem to Section 8.7.4.

8.7.3. Let $C_A(G)$ be the uniform closure of the set of all trigonometric polynomials on G which are of analytic type, i.e., which have the form

$$(1) \qquad f(x) = \sum_{\gamma \geqq 0} a(\gamma)(x, \gamma).$$

It is clear that $C_A(G)$ is a Banach algebra, with respect to pointwise multiplication.

For our present purpose, the following fact is not needed, but it is of interest for its own sake: *$C_A(G)$ consists of all $f \in C(G)$ which are of analytic type.* For if $\mu \in M(G)$ and if $\int f(-x)d\mu(x) = 0$ for all $f \in C_A(G)$, then $\hat{\mu}(\gamma) = 0$ for all $\gamma \geqq 0$; hence, if $g \in C(G)$ and $\hat{g}(\gamma) = 0$ for all $\gamma < 0$, we have $\hat{g}\hat{\mu} = 0$ and so $g * \mu = 0$; by the Hahn-Banach theorem, $g \in C_A(G)$.

Helson [8] observed that some simple facts about $C_A(G)$ extend the classical proof of the M. Riesz theorem to our present context:

If $f = f_1 f_2$, where f_1 and f_2 are of the form (1), then clearly $\hat{f}(0) = \hat{f}_1(0)\hat{f}_2(0)$. It follows that *the map h_0 defined by*

$$(2) \qquad h_0(f) = \hat{f}(0)$$

is a complex homomorphism of $C_A(G)$.

We next claim that *if* $f \in C_A(G)$ *and if* $\operatorname{Re} f(x) > 0$ *for all* $x \in G$, *then* $\operatorname{Re} h(f) > 0$ *for every homomorphism* h *of* $C_A(G)$ *onto the complex field.*

Since h has norm 1 as a linear functional (Appendix D4), h can be extended to a linear functional on $C(G)$ with the same norm, and therefore there exists $\mu \in M(G)$ with $\|\mu\| = 1$, such that $h(f) = \int_G f\,d\mu$ for all $f \in C_A(G)$. But $h(1) = 1$, and so $\int_G d\mu = \|\mu\|$. This implies that $\mu \geqq 0$. Hence

$$\operatorname{Re} h(f) = \operatorname{Re}\int_G f d\mu = \int_G (\operatorname{Re} f) d\mu > 0.$$

Finally, *suppose* $f \in C_A(G)$, $\operatorname{Re} f(x) > 0$ *for all* $x \in G$, p *is a real number, and* $g(x) = [f(x)]^p$ (we take that branch of the pth power which is positive on the positive real axis). *Then* $g \in C_A(G)$ *and*

(3) $$\hat{g}(0) = [\hat{f}(0)]^p.$$

Indeed, our preceding assertion shows that the function $\alpha(z) = z^p$ is analytic on the spectrum of f (Appendix D6). Each $x \in G$ determines a homomorphism $f \to f(x)$ of $C_A(G)$; this shows that $C_A(G)$ is semi-simple. Since $g = \alpha(f)$, it follows that $g \in C_A(G)$ (Appendix D7) and that $h(g) = \alpha(h(f))$ for every complex homomorphism h of $C_A(G)$. Taking $h = h_0$, as defined by (2), we obtain (3).

8.7.4. We turn to the proof of Theorem 8.7.2. Assume first that $1 < p \leqq 2$. Fix p in this range, choose $\delta = \delta_p$ such that $0 < \delta < \pi/2 < p\delta$, and put

(1) $$\alpha = \alpha_p = (\cos p\delta)^{-1}, \qquad \beta = \beta_p = (\cos \delta)^{-p}(1 + |\alpha|).$$

Then $\alpha < 0$, and we claim that

(2) $$1 \leqq \alpha \cos p\theta + \beta\,(\cos\theta)^p \qquad (|\theta| \leqq \pi/2).$$

Indeed, if $\delta \leqq |\theta| \leqq \pi/2$, the right side of (2) is not less than $\alpha \cos p\theta > \alpha \cos p\delta = 1$, and if $|\theta| \leqq \delta$, it exceeds $\beta\,(\cos\delta)^p - |\alpha| = 1$. The idea of using an equality of this type is due to Calderon [1].

Suppose now that u is a positive trigonometric polynomial on G, and put $w = \Psi u$. Then $u = |w| \cos \theta$, where $|\theta| < \pi/2$, and (2) implies that

$$\int |w|^p \leqq \alpha \int |w|^p \cos p\theta + \beta \int |w|^p (\cos \theta)^p = \alpha \operatorname{Re} \int w^p + \beta \int u^p. \tag{3}$$

Since $\hat{w}(0) = \hat{u}(0)$, the final assertion in 8.7.3 shows that

$$\int w^p = \left(\int w\right)^p = \left(\int u\right)^p > 0, \tag{4}$$

and since $\alpha < 0$, (3) and (4) imply $\int |w|^p \leqq \beta \int u^p$. In other words, we have proved that

$$||\Psi u||_p \leqq \beta^{1/p} ||u||_p \tag{5}$$

and hence that

$$||\Phi u||_p \leqq \tfrac{1}{2}(1 + \beta^{1/p}) ||u||_p \tag{6}$$

for all positive trigonometric polynomials u on G.

If $u \geqq 0$ and $u \in L^p(G)$, some sequence $\{u_n\}$ of positive trigonometric polynomials converges to u in the norm of $L^p(G)$. By (6), the functions Φu_n form a bounded set in $L^p(G)$, and a subsequence of them converges weakly to a function $w \in L^p(G)$. The weak convergence also implies that w is the analytic contraction of u, since the map $f \to \hat{f}(\gamma)$ is a continuous linear functional on $L^p(G)$, for every $\gamma \in \Gamma$. Hence (6) holds for every non-negative $u \in L^p(G)$.

If u is real and $u \in L^p(G)$, then $u = u_1 - u_2$, where $u_1 \geqq 0$, $u_2 \geqq 0$, and $u_1 u_2 = 0$. Put $w = \Phi u_1 - \Phi u_2$. Then w is the analytic contraction Φu of u, and since $|u_1| \leqq |u|$, $|u_2| \leqq |u|$, and (6) holds for u_1 and u_2, we obtain

$$||\Phi u||_p \leqq ||\Phi u_1||_p + ||\Phi u_2||_p \leqq (1 + \beta^{1/p}) ||u||_p. \tag{7}$$

Finally, (7) applies to the real and imaginary parts of every $u \in L^p(G)$, so that

$$||\Phi u||_p \leqq A_p ||u||_p \qquad (u \in L^p(G)), \tag{8}$$

where A_p is twice as large as the constant in (7).

To complete the theorem, suppose $2 \leqq q < \infty$. If u and f are trigonometric polynomials on G, then $f * \Phi u = u * \Phi f$. Letting f range over all trigonometric polynomials with $||f||_p \leqq 1$, where $1/p + 1/q = 1$, we therefore obtain

$$||\Phi u||_q = \sup_f \left| \int f(x)(\Phi u)(-x)dx \right| = \sup_f \left| \int u(x)(\Phi f)(-x)dx \right|$$
$$\leqq ||u||_q \cdot ||\Phi f||_p \leqq A_p ||u_q||.$$

Hence the theorem holds for q, and $A_q \leqq A_p$.

If we take the smallest admissible values for A_p and A_q, the last computation can be reversed, and shows that $A_q = A_p$.

8.7.5. Theorem 8.7.2 becomes false if $p = 1$ (hence also if $p = \infty$, by the last computation in 8.7.4) for every non-trivial compact connected G. For if it were true, and if $\chi(\gamma) = 1$ for $\gamma \geqq 0$, $\chi(\gamma) = 0$ for $\gamma < 0$, then χ would be in $B(\Gamma)$, by Theorem 3.8.1. If Z denotes any infinite cyclic subgroup of Γ, it follows that the characteristic function of the set E of all non-negative integers belongs to $B(Z)$. But this is false, since E is not a member of the coset ring of Z. (More direct proofs are also available.)

However, a weaker form of Theorem 8.7.2 still holds:

8.7.6. Theorem. *Suppose $0 < p < 1$. There exist constants A_p and B_p (they do not depend on G) such that the inequalities*

$$(1) \qquad ||\Phi u||_p \leqq A_p ||u||_1, \qquad ||\Psi u||_p \leqq B_p ||u||_1$$

hold for every trigonometric polynomial u on G.

Proof, (Helson [8]): We again assume first that $u > 0$. Putting $w = \Psi u$, we have $u = |w| \cos \theta$, with $|\theta| < \pi/2$. By 8.7.3,

$$||u||_1^p = \left(\int u\right)^p = \left(\int w\right)^p = \int w^p = \int \mathrm{Re}\, w^p = \int |w|^p \cos p\theta,$$

so that

$$(2) \qquad ||\Psi u||_p \leqq \left(\cos \frac{p\pi}{2}\right)^{-1/p} ||u||_1.$$

If now u is a real trigonometric polynomial on G, then $u = u' - u''$ where u', u'' are non-negative continuous functions on G such that $u'u'' = 0$, and there are positive trigonometric polynomials u_n',

u_n'' such that $u_n' \to u'$ and $u_n'' \to u''$ uniformly. The inequality

$$\int |f+g|^p \leqq \int |f|^p + \int |g|^p \qquad (p < 1) \tag{3}$$

which, for $p < 1$, takes the place of the triangle in equality, gives, setting $u_n = u_n' - u_n''$,

$$\int |\Psi u_n|^p \leqq \left(\cos \frac{p\pi}{2}\right)^{-1} \{||u_n'||_1^p + ||u_n''||_1^p\}, \tag{4}$$

since $\Psi u_n = \Psi u_n' - \Psi u_n''$. As $n \to \infty$, $||u_n'||_1 \to ||u'||_1 \leqq ||u||_1$, and similarly for u_n''. Since Ψ is a bounded operator on $L^2(G)$, $\Psi u_n \to \Psi u$ in the norm of $L^2(G)$, hence also in $L^p(G)$ for any $p < 2$. This gives

$$||\Psi u||_p \leqq K_p ||u||_1 \tag{5}$$

for every real trigonometric polynomial u on G, and if we apply (5) to the real and imaginary parts of any trigonometric polynomial, we obtain the second inequality asserted by the theorem. Since

$$2|\Phi u| = |\Psi u + \hat{u}(0)| \leqq |\Psi u| + ||u||_1,$$

the first inequality in (1) also holds, by (3).

8.7.7. The inequality

$$\int |v| \leqq A + B \int |u| \log^+ |u|,$$

where A and B are absolute constants, can also be proved by the preceding methods in the present context. The main point to consider is that if $w = u + iv$ and if $u > 0$, then $z \log z$ is analytic on the spectrum of w, and hence

$$\int w \log w = h_0(w) \log h_0(w) = \hat{w}(0) \log \hat{w}(0) = \hat{u}(0) \log \hat{u}(0).$$

The rest of the proof is as in Zygmund [1], vol. I, p. 254; see also Helson [9].

8.7.8. Theorem 8.7.6 leads to a simple proof of a theorem which, for the case $G = T$, was proved by Paley [1]. Helson [8] extended it to the case $G = T^n$, by a somewhat different method. We refer to 8.7.3 for the definition of $C_A(G)$. As before, G is an arbitrary compact connected abelian group, and $C_A(G)$ is defined with respect to some given ordering of Γ.

THEOREM. *Suppose $\omega(\gamma) \geqq 0$ for all $\gamma \geqq 0$, and suppose that*

(1) $$\sum_{\gamma \geq 0} |\hat{f}(\gamma)|\omega(\gamma) < \infty$$

for all $f \in C_A(G)$. Then

(2) $$\sum_{\gamma \geq 0} \omega^2(\gamma) < \infty.$$

The intuitive content of the theorem is that one cannot say anything stronger about the order of magnitude of the Fourier coefficients $\hat{f}(\gamma)$ of a function in $C_A(G)$ than that $\sum |\hat{f}(\gamma)|^2 < \infty$.

Proof: For $i = 1, 2, 3, \ldots$, let X_i be the set of all $f \in C_A(G)$ for which the left side of (1) does not exceed i. Since the map $f \to |\hat{f}(\gamma)|$ is a continuous function on the Banach space $C_A(G)$, for each $\gamma \in \Gamma$, the sets X_i are closed. Baire's theorem implies that one of the sets X_i has non-empty interior, and it follows from the linearity of the map $f \to \hat{f}\omega$ that there is a constant K (depending on the function ω) such that

(3) $$\sum_{\gamma \geq 0} |\hat{f}(\gamma)|\omega(\gamma) \leqq K\|f\|_\infty \qquad (f \in C_A(G)).$$

Fix non-negative elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, let $r_1, \ldots, r_n$ be the first n Rademacher functions, put

(4) $$g_t(x) = \sum_{i=1}^{n} \omega(\gamma_i) r_i(t)(x, \gamma_i) \qquad (x \in G,\ 0 < t < 1).$$

and let P be a trigonometric polynomial on G, with $\|P\|_1 < 2$ and $\hat{P}(\gamma_i) = 1$ for $1 \leqq i \leqq n$.

By (3), the maps $f \to \sum \hat{f}(\gamma_i)\omega(\gamma_i)r_i(t)$ are bounded linear functionals on $C_A(G)$ whose norm does not exceed K, for all t. Hence there are measures $\mu_t \in M(G)$, with $\|\mu_t\| \leqq K$, such that

(5) $$\sum_{i=1}^{n} \hat{f}(\gamma_i)\omega(\gamma_i)r_i(t) = \int_G f(-x)d\mu_t(x) \qquad (f \in C_A(G),\ 0 < t < 1).$$

Taking a character for f, (5) shows that $\hat{\mu}_t(\gamma_i) = \omega(\gamma_i)r_i(t)$ $(1 \leqq i \leqq n)$, and that $\hat{\mu}_t(\gamma) = 0$ for all other $\gamma \geqq 0$. Hence g_t is the analytic contraction of $P * \mu_t$, and by Theorem 8.7.6 there is an absolute constant B such that

$$(6) \quad \left\{\int_G |g_t(x)|^{1/2} dx\right\}^2 = ||g_t||_{1/2} \leqq B||P * \mu_t||_1 \leqq 2BK \qquad (0 < t < 1).$$

We take square-roots in (6), integrate the resulting inequality with respect to t, interchange the order of the two integrations, and conclude that

$$(7) \qquad \int_0^1 |g_t(x_0)|^{1/2} dt \leqq (2BK)^{1/2}$$

for some $x_0 \in G$. Writing $h(t) = g_t(x_0)$, Hölder's inequality, combined with inequality (6) of 5.7.7, gives

$$(8) \qquad ||h||_1^3 \leqq ||h||_{1/2} \cdot ||h||_2^2 \leqq 2BK \cdot 4||h||_1^2,$$

and so

$$(9) \qquad ||h||_2 \leqq 2||h||_1 \leqq 16BK.$$

By Parseval's formula, (9) implies that

$$\sum_{i=1}^n \omega^2(\gamma_i) = \sum_{i=1}^n |\omega(\gamma_i)(x_0, \gamma_i)|^2 = ||h||_2^2 \leqq 256B^2K^2,$$

and (2) follows, since $\gamma_1, \ldots, \gamma_n$ were arbitrary.

Corollary. *If $\gamma_i \geqq 0$ $(i = 1, 2, 3, \ldots)$ and if $\varepsilon > 0$, there exists $f \in C_A(G)$ such that $\sum |\hat{f}(\gamma_i)|^{2-\epsilon} = \infty$.*

8.7.9. In the preceding theorem, the support of $\hat{f}$ was assumed to be in the positive half of Γ. If the support of $\hat{f}$ is more severely restricted, the theorem and its corollary may become false. The following interesting example is due to Bohr [1] (p. 468).

Consider the infinite-dimensional torus T^ω and its dual Z^∞ (see Section 2.2.5). The elements n of Z^∞ are of the form $n = (n_1, n_2, n_3, \ldots)$ where the n_i are integers, and only finitely many n_i are different from 0 for any n. Let Y be the set of all $n \in Z^\infty$

with $n_i \geqq 0$ for $i = 1, 2, 3, \ldots$; Y is an analogue of the first quadrant in the set of all lattice points in the plane.

Let E be the set of all $n \in Y$ with $\sum n_i = 1$. That is to say, E consists of all $n \in Z^\infty$ with one coordinate equal to 1 and all other coordinates equal to 0.

THEOREM. *If $f \in L^\infty(T^\omega)$ and if $\hat{f}(n) = 0$ for all n not in Y, then*

$$\sum_{n \in E} |\hat{f}(n)| \leqq ||f||_\infty. \tag{1}$$

Proof: Let Λ be the subgroup of Z^∞ consisting of all n with $\sum n_i = 0$. Then $E = Y \cap \Lambda_1$, where Λ_1 is a coset of Λ. There is a measure $\mu \in M(T^\omega)$ such that $\hat{\mu}$ is the characteristic function of Λ_1; clearly $||\mu|| = 1$ (see Section 3.1.2). If f satisfies the hypotheses of theorem and if $g = f * \mu$, it follows that g is an E-function, in the terminology of 5.7.1, and $||g||_\infty \leqq ||f||_\infty$.

Every E-polynomial is of the form $P(x) = \sum c_j e^{in_j x_j}$, and the supremum of this, as x ranges over T^ω, is $\sum |c_j|$. Thus E is a Sidon set in Z^∞, with constant 1. This implies that $\sum |\hat{g}(n)| \leqq ||g||_\infty$. Since $\hat{\mu}(n) = 1$ on E, (1) follows.

Bohr's theorem was stated for Dirichlet series: *If*

$$\phi(s) = \sum_{k=1}^{\infty} c_k/k^s \tag{2}$$

and if $|\phi(s)| \leqq 1$ for all s whose real part is positive, then

$$\sum_p |c_p| \leqq 1, \tag{3}$$

the last sum being extended over all primes.

The connecting link between these two statements is Bohr's observation that every Dirichlet series (2) can be regarded as a trigonometric series on T^ω whose coefficients vanish outside Y. For if $p_1, p_2, p_3, \ldots$ is the sequence of the primes, then each positive integer k has a unique factorization

$$k = \prod_{j=1}^{\infty} p_j^{n_j}, \tag{4}$$

and if we replace each k by the corresponding sequence $\{n_j\}$ of exponents, the series (2) takes the form

$$\sum_{n \in Y} c(n_1, n_2, \ldots) \exp\left\{- s \sum_{j=1}^{\infty} n_j \log p_j\right\}. \tag{5}$$

Writing $z_j = p_j^{-s}$, (5) becomes a power series in infinitely many variables, namely

$$\sum_{n \in Y} c(n_1, n_2, \ldots) z_1^{n_1} z_2^{n_2} \ldots, \tag{6}$$

and if $|z_j| = 1$, i.e. if s is pure imaginary, (6) is a trigonometric series on T^ω.

8.7.10. A closed subset S of the euclidean space R^k will be called a *half-space in* R^k if the boundary of S is a $(k-1)$-dimensional hyperplane Π. The intersection of S with the set Z^k of all lattice points in R^k will be called a *half-space in* Z^k. If Π contains 0 but no other point of Z^k, then S defines an (archimedean) order in Z^k, and the corresponding analytic contraction is a linear operator Φ_S on $L^p(T^k)$, for $1 < p < \infty$, whose norm does not exceed the constant A_p of Theorem 8.7.2.

Explicitly, if χ_S is the characteristic function of S and if $f \in L^p(T^k)$, then $\chi_S \hat{f}$ is the Fourier transform of a function $\Phi_S f$ on T^k, and the inequality

$$\|\Phi_S f\|_p \leqq A_p \|f\|_p \tag{1}$$

holds. The same inequality holds if S is replaced by the half-space $S + n$, for any $n \in Z^k$.

Suppose now that $S_1, \ldots, S_N$ are half-spaces in Z^k, that $E = S_1 \cap \ldots \cap S_N$, and that E is finite. Since E is finite, the boundaries Π_i of the half-spaces S_i can be so moved, if necessary, that (1) holds for each of the sets S_i, and so that E is not affected. Then if $\Phi_E f$ is the trigonometric polynomial whose Fourier transform is the product of $\hat{f}$ with the characteristic function of E, we have

$$\|\Phi_E f\|_p \leqq A_p^N \|f\|_p \qquad (f \in L^p(G),\ 1 < p < \infty). \tag{2}$$

This enables us to prove a theorem about the convergence of the partial sums $\Phi_E f$ of the Fourier series of f:

THEOREM. *Let N be an integer. Suppose $E_1, E_2, E_3, \ldots$ is a sequence of finite subsets of Z^k such that each E_j is the intersection of N half-spaces in Z^k, and such that each $n \in Z^k$ lies in all but finitely many of the sets E_j. If $1 < p < \infty$ and if $f \in L^p(T^k)$, then*

$$\lim_{j\to\infty} \|f - \Phi_{E_j} f\|_p = 0. \tag{3}$$

Proof: Given $\varepsilon > 0$, there is a trigonometric polynomial g on T^k such that $\|f - g\|_p < \varepsilon$. For all large enough j, we have $\Phi_{E_j} g = g$, and (2) implies therefore that

$$\|f - \Phi_{E_j} f\|_p \leqq \|f - g\|_p + \|\Phi_{E_j}(g - f)\|_p < (1 + A_p^N)\varepsilon.$$

The theorem follows.

8.7.11. We conclude this chapter with an extension of Theorem 8.7.2 to Fourier transforms on R^k.

Suppose $1 < p \leqq 2$. If $f \in C_c(R^k)$, then $\|\hat{f}\|_\infty \leqq \|f\|_1$ and $\|\hat{f}\|_2 = \|f\|_2$. The convexity theorem of M. Riesz and Thorin (see Zygmund [1], vol. II, pp. 95, 254) therefore shows that

$$\|\hat{f}\|_q \leqq \|f\|_p \tag{1}$$

where $1/p + 1/q = 1$. Since $C_c(R^k)$ is dense in $L^p(R^k)$, (1) allows us to extend the Fourier transform to a linear map of $L^p(R^k)$ into $L^q(R^k)$, with preservation of (1).

THEOREM. *Suppose $1 < p \leqq 2$. Let χ be the characteristic function of a half-space in R^k. If $f \in L^p(R^k)$, then $\chi\hat{f}$ is the Fourier-transform of a function Φf such that*

$$\|\Phi f\|_p \leqq A_p \|f\|_p, \tag{2}$$

where A_p is as in Theorem 8.7.2.

Proof: Since $L^p(R^k)$ is invariant under rigid motions of R^k, so is the set of its Fourier transforms, and we may therefore assume, without loss of generality, that the boundary Π of our half-space contains 0 but no other point of Z^k.

We introduce an auxiliary function

$$(3)\qquad \hat{\jmath}(y) = \prod_{i=1}^{k} \max\,(1 - |y_i|, 0) \qquad (y = (y_1, \ldots, y_k)).$$

Each factor in (3) is positive-definite. Hence $\hat{\jmath}$ is positive-definite, and since $\hat{\jmath}$ has compact support, $\hat{\jmath}$ is the Fourier-transform of a non-negative continuous function $j \in L^1(R^k)$. Identify T^k with the cube in R^k defined by the inequalities $-\pi \leqq x_i < \pi$ $(1 \leqq i \leqq k)$. The Fourier coefficients of the periodic function

$$J(x) = \sum_{m \in Z^k} j(x + 2\pi m)$$

are, for $n \in Z^k$,

$$\left(\frac{1}{2\pi}\right)^k \int_{T_k} J(x) e^{-in\cdot x}\, dx = \left(\frac{1}{2\pi}\right)^k \int_{R^k} j(x) e^{-in\cdot x}\, dx = \hat{\jmath}(n) = \begin{cases} 1 \text{ if } n = 0, \\ 0 \text{ if } n \neq 0. \end{cases}$$

Thus

$$(4)\qquad \sum_{m \in Z^k} j(x + 2\pi m) = 1 \qquad (x \in R^k).$$

By (3), $\int \hat{\jmath} = 1$, so that $j(0) = 1$, and hence (4) shows that $j(2\pi m) = 0$ if $m \in Z^k$ and $m \neq 0$. A computation quite analogous to the preceding one therefore yields

$$(5)\qquad \sum_{n \in Z^k} \hat{\jmath}(y - n) = 1 \qquad (y \in R^k).$$

Now let f be an infinitely differentiable function on R^k with compact support. Define $(U_r f)(x) = r^k f(rx)$, $r = 1, 2, 3, \ldots$. Take r so large that the support of $U_r f$ lies in the above-mentioned cube. Then $U_r f$ may be regarded as a function on T^k, and Theorem 8.7.2 shows that there exists $g \in L^p(T^k)$ (depending on r) such that

$$(6)\qquad \hat{g}(n) = \hat{f}(n/r)\chi(n) = (\chi \hat{f})(n/r) \qquad (n \in Z^k)$$

and

$$(7)\qquad \|g\|_p \leqq A_p \|U_r f\|_p = A_p r^{k/q} \|f\|_p \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Regard g as a function on R^k, with period 2π in each of the variables $x_1, \ldots, x_k$, and define

$$h_r(x) = r^{-k} g(x/r) j(x/r) \qquad (x \in R^k). \tag{8}$$

By (4), $j \leqq 1$, and

$$\int_{R^k} |h_r(x)|^p \, dx = r^{k(1-p)} \int_{R^k} |g(x)j(x)|^p \, dx \leqq r^{k(1-p)} \int_{R^k} |g(x)|^p j(x) dx$$
$$= r^{k(1-p)} \int_{T^k} |g(x)|^p J(x) dx = r^{k(1-p)} \int_{T^k} |g(x)|^p \, dx.$$

Thus (7) implies

$$||h_r||_p \leqq r^{-k/q} ||g||_p \leqq A_p ||f||_p. \tag{9}$$

By (6), our choice of f shows that the series

$$g(x) = \sum_{n \in Z} \hat{g}(n) e^{in \cdot x}$$

converges absolutely; inserting it into

$$\hat{h}_r(y) = (2\pi)^{-k} \int_{R^k} g(x) j(x) e^{-irx \cdot y} \, dx \qquad (y \in R^k)$$

we obtain

$$\hat{h}_r(y) = \sum_{n \in Z^k} (\chi \hat{f})(n/r) \hat{j}(ry - n) \qquad (y \in R^k). \tag{10}$$

In particular, $\hat{h}_r(n/r) = (\chi \hat{f})(n/r)$ for all $n \in Z^k$.

By (5) and (10), $\hat{h}_r(y)$ is a convex combination of the values of $\chi \hat{f}$ at the vertices of a cube of edge $1/r$ which contains y. Since $\chi \hat{f}$ is continuous, except possibly on the hyperplane Π which bounds our half-space, we see that

$$\lim_{r \to \infty} \hat{h}_r(y) = (\chi \hat{f})(y) \tag{11}$$

uniformly on every compact subset of R^k which does not intersect Π.

By (9), $\{h_r\}$ has a subsequence $\{h_{r_i}\}$ which converges weakly in $L^p(R^k)$ to a function h which also satisfies (9). For any $w \in L^p(R^k)$, (1) implies that $\int h_{r_i} \hat{w} \to \int h \hat{w}$. Since $\int h \hat{w} = \int \hat{h} w$, it follows that

$$\lim_{i \to \infty} \int_{R^k} \hat{h}_{r_i}(y) w(y) dy = \int_{R^k} \hat{h}(y) w(y) dy \qquad (w \in L^p(R^k)). \tag{12}$$

But (11), (9), and (1) imply that $\int \hat{h}_{r_i} w \to \int \chi \hat{f} w$ for all $w \in L^p(R^k)$, and comparison with (12) gives: $\hat{h} = \chi \hat{f}$.

The inequality (2) is thus established for all infinitely differentiable f with compact support. The set of these f is dense in $L^p(R^k)$, and the theorem follows.

CHAPTER 9

Closed Subalgebras of $L^1(G)$

For non-compact LCA groups G, the ideal structure of $L^1(G)$ is so complicated (see Chapter 7) that the larger problem of classifying or describing all closed subalgebras of $L^1(G)$ seems far beyond our reach. Even for compact G, where all closed ideals in $L^1(G)$ are explicitly known (Theorem 7.1.5), our present information about closed subalgebras is very meager; it is contained in Section 9.1. Two types of maximal subalgebras are described in Section 9.2, and Section 9.3 deals with a problem suggested by the Stone-Weierstrass theorem.

9.1. Compact Groups

Suppose G is compact and A is a closed subalgebra of $L^1(G)$. Write $\gamma_1 \sim \gamma_2$ if and only if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in A$. The relation $\sim$ is an equivalence relation R_A in Γ, induced by A. One distinguished equivalence class is the set E_0 which consists of all $\gamma \in \Gamma$ at which $\hat{f}(\gamma) = 0$ for all $f \in A$; E_0 may be infinite. The other equivalence classes, denoted by E_α, where α runs through a suitable index set, must be finite, since $\hat{f} \in C_0(\Gamma)$ for each $f \in A$ and since Γ is discrete.

The question arises whether the closed subalgebras of $L^1(G)$ are characterized by the equivalence relations which they induce in Γ. The answer is unknown even for $G = T$. To obtain a counter example, one has to construct two distinct closed subalgebras of $L^1(G)$ which induce the same equivalence relation in Γ.

We can prove the following, however: If A is as above, there exists a closed subalgebra A_0 of $L^1(G)$ such that $R_{A_0} = R_A$ and such that $A_0 \subset B$ for every closed subalgebra B of $L^1(G)$ for which $R_B = R_A$. This minimal algebra A_0 is the one which is generated by the trigonometric polynomials P_α whose transforms $\hat{P}_\alpha$ are the characteristic functions of the sets E_α $(\alpha \neq 0)$:

THEOREM. *Each of the trigonometric polynomials P_α belongs to A.*

Our question can therefore be rephrased as an approximation problem: If $f \in L^1(G)$ and if $\hat{f}$ is constant on certain sets E_α, can f be approximated in the norm of $L^1(G)$ by trigonometric polynomials P such that $\hat{P}$ is constant on each of the sets E_α?

Proof: Fix α. By definition, there exists $f \in A$ with $\hat{f}(E_\alpha) \neq 0$. Let $\alpha_1, \ldots, \alpha_n$ be the other indices for which $\hat{f}(E_{\alpha_i}) = \hat{f}(E_\alpha)$. There can be only finitely many of these. Since $\hat{f}(E_\alpha)$ is an isolated point of $\hat{f}(\Gamma)$ and since $\hat{f}(\Gamma)$ has no limit point except possibly 0, there is a polynomial ϕ such that $\phi(0) = 0$, $\phi(\hat{f}(E_\alpha)) = 1$, and $|\phi| < \frac{1}{2}$ on the rest of $\hat{f}(\Gamma)$. If $\hat{g} = \phi(\hat{f})$, then $g \in A$, and if $\hat{P}$ is the characteristic function of $E_\alpha \cup E_{\alpha_1} \cup \ldots \cup E_{\alpha_n}$, then $\|\hat{g} - \hat{P}\|_\infty < \frac{1}{2}$. Since $(\hat{g} - \hat{P})^n = \hat{g}^n - \hat{P}$, the spectral radius formula implies that $\|\hat{g}^n - \hat{P}\| < 2^{-n}$ for all large enough n. Hence $P \in A$.

There exist functions $h_i \in A$ with $\hat{h}_i = 1$ on E_α and $\hat{h}_i = 0$ on E_{α_i} $(1 \leqq i \leqq n)$, and the preceding construction yields trigonometric polynomials $P_i \in A$ such that $\hat{P}_i(E_\alpha) = 1$, $\hat{P}_i(E_{\alpha_i}) = 0$. Since $\hat{P}_\alpha = \hat{P} \cdot \hat{P}_1 \cdot \ldots \cdot \hat{P}_n$, $P_\alpha \in A$, and the theorem is proved.

COROLLARY. *If G is compact and if A is a closed subalgebra of $L^1(G)$ such that the Fourier transforms of members of A separate points on Γ, then either $A = L^1(G)$ or A is a maximal ideal in $L^1(G)$.*

Proof: Each equivalence class now consists of exactly one point. If E_0 is empty, the theorem shows that A contains every character on G, hence every trigonometric polynomial on G, and so $A = L^1(G)$. If $E_0 = \{\gamma_0\}$. then A consists of all $f \in L^1(G)$ for which $\hat{f}(\gamma_0) = 0$.

9.2. Maximal Subalgebras

9.2.1. Suppose B is a closed subalgebra of a Banach algebra A, $B \neq A$, and the inclusions $B \subset B_1 \subset A$ (where B_1 is a closed subalgebra of A) imply that either $B_1 = B$ or $B_1 = A$. Under these conditions B is called *a maximal subalgebra* of A.

For any Borel set S in a LCA group G let $L^1(S)$ be the set of all $f \in L^1(G)$ which vanish (almost everywhere) on the complement S' of S. Thus $L^1(S)$ consists of those $f \in L^1(G)$ for which $\int_{S'} |f| = 0$.

It follows that $L^1(S)$ is a closed linear subspace of $L^1(G)$. If S is a semi-group and if $f, g \in L^1(S)$, then $(f * g)(x) = 0$ unless $x \in S + S \subset S$; hence $f * g \in L^1(S)$.

Thus $L^1(S)$ is a closed subalgebra of $L^1(S)$ if S is a semi-group in G.

Wermer [2], [4] has shown that $L^1(S)$ is a maximal subalgebra of $L^1(G)$ if G has an archimedean order and if S is the set of all non-negative elements of G (Theorems 9.2.2, 9.2.3) and Simon [1], [2] showed that these are essentially the only two situations in which $L^1(S)$ is maximal (Theorem 9.2.5).

9.2.2. THEOREM. *Suppose G is a discrete subgroup of R, suppose that A is a semi-simple commutative Banach algebra whose maximal ideal space is Γ (so that A is an algebra of functions on Γ), and suppose that the trigonometric polynomials on Γ are dense in A. If A^+ is the set of all $\phi \in A$ such that*

$$\int_\Gamma \phi(\gamma)(x, \gamma)d\gamma = 0 \tag{1}$$

for all positive $x \in G$, then A^+ is a maximal subalgebra of A.

Special cases of this are of interest. Taking $A = C(\Gamma)$, we see that the algebra $C_A(\Gamma)$ (Section 8.7.3) is maximal in $C(\Gamma)$ (Wermer [1], Hoffman and Singer [1], [2]; the latter paper contains an account of our present knowledge of maximal subalgebras of $C(X)$). Taking $A = A(\Gamma)$, we see that $L^1(G^+)$ is a maximal subalgebra of $L^1(G)$, where G^+ is the set of all non-negative elements of $G \subset R$ (Wermer [2]).

Proof: The letters s, t, u will stand for elements of G (i.e., for real numbers) and it will be convenient to write the continuous characters on Γ in the form χ_s; i.e., $\chi_s(\gamma) = (s, \gamma)$, for $s \in G$ and $\gamma \in \Gamma$.

Suppose $A^+ \subset B \subset A$, $B \neq A$, and B is a closed subalgebra of A. Since $B \supset A^+$, $\chi_t \in B$ for all $t \geqq 0$. Since $B \neq A$, there exists $s > 0$ such that $\chi_{-s} \notin B$. Thus χ_s has no inverse in B, and it follows (Appendix D4(c)) that $h(\chi_s) = 0$ for some homomorphism h of B onto the complex field. From now on, h will be so fixed.

If $t > 0$, there is a positive integer n such that $nt > s$, and if $u = nt - s$, $\chi_u \in B$ and

$$h(\chi_t)^n = h(\chi_{nt}) = h(\chi_s \cdot \chi_u) = h(\chi_s)h(\chi_u) = 0.$$

Hence

(2) $$h(\chi_t) = 0 \qquad (t > 0).$$

The norm of any $\phi \in B$ is the same whether we regard ϕ as an element of B or as an element of A. The spectral radius formula therefore shows that

(3) $$|h(\phi)| \leqq \lim_{n\to\infty} \|\phi^n\|^{1/n} = \sup_{\gamma \in \Gamma} |\phi(\gamma)| \qquad (\phi \in B),$$

and so $h(\phi) = \int \phi d\mu$, where $\mu \in M(G)$ and $\|\mu\| = 1$. By (2),

(4) $$\int_\Gamma (t, \gamma) d\mu(\gamma) = 0 \qquad (t > 0).$$

Since $\|\mu\| \leqq 1 = h(1) = \int d\mu$, μ must be non-negative, and so (4) also holds for all $t < 0$. Hence μ is the Haar measure of Γ, and

(5) $$h(\phi) = \int_\Gamma \phi(\gamma) d\gamma \qquad (\phi \in B).$$

If $\phi \in B$ and $t > 0$, then $\phi \cdot \chi_t \in B$, and

(6) $$\int_\Gamma \phi(\gamma)(t, \gamma) d\gamma = h(\phi \cdot \chi_t) = h(\phi)h(\chi_t) = 0$$

by (5) and (2). Since (6) holds for all $t > 0$, $\phi \in A^+$, and so $B = A^+$. This completes the proof.

9.2.3. Theorem. *Let R^+ be the set of all non-negative real numbers. Then $L^1(R^+)$ is a maximal subalgebra of $L^1(R)$.*

Proof: Wermer [4] showed that this can be reduced to Theorem 9.2.2. We shall given an independent proof.

Define $\alpha(x) = 2e^{-x}$ for $x \geqq 0$, $\alpha(x) = 0$ for $x < 0$, and put $\beta(x) = \alpha(-x)$. Then $\hat{\alpha}(y) = 2(1 + iy)^{-1}$, $\hat{\beta}(y) = 2(1 - iy)^{-1}$, and so

(1) $$\alpha + \beta = \alpha * \beta.$$

The derivatives of $\hat{\alpha}$ are constant multiples of powers of $\hat{\alpha}$. Hence, writing $\alpha^1 = \alpha$ and $\alpha^n = \alpha^{n-1} * \alpha$, we have

$$\alpha^n(x) = c_n x^{n-1}\,\alpha(x) \qquad (n = 1, 2, 3, \ldots), \tag{2}$$

the constants c_n being different from 0.

Suppose $\phi \in L^\infty(R)$, $\phi(x) = 0$ for $x < 0$, and $\int \alpha^n\phi = 0$ for $n = 1, 2, 3, \ldots$. The function

$$F(w) = \int_0^\infty e^{-xw}\,\phi(x)dx \tag{3}$$

is then analytic in the right half-plane, and since

$$\begin{aligned} F^{(n)}(1) &= (-1)^n\int_0^\infty x^n e^{-x}\phi(x)dx \\ &= \frac{(-1)^n}{2c_n}\int_0^\infty \alpha^{n+1}(x)\phi(x)dx = 0 \qquad (n = 0, 1, 2, \ldots), \end{aligned} \tag{4}$$

F is identically 0. In particular, this is so for $F(1 + iy)$, the Fourier transform of $e^{-x}\phi(x)$. Hence $\phi = 0$, and we conclude:

The algebra generated by α is dense in $L^1(R^+)$.

It follows that the algebra generated by α and β is dense in $L^1(R)$.

Suppose now that B is a closed subalgebra of $L^1(R)$, $B \neq L^1(R)$, and $B \supset L^1(R^+)$. If the spectrum of α, regarded as an element of B, did not contain the point 1, then the function $(z - 1)^{-1}$ would be analytic on the spectrum of α, and equation (1) would imply that $\beta \in B$ (Appendix D7). Since the algebra generated by α and β is dense in $L^1(R)$, this contradicts the assumption that $B \neq L^1(R)$.

Hence there is a complex homomorphism h of B such that $h(\alpha) = 1$. The algebra generated by α is dense in $L^1(R^+)$, and so the action of h on $L^1(R^+)$ is determined by the value of $h(\alpha)$. Since $\hat{\alpha}(-i) = 1$, it follows that

$$h(f) = \hat{f}(-i) = \int_0^\infty e^{-x}f(x)dx \qquad (f \in L^1(R^+)). \tag{5}$$

On the other hand, $|h(f)| \leqq \lim \|f^n\|^{1/n} = \|\hat{f}\|_\infty$ for all $f \in B$, and so $h(f) = \int \hat{f}\,d\mu$, where $\mu \in M(R)$ and $\|\mu\| = 1$. Comparison with (5) shows that the Fourier-Stieltjes transform of μ coincides with e^{-x} on R^+. Since $\|\mu\| \leqq 1$ and $\hat{\mu}(0) = 1$, we see that $\mu \geqq 0$; and so $\hat{\mu}(x) = e^{-|x|}$. Hence

$$h(f) = \int_{-\infty}^{\infty} f(x) e^{-|x|}\, dx \qquad (f \in B). \tag{6}$$

The equation $h(f * g) = h(f)h(g)$ leads to the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(t)\left\{e^{-|x+t|} - e^{-|x|-|t|}\right\} dx\, dt = 0 \qquad (f, g \in B). \tag{7}$$

Suppose $f \in B$. Since (7) holds for all $g \in L^1(R^+)$, it follows that

$$\int_{-\infty}^{\infty} f(x)\left\{e^{-|x+t|} - e^{-|x|-t}\right\} dx = 0 \qquad (t > 0). \tag{8}$$

If we multiply (8) by e^t and consider the two possibilities for the sign of $|x + t|$, (8) becomes

$$e^{2t}\int_{-\infty}^{-t} f(x)e^{x}\, dx + \int_{-t}^{-\infty} f(x)e^{-x}\, dx = \int_{-\infty}^{\infty} f(x)e^{-|x|} dx \tag{9}$$

and if we differentiate (9) with respect to t we obtain

$$2e^{2t}\int_{-\infty}^{-t} f(x)e^{x}\, dx = 0 \qquad (t > 0) \tag{10}$$

Hence $f(x) = 0$ almost everywhere on $(-\infty, 0)$.

This implies that $B = L^1(R^+)$ and completes the proof.

9.2.4. The next lemma (due to Simon [2]) shows, for instance, why $L^1(S)$ is not a maximal subalgebra of $L^1(R^2)$ if S is the set of all (x, y) in R^2 with $x \geqq 0$, in spite of the fact that S is a maximal semi-group in R^2.

LEMMA. *If S is a Borel semi-group in G and if $L^1(S)$ is a maximal subalgebra of $L^1(G)$, then $S \cap (-S)$ contains at most one point, namely* 0.

Proof: To get a contradiction, suppose there exists $t \in S$, $t \neq 0$, such that $-t \in S$. Then if $x \in G$, the semi-group property of S shows that $x \in S$ if and only if $x + t \in S$. Since the complement S' of S has positive measure, there exist disjoint sets E and $E + t$ in S', of positive but finite measure. Define $g(x) = 1$ on E, $g(x) = -1$ on $E + t$, and $g(x) = 0$ at all other points of G, and let u be the characteristic function of S'. For any $y \in G$ we then have

$$\int_G g(x-y)u(x)dx = m(S' \cap (E+y)) - m(S' \cap (E+y+t)). \tag{1}$$

Since $x \in S$ if and only if $x + t \in S$, the two sets whose measures appear in (1) are translates of each other, and so the integral in (1) is 0 for all $y \in G$. It follows that $\int (f * g)(x)u(x)dx = 0$ for every $f \in L^1(G)$. Since u evidently annihilates $L^1(S)$, we see that u annihilates the algebra B generated by $L^1(S)$ and g, and so B is not dense in $L^1(G)$. But B is larger than $L^1(S)$, and this contradicts the maximality of $L^1(S)$.

9.2.5. THEOREM. (Simon [2]). *Suppose S is a Borel semi-group in G and $L^1(S)$ is a maximal subalgebra of $L^1(G)$. Then S is contained in a closed semi-group P of G which induces an archimedean order in G.*

Since $L^1(S)$ is maximal, it follows that $L^1(S) = L^1(P)$. Hence the structure theorem 8.1.6 shows that Theorems 9.2.2 and 9.2.3 describe the only situations (up to isomorphisms) in which $L^1(S)$ is maximal.

Proof: Suppose, without loss of generality, that $0 \in S$. (If not, add 0 to S.) Since $L^1(S)$ contains a non-zero element of $L^1(G)$, S contains a set E with $0 < m(E) < \infty$. If χ is the characteristic function of E, then $\chi * \chi \in C(G)$, $\chi * \chi = 0$ outside $S + S \subset S$, and $\int \chi * \chi = m(E)^2 > 0$. It follows that the interior of S is not empty.

The same is true of $-S$, and Lemma 9.2.4 implies that there is a non-empty open set V in $-S$ which does not intersect S.

Consider the family F of all semigroups of G which contain S and do not intersect V, partially order F by set inclusion and apply Zorn's lemma. We conclude that F contains a maximal element P; since V is open and since the closure of a semigroup is a semigroup, P is closed.

Since $L^1(S)$ is maximal, we must have $L^1(P) = L^1(S)$. Lemma 9.2.4 shows that $P \cap (-P) = \{0\}$. By Theorem 8.1.3, our proof will be complete as soon as we show that $P \cup (-P) = G$ and that P is a maximal semi-group in G.

Suppose there exists $x \in G$ such that neither x nor $-x$ are in P. Our choice of P shows that there exist positive integers n_i and elements $p_i \in P$ $(i = 1, 2)$ such that $n_1 x + p_1 = v_1 \in V$ and

$-n_2x+p_2=v_2 \in V$. Multiply the first of these equations by n_2, the second by n_1, and add. The result is

$$(1)\qquad n_2p_1+n_1p_2+n_2(-v_1)+n_1(-v_2)=0.$$

Each of these 4 summands is in P, and since $P \cap (-P)=\{0\}$, their sum can be 0 only if every summand is 0. Also, P contains no elements of finite order. Since v_1 (and v_2) are different from 0, we have a contradiction, and we have proved that $P \cup (-P)=G$.

Finally, fix t in the complement of P, and let Q be the semigroup generated by P and t; being a countable union of translates of P, Q is a Borel set. Since $P \cup (-P)=G$, $-t \in P \subset Q$. Thus Q contains both t and $-t$, and since $L^1(Q) \supset L^1(P)$, Lemma 9.2.4 implies that $L^1(Q)=L^1(G)$. Hence Q is dense in G, and since the interior of Q is not empty (see the first paragraph in this proof), it follows that $Q=G$. Hence P is a maximal semigroup in G, and the proof is complete.

9.2.6. Theorems 9.2.2 and 9.2.3 exhibit "natural" examples of maximal subalgebras of $L^1(G)$. We shall now describe a class of pathological examples.

Suppose G is not compact. Then Γ is not discrete, and so Γ contains a Cantor set E which is also a Helson set (Theorem 5.6.6). That is to say, there is a constant K such that to every $\phi \in C(E)$ there corresponds an $f \in L^1(G)$ such that $\hat{f}(\gamma)=\phi(\gamma)$ on E and such that $\|f\|_1 \leqq K\|\phi\|_\infty$.

It is known (Rudin [5]) that $C(E)$ has a maximal subalgebra A which contains the constants and which separates points on E.

Let B be the set of all $f \in L^1(G)$ such that $\hat{f}$ coincides on E with a member of A. If $f_n \to f$ in the norm of $L^1(G)$ then $\hat{f}_n \to \hat{f}$ uniformly, and since A is uniformly closed, it follows that B is a closed subalgebra of $L^1(G)$. Also, the transforms of the members of B separate points on Γ. *We shall prove that B is a maximal subalgebra of $L^1(G)$.*

Suppose $B_1 \supset B$, $B_1 \neq B$, and B_1 is a closed subalgebra of $L^1(G)$. Choose $f \in L^1(G)$ and $\varepsilon > 0$. The restrictions to E of the transforms of the members of B_1 are dense in $C(E)$, by the maximality of A. Hence there exists $g \in B_1$ such that

$$|\hat{g}(\gamma) - \hat{f}(\gamma)| < \varepsilon/K \qquad (\gamma \in E). \tag{1}$$

Our choice of E shows that there exists $h \in L^1(G)$ such that $\hat{h}(\gamma) = \hat{g}(\gamma) - \hat{f}(\gamma)$ on E and such that $||h||_1 < \varepsilon$. Since $\hat{f} + \hat{h} - \hat{g} = 0$ on E, $f + h - g \in B$, and since $g \in B_1$, it follows that $f+h \in B_1$. Since $||h||_1 < \varepsilon$, and since B_1 is closed, $f \in B_1$, and so $B_1 = L^1(G)$.

Thus B is a maximal subalgebra of $L^1(G)$.

9.3. The Stone-Weierstrass Property

9.3.1. Suppose A is a semi-simple commutative Banach algebra; we regard A as an algebra of functions on its maximal ideal space $\Delta(A)$. A subalgebra B of A is said to be *self-adjoint* if the complex conjugate of each member of B belongs to B.

We say that A is a Stone-Weierstrass algebra (or simply an S-W algebra) if the following is true: *every self-adjoint subalgebra B of A which separates points on $\Delta(A)$ and whose members do not all vanish at any one point of $\Delta(A)$ is dense in A.*

The origin of our terminology is clear: the Stone-Weierstrass Theorem says that $C_0(X)$ is an S-W algebra for every locally compact Hausdorff space X.

We shall consider the question whether $A(\Gamma)$ (or, equivalently, $L^1(G)$) is an S-W algebra. We find (Theorems 9.3.3 and 9.3.5) that this is so if and only if Γ is totally disconnected (Katznelson and Rudin [1]). For discrete Γ we already know this from Section 9.1.

9.3.2. THEOREM. *Every semi-simple commutative Banach algebra A which is spanned by its set of idempotents is a Stone-Weierstrass algebra.*

An element $j \in A$ is *idempotent* if $j^2 = j$, and our hypothesis asserts that the set of all finite linear combinations of the idempotents of A is dense in A. Note that it is not assumed that A is self-adjoint; there are examples (Coddington [1], Katznelson and Rudin [1]) which show that it need not be.

Proof: We regard A as an algebra of functions on $\Delta(A)$. Let B be the closure of a separating self-adjoint subalgebra B_0 of A whose members do not all vanish at any point of $\Delta(A)$. Associate with each complex homomorphism of A its restriction to B. This allows us to consider $\Delta(A)$ as a subset of $\Delta(B)$; moreover, $\Delta(A)$ is closed in $\Delta(B)$ (Loomis [1], p. 76).

The norm of any $f \in B$ is the same whether we regard f as an element of B or as an element of A. Hence the two spectral radii of f (relative to A and to B) are the same, so that

$$\sup_{x \in \Delta(B)} |f(x)| = \sup_{x \in \Delta(A)} |f(x)| \qquad (f \in B). \tag{1}$$

Suppose $f \in B$ and f is real on $\Delta(A)$. For real t, put $F_t = f \exp\{-itf\}$. Then $F_t \in B$, $|F_t| = |f| \exp\{t \operatorname{Im} f\}$, and on $\Delta(A)$ we have $|F_t| = |f|$. Hence (1) implies that

$$|f(x_0)| \exp\{t \operatorname{Im} f(x_0)\} \leqq \sup_{x \in \Delta(A)} |f(x)| \qquad (x_0 \in \Delta(B)). \tag{2}$$

Since (2) holds for all real t, we conclude that $\operatorname{Im} f(x_0) = 0$.

Now take $g \in B_0$. There exists $h \in B_0$ such that $h = \bar{g}$ on $\Delta(A)$. Since $g + h$ is real on $\Delta(A)$, what we have just proved shows that $g + h$ is real on all of $\Delta(B)$. Thus B_0 is self-adjoint as an algebra of functions on $\Delta(B)$.

Since B_0 is dense in B, B_0 separates points on $\Delta(B)$, and the Stone-Weierstrass theorem implies that every function in $C_0(\Delta(B))$ can be uniformly approximated on $\Delta(B)$ by members of B_0. Hence (1) holds for every $f \in C_0(\Delta(B))$, and so $\Delta(A)$ is dense in $\Delta(B)$. But $\Delta(A)$ is closed in $\Delta(B)$, so that $\Delta(B) = \Delta(A)$.

Let j be an idempotent of A. Then $j(x) = 0$ or 1 on $\Delta(A)$, and the Stone-Weierstrass theorem shows that there exists $f \in B_0$ such that $|f(x) - j(x)| < 1/3$, for all $x \in \Delta(A)$. The function ϕ defined by $\phi(z) = 0$ if $|z| < 1/3$, $\phi(z) = 1$ if $|1 - z| < 1/3$ is therefore analytic on the spectrum of f (relative to B, since $\Delta(B) = \Delta(A)$), and hence $\phi(f) \in B$ (Appendix D7). But $\phi(f) = j$. We have shown that B contains every idempotent element of A; the theorem follows.

9.3.3. THEOREM. *If Γ is a totally disconnected LCA group, then $A(\Gamma)$ is a Stone-Weierstrass algebra.*

Proof: By Lemma 2.4.3, Γ has a compact open subgroup Γ_0. Its annihilator G_0 is a compact open subgroup of G. Since G/G_0 is the dual of Γ_0 and since Γ_0 is totally disconnected, G/G_0 has no element of infinite order. Since every compact subset K of G is contained in the union of finitely many cosets of G_0, it follows that K is contained in a compact open subgroup of G.

Hence every $f \in L^1(G)$ can be approximated, in the norm of $L^1(G)$, by a sequence $\{f_n\}$, where each f_n has its support in a compact open subgroup H_n of G, and the restriction of f_n to H_n is a trigonometric polynomial on H_n.

If ϕ is a function on G whose support lies in H_n and whose restriction to H_n is a continuous character of H_n, then ϕ is a constant multiple of an idempotent in $L^1(G)$; the constant depends on the measure of H_n. Hence each of the above functions f_n is a linear combination of idempotents in $L^1(G)$.

It follows that $A(\Gamma)$ is spanned by its set of idempotents, and Theorem 9.3.2 completes the proof.

(If we also assume that Γ is compact, the preceding proof collapses to a triviality.)

9.3.4. LEMMA. *There exists a bounded function β on R, which is positive on a set of positive measure, whose support is a totally disconnected compact set P, such that $|y\hat{\beta}(y)| \leqq 1$ for all $y \in R$.*

We normalize the Haar measures so that $\hat{\beta}(y) = (2\pi)^{-1}\int \beta(x)e^{-iyx}dx$.

Proof: Let Q be the set of all functions in $A(R)$ whose derivative also belongs to $A(R)$. It is clear that Q is a subalgebra of $A(R)$ and that Q consists of the Fourier transforms of all $f \in L^1(R)$ for which $\int |xf(x)|dx < \infty$.

Choose $\alpha_1 \in Q$ so that $\alpha_1 > 0$ in $(0, 1)$, $\alpha_1 = 0$ outside $(0, 1)$, and $|y\hat{\alpha}_1(y)| < \frac{1}{2}$ for all $y \in R$. Choose $\delta_n > 0$ $(n = 1, 2, 3, \ldots)$ so that $\sum \delta_n = 1/4$.

Suppose α_n is constructed so that $0 \leqq \alpha_n \leqq \alpha_1$, $\alpha_n \in Q$, and $|y\hat{\alpha}_n(y)| < 1 - 2^{-n}$ for all $y \in R$. If $\phi_n \in Q$ and $\psi_n = \alpha_n\phi_n$, then $\hat{\psi}_n = \hat{\alpha}_n * \hat{\phi}_n$, so that

$$y\hat{\psi}_n(y) = \int_{-\infty}^{\infty} (y-t)\hat{\alpha}_n(y-t)\hat{\phi}_n(t)dt + \int_{-\infty}^{\infty} \hat{\alpha}_n(y-t)t\hat{\phi}_n(t)dt,$$

or

$$(1)\quad |y\hat{\psi}_n(y)| \leqq \|\hat{\phi}_n\|_\infty \cdot \int_{-\infty}^{\infty} |y\hat{\alpha}_n(y)|dy + \int_{-\infty}^{\infty} |\hat{\alpha}_n(y)|dy \cdot \sup_t |t\hat{\phi}_n(t)|.$$

Let u be an odd function in Q whose support is $[-2, -1] \cup [1, 2]$, which is positive in $(-2, -1)$ and whose integral over $(-2, -1)$ is 1. Put $u_r(x) = ru(rx)$. Then $\hat{u}_r(y) = \hat{u}(y/r)$. Since $\hat{u}(0) = 0$, $\|\hat{u}\|_\infty < 1$, $\hat{u}$ is continuous, and $\hat{u}(y) \to 0$ as $|y| \to \infty$, it follows that there is a sequence of positive integers r_i which increases so rapidly that

$$(2) \quad |\hat{u}_{r_1}(y) + \ldots + \hat{u}_{r_k}(y)| < 1 \qquad (y \in R;\ k = 1, 2, 3, \ldots).$$

Let x_n be the midpoint of the largest interval on which α_n is positive. If

$$(3) \qquad \phi_n(x) = \frac{1}{k}\int_{-\infty}^{x-x_n} \{u_{r_1}(s) + \ldots + u_{r_k}(s)\}ds,$$

then $\phi_n \in Q$, $0 \leqq \phi_n \leqq 1$, the support of ϕ_n lies in $[x_n - 2/r_1, x_n + 2/r_1]$, and $\phi_n(x) = 1$ in $[x_n - 1/r_k, x_n + 1/r_k]$. It follows that $\|\hat{\phi}_n\|_\infty \leqq 2/(\pi r_1)$ and $|t\hat{\phi}_n(t)| < 1/k$, by (2), and hence we can take r_1 and k so large that the support of ϕ_n lies in $[x_n - \delta_n, x_n + \delta_n]$ and so that $|y\hat{\varphi}_n(y)| < 2^{-n-1}$ for all $y \in R$, by (1).

Now define $\alpha_{n+1} = \alpha_n \cdot (1 - \phi_n)$. Then

$$|y\hat{\alpha}_{n+1}(y)| \leqq |y\hat{\alpha}_n(y)| + |y\hat{\varphi}_n(y)| < 1 - 2^{-n} + 2^{-n-1} = 1 - 2^{-n-1},$$

and our induction hypothesis holds, with $n + 1$ in place of n.

The sequence $\{\alpha_n\}$ converges monotonically to a non-negative bounded function β which vanishes outside $(0, 1)$. Since $\hat{\beta}(y) = \lim \hat{\alpha}_n(y)$, we have $|y\hat{\beta}(y)| \leqq 1$ for all $y \in R$.

Our construction of $\{\phi_n\}$ shows that $\beta(x) = 0$ on each of the intervals $[x_n - 1/r_k, x_n + 1/r_k]$, hence $\beta(x) = 0$ on a dense open subset of R, and its support P is totally disconnected. Finally, $\beta(x) = \alpha_1(x) > 0$ at those points on $(0, 1)$ at which $\phi_n(x) = 0$ for $n = 1, 2, 3, \ldots$, i.e., at those points which are not in the union of the intervals $[x_n - \delta_n, x_n + \delta_n]$. The measure of this union does not exceed $2 \sum \delta_n = \frac{1}{2}$. Hence $\beta(x) > 0$ on a set of measure $\frac{1}{2}$ and the proof is complete.

9.3.5. THEOREM. *Suppose Γ is a LCA group which is not totally disconnected. Then $A(\Gamma)$ is not a Stone-Weierstrass algebra.*

Proof: If Γ_1 is a closed subgroup of Γ and if $A(\Gamma_1)$ is not an S-W

algebra, then it is clear that the same is true of $A(\Gamma)$. If Γ does not contain a closed subgroup which is isomorphic to R, then Γ contains a compact connected subgroup which contains a one-parameter subgroup (Theorems 2.4.1, 2.5.6). Hence it is enough to prove the theorem in the following two cases:

Case (*a*). $\Gamma = R$. Take β and P as in Lemma 9.3.4, and let $B[P]$ be the algebra of all functions belonging to Q (see the proof of Lemma 9.3.4) whose derivative vanishes on P. Then $B[P]$ is a self-adjoint subalgebra of $A(R)$ which separates points on R since P is totally disconnected. Since $|y\hat{\beta}(y)| \leqq 1$, the equation

$$\Psi\hat{f} = \int_{-\infty}^{\infty} \hat{f}(y) y\hat{\beta}(y)dy \tag{1}$$

defines a bounded linear functional Ψ on $A(R)$; since $\hat{\beta}$ is not identically 0, Ψ is not the zero functional. For $\hat{f} \in Q$, (1) may be written in the form

$$\Psi\hat{f} = \frac{1}{2\pi}\int_P \beta(x)dx \int_{-\infty}^{\infty} y\hat{f}(y)e^{-iyx}\, dy. \tag{2}$$

The inner integral in (2) is a constant multiple of the derivative of $\hat{f}$ at x. Hence $\Psi\hat{f} = 0$ for all $\hat{f} \in B[P]$. This proves that $B[P]$ is not dense in $A(R)$, and so $A(R)$ is not an S-W algebra.

Case (*b*). Γ is compact and has a dense one-parameter subgroup J. The proof of Theorem 2.5.6(b) shows that the dual group G of Γ is then an infinite subgroup of R_d; we may assume, without loss of generality, that G contains the integers. There is a continuous homomorphism ϕ of R onto J, with the following properties: if $f \in A(\Gamma)$ and if $f^* = f(\phi)$, then

$$f^*(x) = \sum_{t \in G} c_t e^{itx}, \qquad \sum |c_t| < \infty \qquad (x \in R); \tag{3}$$

moreover, all series of the form (3) are obtained in this way.

If $\Gamma \neq T$, then ϕ is one-to-one. If $\Gamma = T$, then G consists of the integers alone, and ϕ is one-to-one on $[0, 2\pi)$. In any case, ϕ is one-to-one on P, the set constructed in Lemma 9.3.4. We also note that $\hat{\beta}(n)$ does not vanish for all integers n, since the support of β is in $[0, 1]$; hence $\hat{\beta}(t) \neq 0$ for some $t \in G$.

Let B be the set of all $f \in A(\Gamma)$ such that $\sum |tc_t| < \infty$ and such that the derivative of f^* is 0 at all points of P, i.e., such that $\sum tc_t e^{itx} = 0$ for all $x \in P$. Then B is a self-adjoint separating subalgebra of $A(\Gamma)$ which contains the constants. Define

$$\Psi f = \sum_{t \in G} c_t \cdot t\beta(-t) \qquad (f \in A(\Gamma)). \tag{4}$$

Then Ψ is a non-zero bounded linear functional on $A(\Gamma)$. For $f \in B$,

$$\Psi f = \frac{1}{2\pi} \int_P \beta(x) \sum_{t \in G} t c_t e^{ixt}\, dx = 0,$$

so that B is not dense in $A(\Gamma)$.

This completes the proof.

9.3.6. Suppose $f \in L^1(R)$ and $[f]$ denotes the smallest closed subalgebra of $L^1(R)$ which contains f. Under what conditions will $[f]$ be a maximal subalgebra of $L^1(R)$? This can happen; an example is furnished by the function α which we used in the proof of Theorem 9.2.3; we saw there that $[\alpha] = L^1(R^+)$.

Suppose $[f]$ is maximal. Put $S = \hat{f}(R) \cup \{0\}$. If S does not separate the plane, then $\hat{f}$ must identify infinitely many pairs of points of R which contradicts the maximality of $[f]$. Hence S separates the plane, $\bar{z}$ cannot be uniformly approximated on S by polynomials in z, and so $\tilde{f} \notin [f]$. This implies:

(a) *If $[f]$ is a maximal subalgebra of $L^1(R)$, then the algebra generated by f and $\tilde{f}$ is dense in $L^1(R)$.*

If the complement of S has two bounded components, let z_0 be a point in one of these. The set of all polynomials in z and $z/(z-z_0)$ is not dense in $C(S)$. It follows that the algebra generated by f and $(f - z_0\delta_0)^{-1} * f$ is not dense in $L^1(R)$, so that $[f]$ was not maximal. We conclude:

(b) *If $[f]$ is a maximal subalgebra of $L^1(R)$, then $\hat{f}$ is one-to-one on R and $\hat{f}(y) \neq 0$ for all $y \in R$.*

The converse of (b) is not true, even if very strong smoothness conditions are imposed on $\hat{f}$. For example, let P be the totally disconnected compact set constructed in Lemma 9.3.4, and let ϕ be an infinitely differentiable function on R such that $\phi = 0$ on P, $\phi > 0$ in the complement of P, $\phi(u) = e^{-|u|}$ for all sufficiently large $|u|$, and $\int \phi(u)du = 2\pi$. Put

$$\hat{f}(y) = 1 - \exp\left\{i\int_{-\infty}^{y} \phi(u)du\right\} \qquad (y \in R).$$

Then $\hat{f}$ and all its derivatives belong to $L^1(R)$, and so $\hat{f} \in A(R)$. Also, $\hat{f}$ is a one-to-one map of R onto the set of all $z \neq 0$ such that $|1 - z| = 1$. The derivative of $\hat{f}$ is 0 at every point of P, and the proof of Theorem 9.3.5 therefore shows that the algebra generated by f and $\bar{f}$ is not dense in $L^1(R)$. By (a), $[f]$ is therefore not maximal.

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Appendices

The appendices are short descriptions of those parts of topology, group theory, and functional analysis which are used in this book. They are intended to provide an easily available reference and to convince the reader that an acquaintance with only the more elementary parts of these subjects will give him the necessary prerequisites. They also contain a record of the exact forms in which certain concepts are used; this is significant in those cases in which the terminology is not standardized.

Since most of this material is quite familiar, it seemed superfluous to document each theorem with a reference to a specific source. Every item may be found in at least one of the following well-known texts: Dunford and Schwartz [1], Halmos [1], Hille and Phillips [1], Kaplansky [2], Kelley [1], Loomis [1], Montgomery and Zippin [1], Pontryagin [1].

A. *Topology*

A1. A family τ of subsets of a set S is called a *topology* on S if (a) S and the empty set belong to τ, (b) τ is closed under the formation of finite intersections and arbitrary unions. If a topology τ is defined on S, then S is called a *topological space* (it would be more accurate to reserve this name for the ordered pair (S, τ), but we shall ignore this distinction, as well as similar ones that occur later) and the members of τ are called *open sets*; their complements are *closed*. The largest open set contained in a set $A \subset S$ is the interior of A. The smallest closed set containing A is the closure $\bar{A}$ of A. If B is the complement of A, then $\bar{A} \cap \bar{B}$ is the *boundary* of A. If $\bar{A} = S$, A is *dense* in S. If some countable set is dense in S, S *is separable*. If p is an interior point of A, then A is a *neighborhood* of p. The set whose only element is p is written $\{p\}$. If $\{p\}$ is open, then p is an *isolated point* of S. If $\{p\}$ is open for every $p \in S$, then S is a *discrete space*.

A2. A family Ω of open subsets of a topological space S is a *base* if every open subset of S is a union of sets belonging to Ω. A family Ω_p of neighborhoods of a point $p \in S$ is a *neighborhood base* at p if every neighborhood of p contains a member of Ω_p. If to every pair p_1, p_2 of distinct points of S there exist neighborhoods N_1, N_2 of p_1, p_2 which are *disjoint* (i.e., whose intersection is empty), then S is called a *Hausdorff space.*

A3. Any subset X of a topological space S is itself a topological space if the open sets of X are defined to be the intersections of the open sets of S with X. This topology is the *relative topology* induced in X by S.

A4. A subset A of S (the case $A = S$ is not excluded) is called *compact* if every family of open sets whose union contains A has a *finite* subfamily whose union contains A. If every point of S has a compact neighborhood, then S is *locally compact.*

Every closed subset of a compact space is compact. Every compact subset of a Hausdorff space is closed. In a locally compact Hausdorff space, every point has a neighborhood base which consists of compact sets.

If Ω is a family of compact sets with the *finite intersection property* (i.e., every finite subfamily of Ω has non-empty intersection), then the intersection of all members of Ω is non-empty.

A5. If S is a topological space, let ∞ denote a point not in S, put $S_\infty = S \cup \{\infty\}$, and call a subset A of S_∞ open either if A is an open subset of S or if the complement of A is a compact subset of S. Then S_∞ is a compact space, and is called the *one-point compactification* of S. If S is compact, $\{\infty\}$ is an isolated point of S_∞. If S is a locally compact Hausdorff space, then S_∞ is a compact Hausdorff space.

A6. A map f of a topological space X into a topological space Y is called *continuous* if $f^{-1}(E)$ is open in X for every open set E in Y; here $f^{-1}(E)$ denotes the set of all $p \in X$ such that $f(p) \in E$. If K is compact, $K \subset X$, and f is continuous, then $f(K)$ is compact.

If $f(E)$ is an open subset of Y whenever E is an open set in X, then f is called an *open map*. If f is one-to-one, if $f(X) = Y$, and if both f and f^{-1} are continuous, then f is a *homeomorphism* of X onto Y.

A7. *If f is a continuous open map of a locally compact Hausdorff space X onto a Hausdorff space Y, and if K is a compact subset of Y, then there is a compact subset C of X such that $K = f(C)$.*

It follows from the hypotheses that there are finitely many points $p_1, \ldots, p_n$ in X with compact neighborhoods $N_1, \ldots, N_n$ such that $K \subset f(N_1) \cup \ldots \cup f(N_n)$. Put $C = f^{-1}(K) \cap \bigcup_1^n N_i$. Since $f^{-1}(K)$ is closed and $\bigcup N_i$ is compact, C is compact.

A8. A set A in a topological space S is *connected* if it is not the union of two disjoint non-empty sets which are open in the relative topology induced in A by S. The *component* of a point $p \in S$ is the union of all connected subsets of S which contain p. Since the closure of a connected set is connected, components are closed sets. If no component of S contains more than one point, S is called *totally disconnected.*

A9. *In a locally compact totally disconnected Hausdorff space, the compact open sets form a base.*

A10. If τ and τ_1 are two topologies on a set S and if $\tau \subset \tau_1$, then τ is said to be *weaker* than τ_1. This terminology does not exclude the case $\tau = \tau_1$.

If F is a family of maps of S into a topological space Y, the collection of all finite intersections of sets of the form $f^{-1}(V)$ ($f \in F$, V open in Y) forms a base for a topology τ_F on S. Each $f \in F$ is evidently continuous with respect to τ_F, and τ_F is the weakest topology on S with this property; τ_F is called the *weak topology induced in S by F*. Of particular importance is the case in which F is a collection of complex-valued functions (i.e., Y is the complex plane).

F is said to *separate points* (or to be *separating*) on S if to every pair of distinct points p_1, p_2 in S there corresponds an $f \in F$ such that $f(p_1) \neq f(p_2)$. If F separates points and if Y is a Hausdorff space, then the weak topology induced by F on S is also a Hausdorff topology.

A11. If S is a topological space, $C(S)$ denotes the set of all bounded continuous complex-valued functions on S. The *support* of a complex function f on S is the closure of the set of all points p at which $f(p) \neq 0$. The set of all $f \in C(S)$ whose support is compact is denoted by $C_c(S)$.

If, for each $\varepsilon > 0$, the inequality $|f(p)| < \varepsilon$ holds for all p in the complement of some compact set, then f is said to ***vanish at infinity***. The set of all $f \in C(S)$ which vanish at infinity is denoted by $C_0(S)$. Each $f \in C_0(S)$ may be extended to a continuous function on S_∞ by setting $f(\infty) = 0$. If S is compact, then $C(S) = C_0(S) = C_c(S)$.

A12. The spaces $C(S)$, $C_0(S)$, $C_c(S)$ are closed under pointwise addition, multiplication, and scalar multiplication: $(f+g)(p) = f(p) + g(p)$; $(fg)(p) = f(p)g(p)$; $(\alpha f)(p) = \alpha f(p)$. Since the usual commutative, associative, and distributive laws hold, these spaces are *algebras* (over the complex field). If we introduce a norm in $C(S)$ by setting

$$\|f\|_\infty = \sup_{p \in S} |f(p)| \qquad (f \in C(S)),$$

the metric $\|f - g\|_\infty$ turns $C(S)$ and $C_0(S)$ into *complete* metric spaces, since they are also closed under the formation of limits of uniformly convergent sequences. In fact, $C(S)$ and $C_0(S)$ are simple examples of *Banach algebras* (Appendix D).

If S is a locally compact Hausdorff space, then $C_c(S)$ is dense in $C_0(S)$.

A13. Tietze's Extension theorem. This theorem is usually stated for real-valued functions, but the following equivalent formulation is better suited to our purpose; we recall that an *arc* is a homeomorphic image of a compact interval of the real line:

Suppose K is a compact subset of the locally compact Hausdorff space S, and f is a continuous map of K into an arc L. Then there exists a continuous map g of S into L such that $g(p) = f(p)$ for all $p \in K$.

A14. The Stone-Weierstrass theorem. *Let S be a locally compact Hausdorff space and let A be a subalgebra of $C_0(S)$ which* *separates points on S, which is self-adjoint* (i.e., $f \in A$ implies $\bar{f} \in A$, where $\bar{f}$ is the complex conjugate of f) *and which contains, for each $p_0 \in S$, a function f such that $f(p_0) \neq 0$. Then A is dense in $C_0(S)$.*

A15. Suppose A is an index set, and S_α is a set, for each $\alpha \in A$. The *cartesian product* $S = \prod_{\alpha \in A} S_\alpha$ is the set of all p which are functions on A such that $p(\alpha) \in S_\alpha$, for all $\alpha \in A$; $p(\alpha)$ may be regarded as the αth coordinate of the point p. If A is finite, say $A = \{1, 2, \ldots, n\}$, the notation $S = S_1 \times S_2 \times \ldots \times S_n$ is also used for S, and the points of S may be regarded as n-tuples $p = (p_1, \ldots, p_n)$ with $p_\alpha \in S_\alpha$.

Suppose now that each S_α is a topological space. For any finite choice of indices, say $\alpha_1, \ldots, \alpha_n$, and for any choice of open sets $V_{\alpha_i} \subset S_{\alpha_i}$ $(1 \leqq i \leqq n)$, let V be the set of all $p \in S$ such that $p(\alpha_i) \in V_{\alpha_i}$ $(1 \leqq i \leqq n)$, and declare a subset E of S to be open if and only if it is a union of such sets V. Then S satisfies the axioms for a topological space, and is called the *topological product of the spaces S_α.*

Each $\alpha \in A$ can be regarded as a function on S whose value at a point $p \in S$ is $p(\alpha)$. If this is done, it becomes evident that *the topology of S is exactly the weak topology induced on S by A.*

If each S_α is a Hausdorff space, it is trivial that S is also a Hausdorff space. The analogous statement for compact spaces lies deeper:

THE TYCHONOFF THEOREM. *The topological product of any collection of compact spaces is compact.*

A16. A topological space is *metrizable* if its topology is induced by a metric. For a compact Hausdorff space S, the following three properties are equivalent:

(a) *S is metrizable;*

(b) *S has a countable base;*

(c) *$C(S)$ is separable.*

A17. If S is a locally compact Hausdorff space, or if S is a complete metric space, the *Baire theorem* holds: *S is not the union* *of countably many closed sets, unless one of them contains a non-empty open set.*

B. Topological Groups

Although many of the statements which follow apply to non-commutative groups as well as to commutative (abelian) ones, we shall confine our attention to the latter class.

B1. An *abelian group* is a set G in which a binary operation, $+$, is defined, with the following properties:

(a) $x + y = y + x$ for all $x, y \in G$;

(b) $x + (y + z) = (x + y) + z$ for all $x, y, z \in G$;

(c) G contains an element 0 such that $x + 0 = x$ for all $x \in G$;

(d) to each $x \in G$ corresponds an element $-x \in G$ such that $x - x = 0$. (We write $x - x$ in place of $x + (-x)$.)

If A and B are subsets of G, $A + B$ denotes the set of all elements of the form $a + b$, with $a \in A$, $b \in B$. Similarly, $-A$ is the set of all elements $-a$, where a ranges over A, and $A - B = A + (-B)$. If $x \in G$, it is customary to write $A + x$ instead of $A + \{x\}$. We call $A + x$ the *translate* of A by x.

A subset H of G which is itself a group, with respect to the same group operation, is a *subgroup* of G. For this it is necessary and sufficient that $H - H \subset H$. If $H \neq G$, then H is a *proper* subgroup of G. If $H = \{0\}$, then H is the *trivial* group.

B2. A *homomorphism* of a group G into a group G_1 is a map ϕ of G into G_1 such that

$$\phi(x + y) = \phi(x) + \phi(y) \qquad (x, y \in G).$$

A homomorphism which is one-to-one is an *isomorphism*. If there is an isomorphism of G *onto* G_1, then G and G_1 are *isomorphic groups*, and for many purposes one need not distinguish between them.

The *kernel* of a homomorphism ϕ is the set $\phi^{-1}(0)$; the kernel is always a subgroup.

If H is a subgroup of G, the sets $H + x$ $(x \in G)$ are the *cosets* of H. Two cosets $H + x$ and $H + y$ are identical if and only if $x - y \in H$; otherwise, $H + x$ and $H + y$ are disjoint. The set of all cosets of H is denoted by G/H, and G/H becomes an abelian group (the *quotient group* of G modulo H) if we define

$$(H + x) + (H + y) = H + x + y \qquad (x, y \in G).$$

The map $x \to H + x$ is a homomorphism of G onto G/H, with kernel H. It is called the *natural homomorphism* of G onto G/H.

Conversely, if ϕ is any homomorphism of G, the group $\phi(G)$ may be regarded as a quotient group of G: $\phi(G) = G/\phi^{-1}(0)$.

The *index* of a subgroup H of G is the number of elements of G/H; it is either a positive integer, or infinite.

B3. If $x \in G$ and n is a positive integer, nx is the element $x + x + \ldots + x$ (n summands). If $nx = 0$ for some n, the smallest positive integer with this property is the *order* of x; if $nx \neq 0$ for all $n > 0$, then x has *infinite order*. If there is an integer q such that $qx = 0$ for all $x \in G$, then G is said to be of *bounded order*.

If $E \subset G$ and if no proper subgroup of G contains E, we say that G is *generated* by E, or that E is *a set of generators*. A group generated by *one* of its elements is *cyclic*.

B4. A *topological abelian group* is a Hausdorff space G which is also an abelian group, provided the map $(x, y) \to x - y$ is a continuous map of the product space $G \times G$ onto G. If, in addition, the topology of G is locally compact, then G is a *locally compact abelian* (LCA) *group*.

(The proof of Theorem B6 (with $H = \{0\}$) shows that the Hausdorff separation axiom can be replaced by the weaker requirement that every point be a closed set, without changing the class of groups so defined. But this is not important for our present purpose.)

It follows that the translation map t_x, defined by $t_x(y) = x + y$, is a homeomorphism of G onto G, for each $x \in G$, and so is the map $x \to -x$. If A is an open set of G and $B \subset G$, then $A + B$ is a union of translates of A and is therefore open. If A and B are compact, then $A + B$ is compact, being the image of the compact set $A \times B$ under the continuous map $(x, y) \to x + y$.

A set $E \subset G$ is *symmetric* if $E = -E$. Since $E \cap (-E)$ is symmetric, it follows that *in every* LCA *group G there is a neighborhood base at* 0 *which consists of compact symmetric sets.* Moreover, the continuity of addition shows that to every neighborhood W of 0 in G there corresponds a neighborhood V of 0 (which may be taken compact and symmetric) such that $V + V \subset W$.

B5. The closure of any subgroup of G is again a subgroup of G. Every closed subgroup of a LCA group is LCA. Every open subgroup is closed; this is so since every coset of an open subgroup H is open, and since H is the complement of the union of all but one of its cosets.

B6. THEOREM. *Suppose G is LCA, ϕ is the natural homomorphism of G onto G/H, where H is a closed subgroup of G, and a subset of G/H is declared open if and only if it is the image under ϕ of an open subset of G. Then G/H is a LCA group.*

Proof: By definition, ϕ is continuous and open, and hence G/H is locally compact. If $x, y \in G$ and $x - y \notin H$, there is a neighborhood W of 0 such that $x + W$ does not intersect $y + H$, since $y + H$ is closed. There exists a symmetric compact neighborhood V of 0 such that $V + V \subset W$, and for this V the sets $x + H + V$ and $y + H + V$ do not intersect. In other words, the points $x + H$ and $y + H$ of G/H have disjoint neighborhoods, and so G/H is a Hausdorff space. The continuity of the group operation in G/H is easily verified.

B7. If $\{G_\alpha\}$ is a collection of abelian groups, their *complete direct sum* is the group G defined as follows: G, as a set, is the cartesian product of the sets G_α, and addition is performed coordinatewise: if $x \in G$ and $y \in G$, $x + y$ is the element of G whose αth coordinate is $x(\alpha) + y(\alpha)$, in the notation of Appendix A15.

The *direct sum* of the groups G_α is the subgroup of their complete direct sum which consists of all x which have $x(\alpha) \neq 0$ for only finitely many α.

If we now introduce the product topologies, the following facts emerge, via the Tychonoff theorem:

The direct sum of any finite collection of LCA *groups is a* LCA *group. The complete direct sum of any collection of compact abelian groups is a compact abelian group.*

If $G = H_1 + H_2$, where H_1 and H_2 are subgroups of G, then G is (isomorphic to) the direct sum $H_1 \oplus H_2$ of these two subgroups if and only if $H_1 \cap H_2 = \{0\}$.

B8. THEOREM. *If G is an abelian group of bounded order, then G is a direct sum of cyclic groups.*

Proof: Each prime p has a largest power p^a ($a \geqq 0$) which divides the order of some $x \in G$. Hence G contains elements x_p of order p^a. If $x^* = \sum x_{p_i}$, the sum being taken over the distinct primes p_i for which $a_i > 0$, then x^* has order $\prod p_i^{a_i}$. Thus G contains an element x^* whose order is a multiple of the order of every $x \in G$.

Suppose H is a proper subgroup of G and

(a) H is a direct sum of cyclic groups;

(b) if n is an integer and if $nx \in H$ for some $x \in G$, then $nx_0 \in H$ for some $x_0 \in H$.

The preceding paragraph, applied to G/H, shows that there exists $y^* \in G$ and an integer m such that $mx \in H$ for all $x \in G$ and such that $ry^* \notin H$ if $0 < r < m$. By (b), $my^* = my$ for some $y \in H$, and if $z = y^* - y$ then z has order m and the group K generated by H and z has property (a) and is larger than H.

Suppose $nx = y + tz$, where $x \in G$, $y \in H$, and n, t are integers. Let d be the greatest common divisor of m and n. Then $(mn/d)x \in H$, hence $(mt/d)z \in H$, hence m divides mt/d, and so d divides t. The congruence $ns \equiv t \pmod{m}$ is therefore solvable for s. Put $k = sz$. Then $nk - tz = (ns - t)\, z = 0$. Hence $n(x - k) = y = ny_0$, for some $y_0 \in H$, since (b) holds for H. We conclude: $n^x = n(y_0 + sz)$. Thus (b) also holds for K.

By Zorn's lemma, there is an H which is maximal with respect to (a) and (b), and the above argument shows that then $H = G$.

B9. Since the topology of a topological abelian group G is translation invariant, it is easy to introduce the notion of *uniform continuity*: A map f from a subset E of G into a metric space with metric d is uniformly continuous on E if to every $\varepsilon > 0$ there exists a neighborhood V of 0 in G such that $d(f(x), f(y)) < \varepsilon$ whenever $x \in E$, $y \in E$, and $y - x \in V$.

THEOREM. *If f is a continuous map of the compact set E in G into a metric space, then f is uniformly continuous on E.*

Proof: Given $\varepsilon > 0$, there corresponds to each $x \in E$ a neighborhood W_x of 0 such that $d(f(x), f(y)) < \varepsilon/2$ if $y \in E \cap (x + W_x)$, and there are symmetric open neighborhoods V_x of 0 such that $V_x + V_x \subset W_x$. Since E is compact, there is a finite set of points $x_1, \ldots, x_n$ in E such that the union of the sets $x_i + V_{x_i}$ covers E. If V is the intersection of these V_{x_i}, and $y - x \in V$, $x \in E$, $y \in E$, then $x \in x_i + V_{x_i}$ for some i, and $y \in x + V \subset x_i + V_{x_i} + V \subset x_i + W_{x_i}$. Hence

$$d(f(x), f(y)) < d(f(x), f(x_i)) + d(f(x_i), f(y)) < \varepsilon.$$

B10. One proves similarly that every $f \in C_0(G)$ is uniformly continuous on G if G is LCA.

C. Banach Spaces

C1. A *normed linear space* X is a vector space over the complex field (i.e., an abelian group in which multiplication by complex numbers is also defined and satisfies the usual distributive laws) in which a non-negative real number $\|x\|$, the *norm* of x, is associated to each $x \in X$, with the following properties:

(a) $\|x\| = 0$ if and only if $x = 0$;

(b) $\|x + y\| \leqq \|x\| + \|y\|$ for all $x, y \in X$;

(c) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in X$ and all complex numbers α.

If X is *complete* with respect to the metric defined by $d(x, y) = \|x - y\|$, i.e., if every Cauchy sequence in X converges, then X is called a *Banach space*.

The topology induced by the above metric is the *norm topology* of X. The set of all $x \in X$ with $||x|| \leqq 1$ is the *unit ball* of X.

C2. If M is a closed linear subspace of a normed linear space X, the quotient space X/M (see Appendix B2) becomes a normed linear space if we introduce the *quotient norm*

$$||x + M|| = \inf_{y \in M} ||x + y|| \qquad (x \in X).$$

If X is a Banach space, so is X/M.

C3. A map T of a normed linear space X into a normed linear space Y is a *linear transformation* if $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha \cdot Tx$ for all $x, y \in X$ and all complex numbers α; in other words, linear transformations are vector-space homomorphisms. The kernel of a linear transformation T is a linear subspace. T is said to be *bounded* if there is a real number C such that $||Tx|| \leqq C||x||$ for all $x \in X$; the smallest C with this property is the *norm* $||T||$ of T. Note that $||T|| = \sup_{x \neq 0} ||Tx||/||x||$.

A linear transformation T is bounded if and only if it is continuous. The set $L(X, Y)$ of all bounded linear transformations of X into Y is itself a normed linear space, and if Y is a Banach space, so is $L(X, Y)$.

If $T \in L(X_0, Y)$, where X_0 is a dense linear subspace of a normed linear space X and Y is a Banach space, then T has a unique extension to an element of $L(X, Y)$, with the same norm. This is a special case of the general metric space theorem which states that any uniformly continuous map into a complete space has a continuous extension to the completion of its domain.

C4. The complex field K, normed by the absolute value, is a Banach space. A bounded linear transformation of X into K is called a *bounded linear functional* on X, and $L(X, K)$ is the *dual space* of X, written X^*.

C5. The Hahn-Banach Theorem. *If M is a linear subspace (not necessarily closed) of the normed linear space X, and if S is a bounded linear functional on M, then there exists $T \in X^*$ such that $Tx = Sx$ for all $x \in M$, and such that $||T|| = ||S||$.*

Corollary. *If $x_0 \in X$ and x_0 is not in the closure of M, then there exists $T \in X^*$ such that $Tx = 0$ for all $x \in M$ but $Tx_0 \neq 0$.*

C6. The next theorem depends in an essential manner on the completeness of the spaces involved and is a consequence of the Baire theorem (Appendix A17):

Theorem. *Suppose X, Y are Banach spaces, $T \in L(X, Y)$, T is one-to-one, and $TX = Y$. Then $T^{-1} \in L(Y, X)$.*

Corollary. *If a vector space X is a Banach space with respect to two norms, say $\|\cdot\|$ and $\|\cdot\|'$, and if there is a constant C such that $\|x\|' \leqq C\|x\|$ for all $x \in X$, then there is a constant C' such that $\|x\| \leqq C'\|x\|'$ for all $x \in X$.*

If these two inequalities hold, the two norms are called *equivalent*. A further consequence is

The Closed Graph Theorem. *If X and Y are Banach spaces, if T is a linear transformation of X into Y, and if the relations $\lim_{n\to\infty} \|x_n - x\| = 0$ and $\lim_{n\to\infty} \|Tx_n - y\| = 0$ imply that $y = Tx$, then T is bounded.*

We need only apply the preceding corollary to the map $x \to (x, Tx)$ of X onto the graph of T which is a Banach space with norm $\|x\| + \|Tx\|$.

C7. Each $x \in X$ may be regarded as a function on X^* whose value at a point $T \in X^*$ is Tx. Then X is a separating family of functions on X^*. The weak topology induced in X^* by X (Appendix A10) is called the weak* topology of X^*.

Theorem. *For any normed linear space X, the unit ball S^* of X^* is a compact Hausdorff space in the weak* topology of X^*.*

Proof: Let D_x be the set of all complex numbers z with $|z| \leqq \|x\|$, where $x \in X$. With X as index set, S^* is then a subset of the topological product D of the discs D_x (Appendix A15). By the Tychonoff theorem, D is a compact Hausdorff space, and since limits of linear functions are linear, S^* is a *closed* subset of D.

C8. If X is separable, then a countable subset of X separates points on X^*. It follows that the weak* topology of X^* has a countable base. We conclude (see Appendix A16):

If X is separable, then the weak topology of the unit ball S^* of X^* is metrizable.*

C9. The following analogue of the corollary to the Hahn-Banach theorem is a direct consequence of the definition of the weak* topology:

If M is a weak-closed linear subspace of X^* and if $T_0 \notin M$, then there exists $x \in X$ such that $T_0 x \neq 0$ but $Tx = 0$ for all $T \in M$.*

C10. Suppose X and Y are Banach spaces, X^* and Y^* are their duals, and $T \in L(X, Y)$. For any $y^* \in Y^*$, the map $x \to y^*(Tx)$ is a bounded linear functional on X; hence there is an element of X^*, which we write T^*y^*, such that $(T^*y^*)(x) = y^*(Tx)$ for all $x \in X$. The map T^* of Y^* into X^* so defined is called the *adjoint* of T. It is easy to see that $T^* \in L(Y^*, X^*)$.

C11. THEOREM. *Suppose X, Y are Banach spaces, $T \in L(X, Y)$, T is one-to-one, and TX is dense in Y. Then each of the following three properties implies the other two:*

(a) $TX = Y$.

(b) *There exists $\delta > 0$ such that $||T^*y^*|| \geqq \delta||y^*||$ for all $y^* \in Y^*$.*

(c) $T^*Y^* = X^*$.

Proof: Let $S_r = \{x \in X: ||x|| \leqq r\}$. If (a) holds, C6 shows $T(S_1)$ contains all $y \in Y$ with $||y|| \leqq \delta$. Hence

$$||T^*y^*|| = \sup_{x \in S_1} |(T^*y^*)(x)| = \sup_{x \in S_1} |y^*(Tx)| \geqq \delta|y^*||$$

for all $y^* \in Y^*$, and (b) holds. If (c) holds, one proves in the same way that $||Tx|| \leqq \alpha||x||$ for some $\alpha > 0$ and all $x \in X$; this implies that TX is norm-closed, and so (a) holds. It remains to show that (b) implies (c).

If (b) holds, then $\Gamma = T^*Y^*$ is norm-closed in Y^*. Moreover, $T^{*-1}(E)$ is bounded for every bounded set E in Γ, and this implies, via C7, that the intersection of Γ with every closed ball in X^* is weak*-compact.

Suppose $\Gamma \neq X^*$. Then there exists $x_0^* \in X^*$ whose distance from Γ exceeds 1. Put $\Gamma_n = \{x^* \in \Gamma: \|x^* - x_0^*\| \leqq n\}$, and if F is any finite subset of X, let F° be the set of all $x^* \in X^*$ such that $|(x^* - x_0^*)(x)| \leqq 1$ for all $x \in F$.

We claim that there exist finite sets $F_n \subset S_{1/n}$ such that

$$(1) \quad F_0^\circ \cap F_1^\circ \cap \ldots F_{n-1}^\circ \cap \Gamma_n \text{ is empty} \qquad (n = 1, 2, 3, \ldots).$$

Since Γ_1 is empty, F_0 may be chosen arbitrarily. Suppose (1) holds for some $n \geqq 1$, $W_n = F_0^\circ \cap \ldots \cap F_{n-1}^\circ$, and $F^\circ \cap W_n \cap \Gamma_{n+1}$ is not empty, no matter what finite set $F \subset S_{1/n}$ we take. Since $W_n \cap \Gamma_{n+1}$ is weak*-compact and since the collection of all sets $F^\circ \cap W_n \cap \Gamma_{n+1}$ has the finite intersection property, there exists $x^* \in W_n \cap \Gamma_{n+1}$ such that $|(x^* - x_0^*)(x)| \leqq 1$ for all $x \in S_{1/n}$. But then $\|x^* - x_0^*\| \leqq n$, or $x^* \in \Gamma_n$, which contradicts (1). Hence F_n exists, and the induction is complete.

Arrange the elements of $\bigcup F_n$ in a sequence $\{x_i\}$. Then $\|x_i\| \to 0$, and

$$(2) \qquad \sup_i |(x^* - x_0^*)(x_i)| > 1 \qquad (x^* \in \Gamma).$$

The map $\Psi : x^* \to \{x^*(x_i)\}$ is a bounded linear transformation of X^* into the space of all sequences which converge to 0, and (2) shows that the distance from $\Psi(x_0^*)$ to $\Psi(\Gamma)$ is positive. Hence there exists $\{\alpha_i\}$, with $\sum |\alpha_i| < \infty$, so that, setting $x = \sum \alpha_i x_i$, we have $x_0^*(x) = \sum \alpha_i x_0^*(x_i) = 1$ but $x^*(x) = \sum \alpha_i x^*(x_i) = 0$ if $x^* \in \Gamma$. The latter condition implies that $y^*(Tx) = (T^*y^*)(x) = 0$ for all $y^* \in Y^*$, so that $Tx = 0$ and hence $x = 0$, which is impossible if $x_0^*(x) = 1$. This contradiction proves that $\Gamma = X^*$, so that (c) holds.

C12. Suppose H is a vector space over the complex field, and suppose that to each ordered pair $x, y \in H$ there is associated a complex number (x, y), called the *inner product* of x and y, with the following properties: (a) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$; (b) $(\alpha x, y) = \alpha(x, y)$; (c) $(y, x) = \overline{(x, y)}$; (d) $(x, x) \geqq 0$; (e) $(x, x) = 0$ only if $x = 0$. Setting $\|x\| = (x, x)^{1/2}$, H then becomes a normed linear space. If H is complete in this norm. H is called a *Hilbert space.*

The *Schwarz inequality* $|(x, y)| \leqq ||x|| \cdot ||y||$ is a consequence of properties (a) to (d) of the inner product. It shows that the map $x \to (x, y)$ is, for each $y \in H$, an element of H^*. Conversely, to each $T \in H^*$ there corresponds a unique $y \in H$ such that $Tx = (x, y)$. Thus H is its own dual.

A set K in H is *convex* if $\alpha x + (1 - \alpha)y \in K$ whenever $x \in K$, $y \in K$, and $0 < \alpha < 1$. Each closed convex set K in a Hilbert space H has a unique element x_0 of minimal norm. If $x_0 + M \subset K$ for some linear subspace M of H, then $(x_0, y) = 0$ for all $y \in M$; in other words, x_0 is *orthogonal* to M. If 0 is the only element in H which is orthogonal to a linear subspace M, then M is dense in H, by the Hahn-Banach theorem and the above characterization of H^*.

D. Banach Algebras

D1. A vector space A over the complex field is a *commutative algebra* if a multiplication is defined in A which satisfies the usual commutative, associative and distributive laws. If a norm is defined in a commutative algebra A which makes A into a Banach space, and if the inequality $||xy|| \leqq ||x|| \cdot ||y||$ holds for all $x, y \in A$, then A is a *commutative Banach algebra.*

In this appendix, the symbol A will always denote a commutative Banach algebra.

There may or may not be a *unit* in A, i.e., an element e such that $xe = x$ for all $x \in A$. If A has a unit, the norm of A can be replaced by an equivalent one (see Appendix C6) such that $||e|| = 1$. The element $x \in A$ is *invertible* if it has a multiplicative inverse, i.e., if there is an element $x^{-1} \in A$ such that $x^{-1}x = e$.

D2. A subalgebra I of A is an *ideal in A* if $xy \in I$ whenever $x \in A$ and $y \in I$. If $I \neq A$, I is a *proper* ideal. *Maximal ideals* are proper ideals which are not contained in any larger proper ideals.

THEOREM. *If A has a unit, then every proper ideal in A is contained in a maximal ideal, and every maximal ideal is closed.*

This is an easy consequence of Zorn's lemma and the following three facts: (a) proper ideals contain no invertible elements, (b) the set of all invertible elements is open, and (c) the closure of an ideal is an ideal.

D3. If I is an ideal in A, a multiplication may be defined in the quotient space A/I by setting

$$(x+I)(y+I) = xy + I \qquad (x, y \in A);$$

this makes A/I into an algebra, the so-called quotient algebra of A modulo I.

If I is closed and if A/I is given the quotient norm (see Appendix C2), *then A/I is a Banach algebra.*

An ideal I in A is said to be *regular* if A/I has a unit (if A has a unit, *every* ideal is regular). Theorem D2 has the following replacement if A has no unit:

Every proper regular ideal in A is contained in a regular maximal ideal, and every regular maximal ideal is closed.

D4. A *complex homomorphism* h of A is a linear functional on A which is also multiplicative: $h(xy) = h(x)h(y)$. Let Δ be the set of all complex homomorphisms of A which are not identically 0. The following statements contain the core of the theory of commutative Banach algebras, as developed by Gelfand [1]:

(a) If I is a regular maximal ideal in A, then A/I is (isometrically isomorphic to) the complex field, and so the canonical homomorphism of A onto A/I belongs to Δ.

(b) Conversely, if $h \in \Delta$, the kernel of h is a regular maximal ideal in A.

(c) If A has a unit, then $x \in A$ is invertible if and only if $h(x) \neq 0$ for all $h \in \Delta$. In any case, the equation $xy = x + y$ is solvable in A if and only if $h(x) \neq 1$ for all $h \in \Delta$.

(d) Each $h \in \Delta$ is a *bounded* linear functional on A, of norm 1. Thus Δ is a subset of the unit ball S^* in the dual space A^* of the Banach space A.

(e) Each $x \in A$ defines a function $\hat{x}$ on Δ, given by

$$\hat{x}(h) = h(x) \qquad (h \in \Delta).$$

The weak topology induced in Δ by the collection of all these functions $\hat{x}$ is called the *Gelfand topology* of Δ. It coincides with the relative topology which Δ has as a subset of A^* if A^* is given the weak* topology. Since $\Delta \subset S^*$, since S^* is weak*-compact (Appendix C7) and since $\Delta \cup \{0\}$ is easily seen to be a closed subset of S^*, it follows that Δ is a locally compact Hausdorff space (usually called the *maximal ideal space of* A) and that each $\hat{x}$ is a member of $C_0(\Delta)$ (Appendix A11).

(f) The map $x \to \hat{x}$ is a homomorphism of A onto a subalgebra $\hat{A}$ of $C_0(\Delta)$, since

$$(\hat{x}\hat{y})(h) = h(xy) = h(x)h(y) = \hat{x}(h)\hat{y}(h) \qquad (x, y \in A;\ h \in \Delta),$$

and similarly for addition and scalar multiplication. Since $\|h\| \leqq 1$ the important inequality

$$\|\hat{x}\|_\infty \leqq \|x\|$$

holds. We call $\hat{x}$ the *Gelfand transform* of x.

(g) If A has a unit e, then Δ is compact, since $\hat{e}(h) = h(e) = 1$ and $1 \in C_0(\Delta)$ only if Δ is compact.

D5. If the Gelfand transformation is an isomorphism, i.e., if $x \neq 0$ implies $\hat{x} \neq 0$ (or, $h(x) \neq 0$ for some $h \in \Delta$), then A is said to be *semi-simple*.

THEOREM. *If A and B are commutative Banach algebras, if B is semi-simple, and if Ψ is a homomorphism of A into B, then Ψ is continuous* (i.e., $\Psi \in L(A, B)$). *If $\Psi \neq 0$, then* $\|\Psi\| \geqq 1$.

Proof: Suppose $x_n \to x_0$ in A and $\Psi x_n \to y_0$ in B, for some sequence $\{x_n\}$ in A. For each $h \in \Delta$, the maximal ideal space of B, the map $x \to h(\Psi x)$ is a complex homomorphism α on A. By D4(d), h and α are continuous, so that

$$h(\Psi x_0) = \alpha(x_0) = \lim \alpha(x_n) = \lim h(\Psi x_n) = h(y_0).$$

Since B is semi-simple, we conclude that $y_0 = \Psi x_0$, and the continuity of Ψ follows from the closed graph theorem.

If $\Psi \neq 0$, the semi-simplicity of B implies that $h(\Psi x) \neq 0$ for some $x \in B$, $h \in \Delta$, and so $\alpha \neq 0$. By D4(d), $\|\Psi\| = \|h\| \cdot \|\Psi\| \geqq \|\alpha\| = 1$.

D6. The *spectrum* of an element $x \in A$ is defined to be the range of the function $\hat{x}$ (with 0 adjoined if A has no unit, so that the spectrum is always a compact subset of the complex plane). The number $\|\hat{x}\|_\infty$ is the *spectral norm* or the *spectral radius* of x. The equation

$$\lim_{n\to\infty} \|x^n\|^{1/n} = \|\hat{x}\|_\infty \qquad (x \in A) \tag{1}$$

is known as the *spectral radius formula*. For Fourier transforms, it was discovered by Beurling [1]. The general case is due to Gelfand [1].

Let α and β be the upper and lower limits of $\{\|x^n\|^{1/n}\}$. Since $|h(x)|^n = |h(x^n)| \leqq \|x^n\|$ for all $h \in \Delta$, we have $\|\hat{x}\|_\infty \leqq \beta$. If λ is not in the spectrum of x, D4(c) shows that there exists $y = y(\lambda) \in A$ such that $-\lambda y + xy = x$; also, $y(\lambda)$ is an analytic function of λ, outside the spectrum of x. If $|\lambda| = C > \|x\|$, then $y(\lambda) = -\sum_1^\infty (x/\lambda)^n$, so that

$$x^n = -\frac{1}{2\pi i}\int_{|\lambda|=C} \lambda^{n-1} y(\lambda)\,d\lambda \qquad (n = 1, 2, 3, \ldots). \tag{2}$$

If $\|\hat{x}\|_\infty < r < R$, the path of integration in (2) can be shrunk to the circle $|\lambda| = r$ without changing the integral. Hence $\|x^n/R^n\| \to 0$, $\alpha \leqq R$, and so $\alpha \leqq \|\hat{x}\|_\infty$. This proves (1).

D7. A similar application of the Cauchy formula shows that analytic functions operate in Banach algebras:

Suppose A is a commutative-semi-simple Banach algebra, $x \in A$, and F is an analytic function defined on an open set which contains the spectrum of x; if A has no unit, we require that $F(0) = 0$. Then there exists a unique $y \in A$ such that $\hat{y}(h) = F(\hat{x}(h))$ for all $h \in \Delta$.

E. Measure Theory

E1. Our discussion will be confined to measures and integrals on locally compact Hausdorff spaces X. Let B be the smallest family of subsets of X which (a) contains all closed subsets of X,

(b) is closed under the formation of countable unions, and (c) is closed under complementation. Then B is also closed under the formation of countable intersections. The members of B are called the *Borel sets* of X.

A *measure on* X is a set function μ, defined for all Borel sets of X, which is *countably additive* (i.e., $\mu(E) = \sum \mu(E_i)$ if E is the union of the countable family $\{E_i\}$ of pairwise disjoint Borel sets of X), and for which $\mu(E)$ is finite if the closure of E is compact.

With each measure μ on X there is associated a set function $|\mu|$, the *total variation* of μ, defined by

$$|\mu|(E) = \sup \sum |\mu(E_i)|, \tag{1}$$

the supremum being taken over all finite collections of pairwise disjoint Borel sets E_i whose union is E. Then $|\mu|$ is also a measure on X (Hewitt [1]). If

$$|\mu|(E) = \sup |\mu|(K) = \inf |\mu|(V), \tag{2}$$

for every Borel set E, where K ranges over all compact subsets of E and V ranges over all open supersets of E, then μ is called *regular*. We put

$$\|\mu\| = |\mu|(X) \tag{3}$$

and define $M(X)$ to be the set of all complex-valued regular measures on X for which $\|\mu\|$ is finite.

It is clear that $M(X)$ is a normed linear space if addition and scalar multiplication are defined by

$$(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E), \qquad (\alpha\mu)(E) = \alpha \cdot \mu(E) \tag{4}$$

for every Borel set E and every complex number α.

We shall also consider *non-negative regular measures on* X; for these, $+\infty$ is an admissible value.

E2. If μ is a measure on X and A is a Borel set, the *restriction* μ_A of μ to A is the measure defined by

$$\mu_A(E) = \mu(A \cap E). \tag{1}$$

If $\mu = \mu_A$, then μ is said to be *concentrated on* A. If two measures μ_1 and μ_2 are concentrated on disjoint sets, the pair (μ_1, μ_2) is said to be *mutually singular*; in that case

$$\|\mu_1 + \mu_2\| = \|\mu_1\| + \|\mu_2\|. \tag{2}$$

If $\mu \in M(X)$, then μ is concentrated on a σ-compact subset of X (i.e., on a set which is a countable union of compact sets) and among all closed subsets of X there is a smallest one, the *support of* μ, on which μ is concentrated.

Every $\mu \in M(X)$ *has a unique decomposition of the form*

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4 \tag{3}$$

in which $\mu_j \geqq 0$, $\mu_j \in M(X)$, *and the pairs* (μ_1, μ_2) *and* (μ_3, μ_4) *are mutually singular*. This is the *Jordan decomposition theorem*.

E3. A measure $\mu \in M(X)$ is called *discrete* if it is concentrated on a countable set; μ is *continuous* if $\mu(E) = 0$ for every countable set E. Every $\mu \in M(X)$ has a unique decomposition $\mu = \mu_d + \mu_c$, where μ_d is discrete and μ_c is continuous.

If $\mu \in M(X)$, if m is a non-negative measure on X, and if $\mu(E) = 0$ whenever $m(E) = 0$, then μ is said to be *absolutely continuous* with respect to m.

THE LEBESGUE DECOMPOSITION THEOREM. *If* $\mu \in M(X)$ *and* $m \geqq 0$, *then* μ *has a unique decomposition* $\mu = \mu_a + \mu_s$, *where* μ_a *is absolutely continuous with respect to* m *and* μ_s *is singular with respect to* m.

E4. If $\mu \in M(X)$, the map $f \to \int_X f\,d\mu$ is a bounded linear functional on the Banach space $C_0(X)$ (see Appendix A11). The converse of this statement is

THE RIESZ REPRESENTATION THEOREM. *To each bounded linear functional* T *on* $C_0(X)$ *there corresponds a unique* $\mu \in M(X)$ *such that*

$$Tf = \int_X f\,d\mu \qquad (f \in C_0(X)). \tag{1}$$

In this generality, the theorem was first proved by Kakutani [1].

Its history is discussed in Dunford and Schwartz [1], pp. 373, 380. Since (Hewitt [1])

$$\text{(2)} \qquad \sup |Tf| = \|\mu\| \qquad (f \in C_0(X),\ \|f\|_\infty \leqq 1),$$

if T and μ are related by (1), we see that (1) establishes an isometric isomorphism between $M(X)$ and the dual of $C_0(X)$. In particular, $M(X)$ is a Banach space.

E5. Another useful version of the Riesz representation theorem is as follows (Halmos [1]):

To each linear functional T on $C_c(X)$ such that $Tf \geqq 0$ if $f \geqq 0$, there corresponds a unique regular non-negative measure m on X such that

$$Tf = \int_X f\,dm \qquad (f \in C_c(X)).$$

E6. A complex function f defined on X is called a *Borel function* if $f^{-1}(V)$ is a Borel set for every open set V in the complex plane. If $\mu \in M(X)$, all bounded Borel functions on X are integrable with respect to μ, and the inequality

$$\left|\int_X f\,d\mu\right| \leqq \|\mu\| \cdot \sup_{x \in X} |f(x)|$$

holds.

E7. If m is a non-negative measure on X and if $0 < p < \infty$, $L^p(m)$ denotes the set of all Borel functions f on X for which the norm

$$\text{(1)} \qquad \|f\|_p = \left\{\int_X |f|^p\,dm\right\}^{1/p}$$

is finite. If we identify functions which differ only on a set E with $m(E) = 0$, $L^p(m)$ becomes a Banach space, normed by (1), if $1 \leqq p < \infty$. $L^2(m)$ is a Hilbert space, with inner product $(f, g) = \int f\bar{g}\,dm$.

$L^\infty(m)$ is the space of all bounded Borel functions on X, normed by

$$\text{(2)} \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{x \in X} |f(x)|;$$

the essential supremum of $|f|$ is, by definition, the smallest number λ such that $m(\{x : f(x) > \lambda\}) = 0$. Again we identify any two members f, g of $L^\infty(G)$ for which $||f - g||_\infty = 0$.

E8. *If m is regular, then $C_c(X)$ is dense in $L^1(m)$.* The set function μ defined by $\mu(E) = \int_E |f|\, dm$ belongs to $M(X)$. Hence, given $\varepsilon > 0$, there is a compact set E such that $\mu(E') < \varepsilon$, where E' is the complement of E. It follows that there is a bounded Borel function g, with compact support E, such that $||f - g||_1 < \varepsilon$.

A theorem of Lusin (Saks [1]) asserts that for every $\delta > 0$ there exists $h \in C_c(X)$ such that $h(x) = g(x)$ except possibly on a set S with $m(S) < \delta$. We may also take h so that $||h||_\infty \leqq ||g||_\infty$. Thus $||g - h||_1 < 2\delta||g||_\infty$, and hence $||f - h||_1 < \varepsilon$ if δ is small enough.

Essentially the same proof holds for $L^p(m)$, if $1 \leqq p < \infty$.

E9. If $f \in L^1(m)$, the measure defined by $\mu(E) = \int_E f\, dm$ belongs to $M(X)$ and is absolutely continuous with respect to m. The converse of this proposition is

THE RADON-NIKODYM THEOREM. *If $\mu \in M(X)$, if m is a non-negative measure on X, and if μ is absolutely continuous with respect to m, then there exists $f \in L^1(m)$ such that*

$$\mu(E) = \int_E f dm$$

for all Borel sets E in X.

Also, $||\mu|| = \int_X |f|\, dm = ||f||_1$.

E10. Suppose $m \geqq 0$, $1 < p < \infty$, and $1/p + 1/q = 1$. The bounded linear functionals T on $L^p(m)$ are in one-to-one correspondence with the members g of $L^q(m)$: each $T \in (L^p)^*$ is of the form

$$Tf = \int fg\, dm \qquad (f \in L^p(m)),$$

and $||T|| = ||g||_q$. Thus $L^q = (L^p)^*$.

If X is the union of a disjoint family of σ-compact sets X_α such that each σ-compact subset of X is contained in the union of a countable subfamily of $\{X_\alpha\}$, then we also have $L^\infty = (L^1)^*$. *This condition is satisfied by every LCA group G.* For if V is a compact symmetric neighborhood of 0 in G, if $V_1 = V$ and $V_{n+1} = V_n + V$, then each V_n is compact, and $H = \bigcup_{n=1}^{\infty} V_n$ is a σ-compact open subgroup of G. The cosets of H have the properties required of the sets X_α in the preceding paragraph.

E11. Suppose μ and λ are regular measures on locally compact Hausdorff spaces X and Y. For any set $A \times B$ in $X \times Y$, where A and B are Borel sets in X and Y, respectively, define

$$(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B).$$

The set function $\mu \times \lambda$ so defined on "rectangles" has a unique extension to a regular measure $\mu \times \lambda$ on the product space $X \times Y$.

FUBINI'S THEOREM. *If $\mu \geqq 0$, $\lambda \geqq 0$, f is a Borel function on $X \times Y$, and $f \geqq 0$, then*

$$\begin{aligned} \int_{X\times Y} fd(\mu \times \lambda) &= \int_X \int_Y f(x, y)d\lambda(y)d\mu(x) \\ &= \int_Y \int_X f(x, y)d\mu(x)d\lambda(y). \end{aligned} \tag{1}$$

If $\mu \in M(X)$, $\lambda \in M(Y)$, f is a Borel function $X \times Y$, and if

$$\int_X \int_Y |f(x, y)|d|\lambda|(y)d|\mu|(x) < \infty,$$

then (1) *also holds.*

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LIST OF SPECIAL SYMBOLS

$L^p(G)$, $L^\infty(G)$, 3
$f * g$, 3
(x, γ), 7
$\hat{f}$, 7
$\tilde{f}$, 9
$A(\Gamma)$, 9
R, T, Z, 12
$M(G)$, 13
$\mu * \lambda$, 13
$\hat{\mu}$, $B(\Gamma)$, 15
$\tilde{\mu}$, 16
$\oplus$, 36, 255
R^n, T^n, Z^n, 36
Z_q, 38, 98
D_q, T^ω, 38
$J(G)$, $S(\mu)$, 59
$F(G)$, 62
$\hat{f} \circ \alpha$, 77
K_q, 98
$B_0(\Gamma)$, $M_0(G)$, 154
$Z(f)$, $Z(I)$, 158
$I(E)$, 158
$I_0(E)$, 161
C^∞, 174
$\alpha(I)$, $\beta(\Phi)$, 184
$\sigma(\Phi)$, $\sigma(\phi)$, 184
$H^p(G)$, 197
$C_A(G)$, 217
$C(X)$, $C_c(X)$, $C_0(X)$, 250
$A+B$, $A+x$, 252
$|\mu|$, $M(X)$, 265
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A Correction

The proof of part (b) of the Pontryagin duality theorem (i.e., of the fact that $\alpha(G)$ is closed in $\hat{\Gamma}$) is incomplete, because the local compactness of $\alpha(G)$ does not by itself guarantee that $\alpha(G) \cap \overline{U}$ is compact, as was asserted on p. 29. The following theorem bridges the gap:

THEOREM. *If* H *is a subgroup of a topological group* G, *and* H *is locally compact* (in the topology induced by G), *then* H *is closed in* G.

LEMMA. *If* X *is a Hausdorff space, and* Y *is a dense subset of* X *which is locally compact* (in the topology induced by X), *then* Y *is open in* X.

Proof: To say that Y is locally compact means that every point of Y lies in an open set V such that the Y-closure of $V \cap Y$ is compact. In other words, the set $K = Y \cap \overline{(V \cap Y)}$ is compact, and is therefore closed in X.

Let W be the set of all points of V that are not in K. The inclusions $W \cap Y \subset V \cap Y \subset K$ show that $W \cap Y$ is empty. Since W is open and Y is dense in X, it follows that W is empty. Thus $V \subset K$, hence $V \subset Y$, and therefore Y is open.

To prove the theorem, let $Y = H$, $X = \overline{H}$. The lemma shows that H is an open subgroup of $\overline{H}$. Since open subgroups are closed (Appendix B5), $H = \overline{H}$.

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