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EULERIAN–LAGRANGIAN RUNGE–KUTTA DISCONTINUOUS GALERKIN METHOD FOR TRANSPORT SIMULATIONS ON UNSTRUCTURED MESHES*

XIAOFENG CAI† AND JING-MEI QIU‡

Abstract. The semi-Lagrangian (SL) approach is attractive in transport simulations, e.g., in climate modeling and kinetic models, due to its numerical stability in allowing extra-large time-stepping sizes. For practical problems with complex geometry, schemes on the unstructured meshes are preferred. However, accurate and mass conservative SL methods on unstructured meshes are still under development and encounter several challenges. For instance, when tracking characteristics backward in time, high order curves are required to accurately approximate the shape of upstream cells, which brings in extra computational complexity. To avoid such computational complexity, we propose an Eulerian–Lagrangian Runge–Kutta discontinuous Galerkin method (EL RK DG) in [X. Cai, J.-M. Qiu, and Y. Yang, *J. Comput. Phys.*, 439 (2021), 110392] as an extension of the SL discontinuous Galerkin (DG) methods. This work is a further extension of the algorithm to unstructured triangular meshes with discussion on the treatment of the inflow boundary condition. We also discuss the discrete geometric conservation law. The nonlinear weighted essentially nonoscillatory (WENO) limiter is applied to control oscillations. Desired properties of the proposed method are numerically verified by a set of benchmark tests.

Key words. Eulerian–Lagrangian, discontinuous Galerkin, unstructured triangular meshes, mass conservation, semi-Lagrangian, characteristics

MSC codes. 65M25, 65M60, 76M10

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1. Introduction.

Transport processes are ubiquitous in a variety of applications such as climate modeling and kinetic models. They can be described by the transport equation

$$u_t + \nabla \cdot (\mathbf{V}u) = 0, \tag{1.1}$$

where $\mathbf{V}$ is the advection coefficient, which could depend on space, time, and the solution u for a nonlinear problem.

In the past decades, extensive mesh-based computational tools such as Eulerian and semi-Lagrangian (SL) approaches have been successfully developed and applied to various areas of science and engineering. For the Eulerian approach, the Runge–Kutta (RK) discontinuous Galerkin (DG) methods [16] are well known for their properties of high resolution, compactness, flexibility for handling complex geometry, high parallel efficiency, and superconvergence for long time integration, which led to successful applications to diverse application fields such as aerodynamics [57], computational geosciences [50], and plasma simulation [14, 48] among many others. One drawback

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†Advanced Institute of Natural Sciences, Beijing Normal University at Zhuhai, 519087, China, and Division of Science and Technology, United International College (BNU-HKBU), Zhuhai, 519087, China (xfcai@bnu.edu.cn, xiaofengcai@uic.edu.cn).
‡Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 USA (jingqiu@udel.edu).

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