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OPTIMAL CONVERGENCE AND SUPERCONVERGENCE OF SEMI-LAGRANGIAN DISCONTINUOUS GALERKIN METHODS FOR LINEAR CONVECTION EQUATIONS IN ONE SPACE DIMENSION

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Abstract. In this paper, we apply semi-Lagrangian discontinuous Galerkin (SLDG) methods for linear hyperbolic equations in one space dimension and analyze the error between the numerical and exact solutions under the L^2-norm. In all the previous works, the theoretical analysis of the SLDG method would suggest a suboptimal convergence rate due to the error accumulation over time steps. However, numerical experiments demonstrate an optimal convergence rate and, if the terminal time is large, a superconvergence rate. In this paper, we will prove optimal convergence and optimal superconvergence rates. There are three main difficulties: 1. The error analysis on overlapping meshes. Due to the nature of the semi-Lagrangian time discretization, we need to introduce the background Eulerian mesh and the shifted mesh. The two meshes are staggered, and it is not easy to construct local projections and to handle the error accumulation during time evolution. 2. The superconvergence of time-dependent terms under the L^2-norm. The error of the numerical and exact solutions can be divided into two parts, the projection error and the time-dependent superconvergence term. The projection strongly depends on the superconvergence rates. Therefore, we need to construct a sequence of projections and gradually improve the superconvergence rates. 3. The stopping criterion of the sequence of projections. The sequence of projections are basically of the same form. We need to show that the projections exist up to some certain order since the superconvergence rate cannot be infinity. Hence, we will seek some "hidden" condition for the existence of the projections. In this paper, we will solve all the three difficulties and construct several local projections to prove the optimal convergence and superconvergence rates. Numerical experiments verify the theoretical findings.

1. Introduction

In this paper, we apply semi-Lagrangian discontinuous Galerkin (SLDG) methods for linear hyperbolic equations in one space dimension. SLDG methods with mass conservation were first proposed for Vlasov applications in [26,27]. Since then, there has been a strong line of research development for theoretical understanding [16] and applications [4,5,19,20] of SLDG methods. Usually, the convergence analysis of semi-Lagrangian methods [17,26] suggest an error estimate of h^p/τ, where h

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