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A CONSERVATIVE SEMI-LAGRANGIAN HYBRID HERMITE WENO SCHEME FOR LINEAR TRANSPORT EQUATIONS AND THE NONLINEAR VLASOV–POISSON SYSTEM*

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Abstract. In this paper, we present a high-order conservative semi-Lagrangian (SL) hybrid Hermite weighted essentially nonoscillatory (HWENO) scheme for linear transport equations and the nonlinear Vlasov–Poisson (VP) system. The proposed SL hybrid HWENO scheme adopts a weak formulation of the characteristic Galerkin method and introduces an adjoint problem for the test function in the same way as the SL discontinuous Galerkin (DG) scheme [W. Guo, R. D. Nair, and J. M. Qiu, *Monthly Weather Rev.*, 142 (2014), pp. 457–475]. Comparing with the original SL DG scheme, we introduce a hybrid moment-based HWENO reconstruction operator in space, bringing at least two benefits. Firstly, with the same order of accuracy, such a reconstruction allows lower degrees of freedom per element in the evolution process. Secondly, it naturally possesses a nonoscillatory property when dealing with discontinuity. In addition, we derive a novel troubled cell indicator which can effectively detect the discontinuous regions for the reconstruction operator. To apply the scheme for 2-D transport equations and the nonlinear VP system, we adopt a fourth-order dimensional splitting method. Positivity-preserving limiters are applied to enforce the positivity of the solution for the system having positive solutions. Finally, we show extensive numerical tests to validate the effectiveness of the proposed SL hybrid HWENO scheme.

Key words. Vlasov–Poisson system, semi-Lagrangian, mass conservation, positivity preservation, hybrid HWENO reconstruction, troubled cell indicator

AMS subject classifications. 65M60, 35L65

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1. Introduction.

The transport equation can be found in a variety of applications such as climate modeling and kinetic description of plasma. It can be written in the form of

$$u_t + \nabla_{\mathbf{x}} \cdot (\mathbf{a}(u, \mathbf{x}, t)u) = 0, \tag{1.1}$$

where $u(\mathbf{x}, t)$ is the scalar density function of a conserved quantity transported in a flow with velocity field $\mathbf{a}(u, \mathbf{x}, t)$ with $\mathbf{x} \in \mathbb{R}^d$. The semi-Lagrangian (SL) approach is popular for solving the transport equation. It uses fixed meshes, like the Eulerian approach, while the information propagates along the characteristics, like

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