Matching and substituability

New MCS

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Matching

Issue

Given a set of agents $\mathcal{X}$ representing demand. And a set of agents $\mathcal{Z}$ representing offer. On may wonder how will these agents pair up at equilibrium. This raises two issues

- How to model this equilibrium in a reasonable manner?
- How to compute that equilibrium?

Example of use cases

- Job market [1]
- Taxi [2]
- Universities

Network of agents

Let $\mathcal{X}$ (resp. $\mathcal{Z}$) the set of types of agents x (resp z). Let $n \in \mathbb{R}_+^{\mathcal{X}}$ (resp. $m \in \mathbb{R}_+^{\mathcal{Z}}$) the number of agent for each type in $\mathcal{X}$ (resp. in $\mathcal{Z}$).

Definition

$\mu \in \mathbb{R}_+^{\mathcal{X} \times \mathcal{Z}}$ is a matching if

$$\sum_x \mu_{xz} \leq m_z, \quad \sum_z \mu_{xz} \leq n_x$$

Modeling the equilibrium

When can a matching be considered stable?
Let $\alpha \in \mathbb{R}^{\mathcal{X}\times\mathcal{Z}}$, a utility vector, i.e. such that α_{xz} quantifies the utility for any agent of type x to pair up with an agent of type z.

Definition
$G : \mathbb{R}^{\mathcal{X}\times\mathcal{Z}} \to \mathbb{R}$ is a *Welfare Function* if

$$\frac{\partial G}{\partial \alpha_{xz}}(\alpha) = \text{aggregate demand for } z \text{ from } x$$

Definition of an equilibrium

Let G (resp. H) be a welfare function for $\mathcal{X}$ (resp. $\mathcal{Z}$). And α (resp. γ) a vector of utilites for agent of type x (resp. z).

Definition
A matching $\mu \in \mathbb{R}_+^{\mathcal{X}\times\mathcal{Z}}$ is an equilibrium if

$$\nabla G(\alpha) = \mu = \nabla H(\gamma)$$

That is demand equals supply.

Reaching an equilibrium

What is the mechanism used to reach an equilibrium?

- Money?
- Money burning?
- Transfers?

Wait and reach an equilibrium

Denote by $\tau_{xz}^{\alpha} \in \mathbb{R}_+$ (resp. τ_{xz}^{γ}) the time agents of type x have to wait in order to have access to agents of type z (resp. waiting time of z for x).
The utilities are now $\alpha - \tau^{\alpha}$ and $\gamma - \tau^{\gamma}$
Why not $\alpha - f(\tau^{\alpha})$ with f weakly increasing?

Aggregate Stable Matching

Note that there cannot be a pair where money is being burned on both sides of the market, i.e., a passenger of type x waiting for a driver of type z while a driver of type z is simultaneously waiting for a passenger of type x. This implies $\min(\tau^\alpha, \tau^\gamma) = 0$.

Definition
$(\mu, \tau^\alpha, \tau^\gamma) \in \mathbb{R}_+^{\mathcal{X}\times\mathcal{Z}}$ is an aggregate stable matching if

- $\nabla G(\alpha - \tau^\alpha) = \mu = \nabla H(\gamma - \tau^\gamma)$
- $\min(\tau^\alpha, \tau^\gamma) = 0$

Construction of a welfare function

Let $\alpha \in \mathbb{R}^{\mathcal{X} \times \mathcal{Z}}$ the vector of utilities as usual. Assume a bit of uncertainty around the utility of each agent of a specific type x. This random shift in taste is represented by $(\epsilon_{xz})_z$ which is a random variable following the law $\mathcal{P}_x$.
Now look at the behaviour of the i-th agent of type x_0

$$\max_z(\alpha_{x_0 z} + \epsilon_{x_0 z}, 0)$$

However since the total number of agent N is large and the shifts in taste are independent.

$$\frac{1}{N} \sum_{i=1}^{n_{x_0} N} \max_z(\alpha_{x_0 z} + \epsilon^i_{x_0 z}, 0) \approx n_{x_0} \mathbb{E}_{\mathcal{P}_{x_0}} \left[\max_z(\alpha_{x_0 z} + \epsilon_{x_0 z}, 0) \right]$$

We obtain the following weighted indirect utility function

$$G(\alpha) = \sum_x n_x \mathbb{E}_{\mathcal{P}_x}\left[\max_z(\alpha_{xz} + \epsilon_{xz}, 0)\right]$$

Proposition
G is a welfare function

It may not be differentiable everywhere but since it is convex observe that

$$\left(n_x \mathcal{P}_x\left(\alpha_{xz} + \epsilon_{xz} \geq \max_{\bar{z}}(\alpha_{x\bar{z}} + \epsilon_{x\bar{z}}, 0)\right)\right)_{xz} \in \partial G(\alpha)$$

We notice that the term on the left is the vector of demand from agents of type x.
Note that G is submodular which will be key to prove the existence of an aggregate stable matching.

Logit case

If for all x the taste shifters $(\epsilon_{xz})_z$ are iid Gumbel variables then

$$G(\alpha) = \sum_x n_x \log \left(1 + \sum_z \exp\left(\alpha_{xz}\right) \right)$$

And the demand for each coupling xz is

$$\frac{\partial G}{\partial \alpha_{xz}} = \frac{n_x \exp(\alpha_{xz})}{1 + \sum_{z'} \exp\left(\alpha_{xz'}\right)}$$

Deferred acceptance algorithm

The proof of existence of an aggregate stable matching relies on an explicit construction given by the following algorithm:

Step 0 Initialize $\mu^{A,0} = \min(n, m)$

Step k Three phases

Proposal phase

$$\mu^{\alpha,k} \in \operatorname*{arg\,max}_{\mu^{\gamma,k-1} \leq \mu \leq \mu^{A,k}} \mu\alpha - G^*(\mu)$$

Disposal phase

$$\mu^{\gamma,k} \in \operatorname*{arg\,max}_{\mu \leq \mu^{\alpha,k}} \mu\gamma - H^*(\mu)$$

Update phase

$$\mu^{A,k+1} = \mu^{A,k} - (\mu^{\alpha,k} - \mu^{\gamma,k})$$

Substituability And Exchangeability

Old MCS

In the indivisible goods case ([3]) we have for c submodular the set of optimal bundles

$$Q\left(p,B\right)\in\arg\max_{Q\subseteq B}\left\{p\left(Q\right)-c\left(Q\right)\right\},$$

satisfies this monotony property

$$R\left(p,B\right)=B\backslash Q\left(p,B\right)\ \text{ increases with } B$$

New MCS

The question we need to answer is the following. How does the following expression behaves as $(\alpha, \bar{\mu})$ increases.

$$r(\alpha, \bar{\mu}) = \bar{\mu} - \arg\max_{\mu \leq \bar{\mu}} \mu\alpha - G^*(\mu)$$

Where G is a convex submodular function. Two things have changed compared to old MCS:

- Goods are divisible, thus the space is continuous
- The function is not submodular but it is the Legendre transform of a submodular function

Link with Topkis

In the case of a convex submodular function G, [6], allows us to describe the behaviour of

$$\arg\max_{\alpha} \mu\alpha - G(\alpha)$$

It is increasing with regard to the following order ([4]):

Definition
We say that a set S is smaller than S' in the Veinott order, $S \leq_p S'$, if:

$$\forall p \in S, \forall p' \in S', p \wedge p' \in S, p \vee p' \in S'$$

Towards an order on functions

Veinott's order can be seen as an order on the indicator functions of the sets.

Definition
We say that ϕ is smaller than ϕ' in the p-order, $\phi \leq_p \phi'$, if

$$\forall p \in \mathrm{dom}\phi, \forall p' \in \mathrm{dom}\phi'$$
$$\phi(p \wedge p') + \phi'(p \vee p') \leq \phi(p) + \phi'(p')$$

Notice that $S \leq_p S'$ is equivalent to $\iota_S \leq_p \iota_{S'}$

Dual order on functions

As we've seen before, G^* is not submodular but G is. This nudges us towards the definition of a dual order.

Definition
We say that ϕ is smaller than ϕ' in the q-order, $\phi \leq_q \phi'$, if

$$\forall q \in \mathrm{dom}(\phi), q' \in \mathrm{dom}(\phi'), \forall \delta_1 \in [0, (x-y)^+], \exists \delta_2 \in [0, (x-y)^-]$$
$$\phi(q - (\delta_1 - \delta_2)) + \phi'(q' + (\delta_1 - \delta_2)) \leq \phi(q) + \phi'(q')$$

Key MCS

We are now ready to state this key theorem

Theorem
Each of these assertions are equivalent whenever c is a convex sci proper function

1. c is exchangeable, that is $c \leq_q c$,
2. c^* is submodular, that is $c^\star \leq_p c^\star$,
3. if $p \leq p'$ then $\pi_{\{p=p'\}} r\left(p, \bar{q}\right) \leq_q \pi_{\{p=p'\}} r\left(p, \bar{q}\right)$ for all $\bar{q}$.

Moreover any of the property above imply: if $\bar{q} \leq \bar{q}'$ then $r\left(p, \bar{q}\right) \leq_q r\left(p, \bar{q}'\right)$.

Idea for $1 \iff 2$

The whole proof relies on the following sequence of (approximative) equivalences

1. $$c \leq_q c$$
2. $$\forall x, y \in \mathrm{dom}(c)$$
$$\sup_{\delta_1 \in [0,(x-y)^+]} \inf_{\delta_2 \in [0,(x-y)^-]} c(x - \delta_1 + \delta_2) + c(y + \delta_1 - \delta_2) \leq f(x) + g(y)$$
3. $$\forall x, y \in \mathrm{dom}(c)$$
$$\sup_{\substack{\lambda \in \mathbb{R}^d \\ \mu \in \mathbb{R}^d}} \mu(x+y) + \lambda^+ x - \lambda^- y - c^*(\mu + \lambda) - c^*(\mu) \leq c(x) + c(y)$$
4. By taking $\lambda = d_f - d_g$ and $\mu = \alpha + d_g$
$$c^* \leq_p c^*$$

Convergence towards the equilibrium

Hypothesis

Let G, H two welfare functions, α, γ the initial utility vectors.

Theorem
Suppose that G, H are submodular, convex, proper, closed, sci functions such that $\min G, \min H > -\infty$. Then there exists an aggregate stable matching $(\mu, \tau^\alpha, \tau^\gamma)$.

We deduce that in DARUM there exists an aggregate stable matching.

Corollary
For any collection of distribution $(\mathcal{P}_x)_x, (\mathcal{Q}_z)_z$. For any utility vectors α, γ. There exists an aggregate stable matching if

$$G(\alpha) = \sum_x n_x \mathbb{E}_{\mathcal{P}_x} \left[\max_z(\alpha_{xz} + \epsilon_{xz}, 0) \right]$$
$$H(\gamma) = \sum_z m_z \mathbb{E}_{\mathcal{Q}_z} \left[\max_x(\gamma_{xz} + \eta_{xz}, 0) \right]$$

3+1 steps proof

Lemma
For all $k \geq 0$ we have

$$\mu^{\alpha,k} \in \operatorname*{arg\,max}_{\mu \leq \mu^{A,k}} \mu\alpha - G^*(\mu)$$

Lemma
For any $\mu \in \operatorname{arg\,max}_{\mu \leq \bar{\mu}} \mu\gamma - H^*(\mu)$

$$T^H(\gamma, \bar{\mu}) = \operatorname*{arg\,min}_{\tau \geq 0} \bar{\mu}\tau + H(\gamma - \tau) = (\gamma - \partial H^*(\mu)) \cap \{. \geq 0\} \cap \left\{.^\top(\bar{\mu} - \mu) = 0\right\}$$

Lemma

Set $\tau^{\alpha,k} = \inf T^G(\alpha, \mu^{A,k})$ *and* $\tau^{\gamma,k} = \inf T^H(\gamma, \mu^{\alpha,k})$*.*
$(\tau^{\alpha,k})_k$ *is weakly increasing,* $(\tau^{\gamma,k})$ *is weakly decreasing.*

Last step

Lemma
$(\mu^{A,k}), (\mu^{\alpha,k}), (\mu^{\gamma,k}), (\tau^{\alpha,k}), (\tau^{\gamma,k})$ *converge up to an extraction.*
Moreover let

$$\lim \mu^{\alpha,k} = \mu = \lim \mu^{\gamma,k}$$

And

$$\lim \tau^{\alpha,k} = \tau^{\alpha}, \qquad \tau^{\gamma,k} = \tau^{\gamma} \tag{1}$$

We have that $(\mu, \tau^{\alpha}, \tau^{\gamma})$ *is an aggregate stable matching*

First step

Proof.
It is clear for $k = 0$. If it is true for $k \geq 0$ we have that

$$\mu^{A,k+1} - \operatorname*{arg\,max}_{\mu \leq \mu^{A,k+1}} \mu\alpha - G^*(\mu) \leq_Q \mu^{A,k} - \operatorname*{arg\,max}_{\mu \leq \mu^{A,k}} \mu\alpha - G^*(\mu)$$

by a classic result from Q-order theory there exists $\mu \in \arg\max_{\mu \leq \mu^{A,k+1}} \mu\alpha - G^*(\mu)$ such that

$$\mu^{A,k+1} - \mu \leq \mu^{A,k} - \mu^{\alpha,k}$$

Thus

$$\mu^{\gamma,k} \leq \mu$$

□

Second step

Proof.
$\mu \in \arg\max_{\mu \leq \bar{\mu}} \mu\gamma - H^*(\mu)$ is a kuhn tucker vector for the minimisation problem thus

$$\min_{\tau \geq 0} \bar{\mu}^\top \gamma + H(\gamma - \tau) = \min_{\tau \geq 0} \bar{\mu}^\top \gamma + H(\gamma - \tau) - \tau^\top (\bar{\mu} - \mu)$$

Following the same steps as in the proof in [5]. For any $\tau \geq 0$ we have

$$\bar{\mu}^\top \gamma + H(\gamma - \tau) - \tau^\top (\bar{\mu} - \mu) \leq \bar{\mu}^\top \gamma + H(\gamma - \tau)$$

And thus if $\tau \in T^H(\gamma, \bar{\mu})$ we have that

$$\tau \in (\gamma - \partial H^*(\mu)) \cap \{. \geq 0\} \cap \left\{ .^\top (\bar{\mu} - \mu) = 0 \right\}$$

And conversely. $\square$

Third step

Proof.
$(\mu^{A,k})$ is decreasing, by [4]
Thanks to the result before we notice that

$$\inf T^H(\gamma, \mu^{\alpha,k}) = \inf T^H(\gamma, \mu^{\gamma,k})$$

Since $\mu^{\gamma,k} \leq \mu^{\alpha,k+1}$ we have that

$$T^H(\gamma, \mu^{\alpha,k+1}) \leq_p T^H(\gamma, \mu^{\gamma,k})$$

and thus $\tau^{T,k+1} \leq \tau^{T,k}$ □

Final step (1/4)

Proof.
$(\mu^{A,k})$is decreasing bounded from below by 0 thus it converges to μ^{A}.
$(\tau^{\alpha,k})$,$(\tau^{\gamma,k})$ are in a compact and are monotone so they converge.
Since $(\mu^{\alpha,k})$ lies in a compact up to an extraction it converges to μ since $\mu^{\alpha,k} - \mu^{\gamma,k} = \mu^{A,k} - \mu^{A,k+1} \to 0$ we have the first equation.

□

Final step (2/4)

(Cont.)
Notice that

$$\min_{\tau \geq 0} \mu^{A,k}\tau + G(\alpha - \tau) = \mu^{A,k}\tau^{\alpha,k} + G(\alpha - \tau^{\alpha,k}) \to \mu^{A}\tau^{\alpha} + G(\alpha - \tau^{\alpha})$$

$$\min_{\tau \geq 0} \mu^{A,k}\tau + G(\alpha - \tau) \to \min_{\tau \geq 0} \mu^{A}\tau + G(\alpha - \tau)$$

Thus $\tau^{\alpha} \in T^{G}(\alpha, \mu^{A})$ similarily $\tau^{\gamma} \in T^{H}(\gamma, \mu)$. We also have that

$$\mu \in \arg\max_{\mu \leq \mu^{A}} \mu\alpha - G^{*}(\mu), \quad \mu \in \arg\max_{\mu \leq \mu} \mu\alpha - G^{*}(\mu)$$

Since μ is a kuhn tucker vector for the two dual problems we have that

$$\partial H(\gamma - \tau^{\gamma}) \ni \mu \in \partial G(\alpha - \tau^{\alpha})$$

□

Final step (3/4)

(Cont.)
Let $(x, z) \in \mathcal{X} \times \mathcal{Z}$ such that $\tau^{\alpha}_{xz}, \tau^{\gamma}_{xz} > 0$, since $(\tau^{\gamma,k})_k$ is weakly decreasing

$$\forall k \geq 0, \tau^{,k}_{xy} > 0$$

Thus by duality

$$\forall k \geq 0, \mu^{\alpha,k}_{xz} = \mu^{\gamma,k}_{xz}$$

Thus by going to the limit we have $\mu^{A}_{xz} = \mu^{A,0}_{xz} = \min(n_x, m_z)$. Since $\tau^{\alpha}_{xz} > 0$ we have once again by duality $\mu^{A}_{xz} = \mu_{xz}$. However since μ is a kuhn-tucker vector

$$T^G(\alpha, \mu^A) = \alpha - \partial G^*(\mu) \cap \{\tau \geq 0\} \cap \{(\mu^A - \mu)\tau = 0\}$$
$$T^H(\gamma, \mu) = \gamma - \partial H^*(\mu) \cap \{\tau \geq 0\}$$

□

Final step (4/4)

(Cont.)
Wlog suppose that $\min(n_x, m_z) = m_z$, we can now show that it is absurd that $\tau^\gamma_{xz} > 0$.f Let $\Delta = \tau^\gamma_{xz} e_{xz}$ and $\tilde{\mu} \in \mathrm{dom} H^*$,

$$\begin{aligned}\mu(\gamma - (\tau^\gamma - \Delta)) - H^*(\mu) = & \quad H(\gamma - \tau^\gamma) + \mu_{xz}\Delta_{xz} \\ \geq & \quad \tilde{\mu}(\gamma - \tau^\gamma) - H^*(\tilde{\mu}) + m_z\Delta_{xz} \\ \geq & \quad \tilde{\mu}(\gamma - (\tau^\gamma - \Delta)) - H^*(\tilde{\mu})\end{aligned}$$

thus $\tau^\gamma - \Delta \in \partial H(\mu)$ the other two conditions follow by construction. We finally have $\tau^\gamma - \Delta \in T^H(\gamma, \mu)$ however $\tau^\gamma - \Delta < \tau^\gamma = \inf T^H(\gamma, \mu)$ which is absurd. Finally $\min(\tau^\alpha, \tau^\gamma) = 0$ $\square$

Refinements

- Really slow converging algorithm $\rightarrow$ Hybrid approach
- Instead of having $\alpha - \tau \rightarrow$ can the impact of "time" be of the form $\alpha - f(\tau)$?
- Is there a way to compute the equilibrium directly?

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