Today

Linear case: special matrices (Z- and M- matrices) and Perron-Froebenius theory

Next time

Nonlinear case: Topkis theorem; submodularity etc.

Study

$Q_z(p) = q_z$

where Q_z is increasing in p_z and weakly decreasing in $p_{z'}$.

$\sum_y \exp(\Phi_{xy} - a_x - b_y) = n_x$
$\sum_x \exp(\Phi_{xy} - a_x - b_y) = m_y$

$p_z = -a_x, b_y$

$Q_x(p) = \sum_y \exp(\Phi_{xy} + p_x - p_y)$
$Q_y(p) = -\sum_x \exp(\Phi_{xy} + p_x - p_y)$

Why? Gross substitute property. Interpret Q as an excess supply function, thus:

* Q_z should be increasing in p_z
* Q_z should be (weakly) decreasing in $p_{z'}$ when $z' \neq z$

Today: focus on the case when Q is linear.

$Q(p) = Qp$ where Q is a $Z \times Z$ matrix.

In the linear case, Gross Substitutes is expressed by:

* $Q_i(p) = \sum_j Q_{ij} p_j$

- increasing in p_i if $Q_{ii} > 0$ ie the diagonal of Q is positive
- weakly decreasing in p_j for $j \neq i$ if $Q_{ij} \leq 0$ ie the off diagonal terms are weakly negative. Q is called a Z-matrix.

How do we determine p^* such that $Qp^* = q$?

At step t assume we know the estimate p^t of p^* and let's find p_i^{t+1} in order to clear the market for good i. We do

$$Q_{ii} p_i^{t+1} + \sum_{j \neq i} Q_{ij} p_j^t = q_i$$

Let's write this in a matrix way. We write

$$Q = \Delta - A$$

where $\Delta = diag\left((Q_{ii})_i\right)$ and $A_{ij} = -Q_{ij} \geq 0$ if $i \neq j$ and 0 else.

The update algorithm above rewrites as

$$\Delta p^{t+1} - Ap^t = q$$

that is

$$\Delta p^{t+1} = Ap^t + q$$

and thus

$$p^{t+1} = \Delta^{-1}Ap^t + \Delta^{-1}q$$

assume p^* exists. Then $Qp^* = q$ and thus $\Delta p^* - Ap^* = q$, therefore

$$p^* = \Delta^{-1}Ap^* + \Delta^{-1}q$$

hence

$$p^{t+1} - p^* = \Delta^{-1}A\left(p^t - p^*\right)$$

and therefore

$$p^t - p^* = \left(\Delta^{-1}A\right)^t\left(p^0 - p^*\right)$$

We are therefore wondering whether

$$\left(\Delta^{-1}A\right)^t \to 0.$$

Example that gross substitutes is not enough for existence of equilibrium price

$$Q = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

As soon as Q is invertible, we will have that Q^{-1} is entrywise positive.