‘math+econ+code’ masterclass on equilibrium transport and matching models in economics

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Summer 2022 session
Special lecture 1. Perron-Froebenius theory

- Bertsekas, Tsitsiklis. Parallel and distributed computation: Numerical methods.
- Berman, Plemmons. Nonnegative matrices in the mathematical sciences.
- Tsatsomeros. Lecture Notes on Matrices with Positive Principal Minors: Theory and Applications. (Online)

- Consider the equilibrium problem

$$Qp = q$$

  where Q is a $n \times n$ matrix. Assume $Q_{ii} > 0$ for each i. (In fact this will be implied by stronger assumptions).
- When Q is invertible (more on this later), and denote p^* the solution of the above equation.
- We shall discuss methods to look for p^*, in the presence of gross substitutes.

- $(Qp)_i$ is interpreted as the supply for good i. $Q_{ii} > 0$ means that when the price of good i increases, the supply for it increases.
- Assume gross substitutes, that is $Q_{ij} \leq 0$ for $i \neq j$. Interpretation: when the price of good j increases, the production of good i decreases because suppliers substitute producing j to producing i.

Definition. One says Q is a *Z-matrix* when $Q_{ij} \leq 0$ for $i \neq j$.

- In the literature, Z-matrices are sometimes referred to as negative *Metzler matrices*. (Metzler matrices are non-negative off-diagonal).

- Recall what the Jacobi algorithm is. Decompose Q as

$$Q = \Delta - A$$

where Δ is diagonal with positive entries, and A has nonnegative terms and zeros on the diagonal.

- Jacobi algorithm rewrites as

$$\Delta p^{k+1} - A p^{k} = q$$

that is

$$p^{k+1} = \Delta^{-1} A p^{k} + \Delta q.$$

- As a result, when p^{*} exists, setting $\delta^{k} = p^{k} - p^{*}$, we have

$$\delta^{k} = \left(\Delta^{-1} A\right)^{k} \delta^{0},$$

and we wonder when Jacobi converges for any starting point p^{0}.

- Consider v an eigenvector of $M = \Delta^{-1} A$ and assume $v_i > 0$ for all i. Then the associated eigenvalue λ is > 0. We have for any δ

$$\left(M\delta\right)_i = \sum_j M_{ij}\delta_j = \sum_j M_{ij} v_j \frac{\delta_j}{v_j} \leq \left(Mv\right)_i \max_j \left(\left|\frac{\delta_j}{v_j}\right|\right) = \lambda v_i \left|\delta\right|_v^\infty$$

where

$$\left|\delta\right|_v^\infty := \max_j \left(\left|\frac{\delta_j}{v_j}\right|\right)$$

- As a result,

$$\left|M\delta\right|_v^\infty \leq \lambda \left|\delta\right|_v^\infty$$

and thus, if $\lambda < 1$, $M^k \delta \to 0$ for any δ.

- Given a norm $|x|$, the *induced norm* $\|.\|$ on matrices is defined as

$$\|M\| = \max\{|Mx| : |x| = 1\}$$

- The *spectral radius* $\rho(M)$ as the maximum modulus of the (complex) eigenvalues of M.
- While the induced norm depends on the norm that is chosen, the spectral radius does not. We have easily

$$\rho(M) \leq \|M\|$$

  for any induced norm, and (less easily) Gelfand's formula

$$\rho(M) = \lim_{k\to\infty} \left\|M^k\right\|^{1/k}$$

- When M is symmetric, the spectral radius coincides with the induced Euclidean norm, which is itself an Euclidian norm on matrices. Thus, the following developments have interest only outside of that case.

- A matrix M is *convergent* if $M^k \to 0$ as $k \to +\infty$. We have:
  Proposition (BT prop. A.20): M is convergent if and only if $\rho(M) < 1$.
- As a result, if $\|M\| < 1$ for some induced norm, then M is convergent, but the converse is not true.
- However, we shall see that when the Perron-Froebenius theorem applies on M, then there exists a norm $|.|$ such that $\rho(M) = \|M\|$ for $\|.\|$ the induced matrix norm.

Before that, we need an important definition.

Definition. A matrix M is irreductible iff for every i and j there is a path $i_0 = i, ...i_p = j$ such that $M_{i_k i_{k+1}} \neq 0$.

Note that in our example with $Q = \Delta - A$, Q has connected strong substitutes if and only if $M = \Delta^{-1} A$ is irreductible.

We have seen that Jacobi converges if and only if $\rho\left(\Delta^{-1}A\right) < 1$.
$\Delta^{-1}A$ being a matrix with nonnegative components, we need a result on spectrum of nonnegative matrices. The Perron-Froebenius applies to that.
Theorem (BT Prop. 6.6). Let M be a $n \times n$ matrix with nonnegative terms with is irreductible. Then:

- $\rho(M)$ is an eigenvalue of M, and there exists a associated right eigenvector v with positive entries (that is, there exists v such that $Mv = \rho(M)\, v$ and $v_i > 0$ for all i).
- v above is (up to rescaling) the only eigenvector of M with positive entries. It is the so-called left Perron eigenvector.
- The rank of $M - \rho(M)\, I$ is $n - 1$.
- Furthermore, considering $|z|_v^\infty = \max\{|z_i/v_i|\}$, and denoting $\|M\|_v^\infty$ the matrix norm induced by that norm, one has

$$\rho(M) = \|M\|_v^\infty .$$

- Let M be a $n \times n$ matrix. This is viewed as the matrix of Markov transitions of a Markov chain on state space$=\{1, ..., n\}$, where M_{ij} is the probability of visiting i at next step conditional on being at j at the current step. We impose therefore that $M_{ij} \geq 0$, and $\sum_i M_{ij} = 1$.
- M is irreductible means that for any $i \neq j$ there is a k such that $\left(M^k\right)_{ij} > 0$ meaning that if you wait long enough, you have a positive probability of visiting every state conditional on being in any state.

- Because $M^{\top}1_n = 1_n$, 1 is an eigenvalue of $M^{\top}$ with associated eigenvector 1_n. By Perron-Froebenius, this implies that 1 is the largest eigenvalue of $M^{\top}$ hence of M, that is $\rho(M) = 1$.
- By Perron-Froebenius again, M has an eigenvector with positive components, call it $\pi_i > 0$ associated with eigenvalue 1. This means

$$M\pi = \pi$$

  and we can impose $\sum_i \pi_i = 1$.
- That is

$$\sum_j M_{ij}\pi_j = \pi_j$$

  hence π can be interpreted as the stationary distribution of the Markov chain.

Definition. A matrix M is *nonreversing* if $\delta \geq 0$ and $M\delta \leq 0$ imply that $\delta = 0$.

Remarks:

- "Nonreversing" is not a standard terminology. In Tsatsomeros' terminology, it is equivalent with "$-M^\top$ is not semipositive".

Definition. A M-matrix is a Z-matrix which is nonreversing.

The Twenty Equivalence theorem. (BP theorem 4.6). Assume M is a Z-matrix. Then the following statements are equivalent to "M is a M-matrix":
(1) M^{-1} is entrywise positive
(2) Jacobi converges from any starting point
(3) $\rho\left(\Delta^{-1}A\right) < 1$.
(4) There exists a vector $w_i > 0$ such that $diag\left(w\right) M$ is diagonally dominating.
plus over 17 equivalences...

- As $\rho(M) = \rho(M^\top)$, the result of the Perron-Froebenius theorem can be applied to $M^\top$, and there is a left eigenvector u with positive entries such that $M^\top u = \rho(M)\, u$.
- What does this entails economically?
- Consider $M = \Delta^{-1}A$ and set $\lambda = \rho(\Delta^{-1}A)$. We have $A^\top \Delta^{-1} u = \lambda u$, and therefore, setting $w = \Delta^{-1}u$, we have

$$A^\top w = \lambda \Delta w$$

and hence

$$w^\top Q = w^\top (\Delta - A) = (1 - \lambda)\, w^\top \Delta$$

- This means that

$$\sum_i w_i Q_{ij} = (1 - \lambda)\, w_j Q_{jj}$$

which implies that the matrix $diag(w)\, Q$ is diagonally dominating.
- This implies the *weighted law of aggregate* supply:

$$\sum_i w_i (Qp)_i = (1 - \lambda) \sum_i w_i Q_{ii} p_i$$

is a increasing function in each of the p_i.