Matrix groups & their homogenous spaces

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Overview

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- Adjoint representation.
- Ad is a 2-fold cover of $\mathrm{SO}(3)$.

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- Ad is a 2-fold cover of $\mathrm{SO}(3)$.

2: Centre via Maximal Tori

- Study of the structure using maximal torus decomposition.

3: Homogeneous Spaces and Quotients

- Properties of homogeneous spaces.
- Identification of classical manifolds as quotients of matrix groups.

Preliminaries: Definitions and Notations

Quaternions: The set of quaternions, denoted by $\mathbb{H}$, is a 4-dimensional non-commutative algebra over $\mathbb{R}$ defined as:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

with basis $\{1, i, j, k\}$ satisfying the multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k, \quad jk = i, \quad ki = j,$$

$$ji = -k, \quad kj = -i, \quad ik = -j.$$

Definition: The set $\mathbb{H}^n$ is a *left vector space* over $\mathbb{H}$, defined as:

$$\mathbb{H}^n = \left\{ \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \;\middle|\; q_i \in \mathbb{H} \text{ for all } i = 1, \ldots, n \right\}$$

with vector addition defined component-wise and scalar multiplication given by:

$$\lambda \cdot \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} \lambda q_1 \\ \vdots \\ \lambda q_n \end{bmatrix}, \quad \text{for } \lambda \in \mathbb{H}.$$

Note: Scalar multiplication is from the left, as $\mathbb{H}$ is non-commutative.

Definition:

A matrix group is a subgroup $G \subseteq GL_n(\mathbb{K})$ *that is closed in* $GL_n(\mathbb{K})$.
where, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

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Notations:

- $SL_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K}) : \det(A) = 1\}$ where $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$.
- $SO(n) = SL_n(\mathbb{R}) \cap O(n)$
- $SU(n) = SL_n(\mathbb{C}) \cap U(n)$
- $O_n(\mathbb{H})$ or $Sp(n) = \{A \in M_n(\mathbb{H}) \ : \ AA^* = A^*A = I\}$

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Definition:

Let $G \subseteq GL_n(\mathbb{K})$ be a matrix group, and let $A \in G$. The tangent space to G at A is:

$$T_A(G) = \{\gamma'(0) \mid \gamma : (-\epsilon, \epsilon) \to G \text{ is differentiable with } \gamma(0) = A\}.$$

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The Lie algebra of a matrix group $G \subseteq GL_n(\mathbb{K})$ is the tangent space to G at the identity I. It is denoted by $\mathfrak{g}(G)$.

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Theorem:

Matrix groups are smooth manifolds.

Adjoint Representation and Fundamental Group of SO(3)

Let G be a matrix group with Lie algebra $\mathfrak{g}$. For all $g \in G$, the conjugation map $C_g : G \to G$ is defined as:

$$C_g(a) = gag^{-1}.$$

This is a smooth isomorphism. The derivative of C_g at I is a vector space isomorphism, which we denote as Ad_g: $\mathrm{Ad}_g = d(C_g)_I$.

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$$\mathrm{Ad}_g(B) = d(C_g)_I(B) = \left.\frac{d}{dt}\right|_{t=0} gb(t)g^{-1} = gBg^{-1}.$$

Let G be a matrix group of dimension d with Lie algebra $\mathfrak{g}$. The adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a vector space isomorphism for each $g \in G$. Once we choose a basis of $\mathfrak{g}$, this isomorphism can be represented by a matrix in $GL_d(\mathbb{R})$. In other words after fixing a basis of $\mathfrak{g}$, we can regard the map $g \to \mathrm{Ad}_g$ as a function from G to $GL_d(\mathbb{R})$.

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Proposition:

If G is a subgroup of $O_n(\mathbb{K})$, then for all $g \in G$ and all $X \in \mathfrak{g}$,

$$|\mathrm{Ad}_g(X)| = |X|,$$

where $|\cdot|$ denotes the Euclidean norm on $M_n(\mathbb{K})$.

Now we study the adjoint representation of $\mathrm{Sp}(1)$:

$$\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{O}(3).$$

Since $\mathrm{Sp}(1)$ is path-connected, $\mathrm{Ad}(\mathrm{Sp}(1))$ is also path-connected, so we in fact have a smooth homomorphism:

$$\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{SO}(3).$$

Our goal is to prove that $\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is surjective, 2-to-1, and a local diffeomorphism.

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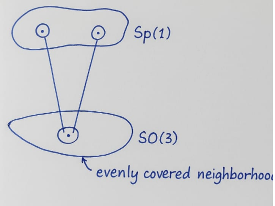

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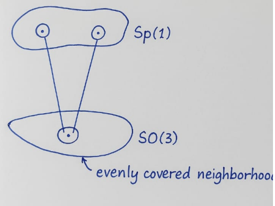

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We know that the number of sheets of a covering space

$$p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$$

with X and $\tilde{X}$ path-connected equals the index of $p^*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$, where p^* is the induced homomorphism from p.

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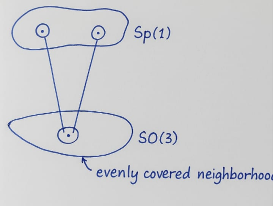

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Here $\mathrm{Sp}(1)$ is simply connected, and the number of sheets is equal to 2. Thus: $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$.

Maximal torus and centre of some matrix groups

Torus:

n-dimensional torus T^n is the group
$T^n = U(1) \times U(1) \times \cdots \times U(1)$ (n copies)

Note: $T^n = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) : \theta_i \in [0, 2\pi]\} \subseteq GL_n(\mathbb{C})$.

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Definition:

Let G be a matrix group. By a torus in G we mean a subgroup of G that is isomorphic to a torus.
A *maximal torus* in G means a torus in G that is not contained in a higher-dimensional torus in G.

Theorem:

Let $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Then each of the following is a maximal torus:

$T = \{\mathrm{diag}(R_{\theta_1}, \ldots, R_{\theta_m}) : \theta_i \in [0, 2\pi]\} \subseteq SO(2m)$.

$\mathrm{T} = \{ \mathrm{diag}(\mathrm{R}_{\theta_1}, \ldots, R_{\theta_m}, 1) : \theta_i \in [0, 2\pi]\} \subseteq SO(2m+1)$.

$\mathrm{T} = \{ \mathrm{diag}(\mathrm{e}^{i\theta_1}, \ldots, e^{i\theta_n}) : \theta_i \in [0, 2\pi]\} \subseteq U(n)$.

$\mathrm{T} = \{ \mathrm{diag}(\mathrm{e}^{i\theta_1}, \ldots, e^{i\theta_n}) : \theta_i \in [0, 2\pi]\} \subseteq Sp(n)$.

$\mathrm{T} = \{ \mathrm{diag}(\mathrm{e}^{i\theta_1}, \ldots, e^{i\theta_{n-1}}, e^{-i(\theta_1+\cdots+\theta_{n-1})}) : \theta_i \in [0, 2\pi]\} \subseteq SU(n)$.

Proposition:

Let $G \in \{SO(n), U(n), SU(n), Sp(n)\}$, and let T be the standard maximal torus of G. Then any element of G that commutes with every element of T must lie in T.

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Theorem:

1. $Z(SO(2m)) = \{I, -I\}$ if $m > 1$.
2. $Z(SO(2m+1)) = \{I\}$.
3. $Z(U(n)) = \{e^{i\theta} I \mid \theta \in [0, 2\pi]\}$.
4. $Z(Sp(n)) = \{I, -I\}$.
5. $Z(SU(n)) = \{\omega I \mid \omega^n = 1\}$.

Coset spaces and Homogeneous Manifolds

Theorem:

Let G be a matrix group and $H \subset G$ a closed subgroup. Let $\mathfrak{h} \subset \mathfrak{g}$ denote their Lie algebras.

Define $p = \mathfrak{h}^{\perp} = \{v \in \mathfrak{g} \mid \langle v, x \rangle_{\mathbb{R}} = 0, \quad \forall x \in \mathfrak{h}\}$ and $p_{\varepsilon} = \{v \in p \mid |v| < \varepsilon\}$.

For every $g \in G$, define the parametrization $\varphi_g : p_{\varepsilon} \to G/H$ as: $\varphi_g(x) = [g \cdot \exp(x)]$.

If ε is sufficiently small, the family of parametrizations $\{\varphi_g : g \in G\}$ determines a manifold structure on the coset space G/H; that is, they are injective and they satisfy the compatibility condition for a manifold.

Proposition:

Define $\pi : G \to G/H$ such that $\pi(g) = [g]$ for all $g \in G$. Where, H is a closed subgroup of G. Then:

(i) π is **smooth**.

(ii) When M is a manifold, a function $f : G/H \to M$ is smooth if and only if $f \circ \pi$ is smooth.

Definition:

A (left or right) action of a group G on a set M is a function $\varphi : G \to \{\text{the set of bijections from } M \text{ to } M\}$ such that for all $g_1, g_2 \in G$,

- (Left action) $\quad \varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$
- (Right action) $\quad \varphi(g_1 g_2) = \varphi(g_2) \circ \varphi(g_1)$

An action of a matrix group G on a manifold M is called **smooth** if the function $G \times M \to M, \quad (g, p) \mapsto \varphi(g)(p)$ is smooth.

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Definition:

A manifold M is called homogeneous if there exists a transitive smooth action of a matrix group on M.

Lemma:

Let φ be a transitive left action of a group G on a set M. Let $p_0 \in M$, and let H be the stabilizer of p_0. Then the function $F : G/H \to M$ defined as $F([g]) = \varphi(g)(p_0)$ is a well-defined bijection.

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Lemma:

In a previous lemma, if M is a manifold and G has only countably many connected components and φ is smooth, then F is a diffeomorphism.

When $H \subset GL_n(\mathbb{K})$ is a subgroup, we will denote:
$S(H) = \{M \in H : \det(M) = 1\}$.
For example, $S(O(n)) = SO(n)$ and $S(U(n)) = SU(n)$.

In each of the following, $n > 1$ and "$=$" means the manifold is diffeomorphic to the coset space:

1. $\mathbb{R}P^n = O(n+1)/(\mathbb{Z}_2 \times O(n))$
   $=$ SO(n+1) / S($Z_2 \times O(n)$)
2. $\mathbb{C}P^n = U(n+1)/(U(1) \times U(n))$
   $=$ SU(n+1) / S(U(1) $\times U(n)$)
3. $\mathbb{H}P^n = Sp(n+1)/(Sp(1) \times Sp(n))$
4. $S^n = O(n+1)/(\{1\} \times O(n))$
   $=$ SO(n+1) / ($\{1\}$ $\times SO(n)$)
5. $S^{2n+1} = U(n+1)/(\{1\} \times U(n))$
   $=$ SU(n+1) / ($\{1\}$ $\times SU(n)$)
6. $S^{4n+3} = Sp(n+1)/(\{1\} \times Sp(n))$.

References

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- Brian C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, Graduate Texts in Mathematics, Vol. 222, Springer, 2nd Edition, 2015.

Thank You!