Pontryagin Duality and Self-Dual Groups

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contents

Recall

Theorem: G be a topological group,
$\mathscr{U} = \{U : U \text{ is open } \& \ e \in U\}$ is a neighborhood system at e, then it satisfies following conditions,

1. for every $U \in \mathscr{U}$, there is an $V \in \mathscr{U}$ such that $V^2 \in \mathscr{U}$.

2. for every $U \in \mathscr{U}$, there is an $V \in \mathscr{U}$ such that $V^{-1} \in \mathscr{U}$.

3. for every $U \in \mathscr{U}$ and for every $x \in U$, there is an $V \in \mathscr{U}$ such that $xV \subset U$.

4. for every $U \in \mathscr{U}$ and $x \in G$, there is an $V \in \mathscr{U}$ such that $xVx^{-1} \subset U$.

Theorem: Let G be a group with identity e. If $\mathscr{U}$ be a collection of subsets of G, each containing e, satisfying the conditions given above and for every U_1, U_2 in $\mathscr{U}$ there exists U_3 in $\mathscr{U}$ such that $U_3 \subseteq U_1 \cap U_2$. Then the collection $\mathscr{B} = \{xU : x \in G \text{ and } \ U \in \mathscr{U}\}$ forms a basis for a topology with respect to which G forms a topological group and $\mathscr{U}$ is the neighborhood system at e.

Characters and Dual Groups

Definition

Let G be an locally compact abelian topological group. Every continuous homomorphism from G to $\mathbb{T}$ is called character of the group G.

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Where $\mathbb{T} = \{z : |z| = 1\}$.

$\widehat{G} = \{\chi : G \to \mathbb{T} \ : \ \chi(x_1 x_2) = \chi(x_1)\chi(x_2) \ \text{ and } \ \chi \ \text{ is continuous } \}.$

Multiplication on $\widehat{G}$ is defined by $\chi_1\chi_2(x) = \chi_1(x)\chi_2(x)$.
With respect to this multiplication operation $\widehat{G}$ forms an abelian group. For any, $\chi \in \widehat{G}$, $\chi^{-1}(x) = \overline{\chi(x)}$ where, $x \in G$ and the identity character is defined as 1. The character group $\widehat{G}$ is a subspace of $C(G, \mathbb{T})$ and so it derives compact open topology.

Theorem

$\widehat{G}$ is a topological group with respect to the topology generated by $\left\{\chi \cdot N(K, V_\epsilon) : N(K, V_\epsilon) \in \mathscr{U} \text{ and } \chi \in \widehat{G}\right\}$.

Where, $\epsilon > 0$, $V_\epsilon = \{z \in \mathbb{T} : |z - 1| < \epsilon\}$,

$N(K, V_\epsilon) = \{\chi \in \widehat{G} : |\chi(x) - 1| < \epsilon \text{ for all } x \in K\}$.

K is a compact set in $\widehat{G}$ and
$\mathscr{U} = \{N(K, V_\epsilon) : K \text{ runs over compact subsets of } \widehat{G} \text{ and } \epsilon > 0\}$.

Properties of Character group

Theorem

If G is locally compact abelian group, then $\widehat{G}$ forms a locally compact abelian group.

Corollary

1. *If G is compact then $\widehat{G}$ is discrete.*
2. *If G is discrete then $\widehat{G}$ is compact.*

Theorem

If G is locally compact abelian and second countable, then $\widehat{G}$ is also second countable.

Theorem

Suppose that G_1 and G_2 are locally compact and 2^{nd}-countable topological groups. Then any surjective homomorphism is an open map.

Examples

1. The character group of $\mathbb{Z}$ is $\mathbb{T}$.
2. The character group of $\mathbb{R}$ is $\mathbb{R}$ itself.

Pontryagin Duality

For an fixed x in G, where G is locally compact abelian group and, $\widehat{G}$ is the character group of G.
Define a map, $\Gamma_x : \widehat{G} \to \mathbb{T}$ by $\Gamma_x(\chi) = \chi(x)$, for $\chi \in \widehat{G}$.

Proposition

1. *The above map Γ_x is well defined and continuous homomorphism from $\widehat{G}$ to $\mathbb{T}$.*
2. *Define a map $\Phi : G \to \widehat{\widehat{G}}$ by $\Phi(x) = \Gamma_x$ for x in G, is a continuous homomorphism.*

Theorem

Let G be compactly generated, locally compact second countable abelian group, then the mapping $\Phi : G \to \widehat{\widehat{G}}$ is a topological isomorphism.

The Correspondence between dual group of subgroups and quotient groups of the dual

Definition

Let G be a locally compact abelian group with character group $\widehat{G}$. Let H be an arbitrary non-empty subset of G. $A(\widehat{G}, H)$ is the subset of $\widehat{G}$ consisting of all $\chi \in \widehat{G}$ such that $\chi(h) = 1$ for all h in H. $A(\widehat{G}, H)$ is called the *annihilator* of H in $\widehat{G}$.

Theorem

Let G be a locally compact abelian group with character group $\widehat{G}$, and let H be a closed subgroup of G. Let Y be the character group of the locally compact abelian group G/H. Then the group Y is topologically isomorphic with the group $A(\widehat{G}, H)$.

Example of Self Dual group

Let $\mathfrak{a}$ be a fixed but arbitrary double infinite sequence of positive integers,

$$\mathfrak{a} = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\}, \quad \text{where each} \quad a_n > 1.$$

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Consider the Cartesian product $\prod_{n\in\mathbb{Z}} \{0, 1, \dots, a_n - 1\} = \prod_{n\in\mathbb{Z}} \mathbb{Z}_{a_n}$.

$$\Omega_{\mathfrak{a}} = \{x \in \prod_{n\in\mathbb{Z}} \mathbb{Z}_{a_n} : x_n = 0, \ \forall n < n_0\}, \ \text{where} \quad n_0 \quad \text{depends on} \quad x.$$

We define **addition** on $\Omega_{\mathfrak{a}}$ as follows :

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We define **addition** on $\Omega_{\mathfrak{a}}$ as follows :
Let, $x, y \in \Omega_{\mathfrak{a}}$ and let n_0 and m_0 are the least integer such that $x_{n_0} \neq 0$ and $y_{m_0} \neq 0$.
Let $p_0 = \min\{n_0, m_0\}$ and $z_n = 0$ for all $n < p_0$.

Then $x_{p_0} + y_{p_0} = a_{p_0} t_{p_0} + z_{p_0}$, where $z_{p_0} \in \{0, 1 \ldots, a_{p_0} - 1\}$ and t_{p_0} is a integer.
Then the next sum is, $x_{p_0+1} + y_{p_0+1} + t_{p_0} = a_{p_0+1} t_{p_0+1} + z_{p_0+1}$, where $z_{p_0+1} \in \{0, 1 \ldots, a_{p_0+1} - 1\}$ and t_{p_0+1} is an integer.

Proceeding similar manner the get the sum of $x + y$ to be the sequence $z = (z_n)$ in $\Omega_{\mathfrak{a}.}$
Define, $0 + x = x + 0 = x$, $\forall\ x \in \Omega_{\mathfrak{a}}$, where 0 is the sequence in $\Omega_{\mathfrak{a}}$, which is identically zero.

Definition

$\Omega_{\mathfrak{a}}$ with the above addition is called $\mathfrak{a}$-adic numbers. The subset

$$\Delta_0 = \{x \in \Omega_{\mathfrak{a}} : x_n = 0, \ \forall n < 0\}$$

of $\Omega_{\mathfrak{a}}$, with respect to the addition is called $\mathfrak{a}$-adic integers.

If all the integers a_n are equal to some fixed integer $r > 1$, we write $\Omega_{\mathfrak{r}}$ and $\Delta_{\mathfrak{r}}$ and called $\mathfrak{r}$-adic numbers and $\mathfrak{r}$-adic integers.

Theorem

The $\mathfrak{a}$-adic numbers $(\Omega_{\mathfrak{a}})$ is an ablian group with respect to the addition and Δ_0 is a subgroup of $\Omega_{\mathfrak{a}}$.

Definition

For each integer k, let $\Lambda_k = \{x \in \Omega_\mathfrak{a} : x_n = 0 \ \forall n < k\}$.
For distinct elements $x, y \in \Omega_\mathfrak{a}$, let $\sigma(x, y) = \frac{1}{2^m}$, where m is the least integer for which $x_m \neq y_m$ and for all $x \in \Omega_\mathfrak{a}$, and let $\sigma(x, x) = 0$.

Theorem

The collection $\mathscr{U} = \{\ldots, \Lambda_{-k}, \ldots, \Lambda_{-1}, \Lambda_0, \Lambda_1, \ldots, \Lambda_k, \ldots\}$ *satisfy the conditions of neighborhood system. Then they defined a topology on* $\Omega_\mathfrak{a}$ *under which* $\Omega_\mathfrak{a}$ *is an topological group. The sets* Λ_k *are compact open subgroups of* $\Omega_\mathfrak{a}$*.* $\Omega_\mathfrak{a}$ *is Hausdorff, locally compact and* σ*-compact and totally disconnected. The function* σ *is an invariant metric on* $\Omega_\mathfrak{a}$*.*

Lemma

Let, u be an element of Λ_k such that $u_n = 0\ \forall n \neq k$ and $u_k = 1$. Then the set, $\{lu\}_{l=-\infty}^{\infty}$ is a dense subgroup of Λ_k.

Theorem

The character group of $\Omega_{\mathfrak{a}}$ is topologically isomorphic with $\Omega_{\mathfrak{a}^*}$, where $a_n^* = a_{-n}$ for all $n \in \mathbb{Z}$.
The character group of $\Omega_{\mathfrak{r}}$ is itself.

Theorem

The character group of $\mathfrak{a}$-adic integers (Δ_0) is isomorphic to the group $\mathbb{Z}(\mathfrak{a}^\infty)$.
Where, $\mathbb{Z}(\mathfrak{a}^\infty) = \{\exp(2\pi i(\frac{l}{a_0 a_1 \ldots a_n})) : l \in \mathbb{Z}, n \in \mathbb{N}\}$.

Solenoid and their Dual

Definition

Consider $\mathbb{R} \times \Delta_0$, the additive locally compact group. $u = (u_n)$ be an element in Δ_0 such that $u_n = 0$, for all $n \neq 0$ and $u_0 = 1$. Let $B = \{(n, nu)\}_{n=-\infty}^{\infty}$ be the subgroup of $\mathbb{R} \times \Delta_0$. Consider the quotient group $\mathbb{R} \times \Delta_0 / B$, we call this group $\mathfrak{a}$-adic solenoid and denote it as $\Sigma_{\mathfrak{a}}$.

Theorem

The group $\Sigma_{\mathfrak{a}}$ is compact, connected, abelian group containing a dense one-parameter subgroup.

Theorem

The dual group of $\mathfrak{a}$-adic solenoid $(\Sigma_{\mathfrak{a}})$ is the discrete additive group of rational $(\mathbb{Q}_d)$.
Where $\mathbb{Q}_d = \{\frac{m}{a_0 a_1 \ldots a_n} : \; n = 0, 1, 2, \ldots \; ; \; m \in \mathbb{Z}\}$.

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Thank You!