Optimization of Plate Fin Heat Sinks

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Objectives

Objective: Minimizing entropy generation to enhance the heat sink performance.

1. Derived conservation equations using RTT
2. Developed 1D/2D fin models with analytical solutions
3. Optimized parameters using Newton-Raphson method

Introduction

- Heat sinks remove heat from electronic devices to maintain performance and longevity.
- The fins increase the surface area of the heat sink, but also the pressure drop.
- Entropy Generation Minimization (EGM) balances thermal performance and pumping power.

Reynolds Transport Theorem (RTT)

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t} \int_{CV} \beta\rho \, dV + \int_{CS} \beta\rho(\mathbf{V} \cdot \mathbf{n}) \, dA$$

Applications:

- Mass conservation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$
- Momentum conservation: $\rho \frac{DV}{Dt} = -\nabla p + \mu \nabla^2 V + \mathbf{f}$ (Navier-Stokes)
- Energy conservation: $\rho C_p \frac{DT}{Dt} = k \nabla^2 T + \Phi$

Heat Transfer Rate

Conduction

Fourier's Law

$$q = -kA\frac{dT}{dx}$$

Convection

Newton's Law

$$q = hA(T_s - T_\infty)$$

Radiation

Stefan-Boltzmann Law

$$q = \epsilon\sigma AT^4$$

1D Fin Analysis

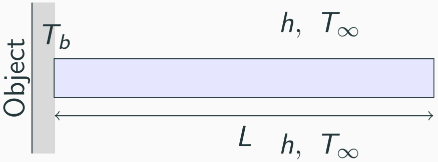


Figure 1: Diagram of a single rectangular fin attached to a heated object.

Governing equation:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \quad m = \sqrt{\frac{hP}{kA}} \tag{1}$$

Solutions for infinite long fin: $T(x) = T_\infty + (T_b - T_\infty)e^{-mx}$

2D Fin Analysis

- Laplace equation: $\nabla^2 T = 0$
- Boundary conditions:

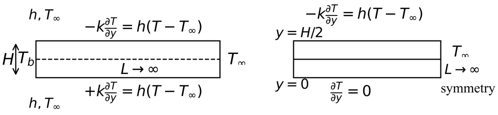


Figure 2: 2D infinite long fin

Solution:

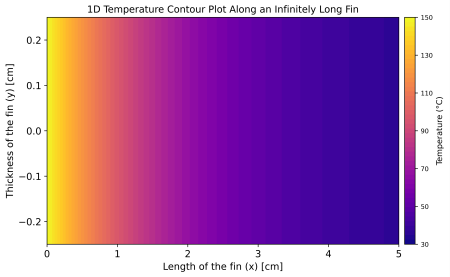
$$T(x, y) = T_\infty + \sum_{n=1}^{\infty} 2(T_b - T_\infty) \frac{Sin(a_n)}{a_n + Sin(a_n)Cos(a_n)} e^{-\lambda_n x} Cos(\lambda_n y).$$

Temperature Distribution

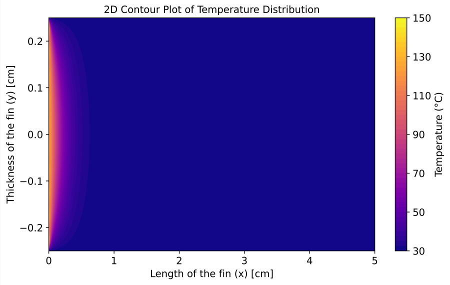
1D: $T(x) = T_\infty + (T_b - T_\infty)e^{-mx}$

2D:

$$T(x,y) = T_\infty + \sum_{n=1}^{\infty} 2(T_b - T_\infty)\frac{Sin(a_n)}{a_n + Sin(a_n)Cos(a_n)} e^{-\lambda_n x} Cos(\lambda_n y)$$



1D Temperature distribution



2D Temperature distribution

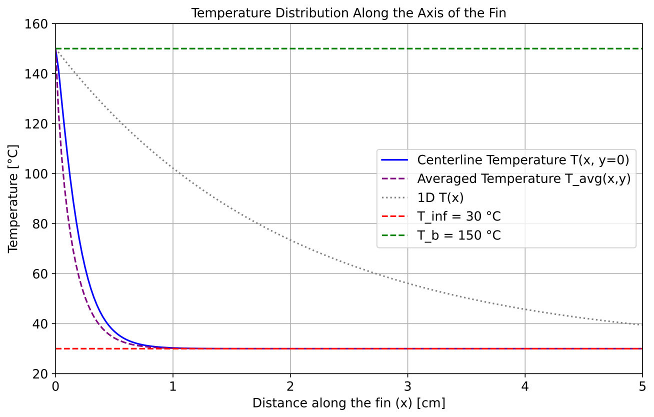


Figure 3: Temperature along the axis

- The temperature distribution along a one-dimensional fin exhibits an exponential decay.
- 2D fin accurately gives temperature distribution across the fin.

Fin Effectiveness & Efficiency

- Fin effectiveness is defined as the ratio of the heat transfer rate from the fin to the heat transfer rate from the base without the fin: $\varepsilon = \frac{q_f}{hA_{c,b}\theta_b}$
- Fin efficiency is the ratio of the actual heat transfer rate from the fin to the ideal heat transfer rate from the fin if the entire fin were at base temperature: $\eta_f = \frac{q_f}{hA_f\theta_b}$
- Optimization criteria: $\varepsilon > 1$, $\eta_f \to 1$

Biot Number (Bi)

Definition: Ratio of internal conduction resistance to external convection resistance

$$Bi_{H/2} = \frac{hH}{2k}$$

Where:

- h: Convective heat transfer coefficient (W/m²K)
- H: Fin thickness (m)
- k: Thermal conductivity of fin material (W/mK)

Key Implications:

- $Bi \ll 1$: Uniform temperature (1D model valid)
- $Bi \approx 1$: Significant internal temperature gradients
- $Bi \gg 1$: Requires 2D/3D analysis

Design Rule: Use 2D models when $Bi > 0.1$

Fixed-Volume Optimization

For an adiabatic-tip fin, the heat transfer rate from the base is:

$$Q = \sqrt{hPkA}\,\theta_b \tanh(mL)$$

Volume constraint: $V = LWt$

Maximize

$$Q = \theta_b W \sqrt{2hk} \cdot \sqrt{t} \cdot \tanh\left(\sqrt{\frac{2h}{k}} \frac{V}{W} t^{3/2}\right)$$

To find optimal dimensions, take $\frac{dQ}{dt} = 0$

Optimal dimensions:

$$t_{\text{opt}} = \left(\frac{V}{W}\right)^{2/3} \left(\frac{h}{k}\right)^{1/3}$$

$$L_{\text{opt}} = \left(\frac{V}{W}\right)^{1/3} \left(\frac{k}{h}\right)^{1/3}$$

Heat Sink

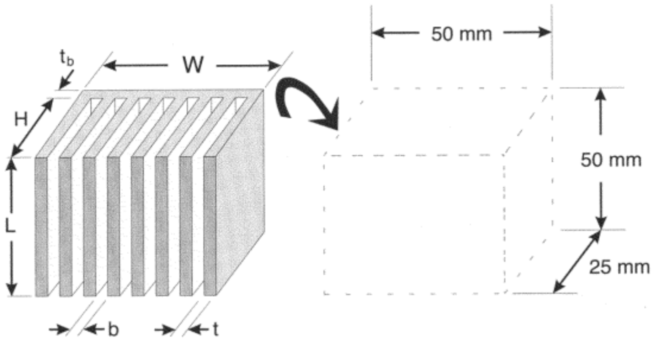


Figure 4: Heat Sink (from J. Culham and Y. Muzychka, "Optimization of plate fin heat sinks using entropy generation minimization,")

Pressure Drop Analysis

Heat Sink Pressure Drop equation is given by:

$$\Delta P = \frac{\rho V^2}{2} \left(4 f_{\text{app}} \frac{L}{D_h} + K_c + K_e \right)$$

P–V tables for $N = 10, 20, 30$ fins at heights $12.5, 25, 50mm$ gives:

- More fins $\rightarrow$ smaller gaps $\rightarrow$ higher velocity $\rightarrow$ higher pressure drop.
- Taller fins, $\rightarrow$ larger channel area $\rightarrow$ lower velocity $\rightarrow$ lower pressure drop.

Entropy Generation Minimization

Total entropy generation of a heat sink:

$$\dot{S}_{\text{gen}} = \frac{Q^2 R_s}{T_\infty^2} + \frac{F_d V_f}{T_\infty}$$

Where:

- R_s = Thermal resistance
- F_d = Drag force
- The first term represents entropy generation due to heat transfer across a temperature difference.
- The second term represents the entropy generation as a result of viscous dissipation from fluid flow.

Optimization Algorithm

Newton-Raphson method:

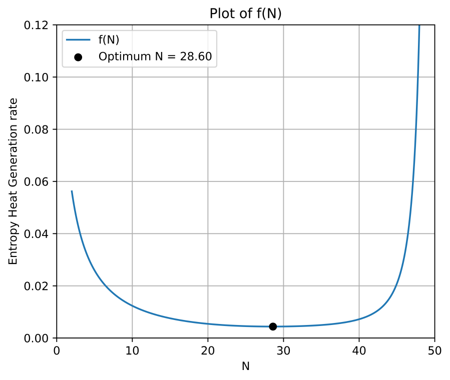
1. Define $\dot{S}_{\text{gen}}(\mathbf{x})$
2. $g_i(\mathbf{x}) \equiv \frac{\partial \dot{S}_{gen}}{\partial x_i} = 0.$
3. Compute Jacobian $J_{ij} = \frac{\partial^2 \dot{S}_{\text{gen}}}{\partial x_i \partial x_j}$
4. Starting from an initial guess $\mathbf{x}^{(0)}$.
   At iteration k: $J(\mathbf{x}^{(k)})\,\delta\mathbf{x} = -\,\mathbf{g}(\mathbf{x}^{(k)}), \quad \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \delta\mathbf{x}.$

Stop when $\|\mathbf{g}\|$ or $\|\delta\mathbf{x}\|$ falls below a prescribed tolerance. Final $\mathbf{x}^{(k+1)}$ minimizes the $\dot{S}_{\text{gen}}$.

Case 1: Optimizing Fin Count (N)

$\dot{S}_{gen}(N)$ wrt N

N	$\dot{S}_{\text{gen}}$	θ_b	ΔP
10	0.0122	35.9	0.436
29	0.0043	11.3	5.574



$\dot{S}_{\text{gen}}$ vs N

- Optimal $N \approx 29$ (after 6 iterations it converges to 28.55)
- U-shaped entropy curve shows optimum solution lies in the range $20 \leq N \leq 35$.

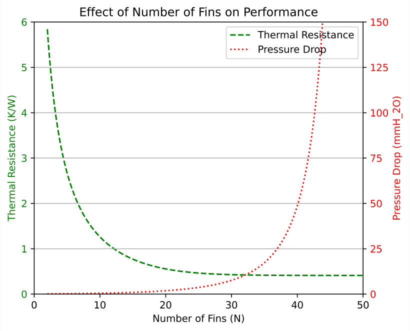


Figure 5: Heat Sink Resistance as a function of N

This plot shows the inverse relationship between heat sink resistance and pressure drop with respect to number of fins

Case 2: Optimizing N & V_f

$\dot{S}_{gen}(N)$ wrt N and V_f

N	V_f	$\dot{S}_{\text{gen}}$	θ_b	ΔP
25	3	0.0041	9.9	5.9
27	2.8	0.0040	9.5	6.923

- Optimum at $N \approx 27$ and $V_f \approx 2.8$ m/s. (converges after 4 iterations)
- Co-optimization reduces entropy

Case 3: Optimizing N, V_f, t

$\dot{S}_{gen}(N)$ wrt N, V_f and t

N	V_f	t	$\dot{S}_{\text{gen}}$	θ_b	ΔP
25	2	0.001	0.0045	12.61	3.4
38	3.3	0.0004	0.0037	8.62	5.82

- Optimal: $N = 38$, $t = 0.4$mm, $V_f = 3.3$m/s (converges after 7 iterations)
- Challenge: Manufacturing thin fins (t is too small)

Case 4: Optimizing N, V_f, H

$\dot{S}_{gen}(N)$ wrt N, V_f and H

N	V_f	H	$\dot{S}_{\text{gen}}$	θ_b	ΔP
25	2	0.025	0.0045	12.61	3.4
24.5	1.49	0.096	0.003	7.24	2.13

- Optimal: $N = 25$, $V_f = 1.5\text{m/s}$, $H = 96.6\text{mm}$ (impractical) (converges after 9 iterations)
- Demonstrates need for constraints

Case 5: Optimizing N, V_f, H, t

$\dot{S}_{gen}(N)$ wrt N, V_f, H and t

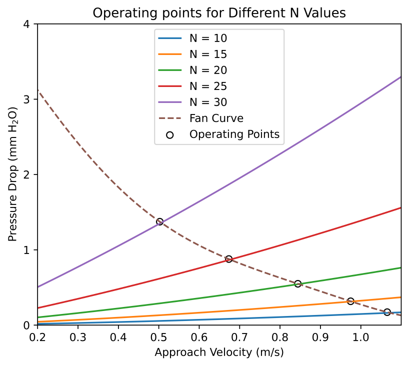
N	V_f	H	t	$\dot{S}_{\mathrm{gen}}$	θ_b	ΔP
25	2	0.025	0.001	0.0045	12.61	3.4
19	1.21	0.122	0.0016	0.0029	7.29	1.88

- Optimal: $N = 19$, $V_f = 1.2$m/s, $H = 122$mm, $t = 1.6$mm (converges after 12 iterations)
- This approach provides optimized heat sink.

Case 6: Optimizing (N) using fan curve

$\dot{S}_{gen}(N)$ wrt N using $V_f(N)$

N	$\dot{S}_{\text{gen}}$	θ_b	ΔP
15	0.0102	30.25	0.31
20	0.0089	26.34	0.55



$\dot{S}_{\text{gen}}$ vs N

- optimum: $N \approx 20$ (after 4 iterations)
- $V_f(N) = 1.119 + 6.758 \times 10^{-3}N - 1.345 \times 10^{-3}N^2 + 1.485 \times 10^{-5}N^3$.
  $\Delta P = 4.56 - 2.31\dot{\Theta} - 2.24\dot{\Theta}^2 + 2.68\dot{\Theta}^3 - 0.983\dot{\Theta}^4 + 0.123\dot{\Theta}^5$

Key Findings

- Boundary conditions and thermal properties are crucial.
- Fins are crucial in improving the thermal performance in various systems.
- Optimization balances heat transfer efficiency and material usage.
- Increasing the number of fins (N) improves heat transfer but increases pressure drop (ΔP).
- Matching the heat sink's pressure drop versus flow rate curve with the fan's characteristic curve gives the operating point, where airflow and pressure drop are balance
- Entropy minimization balances thermal/fluidic losses
- Multi-variable optimization reduces $\dot{S}_{\text{gen}}$.

- Optimal fins improve cooling and reduce power consumption.
- Practical constraints are important for the manufacture of heat sinks, because taller fins (H) reduce ΔP but exceed spatial constraints. Also manufacturing thin fins will be difficult.
- Plots and tables (Temperature distribution and ΔP) gives an idea of complex trade-offs

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Thank you