An Exploration of Foundational Graph Theory

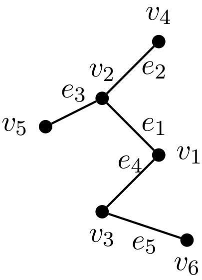
CH V S M Sriram
212305021

May 13, 2025

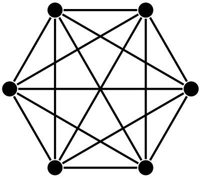
Introduction

Graph $G = (V, E)$

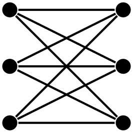
$V(G)$ or V: Vertex Set
$E(G)$ or E: Edge Set



Tree

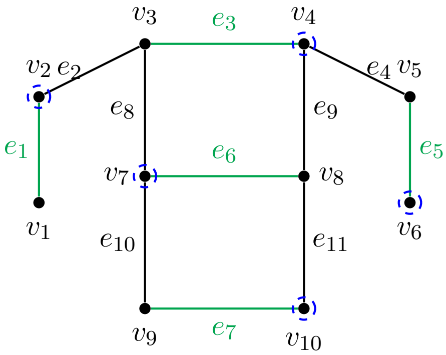


K_6



$K_{3,3}$

Matching



Matching

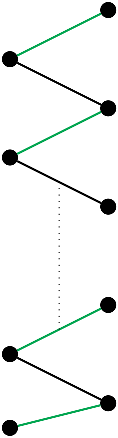
Collection of independent edges in a graph $G = (V, E)$ is called a matching.

$$M = \{e_1, e_3, e_5, e_6, e_7\}$$

$M \subseteq E$ is a matching of G if-

$$\forall v \in V,\ |\{e \in M | v \text{ is incident on } e \in E\}| \leq 1.$$

Alternating Path



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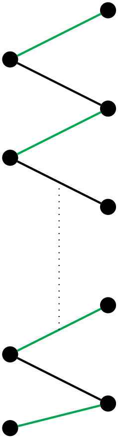
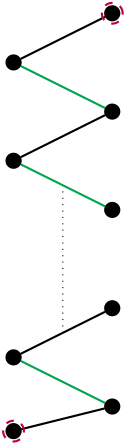


Augmenting Path



Matching

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$S \subseteq V(G)$ is a vertex cover of G(of the edges of the G) if every edge of G is incident with a vertex in S.

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Goal

To establish a relation between Maximum matching and Minimum Vertex Cover in a bipartite graph.

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- Cardinality of a vertex cover cannot be less than maximum matching.
- We will prove that $\exists$ a vertex cover S with $|S| = \alpha'(G)$ (Cardinality of maximum matching)
- For each $e \in M$ if there exists an alternating path that ends in B then take that last vertex ends in B into the vertex cover U else take the vertex in A into vertex cover U.

Matching in General Graph

Tutte

A graph has a 1-factor iff $o_c(G - S) \leq |S| \;\; \forall S \subseteq V(G)$.
$o_c(G)$ denote the number of its odd component i.e. those with odd order.

Packing/ Covering

H-packing

H-packing in G is a collection $V_1, V_2 \ldots V_l$ of vertex-disjoint subgraphs of G, each V_i is isomorphic to H.
H-packing number p_H is the maximum number of such disjoint subgraphs.

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Lower Bound on c_H

$$p_H \leq c_H$$

Can we have an upper bound on c_H as a function of packing number?

Erdős-Pósa Theorem

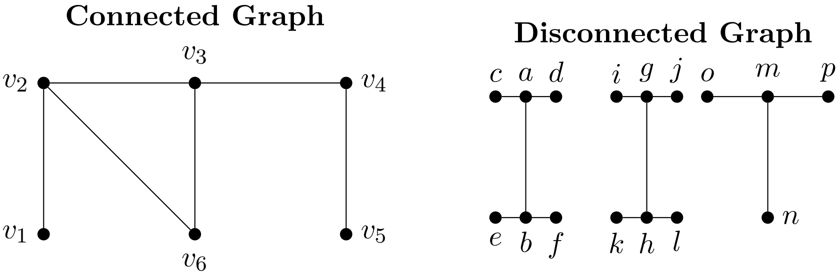
Erdős-Pósa Property

$\mathcal{F}$ be Family of Graphs. Then $\mathcal{F}$ has "Erdős-Pósa property" if $\exists$ a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for given $k \in \mathbb{N}$ and each G,either G has k-vertex disjoint subgraphs; each containing a copy of graph in $\mathcal{F}$ or there exists a set $U \subseteq V$ of cardinality at-most $f(k)$ such that there is no graph of $\mathcal{F}$ in $G - U$.

Connectivity

Connected Graph

A non-empty graph is said to be connected if there exists a path between any two vertices of the graph.



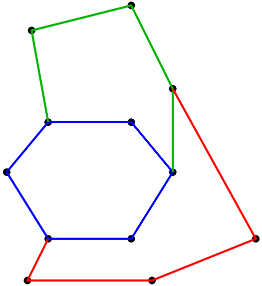
k-Connected Graph

k-Connected Graph

A graph $G = (V, E)$ is said to be k-connected if $|G| \geq k$ $(k \in \mathbb{N} \cup \{0\})$ and $G - X$ is connected for all $X \subseteq V$ with $|X| < k$.

2-Connected Graph Construction

Ear Decomposition



A graph is 2-connected iff it can be constructed starting from a cycle by successively adding H-paths to the existing graph.

3-connected Graph Construction

3-Connectivity-Preserving Operations

$G \dot{-} e$: Removing the edge e from the graph and suppressing any vertex that has a degree 2.
G/e : Contracting both endpoints of an edge e into a new vertex

3-connected Graph Construction (contd.)

Method 1

A graph G is 3-connected iff there exists a sequence

$$G_0, \ldots, G_n$$

of graphs such that G_{i+1} has an edge e such that $G_i = G_{i+1}e \;\; \forall i < n$ with $G_0 = K^4$ and $G_n = G$. Every graph $G_0, \ldots, G_n$ in such a sequence is 3-connected.

3-connected Graph Construction(Contd.)

Method 2

A graph G is 3-connected iff there exists a sequence

$$G_0, \ldots, G_n$$

of graphs such that G_{i+1} has an edge $e = xy$ such that $d(x), d(y) \geq 3$ and $G_i = G_{i+1}/e \;\; \forall i < n$ with $G_0 = K^4$ and $G_n = G$.

Every graph $G_0, \ldots, G_n$ in such a sequence is 3-connected.

Menger's Theorem

G be a graph and $A, B \subseteq V(G)$. Then the minimum number of vertices seperating A fron B in G is equal to the maximum number of internally disjoint $A - B$ paths in G.

- Proof by induction of number of edges $(= m)$.

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- Proof by induction of number of edges $(= m)$.
- Let a minimum $u - v$ seperating set has $k > 1$ vertices. Then G has at-most k internally disjoint $u - v$ paths.
- It remain to show it contains exactly k such paths. For $k = 1$ result is immediate.

Menger's Theorem(Cont.)

- Now we assume the cardinality of mimnimum $u - v$ separating set is $k \geq 2$.
- Based on the adjacency of vertices in $u - v$ separating set with u and v, different cases arise and in each case we apply induction hypothesis and get require k internally disjoint paths.

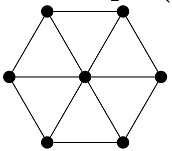
Planarity

Planar Graphs

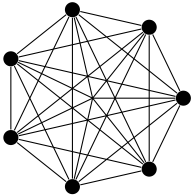
Planar Graph

If a connected graph can be drawn without any edges crossing , it is called as planar.

Planar Graph (W_6)

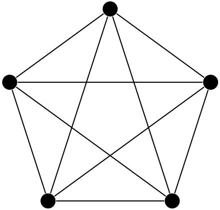


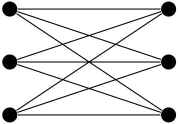
Non-Planar Graph (K^7)



Kuratowski's Theorem

The following are equivalent:
(i) G is planar
(ii) G contains neither K^5 nor $K_{3,3}$ as a minor.
(iii) G contains neither K^5 nor $K_{3,3}$ as a topological minor.





Graphs: K^5, $K_{3,3}$

3-connected planar graph

Lemma

Every 3-connected graph without a K^5 or $K_{3,3}$ minor is planar.

- Applying induction on number of vertices.
- Base case $G = K^4$ is planar.
- Contract an edge xy such that G/xy is is 3-connected. By induction hypothesis G/xy is planar.
- Recover G from G/xy by embedding x and y into a face of G/xy.

Maximal Planar Graph

Lemma

If $|G| > 4$ and G is edge-maximal with TK^5 , $TK_{3,3} \not\subseteq G$, then G is 3-connected.

References

- Diestel, Reinhard. Graph Theory. 5th ed. Berlin: Springer, 2018.

Thank You