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Fixed point theory and Applications

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Some Fixed Point Theorems

Banach Fixed Point Theorem

Let (X, d) be a non-empty complete metric space and let $f : X \to X$ be a contraction mapping. Then f has a unique fixed point $x_0 \in X$.

Brouwer Fixed Point Theorem

Let $K \subset \mathbb{R}^n$ be a compact and convex set, $f : K \to K$ a continuous map. Then f has a fixed point.

Schauder Fixed Point Theorem

Let K be a closed, bounded and convex set in a Banach space V, and $f : K \to K$ a compact map. Then f has a fixed point.

Now, We will see an application of Schauder Fixed Point Theorem to study weak solutions of Partial Differential Equations.

Classical Solutions of PDE's

A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $u(x) = 0$ for every $x \in \partial\Omega$ and the equation $-\Delta u = f(x, u(x))$ is satisfied at every point $x \in \Omega$ is called a Classical Solution of (D).

We note that the Problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

does not have a classical solution for every $f \in C(\overline{\Omega})$.
(Reference: Gilbarg & Trudinger [7.1, Chapter 4]).
Note: The concept of the classical solution is not suitable for the application of many abstract results of nonlinear analysis.
Weak solution is the generalization of the notion of classical solution.

Sobolev space $W^{k,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}_0$, and $1 \le p \le \infty$,
The Sobolev space $W^{k,p}(\Omega)$ is the set of functions $f \in L^p(\Omega)$ whose weak derivatives up to order k also belong to $L^p(\Omega)$. That is,

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \le k\}.$$

- The Sobolev norm on $W^{k,p}(\Omega)$ is defined by:

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and for $p = \infty$,

$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \le k} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

Sobolev space $W_0^{1,2}(\Omega)$

$W_0^{1,2}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ (the space of infinitely differentiable functions with compact support in Ω) with respect to the $W^{1,2}$ norm.

$$W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}.$$

- $\|\nabla u\|_{L^2}$ is an equivalent norm on $W_0^{1,2}(\Omega)$. (By Poincare inequality)

Trace Theorem

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain.
There exists unique continuous linear operator T which assigns to every function $u \in W^{1,p}(\Omega)$ a function $Tu \in L^p(\partial\Omega)$, and has the following property:

$$\text{For } u \in C^\infty(\overline{\Omega}) \text{ we have } Tu = u|_{\partial\Omega}.$$

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : Tu = 0 \text{ in } L^p(\partial\Omega) \right\}.$$

Caratheodory function

Let Ω be an open set in $\mathbb{R}^N$. A function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to have the *Caratheodory property* (notation: $g \in \mathrm{CAR}(\Omega \times \mathbb{R})$) if

(M) for all $y \in \mathbb{R}$ the function $x \mapsto g(x, y)$ is (Lebesgue) measurable on Ω;

(C) for a.e. $x \in \Omega$ the function $y \mapsto g(x, y)$ is continuous on $\mathbb{R}$.

Let $g \in CAR(\Omega \times \mathbb{R})$ and let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain.Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u(x) = g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{D}$$

Definition of Weak Solution

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $g \in CAR(\Omega \times \mathbb{R})$, By a weak solution of the Dirichlet problem,

$$\begin{cases} -\Delta u(x) = g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D}$$

we mean a function $u \in W_0^{1,2}(\Omega)$ such that the integral identity

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega g(x, u(x)) v(x)\, dx$$

holds for every $v \in W_0^{1,2}(\Omega)$.

Well-definedness of Weak Solutions

Theorem(A)

Let $g \in CAR(\Omega \times \mathbb{R})$ and $p, q \in [1, \infty)$. Assume that $r \in L^q(\Omega)$ and $c \in \mathbb{R}$ such that

$$|g(x, y)| \leq r(x) + c|s|^{p/q}, \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}.$$

Then the map $G(\varphi)(x) = g(x, \varphi(x))$, for $x \in \Omega$ satisfies the following:

(i) $G(\varphi) \in L^q(\Omega)$ for all $\varphi \in L^p(\Omega)$,

(ii) G is a continuous mapping from $L^p(\Omega)$ into $L^q(\Omega)$,

(iii) G maps bounded sets in $L^p(\Omega)$ into bounded sets in $L^q(\Omega)$,

Let $g \in CAR(\Omega \times \mathbb{R})$. Assume that $r \in L^2(\Omega)$ and $c > 0$ such that for all $x \in \Omega$ and $\forall s \in \mathbb{R}$

$$|g(x, s)| \leq r(x) + c|s|.$$

Let $u, v \in L^2(\Omega)$. Then by the above Theorem (A),
we get, $g(x, u(x)) \in L^2(\Omega)$.

Using Hölder's inequality:

$$\|g(x, u(x))v(x)\|_1 \leq \|g(x, u(x))\|_{L^2} \cdot \|v(x)\|_{L^2},$$

we conclude that

$$g(x, u(x))v(x) \in L^1(\Omega),$$

therefore; RHS definition is finite.
Since $u, v \in W_0^{1,2}(\Omega)$

$$\left|\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx\right| \leq \left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2} \left(\int_\Omega |\nabla v(x)|^2\, dx\right)^{1/2}.$$

So both sides are finite under the stated assumptions.

Theorem 1

Every classical solution is a weak solution.

Theorem 2

Let Ω be a smooth bounded domain then every weak solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution.

Theorem : *Existence of a weak solution*

Let Ω be a smooth bounded domain and g has sublinear growth with respect to the second variable, i.e
$g \in CAR(\Omega \times \mathbb{R})$ and $r \in L^2(\Omega)$, $c > 0$, and $\delta \in (0, 1)$ such that

$$|g(x, s)| \leq r(x) + c|s|^\delta \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Then there is at least one weak solution u of the Dirichlet boundary value problem,

$$\begin{cases} -\Delta u = g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{d}$$

Proof of the theorem

Step 1

We will find a suitable operator representation for the Dirichelt boundary problem (d)

$$\begin{cases} -\Delta u = g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For fixed $u \in W_0^{1,2}(\Omega)$, let $g \in CAR(\Omega \times \mathbb{R})$, and define:

$$\widehat{L}_u, \widehat{S}_u : W_0^{1,2}(\Omega) \to \mathbb{R}$$

by

$$\widehat{L}_u(v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx, \qquad \widehat{S}_u(v) = \int_\Omega g(x, u(x)) v(x)\, dx.$$

note: Both are continuous linear functionals on the space $W_0^{1,2}(\Omega)$.

Riesz Representation Theorem

Let H be a Hilbert space and let F be a continuous linear form on H. Then there is a unique $f \in H$ such that

$$F(x) = \langle x, f \rangle \quad \forall x \in H$$

By the Riesz Representation Theorem , there exist uniquely determined elements

$$L(u),\ S(u) \in W_0^{1,2}(\Omega) \quad \textit{such that}$$

$$\langle L(u), v \rangle_{W_0^{1,2}(\Omega)} = \widehat{L}_u(v), \qquad \langle S(u), v \rangle_{W_0^{1,2}(\Omega)} = \widehat{S}_u(v)$$

$$\forall v \in W_0^{1,2}(\Omega).$$

so, the Dirichlet problem has at least one weak solution if and only if the operator

$$L(u) = S(u) \quad \textit{in the space } W_o^{1,2}(\Omega).$$

The inner product on $W_0^{1,2}(\Omega)$ as:

$$\langle u, v \rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

Then L is just an identity on $W_0^{1,2}(\Omega)$ $Lu = u$.
Thus it is enough to show that the operator S has a fixed point.

Step 2

We will find a ball $\overline{B(0, R)} \subset W_0^{1,2}(\Omega)$ which maps by S into itself.

For any $u \in W_0^{1,2}(\Omega)$,

$$\|S(u)\| = \sup_{\|v\| \le 1} |\langle Su, v \rangle|$$

$$= \sup_{\|v\| \le 1} \left| \int_\Omega g(x, u(x)) v(x)\, dx \right|$$

$$\le \sup_{\|v\| \le 1} \left(\int_\Omega |g(x, u(x))|^2 dx \right)^{1/2} \left(\int_\Omega |v(x)|^2 dx \right)^{1/2}$$

Compact embedding

$$W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega).$$

There exists a constant $C_{\text{emb}} > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C_{\text{emb}} \|u\|_{W_0^{1,2}(\Omega)} \quad \text{for all } u \in W_0^{1,2}(\Omega).$$

$$\leq C_{\text{emb}} \left(\int_\Omega |g(x, u(x))|^2 \, dx \right)^{1/2}$$

$$\leq C_{\text{emb}} \left(\int_\Omega |r(x) + c|u(x)|^\delta|^2 \, dx \right)^{1/2}$$

$$\leq C_{\text{emb}} \left[\left(\int_\Omega |r(x)|^2 \, dx \right)^{1/2} + c \left(\int_\Omega |u(x)|^{2\delta} \, dx \right)^{1/2} \right] \quad \textit{(By Minkowski inequality).}$$

Using Holder's inequality in 2nd term:

$$\left(\int_\Omega |u(x)|^{2\delta}\,dx\right)^{1/2} = \left(\int_\Omega |u(x)|^{2\delta/\delta}\,dx\right)^{\delta/2}\left(\int_\Omega 1\,dx\right)^{(\frac{1-\delta}{2})}.$$

$$\leq\ C_{\text{emb}}^{\delta}\,(\text{meas}(\Omega))^{\frac{1-\delta}{2}}\|u\|^{\delta}$$

$$\Rightarrow \|S(u)\| \leq C_{\text{emb}}^{\delta}\|r\|_{L^2(\Omega)} + cC_{\text{emb}}^{1-\delta}\,(\text{meas}(\Omega))^{\frac{1-\delta}{2}}\|u\|^{\delta}.$$

For any $u \in \overline{B(0,R)}$, we get:

$$\|S(u)\| \leq R \quad \text{where} \quad C + DR^{\delta} \leq R,$$

Where,

$$C = C_{\text{emb}}^{\delta}\|r\|_{L^2(\Omega)}, D = cC_{\text{emb}}^{1-\delta}\,(\text{meas}(\Omega))^{\frac{1-\delta}{2}}$$

Hence S maps $\overline{B(0,R)}$ into itself if R is large enough.

Step 3: S is a compact map

Let $\{u_n\}_{n=0}^{\infty}$ be a bounded sequence in $\overline{B(0,R)}$.
We will prove $S(u_n)$ has a convergent subsequence in $\overline{B(0,R)}$.

Let

$$S(u_n) = w_n,$$

w_n is an arbitrary sequence in $\overline{B(0,R)}$ therefore,

$$\langle w_n, v\rangle = \int_\Omega g(x, u_n(x))v(x)dx\,.$$

By reflexivity of $W_0^{1,2}(\Omega)$, $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ (upto a subsequence).
Since $W_0^{1,2}(\Omega) \subset L^2(\Omega)$, and

$$(L^2(\Omega))' \subset (W_0^{1,2}(\Omega))'$$

we get,

$$u_n \rightharpoonup u \text{ in } L^2(\Omega).$$

By compact embedding of $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, we get,

$$u_n \to u \text{ in } L^2(\Omega).$$

Then

$$\|S(u_n) - S(u)\|_{W_0^{1,2}(\Omega)} = \sup_{\|v\|\leq 1} |\langle S(u_n) - S(u), v\rangle|$$

$$= \sup_{\|v\|\leq 1} \left| \int_\Omega [g(x, u_n(x)) - g(x, u(x))] v(x)\, dx \right|$$

$$\leq \sup_{\|v\|\leq 1} \left(\int_\Omega |g(x, u_n(x)) - g(x, u(x))|^2\, dx \right)^{1/2} \left(\int_\Omega |v|^2\, dx \right)^{1/2}$$

$$\leq C_{\mathrm{emb}} \|g(\cdot, u_n) - g(\cdot, u)\|_{L^2(\Omega)}$$

By the Theorem (A) (continuity of Nemytskii operator $\phi(x) \to g(x, \phi(x))$) we get g from $L^2(\Omega)$ into $L^2(\Omega)$ is continuous.
Therefore,

$$S(u_n) \to S(u) \quad \text{in } W_0^{1,2}(\Omega).$$

The compactness of the operator S follows.
and continuity will also follow by similar argument.

Step 4

By Schauder's Fixed Point Theorem, S has a fixed point. Therefore, the given problem has at least one weak solution.

Reference

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Thank You...