# The Calculus of Variations

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# Weak derivatives

## Definition (Weak Derivative)

Suppose $u, v \in L^1_{\mathrm{loc}}(U)$ and $\alpha$ is a multi-index. We say that $v$ is the $\alpha$-th weak partial derivative of $u$, written

$$D^\alpha u = v,$$

provided

$$\int_U u\, D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\, \phi \, dx \qquad (1)$$

for all test functions $\phi \in C_c^\infty(U)$.

**Notation:** Where $C_c^\infty(U)$ denote the space of all infinitely differentiable functions $\phi : U \to \mathbb{R}$ with compact support in $U$, where $U$ is a subset of $\mathbb{R}^N$. Any function $\phi$ that belongs to $C_c^\infty(U)$ is referred to as a *test function*.

**Example:** Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } 1 \leq x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } 1 \leq x < 2. \end{cases}$$

Then $u' = v$ in the weak sense.

**Lemma:** *A weak $\alpha$-th partial derivative of $u$, if it exists, is uniquely defined up to a set of measure zero.*

# Sobolev space

## Definition

Fix $1 \leq p \leq \infty$ and let $k$ be a nonnegative integer. The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions

$$u : U \to \mathbb{R}$$

such that for each multi-index $\alpha$ with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(U)$.

## Definition

If $u \in W^{k,p}(U)$, we define its norm by

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left( \sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_{x \in U} |D^\alpha u(x)|, & p = \infty. \end{cases}$$

Let $U \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary $\partial U$. Suppose $L : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ is a smooth function, Define the functional:

$$I[w] = \int_U L(Dw(x), w(x), x)\, dx. \tag{1}$$

For smooth functions $w : \bar{U} \to \mathbb{R}$ satisfying, say, the boundary condition

$$w = g \quad \text{on } \partial U. \tag{2}$$

Now suppose some particular smooth function $u$, satisfying the requisite boundary condition $u = g$ on $\partial U$, happens to be a minimizer of $I[\cdot]$ among all functions $w$ satisfying (2). *Then $u$ automatically a solution of a nonlinear partial differential equation.*

To confirm this, first choose any smooth function $v \in C_c^\infty(U)$ and consider the real-valued function

$$i(\tau) := I[u + \tau v] \qquad (\tau \in \mathbb{R}). \tag{3}$$

Since $u$ is a minimizer of $I[\cdot]$ and $u + \tau v = u = g$ on $\partial U$, we observe that

$i(\cdot)$ has a minimum at $\tau = 0$. Therefore

$$i'(0) = 0. \tag{4}$$

We explicitly compute this derivative (called the *first variation*) by writing out

$$i(\tau) = \int_U L(Du + \tau Dv, u + \tau v, x)\, dx. \tag{5}$$

Let $\tau = 0$, to deduce from (4) that

$$0 = i'(0) = \int_U \left[ \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v \right] dx.$$

Finally, since $v$ has compact support, we can integrate by parts and obtain

$$0 = \int_U \left[ - \sum_{i=1}^{n} (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) \right] v\, dx.$$

This is true for all test function $v$, then $u$ satisfies the equation:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} L_{p_i}(Du, u, x) - L_z(Du, u, x) = 0 \quad \text{in } U.$$

This is known as the Euler–Lagrange equation.

# Example

**Example 1 (Dirichlet's principle).** Take

$$L(p, z, x) = \frac{1}{2}|p|^2.$$

Then $L_{p_i} = p_i$ $(i = 1, \ldots, n)$, $L_z = 0$; and so the Euler–Lagrange equation associated with the functional

$$I[w] := \frac{1}{2}\int_U |Dw|^2\, dx$$

is

$$\Delta u = 0 \quad \text{in } U.$$

This fact is known as the *Dirichlet's principle* $\square$

# Second Variation

We continue in the spirit of the calculations from first variation by computing now the *second variation* of $I[\cdot]$ at the function $u$. This we find by observing that since $u$ gives a minimum for $I[\cdot]$, we must have

$$i''(0) \geq 0,$$

where $i(\cdot)$ is defined as above by (3). In view of (5), we can calculate Setting $\tau = 0$, we derive the inequality

$$\begin{aligned} 0 \leq i''(0) = \int_U \sum_{i,j=1}^{n} L_{p_i p_j}(Du, u, x)\, v_{x_i} v_{x_j} \\ + 2 \sum_{i=1}^{n} L_{p_i z}(Du, u, x)\, v_{x_i} v + L_{zz}(Du, u, x)\, v^2\, dx, \end{aligned} \tag{6}$$

for all test functions $v \in C_c^\infty(U)$. This estimate (6) is also valid for any Lipschitz continuous v vanishing on boundary.

By choosing specific test functions of the form

$$v(x) := \epsilon\rho\left(\frac{x\cdot\xi}{\epsilon}\right)\zeta(x) \quad (x \in U),$$

where $\rho$ is the periodic "zig-zag" function with $|\rho'| = 1$ almost everywhere, and $\zeta \in C_c^\infty(U)$,we deduce the pointwise convexity condition:

$$\sum_{i,j=1}^{n} L_{p_i p_j}(Du, u, x)\xi_i\xi_j \geq 0 \quad (\forall\xi \in \mathbb{R}^n, x \in U).$$

This necessary condition suggests a fundamental convexity assumption on $L$ for the existence theory.

# Existence of minimizers

**Coercivity Condition for Existence of Minimizers:** To ensure the existence of a minimizer for $I[w]$, a coercivity condition on $L$ is required. Specifically, suppose there exist constants $\alpha > 0$, $\beta \geq 0$, and a fixed exponent $q$ satisfying

$$1 < q < \infty,$$

such that

$$L(p, z, x) \geq \alpha |p|^q - \beta$$

for all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $x \in U$. Then we obtain the estimate

$$I[w] \geq \alpha \|Dw\|_{L^q(U)}^q - \gamma,$$

where $\gamma = \beta |U|$. Consequently, $I[w] \to \infty$ as $\|Dw\|_{L^q} \to \infty$, ensuring that $I[w]$ grows sufficiently to prevent arbitrarily large minimizing sequences. This coercivity condition is a key criterion for the existence of minimizers in variational problems.

### Definition

We say that a function $I[\cdot]$ is (sequentially) *weakly lower semicontinuous* on $W^{1,q}(U)$, provided

$$I[u] \;\leq\; \liminf_{k\to\infty} I[u_k]$$

whenever

$$u_k \rightharpoonup u \quad \text{weakly in } W^{1,q}(U).$$

### Theorem (Weak lower semicontinuity)

*Assume that $L$ is smooth , bounded below, and in addition that for each $z \in \mathbb{R}$ and $x \in U$ the mapping*

$$p \;\mapsto\; L(p, z, x)$$

*is convex. Then the functional $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(U)$.*

## Theorem (Uniqueness of minimizer)

*Suppose the integrand* $L = L(p, x)$ *does not depend on* $z$*, and that there exists a constant* $\theta > 0$ *such that for every* $x \in U$*,* $p \in \mathbb{R}^n$*, and all* $\xi \in \mathbb{R}^n$*,*

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p, x)\, \xi_i\, \xi_j \;\geq\; \theta\, |\xi|^2.$$

*In other words,* $p \mapsto L(p, x)$ *is uniformly convex for each* $x$*. Then any minimizer* $u \in \mathcal{A}$ *of the functional* $I[\cdot]$ *is unique.*

# Weak solutions of Euler–Lagrange equation

Now we define

$$\mathcal{A} := \left\{ w \in W^{1,q}(U) \;\middle|\; w = g \text{ on } \partial U \text{ in the trace sense} \right\}$$

to denote this class of *admissible* functions $w$. Note in view of coercivity that $I[w]$ is defined (but may equal $+\infty$) for each $w \in \mathcal{A}$.
We wish next to demonstrate that any minimizer $u \in \mathcal{A}$ of $I[\cdot]$ solves the Euler–Lagrange equation in some suitable sense. We will need some growth conditions on $L$ and its derivatives. Let us hereafter suppose

$$\left|L(p,z,x)\right| \;\leq\; C\left(|p|^q + |z|^q + 1\right), \tag{1}$$

and also

$$\begin{cases} \left|D_p L(p,z,x)\right| \leq C\left(|p|^{q-1} + |z|^{q-1} + 1\right), \\ \left|D_z L(p,z,x)\right| \leq C\left(|p|^{q-1} + |z|^{q-1} + 1\right), \end{cases} \tag{2}$$

for some constant $C$ and all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x \in U$.

We now turn our attention to the boundary-value problem for the Euler–Lagrange PDE associated with our functional $L$. For a (smooth) minimizer $u$, one obtains

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(L_{p_i}(Du, u, x)\right) + L_z(Du, u, x) = 0, & x \in U, \\ u = g, & x \in \partial U. \end{cases} \tag{3}$$

## Definition

We say $u \in A$ is a **weak solution** of the boundary-value problem (3) for the Euler-Lagrange equation provided

$$\int_U \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v \, dx = 0$$

for all $v \in W_0^{1,q}(U)$.

## Theorem (Solution of Euler-Lagrange equation)

*Assume $L$ verifies the growth conditions (1), (2), and $u \in A$ satisfies*

$$I[u] = \min_{w \in A} I[w].$$

*Then $u$ is a weak solution of (3).*

# Constrained Minimization

Now let us consider applications of the calculus of variations to certain constrained minimization problems, and, in particular, discuss the role of Lagrange multipliers in the corresponding Euler-Lagrange PDE.
We investigate first problems with integral constraints. To be specific, let us consider the problem of minimizing the energy functional

$$I[w] := \frac{1}{2}\int_U |Dw|^2 dx$$

over all functions $w$ with, say, $w = 0$ on $\partial U$, but subject now also to the side condition that

$$J[w] := \int_U G(w) dx = 0,$$

where $G : \mathbb{R} \to \mathbb{R}$ is a given, smooth function.
We will henceforth write $g = G'$. Assume now

$$|g(z)| \leq C(|z| + 1),$$

and so

$$|G(z)| \leq C(|z|^2 + 1) \quad (z \in \mathbb{R})$$

for some constant $C$. Let us introduce as well the appropriate admissible class

$$\mathcal{A} := \{w \in H_0^1(U) \mid J[w] = 0\}.$$

We suppose also that the open set $U$ is bounded, connected and has a smooth boundary.

**Theorem (Existence of constrained minimizer)**

*Assume the admissible set $\mathcal{A}$ is nonempty. Then there exists $u \in \mathcal{A}$ satisfying*

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

**Theorem (Lagrange multiplier)**

*Let $u \in \mathcal{A}$ satisfy*

$$I[u] = \min_{w \in \mathcal{A}} I[w]. \tag{4}$$

*Then there exists a real number $\lambda$ such that*

$$\int_U Du \cdot Dv \, dx = \lambda \int_U g(u) v \, dx \tag{5}$$

*for all $v \in H_0^1(U)$.*

# Remark

Thus $u$ is a weak solution of the nonlinear boundary-value problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{6}$$

where $\lambda$ is the Lagrange multiplier corresponding to the integral constraint

$$J[u] = 0. \tag{7}$$

A problem of the form (6) for the unknowns $(u, \lambda)$, with $u \neq 0$, is a non-linear eigenvalue problem. $\square$

# Reference


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