Partial Fractions

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Announcements

1. Homework in MyOpenMath.
2. Office hours every weekday, 10am - 11am.
3. Trig Substitution quiz in Canvas, due Friday.

Today's goal is to learn how to integrate functions such as

$$\int \frac{1}{x^2\left(1+x^2\right)}\,dx$$

or

$$\int \frac{3x}{x^2-x}\,dx$$

in a way that is (arguably) easier than trigonometric substitution.
Trig substitution is still useful and necessary for integrals involving $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$, but when there's no square root in the denominator, what we learn today will be an easier (and actually more useful) method.

Motivation

The main motivation behind what we do today is the following: Let's say we want to evaluate the integral

$$\int \frac{1}{x^2 - 2x}\, dx,$$

and you forgot for a moment (or don't want to do) trig substitution. Well, then doing this integral is kind of hard. But what if I told you that

$$\frac{1}{x^2 - 2x} = \frac{-1}{2x} + \frac{1}{2\,(x-2)}?$$

Motivation

Then, we could find:

$$\int \frac{1}{x^2 - 2x}\, dx = \int \left(\frac{-1}{2x} + \frac{1}{2\,(x-2)} \right) dx$$

And these integrals are much easier to handle! We arrive at the answer

$$\int \frac{1}{x^2 - 2x}\, dx =$$

Partial fraction decomposition

Today we're going to focus on how to break down rational functions like

$$\frac{1}{x^2 - 2x}$$

into

$$\frac{-1}{2x} + \frac{1}{2\left(x - 2\right)}.$$

This process is called finding the ______________________________. Using this method, we're going to be able (in theory) to integrate *any* rational function known to humankind!

The setup

In general, we're trying to find a partial fraction decomposition of a rational function

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $\deg(P(x)) < \deg(Q(x))$.
Why do I have that restriction on the degrees of P and Q?

Example

Use polynomial long division to simplify

$$\frac{x^2 - 3x + 1}{x - 1}.$$

Partial fraction decomposition

So, now that we know that we should only focus on rational functions $P(x)/Q(x)$ with the degree of $P(x)$ less than the degree of $Q(x)$, the most important thing we need to do is factor $Q(x)$ correctly.

Depending on how $Q(x)$ factors, there are a couple of different paths we may take as far as setting up the partial fraction decomposition.

Let's look at the easiest case that might occur: when $Q(x)$ is a product of ______________, none of which repeat.

Non repeating linear factors

Let's say $Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)$. Then, we can write

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)}.$$

The KEY STEP to partial fractions is then writing this as:

$$\frac{P(x)}{(a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)} =$$

(Don't let the n scare you... most of the time $Q(x)$ will only have 2 or 3 factors in this class).

Non repeating linear factors

Let's pause and look at an example before moving on.
Find a partial fraction decomposition for the rational function

$$\frac{2x-3}{x^2-4x-5}.$$

Non repeating linear factors

Application

Find

$$\int \frac{2x - 3}{x^2 - 4x - 5}\, dx.$$

Equating coefficients and strategic substitution

The last example illustrated the two main ways of find the constants in the partial fraction decomposition $A_1, A_2, \ldots, A_n$: Equating coefficients and strategic substitution.
Both methods will always work, but in the case of non repeating linear factors of $Q(x)$, strategic substitution is most of the time more efficient.
We'll see later though, that sometimes equating coefficients is more efficient.

Let's take another break for a second.
How would you calculate

$$\frac{1}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x}?$$

You would get a common denominator of $x(x-1)^2$ and add up to get

Repeating linear factors

From this we can see that when we're trying to find the partial fraction decomposition of

$$\frac{2x^2 - 2x + 1}{x(x-1)^2}$$

that it's not sufficient to only take it of the form

$$\frac{A_1}{x-1} + \frac{A_2}{x} \text{ or } \frac{A_1}{(x-1)^2} + \frac{A_2}{x}$$

but instead, we need to take the decomposition of the form

This illustrates what we need to do in general when $Q(x)$ has repeated linear factors

Repeated linear factors

Rule: Linear factors

If $Q(x)$ has a factor of the form $(ax+b)^n$ where n is a positive integer ≥ 1, then the partial fraction decomposition of $P(x)/Q(x)$ should contain

You can still use the techniques of equating coefficients or strategic substitution to solve for the constants in the partial fraction decomposition.

Example

Find the partial fraction decomposition of the rational function

$$\frac{3x^2 - 3x - 2}{(x^2 - 1)(x - 1)^2}.$$

Example

Example

Find

$$\int \frac{3x^2 - 3x - 2}{(x^2 - 1)(x - 1)^2}\, dx.$$

Nonrepeating quadratic factors

Sometimes when we're factoring $Q(x)$, we might end up with something that looks like

$$Q(x) = (2x+3)\left(x^2+4\right).$$

The issue here is that x^2+4 can't be ______________ anymore using real numbers no matter how hard we try. It's not just a repeated linear factor either: this is a totally new case for us. So, we need to figure out what to do when we have a factorization like this. Fortunately, not much changes. For this specific example, we would have

$$\frac{P(x)}{(2x+3)\left(x^2+4\right)} =$$

and we would again solve for A_1, A_2 and now B_2 using the same methods as before.

Example

Find the partial fraction decomposition of the function

$$\frac{1}{(2x+3)\left(x^2+4\right)}$$

Example

Find

$$\int \frac{1}{(2x+3)\left(x^2+4\right)}\,dx$$

I don't think what's coming next is much of a surprise...

Rule: Quadratic factors

If $Q(x)$ has a repeated quadratic factor $(ax^2 + bx + c)^n$, then the partial fraction decomposition of $P(x)/Q(x)$ should contain terms of the form

Example

Find the partial fraction decomposition of

$$\frac{4x^3 + 2x^2 + 4}{(2x^2 + x + 1)^2}.$$

Example

Example

Find

$$\int \frac{4x^3 + 2x^2 + 4}{(2x^2 + x + 1)^2} \, dx.$$

Putting it all together

Here's an outline of all things partial fractions:

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $\text{degree}(P(x)) < \text{degree}(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$
   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain
   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$
   c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include
   $$\frac{Ax+B}{ax^2+bx+c}.$$
   d. For each repeated irreducible quadratic factor $\left(ax^2+bx+c\right)^n$, the decomposition must include
   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$
   e. After the appropriate decomposition is determined, solve for the constants.
   f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Examples

Find

$$\int \frac{1}{x^3 - 8}\, dx$$

Examples

Examples

Find the volume of the solid generated by rotating the region bounded by the graph of

$$f(x) = \frac{x^2}{(x^2+1)^2}$$

and the $x-$axis between $x = 0$ and $x = 1$ about the $y-$axis.

Examples

Examples

Why only linear and quadratic factors

At this point, you may ask: What if $Q(x)$ has factors that aren't linear or quadratic?
That is a fantastic question, Chase. It turns out though, that any polynomial can be factored completely into (maybe repeated) linear and quadratic factors. Sometimes that factoring might be very very difficult to do, but it's always technically possible. So, we've actually studied all of the cases we need to for partial fractions! Time to celebrate!