Trigonometric Integrals

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Announcements

1. Don't forget about exam corrections.
2. Office hours 10am - 11am.

Today's mission

Our goal today is to tackle three types of trigonometric integrals:

1. Integrals of the form
$$\int \cos^j(x)\sin^k(x)\ dx$$
where k, j are nonnegative integers $(0,1,2,\ldots)$ and not both 0.
2. Integrals of the form
$$\int \sin(ax)\cos(bx)\,dx$$
for real numbers a and b.
3. Integrals of the form
$$\int \tan^j(x)\sec^k(x)\ dx$$
where k, j are nonnegative integers.

Today's mission

Why do we need to cover these?
The main reason we need to cover these is that they will come up quite frequently when we learn about trigonometric substitution tomorrow. Let's begin by looking at 1.
Really the main idea behind most of what we do today lies in a handful of trig identities and trig derivatives, but most importantly:

$$\cos^2(x) + \sin^2(x) = 1,\ \frac{d}{dx}(\sin(x)) = \cos(x),\ \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$1 + \tan^2(x) = \sec^2(x),\ \frac{d}{dx}(\tan(x)) = \sec^2(x),\ \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

Sines and Cosines

There are 4 cases we could have when looking at the integral

$$\int \sin^k(x)\cos^j(x)\, dx$$

We could have

1. k is even and j is even. This is the annoying case.
2. k is even and j is odd. This is a relatively easy case.
3. k is odd and j is even. This is handled almost exactly as the second case.
4. k is odd and j is odd. This is also relatively simple and handled in a similar way as 2 and 3.

Let's look at specific examples, starting from the bottom, of each of these cases and then generalize.

k odd and j odd

Find

$$\int \sin^3(x)\cos^5(x)\ dx$$

k odd and j odd

k even and j odd

Find

$$\int \sin^4(x)\cos^5(x)\ dx$$

k even and j odd

k odd and j even

Find

$$\int \sin^5(x)\cos^4(x)\ dx$$

k odd and j even

k even and j even (not both 0)

Find

$$\int \sin^2(x)\ dx$$

k even and j even (not both 0)

One more example

Find

$$\int \sin^2(x)\cos^2(x)\ dx$$

In general, we can use this problem solving strategy when evaluating integrals of this kind:

PROBLEM-SOLVING STRATEGY

Problem-Solving Strategy: Integrating Products and Powers of sin *x* and cos *x*

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (*Note*: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Other sines and cosines

We've seen how to integrate powers of sines and cosines, but what about an integral of the form

$$\int \sin(2x)\cos(3x)\ dx?$$

We could do integration by parts and arrive back at the same integral (my favorite type of integral), or we could take advantage of some trig identities.

RULE: INTEGRATING PRODUCTS OF SINES AND COSINES OF DIFFERENT ANGLES

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x) \tag{3.3}$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x) \tag{3.4}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x) \tag{3.5}$$

Let's try it

Evaluate

$$\int \sin(2x)\cos(3x)\ dx$$

using trig identities.