Limits of Sequences

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Announcements

1. Homework!
2. Exam Corrections!
3. Project!

The Limit of a Sequence

With sequences, we are usually interested in what happens in what's known as the ______________ of the sequence (i.e. end behaviour):

Definition (Limit of a Sequence)

Suppose $\{a_n\}$ is a sequence of real numbers. When we say

$$\lim_{n\to\infty} a_n = L$$

we mean that we can make a_n as close to L as we like by taking n to be "large enough". If such an L exists, we call the sequence a_n ______________. If no such L exists, we call the sequence ______________.

Let's make some of these ideas a little more precise.

The Limit of a Sequence

Limit Law

If $a_n = f(n)$ for some function f for all $n \geq 1$ (or some starting index) then if there exists L such that $\lim_{x\to\infty} f(x) = L$, then it must be the case that $\lim_{n\to\infty} a_n =$ ________________.

Example

Evaluate the limits of the sequences:

1. $a_n = \frac{1}{2^n}$
2. $b_n = (-1)^n$

Limit Laws

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, where A and B are real numbers. Let c be a real number. Then the following limit laws hold:

1. $\lim_{n\to\infty} c =$
2. $\lim_{n\to\infty} ca_n =$
3. $\lim_{n\to\infty} (a_n \pm b_n) =$
4. $\lim_{n\to\infty}(a_n b_n) =$
5. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) =$

Example

Evaluate

$$\lim_{k\to\infty} \frac{1 - r^k}{1 - r}$$

(Your answer will depend on the value of r.)

Example

Evaluate

$$\lim_{m\to\infty}\left(1-\frac{2}{m}\right)^m.$$

Example

2 More Important Theorems

We'll take the following theorems without proof:

Theorem (Continuous Functions and Convergent Sequences)

Suppose $\{a_n\}$ is a convergent sequence that converges to L and f is a function of a real variable that is continuous at L. Then, the sequence $\{f(a_n)\}$ is ________________ with limit ________________.

Theorem (Squeeze Theorem)

Suppose $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are all sequences that satisfy*

$$a_n \leq b_n \leq c_n$$

for all $n \geq 1$ *(or for all* n *greater than some initial index). If*

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} c_n = L$$

Then

$$\lim_{n\to\infty} b_n =$$

Example

Use the squeeze theorem to show

$$\lim_{k\to\infty} \frac{\sin k}{k} = 0$$

Example

Let

$$S_k = 1 + \frac{1}{2} + \ldots + \frac{1}{2^k}$$

Let's try to find

1. A "nicer" way to write S_k and
2. $\lim_{k\to\infty} S_k$