Power Series!

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Announcements

1. Homework.
2. Project.
3. Office hours canceled today.

Power Series

Definition (Power Series)

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

is called a power series centered at 0. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

is called a ________________.

Example

Which of the following are power series? What is the center of the power series (if it's a power series)?

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}(x-1)^n$
3. $\sum_{n=3}^{\infty} \frac{\sin(nx)}{n}$

A note on definitions

For consistency, we define $0! = 1$ and $x^0 = 1$, even when $x = 0$.

Convergence of a power series

It's always going to be the case that a power series converges at its center $x = a$. There are two other cases that could happen:

Theorem 6.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.

ii. The series converges for all real numbers x.

iii. There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where $|x-a| = R$, the series may converge or diverge.

Proof:

Radius and interval of convergence

The previous theorem leads to the following definitions:

Definition

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (**Figure 6.2**).

Examples

Find the radius of convergence and the interval of convergence for the following power series (make sure to state the center of the series).

1. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$
2. $\displaystyle\sum_{n=3}^{\infty} 2^n (x-3)^n$
3. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2x}{3}\right)^{2n}$

Representing functions using power series

We're one step closer to our ultimate goal, which is representing any (reasonable) function using better and better polynomial approximations. Let's look specifically at the function

$$f(x) = \frac{1}{1-x}.$$

Representing functions using power series

Example

Use the geometric series

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

to find what the following power series converge to as functions. Make sure to find the radius of convergence and the interval of convergence.

1. $\sum_{n=0}^{\infty} (-1)^n x^n$
2. $\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^{2n}$