The Comparison Tests

Chase Mathison[1]

Shenandoah University

11 April 2024

[1] cmathiso@su.edu

Announcements

1. Homework due, and new homework in MyOpenMath.
2. Office hours, 10am - 11am.

The Comparison Test

Now that we have some series for which we can describe their convergence/divergence, let's look at another test for convergence!

The Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$ for all n, and suppose $\sum_{n=1}^{\infty} b_n$ is another series which we know is convergent.

Then, if there is an N such that a_n ____ b_n for all $n \geq N$, then

$$\sum_{n=1}^{\infty} a_n$$

______________.

Similarly, if $\sum_{n=1}^{\infty} b_n$ is a series which we know is divergent, and there is an N such that a_n ____ b_n for all $n \geq N$, then

$$\sum_{n=1}^{\infty} a_n$$

______________.

Example

Discuss the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n!}.$$

Example

Discuss the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$$

Example

Discuss the convergence/divergence of

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

The Limit Comparison Test

The comparison test is nice, but it's a little too simplistic sometimes. For instance, if we want to examine the convergence/divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

then our natural instinct is to compare this series to $\sum_{n=2}^{\infty} \frac{1}{n^2}$, but

$$\frac{1}{n^2 - 1} \qquad\qquad \frac{1}{n^2}$$

so the comparison test wouldn't tell us anything here. This is where the limit comparison test is useful!

The Limit Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $a_n, b_n \geq 0$. Let $M = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then

1. If $M > 0$, is a real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the ______________.
2. If $M = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ ______________.
3. If $M = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ ______________.

Example

Use the limit comparison test to discuss the convergence/divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Example

Use the limit comparison test to discuss the convergence/divergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{2^{\ln(\ln(n))}}$$

Example