Arc Length

Chase Mathison[1]

Shenandoah University

18 March 2024

[1] cmathiso@su.edu

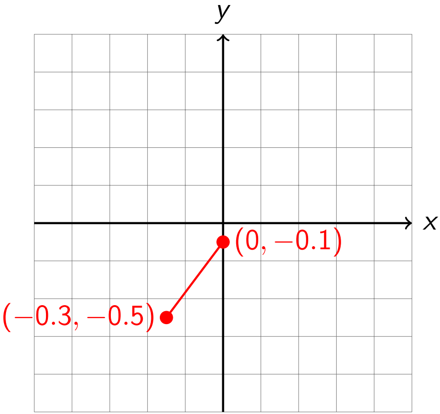
Announcements

1. Exam next week!
2. Office hours, 10am - 11am.

Arc length

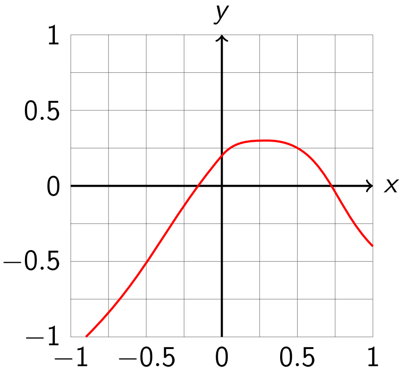
Now we're going to switch back to talking about applications of integration. First up: arc length.

It's easy to calculate the length of a straight line. For instance, what is the length of the line segment given below?



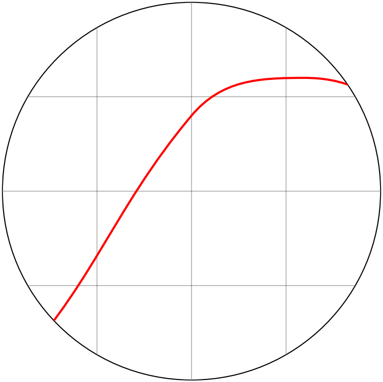
Arc length

Now, how could we try to calculate the length of the following curve $y = f(x)$?



We don't know how this curve is defined, but we can try to approximate the length using some straight lines, which are easy to calculate the length of.

Arc length



Arc length

Arc length

All of this has shown the following:

Theorem (Arc length of the graph of a smooth function)

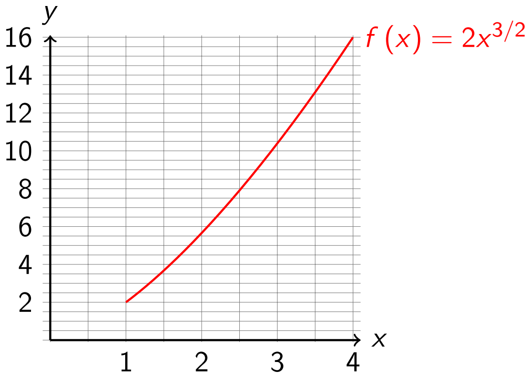
Let $f(x)$ be a differentiable function such that $f'(x)$ is continuous (i.e. f is smooth) over the interval $[a, b]$. The arc length of the portion of the graph of $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

A word of warning: many of the integrals that result from this theorem are very difficult (though not impossible) to evaluate. Some of the methods that we've developed over the last several weeks will help, but we'll still need to use technology for a lot of the resulting integrals.

Example

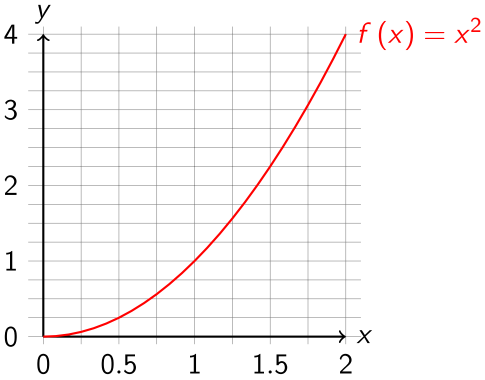
What is the length of the curve defined by the function $y = 2x^{3/2}$ on the interval $[1, 4]$?



Example

Example

What is the length of the parabola defined by $y = x^2$ on the interval $[0, 2]$?



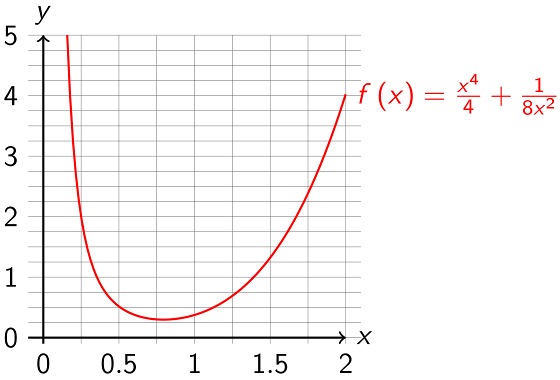
Example

Example

Calculate the length of the curve defined by the function

$$f(x) = \frac{x^4}{4} + \frac{1}{8x^2}$$

over the interval $[1, 2]$.



Example

Arc length for functions of y

If we want to find the arc length of some smooth function of y, say $x = g(y)$, the formula is almost identical:

Theorem (Arc length (function of y))

Suppose $x = g(y)$ is a smooth function of y on the interval $[c, d]$. Then the length of the curved defined by $x = g(y)$ is given by

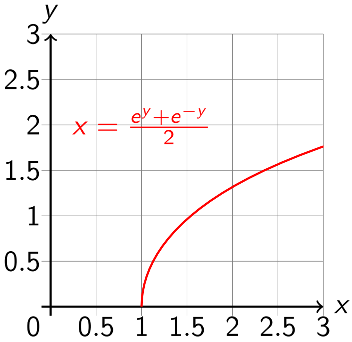
$$s = \int_c^d \sqrt{1 + (g'(y))^2}\, dy.$$

Example

Find the arc length of the curve defined by

$$x = \frac{e^y + e^{-y}}{2}$$

for y on the interval $[0, 1]$.



Example