The Integral Test

Chase Mathison[1]

Shenandoah University

10 April 2024

[1] cmathiso@su.edu

Announcements

1. Homework in MyOpEnMaTh.
2. Office hours, 10am - 11am.

The Integral Test

Let's look at another test that will bring back improper integrals! Let's look at 2 specific examples to illustrate the ______________.

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The Integral Test

The Integral Test

The Integral Test

What we just did works in general, as long as the series in question satisfies a few conditions.

Theorem (The Integral Test)

Suppose $\sum_{n=1}^{\infty} a_n$ *is a series with strictly* ________________ *terms. If there is a continuous function* $f(x)$ *and an integer* N *such that*

1. f *is a decreasing function, and*
2. $f(n) = a_n$ *for all* $n \geq N$*,*

Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx$$

____.

Example

Determine which of the following series converge by using the integral test:

1. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

2. $$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

3. $$\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$$

Example

p—series

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

p—series

Another use for the integral test

In general if $a_n = f(n)$ for a continuous, decreasing function, and $\sum_{n=1}^{\infty} a_n$ converges, it is NOT the case that

$$\sum_{n=1}^{\infty} a_n = \int_{1}^{\infty} f(x)\, dx$$

But, we can still gain information about the value of the series from the value of the improper integral. Let's see how:

Another use for the integral test

Integral test remainder estimate

We've shown the following:

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series that satisfies the criteria for use with the integral test (with corresponding function f).

Let S_N denote the partial sum $\sum_{n=1}^{N} a_n$. Then

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

If we denote $R_N = \sum_{n=1}^{\infty} a_n - S_N$ (called the ________________), then another way to say this is

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

Example

It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

How many terms do we need to use in a partial sum to estimate this series with a remainder (error) of 10^{-3}?

Example