Properties of Power Series

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Announcements

1. Homework in MyOpenMath.
2. Exam next Wednesday.
3. Office hours, 10am - 11am.

Warm up problem

What is the limit of the following recursively defined sequence?

$$x_{n+1} = x_n - 1 + 2e^{-x_n}, x_0 = 1.$$

Properties of Power Series

To find power series representations of other functions, it's going to be convenient to combine power series that we already know

Theorem 6.2: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Example

Find a power series representation centered at 0 for the function

$$f(x) = \frac{x}{1+x^2}.$$

Example

Example

Use the geometric series to find a power series centered at $a = 2$ for the function

$$\frac{1}{(x-1)(x-3)}$$

Example

Differentiation and integration of power series

Theorem 6.4: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a-R,\ a+R)$ for some $R > 0$. Let f be the function defined by the series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ &= c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots \end{aligned}$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R,\ a+R)$ and we can find f' by differentiating the series term-by-term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \\ &= c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \cdots \end{aligned}$$

for $|x-a| < R$. Also, to find $\int f(x)\,dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R,\ a+R)$, and we have

$$\begin{aligned} \int f(x)dx &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \\ &= C + c_0 (x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots \end{aligned}$$

for $|x-a| < R$.

Differentiation and integration of power series

Note that this theorem says $f(x)$ and $f'(x)$ have the same radius of convergence, but does NOT tell us if they have the same endpoint behavior. The same goes for integrals.

Example

Find a power series representation centered at 0 for

1. $\frac{2x}{1-x^2}$
2. $\ln(1+x)$
3. $\arctan(x)$

Uniqueness

One last point to note is that power series representations are

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Theorem 6.5: Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof:

Example

79. $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (*Hint:* Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)