Mass and Moments

Chase Mathison[1]

Shenandoah University

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[1]cmathiso@su.edu

Announcements

1. Exam next Monday.
2. Office hours, 10am - 11am.

Center of mass and moment of masses on a line

We'll start with masses on a line. If we have 2 (not necessarily equal) masses on a line (think of a see-saw) then what would be the point where we could perfectly balance that line? This is called the _______________.
Let's find this:

In general, if we have n point masses at n points, we have the following:

Theorem (Center of mass)

If $m_1, m_2, \ldots m_n$ are masses at the points $x_1, x_2, \ldots, x_n$ respectively, then we have the center of mass

$$\bar{x} =$$

The value $M = m_1 x_1 + m_2 x_2$ is called the ______________ of the system.

Example

Find the center of mass and the moment of the system with respect to the origin for the system of masses:

$$m_1 = 2kg \text{ at } x_1 = 0.2m,\ m_2 = 3kg \text{ at } x_2 = 1m,$$
$$m_3 = 7kg \text{ at } x_3 = -0.5m,\ m_4 = 3.5kg \text{ at } x_4 = -0.75m$$

Example

Center of mass and moments of system in the plane

We can generalize the previous ideas to masses in the $xy-$plane. Let's do this for 2 masses:

Center of mass and moments of system in the plane

Again, we can generalize this to n masses in the plane at n points:

Theorem (Center of mass and moments in the plane)

If $m_1, m_2, \ldots, m_n$ are point masses at the points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ respectively, then we define the quantities

$$M_x =$$

$$M_y =$$

$$\bar{x} =$$

$$\bar{y} =$$

Example

Find the moments M_x, M_y and the center of mass of the system of masses:

$$m_1 = 2kg, \text{ at } (-1, 3)$$
$$m_2 = 6kg, \text{ at } (1, 1)$$
$$m_3 = 4kg, \text{ at } (2, -2)$$

Center of mass of a thin sheet

But what if we want to find the center of mass (or centroid, which is the geometric center of an object) of a 2 dimensional object, like a thin plate or some other thin object defined by a function?

For what follows, we're going to find the moments about the x and y axis and the center of mass for what's called a *lamina*, which is a thin sheet of uniform density ρ represented as a region in the $xy-$plane.

Center of mass of a thin sheet

Can you guess what we're about to do?

Center of mass of a thin sheet

Theorem (Center of mass of a thin sheet)

Suppose R is the region bounded above by the continous function $y = f(x)$, below by $y = 0$ and on the left and right by $x = a$ and $x = b$ respectively. Let ρ (rho) denote the density of the associated lamina. Then we have the following:

1. $m =$
2. $$M_x =$$
   and
   $$M_y =$$
3. $$\bar{x} =$$
   and
   $$\bar{y} =$$

Example

Let R be the region bounded by the curve $y = \sqrt{x}$, the $x-$axis and $x = 4$. This region has a constant density of 2kg/m^2 where x is measured in meters. Find the center of mass of this region (lamina). What is the centroid?

Example

More general lamina

Let's look at what happens if the lamina we are examining is defined by an upper curve $y = f(x)$ and a lower curve $y = g(x)$, again with a constant density ρ which has units of mass per length^2.

More general lamina

In general, assume R is the region bounded above by the graph of $y = f(x)$, below by the graph of $y = g(x)$ and on the left and right by $x = a$ and $x = b$ respectively. Also, suppose that the density of the associated lamina is the constant ρ. Then:

Example

Find the center of mass of the region bounded by the curves $y = 1 - x^2$ and $y = x$, with a constant density of $\rho = 1\text{kg/m}^2$ where x is in m.

Example

Example (The symmetry principal)

Find the center of mass of the region bounded by the curves $y = x^2 - 1$ and $\sqrt{1 - x^2}$, with constant density of $\rho = 3\text{kg/m}^2$ where x is in m.

Example

One more example

Find the center of mass of the region bounded by the curve $y = \cos(x)$ and the $x-$axis between $x = -\pi/2$ and $x = \pi/2$, where the lamina has a constant density of $1 \text{kg}/\text{m}^2$ where x is in meters.