Trigonometric substitution

Chase Mathison[1]

Shenandoah University

27 February 2024

[1]cmathiso@su.edu

Announcements

1. Don't forget about exam corrections!
2. Homework assigned in MyOpenMath.
3. Office hours M-F, 10am - 11am

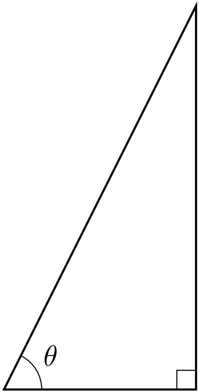
Trig substitution, why do we need it?

Today we will finally learn how to find an antiderivative of the function

$$\sqrt{1-x^2}.$$

Integrating functions that involve $\sqrt{a^2-x^2}$ is one of the key motivations for developing trigonometric substitution.
We'll also use trigonometric substitution to integrate functions that involve $\sqrt{x^2-a^2}$ and $\sqrt{a^2+x^2}$. So almost anytime that we deal with integrating a function that has one of these three forms in it, you can pretty safely bet that trigonometric substitution will be involved.

A reminder: SOHCAHTOA



The idea

I'll illustrate the main idea for integrating functions that involve $\sqrt{a^2 - x^2}$ with the following example.

$$\int \sqrt{1 - x^2}\, dx$$

Integrals involving $\sqrt{a^2 - x^2}$

In general, to integrate a function that involves $\sqrt{a^2 - x^2}$, use the following strategy:

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}} dx$, $\int \frac{x}{\sqrt{a^2 - x^2}} dx$, and $\int x\sqrt{a^2 - x^2}\, dx$, they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a \sin\theta$ and $dx = a\cos\theta d\theta$. *Note*: This substitution yields $\sqrt{a^2 - x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 3.4 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

Example

Evaluate

$$\int \frac{x^2}{\sqrt{4 - x^2}}\,dx$$

Example

Evaluate

$$\int_0^{\frac{1}{2}} \frac{1}{1 - x^2} \, dx$$

Example

Integrals involving $\sqrt{a^2+x^2}$

Integrating functions that involve $\sqrt{a^2+x^2}$ works in a very similar way, we just need to make a slightly different trigonometric substitution:

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 3.7 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (*Note*: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

Example

Evaluate
$$\int \frac{1}{\sqrt{3+x^2}}\,dx$$

Example