1. 抽象测度空间. $(X, \mathcal{M}, \mu)$.

A measure space consists of a set X equipped with two fundamental objects

(I) A σ-algebra $\mathcal{M}$ of "measurable" sets

(II) A measure $\mu: \mathcal{M} \to [0, \infty)$ with the following defining property:

① $\mu(\phi) = 0$,

② 对 $E_1, \cdots, E_n, \cdots$ is a countable family of disjoint sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

$(X, \mathcal{M}, \mu)$ is called measure space. (X, μ).

称 $(X, \mathcal{M}, \mu)$ 是完备的, 如果 $\mu(E) = 0$, $F \subset E$, 则 F 可测.

Example 1. $X = \{a_1, a_2, \cdots\}$ countable sets. $\mathcal{M} = P(X)$ (X 的子集的全体).

令 $\mu(\{x_n\}) = \mu_n$, $\{\mu_n\}$ 是给定的非负实数列. $\underline{\mu(E) = \sum_{a_n \in E} \mu_n}$.

则 $(X, \mathcal{M}, \mu)$ 是测度空间. 若 $\mu_n \equiv 1$, $\forall n$. μ 称为 counting measure.

Example 2. $X = R^d$, $\mathcal{M} = \{\text{Lebesgue 可测集}\}$. 令 $f \geq 0$ 是 R^d 上的可测函数. 定义 $\mu(E) = \int_E f \, dm$, 则 $(X, \mathcal{M}, \mu)$ 是测度空间.

特别地, 如 $f \equiv 1$, 则 μ 是 Lebesgue 测度.

目标: 构造测度空间.

X

Step 1. 由外测度构造测度空间.

An outer measure on a non-empty set X is a function $\mu^*: P(X) \to [0, +\infty]$ that satisfies

- $\mu^*(\phi) = 0$.
- $\mu^*(A) \leq \mu^*(B)$, $A \subset B$.
- $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

定义. 若 μ^* 是 X 上的外测度. 集合 $A\subset X$ 称为 μ^*-可测, 如果

$$\underline{\mu^*(E)=\mu^*(E\cap A)+\mu^*(E\cap A^c),\ \forall E\subset X.}\qquad E=(E\cap A)\cup(E\cap A^c).$$

Thm 1 (Caratheodory). If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Pf. ① $\mathcal{M}$ 在补集, 可数并, 可数交运算下封闭.

· 设 $E_1,\cdots,E_N\in\mathcal{M}$. 要证 $\bigcup_{j=1}^{N}E_j\in\mathcal{M}$.

$E_1,E_2\in\mathcal{M}$ $\Rightarrow$ $\underline{E_1\cup E_2\in\mathcal{M}}$.

$\forall E\subset X$,

$$\mu^*(E)=\mu^*(E\cap E_1)+\mu^*(E\cap E_1^c)$$

$$=\mu^*(E\cap E_1\cap E_2)+\mu^*(E\cap E_1\cap E_2^c)+\mu^*(E\cap E_1^c\cap E_2)+\mu^*(E\cap E_1^c\cap E_2^c)$$

$$\geq\mu^*(E\cap(E_1\cup E_2))+\mu^*(E\cap(E_1\cup E_2)^c).$$

$$\mu^*(E\cap(E_1\cup E_2)^c)=\mu^*(E\cap E_1^c\cap E_2^c).$$

② 设一列 $E_1,E_2,\cdots\in\mathcal{M}$ 不交

定义 $G=\bigcup_{j=1}^{\infty}E_j$, $\underline{G_n=\bigcup_{j=1}^{n}E_j}$. 则由①可知, $\forall E\subset X$,

$$\mu^*(E)=\mu^*(E\cap G_n)+\mu^*(E\cap G_n^c).$$

$$=\mu^*(E\cap G_n\cap E_n)+\mu^*(E\cap G_n\cap E_n^c)+\mu^*(E\cap G_n^c).$$

$$=\mu^*(E\cap E_n)+\mu^*(E\cap G_{n-1})+\mu^*(E\cap G_n^c)$$

$$=\sum_{j=1}^{n}\mu^*(E\cap E_j)+\mu^*(E\cap G_n^c).\qquad G_n\subset G.$$

$$\geq\sum_{j=1}^{n}\mu^*(E\cap E_j)+\mu^*(E\cap G^c)\qquad G_n^c\supset G^c.$$

令 $n\to\infty$. 则有 $\underline{\mu^*(E)}\geq\sum_{j=1}^{\infty}\mu^*(E\cap E_j)+\underline{\mu^*(E\cap G^c)}.$

$$\geq\mu^*(E\cap G)+\mu^*(E\cap G^c).$$

故 G 可测.

特别地, 取 $E=G$, 则 $\mu^*(G)=\sum_{j=1}^{\infty}\mu^*(E_j)$.

特别地，取 $E=G$，则 $\mu^*(G)=\sum_{j=1}\mu^*(E_j)$.

最后证明完备性．若 $\mu^*(E)=0$，则 $\forall A\subset X$，

$$\mu^*(A)\le \mu^*(A\cap E)+\mu^*(A\cap E^c)\le \mu^*(E)+\mu^*(A\cap E^c)\le \mu^*(A)$$

故 E 可测．

Step 2. 构造外测度． premeasure.

- $\mathcal{A}\subset \mathcal{P}(X)$ is an algebra: $\mathcal{A}$ is closed under complement, finite union, finite intersection.
- 称 $\mathcal{E}\subset\mathcal{P}(X)$ 是 elementary family，如果 ① $\phi\in\mathcal{E}$．

  ② $E,F\in\mathcal{E}\Rightarrow E\cap F\in\mathcal{E}$． ③ $E\in\mathcal{E}\Rightarrow E^c$ 是 $\mathcal{E}$ 中元素的不交并．

Prop (F-Prop 1.7). If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint union of members of $\mathcal{E}$ is an algebra.

- $\mu_0:\mathcal{A}\to[0,+\infty]$ is called a premeasure 若

  ① $\mu_0(\phi)=0$.

  ② If $\{A_j\}$ is a sequence of disjoint sets in $\mathcal{A}$, s.t. $\bigcup_{j=1}^{\infty}A_j\in\mathcal{A}$

  then $\mu_0(\bigcup_{j=1}^{\infty}A_j)=\sum_{j=1}^{\infty}\mu_0(A_j)$.

Prop (F-Prop 1.10) Let $\mathcal{E}\subset\mathcal{P}(X)$, and $\rho:\mathcal{E}\to[0,\infty]$ be such that $\phi\in\mathcal{E}$, $X\in\mathcal{E}$, $\rho(\phi)=0$. For any $A\subset X$, define

$$\mu^*(A)=\inf\left\{\sum_{j=1}^{\infty}\rho(E_j): E_j\in\mathcal{E},\ A\subset\bigcup_{j=1}^{\infty}E_j\right\}.$$

Then μ^* is an outer measure.

Pf. • $\mu^*(\phi)\le\rho(\phi)=0$

• $A\subset B$, $\forall\{E_j\}\supset B\supset A$, $\mu^*(A)\le\sum_{j=1}^{\infty}\rho(E_j)\Rightarrow \mu^*(A)\le\mu^*(B)$.

• $\mu^*(\bigcup_{j=1}^{\infty}A_j)\le\sum_{j=1}^{\infty}\mu^*(A_j)$

$\forall\varepsilon>0$, $\exists E_{jk}\in\mathcal{E}$ s.t. $A_j\subset\bigcup_{k=1}^{\infty}E_{j,k}$ 且 $\sum_{k=1}^{\infty}\rho(E_{jk})\le\mu^*(A_j)+\frac{\varepsilon}{2^j}$.

则 $\bigcup_{j,k=1}^{\infty} E_{j,k} \supset \bigcup_{j=1}^{\infty} A_j$

且 $\sum_{j,k=1}^{\infty} \rho(E_{j,k}) \le \sum_j \left(\sum_k \rho(E_{j,k})\right) \le \sum_j \left(\mu^*(A_j) + \frac{\varepsilon}{2^j}\right)$

$$\le \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon$$

由 ε 的任意性, $\mu^*(\bigcup A_j) \le \sum_{j=1}^{\infty} \mu^*(A_j)$.

If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, define $\forall E \subset X$,

$$\mu^*(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A},\ E \subset \underbrace{\bigcup_{j=1}^{\infty} A_j}_{\in \mathcal{A}} \right\} \quad (1.12)$$

then we have

Prop (F-Prop 1.13) If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by (1.12), then

a. $\mu^*|_{\mathcal{A}} = \mu_0$

b. every set in $\mathcal{A}$ is μ^*-measurable.

Pf. a). 只要证明若 $E \in \mathcal{A}$, 则 $\mu^*(E) = \mu_0(E)$.

由定义可知, $\mu^*(E) \le \mu_0(E)$. 只要证明 $\mu_0(E) \le \mu^*(E)$

若 $E \subset \bigcup_{j=1}^{\infty} A_j$, $A_j \in \mathcal{A}$. 令 $B_n = A_n - \bigcup_{j=1}^{n-1} A_j$, 则 B_n 不交.

且 $\mathcal{A} \ni \bigcup_{j=1}^{\infty} A_j = \bigcup_{n=1}^{\infty} B_n \supset E$. 于是,

$$\underline{\mu_0(E) \le \mu_0\left(\bigcup_{n=1}^{\infty} B_n\right)} = \sum_{n=1}^{\infty} \mu_0(B_n) \le \sum_{n=1}^{\infty} \mu_0(A_n).$$

故 $\mu_0(E) \le \mu^*(E)$.