$(X, \mathcal{M}, \mu)$

→ 若μ^*是一个外测度. 称$E\subset X$是μ^*可测, 如果$\forall A\subset X$, $\mu^*(A)=\mu^*(E\cap A)+\mu^*(E^c\cap A)$.
μ^*-可测集的全体构成一个σ-代数$\mathcal{M}$. 令$\mu=\mu^*|_{\mathcal{M}}$, 则$(X,\mathcal{M},\mu)$是一个测度空间.

$\boxed{\mu^*(A)\geq \mu^*(E\cap A)+\mu^*(E^c\cap A)}$.

→ $\mathcal{E}$基本族 ⟿ 代数$\mathcal{A}$ $\mu_0: \mathcal{A}\to[0,+\infty]$ premeasure

$$\mu^*(E)=\inf\left\{\sum_{j=1}^{\infty}\mu_0(A_j)\ \middle|\ A_j\in\mathcal{A},\ E\subset\bigcup_{j=1}^{\infty}A_j\in\mathcal{A}\right\} \quad (1.12)$$

则μ^*是外测度.

Prop. If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by (1.12), then

a). $\mu^*|_{\mathcal{A}}=\mu_0$

b) every set in $\mathcal{A}$ is μ^*-measurable.

Pf. b). 令$A\in\mathcal{A}$, 要证A是μ^*-可测. 即只要证明$\forall E\subset X$,

$$\mu^*(E)\geq\mu^*(A\cap E)+\mu^*(A^c\cap E).$$

不妨设$\mu^*(E)<+\infty$ 则对$\forall\varepsilon$, 存在$B_j\in\mathcal{A}$, $\bigcup_{j=1}^{\infty}B_j\supset E$, s.t.

$$\sum_{j=1}^{\infty}\mu_0(B_j)\leq\mu^*(E)+\varepsilon.$$

于是$\bigcup_{j=1}^{\infty}(B_j\cap A)\supset A\cap E$, $\bigcup_{j=1}^{\infty}(B_j\cap A^c)\supset A^c\cap E$

故由(1.12). $\mu^*(A\cap E)+\mu^*(A^c\cap E)\leq\sum_{j=1}^{\infty}\mu_0(B_j\cap A)+\sum_{j=1}^{\infty}\mu_0(B_j\cap A^c)$

$$=\sum_{j=1}^{\infty}\mu_0(B_j)\leq\mu^*(E)+\varepsilon.$$

由ε的任意性, $\mu^*(A\cap E)+\mu^*(A^c\cap E)\leq\mu^*(E)$.

Thm 1.14 ① Let $\mathcal{A}\subset\mathcal{P}(X)$ be an algebra, μ_0 is a premeasure on $\mathcal{A}$ and $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. There exits a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0. Namely $\mu=\mu^*|_{\mathcal{M}}$, μ^* is defined by (1.12).

Pf. 由Cara-thm, μ^*可测集构成一个σ-代数$\mathcal{M}'$, 由前述命

题, $\mu^*|_{\mathcal{A}} = \mu_0$. 且 $\mathcal{M}^* \supset \mathcal{A}$, 由于 $\mathcal{M}$ 是由 $\mathcal{A}$ 生成的, $\mathcal{M} \subset \mathcal{M}^*$.

故 $\mu^*|_{\mathcal{M}}$ 是一个测度. 且 $\mu^*|_{\mathcal{A}} = \mu_0$.

② If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \le \mu(E)$ $\forall E \in \mathcal{M}$ with equality when $\mu(E) < +\infty$.

Pf. 只需证明 $\nu(E) \le \mu^*(E)$. $\forall \bigcup_j E_j \supset E$, $E_j \in \mathcal{A}$, 则 $E_j \in \mathcal{M}$, 且 $\bigcup_j E_j \in \mathcal{M}$. 于是,

$$\boxed{\mu^*(E) \le \sum_{j=1}^{\infty} \mu_0(E_j)}$$

$$\nu(E) \le \nu\left(\bigcup_j E_j\right) \le \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu_0(E_j).$$

于是, $\nu(E) \le \mu^*(E) = \mu(E)$.

若 $\mu^*(E) < +\infty$. 要证 $\mu(E) \le \nu(E)$.

由于 $\mu^*(E) < +\infty$, $\forall \varepsilon > 0$, $\exists \bigcup E_j \supset E$, $E_j \in \mathcal{A}$, s.t. $\sum_{j=1}^{\infty} \mu(E_j) \le \mu(E) + \varepsilon$.

于是 $\mu\left(\bigcup_{j=1}^{\infty} E_j - E\right) \le \varepsilon$.

因此 $\mu(E) \le \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \overset{?}{=} \nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \nu\left(\bigcup_{j=1}^{\infty} E_j - E\right) + \nu(E)$

$$\le \mu\left(\bigcup_{j=1}^{\infty} E_j - E\right) + \nu(E) \le \varepsilon + \nu(E)$$

"?" $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \mu_0\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \nu\left(\bigcup_{j=1}^{n} E_j\right)$

$$= \nu\left(\bigcup_{j=1}^{\infty} E_j\right).$$

③ If μ_0 is σ-finite ($X = \bigcup_{j=1}^{\infty} E_j$, $E_j \in \mathcal{A}$. s.t. $\mu_0(E_j) < +\infty$), then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

Pf. 若 μ_0 是 σ-有限的. 设 $X = \bigcup_{j=1}^{\infty} E_j$, $\mu_0(E_j) < +\infty$. 不妨设 E_j 不交.

假设 ν 是 $\mathcal{M}$ 上的另一个测度, 且它是 μ_0 的延拓. $E_n = \bigcup_{j=1}^{n} E_j - \bigcup_{j=1}^{n-1} E_j$.

则 $\forall F \in \mathcal{M}$,

$$\mu(F) = \mu\left(\bigcup_{j=1}^{\infty} (F \cap E_j)\right) = \sum_{j=1}^{\infty} \mu(F \cap E_j) \overset{②}{=} \sum_{j=1}^{\infty} \nu(F \cap E_j)$$

则 $\forall F\in\mathcal{M}$,

$$\mu(F)=\mu\Big(\bigcup_{j=1}^{\infty}(F\cap E_j)\Big)=\sum_{j=1}^{\infty}\mu(F\cap E_j)\overset{②}{=}\sum_{j=1}^{\infty}\nu(F\cap E_j)$$

$$=\nu(F).$$

Metric exterior measure.

(X,d) metric space. $d: X\times X\longrightarrow[0,+\infty)$, s.t.

① $d(x,y)=0 \iff x=y$.

② $d(x,y)=d(y,x)$.

③ $d(x,y)\le d(x,z)+d(z,y)$.

$B_r(x)=\{y\in X \mid d(y,x)<r\}$

称 $O\subset X$ 是开集，如果 $\forall x\in O$, $\exists r>0$, s.t. $B_r(x)\subset O$. 称集合F是闭集如果它的补集是开集. 称包含X中所有开集的最小的σ-代数为Borel代数，记为 $\mathcal{B}_X$. $\mathcal{B}_X$ 中的元素被称为Borel集.

若 $A,B\subset X$, 定义 $d(A,B)=\inf\{d(x,y)\mid x\in A, y\in B\}$.

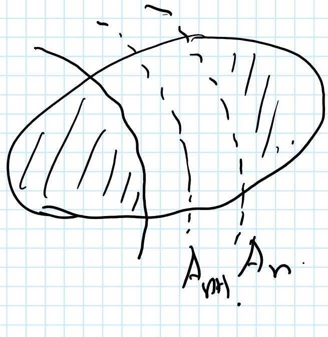
X上的外测度 μ^* 称为是度量外测度，如果 $\mu^*(A\cup B)=\mu^*(A)+\mu^*(B)$, $\forall d(A,B)>0$.

Thm (S-Thm 1.2) If μ^* is a metric exterior measure on a metric space X, then the Borel sets in X are measurable. Hence μ^* restricted to $\mathcal{B}_X$ is a measure.

Pf. 由于 cara-可测集是一个σ-代数，只需证明所有开集都是cara-可测的. 即所有闭集都可测. 令 $F\subset X$ 是闭集. 要证 $\forall A\subset X$,

$$\mu^*(A)\ge\mu^*(A\cap F)+\mu^*(A\cap F^c).$$

不妨设 $\mu^*(A)<+\infty$.



令 $A_n=\{x\in A\cap F^c \mid d(x,F)\ge\frac{1}{n}\}$, 则 $A_n\subset A_{n+1}$

且 $d(A_n, A\cap F)\ge\frac{1}{n}$

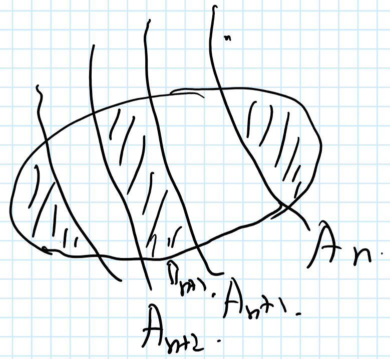
由于 μ^* 是度量外测度,

$$\mu^*(F)\ge\mu^*((A\cap F)\cup A_n)=\mu^*(A\cap F)+\mu^*(A_n)$$

claim: $\lim \mu^*(A_n)=\mu^*(A\cap F^c)$. (*)

claim: $\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A\cap F^c)$. (*)

则 $\mu^*(F) \geq \mu^*(A\cap F) + \mu^*(A\cap F^c)$.



再来证明(*). 令 $B_n = A_{n+1} - A_n$, 则 B_{n+1} 与 A_n 较

远, $d(B_{n+1}, A_n) \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

故由度量外测度的定义,

$\mu^*(A) \geq \mu^*(A_{2k+1}) \geq \mu^*(B_{2k} \cup A_{2k-1}) = \mu^*(B_{2k}) + \mu^*(A_{2k-1})$

$\geq \sum_{j=1}^{k} \mu^*(B_{2j})$.

由 $A_{2k+1} \subset A\cap F^c \subset A$ 知 $\mu^*(A_{2k+1}) \leq \mu^*(A) < +\infty$.

因此, $\sum_{j=1}^{\infty} \mu^*(B_{2j})$ 收敛. 同理, $\sum_{j=1}^{\infty} \mu^*(B_{2j-1})$ 收敛.

于是, 由 $A\cap F^c = A_n \cup \bigcup_{j=n}^{\infty} B_j$. 可知

$\mu^*(A_n) \leq \mu^*(A\cap F^c) \leq \mu^*(A_n) + \sum_{j=n}^{\infty} \mu^*(B_j)$

即 $\left|\mu^*(A\cap F^c) - \mu^*(A_n)\right| \leq \sum_{j=n}^{\infty} \mu^*(B_j) \xrightarrow{n\to\infty} 0$.

因此, $\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A\cap F^c)$.

Prop (S-Prop 1.3) Suppose that the Borel measure μ is finite on all balls in X of finite radius. Then for any Borel set E and $\varepsilon>0$, there are an open set O and a closed set F such that $O\supset E$, $F\subset E$, and $\mu(O-E)<\varepsilon$, $\mu(E-F)<\varepsilon$.

Pf. Step 1. 令 F_k 闭, $F^* = \bigcup_{k=1}^{\infty} F_k$. 则 $\forall \varepsilon>0$, $\exists$ 闭 $F\subset F^*$ s.t. $\mu(F^*-F)<\varepsilon$.

固定 $x_0\in X$, 令 $B_n = \{x \mid d(x,x_0)<n\}$. 则 B_n 开 且 $X = \bigcup_{n=1}^{\infty} B_n$.

立得 $X = \bigcup_{n=1}^{\infty} (\overline{B}_n - B_{n-1})$. 于是 $F^* = \bigcup_{n=1}^{\infty} (F^* \cap (\overline{B}_n - B_{n-1}))$

由于 $F^* \cap (\overline{B}_n - B_{n-1}) = \bigcup_{k=1}^{\infty} F_k \cap (\overline{B}_n - B_{n-1})$, $\mu(\overline{B}_n) < +\infty$. 不妨设 $F_k \nearrow F^*$

由于 $F^* \cap (\overline{B}_n - B_{n-1}) = \bigcup_{k=1}^{\infty} F_k \cap (\overline{B}_n - B_{n-1})$ $\mu(\overline{B}_n) < +\infty$. 而由前 $F_k \nearrow F^*$

故 $\lim_{n\to\infty} \mu(F_k \cap (\overline{B}_n - B_{n-1})) = \mu(F^* \cap (\overline{B}_n - B_{n-1}))$.

于是 $\forall \varepsilon > 0$, $\exists N(n)$, s.t. $\mu(F^* \cap (\overline{B}_n - B_{n-1}) - F_{N(n)} \cap (\overline{B}_n - B_{n-1})) < \varepsilon/2^n$.

令 $F = \bigcup_{n=1}^{\infty} F_{N(n)} \cap (\overline{B}_n - B_{n-1}) \subset F^*$, 则 $\mu(F^* - F) \le \sum_{n=1}^{\infty} \mu(F^* \cap (\overline{B}_n - B_{n-1})$

$- F_{N(n)} \cap (\overline{B}_n - B_{n-1})) \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$.

由于 $F \cap \overline{B}_k$ closed, $\forall k$. 故 F 是闭集.

Step 2. 令 $\mathcal{C} = \{E \subset X \mid \forall \varepsilon > 0, \exists$ 开 $O \supset E$, 闭 $F \subset E$, s.t. $\mu(O - E) < \varepsilon$, $\mu(E - F) < \varepsilon\}$.

要证 $\mathcal{C}$ 是一个 σ-代数.

- 若 $E \in \mathcal{C}$, 则 $E^c \in \mathcal{C}$. $\mu(O - E) = \mu(E^c - O^c) < \varepsilon$.

- 若 $E_j \in \mathcal{C}$, 要证 $\bigcup_{j=1}^{\infty} E_j \triangleq E \in \mathcal{C}$.

由于 $E_j \in \mathcal{C}$, $\forall \varepsilon > 0$, $\exists$ 开 $O_j \supset E_j$, s.t. $\mu(O_j - E_j) < \frac{\varepsilon}{2^j}$.

令 $O = \bigcup_{j=1}^{\infty} O_j \supset E$, 则 O 是开集. 且 $\mu(O - E) \le \sum_j \mu(O_j - E_j) < \sum_j \frac{\varepsilon}{2^j} = \varepsilon$.

$\forall \varepsilon > 0$, $\exists$ 闭集 $F_j \subset E_j$, s.t. $\mu(E_j - F_j) < \frac{\varepsilon}{2^{j+1}}$. 令 $F^* = \bigcup_{j=1}^{\infty} F_j$.

则 $\mu(E - F^*) < \varepsilon/2$. 由 Step 1, $\exists F$ 闭, $F \subset F^*$, s.t. $\mu(F^* - F) < \varepsilon/2$

故 $F \subset E$, 且 $\mu(E - F) \le \mu(E - F^*) + \mu(F^* - F) \le \varepsilon$.

Step 3. $\mathcal{C}$ 包含所有开集

设 O 是开集, 只要证明 $\forall \varepsilon > 0$, $\exists$ 闭集 $F \subset O$, s.t. $\mu(O - F) < \varepsilon$.

令 $F_k = \{x \in \overline{B}_k \mid d(x, O^c) \ge \frac{1}{k}\}$. 则 F_k 是闭集. 且 $O = \bigcup_{k=1}^{\infty} F_k$

由 Step 1, 存在 F 闭 $\subset O$, s.t. $\mu(O - F) < \varepsilon$.

3 Examples.

3.1. Product measures

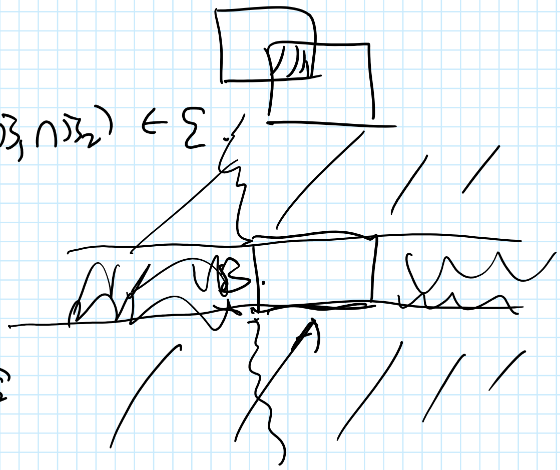
设 $(X_1, \mathcal{M}_1, \mu_1)$，$(X_2, \mathcal{M}_2, \mu_2)$ 是测度空间.

令 $X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1,\ x_2 \in X_2\}$.

要在 $X_1 \times X_2$ 上构造测度空间.

可测矩形 $\{A \times B \mid A \in \mathcal{M}_1,\ B \in \mathcal{M}_2\} \triangleq \mathcal{E}$.

- $\phi \in \mathcal{E}$.
- $A_1 \times B_1 \cap A_2 \times B_2 = (A_1 \cap A_2) \times (B_1 \cap B_2) \in \mathcal{E}$.
- $(A \times B)^c = (X \times B^c) \cup A^c \times B$



则 $\mathcal{E}$ 是基本族.

由 Prop 1.7，$\mathcal{E}$ 中元素有限交并的集合

是一个代数 $\mathcal{A}$.

在 $\mathcal{A}$ 上定义 μ_0：$\underline{\mu_0(A \times B) = \mu_1(A)\,\mu_2(B)}$.

claim: 若 $A \times B = \bigcup_{j=1}^{\infty} A_j \times B_j$（不交并），则 $\underline{\mu_0(A \times B) = \sum_{j=1}^{\infty} \mu_0(A_j \times B_j)}$.

$$\chi_A(x)\chi_B(y) = \chi_{A \times B}(x,y) = \sum_{j=1}^{\infty} \chi_{A_j \times B_j}(x,y) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\chi_{B_j}(y)$$

对 x 积分，$\mu_1(A)\chi_B(y) = \sum_{j=1}^{\infty} \mu_1(A_j)\chi_{B_j}(y)$

再对 y 积分，$\mu_1(A)\mu_2(B) = \sum_{j=1}^{\infty} \mu_1(A_j)\mu_2(B_j)$

$$\Longleftrightarrow \quad \mu_0(A \times B) = \sum_{j=1}^{\infty} \mu_0(A_j \times B_j)$$

故 μ_0 是 $\mathcal{A}$ 上的 premeasure.

于是，由 F-Thm 1.14，存在 $X_1 \times X_2$ 上的 $\underline{\sigma\text{-代数 } \mathcal{M}_1 \otimes \mathcal{M}_2}$，测度 μ

s.t. $\mu|_{\mathcal{A}} = \mu_0$. 记 $\mu = \mu_1 \times \mu_2$. 于是，我们构造了测度空间

$(X_1 \times X_2,\ \mathcal{M}_1 \otimes \mathcal{M}_2,\ \mu_1 \times \mu_2)$.