# STOCHASTICS : PROBABILITY

## Q1:

a) $\Omega = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}$

b) $\omega = hht$ , $E = \{hhh, thh, tth\}$

c) $E_1 = \{hhh, ttt\}$

$E_2 = \{hht, hth, thh\}$ $\quad E_1 \cap E_2 = \emptyset$

$E_3 = \{hhh, ttt\}$

$E_4 = \{hhh, hht, hth, thh\}$ $\quad E_3 \cap E_4 = \{hhh\}$

d) $E_1 = \{hhh, ttt\}$ $\quad \overline{E_1} = \{hht, hth, thh, tth, tht, htt\}$

e) $E_2 = \{htt, tht, tth\}$ $\quad \overline{E_2} = \{hhh, hht, hth, thh, ttt\}$

$E_3 = \{hhh, hht, hth, hht\}$ $\quad \overline{E_3} = \{ttt, tth, tht, htt\}$

$E_4 = E_3$ $\quad \overline{E_4} = \overline{E_3}$

f) $E_1 \cap E_3 = \{hhh\}$

g) $E_1 \cup E_3 = \{hhh, hht, hth, hht, ttt\}$

## Q2:

$P(a) = \frac{7}{25}$ $\quad P(b) = \frac{4}{25}$ $\quad P(c) = \frac{6}{25}$

$P(d) = \frac{8}{25}$

Ex 3:

a) All events have positive probability.

b) $P(\{1,2,3,4,5,6\}) = 1$ i.e. we certainly throw a number.

c) $P(\{1\} \cup \{2\}) = \frac{1}{3} = \frac{1}{6} + \frac{1}{6} = P(\{1\}) + P(\{2\})$

1) $P(\emptyset) = 0$ i.e. it is impossible that we do not throw any number.

2) $P(\overline{\{1\}}) = P(\{2,3,4,5,6\}) = \frac{5}{6} = 1 - \frac{1}{6} = 1 - P(\{1\})$

3) $P(\{\text{even}\} \cup \{\text{multiple of } 3\}) = P(\{2,3,4,6\}) = \frac{4}{6}$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = P(\{\text{even}\}) + P(\{\text{multiple of } 3\}) - P(\{3\})$

Ex 4: $E \cup F = (E \setminus F) \cup (E \cap F) \cup (F \setminus E)$

mutually exclusive

$E = (E \setminus F) \cup (E \cap F)$

$F = (F \setminus E) \cup (E \cap F)$

$\overset{c)}{\Rightarrow}$ $P(E \cup F) = P(E \setminus F) + P(E \cap F) + P(F \setminus E)$

$P(E) = P(E \setminus F) + P(E \cap F)$

$\Rightarrow P(E \setminus F) = P(E) - P(E \cap F)$

$P(F) = P(F \setminus E) + P(E \cap F)$

$\Rightarrow P(F \setminus E) = P(F) - P(E \cap F)$

$\Rightarrow P(E \cup F) = P(E) - P(E \cap F) + P(F) - P(E \cap F) + P(E \cap F)$

$= P(E) + P(F) - P(E \cap F)$

2/8

Ex 5:

- Yes, all sides are thrown equally likely.
- No, sides are not equal.
- Yes.

Ex 6:

G4b:

$P(\{R\}) = \frac{1}{11}$

$P(\{E\}) = 0$

$P(\{P, R, B, C, T, Y\}) = \frac{2}{11}$

$P(\{O, A, I\}) = \frac{4}{11}$

$P(\{L, E\}) = \frac{1}{11}$

G4c:

# PROBABILITY

## Ex 7

a) $P(\{\text{at least once a } 1\}) = 1 - P(\{\text{never a } 1\}) = 1 - \frac{5\cdot5\cdot5}{6\cdot6\cdot6} = \frac{91}{216} = \underline{\underline{0.42}}$

b) $P(\{3 \text{ at most twice}\}) = 1 - P(\{3\text{x a } 3\}) = 1 - \frac{1\cdot1\cdot1}{6\cdot6\cdot6} = \frac{215}{216} = \underline{\underline{0.995}}$

c) $P(\{\text{sum is not } 5\}) = 1 - P(\{\text{sum is } 5\}) = 1 - 3\cdot\left(\frac{1}{6^3} + \frac{1}{6^3}\right)$

$5 = 1+1+3$ $\qquad = \frac{35}{36} = \underline{\underline{0.972}}$

$5 = 2+2+1$

## Ex 8;

$P(\{\text{a winner}\}) = \left(1-\frac{1}{2}\right)\cdot 0.2 = 0.1 = \underline{\underline{10\%}}$

$P(\{\text{a consolation prize}\}) = 1 - 50\% - 10\% = \underline{\underline{40\%}}$

$P(\{\text{a loser}\}) = 50\%$

$P(\{\text{not a loser}\}) = 1 - 50\% = 50\%$

The 400 tickets information is irrelevant.

## Ex 9;

a) $P(\{\text{all 6 numbers correct}\}) = \frac{1}{\binom{42}{6}} = 1.9\cdot10^{-7}$

b) $P(\{\text{exactly 4 out of 6 correct}\}) = \frac{\binom{6}{4}\cdot\binom{36}{2}}{\binom{42}{6}} = 0.00180$

c) $P(\{\text{at least 4 out of 6 correct}\}) = \frac{\binom{6}{4}\binom{36}{2}}{\binom{42}{6}} + \frac{\binom{6}{5}\binom{36}{1}}{\binom{42}{6}} + \frac{1}{\binom{42}{6}}$

$= 0.001843$

**Ex 10:**

$P(\{\text{two students have same birth date day}\})$

$= 1 - P(\{\text{all students have on different days}\})$

$$= 1 - \frac{\frac{365!}{(365-18)!}}{365^{18}} = \underline{\underline{0.35}}$$

**Ex 11**

No: If there is a non-ordered with repetion case, it is not a Laplace experiment: The outcome "all the same" is not as likely as the outcome "of each one" since in the first case they all have to be the same, i.e. 1 possibility, in the second case there are all permutations.

If it is ordered, the chances for each case are the same. If it is not ordered but without rep. the same holds.

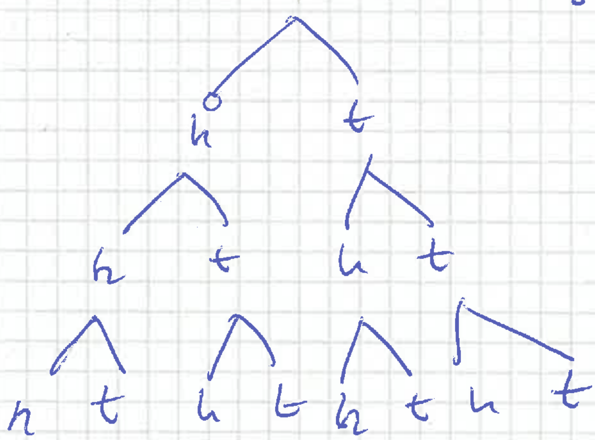
**Ex 12:**

$P(\{\text{no head}\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$P(\{\text{exactly one head}\}) = P(\{htt, tht, tth\}) = 3 \cdot \frac{1}{8} = \frac{3}{8}$

$P(\{\text{exactly two head}\}) = \frac{3}{8}$

$P(\{\text{at least two head}\}) = \frac{3}{8} + \frac{1}{8} = \underline{\underline{\frac{1}{2}}}$



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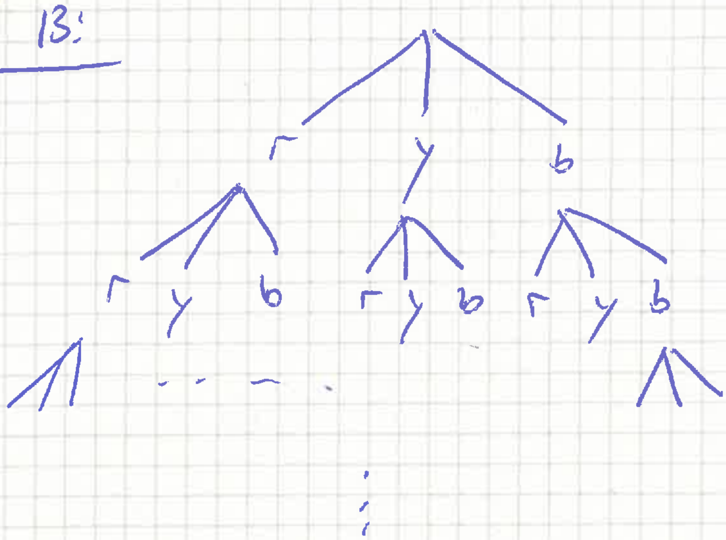
Ex 13:

In total $3^7$ paths.




$P(\{\text{all red}\}) = \frac{1}{3^7} = \frac{1}{2187} = 0.000457$

$P(\{\text{all same color}\}) = P(\{\text{all red}\}) + P(\{\text{all } y\}) + P(\{\text{all } b\}) = 3 \cdot \frac{1}{3^7} = \frac{1}{3^6}$

$P(\{\text{no blue}\}) = \frac{2^7}{3^7} = \left(\frac{2}{3}\right)^7 = \underline{\underline{0.059}}$

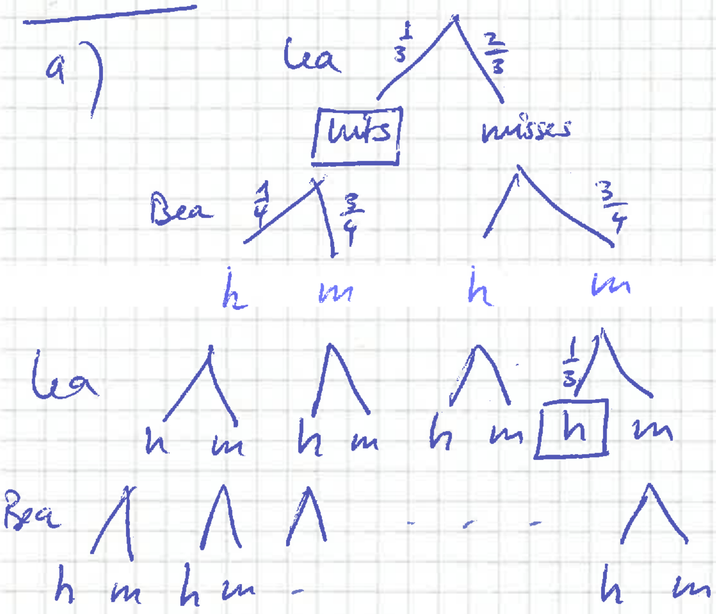
$P(\{\text{exactly one plate empty}\}) = 3 \cdot \left(\frac{2}{3}\right)^7 = \frac{2^7}{3^6} = \underline{\underline{0.176}}$

$P(\{\text{at least 1 plate empty}\}) = \underbrace{\frac{2^7}{3^6}}_{\text{exactly one empty}} + \underbrace{\frac{1}{3^6}}_{\text{exactly two empty} = \text{all same color}} = \frac{2^7+1}{3^6} = \underline{\underline{0.177}}$

$P(\{\text{no plate empty}\})$
$= 1 - P(\{\text{at least 1 plate empty}\}) = 1 - \frac{2^7+1}{3^6}$
$= \frac{3^6 - 2^7 - 1}{3^6} = \underline{\underline{0.823}}$

## Q 14:

a)




Lea hits first: h or mm h

$P(\{h\}) = \frac{1}{3}$

$P(\{mmh\}) = \cancel{\frac{2}{3}} \cdot \cancel{\frac{3}{4}} \cdot \frac{1}{3} = \frac{1}{6}$

$\Rightarrow P(\{\text{Lea hits first and wins}\}) = \frac{1}{3} + \frac{1}{6}$

$= \frac{1}{2}$

b) $P(\{\text{Bea wins}\}) = P(\{mh\}) + P(\{mmmh\})$

$= \frac{2}{3} \cdot \frac{1}{4} + \cancel{\frac{2}{3}} \cdot \cancel{\frac{3}{4}} \cdot \frac{2}{3} \cdot \frac{1}{4}$

$= \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

(in the rest of the cases no one wins)

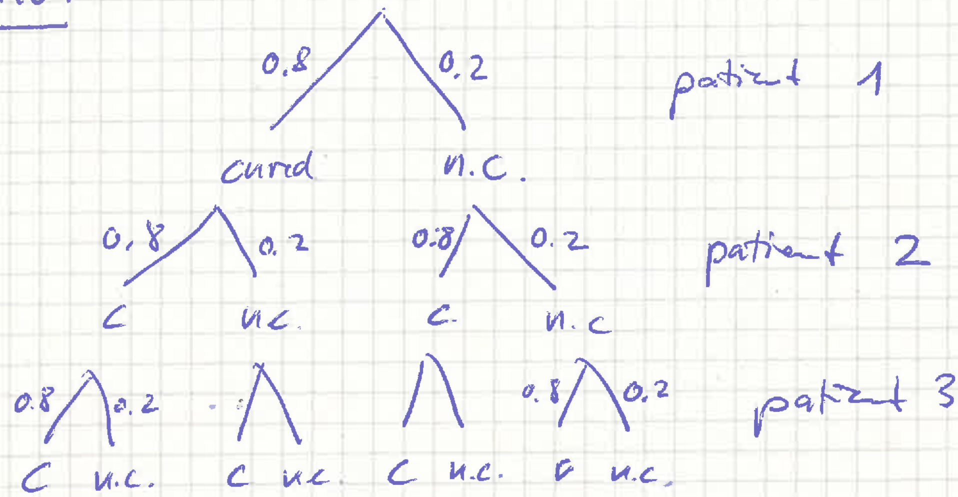
c) $P(\{\text{Lea wins}\}) = \sum_{k=0}^{\infty} \left(\frac{2}{3} \cdot \frac{3}{4}\right)^k \cdot \frac{1}{3} = \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

$= \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{2}{3}$

$P(\{\text{Bea wins}\}) = 1 - \frac{2}{3} = \frac{1}{3}$

5/8

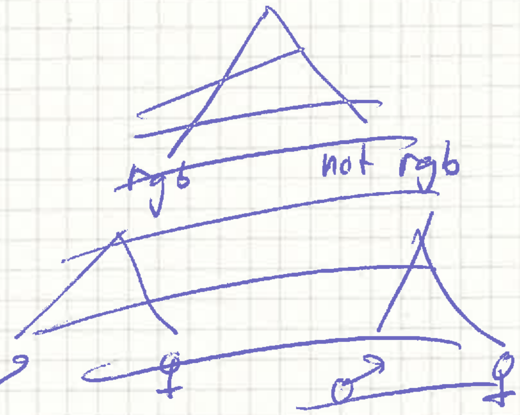
## Ex 15:

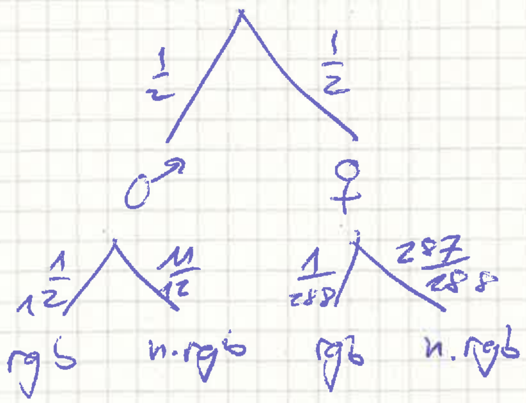



$$P(\{ccc\}) = 0.8^3 = \underline{\underline{0.512}}$$

$$P(\{\text{at least } 2\}) = 0.8^3 + 3 \cdot 0.8^2 \cdot 0.2 = \underline{\underline{0.896}}$$

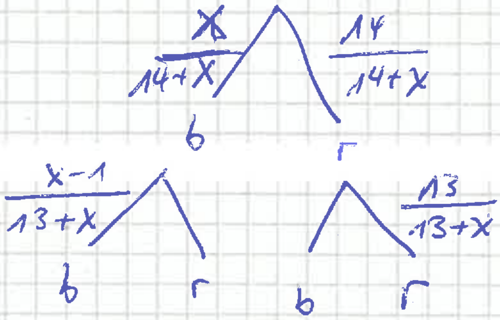
## Ex 16: $P(\{\text{man} \mid \text{red-green-blind}\}) = ?$




$$P(\{♂ \mid rgb\}) = \frac{\frac{1}{12} \cdot \frac{1}{2}}{\frac{1}{12} \cdot \frac{1}{2} + \frac{1}{288} \cdot \frac{1}{2}} = \underline{\underline{0.96}}$$

Ex 17: $P(\{\text{two blue balls} \mid \text{two same-colored balls}\}) = \frac{3}{16}$.




$$P(\{bb \mid \text{same color}\}) = \frac{\frac{x}{14+x} \cdot \frac{x-1}{13+x}}{\frac{x(x-1)}{(14+x)(13+x)} + \frac{14}{14+x} \cdot \frac{13}{13+x}} \overset{!}{=} \frac{3}{16}$$

$\xrightarrow[\text{solve}]{\text{CAS}}$ $x = -6$ or $x = 7$

$\downarrow$ does not make sense! $\Longrightarrow$ 7 blue balls.

Ex 18: a) $P(\{5 \text{ or } 6 \text{ at least once}\})$

$= 1 - P(\{\text{never } 5 \text{ or } 6\}) = 1 - \left(\frac{4}{6}\right)^7 = 0.941$

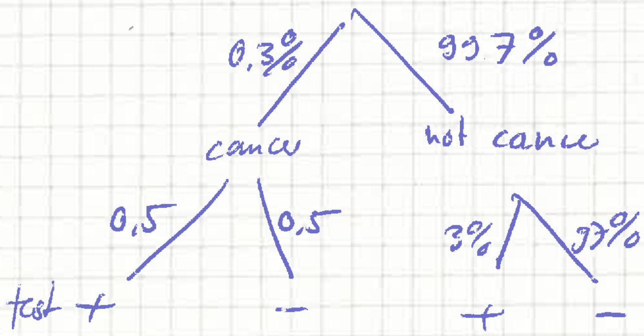
b) $1 - \left(\frac{4}{6}\right)^n \geq 0.995$

$1 - \left(\frac{2}{3}\right)^2 = 0.995 \Rightarrow n = 13.067$

$\Rightarrow$ 14 times

## Ex 19:

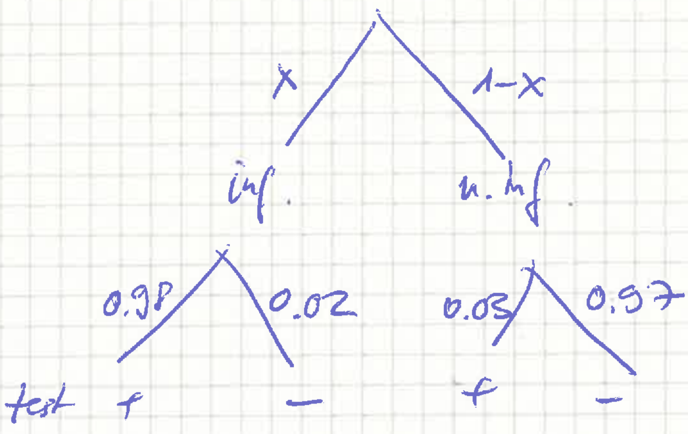
$P(\{\text{colorectal cancer} \mid \text{positive test}\}) = ?$




$$\Rightarrow P(\{\text{cancer} \mid +\}) = \frac{0.003 \cdot 0.5}{0.003 \cdot 0.5 + 0.997 \cdot 0.03} \approx 0.048 = \underline{\underline{4.8\%}}$$

## Ex 20:

$P(\{\text{infected} \mid \text{positive test}\}) = 0.89$




$$0.89 = \frac{x \cdot 0.98}{x \cdot 0.98 + (1-x) \cdot 0.03} \overset{\text{CAS}}{\Longrightarrow} \underline{\underline{x}} = 0.199 \approx \underline{\underline{20\%}}$$

## Ex 21:

a) $X: \{\text{rainfall on certain day}\} \longrightarrow [0, 200]$

$x \longmapsto$ height of water in gauge.

continuous or discrete if rounded up.

b) $X: \{\text{stopping distance}\} \longrightarrow [0, 100]$

$x \longmapsto$ length of braking distance.

continuous.

c) $X: \{\text{reliability of light switch}\} \longrightarrow [0, 10'000]$

$x \longmapsto$ # turn off/on till fail

discrete. or $X: \{\text{reliability}\} \longrightarrow [0, 10'000]$

$x \longmapsto$ Time till it fails.

## Ex 22:

a) $0 \leq x \leq 4$

b)

| | | | | |
|---|---|---|---|---|
| YYYY | YYYN | YYNN | YNNN | NNNN |
| | YYNY | YNYN | NYNN | |
| | YNYY | NYYN | NNYN | |
| | NYYY | YNNY | NNNY | |
| | | NYNY | | |
| | | NNYY | | |
| 4 | 3 | 2 | 1 | 0 |

Two accurate: $x = 2$

At least two accurate: $x \in \{2, 3, 4\}$.

7/8

Ex 23: a) $0.3+k+0.5 \overset{!}{=} 1 \Rightarrow \underline{\underline{k=0.2}}$

b) $k+2k+3k+k=1 \Rightarrow 7k=1 \Rightarrow \underline{\underline{k=\frac{1}{7}}}$

Ex 24:

| X-values | Two head | 1 head 1 tail | 2 tail |
|---|---|---|---|
| $P(x_i)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

Ex 25:

a) $$\sum_{x=1}^{4} \frac{x^2+1}{34} = \frac{2}{34} + \frac{5}{34} + \frac{10}{34} + \frac{17}{34} = \frac{34}{34} = 1$$

$\frac{2}{34} < 1$, $\frac{5}{34} < 1$, $\frac{10}{34} < 1$, $\frac{17}{34} < 1$

b) $$\sum_{x=0}^{3} \underbrace{\binom{3}{x}\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{3-x}}_{<1 \;\forall x.} = \left(\frac{3}{5}+\frac{2}{5}\right)^3 = 1^3 = 1 \quad \checkmark$$

Ex 26:

| X-values | 0 red | 1 red | 2 red |
|---|---|---|---|
| $P(x_i)$ | $\underbrace{\frac{2}{6}\cdot\frac{1}{5}}_{=\frac{1}{15}}$ | $\underbrace{\frac{4}{6}\cdot\frac{2}{5}\cdot 2}_{\frac{8}{15}}$ | $\underbrace{\frac{4}{6}\cdot\frac{3}{5}}_{\frac{6}{15}}$ |

## Ex 27:

$$P(\text{no faulty component}) = P(0) = \binom{10}{0} \cdot 0.4^0 \cdot 0.96^{10} = 0.96^{10}$$

$$\approx 0.66 \approx \underline{\underline{66\%}}$$

$$P(\text{at least one faulty}) = 1 - P(0) = 1 - 0.96^{10} \approx 0.34 = \underline{\underline{34\%}}$$

## Ex 28:

$$P(0) = e^{-0.2} \approx 0.8187$$

$$P(1) = \frac{0.2 \cdot e^{-0.2}}{1} \approx 0.1637$$

$$P(2) = \frac{0.2^2 \cdot e^{-0.2}}{2} \approx 0.0164$$

$$P(\text{at least } 3) = 1 - P(0) - P(1) - P(2)$$

$$= 0.0011 \quad (\text{almost } 0 \ldots)$$

## Ex 29:

$0.28 \cdot 365 = 102.2$ i.e. about 102 days.

## Ex 30:

In $\frac{6}{36}$ of the cases we throw twice the same number of pips, i.e. in $\frac{1}{6} \cdot 180 = 30$ cases.

## Ex 31:

| # head | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

$$E(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = 1.5$$

i.e. the expected value is less than the cost of the game, so I would not play the game.

8/8

## Ex 32:

$$E(x) = 0 \cdot \frac{5}{8} + 1 \cdot \frac{1}{8} + 2 \cdot \frac{2}{8} = \frac{9}{8} = 1.125$$

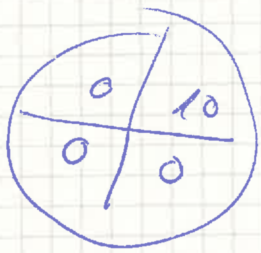
## Ex 33:

a) $E(X) = 0 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{4} = \frac{14}{4} = 3.5$

i.e. The game should cost chf 3.50.

b) Many possibilities

For example

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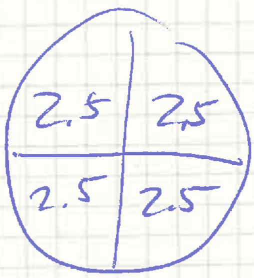


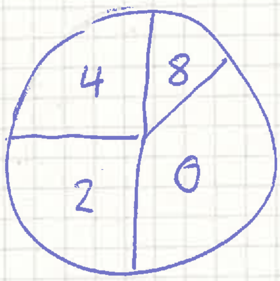


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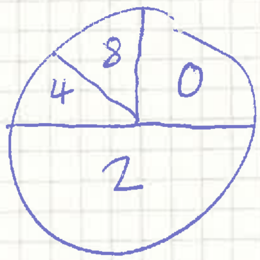

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