Variations on Stochastic Gradient Descent

Computational Statistics

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Errata

- There is rationale for step size differences in least squared loss and log-likelihood loss: gradients are larger in former.

Last Time

SGD
Introduced stochastic gradient descent (SGD) and mini-batch version thereof.

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Problems

We indicated that there were problems with vanilla SGD: poor convergence, erratic behavior.

Algorithm 1: Mini-Batch SGD

Data: $\gamma_0 > 0$

for $k \leftarrow 1, 2, \ldots$ **do**

  $A_k \leftarrow$ random mini-batch of m samples;

  $x_k \leftarrow x_{k-1} - \frac{\gamma_k}{|A_k|} \sum_{i \in A_k} \nabla f_i(x_{k-1})$;

Today

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Momentum
Base update on combination of gradient step and previous point.

Two versions: Polyak and Nesterov momentum

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Base update on combination of gradient step and previous point.

Two versions: Polyak and Nesterov momentum

Adaptive Gradients

Adapt learning rate to particular feature.

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Basic Idea
Give the particle **momentum**: like a heavy ball

Not specific to stochastic GD!

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Polyak Momentum

Classical version

$\mu \in [0, 1)$ decides strength of momentum; $\mu = 0$ gives standard gradient descent

Typically let $x_{-1} = x_0$.

Guaranteed convergence for quadratic functions

Algorithm 2: GD with Polyak Momentum

Data: $\gamma > 0$, $\mu \in [0, 1)$

for $k \leftarrow 1, 2, \ldots$ **do**

$\quad x_k \leftarrow x_{k-1} - \gamma \nabla f(x_{k-1}) + \mu(x_{k-1} - x_{k-2})$;

Polyak Momentum in Practice

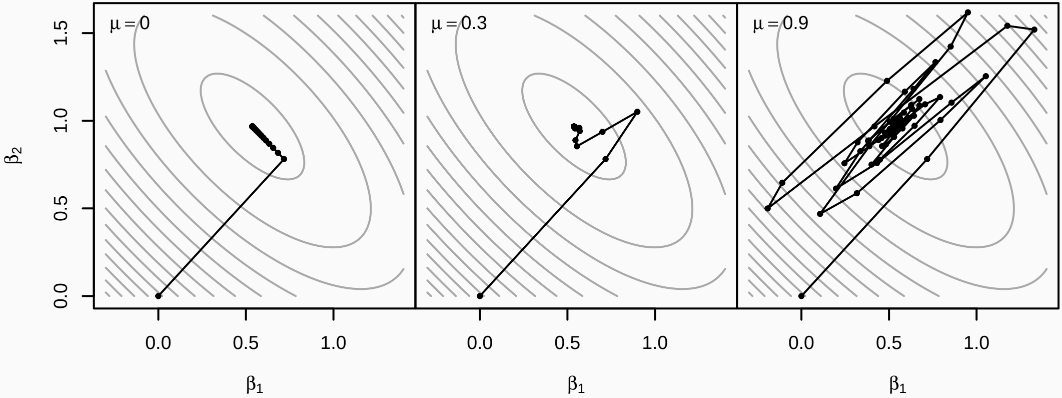

Figure 1: Trajectories of GD for different momentum values for a least-squares problem

Local Minima

For nonconvex f, gradient descent may get stuck at a local minimum.

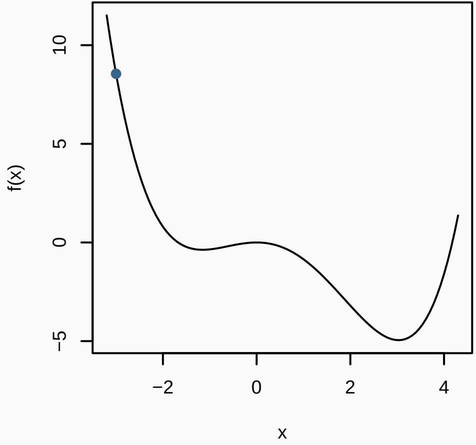

Figure 2: $\mu = 0$

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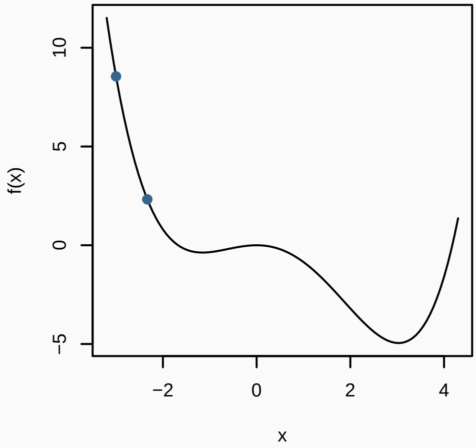

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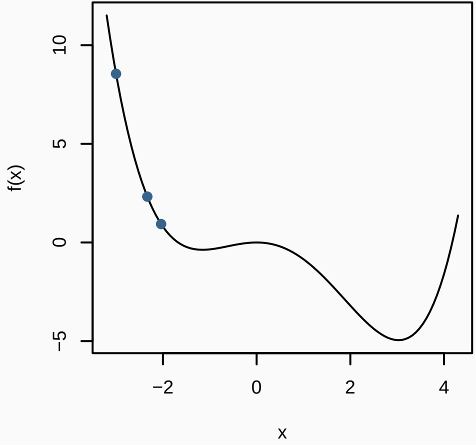

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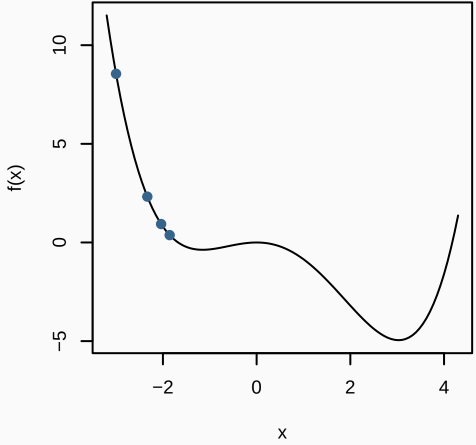

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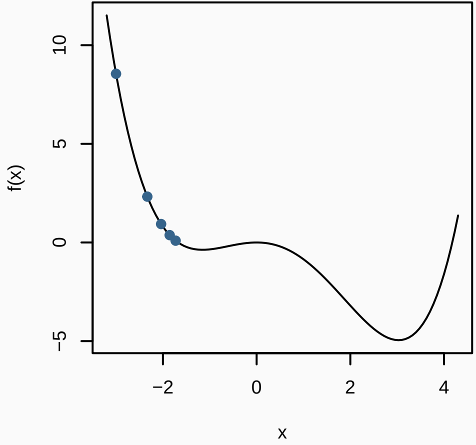

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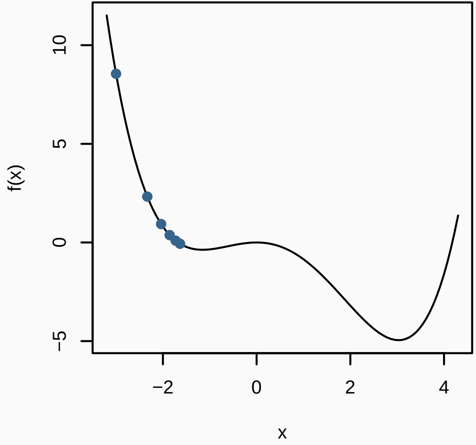

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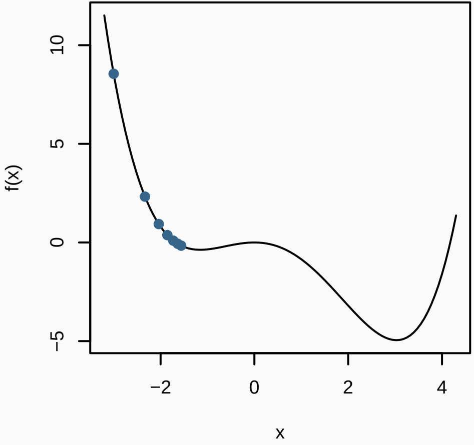

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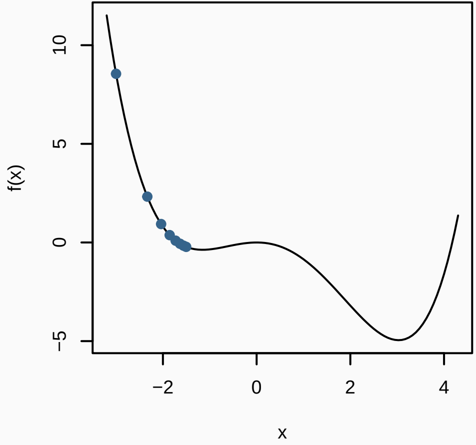

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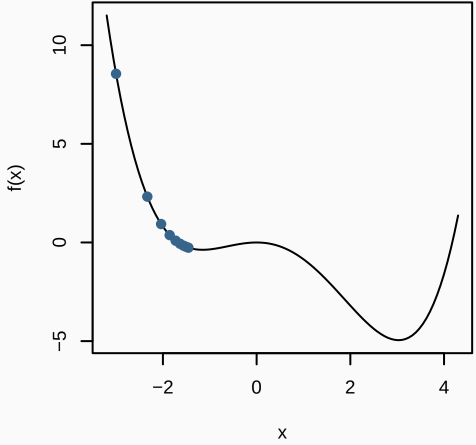

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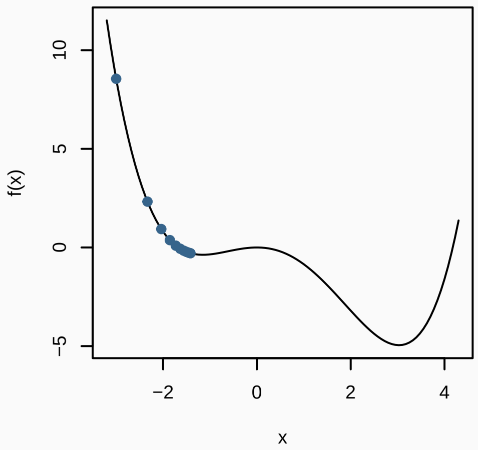

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Escaping Local Minima

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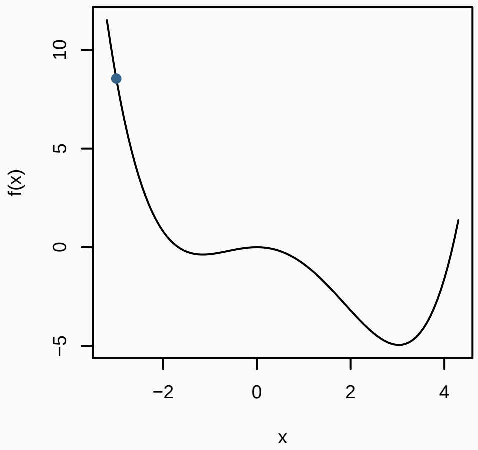

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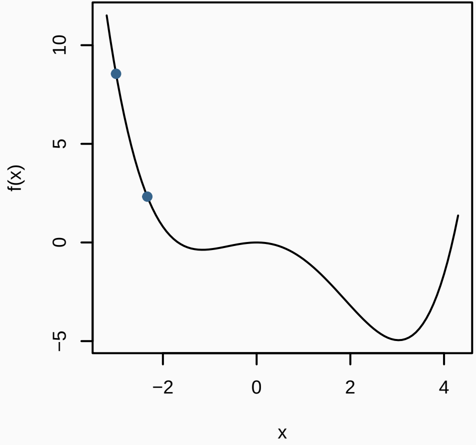

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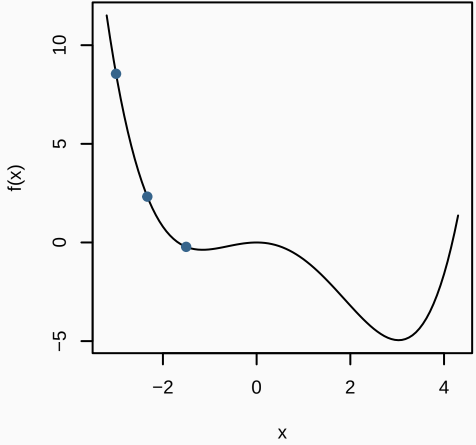

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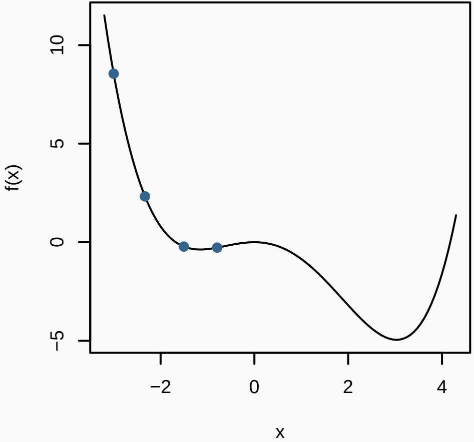

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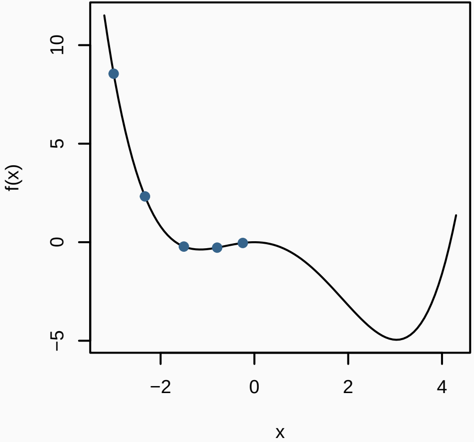

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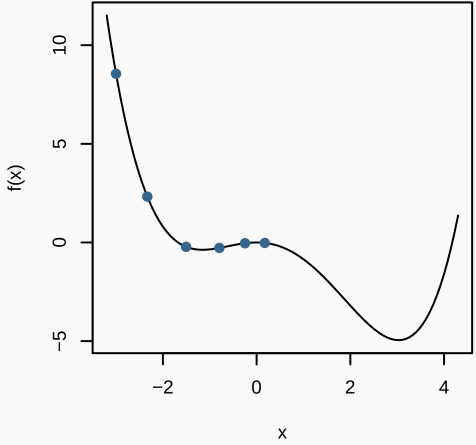

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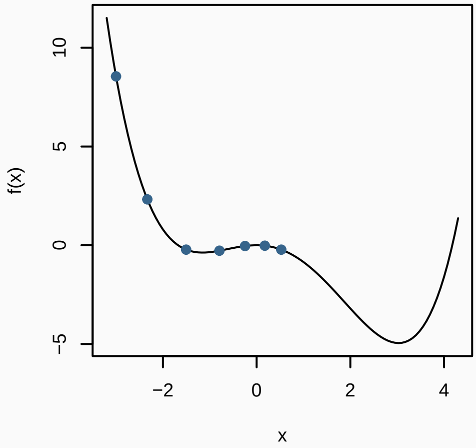

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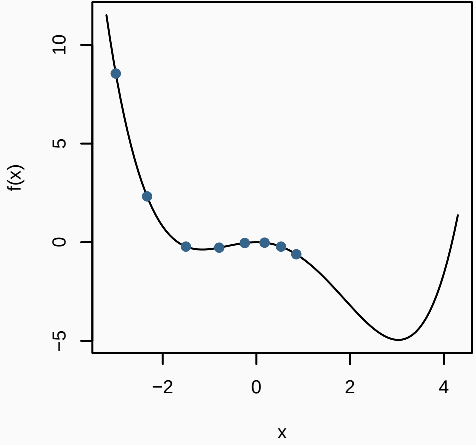

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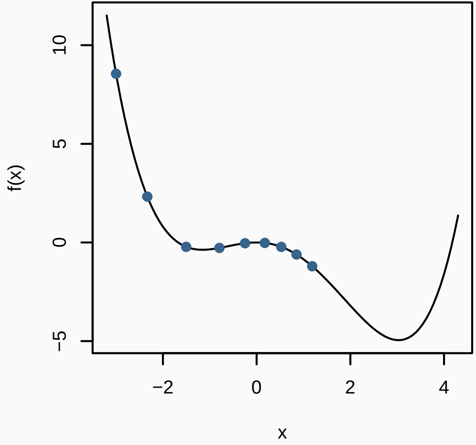

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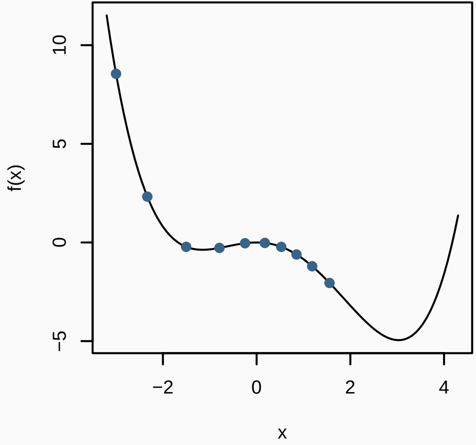

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Consider

$$f(x) = \begin{cases} \frac{25x^2}{2} & \text{if } x < 1, \\ \frac{x^2}{2} + 24x - 12 & \text{if } 1 \leq x < 2, \\ \frac{25x^2}{2} - 24x + 36 & \text{if } x \geq 2. \end{cases}$$

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For an "optimal" step size $1 = 1/L$ with $L = 25$, GD momentum steps converge to three limit points.

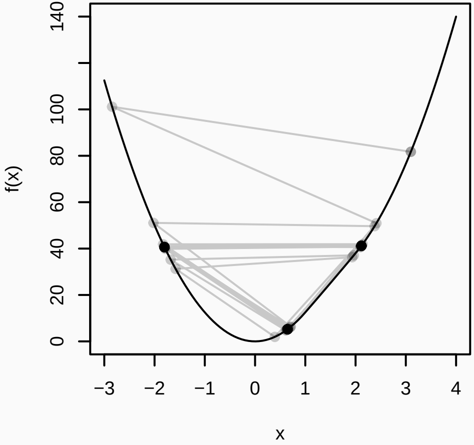

Figure 4: Initialized at $x_0 = 3.2$, the algorithm fails to converge.

Nesterov Momentum

Algorithm 3: GD with Nesterov Momentum

Data: $\gamma > 0$, $\mu \in [0, 1)$

for $i \leftarrow 1, 2, \dots$ **do**

$\quad v_k \leftarrow x_{k-1} - \gamma \nabla f(x_{k-1})$;

$\quad x_k \leftarrow v_k + \mu(v_k - v_{k-1})$;

Overcomes convergence problem of classical (Polyak) momentum.

Nesterov: Sutskever Perspective

Consider two iterations of Nesterov algorithm:

$$
\begin{aligned}
v_k &= x_{k-1} - \gamma \nabla f\left(x_{k-1}\right) \\
x_k &= v_k + \mu\left(v_k - v_{k-1}\right) \\
v_{k+1} &= x_k - \gamma \nabla f\left(x_k\right) \\
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Reindex:

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$$

But since $x_k = v_k$ for $k = 1$ by construction, we can swap x_k for v_k and get the update

$$
x_k = x_{k-1} + \mu(x_{k-1} - x_{k-2}) - \gamma \nabla f(x_k + \mu(x_{k-1} - x_{k-2})).
$$

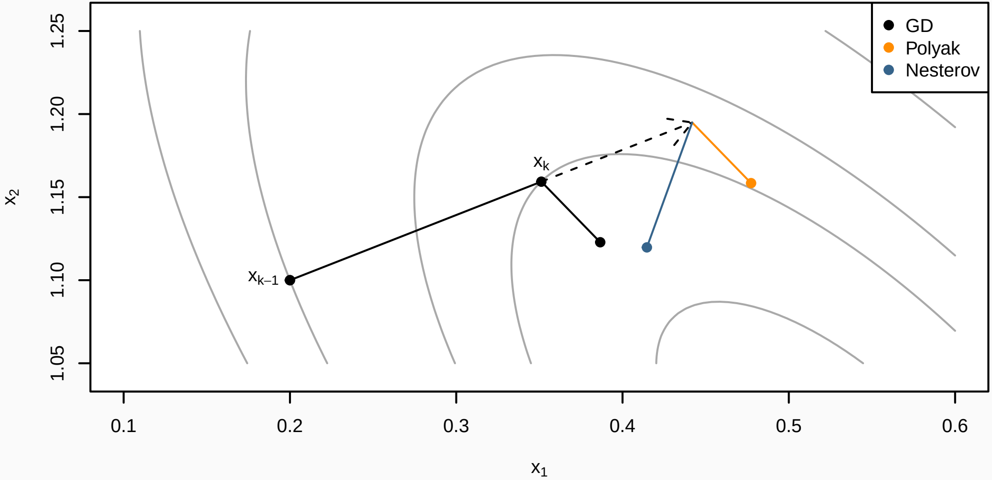

Figure 5: Illustration of Nesterov and Polyak momentum

Optimal Momentum

For gradient descent with $\gamma = 1/L$, the optimal choice of μ_k for general convex and smooth f is

$$\mu_k = \frac{a_{k-1} - 1}{a_k}$$

for a series of

$$a_k = \frac{1 + \sqrt{4a_{k-1}^2 + 1}}{2}$$

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First step ($k = 1$) is just standard gradient descent

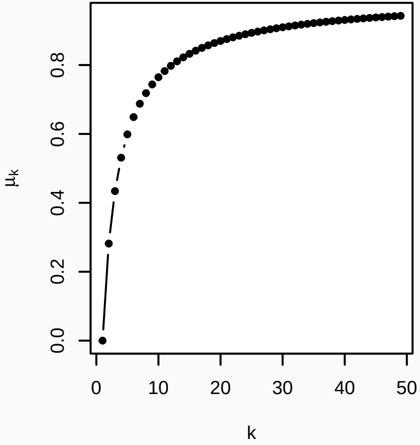

Figure 6: Optimal momentum for Nesterov acceleration (for GD).

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Convergence improves further for quadratic and strongly convex!

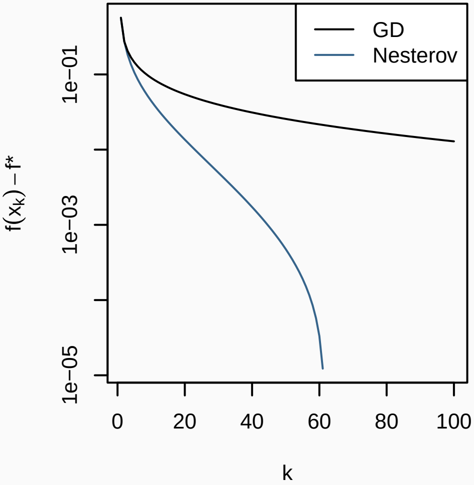

Figure 7: Suboptimality plot for a logistic regression problem with $n = 1\,000$, $p = 100$.

Exercise: Rosenbrock's Banana

Steps

1. Minimize
$$f(x_1, x_2) = (a - x_1)^2 + b(x_2 - x_1^2)^2$$
with $a = 1$ and $b = 100$ using GD with Polyak momentum. Optimum is (a, a^2).
2. Implement gradient descent.
3. Add Polyak momentum.

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Plot Contours

```
x1 <- seq(-2, 2, length.out =
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x2 <- seq(-1, 3, length.out =
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z <- outer(x1, x2, f)
contour(x1, x2, z, nlevels =
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What About SGD?

So far, we have mostly talked about standard GD, but we can use momentum (Polyak or Nesterov) for SGD as well.

For standard GD, Nesterov is the dominating method for achieving acceleration; for SGD, Polyak momentum is actually quite common.

In term of convergence, all bets are now off.

No optimal rates anymore, just heuristics.

Adaptive Gradients

General Idea

Some directions may be important, but feature information is sparse.

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AdaGrad
Store matrix of gradient history,

$$G_k = \sum_{i=1}^{k} \nabla f(x_k) \nabla f(x_k)^\intercal,$$

and update by multiplying gradient with $G_k^{-1/2}$.

Algorithm 5: AdaGrad

Data: $\gamma > 0$
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Effects

Larger learning rates for sparse features

Step-sizes adapt to curvature.

AdaGrad In Practice

Simplified Version

Computing $\nabla f \nabla f^\intercal$ is $O(p^2)$; expensive!

Replace $G_k^{-1/2}$ with $\mathrm{diag}(G_k)^{-1/2}$

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Algorithm 6: Simplified AdaGrad

Data: $\gamma > 0$, $\epsilon > 0$

for $k \leftarrow 1, 2, \ldots$ **do**

$\quad G_k \leftarrow G_{k-1} + \operatorname{diag}\left(\nabla f(x_{k-1})^2\right)$;

$\quad x_k \leftarrow x_{k-1} - \gamma \operatorname{diag}\left(\epsilon I_p + G_k\right)^{-1/2} \nabla f(x_{k-1})$;

RMSProp

Acronym for **R**oot **M**ean **S**quare **Prop**agation.

Idea

Divide learning rate by running average of magnitude of recent gradients:

$$v(x,k) = \xi v(x,k-1) + (1-\xi)\nabla f(x_k)^2$$

where ξ is the forgetting factor.

Similar to AdaGrad, but uses forgetting to gradually decrease influence of old data.

Algorithm 7: RMSProp

Data: $\gamma > 0$, $\xi > 0$

for $k \leftarrow 1, 2, \ldots, \xi \in [0,1)$ **do**

  $v_k = \xi v_{k-1} + (1-\xi)\nabla f(x_{k-1})$;

  $x_k \leftarrow x_{k-1} - \frac{\gamma}{\sqrt{v_k}} \odot \nabla f(x_{k-1})$;

ADAM

Acronym for **Ada**ptive **m**oment estimation

Basically RMSProp + *momentum* (for both gradients and second moments theorof)

Popular and still in much use today.

Loops

Any language (e.g. R) that imposes overhead for loops, will have a difficult time with SGDS

Implementation Aspects of SGD

Loops

Any language (e.g. R) that imposes overhead for loops, will have a difficult time with SGDS

Storage Order

In a regression setting, when indexing a single obervation at a time, slicing rows is not efficient n is large.

We can either transpose first or use a row-major storage order (not possible in R).

Example: Nonlinear Least Squares

Let's assume we're trying to solve a least-squares type of problem:

$$f(\theta) = \frac{1}{2n} \sum_{i=1}^{n} (y_i - g(\theta; x_i, y_i))^2$$

with $\theta = (\alpha, \beta)$ and

$$g(\theta; x, y) = \alpha \cos(\beta x).$$

Example: Nonlinear Least Squares

Let's assume we're trying to solve a least-squares type of problem:

$$f(\theta) = \frac{1}{2n}\sum_{i=1}^{n}(y_i - g(\theta; x_i, y_i))^2$$

with $\theta = (\alpha, \beta)$ and

$$g(\theta; x, y) = \alpha \cos(\beta x).$$

Then

$$\nabla_\theta f(\theta) = \begin{bmatrix} \cos(\beta x) \\ -\alpha x \sin(\beta x) \end{bmatrix}.$$

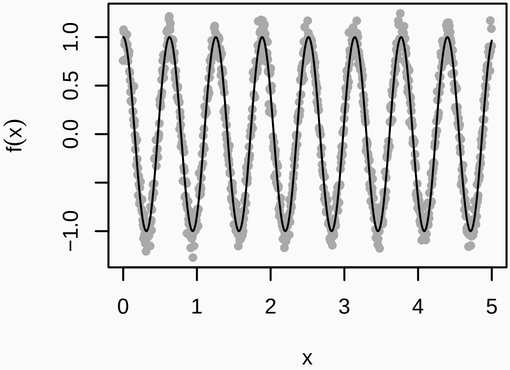


Figure 8: Simulation from problem

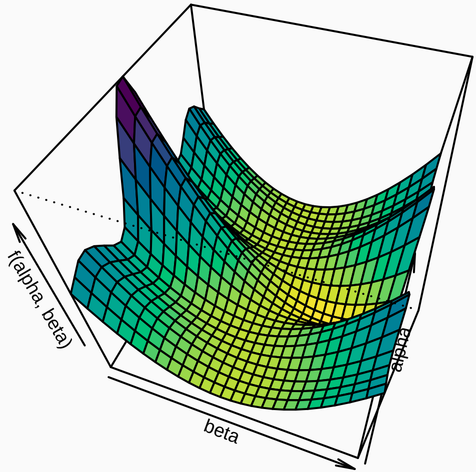


Figure 9: Perspective plot of function

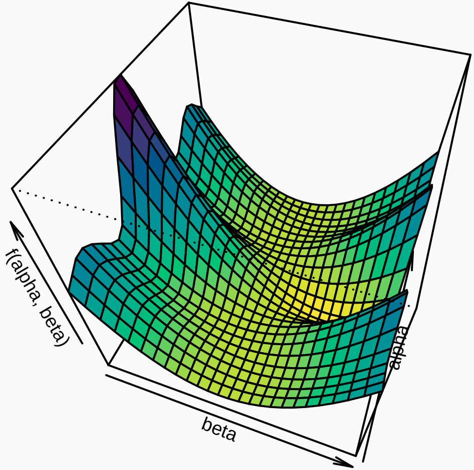


Figure 9: Perspective plot of function

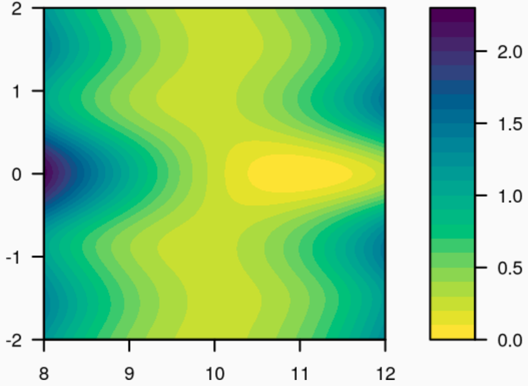


Figure 10: Contour plot of f

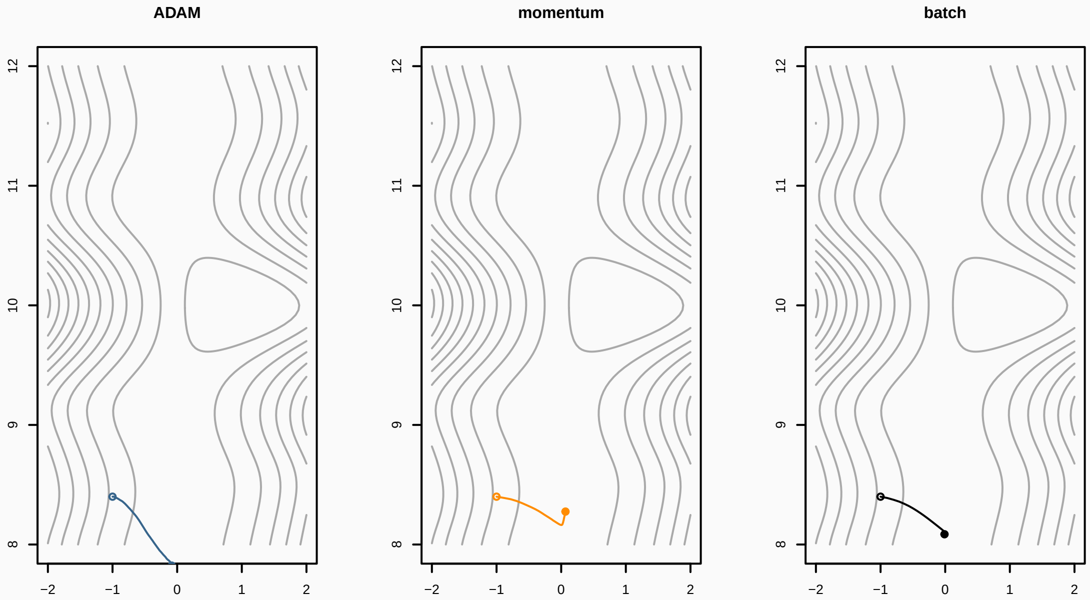


Figure 11:

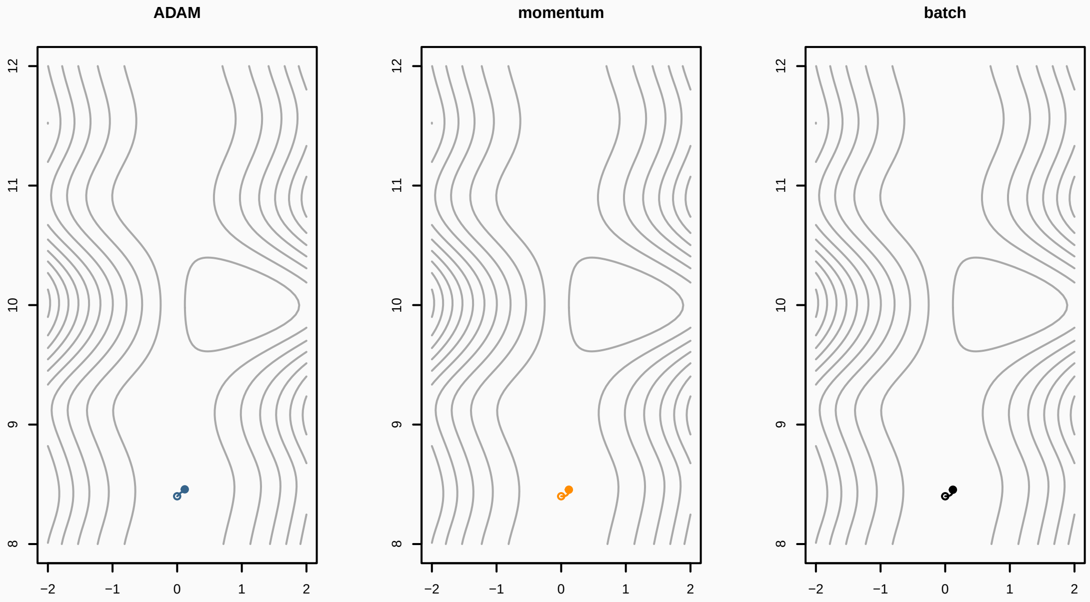


Figure 11:

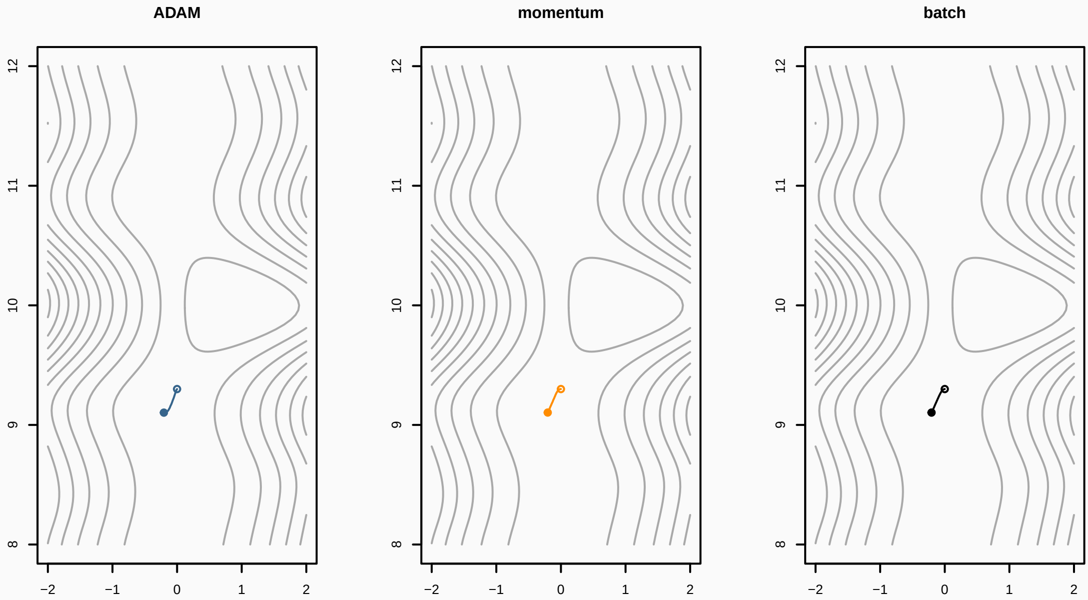
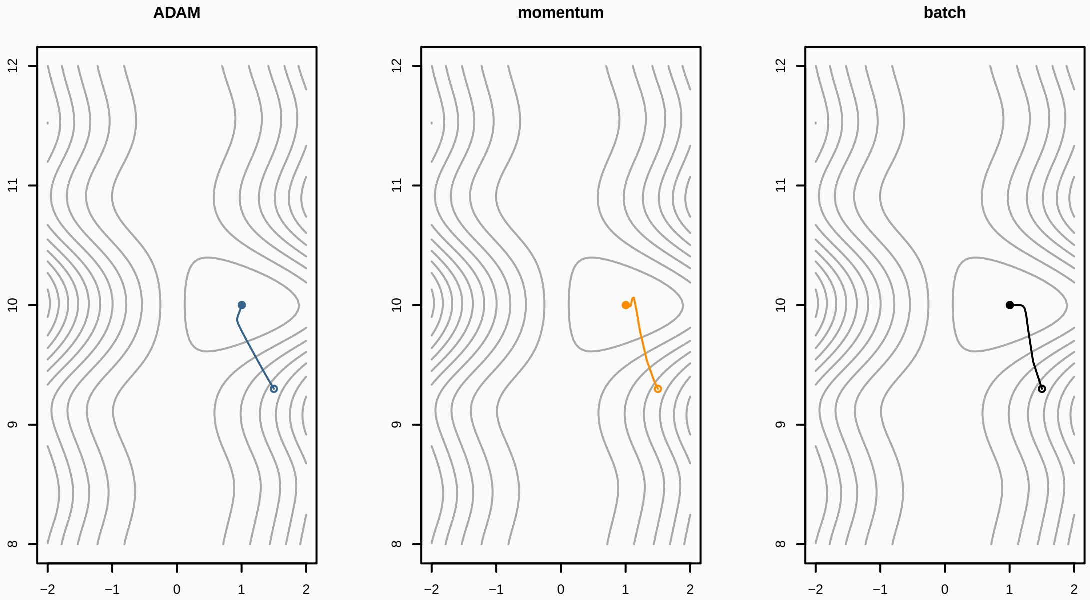


Figure 11:



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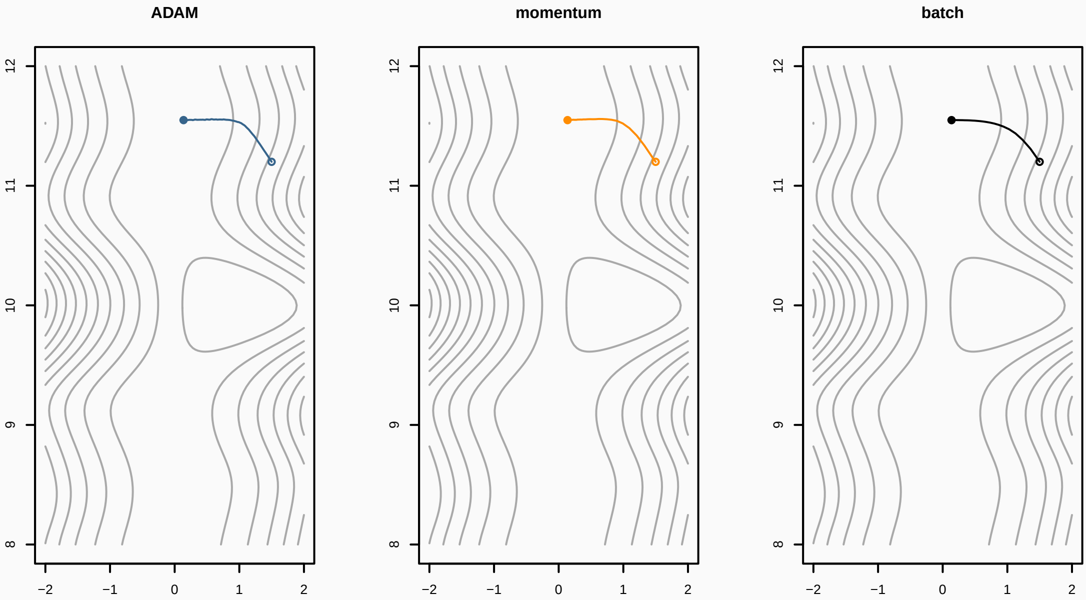

Figure 11:

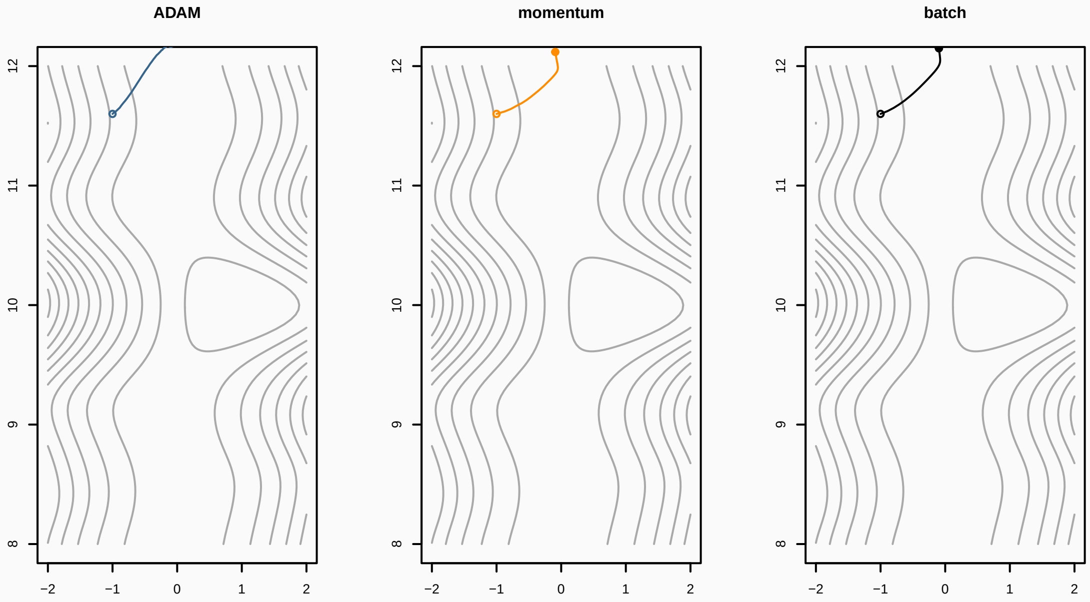

Figure 11:

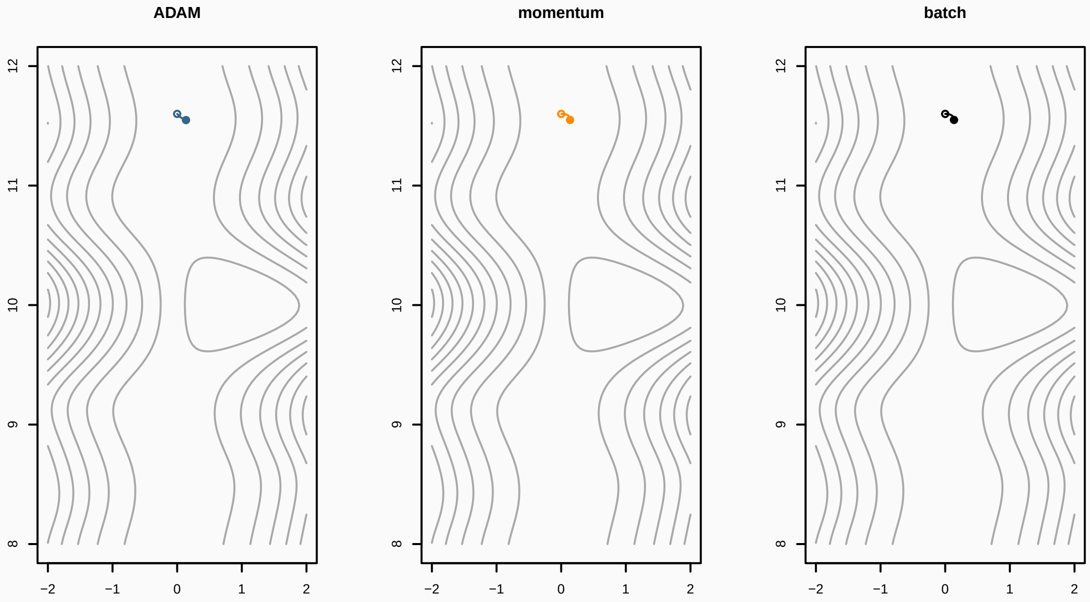

Figure 11:

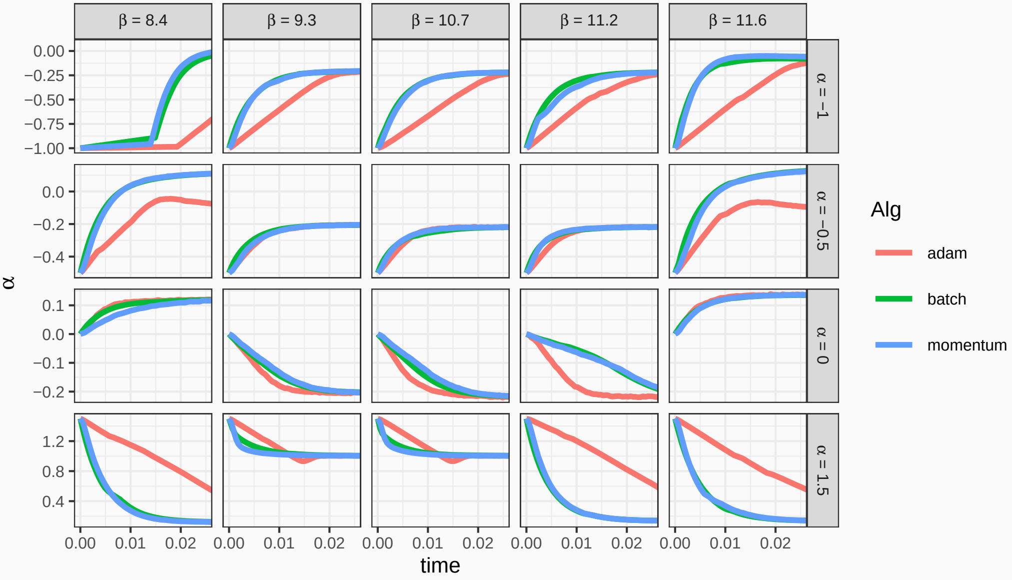

Figure 12: Updates of α parameter over time for the different algorithms over different starting values.

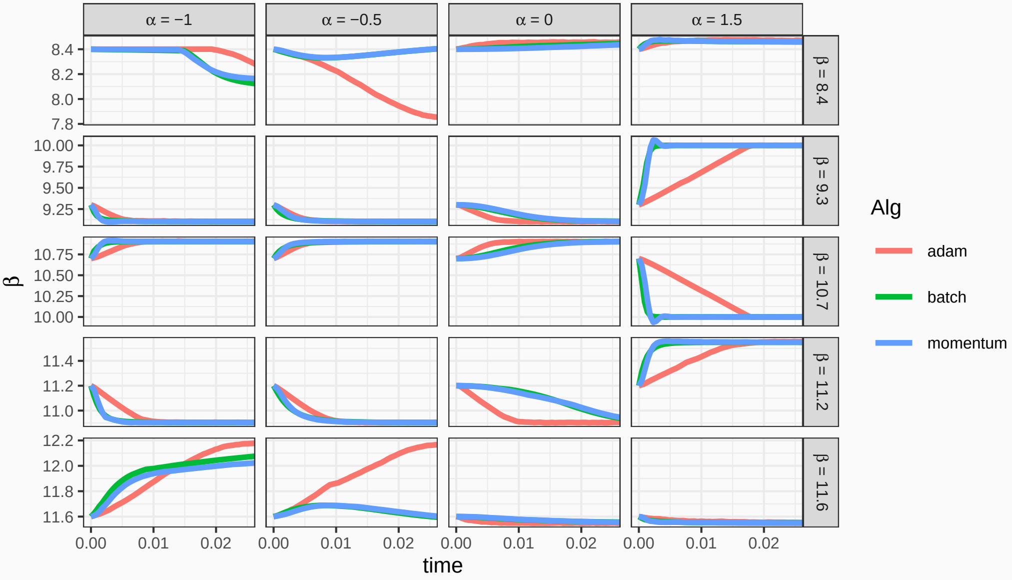

Figure 13: Updates of β parameter over time for the different algorithms over different starting values.

Rcpp

Very attractive for stochastic methods due to all the loop constructs and slicing.

However, Rcpp lacks linear algebra functions.

Approaches

- Still use only Rcpp (but then you need to write your own linear algebra functions[1]
- Use RcppEigen or RcppArmadillo.

[1] Not recommended!

Exercise: Rosenbrock Revisited

Steps

1. Convert your gradient descent algorithm to C++ through Rcpp.
2. Modify it to be a stochastic gradient descent algorithm instead.

Exercise: Rosenbrock Revisited

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1. Convert your gradient descent algorithm to C++ through Rcpp.
2. Modify it to be a stochastic gradient descent algorithm instead.

Hints

- Use the Rcpp function `Rcpp::sugar()` to sample indices.
- Don't bother with a stopping criterion to begin with; just set a maximum number of iterations.
- You can return a list by calling `Rcpp::List::create(Rcpp::named("name") = x)`.
- Use a pure Rcpp implementation.

Summary

We introduced several new concepts:

- Polyak momentum,
- Nesterov acceleration (momentum), and
- adapative gradients (AdaGrad),

We practically implemented versions of gradient descent and stochastic gradient descent with momentum.

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Additional Resources

- Goh (2017) is a article on momentum in gradient descent with lots of interactive visualizations.

Next Time

Reproducibility
How to make your code reproducible

We build an R package.

Next Time

Reproducibility

How to make your code reproducible

We build an R package.

Summary

We summarize the course.

Next Time

Reproducibility
How to make your code reproducible

We build an R package.

Summary
We summarize the course.

Exam Advice
We talk about the upcoming oral examinations.

Thank you!

References i

Goh, Gabriel (2017). "Why Momentum Really Works". In: *Distill*. DOI: 10.23915/distill.00006. URL: http://distill.pub/2017/momentum.