Scissors beats Rock & other failures of Transitive Models using Paired Comparisons

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This poster was created using my revived version of the `posterdownLaTeX` `R` package. The code needed to produce this poster can be found at **github.com/math-mcshane/IMS-NRC-2022-Poster-and-Presentation**

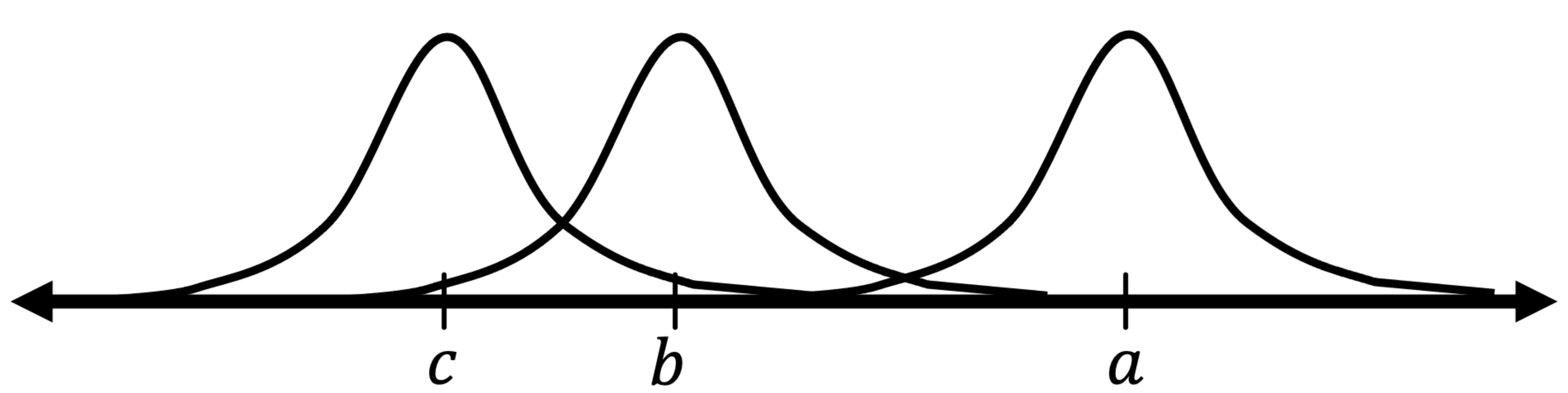
Abstract

If the Utah Jazz beat the Portland Trailblazers and the Trailblazers beat the Denver Nuggets, does that mean that the Jazz are better than the Nuggets, or could the Nuggets actually be better than the Jazz? To answer this question, we investigate the assumption of transitivity. We propose a novel linear model (CRSP) whose latent bilinear fixed effect allows us to estimate deviations from our transitive model (C). Additionally, we consider extensions.

Paired Comparisons and (In)transitivity

Classically [1] n objects are compared [teams compete] one pair at a time, until all $\binom{n}{2}$ comparisons are made. Then, object [team] ratings are derived and subsequently ranks. First documented in 1283 to select a pope [2]. $\longrightarrow$

Transitive Property of Inequality: Let a, b, c be real numbers (team ratings). If $a > b$ and $b > c$, then $a > c$.
Weak Stochastic Transitivity Property: Let A, B, C be teams. If $P(A \to B)$, $P(B \to C) > 0.5$, then $P(A \to C) > 0.5$.



Every common paired comparisons model (Bradley-Terry, Elo, Glicko, Trueskill, etc) assumes at least the WST Property [3].

We introduce a model (CRSP) that does allow intransitivity. For the following, assume that a double round robin has been observed — all teams i play all teams j, $i \neq j$, $n_{ij} = n_{ji} = 1$ times.

Chain Model (Transitive)

We want to model Y_{ij}, the score of home team i minus the score of away team j. We consider several models of the form $Y_{ij} = \mu_{ij} + \varepsilon_{ij}$, where ε_{ij} independent and σ_ε constant. The first is the **Chain model** with fixed effect strength parameters α_i for team i and home effect h, for the expected score differential μ_{ij} when team i hosts team j:

$$\mu_{ij} = h + \alpha_i - \alpha_j.$$

Finding solutions for $\hat{h}$ and $\hat{\boldsymbol{\alpha}}$ is straight forward:

$$\hat{h} = (\sqrt{2}n)^{-2}\mathbf{1}^T(\boldsymbol{Y} - \boldsymbol{Y}^T)\mathbf{1} = \bar{y}_{..},$$
$$\hat{\alpha}_i = 2^{-1}(\bar{y}_{i.} - \bar{y}_{.i}) \Rightarrow \hat{\boldsymbol{\alpha}} = (2n)^{-1}\left[\boldsymbol{Y} - \boldsymbol{Y}^T\right]\mathbf{1}.$$

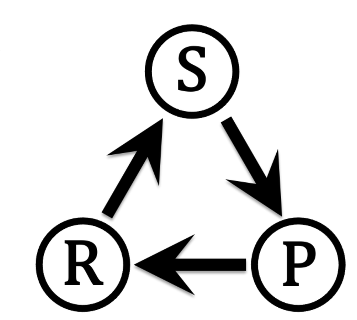
CRSP (Chain Rock-Scissors-Paper) Model

We can make the μ_{ij} intransitive by including a fixed interaction effect β_{ij}, where $\beta_{ij} = -\beta_{ji}$ and $\Sigma_j \beta_{ij} = 0\ \forall i$ (this will overfit when n_{ij} small):

$$\mu_{ij} = h + \alpha_i - \alpha_j + \beta_{ij}$$

We instead introduce the notion of circular triads, $\boldsymbol{T}$, from [4]:

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$



We propose an alternative to β_{ij}, $f(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)$, with $\boldsymbol{\beta}_i = \begin{bmatrix} r_i & s_i & p_i \end{bmatrix}$ and let $r_i + s_i + p_i = 0$,

$$f(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j) = \boldsymbol{\beta}_i^T \boldsymbol{T} \boldsymbol{\beta}_j \propto r_i s_j - s_i r_j.$$

This gives us our **CRSP model** with $\boldsymbol{r}, \boldsymbol{s}$ orthogonal

$$\mu_{ij} = h + \alpha_i - \alpha_j + r_i s_j - r_j s_i.$$

Extension: **Generalized CRSP model**, $K \in \{0, 1, ..., \left[\frac{n-1}{2}\right]\}$:

$$\mu_{ij} = h + \alpha_i - \alpha_j + \Sigma_{k=1}^K r_{ik} s_{jk} - s_{ik} r_{jk}.$$

Finding solution to $\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}$, where $\boldsymbol{C}$ is the residual matrix:

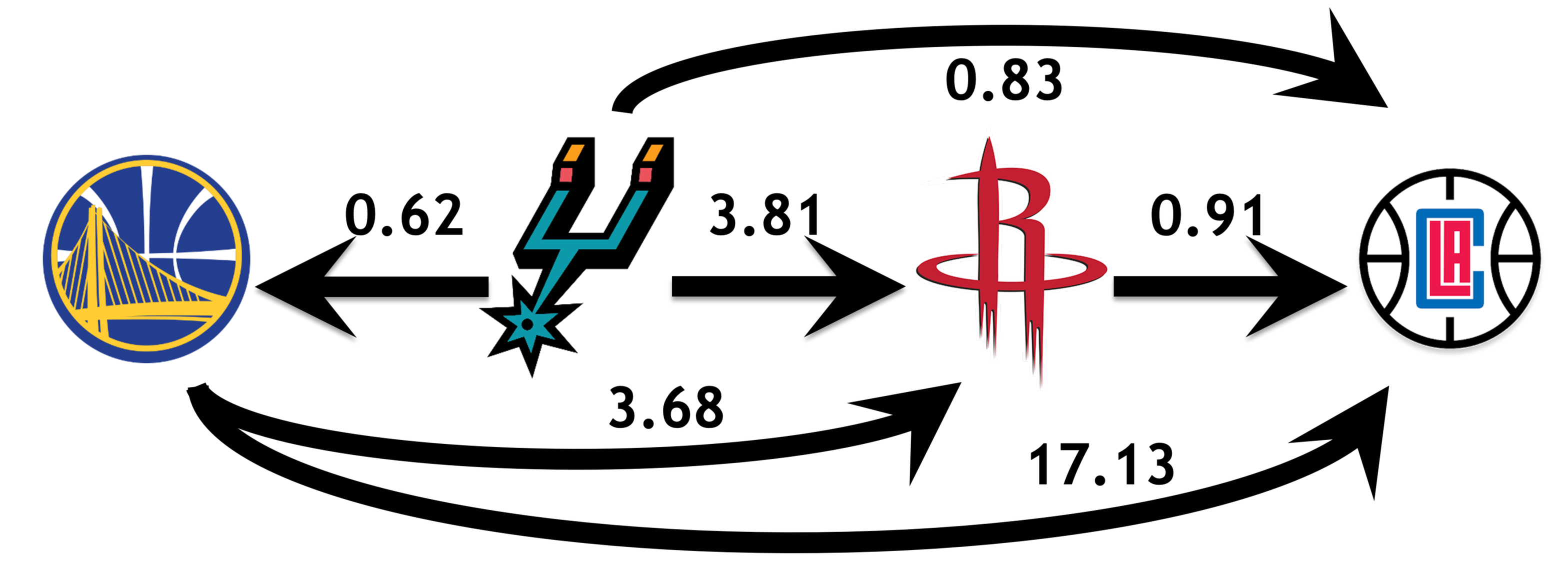
$$\boldsymbol{C} = 2^{-1}(\boldsymbol{Y} - \boldsymbol{Y}^T) - (\hat{\boldsymbol{\alpha}}\mathbf{1}^T - (\hat{\boldsymbol{\alpha}}\mathbf{1}^T)^T) = \Sigma_{k=1}^K \boldsymbol{r}_k \boldsymbol{s}_k^T - \boldsymbol{s}_k \boldsymbol{r}_k^T.$$

$\boldsymbol{r}_k$ and $\boldsymbol{s}_k$ are the eigenvectors of $\boldsymbol{C}\boldsymbol{C}^T$. The $\boldsymbol{r}_k, \boldsymbol{s}_k$ with the largest eigenvalue pair are the solutions to the CRSP model, second largest eigenvalues to the second intransitive vector pair in generalized CRSP, etc. See Ch. 4 of [3] for more details.

Chain and CRSP Example (2017 NBA Data)

We use all data from the 2016-2017 NBA season, where repeated games are averaged. We estimate $\hat{h} = 2.89$, $\hat{\alpha}_{GSW} = 11.27, \hat{\alpha}_{SAS} = 7.91,\ \hat{\alpha}_{HOU} = 4.93,\ \hat{\alpha}_{LAC} = 4.33, \ldots$. Displayed are the relative differences of team strengths, $\alpha_i - \alpha_j$.

This is the chain model. Note that $\hat{\alpha}_{GSW} - \hat{\alpha}_{HOU} = (\hat{\alpha}_{GSW} - \hat{\alpha}_{SAS}) + (\hat{\alpha}_{SAS} - \hat{\alpha}_{HOU}) = 6.34$. Now, we take the residuals, $\boldsymbol{C}$, and find the eigendecomposition of $\boldsymbol{C}\boldsymbol{C}^T$. We get, for example, $\hat{r}_{GSW} = -3.40, \hat{r}_{SAS} = 0.86, \hat{s}_{GSW} = 0.29, \hat{s}_{SAS} = 1.10$. Then, if $h = 0$, $\hat{\mu}_{SAS,GSW} = \hat{\alpha}_{SAS} - \hat{\alpha}_{GSW} + \hat{r}_{SAS}\hat{s}_{GSW} - \hat{r}_{GSW}\hat{s}_{SAS} = 0.62$.



Ongoing Work

1. EM Algorithm for incomplete cases, 2. How to pick K in Generalized CRSP, 3. Weights ω_{ijh} for game h b/w i and j, 4. Clustering players and line-ups, use CRSP at line-up level, and identify team-level intransitivity causes. 5. Smash N64 application — player γ_ℓ, character $\alpha_i, r_i s_j - r_j s_i$. 6. Possible digraphs.

References

[1] L. L. Thurstone. "A Law of Comparative Judgment". In: *Psychological Review* 34.4 (1927), p. 273. DOI: 10.1037//0033-295x.101.2.266.

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[3] R. P. A. McShane. "Modeling Stochastically Intransitive Relationships in Paired Comparison Data". PhD thesis. Southern Methodist University, 2019. URL: https://scholar.smu.edu/hum_sci_statisticalscience_etds/13.

[4] M. G. Kendall and B. B. Smith. "On the method of paired comparisons". In: *Biometrika* 31.3/4 (1940), pp. 324–345. DOI: 10.2307/2332613.