properties of symmetric difference

Recall that the symmetric difference of two sets A, B is the set $A \cup B - (A \cap B)$. In this entry, we list and prove some of the basic properties of $\triangle$.

1. (commutativity of $\triangle$) $A \triangle B = B \triangle A$, because $\cup$ and $\cap$ are commutative.
2. If $A \subseteq B$, then $A \triangle B = B - A$, because $A \cup B = B$ and $A \cap B = A$.
3. $A \triangle \emptyset = A$, because $\emptyset \subseteq A$, and $A - \emptyset = A$.
4. $A \triangle A = \emptyset$, because $A \subseteq A$ and $A - A = \emptyset$.
5. $A \triangle B = (A - B) \cup (B - A)$ (hence the name symmetric difference).

   Proof.
   $A \triangle B = (A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)' = (A \cup B) \cap (A' \cup B') = ((A \cup B) \cap A') \cup ((A \cup B) \cap B') = (B \cap A') \cup (A \cap B') = (B - A) \cup (A - B)$
   . ∎
6. $A' \triangle B' = A \triangle B$, because $A' \triangle B' = (A' - B') \cup (B' - A') = (A' \cap B) \cup (B' \cap A) = (B - A) \cap (A - B) = A \triangle B$.
7. (distributivity of $\cap$ over $\triangle$) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

   Proof. $A \cap (B \triangle C) = A \cap ((B \cup C) - (B \cap C))$, which is $(A \cap (B \cup C)) - (A \cap (B \cap C))$, one of the properties of set difference (see proof here (`http://planetmath.org/PropertiesOfSetDifference`)). This in turns is equal to $((A \cap B) \cup (A \cap C)) - ((A \cap B) \cap (A \cap C)) = (A \cap B) \triangle (A \cap C)$.
   ∎
8. (associativity of $\triangle$) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

   Proof. Let U be a set containing A, B, C as subsets (take $U = A \cup B \cup C$ if necessary). For a given B, let $f : P(U) \times P(U) \to P(U)$ be a function defined by $f(A, C) = (A \triangle B) \triangle C$. Associativity of $\triangle$ is then then same as showing that $f(A, C) = f(C, A)$, since
   $A \triangle (B \triangle C) = (B \triangle C) \triangle A = (C \triangle B) \triangle A$.

   By expanding $f(A, C)$, we have

$$\begin{aligned}
(A \triangle B) \triangle C &= ((A \triangle B) - C) \cup (C - (A \triangle B)) \\
&= (((A - B) \cup (B - A)) \cap C') \cup (C - ((A \cup B) - (A \cap B))) \\
&= (((A \cap B') \cup (B \cap A')) \cap C') \cup ((C \cap A \cap B) \cup (C - (A \cup B)) \\
&= ((A \cap B' \cap C') \cup (B \cap A' \cap C')) \cup ((C \cap A \cap B) \cup (C \cap A' \cap B')) \\
&= (B \cap A' \cap C') \cup (B \cap A \cap C) \cup (B' \cap A \cap C') \cup (B' \cap A' \cap C).
\end{aligned}$$

   It is now easy to see that the last expression does not change if one exchanges A and C. Hence, $f(A, C) = f(C, A)$ and this shows that $\triangle$ is associative. ∎

Remark. All of the properties of $\triangle$ on sets can be generalized to $\triangle$ (`http://planetmath.org/DerivedBooleanOperations`) on Boolean algebras.