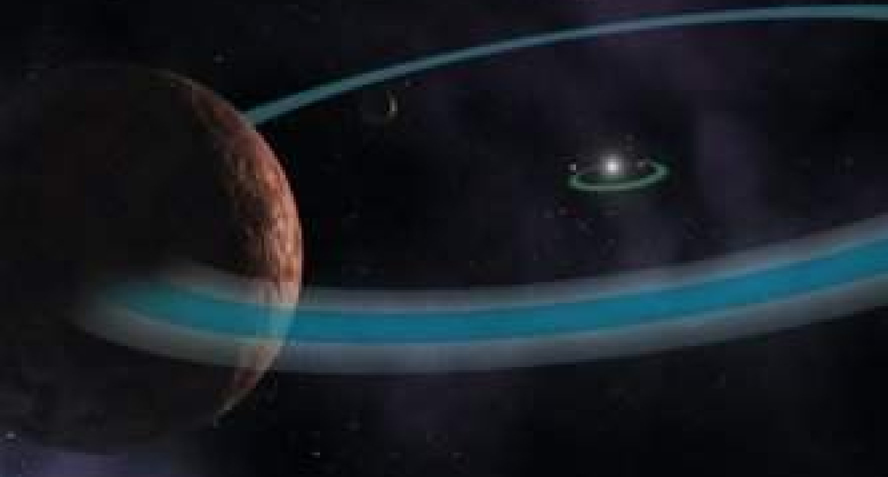
A Student's Guide to the Mathematics of Astronomy



Daniel Fleisch
and Julia Kregenow

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A Student's Guide to the Mathematics of Astronomy

The study of astronomy offers an unlimited opportunity for us to gain a deeper understanding of our planet, the Solar System, the Milky Way galaxy, and the known Universe.

Using the plain-language approach that has proven highly popular in Fleisch's other *Student's Guides*, this book is ideal for non-science majors taking introductory astronomy courses. The authors address topics that students find most troublesome, on subjects ranging from stars and light to gravity and black holes. Dozens of fully worked examples and over 150 exercises and homework problems help readers get to grips with the concepts presented in each chapter.

An accompanying website, available at www.cambridge.org/9781107610217, features a host of supporting materials, including interactive solutions for every exercise and problem in the text and a series of video podcasts in which the authors explain the important concepts of every section of the book.

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CAMBRIDGE
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University Printing House, Cambridge CB2 8BS, United Kingdom

Published in the United States of America by Cambridge University Press, New York

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107610217

First published 2013

Printing in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Fleisch, Daniel A.
A student's guide to the mathematics of astronomy / Daniel Fleisch and Julia Kregenow.
pages cm.
ISBN 978-1-107-61021-7 (pbk.)
1. Astronomy – Mathematics – Textbooks. I. Kregenow, Julia. II. Title.
QB51.3.M38F54 2013
520.1′51–dc23
2013008432

ISBN 978-1-107-03494-5 Hardback
ISBN 978-1-107-61021-7 Paperback

Additional resources for this publication at www.cambridge.org/9781107610217

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Preface

This book has one purpose: to help you understand and apply the mathematics used in college-level astronomy. The authors have instructed several thousand students in introductory astronomy courses at large and small universities, and in our experience a common response to the question “How’s the course going for you?” is “I’m doing fine with the concepts, but I’m struggling with the math.” If you’re a student in that situation, or if you’re a life-long learner who’d like to be able to delve more deeply into the many wonderful astronomy books and articles in bookstores and on-line, this book is here to help.

We want to be clear that this book is not intended to be your first exposure to astronomy, and it is not a comprehensive treatment of the many topics you can find in traditional astronomy textbooks. Instead, it provides a detailed treatment of selected topics that our students have found to be mathematically challenging. We have endeavored to provide just enough context for those topics to help foster deeper understanding, to explain the meaning of important mathematical relationships, and most of all to provide lots of illustrative examples.

We’ve also tried to design this book in a way that supports its use as a supplemental text. You’ll notice that the format is modular, so you can go right to the topic of interest. If you’re solid on gravity but uncertain of how to use the radiation laws, you can skip Chapter 2 and dive right into Section 3.2 of Chapter 3. Additionally, we’ve put a detailed discussion of four foundational topics right up front in Chapter 1, so you can work through those if you’re in need of some review on unit conversions, using ratios, rate problems, or scientific notation.

To help you use this book actively (rather than just passively reading the words), we’ve put one or more exercises at the end of most subsections. These exercises are usually drills of a single concept or mathematical operation just discussed, and you’ll find a full solution to every exercise on the book’s website. Additionally, at the end of each chapter you'll find approximately 10 problems. These chapter-end problems are generally more comprehensive and challenging than the exercises, often requiring you to synthesize multiple concepts and techniques to find the solution. Full solutions for all problems are available on the book's website, and those solutions are interactive. That means you'll be able to view the entire solution straightaway, or you can request a hint to help you get started. Then, as you work through the problem, if you get stuck you can ask for additional hints (one at a time) until you finally reach the full solution.

Another resource on the book's website is a series of video podcasts in which we work through each section of the book, discussing important concepts and techniques and providing additional explanations of equations and graphs. In keeping with the modular nature of the book, we've made these podcasts as stand-alone as possible, so you can watch them all in order, or you can skip around and watch only those podcasts on the topics with which you need help.

The book's website also provides links to helpful resources for topics such as the nature of light, the center of mass, conic sections, potential energy, and significant figures (so you'll know when you should keep lots of decimal places and when it's safe to round your results).

So if you're interested in astronomy and have found mathematics to be a barrier to your learning, we're here to help. We hope this book and the supporting materials will help you turn that barrier into a stepping stool to reach a higher level of understanding. Whether you're a college student seeking additional help with the mathematics of your astronomy course or a life-long learner working on your own, we commend your initiative.

Acknowledgements

This book grew out of conversations and help sessions with many astronomy students over the years. The initiative of those students in asking thoughtful questions, often in the face of deep-seated math anxiety, inspired us not only to write this book, but to make every explanation as clear and complete as possible. In addition to inspiration, our students have provided detailed feedback as to which topics are most troublesome and which explanations are most helpful, and those are the topics and explanations that appear in this book. For this inspiration and guidance, we thank our students.

Julia also thanks Jason Wright for his moral support throughout the project and for sharing his technical expertise on stars, and she thanks Mel Zernow for his helpful comments on an early draft.

Dan thanks Gracie Winzeler for proving that every math problem can be overcome by persistence and determination. And as always, Dan cannot find the words to properly express his gratitude to the galactically terrific Jill Gianola.

1

Fundamentals

This chapter reviews four important mathematical concepts and techniques that will be helpful in many quantitative problems you're likely to encounter in a college-level introductory astronomy course or textbook. As with all the chapters in the book, you can read the sections within this chapter in any order, or you can skip them entirely if you're already comfortable with this material. But if you're working through one of the later chapters and you find that you're uncertain about some aspect of unit conversion, the ratio method, rate problems, or scientific notation, you can turn back to the relevant section of this chapter.

1.1 Units and unit conversions

One of the most powerful tools you can use in solving problems and in checking your solutions is to consistently include *units* in your calculations. As you may have noticed, among the first things that physics and astronomy professors look for when checking students' work is whether the units of the answer make sense. Students who become adept at problem-solving develop the habit of checking this for themselves.

Understanding units is important not just in science, but in everyday life as well. That's because units are all around you, giving meaning to the numbers that precede them. Telling someone "I have a dozen" is meaningless. A dozen what? Bagels? Minutes to live? Spouses? If you hope to communicate information about quantities to others, numbers alone are insufficient. Nearly every number must have units to define its meaning. So a very good habit to start building mastery is to always include the units of any number you write down.

Of course, some numbers are inherently "unitless." As an example of such a number, consider what happens when you divide the mass of the Sun

(2×10^{30} kg) by the mass of the Earth (6×10^{24} kg) in order to compare their values. The result of this division is approximately 333,333. Not 333,333 kg, just 333,333, because the units of kilograms in the numerator and denominator cancel, as explained later in this section. This unit cancellation happens whenever you divide two numbers with the same units, so you'll see several unitless numbers in Section 1.2 of this chapter.

If keeping track of units is the vital first step in solving astronomy problems, knowing how to reliably convert between different units is a close second. When you travel to a country that uses a different currency, you learn firsthand the importance of unit conversions. If you come upon a restaurant offering a full dinner for 500 rupees, is that a good deal? You'll have to do a unit conversion to find out. And to do that conversion, you'll need two things: (1) a conversion factor between currencies, such as those shown in Figure 1.1; and (2) knowledge of how to use conversion factors.

To understand the process of unit conversion, it's best to start with simple cases using everyday units, because you probably have an intuitive sense of how to perform such conversions. For example, if a movie lasts 2 hours, you know that is 120 minutes, because there are 60 minutes in 1 hour. But think about the process you used to convert hours to minutes: you intuitively multiplied 2 hours by 60 minutes in each hour.

Unfortunately, unit conversion becomes less intuitive when you're using units that are less familiar to you, or when you're using large numbers that can't be multiplied in your head. In such cases, students sometimes resort to guessing whether to multiply or divide the original quantity by the conversion factor. After a short discussion of conversion factors, we'll show you a foolproof method for setting up any unit conversion problem that will ensure you always know whether to multiply or divide.

1.1.1 Conversion factors

So exactly what *is* a conversion factor? It's just a statement of the equivalence between expressions with different units, and that statement lets you translate between those units in either direction. How can two expressions with different numbers be equivalent? Well, the distance represented by 1 meter is exactly the same as the distance represented by 100 cm. So it's the *underlying quantity* that's the same, and that quantity is represented by the *combination* of the number and the unit.

This means that a conversion factor is always a statement that some number of one unit is equivalent to a different number of another unit. Conversion factors are usually written in one of two ways: either as an equivalence relation

Figure 1.1 Currency exchange rates on a bank board. Each entry is a conversion factor between one unit and another.

or as a fraction. For example, 12 inches of length is equivalent to 1 foot, 60 minutes of time is equivalent to 1 hour, and the astronomical distance unit of 1 parsec (pc) is equivalent to 3.26 light years (ly). Each of these conversion factors can be expressed in an equivalence relation, which we signify using a double-headed arrow ($\leftrightarrow$):

$$12\text{ in} \leftrightarrow 1\text{ ft}, \qquad 1\text{ hr} \leftrightarrow 60\text{ min}, \qquad 3.26\text{ ly} \leftrightarrow 1\text{ pc}.$$

For convenience, one of the numbers in a conversion factor is often chosen to be 1, but it doesn't have to be. For example, 36 inches $\leftrightarrow$ 3 feet is a perfectly valid conversion factor.

It is convenient to represent the conversion factor as a fraction, with one set of units and its corresponding number in the numerator, and the other set in the denominator. Representing the example conversion factors shown above as fractions, you have

$$\frac{12\text{ in}}{1\text{ ft}} \text{ or } \frac{1\text{ ft}}{12\text{ in}}, \qquad \frac{60\text{ min}}{1\text{ hr}} \text{ or } \frac{1\text{ hr}}{60\text{ min}}, \qquad \frac{3.26\text{ ly}}{1\text{ pc}} \text{ or } \frac{1\text{ pc}}{3.26\text{ ly}}.$$

Because the two quantities in the conversion factor must represent the same amount, representing them as a fraction creates a numerator and a denominator that are equivalent, and thus the intrinsic *value* of the fraction is 1. You can multiply other values by this fraction with impunity, since multiplying any quantity by 1 does not change the amount – but it does change the way it looks. This is the goal of unit conversion: to change the units in which a quantity is expressed while retaining the underlying physical quantity.

Exercise 1.1. Write the following equivalence relations as fractional conversion factors:
1 in $\leftrightarrow$ 2.54 cm, 1.6 km $\leftrightarrow$ 1 mile, 60 arcmin $\leftrightarrow$ 3,600 arcsec.

1.1.2 Setting up a conversion problem

The previous section explains *why* unit conversion works; here's a foolproof way to do it:

- Find the conversion factor that contains both units – the units you're given and the units to which you wish to convert.
- Write the expression you're given in the original units followed by a $\times$ symbol followed by the relevant conversion factor in fractional form.
- Multiply all the numbers and all the units of the original expression by the numbers and the units of the conversion factor. Grouping numbers and terms allows you to treat them separately, making this step easier.

You can see this method in action in the following example.

Example: Convert 1,000 minutes to hours.

The fractional forms of the relevant conversion factor (that is, the conversion factor containing hours and minutes) are $\frac{1\text{ hr}}{60\text{ min}}$ and $\frac{60\text{ min}}{1\text{ hr}}$. But how do you know which of these to use? Both are proper conversion factors, but one will help you solve this problem more directly.

To select the correct form of the conversion factor, look at the original units you're given. If those units are standing alone (as are the units of minutes in the expression "1,000 minutes"), use the conversion factor with the units you're trying to get rid of in the denominator and the units that you're trying to obtain in the numerator. That way, when you multiply, the units you don't want will cancel, and the units you want will remain. This works because you can cancel units that appear in both the numerator and the denominator of a fraction in the same way you can cancel numerical factors.

In this example, since the units you're given (minutes) appear standing alone and you want to convert to units of hours, the correct form of the conversion factor has minutes in the denominator and hours in the numerator. That factor is $\frac{1\text{ hr}}{60\text{ min}}$. With that conversion factor in hand, you're ready to write down the given quantity in the original units and multiply by the conversion factor. Here's how that looks with the $\boxed{\text{conversion factor}}$ boxed:

$$1{,}000\text{ min} \times \boxed{\frac{1\text{ hr}}{60\text{ min}}}.$$

To simplify this expression, it helps to realize that there is an implicit multiplication between each number and its unit, and to remember that multiplication is commutative – so you can rearrange the order of the terms in both the numerator and denominator. That lets you multiply the numerical parts together and the units together, canceling units that appear on both top and bottom. Then you can simplify the numbers and express your answer in whatever units remain uncanceled:

$$1{,}000\text{ min} \times \boxed{\frac{1\text{ hr}}{60\text{ min}}} = \frac{(1{,}000 \times 1)(\cancel{\text{min}} \times \text{hr})}{60\ \cancel{\text{min}}} = \frac{1{,}000\text{ hr}}{60} = 16.7\text{ hr}$$

So a time value of 1,000 minutes represents the same amount of time as 16.7 hours.

Here's another example that uses the common astronomical distance units of parsecs and light years:

Example: Convert 1.29 parsecs, the distance of the closest star beyond our Sun, to light years.

In most astronomy texts, you'll find the conversion factor between parsecs and light years given as 3.26 ly $\leftrightarrow$ 1 pc, or equivalently 0.3067 pc $\leftrightarrow$ 1 ly.

In this case, since the quantity you're given has units of parsecs standing alone, you'll need the fractional conversion factor with parsecs in the denominator and light years in the numerator. Using that factor, your multiplication should look like this, again with the conversion factor boxed:

$$1.29\text{ pc} = 1.29\text{ pc} \times \boxed{\frac{3.26\text{ ly}}{1\text{ pc}}} = \frac{(1.29 \times 3.26)(\cancel{\text{pc}} \times \text{ ly})}{1\ \cancel{\text{pc}}} = \frac{4.21\text{ ly}}{1} = 4.21\text{ ly}.$$

Notice that the original quantity of 1.29 pc may be written as the fraction $\frac{1.29\text{ pc}}{1}$ in order to remind you to multiply quantities in both the numerator and in the denominator. The result of this unit conversion tells you that 4.21 light years represent the same amount of distance as 1.29 parsecs. Thus, the light from the nearest star beyond the Sun (a star called Proxima Centauri) takes over 4 years to travel to Earth.

An additional benefit of this method of unit conversion is that it helps you catch mistakes. Consider what would happen if you mistakenly used the conversion factor upside-down; the units of your answer wouldn't make sense. Here are incorrect setups for the previous two examples:

$$\begin{aligned} 1{,}000 \text{ min} \times \frac{60 \text{ min}}{1 \text{ hr}} &= \frac{(1{,}000 \times 60)(\text{min} \times \text{min})}{1 \text{ hr}} \\ &= 60{,}000 \frac{\text{min}^2}{\text{hr}} (\text{INCORRECT}) \end{aligned}$$

and

$$1.29 \text{ pc} \times \frac{1 \text{ pc}}{3.26 \text{ ly}} = \frac{(1.29 \times 1)(\text{pc} \times \text{pc})}{3.26 \text{ ly}} = 0.40 \frac{\text{pc}^2}{\text{ly}} \text{ (INCORRECT).}$$

Since these units are not the units to which you're trying to convert, you know you must have used conversion factors incorrectly.

Exercise 1.2. Perform the following unit conversions (you can find the relevant conversion factors in most astronomy texts or on the Internet).

(a) Express 12 inches in centimeters.
(b) Express 100 cm in inches.
(c) Express 380,000 km in miles (this is roughly the distance from the Earth to the Moon).
(d) Express 93,000,000 miles in kilometers (this is roughly the distance from the Earth to the Sun).
(e) Express 0.5 degrees in arcseconds (this is roughly the angular size of the full Moon viewed from Earth).

1.1.3 Checking your answer

Whenever you do a unit conversion (or other problems in astronomy, or any other subject for that matter), you should always give your answer a sanity check. That is, you should ask yourself "Does my answer make sense? Is it reasonable?" For example, in the incorrect version of the conversion from minutes to hours, you can definitely tell from the numerical part of your answer that something went wrong. After all, since 60 minutes are equivalent to 1 hour, then for any amount of time the number of minutes must be greater than the equivalent number of hours. So if you were to convert 1,000 minutes to hours and obtain an answer of 60,000 hours, the number of minutes would be smaller than the number of hours. That means these two quantities can't possibly be equivalent, which alerts you to a mistake somewhere.

Of course, if the units are outside your common experience (such as parsecs and light years in the previous example), you might not have a sense of what is or isn't reasonable. But you'll develop that sense with practice, so be sure to always take a step back from your answer to see if it makes sense. And remember that whenever you're converting to a *larger* unit (such as minutes to hours), the numerical part of the answer should get *smaller* (so that the combination of the number and the units represents the same quantity).

Exercise 1.3. How do you know that your answers to each of the unit conversion problems in the previous exercise make sense? Give a brief explanation for each.

1.1.4 Multi-step conversions

Up to this point, we've been working with quantities that have single units, such as meters, hours, or light years. But many problems in astronomy involve quantities with multiple units, such as meters per second or watts per square meter. Happily, the conversion-factor approach works just as well for multi-unit quantities.

Example: Convert from kilometers per hour to meters per second.

Since this problem statement doesn't tell you how many km/hr, you can use 1 km/hr. To convert quantities which involve two units (kilometers and hours in this case), you can use two conversion factors in immediate succession: one to convert kilometers to meters and another to convert hours to seconds. Here's how that looks:

$$\frac{1\text{ km}}{\text{hr}} \times \boxed{\frac{1000\text{ m}}{1\text{ km}}} \times \boxed{\frac{1\text{ hr}}{3600\text{ s}}} = \frac{(1\times 1{,}000\times 1)(\cancel{\text{km}}\times\ \text{m}\times\ \cancel{\text{hr}})}{(1\times 3{,}600)(\ \cancel{\text{hr}}\times\ \cancel{\text{km}}\times\ \text{s})},$$

$$1\frac{\text{km}}{\text{hr}} = \frac{1{,}000}{3{,}600}\frac{\text{m}}{\text{s}} = 0.28\text{ m/s}.$$

Alternatively, you could have done two separate conversions in succession, such as km/hr to km/s, and then km/s to m/s.

You may also encounter problems in which you need to break a single conversion into multiple steps. This may occur, for example, if you don't know the conversion factor directly from the given units to the desired units, but you do know the conversions for intermediate units. This is illustrated in the following example:

Example: How many seconds old were you on your first birthday?

Even if you don't know how many seconds are in a year, you can break this problem up into years to days, then days to hours, hours to minutes, and finally minutes to seconds. So to convert between years and seconds, you could do the following:

$$1\ \cancel{\text{yr}} \times \boxed{\frac{365\ \cancel{\text{d}}}{1\ \cancel{\text{yr}}}} \times \boxed{\frac{24\ \cancel{\text{hr}}}{1\ \cancel{\text{d}}}} \times \boxed{\frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hr}}}} \times \boxed{\frac{60\ \text{s}}{1\ \cancel{\text{min}}}} = \frac{(365 \times 24 \times 60 \times 60)\ \text{s}}{1}$$

$$= 31{,}536{,}000\ \text{s}.$$

By determining that there are about 31.5 million seconds in a year, you've derived the conversion factor between seconds and years. With the fractional conversion factor $\frac{31{,}536{,}000\ \text{s}}{1\ \text{yr}}$ in hand you can, for example, find the number of seconds in 30 years in a single step:

$$30\ \cancel{\text{yr}} \times \boxed{\frac{31{,}536{,}000\ \text{s}}{1\ \cancel{\text{yr}}}} = \frac{30 \times 31{,}536{,}000\ \text{s}}{1} = 946{,}080{,}000\ \text{s},$$

which is just under 1 billion. This gives you a sense of how large a billion is – you've lived a million seconds when you're about 11.5 days old, but even 30 years later you still haven't lived for a billion seconds.

Exercise 1.4. Perform the following unit conversions.

- **(a) Convert 60 mph (miles per hour) to meters per second.**
- **(b) Convert 1 day to seconds.**
- **(c) Convert dollars per kilogram to cents per gram (100 cents $\leftrightarrow$ 1 dollar).**
- **(d) Convert 1 mile to steps, assuming 1 step $\leftrightarrow$ 30 inches (there are 1,760 yards in 1 mile, 3 ft in 1 yard, and 12 inches in 1 ft).**

1.1.5 Converting units with exponents

Sometimes when doing a unit conversion problem, you will need to convert a unit that is raised to a power. In these cases, you must be sure to raise the conversion factor to the same power, and apply that power to all numbers *and* units in the conversion factor.

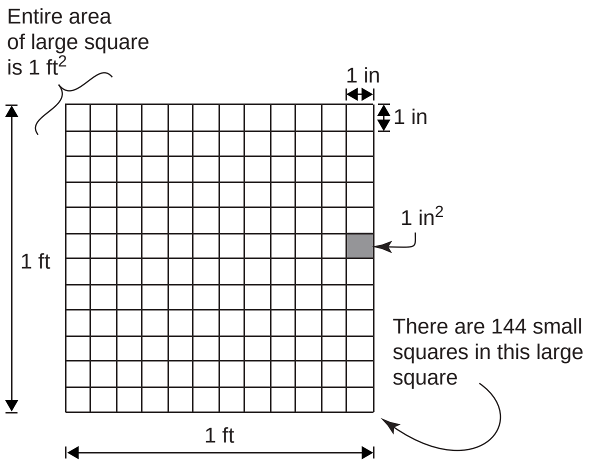

Figure 1.2 One square foot (ft^2), composed of 12 in × 12 in = 144 in^2.

Example: Convert 1 square foot (1 ft^2) to square inches (in^2).

You already know that there are 12 inches in 1 foot. Feet and inches are both units of one-dimensional length, or *linear* dimension. *Square* feet and inches, however, are units of two-dimensional *area*. The illustration in Figure 1.2 makes it clear that one square foot is not equal to just 12 square inches, but rather 12^2, or 144 square inches.

To perform this unit conversion mathematically, without having to draw such a picture, you'd write:

$$1\ \text{ft}^2 = 1\ \text{ft}^2 \left(\frac{12\ \text{in}}{1\ \text{ft}}\right)^2 = 1\ \text{ft}^2 \left(\frac{12^2\ \text{in}^2}{1^2\ \text{ft}^2}\right) = 1\ \cancel{\text{ft}^2} \left(\frac{144\ \text{in}^2}{1\ \cancel{\text{ft}^2}}\right) = 144\ \text{in}^2.$$

Notice that when you raise the conversion factor $(\frac{12\ \text{in}}{1\ \text{ft}})$ to the second power, both the numerical parts and the units, in both numerator and denominator, get squared.

Example: How many cubic centimeters (cm^3) are in 1 cubic meter (m^3)?

You know that there are 100 cm in 1 m, and both centimeters and meters are units of one-dimensional length. A cubic length unit, however, is a unit of three-dimensional volume. When you multiply by the appropriate conversion factor that converts between centimeters and meters, you must raise that factor to the third power, applying that power to all numbers and units separately.

$$1\ \text{m}^3 = 1\ \text{m}^3 \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3 = 1\ \cancel{\text{m}^3} \left(\frac{100^3\ \text{cm}^3}{1^3\ \cancel{\text{m}^3}}\right) = 1{,}000{,}000\ \text{cm}^3,$$

so there are 1 million cubic centimeters in 1 cubic meter.

Example: Convert 9.8 m/s^2 to km/hr^2.

One conversion factor is needed to convert length from meters to kilometers, and another to convert time from seconds to hours. The time conversion factor needs to be squared, but the length conversion factor does not.

$$9.8\frac{\text{m}}{\text{s}^2} = 9.8\frac{\cancel{\text{m}}}{\cancel{\text{s}^2}}\left(\frac{1\text{ km}}{1{,}000\cancel{\text{m}}}\right)\left(\frac{3{,}600\cancel{\text{s}}}{1\text{ hr}}\right)^2 = \frac{9.8\times 3{,}600^2\text{ km}}{1{,}000\text{ hr}^2} = 127{,}000\frac{\text{km}}{\text{hr}^2}.$$

Exercise 1.5. Perform the following unit conversions.

(a) How many square feet are in 1 square inch?
(b) Convert 1 cubic foot to cubic inches.
(c) How many square centimeters are in a square meter?
(d) Convert 1 cubic yard to cubic feet (3 feet $\leftrightarrow$ 1 yard).

1.1.6 Compound units

A handful of units that you're likely to encounter in an astronomy class are actually compound units, meaning that they are combinations of more basic[1] units. For example, the force unit of newtons (N) is defined as a mass in kilograms times a distance in meters divided by the square of the time in seconds: 1 N = 1 kg·m/s^2. This means that wherever you see units of newtons (N), you are free to replace that unit with its equivalent, kg·m/s^2, without changing the number in front of the unit. Put another way, you can use 1 N $\leftrightarrow$ 1 kg·m/s^2 to make the conversion factor $\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}$ or its inverse, which you can multiply by your original quantity in order to get it into a new set of units.

The energy unit joules is another example. Energy has dimensions of force (SI units of newtons) times distance (SI units of meters), so 1 J $\leftrightarrow$ 1 N·m.

As one final example of compound units, the power units of watts (W) are defined as energy (SI units of joules) per time (SI units of seconds). Therefore 1 W $\leftrightarrow$ 1 J/s.

Example: Express the compound unit watts in terms of the base units kilograms (kg), meters (m), and seconds (s).

The definition of watts is given just above: energy per unit time, with SI units of joules per second:

$$1\text{ W} \quad \leftrightarrow \quad 1\text{ J/s}.$$

[1] The *base units* you will encounter in this book are those of the International System of Units ("SI"): meters for length, kilograms for mass, seconds for time, and kelvins for temperature. Many astronomers (and some astronomy texts) use the "cgs" system in which the standard units are centimeters for length, grams for mass, and seconds for time.

But joules are compound units as well:

$$1\ \mathrm{W} = 1\ \mathrm{J/s} = 1\ (\mathrm{N{\cdot}m})/\mathrm{s},$$

and newtons are compound units:

$$1\ \mathrm{W} = 1\ \mathrm{N{\cdot}m/s} = 1\ (\mathrm{kg{\cdot}m/s^2}){\cdot}\mathrm{m/s}.$$

This simplifies to

$$1\ \mathrm{W} = 1\ \frac{\mathrm{kg{\cdot}m}}{\mathrm{s^2}}{\cdot}\frac{\mathrm{m}}{\mathrm{s}} = 1\ \frac{\mathrm{kg{\cdot}m^2}}{\mathrm{s^3}}.$$

This is the expression of watts in terms of SI base units. Compound units are often more convenient to use because they keep the units simpler and more compact.

Exercise 1.6. Express the following compound units in terms of base units kilograms, meters, and seconds.

(a) Pressure: $\mathrm{N/m^2}$ (note 1 $\mathrm{N/m^2}$ is defined as 1 pascal, or 1 Pa).
(b) Energy density: $\mathrm{J/m^3}$.

The exercises throughout the section should help you practice the individual concepts and operations needed for doing unit conversions. If you're ready for some more-challenging questions that require synthesizing tools from this and other sections, take a look at the problems at the end of this chapter and the on-line solutions.

1.2 Absolute and ratio methods

On the first day of some astronomy classes, students are surprised to learn that the use of a calculator is prohibited, or at least discouraged, by the instructor. In other astronomy classes, calculators may be encouraged or even required. So what's the best way to solve problems in astronomy?

As is often the case, there is no one way that works best for everyone. There are, however, two basic methods that you're likely to find helpful. Those two methods will be referred to in this book as the *absolute method* and the *ratio method*. And although either of these methods may be used with or without a calculator, it's a good bet that if your instructor intends for you to use only the ratio method, calculators may be prohibited or discouraged.

In this book, you'll find that both the absolute method and the ratio method are used throughout the examples and problems. That way, no matter which type of class you're in (or which method you prefer to use), you'll be able to see plenty of relevant examples.

So exactly what are the absolute and ratio methods? The short answer is that the absolute method is a way to determine the absolute numeric value of a quantity in the relevant units (such as a distance of 3 meters, time duration of 15 seconds, or mass of 2 million kilograms), and the ratio method is a way to find the unitless relative value of a quantity (such as a distance that is twice as far, time duration that is three times as long, or mass that is 50 times greater). Of course, for the relative value to have meaning, you must specify the reference quantity as well (twice as far as what, for example).

1.2.1 Absolute method

The absolute method is probably the way you first learned to solve problems: using an equation with an "equals" sign, just get the variable you're trying to find all by itself on the left side of the equation and then plug in the values (with units!) on the right side of the equation. So if you're trying to find the area (A) of a circle of given radius (R), you can use the equation

$$A = \pi R^2.$$

If the radius (R) is 2 meters, the area is

$$A = (3.1416)(2\text{ m})^2 = 12.6\text{ m}^2.$$

The units of the answer (square meters in this case) come directly from the units attached to the variables on the right side of the equation. Notice that when the radius gets squared, you have to square both the number and the unit, so $(2\text{ m})^2$ is 2^2 m^2, or 4 m^2.

Exercise 1.7. Calculate the following quantities for Earth, assuming a radius (R) of 6371 km. Be sure to include units with your answer.

(a) The circumference (C) of the Earth's equator ($C_{circle} = 2\pi R$).
(b) The surface area (SA) of Earth ($SA_{sphere} = 4\pi R^2$).
(c) The volume (V) of Earth ($V_{sphere} = \frac{4}{3}\pi R^3$).

1.2.2 Comparing two quantities

In everyday life, comparisons between two quantities are usually made in two ways: either by subtracting or by dividing the quantities. For example, if one city is 250 km away from your location, and a second city is 750 km away from your location, you could say that the second city is 500 km farther than the first (since 750 km −250 km = 500 km). But you could also say that the second city is three times farther away than the first (since 750 km/250 km = 3). Both of these statements are correct, but which is more useful depends on the situation. In astronomy, the values of many quantities (such as the mass of a planet, the luminosity of a star, or the distance between galaxies) are gigantic, and subtraction of one extremely large number from another can lead to results that are uncertain and difficult to interpret. In such cases, comparison by dividing is far more useful than comparison by subtracting.

For example, saying that the distance to the star Rigel is approximately 4.4 quadrillion miles (which is 4.4 million billion miles) greater than the distance to the star Vega may be useful in some situations, but saying that Rigel is about 31 *times farther* than Vega is more helpful for giving a sense of scale (and is also easier to remember). Of course, it's always possible to convert the difference in values to the ratio of the values and vice versa, provided you have the required reference information (for example, the distance to Vega). But since it's easiest to just do one comparison rather than both, your best bet is to compare using a ratio unless explicitly instructed otherwise.

It's the utility of this "comparing by dividing" idea that makes the ratio method so useful in astronomy.

Example: Compare the area of the circle you found in Section 1.2.1 (call it circle 1) to the area of another circle (call this one circle 2) with three times larger radius (so R = 6 meters for circle 2).

If you want to know how many times bigger the area of circle 2 is compared to circle 1, you could use the absolute method and calculate the area of each circle separately:

$$A_1 = \pi R_1^2 = (3.1416)(2\text{ m})^2 = 12.6\text{ m}^2,$$
$$A_2 = \pi R_2^2 = (3.1416)(6\text{ m})^2 = 113.1\text{ m}^2.$$

To compare these areas by dividing, you would then do the following

$$\frac{A_2}{A_1} = \frac{113.1\text{ m}^2}{12.6\text{ m}^2} = 8.98 \approx 9,$$

so the area of circle 2 is about nine times greater[2] than that of circle 1.

Notice that in addition to giving you the answer to the question "how many times bigger," this absolute method also provides the value of the area of each of the two circles (113.1 m^2 for circle 2 and 12.6 m^2 for circle 1). But if you're

[2] You could have taken A_1/A_2, in which case you would have gotten $\frac{A_1}{A_2} = \frac{12.6\text{ m}^2}{113.1\text{ m}^2} = \frac{1}{8.98} \approx \frac{1}{9}$, which is an equivalent result.

only interested in *comparing* these areas, the ratio approach can give you the answer much more quickly and easily.

Exercise 1.8. Compare the two numbers in each of the following situations using both methods of subtraction and division. Use your results to decide which method is useful in that situation, or if both might be useful.

(a) The tallest building in Malaysia is 452 m tall. A typical person is about 1.7 m tall. How much taller is the tall building than a person?
(b) A man weighed 220 lb. After dieting, his weight dropped to 195 lb. How much more did he weigh before he lost the weight?
(c) A typical globular cluster of stars might have 400,000 stars. A typical galaxy might have 200,000,000,000 stars. How many more stars are in the galaxy?

1.2.3 Ratio method

To understand how comparing with ratios works, try writing the equations for the areas of circles 1 and 2 from the previous example as a fraction:

$$\frac{A_2 = \pi R_2^2}{A_1 = \pi R_1^2}, \tag{1.1}$$

which is

$$\frac{A_2}{A_1} = \frac{\cancel{\pi} R_2^2}{\cancel{\pi} R_1^2} = \frac{R_2^2}{R_1^2}$$

or

$$\frac{A_2}{A_1} = \left(\frac{R_2}{R_1}\right)^2. \tag{1.2}$$

Look at the simplicity of this last result: to know the ratio of area A_2 to area A_1, simply find the ratio of radius R_2 to radius R_1 and then square that value. Since you know that R_2 is three times larger than R_1, the ratio of the areas (A_2/A_1) must be nine (because $3^2 = 9$). Notice that this was the same result obtained in the previous section using the absolute method, without going through the steps necessary to individually determine the values of A_2 and A_1 and then dividing those values. The ratio method also gave you the exact answer of 9, instead of the approximate answer obtained by rounding the value of π and the values of the areas before dividing them.

Of course, in this example, those extra steps were fairly simple (squaring each radius and multiplying by π), so using the ratio method saved you only three steps – a small amount of work. But in other problems using ratios may save you many steps, so we strongly encourage you to use the ratio method whenever possible. Remember that minimizing the number of calculations you do when working a problem reduces the opportunities for errors.

Example: Compare the volumes of two spheres, one of which has three times the radius of the other.

As you may recall, the volume (V) of a sphere can be found from the sphere's radius (R) using the equation

$$V = \frac{4}{3}\pi R^3$$

in which the volume comes out in units of cubic meters (m^3) if the radius has units of meters. If you know the radius of each sphere, you could use the absolute method to find the volume of each, and then by dividing the larger volume by the smaller you could specify how many times larger that volume is. But it is preferable to use the ratio method as in the previous example:

$$\frac{V_2 = \frac{4}{3}\pi R_2^3}{V_1 = \frac{4}{3}\pi R_1^3}, \tag{1.3}$$

and, as before, all the constants in the numerator cancel with those in the denominator, leaving

$$\frac{V_2}{V_1} = \frac{R_2^3}{R_1^3}$$

or

$$\frac{V_2}{V_1} = \left(\frac{R_2}{R_1}\right)^3. \tag{1.4}$$

So to determine the ratio of the volumes you simply *cube* the ratio of the radii. You know that the larger sphere has three times the radius of the smaller, and $3^3 = 27$, so you can be certain that the larger sphere's volume is 27 times greater.

There's another way to carry through the mathematical steps to solve this type of problem. If sphere 2 has the larger radius, the relationship between the radii is $R_2 = 3R_1$. Now wherever R_2 appears in Eq. 1.4, you can replace it with $3R_1$:

$$\frac{V_2}{V_1} = \left(\frac{R_2}{R_1}\right)^3 = \left(\frac{3\cancel{R_1}}{\cancel{R_1}}\right)^3 = \left(\frac{3}{1}\right)^3 = 3^3 = 27. \tag{1.5}$$

One powerful aspect of the ratio method is that you can determine the ratio of the volumes without knowing the radius of either sphere, as long as you know the ratio of those radii. So if the large sphere has three times the radius of the small sphere its volume must be 27 times larger, irrespective of the values of the radii. The spheres could have radii of 3 meters and 9 meters, or radii of 100 meters and 300 meters, or radii of 6,000 km and 18,000 km – in every case, if the ratio of the radii is 3, the ratio of the volumes is 27.

Exercise 1.9. The Sun's radius is 109 times larger than Earth's. Use the ratio method to make the following comparisons.

(a) How many times bigger is the Sun's circumference than Earth's?
(b) How many times bigger is the Sun's surface area than Earth's?
(c) How many times bigger is the Sun's volume than Earth's?

1.2.4 Interpreting ratio answers

The last step of a ratio problem is crucial: interpreting your answer. Many students find a result but don't know what to conclude from all their work. Your result from a ratio problem typically takes the form of an equation with two variables, an equals sign, and a number, such as Eq. 1.5. You can understand your result in one of two ways. First, you can look at the ratio you were calculating (V_2/V_1) and check if its numerical equivalent ($= 27$) is larger or smaller than 1. In this case 27 is larger than 1, so you know the quantity in the numerator (V_2) is larger than the quantity in the denominator (V_1), and by how many times (27). In other words, the volume of sphere 2 is 27 times larger than the volume of sphere 1. If the answer had been less than 1 (as would have happened if you had established sphere 1 as having the larger radius), you would have concluded the opposite, that the volume of sphere 2 was smaller, as shown in this example.

Example: You obtain the following result from a ratio problem comparing the radius of two spheres: $\frac{R_a}{R_b} = \frac{1}{5}$. *Which sphere is bigger, sphere a or sphere b, and by how many times?*

Inspection of the right side of the equation $\frac{R_a}{R_b} = \frac{1}{5}$ shows that the denominator (5) is larger than the numerator (1) by five times. Since this is true on the right side, it must also be true on the left, so you know that sphere b (R_b) must be larger than sphere a (R_a) by five times.

The second way to interpret a ratio answer is to make one more mathematical step of rearranging the answer, and then translate the math into words. Rearrange $\frac{V_2}{V_1} = 27$ to get $V_2 = 27\ V_1$. This small equation is a mathematical sentence that conveys information. It can be mapped term by term into a complete[3] sentence in words: *"The volume of sphere 2 is 27 times the volume of sphere 1."* This gives you the physical insight you need to understand what your answer means. Understanding and being able to translate an equation into meaningful words in your native language is a very important skill.

Example: Translate the mathematical result from the previous example into words: $\frac{R_a}{R_b} = \frac{1}{5}$.

First rearrange the equation to get one variable on each side:

$$\frac{R_a}{\cancel{R_b}} \times \cancel{R_b} = \frac{1}{5} \times R_b$$

$$R_a = \frac{1}{5} R_b.$$

Now translate this term-by-term into words: "The radius of sphere a" (R_a) "is" (=) "one fifth" $\left(\frac{1}{5}\right)$ "of" ($\times$, which is implicit on the right side) "the radius of sphere b" (R_b). That is, sphere a is one-fifth as large as sphere b, so sphere *a* is smaller.

Notice that your answer would look superficially different but have the same underlying meaning if you had chosen to rearrange the equation with R_b on the left and R_a on the right. To see this, start by taking the reciprocal of both sides:

$$\left(\frac{R_a}{R_b}\right) = \frac{1}{5},$$

$$\left(\frac{R_a}{R_b}\right)^{-1} = \left(\frac{1}{5}\right)^{-1},$$

$$\left(\frac{R_b}{R_a}\right) = \frac{5}{1},$$

and then multiply both sides by R_a to get R_b by itself:

$$\left(\frac{R_b}{\cancel{R_a}}\right) \times \cancel{R_a} = \left(\frac{5}{1}\right) \times R_a,$$

$$R_b = 5R_a.$$

Translate this term-by-term into a complete sentence of words: "The radius of sphere b is five times as large as the radius of sphere a." Saying that sphere b is five times as large as sphere a is mathematically identical to saying that sphere a is one-fifth as large as sphere b. Both phrasings make it clear that a is the smaller sphere and b is the larger, by a factor of five. So it does not matter how

[3] Making it a *complete* sentence ensures that you don't leave any parts out.