An eigenbasis study of linear homogeneous recurrence relations

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Abstract This article was meant to accompany a course project presented for CS691-XVIII, Applications of Linear Algebra, at IIT Gandhinagar during the summer of 2024. The objective is to produce mathematically motivated visual arguments to demonstrate the use of techniques from linear algebra to address certain problems from discrete mathematics, following the material from the book *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra*. The primary focus in the first part of this submission is to address Miniatures 1 and 2 from this book. Although we do resort to using slightly different representations, namely diagonalization of linear transformations that represent recurrence relations, efforts have been made to conform to the underlying principles insofar that the same results are obtained.

The second part of this submission delves into the periodic of behavior of second order recurrence relations. In the remainder of this work, contrary to convention, we use the word *periodic* to mean sequences that exhibit periodic sign changes and not necessarily those that satisfy $f(x + t) = f(x)$. It is for this reason we preface it with *periodic behaviour* where necessary. Here we propose a proof for the existence of periodic zeros in specific sequences and derive the conditions under which such sequences exhibit periodic behavior. We achieve this by deconstructing linear transformations that represent such sequences through their eigenbasis, in keeping with the theme of this submission.

1 Part I

1.1 Linear recurrence relations

Given below is the general form of a linear homogeneous recurrence relation. [1]

$$S_n = a_k S_{n-1} + a_{k-1} S_{n-2} + ... + a_1 S_{n-k} \ni a_i \in \mathbb{R}$$

Any term is a linear combination of the preceding k terms and first k terms S_1 to S_k are given by the initial values b_1 to b_k. Given a set of k terms in such a sequence, the process of obtaining subsequent terms can be represented as a linear transformation of vectors in $\mathbb{R}^k$. Consider the following vectors.

$$v_{i-1} = \begin{bmatrix} S_{i+k-1} \\ S_{i+k-2} \\ . \\ . \\ . \\ S_i \end{bmatrix} \text{ and } v_i = \begin{bmatrix} S_{i+k} \\ S_{i+k-1} \\ . \\ . \\ . \\ S_{i+1} \end{bmatrix} \ni v_i = Av_{i-1}, \text{ where } A \text{ is a linear transformation.}$$

The entries of A would depend on the coefficients [2] a_1 to a_k from the sequence.

$$\text{For this to be true, } A = \begin{bmatrix} a_k & a_{k-1} & a_{k-2} & ... & a_1 \\ 1 & ... & 0 & 0 & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & ... & 1 & 0 & 0 \\ 0 & ... & 0 & 1 & 0 \end{bmatrix}$$

1.2 Eigenbasis of A

The eigenvalues of A are the roots of the polynomial $P(\lambda) = |A - \lambda I_k| = 0$. Using the eigenvalues of A, λ_1 to λ_k and their corresponding eigenvectors ϕ_1 to ϕ_k, the transformation A can be written as PDP^{-1} when digonalizable, where

$$P = \begin{bmatrix} \phi_1 & \phi_2 & ... & \phi_k \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & ... & \lambda_k \end{bmatrix}.$$

This gives us $v_n = A^n v_0 = PD^n P^{-1} v_0$. In this work, we only consider this method for transformations A that are diagonalizable. Particularly, the non-diagonalizable transformations corresponding to linear homogeneous recurrence relations of the second order can be handled separately as we will see in §2.1.2.

1.3 Classic examples

1.3.1 $S_n = S_{n-2} + S_{n-1}, \qquad \{S_1 = S_2 = 1\}$

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

The first example [3] we'll look at is the Fibonacci sequence (OEIS A000045). From the definition, we have $a_1 = a_2 = 1$ and $S_1 = S_2 = 1$. Therefore, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $v_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$P(\lambda) = |A - \lambda I_2| = 0 \implies \lambda^2 - \lambda - 1 = 0$, giving us $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Solving for λ and ϕ,

$$\left[\begin{array}{cc|c} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1-\lambda & 1 & 0 \\ 0 & \frac{\lambda^2-\lambda-1}{1-\lambda} & 0 \end{array}\right] \implies \phi = \begin{bmatrix} 1 \\ \frac{1}{\lambda} \end{bmatrix}$$

Plugging in λ_1 and λ_1, we find that $\phi_1 = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1} \end{bmatrix}$ and $\phi_2 = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2} \end{bmatrix}$. Hence,

$$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}, \; D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } P^{-1} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ \frac{-1}{\lambda_1} & 1 \end{bmatrix}$$

Since $v_n = A^n v_0 = PD^nP^{-1}v_0$, we look at the first coordinate of $PD^{n-2}P^{-1}v_0$ to obtain the n^{th} term of the sequence. This gives us,

$$\begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_1} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix} \begin{bmatrix} \lambda_1^{n-2} & 0 \\ 0 & \lambda_2^{n-2} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ \frac{-1}{\lambda_1} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1\lambda_2} + \lambda_2^{n-2} - \lambda_1^{n-2} \\ \frac{\lambda_1^{n-2} - \lambda_2^{n-2}}{\lambda_1\lambda_2} + \lambda_2^{n-3} - \lambda_1^{n-3} \end{bmatrix}$$

Extracting S_n,

$$S_n = \frac{1}{\lambda_1 - \lambda_2}((\lambda_1^{n-1} - \lambda_2^{n-1}) + (\lambda_1^{n-2} - \lambda_2^{n-2}))\{\because \lambda_1\lambda_2 = -1\}$$

$$= \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^n - \lambda_2^n)\{\because \lambda^{n-2} + \lambda^{n-1} = \lambda^n\}$$

Plugging in the values of λ_1 and λ_2 produces the following formula.

$$S_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

1.3.2 $S_n = S_{n-2} + 2S_{n-1}, \qquad \{S_1 = S_2 = 1\}$

$$1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, ...$$

This is sequence A001333 in the OEIS, closely related to the Pell Numbers. Each term is the sum of twice the preceding term and once the term before that. The base cases are identical to that of the Fibonacci sequence. The transformation for this sequence is $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and the characteristic polynomial $\lambda^2 - 2\lambda - 1 = 0$. The eigenvalues and eigenvectors of A are $\lambda_1 = \frac{2+\sqrt{8}}{2}$, $\lambda_2 = \frac{2-\sqrt{8}}{2}$, $\phi_1 = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1} \end{bmatrix}$ and $\phi_2 = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2} \end{bmatrix}$.

$$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}, \; D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } P^{-1} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ \frac{-1}{\lambda_1} & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1\lambda_2} + \lambda_2^{n-2} - \lambda_1^{n-2} \\ \frac{\lambda_1^{n-2} - \lambda_2^{n-2}}{\lambda_1\lambda_2} + \lambda_2^{n-3} - \lambda_1^{n-3} \end{bmatrix}$$

$$\implies S_n = \frac{1}{\lambda_1 - \lambda_2}((\lambda_1^{n-1} - \lambda_2^{n-1}) + (\lambda_1^{n-2} - \lambda_2^{n-2}))$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1^n\left(1 - \frac{1}{\lambda_1}\right) - \lambda_2^n\left(1 - \frac{1}{\lambda_2}\right)\right)$$

which simplifies to the following formula.

$$S_n = \frac{1}{2+\sqrt{8}}\left(1+\sqrt{2}\right)^n + \frac{1}{2-\sqrt{8}}\left(1-\sqrt{2}\right)^n$$

1.3.3 $S_n = S_{n-2} + mS_{n-1}$

As a fun aside, varying the coefficient of S_{n-1} produces different sequences characterized by the following transformation.

$$A = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues of A are the roots to the equation $\lambda^2 - m\lambda - 1 = 0$, with $\lambda = \frac{m \pm \sqrt{m^2+4}}{2}$. The positive root to these equations are known as the metallic means [4] and they have the following general form.

$$\varphi_m = \frac{m + \sqrt{m^2+4}}{2}$$

When the first metallic mean (golden ratio) corresponds to the case $m = 1$. The mean so obtained from example 1.3.2 is the silver ratio. Although there isn't any formal nomenclature, the general consensus is that the third metallic is called the bronze ratio, followed by copper and nickel. We shall refer to sequence corresponding to the k^{th} metallic ratio as the k-Fibonacci sequence. The n^{th} term in such a sequence is $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$. We assume base cases in these sequences to be $F_{k,1} = 1$ and $F_{k,2} = 1$.

1.3.4 General solution for second order recurrence relations

$$S_n = a_1 S_{n-2} + a_2 S_{n-1}, \qquad \{S_1 = b_1, S_2 = b_2\}$$

In the general case when the n^{th} term is a linear combination of the preceding two terms with coefficients a_1 and a_2 and base cases b_1 and b_2, we represent the corresponding transformation by $A = \begin{bmatrix} a_2 & a_1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues of A are obtained by solving $\lambda^2 - a_2\lambda - a_1 = 0$, from where $\lambda_1 = \frac{a_2+\sqrt{a_2^2+4a_1}}{2}$ and $\lambda_2 = \frac{a_2-\sqrt{a_2^2+4a_1}}{2}$. Solving $\left[\begin{array}{cc|c} a_2-\lambda & a_1 & 0 \\ 1 & -\lambda & 0 \end{array}\right]$

$$\rightsquigarrow \left[\begin{array}{cc|c} a_2-\lambda & a_1 & 0 \\ 0 & \frac{\lambda^2-a_2\lambda-a_1}{a_2-\lambda} & 0 \end{array}\right] \implies \phi_1 = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1} \end{bmatrix} \text{ and } \phi_2 = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}, \; D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \; \text{and } P^{-1} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ \frac{-1}{\lambda_1} & 1 \end{bmatrix}$$

We obtain v_{n-2} from the following expression.

$$\begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_1} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix} \begin{bmatrix} \lambda_1^{n-2} & 0 \\ 0 & \lambda_2^{n-2} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ \frac{-1}{\lambda_1} & 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$$

from where

$$S_n = \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1^{n-2} \left(a_1 b_1 + b_2 \lambda_1 \right) - \lambda_2^{n-2} \left(a_1 b_1 + b_2 \lambda_2 \right) \right)$$

This produces the following formula.

$$\begin{aligned} S_n = & \left(\frac{2a_1b_1 + a_2b_2 + b_2\sqrt{a_2^2+4a_1}}{2\sqrt{a_2^2+4a_1}} \right) \left(\frac{a_2 + \sqrt{a_2^2+4a_1}}{2} \right)^{n-2} \\ & - \left(\frac{2a_1b_1 + a_2b_2 - b_2\sqrt{a_2^2+4a_1}}{2\sqrt{a_2^2+4a_1}} \right) \left(\frac{a_2 - \sqrt{a_2^2+4a_1}}{2} \right)^{n-2} \end{aligned}$$

1.4 Higher order sequences

1.4.1 $S_n = S_{n-3} + S_{n-2} + S_{n-1}, \qquad \{S_1 = S_2 = 0, S_3 = 1\}$

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$$

This is often referred to as the Tribonacci sequence (OEIS A000073). With some care, it is possible to repeat the above process to obtain a closed form for this sequence as well. Unlike the previous examples, complex numbers make their debut in this example. We shall quickly go through much of the same steps as before, but without going into a deeper theory for higher order recurrences or attempting to provide closed forms for those.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A - \lambda I = \left[\begin{array}{ccc|c} 1-\lambda & 1 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1-\lambda & 1 & 1 & 0 \\ 0 & \frac{\lambda^2-\lambda-1}{1-\lambda} & \frac{1}{\lambda-1} & 0 \\ 0 & 0 & \frac{-\lambda^3+\lambda^2+\lambda+1}{\lambda^2-\lambda-1} & 0 \end{array}\right] \implies \phi = \begin{bmatrix} 1 \\ \frac{1}{\lambda} \\ \frac{1}{\lambda^2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \\ \frac{1}{\lambda_1^2} & \frac{1}{\lambda_2^2} & \frac{1}{\lambda_3^2} \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

The eigenvalues are the roots of $\lambda^3 - \lambda^2 - \lambda - 1 = 0$, whose approximate values are $\lambda_1 = 1.839286$, $\lambda_2 = -0.419643 + 0.606290i$ and $\lambda_3 = -0.419643 - 0.606290i$. Substituting these values in the equation $\begin{bmatrix} S_n \\ S_{n-1} \\ S_{n-2} \end{bmatrix} = A^{n-3}v_0 = PD^{n-3}P^{-1}v_0$, we have

$$\begin{bmatrix} S_n \\ S_{n-1} \\ S_{n-2} \end{bmatrix} \approxeq \begin{bmatrix} 1 & 1 & 1 \\ 0.543689 & -0.771845 - 1.115143i & -0.771845 + 1.115143i \\ 0.295598 & -0.647799 + 1.721433i & -0.647799 - 1.721433i \end{bmatrix}$$

$$\times \begin{bmatrix} 1.839287^{n-3} & 0 & 0 \\ 0 & (-0.419643 + 0.606291i)^{n-3} & 0 \\ 0 & 0 & (-0.419643 - 0.606291i)^{n-3} \end{bmatrix}$$

$$\times \begin{bmatrix} 0.618420 & 0.519032 & 0.336228 \\ 0.190790 - 0.018701i & -0.259516 + 0.142222i & -0.168114 - 0.198324i \\ 0.190790 + 0.018701i & -0.259516 - 0.142222i & -0.168114 + 0.198324i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

from where

$$S_n \approxeq 0.618420 \times (1.839287)^{n-3}$$
$$+(0.190790 - 0.018701i) \times (-0.419643 + 0.606291i)^{n-3}$$
$$+(0.190790 + 0.018701i) \times (-0.419643 - 0.606291i)^{n-3}$$

1.4.2 $S_n = 3S_{n-1} - 3S_{n-2} + S_{n-3}, \qquad \{S_1 = 1, S_2 = 4, S_3 = 9\}$

Another example of a third order sequence is the squares of natural numbers [1]. The choice of coefficients follows from

$$\begin{aligned}
(n+1)^2 &= n^2 + 2n + 1 \\
(n-1)^2 &= n^2 - 2n + 1 \\
(n-2)^2 &= n^2 - 4n + 4 \\
\implies n^2 + 2n + 1 &= n^2 - 4n + 4 - 3n^2 + 6n - 3 + 3n^2 \\
&= (n^2 - 4n + 4) - 3(n^2 - 2n + 1) + 3(n^2) \\
&= (n-2)^2 - 3(n-1)^2 + 3n^2
\end{aligned}$$

With $S_n = n^2$,

$$S_n = 3S_{n-1} - 3S_{n-2} + S_{n-3}$$

The sequence of cubes of natural numbers is one of order 4. [1]p. 10

1.5 A few other sequences

1.5.1 Arithmetic and geometric progressions

for an arithmetic progression whose n^{th} term is given by $c + dn$, for any two consecutive terms S_n and S_{n+1}, the common difference is $d = S_{n+1} - Sn$. The term following S_{n+1} is $S_{n+2} = S_{n+1} + d = 2S_{n+1} - S_n \implies S_n = 2S_{n-1} - S_{n-2}$. The two consecutive terms required to describe the progression are the initial conditions $b_1 = c$ and $b_2 = c + d$. It is therefore possible to generate arbitrary arithmetic progressions with a recurrence relation of the second order. Given below are some examples.

- Natural numbers : $b_1 = 1$ and $b_2 = 2$
- Negative integers : $b_1 = -1$ and $b_2 = -2$
- Positive even numbers : $b_1 = 2$ and $b_2 = 4$

For consecutive terms a and ar in geometric progression, we can use the following relation.

$$S_n = rS_{n-1}$$

In this case $a_1 = 0$ and the initial values would be $b_1 = S_{n-2}$ and $b_2 = S_{n-1}$. The value of a_2 is what produces the sequence. For example, the GP $5, 15, 45, 135, ...$ is 5×3^n and is equivalent to the relation $S_n = 3S_{n-1}$ with initial values 5 and 15.

1.5.2 Univariate polynomials

All univariate polynomials can be expressed as a linear recurrence. Consider the following polynomial in x.

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

We wish to rewrite it as a recurrence $P(x) = \sum_{i=1}^{n+1} c_i P(x-i)$. In $P(x)$, the term a_0 is a constant. We therefore require the following to hold.

$$\sum_{i=1}^{n+1} c_i = 1$$

Since each term of the form $c_i P(x-i)$ in the recurrence contributes terms of the form $c_i((-1)^k a_j \binom{j}{k} i^k x^{j-k}) \mid 0 < k \leq j$ and $0 < j \leq n$ in excess of the terms in just $P(x)$, we need the sum of these terms over i to be zero. We shall call this sum S. This can be restated as follows.

$$S = \sum_{i=1}^{n+1} c_i \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^k \binom{j}{k} i^k = 0$$

We propose a candidate solution for c_i's by observing the following identity.

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k = 0 \ \forall k \mid 0 \leq k < n$$

Suppose $c_i = (-1)^i \binom{n+1}{i}$. Then fixing j and k, running the sum over i, we get

$$S_{(j,k)} \binom{j}{k} \sum_{i=1}^{n+1} (-1)^{(i+k)} \binom{n+1}{i} i^k = 0$$

since shifting the exponent of (-1) just changes the sign of the sum. Since each $S = \sum_{j=1}^{n} \sum_{k=1}^{j} S_{(i,j)}$, this assignment also satisfies $S = 0$. We finally need to make sure that $\sum_{i=1}^{n+1} c_i = 1$, hence we shift the exponent of (-1) in each c_i by 1. We now have values for the coefficients that satisfy all the requirements.

$$c_i = (-1)^{i+1} \binom{n+1}{i}$$

This gives us a way to write polynomials in x of degree n as a recurrence over preceding $n+1$ terms as follows.

$$P(x) = \sum_{i=1}^{(n+1)} (-)^{i+1} \binom{n+1}{i} P(x-i)$$

2 Part II

2.1 Periodic behavior in second order sequences

In a sequence of the form $S_n = a_2 S_{n-1} + a_1 S_{n-2}$, some choices of a_1 and a_2 produce complex eigenvalues. We consider an example where the initial values are $b_1 = b_2 = 1$. Given below is one such sequence.

$$S_n = S_{n-1} - 2S_{n-2}$$

This is represented by a transformation whose eigenvalues are $\lambda_1 = \frac{1+i\sqrt{7}}{2}$ and $\lambda_1 = \frac{1-i\sqrt{7}}{2}$. The sequence thus generated is the following.

$$1, 1, -1, -3, -1, 5, 7, -3, -17, -11, 23, 45, -1, -91, -89$$

Another example is $S_n = 2S_{n-1} - 2S_{n-2}$ with eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, producing the following sequence.

$$1, 1, 0, -2, -4, -4, 0, 8, 16, 16, 0, -32, -64, -64, 0$$

It can be observed that in the above two examples, the signs of the terms oscillate. In the second sequence, 0 occurs more than once and they appear to occur at positions $4n + 3$. What lies ahead is an attempt to address the following questions.

- Does this pattern of zeros continue?
- Under what conditions do periodic zeros occur?

2.1.1 Where zeros appear in $S_n = 2S_{n-1} - 2S_{n-2}$

From the closed form formula for this sequence, we find that the term at position n is

$$\left(\frac{2a_1b_1 + a_2b_2 + b_2\sqrt{a_2^2 + 4a_1}}{2\sqrt{a_2^2 + 4a_1}}\right)(1+i)^{n-2}$$
$$-\left(\frac{2a_1b_1 + a_2b_2 - b_2\sqrt{a_2^2 + 4a_1}}{2\sqrt{a_2^2 + 4a_1}}\right)(1-i)^{n-2}$$

Substituting the values of a_1, a_2, b_1 and b_2 and regrouping the terms results in the following expression.

$$S_n = (\frac{i}{2})((1+i)^{n-2} - (1-i)^{n-2}) + (\frac{1}{2})((1+i)^{n-2} + (1-i)^{n-2})$$

Solving for $S_n = 0$, we have

$$i((1+i)^{n-2} - (1-i)^{n-2}) = -((1+i)^{n-2} + (1-i)^{n-2})$$

$$\implies i\left(e^{i(\frac{(n-2)\pi}{4})} - e^{-i(\frac{(n-2)\pi}{4})}\right) = -\left(e^{i(\frac{(n-2)\pi}{4})} + e^{-i(\frac{(n-2)\pi}{4})}\right)$$

$$\implies \cos\left(\frac{(n-2)\pi}{4}\right) = \sin\left(\frac{(n-2)\pi}{4}\right)$$

Since $\cos\theta = \sin\theta \implies 4\theta = \pi(4k+1) \ni k \in \mathbb{Z}$, $(n-2) = 4k+1$, from where

$$n = 4k+3$$

Therefore, for the sequence $S_n = 2S_{n-1} - 2S_{n-2}$ with initial values $b_1 = b_2 = 1$, every term at position $4k+3$ is zero $\forall k \in \mathbb{Z}$.

2.1.2 Necessary conditions for periodic zeros

A second order recurrence relation has the following general form.

$$S_n = a_2 S_{n-1} + a_1 S_{n-2}$$

Suppose $S_n = 0$ for some n. If this sequence were to produce periodic zeros, $\exists k \ni k \in \mathbb{Z}$ and $S_{n+k} = 0$. Stated differently, if A is the transformation corresponding to the sequence, then

$$A^k \begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix} = A^k \begin{bmatrix} 0 \\ S_{n-1} \end{bmatrix} = \begin{bmatrix} S_{n+k} \\ S_{n+k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ S_{n+k-1} \end{bmatrix}$$

This would mean that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A^k with eigenvalue τ_1. Similarly from

$$A^k \begin{bmatrix} S_{n+1} \\ S_n \end{bmatrix} = A^k \begin{bmatrix} S_{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} S_{n+k+1} \\ S_{n+k} \end{bmatrix} = \begin{bmatrix} S_{n+k+1} \\ 0 \end{bmatrix}$$

we find that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A^k with eigenvalue τ_2. Hence A^k has the following form.

$$A^k = \begin{bmatrix} a_2 & a_1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} \tau_2 & 0 \\ 0 & \tau_1 \end{bmatrix}$$

Rewriting A^k as PD^kP^{-1}, we get

$$A^k = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ \frac{-1}{\lambda_1} & 1 \end{bmatrix} = \begin{bmatrix} \tau_2 & 0 \\ 0 & \tau_1 \end{bmatrix}$$

$$= \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{\lambda_1^k}{\lambda_2} - \frac{\lambda_2^k}{\lambda_1} & \lambda_2^k - \lambda_1^k \\ \frac{\lambda_1^{k-1}}{\lambda_2} - \frac{\lambda_2^{k-1}}{\lambda_1} & \lambda_2^{k-1} - \lambda_1^{k-1} \end{bmatrix}$$

$$\implies \lambda_1^k = \lambda_2^k$$

$$\implies A^k = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1^k\left(\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}\right) & 0 \\ 0 & \lambda_2^k\left(\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}\right)\end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$

$$\implies \tau_1 = \tau_2 = \lambda_1^k = \lambda_2^k$$

We now have a few concrete cases to analyze. λ_1 and λ_2 are eigenvalues of A and have the following form.

$$\lambda_1 = \frac{a_2 + \sqrt{a_2^2 + 4a_1}}{2} \text{ and } \lambda_2 = \frac{a_2 - \sqrt{a_2^2 + 4a_1}}{2}$$

Since $\lambda_1^k = \lambda_2^k$, if these eigenvalues are real, then either $a_2 = 0$ or $\sqrt{a_2^2 + 4a_1} = 0$. If $\sqrt{a_2^2 + 4a_1} = 0$, then $a_1 = -\frac{a_2^2}{4}$ and, $\phi = \begin{bmatrix} 1 \\ \frac{1}{\lambda} \end{bmatrix}$.

If $\sqrt{a_2^2 + 4a_1} = 0$, then $\lambda_1 = \lambda_2$, $\phi_1 = \phi_2$ and P is singular. This would mean that the period of zeros k is undefined, i.e., the sequence is not periodic. Meanwhile, $a_2 = 0$ is acceptable as $S_n = 0 \implies S_{n-2} = 0$, since $a_1 \neq 0$. This would generate a sequence where alternate terms are zeros, i.e., of period $k = 2$.

If λ_1 and λ_2 are complex, then they must occur as a conjugate pair of the form $|r|e^{i\theta}$ and $|r|e^{-i\theta}$ satisfying $|r^k|e^{ik\theta} = |r^k|e^{-ik\theta}$. Since powers of a conjugate pair also form a conjugate pair,

$$e^{ik\theta} = e^{-ik\theta} \implies \theta = q\pi, q \in \mathbb{Q}$$

Since θ cannot be a whole number multiple of π for otherwise λ_1 and λ_2 would be real, we must also assert that $q \notin \mathbb{Z}$. Therefore,

$$\tan(\theta) = \left|\frac{\sqrt{|a_2^2 + 4a_1|}}{a_2}\right| = \tan(q\pi), \quad q \in \mathbb{Q} \setminus \mathbb{Z}$$

$$\implies a_1 = -\frac{a_2^2(\tan^2(q\pi) + 1)}{4}, \quad 0 < \tan^2(q\pi)$$

Therefore, for a second order sequence $S_n = a_2 S_{n-1} + a_1 S_{n-2}$, if there exists some $n \ni S_n = 0, 1 < n$, then the zeros of S_n are periodic only if

$$\text{either } a_2 = 0$$

$$\text{or } a_1 = -\frac{a_2^2(\tan^2(q\pi) + 1)}{4}, 0 < \tan^2(q\pi)$$

These conditions, although necessary, are not sufficient to guarantee periodic zeros. The initial values S_1 and S_2 must be such that $A^n \begin{bmatrix} S_2 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 \\ S_{n+1} \end{bmatrix}$ for some $n \in \mathbb{Z}$. This can be ensured by assuming without loss of generality, $n = 0$, i.e., $S_2 = 0$ and $S_1 = 1$. Hence when a_1 and a_2 meet the necessary conditions, then setting the initial values $\begin{bmatrix} S_2 \\ S_1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does the job.

Closing remarks

The original goal of this project was to explore just the first two miniatures from the book. The departure from the Fibonacci sequence into other sequences, followed by a study of the general structure of such sequences of the second order has been exhilarating. A closer look at the periodic behavior of these sequences was warranted around the half way point in our work on this project when we decided to include it. To our delight, diagonalization turned out to provide a intuitive explanation for the presence of periodic zeros. Furthermore, it gave us the means to articulate the conditions necessary for their existence in the case of second order sequences using familiar arguments.

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