There is another type of ordinary differential equation, other than the ones which are linear and with constant coefficients, which we can solve.

Definition 1

A n-th order **Cauchy-Euler equation** is an ordinary differential equation of the form

$$a_n t^n y^{(n)} + a_{n-1} t^{n-1} y^{(n-1)} + \cdots + a_1 t y' + a_0 y = f,$$

where $a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$.

In other words, the difference between this and the linear ordinary differential equations with constant coefficients we have dealt so far is that, in the left hand side, we have a linear combination of $t^k y^{(k)}$'s instead of $y^{(k)}$'s. Examples of second order are

$$t^2 y'' - 4ty' + 2y = 0, \quad 2t^2 y'' + 5ty' = \mathrm{e}^t, \quad t^2 y'' + 6y = \sin(2t), \quad \text{etc.}$$

To deal with them, we proceed as usual, *trying* a certain "basic" solution. With linear homogeneous differential equations with constant coefficients, one would try $y = \mathrm{e}^{rt}$, and find the values of r for which this is actually a solution of the equation, by studying a characteristic equation. Here, instead of $y = \mathrm{e}^{rt}$, we try $y = t^p$, and find values of p for which this is a solution. The characteristic equation in the Cauchy-Euler case, however, is not as simple as replacing derivatives of y with powers of r.

Let's firstly consider homogeneous Cauchy-Euler equations and deduce what the new characteristic equation must be. If $y = t^p$, then

$$y' = pt^{p-1}, \quad y'' = p(p-1)t^{p-2}, \quad y''' = p(p-1)(p-2)t^{p-3}, \ldots,$$

and, in general,

$$y^{(k)} = p(p-1)(p-2)\cdots(p-k+1)t^{p-k}.$$

Substituting this into the original equation, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{n} a_k t^k y^{(k)} = \sum_{k=0}^{n} a_k t^k p(p-1)\cdots(p-k+1)t^{p-k} \\ &= \sum_{k=0}^{n} a_k p(p-1)\cdots(p-k+1)t^p = \left(\sum_{k=0}^{n} a_k p(p-1)\cdots(p-k+1)\right) t^p \end{aligned}$$

In other words, if $y = t^p$ is to be a solution of the original Cauchy-Euler equation, then p must be a solution of the characteristic equation

$$\left(\sum_{k=0}^{n} a_k p(p-1)\cdots(p-k+1)\right) = 0.$$

The point here is that the powers t^k in the original equation match up with the powers t^{p-k} affected by derivatives in each term, allowing us to obtain t^p in all terms, which

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is then factored and cancelled. Now, this is obviously much more complicated than the characteristic equations in the case where we deal with linear ordinary differential equations with constant coefficients. You should **not** try to memorize this new characteristic equation, but instead derive it again each time you need to. It doesn't sound like it, but redoing it each time is faster.

Example 1 (Two distinct roots)

Find the general solution of $t^2y'' + ty' - y = 0$.

Start trying $y = t^p$, so that $y' = pt^{p-1}$ and $y'' = p(p-1)t^{p-2}$. Plug this into the equation to obtain

$$t^2p(p-1)t^{p-2} + tpy^{p-1} - t^p = 0,$$

and factor out t^p, so

$$(p(p-1) + p - 1)t^p = 0.$$

The resulting characteristic equation is $p(p-1) + p - 1 = 0$, which may be directly factored as $(p+1)(p-1) = 0$. Therefore $y_1 = t$ and $y_2 = t^{-1}$ are two linearly independent solutions, and so the general solution must be an arbitrary linear combination of those two solutions:

$$y = c_1t + c_2t^{-1}, \quad c_1, c_2 \in \mathbb{R}.$$

Example 2 (Double root)

Find the general solution of $t^2y'' + 3ty' + y = 0$.

Start trying $y = t^p$, so that $y' = pt^{p-1}$ and $y'' = p(p-1)t^{p-2}$. Plug this into the equation to obtain

$$t^2p(p-1)t^{p-2} + 3tpy^{p-1} + t^p = 0,$$

and factor out t^p, so

$$(p(p-1) + 3p + 1)t^p = 0.$$

The resulting characteristic equation is $p(p-1) + 3p + 1 = 0$, which may be rewritten as $p^2 + 2p + 1 = 0$, so $(p+1)^2 = 0$ and $p = -1$ is a double root. Then $y_1 = t^{-1}$ is a solution, and to obtain a second linearly independent solution, we multiply by $\ln t$ as opposed to t, so that $c_2 = t^{-1}\ln t$ is also a solution. Therefore, the general solution is

$$y = c_1t^{-1} + c_2t^{-1}\ln t, \quad c_1, c_2 \in \mathbb{R}.$$

To briefly justify why $\ln t$ appears instead of t in the double root case, one can set $x = \ln t$, so the Cauchy-Euler equation for $y(t)$ becomes a constant coefficient equation for $y(x)$. Double roots "here" correspond to double roots "there", and that's that.

Example 3 (Complex roots)

Find the general solution of $t^2y'' - ty' + 5y = 0$.

Start trying $y = t^p$, so that $y' = pt^{p-1}$ and $y'' = p(p-1)t^{p-2}$. Plug this into the equation to obtain

$$t^2p(p-1)t^{p-2} - tpy^{p-1} + 5t^p = 0,$$

and factor out t^p, so

$$(p(p-1) - p + 5)t^p = 0.$$

The resulting characteristic equation is $p(p-1) - p + 5 = 0$, which may be rewritten as $p^2 - 2p + 5 = 0$, whose solutions are $p = 1 \pm 2\mathrm{i}$. This says that $y = t^{1+2\mathrm{i}}$ is a complex solution, but we would like to deal with real solutions instead. So we use Euler's formula together with the oldest trick in the book from single-variable calculus, namely, whenever $\star$ is any (say, positive) quantity, we have that $\star = \mathrm{e}^{\ln \star}$. Thus

$$\begin{aligned} t^{1+2\mathrm{i}} &= t \cdot t^{2\mathrm{i}} = t\mathrm{e}^{\ln(t^{2\mathrm{i}})} \\ &= t\mathrm{e}^{2\mathrm{i}\ln t} = t(\cos(2\ln t) + \mathrm{i}\sin(2\ln t)) \\ &= t\cos(2\ln t) + \mathrm{i}t\sin(2\ln t). \end{aligned}$$

Since real and imaginary parts of complex solutions are real solutions, we obtain that $y_1 = t\cos(2\ln t)$ and $y_2 = t\sin(2\ln t)$ are two linearly independent real solutions. So, the generic solution must be an arbitrary linear combination of those two solutions:

$$y = c_1t\cos(2\ln t) + c_2t\sin(2\ln t), \quad c_1, c_2 \in \mathbb{R}.$$

This covers how to treat homogeneous Cauchy-Euler equations. One can solve higher order Cauchy-Euler equations in the same way, provided one can solve the associated characteristic equation. One multiplies obtained solutions by $\ln t$ as opposed to t whenever roots with multiplicity appear, and Euler's formula generates real solutions from complex solutions by taking real and imaginary parts.

For the non-homogeneous case, the method of indeterminate coefficients to find particular solutions y_p still works (with $\ln t$ used instead of t when resonance happens), and the general solution of a non-homogeneous Cauchy-Euler equation is still $y = y_h + y_p$, where y_h is the general solution of the associated homogeneous Cauchy-Euler equation, and y_p is any particular solution of the original equation. To solve Initial Value Problems (IVP's), the strategy also remains the same, using the given initial conditions to find the values of c_1 and c_2 (in the case of second order equations).