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CONFORMAL FLATNESS OF COMPACT THREE-DIMENSIONAL COTTON-PARALLEL MANIFOLDS

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Abstract. A three-dimensional pseudo-Riemannian manifold is called essentially conformally symmetric (ECS) if its Cotton tensor is parallel but nowhere-vanishing. In this note we prove that three-dimensional ECS manifolds must be noncompact or, equivalently, that every compact three-dimensional Cotton-parallel pseudo-Riemannian manifold must be conformally flat.

1. Introduction and main result

Pseudo-Riemannian manifolds of dimensions $n \geq 4$ whose Weyl tensor is parallel are called *conformally symmetric* [3]. Those which are not locally symmetric or conformally flat are called *essentially conformally symmetric* (*ECS*, in short).

It has been shown by Roter in [9, Corollary 3] that ECS manifolds do exist in all dimensions $n \geq 4$, and in [4, Theorem 2] that they necessarily have indefinite metric signature. The local isometry types of ECS manifolds were described by Derdzinski and Roter in [6]. Compact ECS manifolds exist in all dimensions $n \geq 5$ and realize all indefinite metric signatures – see [8] and [7]. It is not currently known if compact four-dimensional ECS manifolds exist.

When the dimension of M is $n \leq 3$, the Weyl tensor vanishes and this discussion becomes meaningless. In dimension $n = 3$, however, conformal flatness is encoded in the Cotton tensor as opposed to the Weyl tensor, and so the following natural definition has been proposed in [1]: a three-dimensional pseudo-Riemannian manifold is called *conformally symmetric* if its Cotton tensor is parallel, and those which are not conformally flat are then called *ECS* (note that every three-dimensional locally symmetric manifold is conformally flat). There, it is also shown [1, Theorem 1] that, reversing the metric if needed, any point in a three-dimensional ECS manifold has a neighborhood isometric to an open subset of

$$(\widehat{M}, \widehat{\mathfrak{g}}) = \big(\mathbb{R}^3, (x^3 + \mathfrak{a}(t)x)\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \mathrm{d}x^2\big), \tag{1.1}$$

for some suitable smooth function $\mathfrak{a}\colon \mathbb{R} \to \mathbb{R}$. The coordinates t and s of $\widehat{M}$ are called y and t in [1], respectively, but have been renamed here as to make (1.1) directly resemble the corresponding local model given in [6, Section 4] for $n \geq 4$.

The pursuit of compact three-dimensional ECS manifolds quickly comes to an end in view of the following result, interesting on its own right without reference to ECS geometry:

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Theorem A. *A compact three-dimensional pseudo-Riemannian manifold with parallel Cotton tensor must be conformally flat.*

While the compactness assumption here is crucial, Theorem A may be seen as a close relative (in general signature) of [2, Theorem 1]: compact Riemannian *Cotton solitons* are conformally flat, but nontrivial compact Lorentzian ones do exist.

2. Preliminaries

Throughout this paper, we work in the smooth category and all manifolds considered are connected.

2.1. **Symmetries of the Cotton tensor.** The *Cotton tensor* of an n-dimensional pseudo-Riemannian manifold $(M, \mathtt{g})$ is the three-times covariant tensor field C on M defined by

$$\mathrm{C}(X,Y,Z) = (\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z), \quad \text{for } X,Y,Z \in \mathfrak{X}(M). \tag{2.1}$$

Here, P is the *Schouten tensor* of $(M, \mathtt{g})$, given by

$$P = \mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)}\mathtt{g}, \tag{2.2}$$

where Ric and s stand for the Ricci tensor and scalar curvature of $(M, \mathtt{g})$, respectively. The Cotton tensor satisfies the following symmetries:

$$\begin{aligned} &\text{(i)} \ \ \mathrm{C}(X,Y,Z) + \mathrm{C}(Y,X,Z) = 0 \\ &\text{(ii)} \ \ \mathrm{C}(X,Y,Z) + \mathrm{C}(Y,Z,X) + \mathrm{C}(Z,X,Y) = 0 \\ &\text{(iii)} \ \ \mathrm{tr}_{\mathtt{g}}\big((X,Z) \mapsto \mathrm{C}(X,Y,Z)\big) = 0 \end{aligned} \tag{2.3}$$

for all $X,Y,Z \in \mathfrak{X}(M)$. Symmetry (i) is obvious, while (ii) follows from a straightforward computation (six terms cancel in pairs), and (iii) from $\operatorname{div} P = \mathrm{d}(\mathrm{tr}_{\mathtt{g}} P)$ (which, in turn, is a consequence of the twice-contracted differential Bianchi identity $\operatorname{div}\mathrm{Ric} = \mathrm{ds}/2$).

2.2. **Algebraic structure in dimension 3.** A routine computation shows that

$$\begin{aligned} &\text{the Ricci and Cotton tensors of (1.1) are given} \\ &\text{by } \mathrm{Ric} = -3x\,\mathrm{d}t \otimes \mathrm{d}t \text{ and } \mathrm{C} = 3\,(\mathrm{d}t \wedge \mathrm{d}x) \otimes \mathrm{d}t. \end{aligned} \tag{2.4}$$

The expression for C motivates the following result, analogous to [5, Lemma 17.1]:

Theorem 2.1. *Let $(V, \langle\cdot,\cdot\rangle)$ be a three-dimensional pseudo-Euclidean space, and C be a nonzero* Cotton-like tensor *on V, i.e., a three-times covariant tensor on V which formally satisfies* (2.3)*, and consider $\mathcal{D} = \{u \in V \mid \mathrm{C}(u,\cdot,\cdot) = 0\}$. Then:*

(a) *$\mathcal{D}$ consists only of null vectors, and hence $\dim \mathcal{D} \le 1$.*
(b) *$\dim \mathcal{D} = 1$ if and only if $\mathrm{C} = (u \wedge v) \otimes u$ for some $u \in \mathcal{D} \smallsetminus \{0\}$ and unit $v \in \mathcal{D}^\perp$.*
(c) *In* (b)*, u is unique up to a sign, while v is unique modulo $\mathcal{D}$.*

Here, we identify $V \cong V^*$ with the aid of $\langle\cdot,\cdot\rangle$.

Proof. For (a), assuming by contradiction the existence of a unit vector $e_1 \in \mathcal{D}$, we will show that $\mathrm{C} = 0$. Considering an orthonormal basis $\{e_1, e_2, e_3\}$ for $(V, \langle\cdot,\cdot\rangle)$ and using (2.3-i) and (2.3-ii), we see that

$$\mathrm{C}_{ijk} \text{ is only possibly nonzero when } \{i,j,k\} = \{2,3\} \text{ with } i \neq j. \tag{2.5}$$

Now $\mathrm{C}_{322} = -\mathrm{C}_{232}$ and $\mathrm{C}_{323} = -\mathrm{C}_{233}$, while $\mathrm{tr}_{\langle\cdot,\cdot\rangle}\big((w,w') \mapsto \mathrm{C}(e_j,w,w')\big) = 0$ for $j = 2$ and $j = 3$ readily yields $\mathrm{C}_{233} = 0$ and $\mathrm{C}_{322} = 0$, respectively. Hence $\mathrm{C} = 0$, as claimed. As for (b), assume that $\dim \mathcal{D} = 1$, fix a null vector $e_1 \in \mathcal{D} \smallsetminus \{0\}$, and complete it to a basis $\{e_1, e_2, e_3\}$ of V satisfying the relations

$$\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_3\rangle = 0 \quad \text{and} \quad \langle e_1, e_3\rangle = \langle e_2, e_2\rangle = (-1)^{q+1}, \tag{2.6}$$

where $q \in \{1, 2\}$ is the index of $\langle\cdot,\cdot\rangle$. By the same argument as in (a), we again obtain (2.5), but this time $\mathrm{tr}_{\langle\cdot,\cdot\rangle}\big((w, w') \mapsto \mathrm{C}(e_3, w, w')\big) = 0$ reduces to $\mathrm{C}_{232} = 0$ in view of (2.6). Writing $a = \mathrm{C}_{323} \neq 0$ for the last essential component of C, it follows that $\mathrm{C} = a(e^3 \wedge e^2) \otimes e^3$, where $\{e^1, e^2, e^3\}$ is the basis of V^* dual to $\{e_1, e_2, e_3\}$. Applying the isomorphism $V \cong V^*$ and setting $u = |a|^{1/2} e_1$ and $v = \mathrm{sgn}(a) e_2$, we obtain the required expression $\mathrm{C} = (u \wedge v) \otimes u$. Conversely, it is straightforward to verify that the tensor $(u \wedge v) \otimes u$ with u null and v unit and orthogonal to u is Cotton-like with $\mathcal{D} = \mathbb{R}u$ and $\mathcal{D}^\perp = \mathbb{R}u \oplus \mathbb{R}v$. Finally, (c) is clear from (b). $\square$

As a consequence, whenever (M, g) is a three-dimensional pseudo-Riemannian manifold, we may assign to each point $x \in M$ the kernel $\mathcal{D}_x$ of C_x in (T_xM, g_x). In the ECS case, we have that

(2.7) $\mathcal{D}$ is a smooth rank-one parallel distribution on M, which contains the image of the Ricci endomorphism of (M, g).

Indeed, we may note that (2.7) holds in model (1.1) (as (2.4) gives us that $\mathcal{D}$ is spanned by the coordinate vector field ∂_s, $\widehat{\mathsf{g}}$-dual to $\mathrm{d}t$ up to a factor of 2), and invoke [1, Theorem 1].

3. Proof of Theorem A

In this section, we fix a compact three-dimensional ECS manifold (M, g) and its universal covering manifold $\pi\colon \widetilde{M} \to M$, which equipped with the natural pull-back metric $\widetilde{\mathsf{g}} = \pi^*\mathsf{g}$ becomes an ECS manifold. We will use the same symbols Ric, P, C, ∇, and $\mathcal{D}$ for the corresponding objects in both (M, g) and $(\widetilde{M}, \widetilde{\mathsf{g}})$. Observe that

(3.1) the fundamental group $\Gamma = \pi_1(M)$ acts properly discontinuously on $(\widetilde{M}, \widetilde{\mathsf{g}})$ by deck isometries, with quotient $\widetilde{M}/\Gamma \cong M$.

As $\widetilde{M}$ is simply connected, we may fix two globally defined smooth vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\mathrm{C} = (\boldsymbol{u} \wedge \boldsymbol{v}) \otimes \boldsymbol{u}$ on $\widetilde{M}$. Now, as $\mathcal{D}$ is parallel, item (c) of Theorem 2.1 gives us that

(3.2)
(i) $\boldsymbol{u}$ is a null parallel vector field spanning $\mathcal{D}$;
(ii) every $\gamma \in \Gamma$ either pushes $\boldsymbol{u}$ forward onto itself or onto its opposite.

Next, as the Ricci endomorphism of $(\widetilde{M}, \widetilde{\mathsf{g}})$ is self-adjoint, (2.7) allows us to write

$$\mathrm{Ric} = -f\, \boldsymbol{u} \otimes \boldsymbol{u}, \text{ for some smooth function } f\colon \widetilde{M} \to \mathbb{R}. \tag{3.3}$$

By (3.3) and (3.2-i), $(\widetilde{M}, \widetilde{\mathsf{g}})$ is scalar-flat, and so $P = \mathrm{Ric}$. Combining this with (3.2-i) again to compute C via (2.1), we obtain that

$$\mathrm{C} = (\boldsymbol{u} \wedge \nabla f) \otimes \boldsymbol{u}, \text{ where } \nabla f \text{ is the } \widetilde{\mathsf{g}}\text{-gradient of } f. \tag{3.4}$$

However, it follows from (3.2-ii) and (3.3) that f is Γ-invariant, and so it has a critical point due to (3.1) and compactness of M. Such a critical point is in fact a zero of C by (3.4), and therefore $\mathrm{C} = 0$. This is the desired contradiction: (M, g) must be either noncompact, or conformally flat.

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