MATH1152/1172 — WHAT YOU NEED TO KNOW ABOUT CROSS PRODUCTS

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Here we'll denote vectors by bold letters (such as $\boldsymbol{u}$, $\boldsymbol{v}$, etc.), and to avoid confusion between the absolute value of a number and the magnitude of a vector, we'll use double bars[1] $\|\cdot\|$ for the latter, e.g., $\|\boldsymbol{u}\|$ (so we have the formula $\|\lambda\boldsymbol{v}\| = |\lambda|\|\boldsymbol{v}\|$). In the plane ($\mathbb{R}^2$) we write $\boldsymbol{i} = \langle 1,0\rangle$ and $\boldsymbol{j} = \langle 0,1\rangle$, and in the space ($\mathbb{R}^3$) we write $\boldsymbol{i} = \langle 1,0,0\rangle$, $\boldsymbol{j} = \langle 0,1,0\rangle$ and $\boldsymbol{k} = \langle 0,0,1\rangle$. This is a standard notation, so everytime you see anyone write $\boldsymbol{i}$, $\boldsymbol{j}$ or $\boldsymbol{k}$ for a vector, they mean one of those previous vectors ($\boldsymbol{i}$ is always the first one, $\boldsymbol{j}$ always the second, $\boldsymbol{k}$ always the third, if you're in $\mathbb{R}^3$).

Definition. Let $\boldsymbol{u} = \langle u_1, u_2, u_3\rangle$ and $\boldsymbol{v} = \langle v_1, v_2, v_3\rangle$ be two vectors in the space. The *cross product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined to be

$$\boldsymbol{u}\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle.$$

You should not waste time trying to memorize the expression in the right. Compute the determinant everytime using whichever way you prefer (Sarrus' rule, row expansion, or whatever method you may have learned before in your life).

Examples.

(1) $\langle 1,2,0\rangle\times\langle 0,1,1\rangle = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} = \langle 2,-1,1\rangle.$

(2) $\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j}$.

General properties: If the word "determinant" above is too scary, here are some basic algebraic properties to help you avoid it:

(i) $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\times\boldsymbol{v} + \boldsymbol{u}\times\boldsymbol{w}$;

(ii) $(\boldsymbol{u}+\boldsymbol{w})\times\boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{v} + \boldsymbol{w}\times\boldsymbol{v}$;

(iii) $(\lambda\boldsymbol{u})\times\boldsymbol{v} = \lambda\,\boldsymbol{u}\times\boldsymbol{v}$;

(iv) $\boldsymbol{u}\times(\lambda\boldsymbol{v}) = \lambda\,\boldsymbol{u}\times\boldsymbol{v}$;

(v) $\boldsymbol{v}\times\boldsymbol{u} = -\boldsymbol{u}\times\boldsymbol{v}$.

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[1] This is also a very standard notation in mathematics. I don't know why people insist on using single bars for everything when teaching basic Calculus courses. It's likely to cause confusion. Go figure.

Example. Let's redo example (1) on the previous page without explictly computing a determinant, but using the results from (2) instead. Write $\langle 1, 2, 0\rangle = \boldsymbol{i} + 2\boldsymbol{j}$ and $\langle 0, 1, 1\rangle = \boldsymbol{j} + \boldsymbol{k}$. So:

$$(\boldsymbol{i} + 2\boldsymbol{j}) \times (\boldsymbol{j} + \boldsymbol{k}) = (\boldsymbol{i} \times \boldsymbol{j}) + (\boldsymbol{i} \times \boldsymbol{k}) + (2\boldsymbol{j} \times \boldsymbol{j}) + (2\boldsymbol{j} \times \boldsymbol{k}) = \boldsymbol{k} - \boldsymbol{j} + \boldsymbol{0} + 2\boldsymbol{i}.$$

It is the same thing as before, up to order.

Geometric features:

(i) $\boldsymbol{u} \times \boldsymbol{v}$ is always orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$. This means that if a problem asks you to find a vector orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$ with magnitude, say, 10, you compute $\boldsymbol{u} \times \boldsymbol{v}$, normalize it to get a unit vector $\hat{\boldsymbol{w}} = (\boldsymbol{u} \times \boldsymbol{v})/\|\boldsymbol{u} \times \boldsymbol{v}\|$, and multiply by 10 to get $\boldsymbol{w} = 10\hat{\boldsymbol{w}}$.

(ii) $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ *precisely* when $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.

(iii) Right hand rule: to visualize $\boldsymbol{u} \times \boldsymbol{v}$, do the following: point your index finger in the direction of $\boldsymbol{u}$, your middle finger in the direction of $\boldsymbol{v}$, and stick out your thumb. The direction of your thumb is the direction of $\boldsymbol{u} \times \boldsymbol{v}$.

(iv) The magnitude of $\boldsymbol{u} \times \boldsymbol{v}$ is $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. It is similar to the formula $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$ previously seen, but we need the magnitude on the left side now, since we cannot compare a vector with a number. Observe here that the previous remark about $\boldsymbol{u} \times \boldsymbol{v}$ being the zero vector $\boldsymbol{0}$ precisely when $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel is also a consequence of this formula, as $\theta = 0$ for parallel vectors, so $\|\boldsymbol{u} \times \boldsymbol{v}\| = 0$ implies $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$.

(v) The magnitude $\|\boldsymbol{u} \times \boldsymbol{v}\|$ is the area of the parallelogram spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$. This can also be used to compute areas of triangles. For example, let's say you have three points in space, P_1, P_2 and P_3, and you're asked to find the area of the triangle whose vertices are these points. Let $\boldsymbol{u} = P_2 - P_1$, $\boldsymbol{v} = P_3 - P_1$, and compute $A = \|\boldsymbol{u} \times \boldsymbol{v}\|/2$ (we are dividing by two because a triangle is half of a parallelogram).

(vi) Cross products can also be used to compute areas of parallelograms in the plane. If you're given vectors in the plane, pretend that they are in space by adding a third zero component. For example, to find the area of the triangle with vertices $P_1 = (0, 1)$, $P_2 = (1, 2)$ and $P_3 = (1, 3)$, we compute $\boldsymbol{u} = P_2 - P_1 = \langle 1, 1\rangle$ and $\boldsymbol{v} = P_3 - P_1 = \langle 1, 2\rangle$. Regard them in the space as $\boldsymbol{u} = \langle 1, 1, 0\rangle$ and $\boldsymbol{v} = \langle 1, 2, 0\rangle$, compute

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \boldsymbol{k} = \langle 0, 0, 1\rangle,$$

so that $A = \|\boldsymbol{u} \times \boldsymbol{v}\|/2 = \|\boldsymbol{k}\|/2 = 1/2$.