Nijenhuis geometry of parallel tensors

Andrzej Derdzinski[1] · Paolo Piccione[2,3] · Ivo Terek[1,4]

Abstract

A tensor—meaning here a tensor field Θ of any type (p, q) on a manifold—may be called integrable if it is parallel relative to some torsion-free connection. We provide analytical and geometric characterizations of integrability for differential q-forms, $q = 0, 1, 2, n-1, n$ (in dimension n), vectors, bivectors, symmetric $(2, 0)$ and $(0, 2)$ tensors, as well as complex-diagonalizable and nilpotent tensors of type $(1, 1)$. In most cases, integrability is equivalent to algebraic constancy of Θ coupled with the vanishing of one or more suitably defined Nijenhuis-type tensors, depending on Θ via a quasilinear first-order differential operator. For $(p, q) = (1, 1)$, they include the ordinary Nijenhuis tensor.

Keywords Integrable tensor field · Parallel tensor field · Nijenhuis tensor

Mathematics Subject Classification Primary 53C15; Secondary 53D17

1 Introduction

We refer to a tensor field Θ of any type on a manifold M as *algebraically constant* when, for any $x, y \in M$, some linear isomorphism $T_x M \to T_y M$ sends Θ_x to Θ_y. The algebraic

✉ Andrzej Derdzinski
andrzej@math.ohio-state.edu

Paolo Piccione
paolo.piccione@usp.br

Ivo Terek
it3@williams.edu

1   Department of Mathematics, The Ohio State University, 231 W.18th Avenue, Columbus, OH 43210, USA

2   Department of Mathematics School of Sciences, Great Bay University, Dongguan 523000, Guangdong, China

3   Departamento de Matemática Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP 05508-900, Brazil

4   Present Address: Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, USA

constancy amounts to being constant for functions, to vanishing nowhere or everywhere in the case of vector fields and 1-forms, and to having constant rank for symmetric or skew-symmetric $(0, 2)$ and $(2, 0)$ tensors.

We call a tensor field Θ *integrable* if some torsion-free connection makes it parallel, and *locally constant* if it has constant components in suitable local coordinates around each point. As one sees using a partition of unity, for integrability of Θ it suffices that such torsion-free connections exist locally. Consequently,

$$\text{the local constancy of } \Theta \text{ implies its integrability (but not conversely),} \qquad (1.1)$$

counterexamples to the converse being nonflat pseudo-Riemannian metrics.

Given an algebraically constant tensor Θ on a manifold M and a distribution $\mathcal{D} \subseteq TM$ naturally associated with it, as $\mathcal{D}$ is obviously ∇-parallel when $\nabla\Theta = 0$,

$$\text{the integrability of } \Theta \text{ implies the distribution-integrability of } \mathcal{D}. \qquad (1.2)$$

The local constancy of an algebraically constant tensor is nothing else than integrability, in the sense of [17, Prop. 1.1], of the corresponding G-structure (Remark 6.1).

With a $(1, 1)$ tensor Θ on a manifold M one associates its Nijenhuis tensor N, introduced by Nijenhuis [21] and studied by many others [4, 5, 7, 11, 12, 14, 16, 19, 28], which sends vector fields v, w to the vector field

$$N(v, w) = \Theta[\Theta v, w] + \Theta[v, \Theta w] - [\Theta v, \Theta w] - \Theta^2[v, w]. \qquad (1.3)$$

As pointed out by several authors [7, Sect. 2.3], [4, Definition 2.2], $N = 0$ identically whenever Θ is integrable since, for any torsion-free connection ∇ on M,

$$N(v, w) = [\Theta\nabla_v\Theta - \nabla_{\Theta v}\Theta]w + [\nabla_{\Theta w}\Theta - \Theta\nabla_w\Theta]v. \qquad (1.4)$$

Various generalizations of the Nijenhuis tensor have been proposed [3, 18, 24, 25, 27]. Below, after stating Proposition F, we elaborate on such generalizations that are of interest to us and have therefore been introduced in this paper.

Complex-diagonalizability of a linear endomorphism of a finite-dimensional real vector space V means, as usual, diagonalizability of its complex-linear extension to the complexification of V. Since any endomorphism of V is, uniquely, the sum of a complex-diagonalizable and a nilpotent one [15, Sect. 4.2], it is natural to deal with these two classes of endomorphisms separately.

In Sect. 5, 8–9, 11, 13, 14 and 15–16 we prove our six main results, stated below. We begin with a fact due to Kurita [19, Theorem 9], which also easily follows (see Sect. 3) from a theorem of Bolsinov, Konyaev and Matveev [4, Theorem 3.2]:

Remark A For an algebraically constant complex-diagonalizable $(1, 1)$ tensor Θ on a manifold M of dimension $n \geq 1$, the vanishing of N is equivalent to the integrability of Θ, as well as to its local constancy.

Algebraically constant tensors Θ of type $(1, 1)$ give rise to the vector subbundles $\mathcal{Z}^i = \operatorname{Ker} \Theta^i$ and $\mathcal{B}^i = \operatorname{Im} \Theta^i$ of TM, for integers $i \geq 0$.

Theorem B *Given an algebraically constant nilpotent $(1, 1)$ tensor Θ on a manifold M of dimension $n \geq 1$, the following four conditions are equivalent.*

  (i) *$N = 0$ and $\mathcal{Z}^i = \operatorname{Ker} \Theta^i$ is integrable for every $i = 1, \ldots, n$.*
  (ii) *In some commuting local frame $e_1, \ldots, e_n$ around each point, Θ has the Jordan normal form, with $\Theta e_1 = 0$ and $\Theta e_i = 0$ or $\Theta e_i = e_{i-1}$ if $i > 1$.*

(iii) Θ *is locally constant.*
(iv) Θ *is integrable.*

The Jordan normal form of an algebraically constant nilpotent $(1, 1)$ tensor Θ may be represented by

$$\text{a weakly decreasing string } d_1 \ldots d_m \text{ of positive integers,} \tag{1.5}$$

each d_q standing for a $d_q \times d_q$ Jordan block matrix with ones immediately above the diagonal and zeros everywhere else. Of interest to us are the Jordan normal forms $d_1 \ldots d_m$ such that $d_1 = \ldots = d_{m-1}$. In other words,

$$\text{either all blocks have the same length, or they represent exactly} \atop \text{two different lengths, with the shorter one occurring just once.} \tag{1.6}$$

We say that a given algebraic type of an algebraically constant nilpotent $(1, 1)$ tensor Θ is *controlled by the Nijenhuis tensor* if the vanishing of N implies, on any underlying manifold, the local constancy of Θ.

Theorem C *Condition* (1.6) *imposed on the Jordan normal form of an algebraically constant nilpotent* $(1, 1)$ *tensor* Θ *on a manifold* M *of dimension* $n \geq 1$ *is necessary and sufficient for the algebraic type of* Θ *to be controlled by its Nijenhuis tensor.*

Theorem C would be true as stated even if our definition of being controlled by N referred to integrability rather than local constancy. Namely, Proposition 8.1—our proof of the necessity of (1.6)—realizes any Θ not satisfying (1.6) as a left-invariant tensor with $N = 0$ on a step 2 nilpotent Lie group, which fails the integrability test (1.1) due to having nonintegrable Ker Θ^p for some integer $p \geq 1$.

For nilpotent $(1, 1)$ tensors Θ which are *generic*, that is, $\dim \text{Ker } \Theta = 1$ or, equivalently, the Jordan normal form of Θ is the one-term string n (a single Jordan block), the sufficiency of (1.6) in Theorem C is a result of Kobayashi [16, Sect. 3]. See also [12, Cor. 2.4], [28, Theorem 1], [5, Theorem 1.3, Cor. 1.5], [4, Theorem 4.6]. Kobayashi [16, Sect. 5] further illustrated the necessity of (1.6) by an example, with $n = 4$ and the Jordan normal form 211, cited in [4, Example 2.1].

Section 10 exhibits a special case of Theorem C by means of an affine-bundle construction, resulting in nonzero algebraically constant nilpotent $(1, 1)$ tensors Θ with $N = 0$, satisfying the condition $\Theta^2 = 0$ (equivalent to $\text{Im } \Theta \subseteq \text{Ker } \Theta$, that is, to having the Jordan normal form $2 \ldots 2$ or $2 \ldots 21 \ldots 1$).

For the normal form $2 \ldots 2$, also characterized by the equality $\text{Ker } \Theta = \text{Im } \Theta$, corresponding to the *almost-tangent structures* [31], the assertion of Theorem C is due to Goel [11, Theorem 2.4], while our affine-bundle construction becomes that of Crampin and Thompson [9]. Our construction is "locally universal" (Theorem 10.1), which generalizes the local version of [9, Theorem on p. 69].

We justify the following observation in Sect. 11.

Proposition D *The closedness of an algebraically constant differential* q-*form on an* n-*manifold,* $q = 0, 1, 2, n - 1, n$, *implies its local constancy.*

The converse implication (closedness from integrability, and hence also from local constancy) is obviously true for forms of all degrees.

Even weakened by the replacement of local constancy with integrability—cf. (1.1)—Proposition D fails to hold for differential forms of other degrees: as we verify in Sect. 12,

for any dimension $n \geq 5$ and any $q \in \{3, \ldots, n-2\}$, in local coordinates $x^1, \ldots, x^n$, the following formula defines a differential q-form ζ which is algebraically constant and closed, but not integrable:

$$\zeta = (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge (dx^5 + x^1 dx^2 - x^3 dx^4) \wedge dx^6 \wedge \ldots \wedge dx^{q+2}. \quad (1.7)$$

Constant-rank (skew)symmetric $(0, 2)$ and $(2, 0)$ tensors, being bundle morphisms $TM \to T^*M$ or $T^*M \to TM$, have well-defined unique kernels and images. The next displayed condition uses the natural concept of projectability, presented in Sect. 2: for the integrability of a constant-rank symmetric $(0, 2)$ tensor g on a manifold, it is necessary and sufficient—as we justify in Sect. 13—that

$$\text{the distribution } \operatorname{Ker} g \text{ be integrable, and } g \text{ projectable along } \operatorname{Ker} g. \quad (1.8)$$

Condition (1.8), rephrased as $\pounds_v g = 0$ for every local section v of $\operatorname{Ker} g$, is well known to be an integrability test for g. To the best of our knowledge, this fact goes back to Moisil [20] and Vrănceanu [30]. See also [8, 22, 26], [10, Theorem 5.1].

The sweeping recent result of Bandyopadhyay, Dacorogna, Matveev and Troyanov [1, Theorem 4.4] provides a characterization of local constancy for $(0, 2)$ tensors *without any symmetry/skew-symmetry assumptions*. The criterion (1.8), much more modest in scope, focuses on the symmetric case and integrability (as opposed to local constancy); what we gain is simplicity of the resulting conditions.

In contrast with 1-forms (Proposition D), the local constancy of a vector field obviously follows just from its algebraic constancy. An analogous difference occurs between symmetric $(2, 0)$ tensors and symmetric $(0, 2)$ tensors: the former—unlike the latter—require no projectability condition to guarantee integrability.

Proposition E *The integrability of a constant-rank symmetric $(2, 0)$ tensor Θ on a manifold is equivalent to the integrability of* $\operatorname{Im} \Theta$.

Such tensors Θ can be naturally identified (see Remark 2.3) with *sub-pseudo-Riemannian metrics* [13], which include the sub-Riemannian ones [2], such as the Galilei spacetime metric. For a bivector, that is, a skew-symmetric $(2, 0)$ tensor Θ, assumed to have constant rank, formula (2.7) defines the *restriction* of Θ to $\mathcal{B} = \operatorname{Im} \Theta$, which is a nondegenerate section of $\mathcal{B}^{\wedge 2}$, thus giving rise to its inverse, a section of $[\mathcal{B}^*]^{\wedge 2}$.

Proposition F *A constant-rank bivector Θ on a manifold is locally constant or—equivalently— integrable if and only if the distribution $\operatorname{Im} \Theta$ is integrable and the inverse of the restriction of Θ to $\operatorname{Im} \Theta$ is closed along each leaf of $\operatorname{Im} \Theta$.*

The generalizations of the Nijenhuis tensor which are of interest to us are motivated by Remark A, Theorem C and Proposition D: we want to associate with a given tensor Θ one (or more) Nijenhuis-type tensor(s), each depending on Θ via a quasilinear first-order differential operator, in such a way that, if Θ algebraically constant, the vanishing of these tensors is equivalent to the integrability of Θ.

As an example, N given by (1.3) serves in this capacity for complex-diagonalizable $(1, 1)$ tensors and nilpotent $(1, 1)$ tensors with the property (1.6); its quasi-linearity is immediate from (1.4). In the case of differential q-forms ζ in dimension n, where $q \in \{0, 1, 2, n-1, n\}$ (but not—see (1.7)—for other degrees), the exterior derivative $d\zeta$ is a Nijenhuis-type tensor in our sense, while an analogous role for vector fields is played by the zero tensor.

For any symmetric $(0, 2)$ tensor g of constant rank r on a manifold M, we introduce two Nijenhuis-type tensors N' and N'', both of type $(0, 2r+3)$, defined as follows: N' (or, N'')

sends vector fields $v, v_1, \ldots, v_r$ (or, $w, u, v_1, \ldots, v_r$) to the $(r+2)$-form, or $(r+1)$-form

$$
\begin{aligned}
&\text{a) } N'(v, v_1, \ldots, v_r) = d[g(v, \cdot)] \wedge g(v_1, \cdot) \wedge \ldots \wedge g(v_r, \cdot), \\
&\text{b) } N''(w, u, v_1, \ldots, v_r) = \{[\pounds g](w, u)\} \wedge g(v_1, \cdot) \wedge \ldots \wedge g(v_r, \cdot),
\end{aligned}
\tag{1.9}
$$

$[\pounds g](w, u)$ being treated here, formally, as a 1-form sending any vector field v to the function $[\pounds_v g](w, u)$. The word 'formally' reflects the fact that $[\pounds g](w, u)$ is not tensorial in v. Nevertheless, in Sect. 16 we point out that N' and N'' are well-defined tensors, and prove the following result.

Theorem G *The vanishing of both N' and N'' is necessary and sufficient for the integrability of the given symmetric $(0, 2)$ tensor g of constant rank r.*

The lines following formula (15.3) in Sect. 15 provide a set of d_1 Nijenhuis-type tensors for an algebraically constant nilpotent $(1, 1)$ tensor Θ with the Jordan normal form $d_1 \ldots d_m$. This set consists of N—see (1.3)—and $d_1 - 1$ additional tensors responsible for integrability of $\operatorname{Ker} \Theta^i$, $1 \leq i < d_1$ (which makes them redundant in the case (1.6), due to Theorem C).

Finally, formulae (15.4)—(15.5) in Sect. 15 define, for (skew)symmetric $(2, 0)$ tensors Θ of constant rank r, a Nijenhuis-type $(2r + 3, 0)$ tensor $\widehat{N}$ such that $\widehat{N} = 0$ if and only if $\operatorname{Im} \Theta$ is integrable. When Θ is symmetric, vanishing of $\widehat{N}$ thus amounts, by Proposition E, to integrability of Θ. However, for a rank r *bivector* Θ, the condition $\widehat{N} = 0$, despite still being necessary, is not sufficient in order that Θ be integrable. Obvious examples illustrating the last claim arise, cf. Proposition F, on a product manifold $M = \Sigma \times \Sigma'$, with Θ obtained as the trivial extension to M of the inverse of a nonclosed nondegenerate 2-form on Σ.

2 Preliminaries

Manifolds (by definition connected) and mappings, including sections of bundles, are always assumed to be smooth. Tensor fields will usually be referred to as *tensors*. All vector spaces are real (except in Sect. 3) and finite-dimensional.

Given a manifold M and vector subbundles $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ of TM, we write

$$
[\mathcal{D}, \mathcal{D}'] \subseteq \mathcal{D}''
\tag{2.1}
$$

when $[w, w']$ is a local section of $\mathcal{D}''$ for any local sections w of $\mathcal{D}$ and w' of $\mathcal{D}'$.

Lemma 2.1 *For $M, \mathcal{D}, \mathcal{D}'$ as above, suppose that $\mathcal{D}$ contains $\mathcal{D}'$ with codimension one, and $[\mathcal{D}, \mathcal{D}'] \subseteq \mathcal{D}$. Then $\mathcal{D}$ is integrable.*

Proof As $[\mathcal{D}', \mathcal{D}'] \subseteq \mathcal{D}$, the relation $[\mathcal{D}, \mathcal{D}] \subseteq \mathcal{D}$ follows if we note that, locally, sections of $\mathcal{D}$ have the form $v + \phi w$ for various sections v of $\mathcal{D}'$, functions ϕ, and one fixed section w of $\mathcal{D}$. $\qquad \square$

Let $\pi : M \to \Sigma$ be a mapping between manifolds. We say that a vector field w (or, a distribution $\mathcal{Z}$) on M is π-*projectable* if $d\pi_x w_x = u_{\pi(x)}$ or, respectively, $d\pi_x(\mathcal{Z}_x) = \mathcal{W}_{\pi(x)}$ for all $x \in M$ and some vector field u (or, distribution $\mathcal{W}$) on Σ. If this is the case,

$$
\text{the integrability of } \mathcal{Z} \text{ implies that of } \mathcal{W},
\tag{2.2}
$$

since π restricted to any leaf of $\mathcal{Z}$ is, locally, a submersion onto an integral manifold of $\mathcal{W}$. We also define π-projectability of a $(0, q)$ tensor field Θ on M by requiring Θ to be the π-pullback of a $(0, q)$ tensor field on Σ.

Given an integrable distribution $\mathcal{V}$ on a manifold M, every point of M has a neighborhood U such that, for some manifold Σ, the leaves of $\mathcal{V}$ restricted to U are the fibres of a bundle projection $\pi : U \to \Sigma$.

Let $\mathcal{V}$ be an integrable distribution on a manifold M. By $\mathcal{V}$-*projectability* of a vector field on an open set $U' \subseteq M$ (or, of a distribution on U', or of a $(0, q)$ tensor field on U') we mean its π-projectability for any π, U, Σ as in the last paragraph such that $U \subseteq U'$. Then, for a vector field w on M,

$$w \text{ is } \mathcal{V}\text{-projectable if and only if, for every section} \atop v \text{ of } \mathcal{V}, \text{ the Lie bracket } [v, w] \text{ is also a section of } \mathcal{V}. \tag{2.3}$$

(This is obvious in local coordinates for M turning π as above into a Cartesian-product projection.) It is also clear that, given a $(0, q)$ tensor field Θ,

$$\Theta \text{ is } \mathcal{V}\text{-projectable if and only if } d_v[\Theta(w_1, \ldots, w_q)] = 0 \text{ for all sec-} \atop \text{tions } v \text{ of } \mathcal{V} \text{ and all } \mathcal{V}\text{-projectable local vector fields } w_1, \ldots, w_q. \tag{2.4}$$

Lemma 2.2 *For an integrable distribution $\mathcal{V}$ and any distribution $\mathcal{Z}$ on an n-dimensional manifold M, the following two conditions are equivalent.*

(a) *$\mathcal{Z}$ is $\mathcal{V}$-projectable,*
(b) *$\mathcal{Z} \cap \mathcal{V}$ has a constant dimension and $[\mathcal{V}, \mathcal{Z}] \subseteq \mathcal{V} + \mathcal{Z}$.*

Under the additional assumption that $\mathcal{V} \subseteq \mathcal{Z}$,

(c) *$\mathcal{Z}$ is $\mathcal{V}$-projectable if and only if $[\mathcal{V}, \mathcal{Z}] \subseteq \mathcal{Z}$.*

Proof The equivalence of (a) and (b), once established, trivially implies (c) when $\mathcal{V} \subseteq \mathcal{Z}$. We proceed, however, by first proving (c), in the case where $\mathcal{V} \subseteq \mathcal{Z}$. It will then clearly follow from (c) that (a) and (b) are equivalent, since $\mathcal{V}$-projectability of $\mathcal{Z}$ amounts to $\mathcal{V}$-projectability of $\mathcal{V} + \mathcal{Z}$, and $\mathcal{V} \subseteq \mathcal{V} + \mathcal{Z}$.

If $\mathcal{V} \subseteq \mathcal{Z}$ and $\mathcal{Z}$ is $\mathcal{V}$-projectable, onto some distribution $\mathcal{W}$ on a local leaf space of $\mathcal{V}$, then $\mathcal{Z}$ is spanned by $\mathcal{V}$-projectable sections obtained as lifts of sections of $\mathcal{W}$ (including sections of $\mathcal{V}$, which are lifts of 0). As any section of $\mathcal{Z}$ is a functional combination of $\mathcal{V}$-projectable ones, (2.3) yields $[\mathcal{V}, \mathcal{Z}] \subseteq \mathcal{Z}$.

Conversely, let $\mathcal{V} \subseteq \mathcal{Z}$ and $[\mathcal{V}, \mathcal{Z}] \subseteq \mathcal{Z}$. We choose local coordinates $x^1, \ldots, x^n$ such that $\mathcal{V}$ is spanned by the coordinate vector fields ∂_i, $i = 1, \ldots, m$, and a local trivialization of the subbundle $\mathcal{Z}$ of TM having the form $\partial_1, \ldots, \partial_m, w_{m+1}, \ldots, w_s$. Using the index ranges $1 \le i, j, k \le m < a, b, c \le s$, we obtain, since $[\mathcal{V}, \mathcal{Z}] \subseteq \mathcal{Z}$,

$$[\partial_i, w_a] = \Gamma_{ia}^{\ j} \partial_j + \Gamma_{ia}^{\ b} w_b \tag{2.5}$$

for some functions $\Gamma_{ia}^{\ j}$, $\Gamma_{ia}^{\ b}$. The w_b-component of the Jacobi identity $[\partial_i, [\partial_j, w_a]] = [\partial_j, [\partial_i, w_a]]$, with $[\partial_i, \partial_j] = 0$, now implies *symmetry of* $\partial_i \Gamma_{ja}^{\ b} + \Gamma_{ic}^{\ b} \Gamma_{ja}^{\ c}$ *in* i, j. This symmetry amounts to the vanishing of the curvature, that is, flatness, for the linear connection with the components $\Gamma_{ia}^{\ b}$ in a rank $s - m$ vector bundle over a manifold with the coordinates x^i, $i = 1, \ldots, m$. The equations $\partial_i \psi^b + \Gamma_{ic}^{\ b} \psi^c = 0$, stating that ψ^a, with $m < a \le s$, are the components of a parallel section ψ, is thus locally solvable with any prescibed initial value at a given point z. Let us choose such a section ψ_a, for $a = m + 1, \ldots, s$, with the initial value $(\delta_a^1, \ldots, \delta_a^m)$ at z, so that

$$\partial_i \psi_a^b + \Gamma_{ic}^{\ b} \psi_a^c = 0, \quad \psi_a^b(z) = \delta_a^b. \tag{2.6}$$

Setting $u_a = \psi_a^b w_b$, we obtain a new local trivialization $\partial_1, \ldots, \partial_m, u_{m+1}, \ldots, u_s$ of $\mathcal{Z}$ while, by (2.5) and (2.6), $[\partial_i, u_a]$ are sections of $\mathcal{Z}$. Therefore, due to (2.3), our new local trivialization of $\mathcal{Z}$ consists of $\mathcal{V}$-projectable sections, which makes $\mathcal{Z}$ itself $\mathcal{V}$-projectable. □

Given a symmetric or skew-symmetric $(2, 0)$ tensor Θ in a vector space V, let $\mathrm{Ker}\, \Theta$ and $\mathrm{Im}\, \Theta$ be the kernel and image of $V^* \ni \xi \mapsto \Theta(\xi, \cdot) \in V$. By the *restriction* of Θ to $W = \mathrm{Im}\, \Theta$ we mean the $(2, 0)$ tensor Θ_W in W given by

$$\begin{aligned} &\Theta_W(\eta, \eta') = \Theta(\xi, \xi') \text{ for any } \eta, \eta' \in W^* \text{ and} \\ &\text{any extensions } \xi, \xi' : V \to \mathbb{R} \text{ of } \eta, \eta' \text{ to } V. \end{aligned} \tag{2.7}$$

As ξ and ξ' are unique up to adding elements of $W' = \mathrm{Ker}\, \Theta \subseteq V^*$, the polar space of W, the restriction is well defined. In other words, since $W' = \mathrm{Ker}\, \Theta$, the bilinear form Θ on V^* descends to one on $V^*/W' = W^*$, which is our Θ_W. (The natural identification of W^* with V^*/W' sends $\eta \in W^*$ to the W'-coset of any extension of η to V.) In addition, Θ is the image of Θ_W under the linear operator $W^{\odot 2} \to V^{\odot 2}$ or $W^{\wedge 2} \to V^{\wedge 2}$ induced by the inclusion $W \to V$. Finally,

$$\text{the restriction } \Theta_W \text{ is nondegenerate,} \tag{2.8}$$

as any $\eta \in W \smallsetminus \{0\}$ has an extension ξ to V not lying in $W' = \mathrm{Ker}\, \Theta$, and hence $\Theta(\xi, \xi') \neq 0$ for some $\xi' \in V^*$.

Remark 2.3 Constant-rank symmetric $(2, 0)$ tensors Θ on a manifold M are naturally identified with sub-pseudo-Riemannian metrics on M, that is, pseudo-Riemannian fibre metrics h on vector subbundles $\mathcal{B}$ of TM. In fact, one may set $\mathcal{B} = \mathrm{Im}\, \Theta$ and, using (2.7)–(2.8), declare h to be the inverse of the restriction of Θ to $\mathcal{B}$. Thus, cf. the lines preceding (2.8), Θ is the image of the inverse of h under the bundle morphism $\mathcal{B}^{\odot 2} \to [TM]^{\odot 2}$ induced by the inclusion $\mathcal{B} \to TM$.

The $(r-1)$-fold contraction of two $(r, 0)$-tensors Θ, Π on a manifold with a fixed Riemannian metric g, appearing in (i) below, is

$$\text{the } (2, 0) \text{ tensor } \beta \text{ given by } \beta^{ij} = \Theta^{i i_2 \ldots i_r} \Pi^{j j_2 \ldots j_r} g_{i_2 j_2} \cdots g_{i_r j_r}. \tag{2.9}$$

Remark 2.4 Let V be a Euclidean n-space with the inner product $\langle \cdot, \cdot \rangle$.

(i) The $(r-1)$-fold contraction (2.9) against itself of a nonzero decomposable r-vector $v_1 \wedge \ldots \wedge v_r \in V^{\wedge r}$ yields a $(2, 0)$ tensor which, viewed with the aid of $\langle \cdot, \cdot \rangle$ as an endomorphism of V, equals a nonzero multiple of the orthogonal projection onto the span of $v_1, \ldots, v_r$. (To see this, we are free to assume that $v_1, \ldots, v_r$ are orthonormal.)

(ii) If V is oriented, $*(e_1 \wedge \ldots \wedge e_r) = e_{r+1} \wedge \ldots \wedge e_n$ for the Hodge star $* : V^{\wedge r} \to V^{\wedge(n-r)}$ and any positive orthonormal basis $e_1, \ldots, e_n$ of V.

Remark 2.5 In an $s \times (n-s)$ product n-dimensional manifold M with global product coordinates x^i, x^a (index ranges $1 \leq i \leq s < a \leq n$), let the component functions g_{ij}, Θ^{ij} and $\Gamma_{ij}^k = \Gamma_{ji}^k$ represent families of $(0, 2)$ tensors, $(2, 0)$ tensors and torsion-free connections on the leaves of the integrable distribution spanned by the coordinate vector fields ∂_i. Suppose that each tensor is parallel relative to the corresponding connection on the leaf: $\partial_i g_{jk} = \Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl}$ and $\partial_i \Theta^{jk} = -\Gamma_{il}^j \Theta^{lk} - \Gamma_{il}^k \Theta^{jl}$. Setting $g_{\lambda\mu} = \Theta^{\lambda\mu} = \Gamma_{\lambda\mu}^\nu = 0$

whenever at least one of the indices $\lambda, \mu, \nu \in \{1, \ldots, n\}$ is in the a range, we extend the above data to their analogs defined on M, namely, a $(0, 2)$ tensor g, a $(2, 0)$ tensor Θ and a torsion-free connection ∇, in such a way that, obviously, $\nabla g = \nabla \Theta = 0$.

3 The complex-diagonalizable case

By (1.3), for a $(1, 1)$ tensor field Θ and any $a \in \mathbb{R}$,

$$\Theta \quad \text{and} \quad \Theta - a \, \text{Id} \quad \text{have the same Nijenhuis tensor.} \tag{3.1}$$

To justify Remark A, we invoke a result of Bolsinov, Konyaev and Matveev [4, Theorem 3.2]. It states that, if a $(1, 1)$ tensor Θ with $N = 0$ on a manifold M has complex characteristic roots of constant (algebraic) multiplicities, then M and Θ are, locally, decomposed into Cartesian products of factor manifolds/tensors with $N = 0$, where each factor corresponds to (and realizes) a real eigenvalue function of Θ, or a conjugate pair of its (nonreal) complex characteristic-root functions.

Under the assumption made in Remark A, the complex characteristic roots of Θ are all constant. Let the symbols M and Θ now stand for one of of factor manifolds/tensors with $N = 0$, mentioned above.

If the (constant) eigenvalue realized by this Θ is real, our claim follows: Θ equals a constant multiple of Id.

Otherwise, the characteristic roots realized by Θ are $a \pm bi$, with $a, b \in \mathbb{R}$ and $b \neq 0$. Let $J = b^{-1}(\Theta - a \, \text{Id})$. By (3.1), J still has $N = 0$, while the characteristic roots of J (and hence those of its complexification $\hat{J}$) are i and $-i$. As $\hat{J}$ is diagonalizable—due to our assumption—we get $\hat{J}^2 = -\text{Id}$. Thus, $J^2 = -\text{Id}$, and the local constancy of Θ follows from the Newlander-Nirenberg theorem.

The more modest goal of establishing a weaker version of Remark A, with integrability of Θ replacing its local constancy, is easily achieved as follows. Rather than invoking the Newlander-Nirenberg theorem, one shows that an almost-complex structure J or, more generally, a $(1, 1)$ tensor J with $N = 0$ and $J^2 = c \, \text{Id}$, where $c \in \mathbb{R} \smallsetminus \{0\}$, has $\hat{\nabla} J = 0$ for some torsion-free connection $\hat{\nabla}$.

We start from any torsion-free connection ∇. By (1.4), $4c[B_v w - B_w v] = N(v, w)$ for $B_v w$ given by $4cB_v w = 2J[\nabla_v J]w + J[\nabla_w J]v + [\nabla_{Jw} J]v$, that is, $4cB_v w = (J[\nabla_v J]w + J[\nabla_w J]v) + (J[\nabla_w J]v + [\nabla_{Jw} J]v)$. Thus, the vanishing of N for J amounts to symmetry of $B_v w$ in v, w, while $[J, B_v] = \nabla_v J$ since, J^2 being parallel, $\nabla_v J$ anticommutes with J. This is precisely the relation $\hat{\nabla} J = 0$ for the torsion-free connection $\hat{\nabla}$ characterized by $\hat{\nabla}_v = \nabla_v + B_v$.

The assignment $\nabla \mapsto \hat{\nabla} = \nabla + B$ appearing above is a natural projection of the affine space of all torsion-free connections on the manifold in question onto the affine subspace formed by those connections which make J parallel.

The above conclusion is due to Clark and Bruckheimer [7, Theorem 6]. Our argument is a concise version of one used, in a more general situation, by Hernando, Reyes and Gadea [14, Theorems 3.4 and 7.1].

4 Tensors of type $(1, 1)$

For the reader's convenience, we repeat here the definition, due to Nijenhuis [21], of the Nijenhuis tensor (1.3) associated with a $(1, 1)$ tensor Θ on a manifold:

$$N(v, w) = \Theta[\Theta v, w] + \Theta[v, \Theta w] - [\Theta v, \Theta w] - \Theta^2[v, w]. \tag{4.1}$$

Applying Θ^i to both sides, with any integer $i \geq 0$, one obviously obtains

$$\Theta^{i+1}(\Theta[v, w] - [v, \Theta w]) = \Theta^i(\Theta[\Theta v, w] - [\Theta v, \Theta w]) - \Theta^i[N(v, w)]. \tag{4.2}$$

Let $N = 0$. For any vector fields v, w and integers $i, j \geq 0$,

$$\text{if } \Theta^i v = 0, \text{ then } \Theta^i \text{ also annihilates } \Theta^j[v, w] - [v, \Theta^j w]. \tag{4.3}$$

Namely, let $R(i, j)$ be the assertion (4.3), and $R(j)$ the claim that $R(i, j)$ holds for all $i \geq 1$. Now $R(1, 1)$ is immediate from (4.1), while, assuming $R(i, 1)$, and choosing any v with $0 = \Theta^{i+1} v = \Theta^i \Theta v$, we get, from $R(i, 1)$ for Θv (not v), zero on the right-hand side of (4.2), and hence also on the left-hand side, which yields $R(i + 1, 1)$ and, by induction on i, establishes $R(i, 1)$ for all $i \geq 1$, that is $R(1)$. If we now assume $R(j)$, and use any $i \geq 1$, we see that $\Theta^i[v, \Theta^{j+1} w] = \Theta^{i+1}[v, \Theta^j w]$ when $\Theta^i v = 0$ (from $R(i, 1)$ applied to $\Theta^j w$ rather than w), which in turn equals $\Theta^{i+j+1}[v, w]$ (due to $R(i+1, j)$, a consequence of $R(j)$). One thus has $R(j + 1)$, which completes the proof of (4.3).

When, again, $N = 0$ in (4.1) and i, j, k are nonnegative integers,

$$\begin{aligned} &\text{a) } [\mathcal{B}^i, \mathcal{B}^i] \subseteq \mathcal{B}^i, \quad \text{b) } [\mathcal{Z}^i, \mathcal{B}^j] \subseteq \mathcal{Z}^i + \mathcal{B}^j, \quad \text{c) } [\mathcal{Z}^i, \mathcal{Z}^j] \subseteq \mathcal{Z}^{i+j}, \\ &\text{d) } [\mathcal{Z}^i, \mathcal{Z}^k] \subseteq \mathcal{Z}^k \text{ if } \mathcal{Z}^i \text{ is integrable and } k \geq i \end{aligned} \tag{4.4}$$

—notation of (2.1)—with Θ assumed algebraically constant. In fact, (4.4-a), that is, the integrability of each $\mathcal{B}^i$, follows via induction on i, from (4.1) with $N = 0$ and with v, w replaced by $\Theta^i v, \Theta^i w$. (The third Lie bracket in (4.1) then is a section of $\mathcal{B}^{i+1}$, once we assume that $[\mathcal{B}^i, \mathcal{B}^i] \subseteq \mathcal{B}^i$.) For (4.4-b), note that the Lie bracket of sections v of $\mathcal{Z}^i$ and $\Theta^j w$ of $\mathcal{B}^j$ equals, by (4.3), $\Theta^j[v, w]$ plus a section of $\mathcal{Z}^i$. Finally, (4.4-c) and (4.4-d) are further consequences of (4.3): given sections v of $\mathcal{Z}^i$ and w of $\mathcal{Z}^j$, (4.3) reads $\Theta^{i+j}[v, w] = 0$, while (4.3) with $j = k - i$, for sections v of $\mathcal{Z}^i$ and w of $\mathcal{Z}^k$ (which makes $\Theta^j w$ and $[v, \Theta^j w]$ sections of $\mathcal{Z}^i$—the latter due to the assumed integrability of $\mathcal{Z}^i$), yields $\Theta^k[v, w] = 0$.

The conclusion (4.4-a) is due to Bolsinov, Konyaev and Matveev [4, Cor. 2.5.].

5 Proof of Theorem B

We use induction on the dimension, with the following induction step.

Lemma 5.1 *Let Θ be an algebraically constant $(1, 1)$ tensor with $N = 0$ in (1.3) on a manifold M such that the distribution $\mathcal{Z} = \operatorname{Ker} \Theta$ is integrable.*

(a) *Θ-images of $\mathcal{Z}$-projectable local vector fields in M are themselves $\mathcal{Z}$-projectable, so that Θ naturally descends to a $(1, 1)$ tensor $\widehat{\Theta}$ on any local leaf space Σ of $\mathcal{Z}$, and $\widehat{\Theta}$ also has $N = 0$.*

(b) *If local vector fields v, w, and hence also $\Theta v, \Theta w$, are $\mathcal{Z}$-projectable and the projected images of $v, w, \Theta v, \Theta w$ all commute, then $[\Theta v, \Theta w] = 0$.*

(c) *Nilpotency of Θ, or integrability of the distributions $\operatorname{Ker} \Theta^i$ for all $i \geq 1$, implies the same property for $\widehat{\Theta}$.*

Proof Applying (1.3) to v with $\Theta v = 0$ and w projectable along $\mathcal{Z}$, we obtain $\Theta[v, \Theta w] = 0$, as $\Theta[v, w]$, and hence $\Theta^2[v, w]$, vanishes due to projectability of w and (2.3). By (2.3),

this proves the first part of (a), with an obvious definition of $\widehat{\Theta}$. Evaluating (1.3) on projectable vector fields, or applying Θ to them, we get $N = 0$ for $\widehat{\Theta}$ or, respectively, the claim about nilpotency in (c).

Under the assumptions of (b), $[\Theta v, w], [v, \Theta w]$ and $\Theta[v, w]$ are—by (a)—projectable onto 0, which makes them sections of $\mathcal{Z} = \mathrm{Ker}\,\Theta$, so that (1.3) with $N = 0$ yields $[\Theta v, \Theta w] = \Theta([\Theta v, w] + [v, \Theta w] - \Theta[v, w]) = 0$.

If the distributions $\mathcal{Z}^i = \mathrm{Ker}\,\Theta^i$ are all integrable, projectable vector fields that project onto sections of $\mathrm{Ker}\,\widehat{\Theta}^i$ span the distribution $\mathcal{Z}^{i+1}$ (the Θ^i-preimage of the vertical distribution $\mathcal{Z}$). Projectability of each $\mathcal{Z}^{i+1}$, immediate from that of Θ, or from (4.4-d) and Lemma 2.2(c), combined with (2.2), proves (c). □

The assertion $N = 0$ in (a) is also a special case of [4, Prop. 2.4].

Proof of Theorem B As the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) are obvious—the last two from from (1.1), (1.2) and (1.4)—we now just proceed to show that (ii) holds whenever (i) does, using induction on $n \geq 1$. The case $n = 1$ being trivial, we now fix $n > 1$ and assume that (i) implies (ii) in dimensions less than n, while (i) is satisfied on an n-manifold M, with $\Theta \neq 0$. Using $\widehat{\Theta}$ and a local leaf space Σ arising from Lemma 5.1(a), and replacing M by a suitable neighborhood of a given point, we get a bundle projection $\pi : M \to \Sigma$ with the vertical distribution $\mathcal{Z} = \mathrm{Ker}\,\Theta$, while (i), and hence (ii), holds for $\widehat{\Theta}$, on Σ, since $\dim \Sigma < n$. The resulting commuting Jordan-form frame for $\widehat{\Theta}$ is split into $\widehat{\Theta}$-orbits $u_1, \ldots, u_d$ of various *lengths* $d \geq 1$, with the *initial* vector (field) u_1 lying in $\mathrm{Ker}\,\widehat{\Theta}$, the *final* vector u_d outside of $\mathrm{Im}\,\widehat{\Theta}$, and $u_i = \widehat{\Theta}^{d-i} u_d$ for $i = 1, \ldots, d$.

We now associate with every given $\widehat{\Theta}$-orbit $u_1, \ldots, u_d$ the corresponding Θ-orbit $v_0, v_1, \ldots, v_d$ of length $d + 1$ in M. First, we choose each final vector field v_d, on M, so that it projects onto u_d under π, and set $v_i = \Theta^{d-i} v_d$, $i = 0, \ldots, d - 1$. We call v_0 the *pre-initial* vector. Our v_d is only unique up to adding sections of $\mathcal{Z} = \mathrm{Ker}\,\Theta$ (and will be modified later); this is also the obvious reason why *the nonfinal vectors* $v_0, v_1, \ldots, v_{d-1}$ *are uniquely determined*. Due to π-projectability of Θ onto $\widehat{\Theta}$ in Lemma 5.1(a), the resulting Θ-orbit, with the pre-initial vector removed, projects onto the original $\widehat{\Theta}$-orbit, while the pre-initial vectors are sections of $\mathcal{Z} \cap \mathrm{Im}\,\Theta$, projecting onto zero. Also, by Lemma 5.1(b), *the nonfinal vectors from the union of all the* Θ-*orbits commute with one another* (which includes the pre-initial ones). Denoting by k the total number of these commuting vectors, we see that they generate

$$\text{a free local action of } \mathbb{R}^k \text{ in } M. \tag{5.1}$$

The union of all the Θ-orbits forms a linearly independent system at every point: the non-pre-initial ones project onto a frame in Σ, which makes them linearly independent over $\mathcal{Z} = \mathrm{Ker}\,\Theta$ (meaning linear independence of their images in $TM/\mathcal{Z}$), while the pre-initial ones, lying in $\mathcal{Z}$, are linearly independent, being the Θ-images of the initial vectors, linearly independent over $\mathcal{Z}$.

Next, we modify—as announced above—the final vectors e_a chosen in M, and augment the union of all the Θ-orbits with some sections e_λ, so as to obtain a commuting frame in M which, automatically, will be a Jordan-form frame for $\widehat{\Theta}$. (The indices a, λ have some appropriate ranges.) To this end, we identify M, locally, with a Cartesian product of a horizontal factor (our leaf space Σ) and a vertical factor, tangent to $\mathcal{Z}$. Our e_a and e_λ are suitable systems of commuting vector fields on the factor manifolds, trivially extended to vector fields in M (which causes e_a to commute with e_λ). In a first step, for e_a we choose the final vectors of our Jordan-form frame for $\widehat{\Theta}$, and for e_λ some vertical coordinate vector

fields chosen, locally, so as to be linearly independent over $\mathcal{Z} \cap \mathrm{Im}\,\Theta$ and represent, under the quotient-bundle projection, a local trivialization of $\mathcal{Z}/(\mathcal{Z} \cap \mathrm{Im}\,\Theta)$. Let Q now be one leaf of the integrable distribution spanned by all e_a and e_λ. Thus, Q has codimension k in M and is transverse to the orbits of the local free action (5.1). We now modify all e_a and e_λ further, by using the action (5.1) to spread their restrictions to Q from Q to a neighborhood of Q in M. Due to equivariance of π relative to the action (5.1) and the analogous free action in Σ generated by the nonfinal vectors from the union of all the $\widehat{\Theta}$-orbits, and the invariance of the final vectors in Σ under the latter action, the modified e_a still project onto the final vectors (and e_λ onto 0, as the action leaves $\mathcal{Z}$ invariant). Finally, e_a and e_λ commute both with the nonfinal vectors from the union of the Θ-orbits, and with one another: the former follows from their $\mathbb{R}^k$-invariance, the latter since their restrictions to Q commute. This completes the proof. □

6 Algebraic constancy and connections

Given a real vector bundle E of rank k over a manifold M and integers $p, q \geq 0$, we say that a smooth section Θ of $E^{\otimes p} \otimes [E^*]^{\otimes q}$ is *algebraically constant* when, for any $x, y \in M$, some linear isomorphism $E_x \to E_y$ sends Θ_x to Θ_y. In this case, fixing $z \in M$ and an ordered basis $\mathbf{e} = (e_1, \dots, e_k)$ of E_z, let us

$$\text{denote by } \mathbf{e}\Theta_z \text{ the system of components of } \Theta_z \text{ in the basis } \mathbf{e}, \qquad (6.1)$$

that is, the (p, q) tensor in $\mathbb{R}^k$ arising as the image of Θ_z under the linear isomorphism $E_z \to \mathbb{R}^k$ associated with $\mathbf{e}$. We now define two objects, the first being the matrix group $G \subseteq \mathrm{GL}(k, \mathbb{R})$ formed by all transition matrices between $\mathbf{e}$ and all ordered bases $\bar{\mathbf{e}}$ of E_z such that $\bar{\mathbf{e}}\Theta_z = \mathbf{e}\Theta_z$. In other words, G is the isotropy group of $\mathbf{e}\Theta_z$ for the obvious action of $\mathrm{GL}(k, \mathbb{R})$ on (p, q) tensors in $\mathbb{R}^k$.

The second one, a G-principal bundle P over M, is contained in the $\mathrm{GL}(k, \mathbb{R})$-principal bundle Q over M naturally associated with E, and the fibre of P over any $x \in M$ consists of the ordered bases $\tilde{\mathbf{e}}$ of E_x having $\tilde{\mathbf{e}}\Theta_x = \mathbf{e}\Theta_z$.

Smoothness of P follows since P is the preimage of the point $\mathbf{e}\Theta_z$ under the submersion $\Phi : Q \to \Sigma$ sending any ordered basis $\hat{\mathbf{e}}$ of E_x, at any $x \in M$, to $\hat{\mathbf{e}}\Theta_x$, with Σ denoting the $\mathrm{GL}(k, \mathbb{R})$-orbit of $\mathbf{e}\Theta_z$ viewed, again, as a (p, q) tensor in $\mathbb{R}^k$. The submersion property of Φ is obvious: even the restriction of Φ to any fibre Q_x of Q is a submersion, diffeomorphically equivalent to the projection $\mathrm{GL}(k, \mathbb{R}) \to \mathrm{GL}(k, \mathbb{R})/G$.

Thus, a smooth section Θ of $E^{\otimes p} \otimes [E^*]^{\otimes q}$ is parallel relative to some linear connection ∇ in E if and only if it is algebraically constant [19, Theorems 1–2], the 'only if' (or, 'if') claim being obvious since M is assumed connected or, respectively, since ∇ induced by any principal G-connection in P clearly makes Θ parallel. Such connections are precisely the linear connections in E characterized by vanishing of their *inner torsion* in the sense of [23, Sect. 5].

Remark 6.1 Our construction depends on the choice of $z \in M$ and an ordered basis $\mathbf{e}$ of E_z. However, different choices lead to equivariantly equivalent objects. The case of importance to us is $E = TM$, where P is the G-structure associated with the given algebraically constant (p, q) tensor Θ. When $(p, q) = (1, 1)$ and Θ is nilpotent, *we will always use z and $\mathbf{e}$ realizing the Jordan normal form $d_1 \dots d_m$ of Θ*, defined as in (1.5).

7 The Lie brackets of a local Jordan frame

Recall our convention (1.5) about representing the Jordan normal forms of nilpotent $(1, 1)$ tensors in dimension n as weakly decreasing strings $d_1 \ldots d_m$ of positive integers, so that $d_1 + \ldots + d_m = n$, and $\Theta = 0$ has the Jordan normal form $1 \ldots 1$, each 1 being the 1×1 block matrix $[0]$, while n is the single-block Jordan normal form of a *generic* nilpotent $(1, 1)$ tensor in dimension n. The Jordan normal form $2 \ldots 2$ characterizes the case $\operatorname{Ker} \Theta = \operatorname{Im} \Theta$.

If an algebraically constant nilpotent $(1, 1)$ tensor Θ on an n-manifold has the Jordan normal form $d_1 \ldots d_m$, then each subbundle $\operatorname{Ker} \Theta^i$ clearly has the fibre dimension $\min(i, d_1) + \ldots + \min(i, d_m)$, and hence

$$\operatorname{rank} \Theta^i = n - \min(i, d_1) - \ldots - \min(i, d_m). \tag{7.1}$$

Let us fix an algebraically constant nilpotent $(1, 1)$ tensor Θ on an n-manifold M and a local frame field realizing the Jordan normal form $d_1 \ldots d_m$ of Θ. (See Remark 6.1.) We focus on three (not necessarily distinct) Θ-orbits

$$(e_1, \ldots, e_p), \ (\tilde{e}_1, \ldots, \tilde{e}_q), \ (\hat{e}_1, \ldots, \hat{e}_r), \tag{7.2}$$

by which we mean portions of our frame field corresponding to three of the entries $d_1, \ldots, d_m$. Setting $e_i = \tilde{e}_j = \hat{e}_k = 0$ for nonpositive integers i, j, k, we obtain $(\Theta e_i, \Theta \tilde{e}_j, \Theta \hat{e}_k) = (e_{i-1}, \tilde{e}_{j-1}, \hat{e}_{k-1})$ for all integers i, j, k not exceeding, respectively, p, q or r. Finally, we denote by $C_{i,j}^k$ the coefficient of $\hat{e}_k$ in the expansion of the Lie bracket $[e_i, \tilde{e}_j]$ as a (functional) combination of our frame, and also set $C_{i,j}^k = 0$ if $k > r$ or one of i, j, k is nonpositive; thus $C_{i,j}^k$ is well defined for integers i, j, k with $i \leq p$ and $j \leq q$. Now $N = 0$ in (1.3) if and only if

$$C_{i,j}^k + C_{i-1,j-1}^{k-2} = C_{i-1,j}^{k-1} + C_{i,j-1}^{k-1} \text{ whenever } k \geq 3, \ i \leq p \text{ and } j \leq q, \tag{7.3}$$

with $C_{i,j}^k = 0$ for $k > r$. Namely, $N(e_i, \tilde{e}_j)$ evaluated from (1.3), and then projected onto the span of $(\hat{e}_1, \ldots, \hat{e}_r)$, obviously equals

$$(C_{i-1,j}^k + C_{i,j-1}^k)\hat{e}_{k-1} - C_{i-1,j-1}^k \hat{e}_k - C_{i,j}^k \hat{e}_{k-2} \text{ summed over all } k \leq r. \tag{7.4}$$

The vanishing of the terms involving $\hat{e}_r$ (or $\hat{e}_{r-1}$, if $r \geq 2$) means that $C_{i-1,j-1}^r = 0$ (or, respectively, $C_{i-1,j-1}^{r-1} = C_{i-1,j}^r + C_{i,j-1}^r$), both of which are special cases of (7.3), with $k \in \{r + 1, r + 2\}$. Leaving these terms aside, we see that the equality in (7.3) multiplied by $\hat{e}_{k-2}$ follows if we shift the summation index from k or $k - 1$ to $k - 2$ in the first two terms of (7.4), which yields (7.3) as $\hat{e}_{k-2} = 0$ unless $k \geq 3$. In terms of $E_{i,j}^s = C_{i,j}^{i+j-s+1}$, or $C_{i,j}^k = E_{i,j}^{i+j-k+1}$, (7.3) can be rewritten as

$$E_{i,j}^s + E_{i-1,j-1}^s = E_{i-1,j}^s + E_{i,j-1}^s \text{ if } i + j \geq s + 2, \ i \leq p \text{ and } j \leq q, \tag{7.5}$$

while $E_{i,j}^s = 0$ whenever $i + j \geq s + r$. The reason why we prefer to switch from the integer variables i, j, k to i, j, s with $s = i + j - k + 1$, or $k = i + j - s + 1$, is that (7.5) uses a fixed value of s, allowing us to treat different values of s as completely unrelated. Our conclusions may be summarized as follows.

Lemma 7.1 *Given* Θ *and the frame field as above, the Nijenhuis tensor* (1.3) *vanishes if and only if, for any ordered triple of not necessarily distinct* Θ*-orbits* (7.2)*, one has* (7.5) *along with*

$$E_{i,j}^s = 0 \text{ if } i+j < s, \text{ or } i+j \geq s+r, \text{ or } i \leq 0, \text{ or } j \leq 0, \text{ and}$$
$$\text{our } E_{i,j}^s \text{ are defined for all } i, j, s \in \mathbf{Z} \text{ such that } i \leq p \text{ and } j \leq q. \tag{7.6}$$

Proof We already saw that (7.5) is equivalent to (7.3), while (7.6) is clearly nothing else than the obvious boundary conditions ($C_{i,j}^k = 0$ if $k \leq 0$, or $k > r$, or $i \leq 0$, or $j \leq 0$) coupled with our convention about when $C_{i,j}^k$ makes sense. $\qquad\square$

Next, $\mathcal{Z}^l = \mathrm{Ker}\,\Theta^l$ is integrable (which may or may not be the case) if and only if $C_{i,j}^k = 0$ whenever $i, j \leq l < k$, that is,

$$E_{i,j}^s = 0 \text{ for all } i, j, s \text{ with } i, j \leq l \text{ and } i+j \geq s+l, \tag{7.7}$$

and for all ordered triples of (not necessarily distinct) Θ-orbits (7.2). With the last clause repeated, the integrability of $\mathcal{Z}^l = \mathrm{Ker}\,\Theta^l$ for *all* $l \geq 0$ clearly amounts to

$$E_{i,j}^s = 0 \text{ whenever } i, j \geq s, \tag{7.8}$$

since the condition $i, j \leq i+j-s$ is nothing else than $i, j \geq s$.

Remark 7.2 The equality in (7.5) obviously holds, for all $i, j, s \in \mathbf{Z}$, if $E_{i,j}^s$ is a function of i alone, or of j alone, or equals $i+j$ plus a function of s.
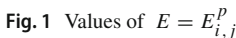
8 Proof of Theorem C: the necessity of (1.6)

For the Jordan normal form of an algebraically constant nilpotent $(1, 1)$ tensor, *not* being of type (1.6) clearly means that it

$$\text{contains three different Jordan blocks of lengths } p, q, r \text{ with } p \leq q < r. \tag{8.1}$$

Proposition 8.1 *In any dimension* $n \geq 1$*, the condition* (8.1)*, imposed on the Jordan normal form of an algebraically constant nilpotent* $(1, 1)$ *tensor* Θ*, implies that the algebraic type of* Θ *is not controlled by the Nijenhuis tensor* (1.3)*. More precisely,* Θ *can be realized as a left-invariant* $(1, 1)$ *tensor on a Lie group, in such a way that* $N = 0$*, but* $\mathrm{Ker}\,\Theta^p$ *is nonintegrable for some integer* $p \geq 1$*. One may choose* p *to be the shortest block length in the Jordan normal form of* Θ*.*

Proof We identify a local frame field for Θ, chosen as in Sect. 7, with a basis of a Lie algebra $\mathfrak{g}$ formed by left-invariant vector fields on a Lie group G. This is achieved by requiring (7.3) and the boundary conditions ($C_{i,j}^k = 0$ if $k \leq 0$, or $k > r$, or $i \leq 0$, or $j \leq 0$) to be satisfied by *constants* $C_{i,j}^k$ or, equivalently, finding constants $E_{i,j}^s$ with (7.5)–(7.6). (Our choice will cause all brackets to lie in the center, thus implying the Jacobi identity.) Our Θ then becomes a left-invariant $(1, 1)$ tensor field on G acting as an endomorphism of the tangent bundle which sends each frame vector field either to the preceding one, or to zero. As a consequence of (8.1), our local frame contains

$$\text{three different } \Theta\text{-orbits (7.2) of lengths } p, q, r \text{ with } p \leq q < r. \tag{8.2}$$

Fig. 1 Values of $E = E_{i,j}^p$

Fixing such Θ-orbits, we now set, in the discussion of Sect. 7, $E_{i,j}^s = 0$ for all integers i, j, s, with the exception of (i, j, p) from the set $[1, p] \times [1, q] \times \{p\}$ contained in the range $[1, p] \times [1, q] \times [1, r]$ corresponding to our three Θ-orbits (8.2).

Given integers $i \leq p$ and $j \leq q$, we define $E_{i,j}^p$ (as illustrated in Fig. 1) by

$$E_{i,j}^p = \max(0, \, i + \min(0, \, j - p)). \tag{8.3}$$

Speaking below of rectangles, triangles, lines and line segments, we always mean their intersections with $\mathbf{Z}^2$, while (sub)rectangles are occasionally reduced to segments or single points. Restricted to (i, j) ranging over the rectangle $[1, p] \times [1, q]$, our $E_{i,j}^p$ equals 0 on the triangle with vertices $(1, 1), (1, p-1), (p-1, 1)$ (treated as the empty set when $p = 1$, or the single point $(1, 1)$ for $p = 2$), and $E_{i,j}^p = p$ on the segment $\{p\} \times [p, q]$ (a point when $p = q$); the latter claim is obvious, the former immediate from the equality

$$i + \min(0, \, j - p) = \min(i, \, i + j - p). \tag{8.4}$$

If $p > 1$, then, for any $l \in \{1, \ldots, p - 1\}$, we have $E_{i,j}^p = l$ on the two-segment broken line joining the points $(l, q), (l, p), (p, l)$ (reduced to a segment when $p = q > 1$); cf. (8.4). (This is particularly simple for $p = q = 1$, with $E_{1,1}^1 = 1$.)

The corresponding Nijenhuis tensor (1.3) vanishes identically, by Lemma 7.1, since—as we now proceed to show—our $E_{i,j}^p$ satisfy (7.5) and (7.6). First, (7.6) holds, as nonpositivity of i, j or $i + j - s = i + j - p$ in (8.3) yields $E_{i,j}^p = 0$, by (8.4), and the remaining implication is vacuous: $i + j \leq p + q = s + q < s + r$.

Next, (7.5) "essentially" follows from Remark 7.2: $E_{i,j}^{p} = \max(0, i)$, which is a function of i, on the subrectangle $[1, p] \times [p, q]$ (a segment when $p = q$). On $[1, p] \times [1, p]$, (8.3) in turn gives $E_{i,j}^{p} = \max(0, i + j - p)$, which coincides with $i + j - p$ on the subtriangle given by $i + j \geq s + 2 = p + 2$.

To dispel any doubts, we now establish (7.5) rigorously, for $s = p$. Of interest to us are integers i, j with $i + j \geq p + 2$, $i \leq p$ and $j \leq q$. We are also free to assume that $i, j \geq 1$, since otherwise, by (7.6), all four terms in (7.5) equal 0. If $j > p$, (8.3) gives $E_{i,j}^{p} = E_{i,j-1}^{p} = i$ for all $i \geq 0$. Now the four terms in (7.5) are $i, i - 1, i - 1, i$ (whenever $i \geq 1$), and the required equality follows. When $j \leq p$ (and hence $j - 1 < p$), given $i \in \mathbb{Z}$, (8.3) reads $E_{i,j}^{p} = \max(0, i + j - p)$ and, similarly, $E_{i,j-1}^{p} = \max(0, i + j - p - 1)$ with j replaced by $j - 1$. In the case of interest to us, $i + j \geq p + 2$ (see the beginning of this paragraph), so that $E_{i,j}^{p} = i + j - p$ and $E_{i,j-1}^{p} = i + j - p - 1$, for all i, and we get the equality in (7.5): $(i + j - p) + (i - 1 + j - 1 - p) = (i - 1 + j - p) + (i + j - 1 - p)$.

Finally, since $E_{p,p}^{p} = p \neq 0$, (7.7) applied to $i = j = s = l = p$ shows that $\operatorname{Ker} \Theta^{p}$ is not integrable. $\qquad\square$

9 Proof of Theorem C: the sufficiency of (1.6)

We now show that, given an algebraically constant nilpotent $(1, 1)$ tensor Θ on a manifold M of dimension $n \geq 1$, with $N = 0$, and with the Jordan normal form $d_1 \ldots d_m$ satisfying condition (1.6), Θ must also have the property (i) in Theorem B, and hence be locally constant. To this end, we choose a local frame field realizing the Jordan normal form of Θ. See Remark 6.1.

In the first case of (1.6), $d_1 = \ldots = d_m = d$ for some $d \geq 1$, and our local frame field splits into disjoint Θ-orbits of the form $v_1, \ldots, v_d$, all of length d, while $v_i = \Theta^{d-i} v_d$ for $i = 1, \ldots, d - 1$, and the final vector v_d lies outside of $\operatorname{Im} \Theta$. Thus, $\mathcal{Z}^i$, $i \geq 0$, is obviously equal to either TM (when $i \geq d$), or to $\mathcal{B}^{d-i}$ (if $1 \leq i < d$), and (4.4-a) yields our claim.

Consider now the second case of (1.6): $d_1 = \ldots = d_{m-1} = d > d' = d_m$ for some $d, d' \geq 1$, with $m > 1$, leading to Θ-orbits $v_1, \ldots, v_d$ of length d, of which there are $m - 1$, and to one Θ-orbit of length d'.

We first prove the integrability of $\mathcal{Z}^i$ when $1 \leq i \leq d'$, using induction on i. As $\mathcal{Z}^1$ is spanned by the m initial vectors from all Θ-orbits, taken one from each, and $\mathcal{B}^{d-1}$ by the $m - 1$ initial vectors from all Θ-orbits of length d, the latter subbundle of TM is contained in the former with codimension one. Thus, (4.4-b) and Lemma 2.1 yield the integrability of $\mathcal{Z}^1$. For the induction step, if $1 \leq i < d'$ and $\mathcal{Z}^i$ is integrable, $\mathcal{Z}^{i+1}$ is spanned by $m(i + 1)$ vectors: $v_1, \ldots, v_{i+1}$ from all the Θ-orbits combined (if one writes the Θ-orbits as $v_1, \ldots, v_d$ or $v_1, \ldots, v_{d'}$), and so $\mathcal{Z}^{i+1}$ contains, with codimension one, the span $\mathcal{Z}^i + \mathcal{B}^{d-i-1}$ of $\mathcal{Z}^i$ and $\mathcal{B}^{d-i-1}$. By (4.4-b) and (4.4-d) $[\mathcal{Z}^i + \mathcal{B}^{d-i-1}, \mathcal{Z}^{i+1}] \subseteq \mathcal{Z}^{i+1} + \mathcal{B}^{d-i-1} \subseteq \mathcal{Z}^{i+1}$, and Lemma 2.1 completes the induction step.

Finally, let $d' < i < d$ and $k = d' - 1$. This time $\mathcal{Z}^i$ contains $\mathcal{Z}^k + \mathcal{B}^{d-i}$ with co-dimension one: the former is spanned by $(m - 1)i + d'$ vectors (the initial i ones from all Θ-orbits of length d, plus the whole Θ-orbit of length d'), the latter—by the same vectors except the last one in the length d' orbit. Once again, (4.4-b) and (4.4-d) give $[\mathcal{Z}^k + \mathcal{B}^{d-i}, \mathcal{Z}^i] \subseteq \mathcal{Z}^i + \mathcal{B}^{d-i} \subseteq \mathcal{Z}^i$, and we can use Lemma 2.1.

10 Generalized almost-tangent structures

The following construction provides—as shown below—a local description of all algebraically constant $(1, 1)$ tensors Θ such that $\Theta^2 = 0$ and the Nijenhuis tensor (1.3) vanishes identically.

Given a distribution $\mathcal{D}$ on a manifold Σ, let M be the total space of an affine bundle over Σ associated with the quotient vector bundle $T\Sigma/\mathcal{D}$. Using the bundle projection $\pi : M \to \Sigma$ and the quotient-bundle projection morphism $T\Sigma \ni v \mapsto [v] \in T\Sigma/\mathcal{D}$, we define a $(1, 1)$ tensor Θ on M by

$$\Theta_x v = [d\pi_x v] \in T_y\Sigma/\mathcal{D}_y = T_x M_y, \text{ if } x \in M_y = \pi^{-1}(y), \tag{10.1}$$

whenever $x \in M$ and $v \in T_x M$. Then $\Theta^2 = 0$, since all Θ-images are vertical. Also, $N = 0$ in (1.3). In fact, Im Θ is the vertical distribution $\mathcal{V} = \text{Ker } d\pi$. Evaluating (1.3), without loss of generality, on π-projectable vector fields, we see that, by (2.3), the first, second and fourth terms on the right-hand side of vanish as $\Theta^2 = 0$. So does the third term: Θv, Θw restricted to each fibre are affine-space translations, and consequently commute.

Theorem 10.1 *Every algebraically constant $(1, 1)$ tensor Θ with $\Theta^2 = 0$ and vanishing Nijenhuis tensor (1.3) arises, locally, from the above construction, and the fibre dimension of $\mathcal{D}$ equals the codimension of* Im Θ *in* Ker Θ*, while*

$$\Theta \text{ is integrable if and only if so is the distribution } \mathcal{D}. \tag{10.2}$$

Proof Suppose that $\Theta^2 = 0$ and $N = 0$ in (1.3). By (4.4-a), Im Θ is an integrable distribution, while Im $\Theta \subseteq$ Ker Θ. Due to (2.3) and (1.3) with $\Theta^2 = 0$,

$$\text{any two } (\text{Im } \Theta)\text{-projectable vector fields have commuting } \Theta\text{-images.} \tag{10.3}$$

By (4.4-b) for $i = j = 1$ and Lemma 2.2(c), on an open set $M' \subseteq M$ with a bundle projection $\pi : M' \to \Sigma$ having Im Θ as the vertical distribution, Ker Θ is π-projectable onto a distribution $\mathcal{D}$ on Σ, with (10.2) obvious from Theorem B and (2.2). Any π-projectable lift, along the fibre $\pi^{-1}(y)$, of any vector w tangent to Σ at $y \in \Sigma$, is mapped by Θ onto the "vertical lift" of w, a vector field tangent to $\pi^{-1}(y)$, which vanishes precisely when w is tangent to $\mathcal{D}$. By (10.3) the vertical lifts of any $w, w' \in T_y\Sigma$ commute. This turns $\pi^{-1}(y)$, locally, into an affine space having the translation vector space $T_y\Sigma/\mathcal{D}_y$, with Θ given by (10.1). □

Theorem 10.1 illustrates a special case of Theorem C: the condition $\Theta^2 = 0$ corresponds to the Jordan normal forms $2 \ldots 2$ and $2 \ldots 21 \ldots 1$ (plus $1 \ldots 1$, for $\Theta = 0$). Of these, only $2 \ldots 2$, $2 \ldots 21$ and $1 \ldots 1$ satisfy (1.6), reflecting the fact that $\mathcal{D}$ is necessarily integrable only if it has the fibre dimension 0, 1 or dim Σ.

When Ker $\Theta = $ Im Θ, that is, $\mathcal{D}$ is the zero distribution, our construction gives rise to what is referred to as *almost-tangent structures* [11, 31], and Theorem 10.1 becomes the local version of [9, Theorem on p. 69].

11 Differential q-forms on an n-manifold, $q = 0, 1, 2, n - 1, n$

We now prove Proposition D. Let ζ be an algebraically constant differential q-form on an n-dimensional manifold, $q = 0, 1, 2, n - 1, n$, with $d\zeta = 0$ (the last condition being obviously redundant if $q = n$ or—as ζ is constant—if $q = 0$).

The cases $q = 0$ and $q = 1$ are obvious: the 1-form ζ (if nonzero), being locally exact, equals dx^1 in suitable local coordinates $x^1, \ldots, x^n$.

When $q = 2$, algebraic constancy amounts to constant rank, and our claim follows as Darboux's theorem [6, p. 40] gives $\zeta = dx^1 \wedge dx^2 + \ldots + dx^{2r-1} \wedge dx^{2r}$ in some local coordinates $x^1, \ldots, x^n$, with $2r = \operatorname{rank} \zeta \geq 0$.

If $q = n$ and $\zeta \neq 0$, we have, in suitable local coordinates $x^1, \ldots, x^n$,

$$\zeta = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \text{ where } x^2, \ldots, x^n \text{ can be arbitrary,} \tag{11.1}$$

as long as $dx^2 \wedge \ldots \wedge dx^n \neq 0$. In fact, starting from $\zeta = \phi \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$ for a function ϕ without zeros, and choosing ψ with $\partial_1 \psi = \phi$, we see that $d\psi$ equals $\phi \, dx^1$ plus a functional combination of $dx^2, \ldots, dx^n$ and so $\zeta = d\psi \wedge dx^2 \wedge \ldots \wedge dx^n$.

Finally, let $q = n - 1$. Assuming ζ to be nonzero, and fixing a nonzero n-form ω, we get $\zeta = \omega(v, \cdot, \ldots, \cdot)$, for a unique (nonzero) vector field v. Then, by (11.1), $\omega = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$ in some local coordinates $x^1, \ldots, x^n$, with $x^2, \ldots, x^n$ chosen so that $dx^2(v) = \ldots = dx^n(v) = 0$. Now $\zeta = \chi \, dx^2 \wedge \ldots \wedge dx^n$ for $\chi = dx^1(v)$, and $\partial_1 \chi = 0$ as $d\zeta = 0$. Our ζ, being thus a top-degree form in $n - 1$ variables, equals, by (11.1), $dy^2 \wedge \ldots \wedge dy^n$ in suitable coordinates $y^1, \ldots, y^n$.

12 Differential forms of other degrees

We now proceed to verify the statement preceding formula (1.7). The algebraic constancy of ζ is clear as $\zeta = (\xi^1 \wedge \xi^2 + \xi^3 \wedge \xi^4) \wedge \xi^5 \wedge \ldots \wedge \xi^{q+2}$, with linearly independent 1-forms $\xi^1, \ldots, \xi^{q+2}$, and its closedness since $d\zeta$ is the exterior product of $(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge (dx^1 \wedge dx^2 - dx^3 \wedge dx^4)$ (obviously equal to 0) and $dx^6 \wedge \ldots \wedge dx^{q+2}$. Being algebraically constant, ζ gives rise to the vector subbundle $\mathcal{F}$ of T^*M such that the sections of $\mathcal{F}$ are those 1-forms ξ for which $\xi \wedge \zeta = 0$. The sections ξ of $\mathcal{F}$ also coincide with functional combinations of the 1-forms

$$\eta, \, dx^6, \ldots, dx^{q+2}, \text{ where } \eta = dx^5 + x^1 dx^2 - x^3 dx^4. \tag{12.1}$$

In fact, writing $\xi = \xi_i \, dx^i$, we see that $\xi \wedge \zeta$ contains no contributions from the terms $\xi_i \, dx^i$ (no summation) with $6 \leq i \leq q + 2$ (making $\xi_6, \ldots, \xi_{q+2}$ completely arbitrary) while for $\theta = (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge (dx^5 + x^1 dx^2 - x^3 dx^4)$ one has

$$\theta = dx^1 \wedge dx^2 \wedge dx^5 + dx^3 \wedge dx^4 \wedge dx^5 - x^3 dx^1 \wedge dx^2 \wedge dx^4 + x^1 dx^2 \wedge dx^3 \wedge dx^4.$$

and so each term $\xi_i \, dx^i$ (no summation, again) with $q + 2 < i \leq n$ contributes to $\xi \wedge \zeta$ the expression $\xi_i \, dx^i \wedge \theta \wedge dx^6 \wedge \ldots \wedge dx^{q+2}$ (no summation) comprising all the terms in $\xi \wedge \zeta$ involving the factor dx^i. Linear independence of the differentials $dx^1, \ldots, dx^n$ now gives $\xi_i = 0$ whenever $q + 2 < i \leq n$. Finally, the exterior products of $\xi_1 dx^1, \xi_2 dx^2, \xi_3 dx^3, \xi_4 dx^4, \xi_5 dx^5$ with θ are

$$\begin{aligned}
&\xi_1 (dx^1 \wedge dx^3 \wedge dx^4 \wedge dx^5 + x^1 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4), \\
&\xi_2 (dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5), \\
&\xi_3 (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5 - x^3 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4), \\
&\xi_4 (dx^1 \wedge dx^2 \wedge dx^4 \wedge dx^5), \\
&\xi_5 (x^3 dx^1 \wedge dx^2 \wedge dx^4 \wedge dx^5 - x^1 dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5).
\end{aligned} \tag{12.2}$$

The condition $\xi \wedge \zeta = 0$ means, after the cancellation of $dx^6 \wedge \ldots \wedge dx^{q+2}$, that the sum of the five lines of (12.2) equals 0. Writing $[ijkl]$ for $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$, we see that $[1345]$

and [1235] occur just once each, giving $\xi_1 = \xi_3 = 0$, while the sum of the remaining three lines equals $(\xi_4 + \xi_5 x^3)[1245] + (\xi_2 - \xi_5 x^1)[2345]$. Thus, $\xi_4 + \xi_5 x^3 = 0 = \xi_2 - \xi_5 x^1$, and the sum of $\xi_i \, dx^i$ over $i = 1, \ldots, 5$ equals a function times the 1-form η in (12.1), proving our claim about (12.1).

If ζ were integrable, so would be—according to (1.2)—the simultanous kernel of the 1-forms (12.1) (that is, of all sections of $\mathcal{F}$), naturally determined by ζ. This is not the case, as $d\eta \wedge \eta \wedge dx^6 \wedge \ldots \wedge dx^{q+2}$ is nonzero, being equal to $(dx^1 \wedge dx^2 \wedge dx^5 - dx^3 \wedge dx^4 \wedge dx^5 - x^3 dx^1 \wedge dx^2 \wedge dx^4 - x^1 dx^2 \wedge dx^3 \wedge dx^4) \wedge dx^6 \wedge \ldots \wedge dx^{q+2}$.

13 Symmetric $(0, 2)$ and $(2, 0)$ tensors

Necessity and sufficiency of (1.8) Let g be integrable, with $\nabla g = 0$ for a fixed torsion-free connection ∇. The integrability of the distribution $\mathcal{V} = \mathrm{Ker}\, g$, due to (1.2), allows us to choose local coordinates and index ranges for i, a, λ, μ, ν as in Remark 2.5, so that $\mathcal{V}$ is spanned by the coordinate vector fields ∂_a. As $\mathcal{V}$ is obviously ∇-parallel, $\Gamma_{ia}^k = \Gamma_{ab}^k = 0$, while $g_{ia} = g_{ab} = 0$, so that $\partial_a g_{ij} = \Gamma_{ai}^k g_{kj} + \Gamma_{aj}^k g_{ik} = 0$, and projectability of g along $\mathcal{V}$ follows from (2.4).

Conversely, suppose that g is projectable along the integrable distribution $\mathcal{V} = \mathrm{Ker}\, g$. As before, we invoke Remark 2.5, selecting local coordinates with index ranges for i, a, λ, μ, ν so as to make $\mathcal{V}$ the span of the coordinate fields ∂_a. As projectability of g along $\mathcal{V}$ gives $\partial_a g_{ij} = 0$, while $g_{ia} = g_{ab} = 0$, the components g_{ij} represent a pseudo-Riemannian metric in the factor manifold with the coordinates x^i. Denoting by Γ_{ij}^k the components of its Levi–Civita connection, we now use Remark 2.5 to define the required torsion-free connection ∇ with $\nabla g = 0$. □

Proof of Proposition E Integrability of the former implies that of the latter by (1.2). Conversely, let the distribution $\mathcal{B} = \mathrm{Im}\, \Theta$ be integrable. Using Remark 2.5, we fix local coordinates and index ranges for i, a, λ, μ, ν so that $\mathcal{B}$ is the span of the coordinate fields ∂_i. Thus, $\Theta^{ia} = \Theta^{ab} = 0$, as the 1-forms dx^a annihilate each ∂_i, and hence are sections of the subbundle $\mathrm{Ker}\, \Theta \subseteq T^*M$. On each leaf of $\mathcal{B}$, the restriction of Θ is nondegenerate—see (2.8)—and so it is the reciprocal of a pseudo-Riemannian metric on the leaf. Its Levi–Civita connection, with the components Γ_{ij}^k (possibly depending on the variables x^a), makes the restriction of Θ parallel. Thus, we may again invoke Remark 2.5 to obtain a torsion-free connection ∇ such that $\nabla \Theta = 0$. □

14 Local constancy of bivector fields

Proof of Proposition F The 'only if' part is immediate: for a torsion-free connection ∇ on the given manifold having $\nabla \Theta = 0$, the distribution $\mathcal{B} = \mathrm{Im}\, \Theta$ is ∇-parallel and hence integrable, cf. (1.2), and the torsion-free connections induced by ∇ on the leaves of $\mathcal{B}$ make the restriction of Θ to each leaf parallel, which implies the same (and hence also closedness) for their inverses.

Let us now assume that $\mathcal{B} = \mathrm{Im}\, \Theta$ is integrable and the inverses of the restrictions of Θ to the leaves of $\mathcal{B}$ are all closed. These inverses are symplectic forms ζ on the leaves, and the Darboux theorem with parameters [1, Lemma 3.10] allows us to choose functions x^i which, restricted to each leaf, form local coordinates with $\zeta = dx^1 \wedge dx^2 + \ldots + dx^{2r-1} \wedge dx^{2r}$, where $2r = \mathrm{rank}\, \zeta = \mathrm{rank}\, \Theta$. We may also choose functions x^a, with the index ranges

$1 \leq i \leq 2r < a \leq \dim M$, such that the differentials dx^a form a local trivialization of the subbundle $\operatorname{Ker} \Theta \subseteq T^*M$. In the resulting product coordinates x^i, x^a the components of Θ are all constant: $\Theta^{ia} = \Theta^{ab} = 0$, while $\Theta^{ij} = 1$ (or, $\Theta^{ij} = -1$) if (j, i), or (i, j), is one of the pairs $(1, 2), (3, 4), \ldots, (2r - 1, 2r)$, and $\Theta^{ij} = 0$ otherwise. $\qquad\square$

15 Integrability of the kernels and images

For any vector bundle $\mathcal{L}$ over a manifold M and a vector-bundle morphism $\Theta : TM \to \mathcal{L}^*$ of constant rank r into its dual $\mathcal{L}^*$, the resulting dual morphism $\Theta^* : \mathcal{L} \to T^*M$, which also has rank $\Theta^* = r$, gives rise to a tensor-like object $\widetilde{N}$ (specifically, a section of $\operatorname{Hom}(\mathcal{L} \otimes \mathcal{L}^{\wedge r}, [T^*M]^{\wedge(r+2)}))$, sending sections $v, v_1, \ldots, v_r$ of $\mathcal{L}$ to the $(r + 2)$-form

$$\widetilde{N}(v, v_1, \ldots, v_r) = [d(\Theta^*v)] \wedge \Theta^*v_1 \wedge \ldots \wedge \Theta^*v_r. \tag{15.1}$$

Here $d[\Theta^*(fv)] = fd(\Theta^*v) + df \wedge \Theta^*v$ for a function f. However, $\widetilde{N}$ itself is tensorial: the nontensorial term $df \wedge \Theta^*v$ in the last equality has zero exterior product with $\Theta^*v_1 \wedge \ldots \wedge \Theta^*v_r$, since $\operatorname{rank} \Theta^* = r$. Furthermore,

$$\widetilde{N} = 0 \text{ identically if and only if } \operatorname{Ker} \Theta \text{ is integrable.} \tag{15.2}$$

In fact, as $\operatorname{Ker} \Theta$ is the simultanous kernel of the 1-forms Θ^*v, for all sections v of $\mathcal{L}$, its integrability amounts to d-closedness of the ideal generated by all such 1-forms which, as $\operatorname{rank} \Theta^* = r$, is nothing else than the vanishing of $\widetilde{N}$.

In the case of $\mathcal{L} = TM$ and a (possibly nonsymmetric) $(0, 2)$ tensor g of constant rank r on M, treated as a morphism $\Theta : TM \to T^*M$ sending a vector field w to the 1-form $g(\cdot, w)$, the dual Θ^* acts via $v \mapsto g(v, \cdot)$. Then

$$\widetilde{N} \text{ in } (15.1) \text{ becomes } N' \text{ in } (1.9\text{-a}), \text{ so that } N' \text{ is tensorial.} \tag{15.3}$$

Let Θ now be an algebraically constant nilpotent $(1, 1)$ tensor on an n-manifold with the Jordan normal form $d_1 \ldots d_m$, cf. (1.5). Integrability of Θ, as well as its local constancy, is equivalent, by Theorem B, to the simultaneous vanishing of the Nijenhuis tensor N in (1.3) along with further $d_1 - 1$ Nijenhuis-type tensors N^i, where $1 \leq i < d_1$, such that $N^i = 0$ if and only if $\mathcal{Z}^i = \operatorname{Ker} \Theta^i$ is integrable. Specifically, this follows from (15.2) if we define N^i to be $\widetilde{N}$ in (15.1) with $\mathcal{L} = TM$ and Θ replaced by Θ^i, where r equals (7.1), and a fixed Riemannian metric on M has been used to identify TM with T^*M, thus turning each Θ^i separately into a vector-bundle morphism $TM \to T^*M = \mathcal{L}^*$.

Finally, given a (skew)symmetric $(2, 0)$ tensor Θ of constant rank r on a manifold M, we associate with Θ a Nijenhuis-type $(2r + 3, 0)$ tensor $\widehat{N}$, testing the integrability of the image distribution $\mathcal{V} = \operatorname{Im} \Theta \subseteq TM$. (Note that Θ is a bundle morphisms $T^*M \to TM$ acting via $\xi \mapsto \Theta\xi = \Theta(\cdot, \xi)$.) To define $\widehat{N}$, we again fix a Riemannian metric on M, which allows us to use contractions and the Hodge star operator $*$ (as the latter enters our formula quadratically, M need not be oriented). With $\Theta\xi = \Theta(\cdot, \xi)$ as above, for 1-forms ξ on M, we set

$$\widehat{N}(\xi, \xi^1, \ldots, \xi^r, \eta, \eta^1, \ldots, \eta^r) = \Omega[\Theta\xi, \Theta\eta], \tag{15.4}$$

where $\xi, \xi^1, \ldots, \xi^r, \eta, \eta^1, \ldots, \eta^r$ are any 1-forms on M, and

$$\Omega \text{ denotes the result of an } (r - 1)\text{-fold contraction} \atop (2.9) \text{ of } *(\Theta\xi^1 \wedge \ldots \wedge \Theta\xi^r) \text{ against } *(\Theta\eta^1 \wedge \ldots \wedge \Theta\eta^r). \tag{15.5}$$

Clearly, at points where the r-tuples $\Theta\xi^1, \ldots, \Theta\xi^r$ and $\Theta\eta^1, \ldots, \Theta\eta^r$ of vector fields are both linearly independent, Ω is, by Remark 2.4, a nonzero functional multiple of the orthogonal projection onto the orthogonal complement of $\mathcal{V}$ and, applied to the Lie brackets $[\Theta\xi, \Theta\eta]$, tests the integrability of $\mathcal{V}$.

16 Twice-covariant symmetric tensors

The tensoriality of N' in (1.9-a) was established in (15.3). For N'', since

$$[\pounds_{\phi v} g](w, u) = \phi[\pounds_v g](w, u) + (d_w\phi)g(v, u) + (d_u\phi)g(w, v) \tag{16.1}$$

for any function ϕ on the given manifold M, the resulting nontensorial contribution to (1.9 -b) equals the sum $(d_w\phi)g(u, \cdot) + (d_u\phi)g(w, \cdot)$ of the last two terms in (16.1). Its exterior product with $g(v_1, \cdot) \wedge \ldots \wedge g(v_r, \cdot)$ vanishes, being a sum of $(r+1)$-fold exterior products of sections of a rank r subbundle of T^*M, namely, the image of the morphism sending each vector field v to $g(v, \cdot)$.

Proof of Theorem G We derive our conclusion from (1.8), by showing that the vanishing of N' (or N''), is equivalent to the integrability of the distribution $\mathcal{V} = \operatorname{Ker} g$ (or, respectively, to projectability of g along $\mathcal{V}$).

The first of these claims is obvious from (15.2) and (15.3). It thus obviously suffices to show that *the second equivalence holds if $\mathcal{V}$ is integrable.*

Clearly, with $\mathcal{V} = \operatorname{Ker} g$ from now on assumed integrable,

$$\begin{aligned}&N'' = 0 \text{ if and only if } N''(w, u, v_1, \ldots, v_r) = 0 \text{ for all}\\&\text{local vector fields } w, u, v_1, \ldots, v_r \text{ projectable along } \mathcal{V}.\end{aligned} \tag{16.2}$$

Although $[\pounds g](w, u)$ in (1.9-b) is not a genuine 1-form on the manifold M in question, we now artificially turn it into one, by fixing a local trivialization of TM, containing a local trivialization of $\mathcal{V}$, and declaring $[\pounds g](w, u)$ to be 1-form acting by $v \mapsto [\pounds_v g](w, u)$ on our selected (finitely many) vector fields v trivializing TM. As $[\pounds_v g](w, u) = d_v[g(w, u)] - g([v, w], u) - g(w, [v, u])$, projectability of w, u and (2.3) imply that

$$[\pounds_v g](w, u) = d_v[g(w, u)] \text{ whenever } v \text{ is a section of } \mathcal{V} = \operatorname{Ker} g. \tag{16.3}$$

If $N'' = 0$, the $(r+1)$-form $\zeta = N''(w, u, v_1, \ldots, v_r)$ vanishes, and hence so does $d_v[g(w, u)]$ in (16.3), for sections v of $\mathcal{V}$, as $\zeta(v, \cdot, \ldots, \cdot)$ equals $d_v[g(w, u)]$ times the the exterior product $g(v_1, \cdot) \wedge \ldots \wedge g(v_r, \cdot)$ (and the latter r-form may be chosen nonzero since rank $g = r$). Thus, by (2.4), g is projectable along $\mathcal{V}$.

Conversely, let us assume projectability of g along $\mathcal{V}$. Now in (16.2)–(16.3) $d_v[g(w, u)] = 0$, and hence $[\pounds_v g](w, u) = 0$ for all sections v of $\mathcal{V}$. The 1-form $[\pounds g](w, u)$ vanishes on $\mathcal{V} = \operatorname{Ker} g$, and so obviously do $g(v_1, \cdot), \ldots, g(v_r, \cdot)$ in (16.2). As rank $g = r$, the 1-forms vanishing on $\mathcal{V}$ constitute a vector subbundle of fibre dimension r in T^*M. Thus, $N'' = 0$ by (16.2). □

References

1. Bandyopadhyay, S., Dacorogna, B., Matveev, V.S., Troyanov, M.: Bernhard Riemann 1861 revisited: existence of flat coordinates for an arbitrary bilinear form. Math. Z. **305**, 12 (2023)
2. Bejancu, A.: Curvature in sub-Riemannian geometry. J. Math. Phys. **53**, 023513 (2012)

3. Blaga, A.M., Nannicini, A.: Statistical structures, α-connections and generalized geometry. Riv. Math. Univ. Parma (N.S.) **13**, 283–296 (2022)

4. Bolsinov, A.V., Konyaev, A.Y., Matveev, V.S.: Nijenhuis geometry. Adv. Math. **394**, 108001 (2022)

5. Boubel, C.: An integrability condition for fields of nilpotent endomorphisms (2011) **(preprint)**

6. Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., Griffiths, P.A.: Exterior Differential Systems, Math. Sci. Res. Inst. Publ., vol. 18. Springer, New York (1991)

7. Clark, R.S., Bruckheimer, M.: Tensor structures on a differentiable manifold. Ann. Mat. Pura Appl. (4) **54**, 123–141 (1961)

8. Crampin, M.: On differentiable manifolds with degenerate metrics. Proc. Camb. Philos. Soc. **64**, 307–316 (1968)

9. Crampin, M., Thompson, G.: Affine bundles and integrable almost tangent structures. Math. Proc. Camb. Philos. Soc. **98**, 61–71 (1985)

10. Duggal, K.L., Bejancu, A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Mathematics and its Applications, 364. Kluwer Academic Publishers Group, Dordrecht (1996)

11. Goel, D.S.: Almost tangent structures. Kodai Math. Sem. Rep. **26**, 187–193 (1974/75)

12. Grifone, J., Mehdi, M.: Existence of conservation laws and characterization of recursion operators for completely integrable systems. Trans. Am. Math. Soc. **349**, 4609–4633 (1997)

13. Grochowski, M., Kryński, W.: On contact sub-pseudo-Riemannian isometries. ESAIM Control Optim. Calc. Var. **23**, 1751–1765 (2017)

14. Hernando, J.M., Reyes, E., Gadea, P.M.: Integrability of tensor structures of electromagnetic type. Publ. Inst. Math. (Beograd) (N.S.) **37**, 113–122 (1985)

15. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math., vol. 9, 3rd edn. Springer, New York (1980)

16. Kobayashi, E.T.: A remark on the Nijenhuis tensor. Pac. J. Math. **12**, 963–977 (1962)

17. Kobayashi, S.: Transformation Groups in Differential Geometry, Classics in Mathematics. Springer, Berlin (1995). (**reprint of the 1972 edition**)

18. Kosmann-Schwarzbach, Y.: Beyond recursion operators. In: Geometric Methods in Physics XXXVI, Trends Math., pp. 167–180. Birkhäuser/Springer, Cham (2019)

19. Kurita, M.: Tensor fields and their parallelism. Nagoya Math. J. **18**, 133–151 (1961)

20. Moisil, G.C.: Sur les géodésiques des espaces de Riemann singuliers. Bull. Math. Soc. Roumaine Sci. **42**, 33–52 (1940)

21. Nijenhuis, A.: X_{n-1}-forming sets of eigenvectors. Indag. Math. **13**, 200–212 (1951)

22. Oproiu, V.: Degenerate Riemannian and degenerate conformal connexions. An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **16**, 357–376 (1970)

23. Piccione, P., Tausk, D.V.: An existence theorem for G-structure preserving affine immersions. Indiana Univ. Math. J. **57**, 1431–1465 (2008)

24. Reyes, D., Tempesta, P., Tondo, G.: Generalized Nijenhuis torsions and block-diagonalization of operator fields. J. Nonlinear Sci. **33**, 35 (2023)

25. Studený, V.: General Nijenhuis tensor. An example of a secondary invariant. In: J. Bureš et al. (eds.), Proceedings of the Winter School on Geometry and Physics, Srní, Czech Republic, January 1994. Palermo: Circolo Matematico di Palermo, Suppl. Rend. Circ. Mat. Palermo, II, Ser. 39:133–141 (1996)

26. Sulik, K.: A note on degenerate pseudo-Riemannian manifolds. Univ. Iagel. Acta Math. **24**, 67–75 (1984)

27. Tempesta, P., Tondo, G.: Haantjes algebras and diagonalization. J. Geom. Phys. **160**, 103968 (2021)

28. Thompson, G.: The integrability of a field of endomorphisms. Math. Bohem. **127**, 605–611 (2002)

29. Tondeur, P.: Affine Zusammenhänge auf Mannigfaltigkeiten mit fast-symplektischer Struktur. Comment. Math. Helv. **36**, 234–244 (1961)

30. Vrănceanu, G.: Sur les invariants des espaces de Riemann singuliers. Disquisit. Math. Phys. **2**, 253–281 (1942)

31. Yano, K., Davies, E.T.: Differential geometry on almost tangent manifolds. Ann. Mat. Pura Appl. (4) **103**, 131–160 (1975)