MATH1150 - AU20 - Completing squares

The key thing to have in mind here is the identity

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

as well as its close cousin

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

Completing the square is a perhaps useful device for factoring certain expressions. Usually one already has the first squared term and a "mixed" term to begin with. The trick is then to *add and subtract* whatever is left for you to actually get a perfect square.

Example 1. Consider the expression $x^2 - 2x - 4$. We already have x^2, and we have a "mixed" term $-2x$. What about that -4? Whatever happens to it, happens. You know that the mixed term is supposed to have the form

$$-2 \cdot (\text{something}) \cdot (\text{something}).$$

Compare the above with $-2x$. One of the "somethings" will be x. The other one can be 1. So:

$$\begin{aligned} x^2 - 2x - 4 &= x^2 - 2x + 1 - 1 - 4 \\ &= (x-1)^2 - 1 - 4 \\ &= (x-1)^2 - 5. \end{aligned}$$

Example 2. Now look at $x^2 + 5x + 7$. Again we already have the first squared term, x^2 (and this will probably be the case at least 99% of the time). Now look at the second term, $+5x$. Write

$$5x = 2 \cdot \frac{5}{2} \cdot x.$$

This is done on purpose to *create* the coefficient 2 that appears in the formulas given in the beginning of this discussion. This means that

$$\begin{aligned} x^2 + 5x + 7 &= x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 7 \\ &= \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 7 \\ &= \left(x + \frac{5}{4}\right)^2 - \frac{53}{4}. \end{aligned}$$

terekcouto.1@osu.edu

Example 3. One last direct example: $x^2 - 4x + 9$. The term x^2 is ready to be used. Now write

$$-4x = -2 \cdot 2 \cdot x.$$

This indicated that we should complete the square as

$$\begin{aligned} x^2 - 4x + 9 &= x^2 - 4x + 2^2 - 2^2 + 9 \\ &= (x-2)^2 - 4 - 9 \\ &= (x-2)^2 - 13. \end{aligned}$$

Doing this procedure allows us to find the vertex coordinates of a parabola completely bypassing the $h = -b/2a$ formula.

Example 4. Consider $f(x) = 3x^2 - 9x + 12$. Based on the previous examples, we would like to deal with x^2 instead of $3x^2$, so we *factor out this* 3 on everything except the constant term (because again, whatever happens to it, happens). So write

$$3x^2 - 9x + 12 = 3(x^2 - 3x) + 12$$

and let's focus on what is in the parentheses. Writing

$$-3x = -2 \cdot \frac{3}{2} \cdot x,$$

we see that

$$\begin{aligned} 3x^2 - 9x + 12 &= 3(x^2 - 3x) + 12 \\ &= 3\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right) + 12 \\ &= 3\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right) + 12 \\ &= 3\left(x - \frac{3}{2}\right)^2 - \frac{27}{4} + 12 \\ &= 3\left(x - \frac{3}{2}\right)^2 + \frac{21}{4}. \end{aligned}$$

This means that the coordinates (h, k) of the vertex of f are

$$h = \frac{3}{2} \quad \text{and} \quad k = \frac{21}{4}.$$

Sanity-check: we have

$$h = -\frac{b}{2a} = -\frac{(-9)}{2 \cdot 3} = \frac{9}{6} = \frac{3}{2}$$

and also

$$k = f(h) = f\left(\frac{3}{2}\right) = 3\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) + 12 = \frac{27}{4} - \frac{54}{4} + \frac{48}{4} = \frac{21}{4}.$$

Let's now try to understand the general deduction. In general, consider a quadratic function $f(x) = ax^2 + bx + c$, with $a \neq 0$ (or else this would be a line to begin with — this condition is natural not only geometrically, but it makes sense algebraically since in all of what will happen next divisions by a will be required, and one cannot divide by zero). First write

$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a}\right) + c,$$

so that we can focus on what happens inside the parentheses. To create the coefficient 2 required in the notable product, observe that

$$\frac{bx}{a} = 2 \cdot \frac{b}{2a} \cdot x.$$

Now:

$$\begin{aligned} ax^2 + bx + c &\overset{(1)}{=} a\left(x^2 + \frac{bx}{a}\right) + c \\ &\overset{(2)}{=} a\left(x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\ &\overset{(3)}{=} a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\ &\overset{(4)}{=} a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\ &\overset{(5)}{=} a\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a}. \end{aligned}$$

Here's what happened in each step:

(1) We factored a out in the first two terms, so we could complete the squares inside the parentheses there.

(2) We're effectively completing the square, i.e., adding and subtracting what was left for us to get a perfect square (what to add and subtract was deduced before).

(3) We have gathered the first three terms inside the parentheses in a perfect square, and computed the square of the remaining terms there.

(4) Foil. Note a cancellation of a turned the denominator $4a^2$ into $4a$.

(5) We put the last two terms under a common denominator.

This tells us that

$$\boxed{h = -\frac{b}{2a} \quad \text{and} \quad k = -\frac{(b^2 - 4ac)}{4a}}$$

Note that $b^2 - 4ac$ in the numerator of k is the same one that appears in the quadratic formula

$$\boxed{x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

In fact, we are in position to *prove* that the quadratic formula is true. Solving the equation $ax^2 + bx + c = 0$ is the same, by the above work, as solving

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a} = 0.$$

Reorganize:

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}.$$

Divide by a:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Take roots:

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Solve for x:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Put everything under the common denominator $2a$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

□

Remark. If x_1 and x_2 are the roots of $ax^2 + bx + c$, then we have that

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a},$$

and also that

$$x_1 x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

The relations

$$\boxed{x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad x_1 x_2 = \frac{c}{a}}$$

are called *Vieta's formulas* and sometimes are useful in factoring quadratic expressions without dealing with directly with formulas. For example, consider $x^2 - 5x + 6$. Can you find two numbers (x_1 and x_2) that add to 5 (which is $-b/a$), whose product is 6 (which is c/a)? After a moment of thought we see that 2 and 3 fit the bill. So we conclude that $x^2 - 5x + 6 - (x - 2)(x - 3)$.