Quiz 1

MATH 2173, FALL 2019

RECITATION TIME: ———   NAME: SOLUTION KEY

Write down the algebraic process and *circle the final answers*. No electrical devices and notes can be used in this quiz. Any correct answer without process will get only ONE POINT. You will have 20 minutes for 3 problems.

Problem 1. (3 points) Find the value of the directional derivative of $f(x,y) = 2x^2 - xy$ at the point $(1,1)$ in the direction $\langle \cos\theta, \sin\theta \rangle$, where $\theta = \pi/3$.

Solution: We have $\nabla f(x,y) = \langle 4x - y, -x \rangle$, so

$$\nabla f(1,1) \cdot \langle \cos\theta, \sin\theta \rangle = \langle 3, -1 \rangle \cdot \langle 1/2, \sqrt{3}/2 \rangle = \frac{3}{2} - \frac{\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2}.$$

Problem 2. (3 points) Consider the function $f(x,y) = \sin(x+2y)$ and the point $(0, \pi/2)$. Find the unit vectors that give the direction of steepest ascent and steepest descent at that point.

Solution: First, we compute $\nabla f(x,y) = \langle \cos(x+2y), 2\cos(x+2y) \rangle$. Evaluating, it follows that $\nabla f(0,\pi/2) = \langle -1, -2 \rangle$. Normalizing, we get that

$$\text{steepest ascent} = \langle -1/\sqrt{5}, -2/\sqrt{5} \rangle \quad \text{and} \quad \text{steepest descent} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle.$$

Problem 3. (4 points) Find the critical points of the following function. Use the Second Derivative Test to determine whether each critical point corresponds to a local maximum, local minimum, or saddle point:

$$f(x,y) = x^4 - \frac{1}{2}x - y^3 + 3y.$$

Solution: First we compute

$$\nabla f(x,y) = \langle 4x^3 - 1/2, -3y^2 + 3 \rangle,$$

from where we see that $\nabla f(x,y) = \langle 0,0 \rangle$ is equivalent to $4x^3 = 1/2$ and $3y^2 = 3$. So the critical points of f are $(1/2, 1)$ and $(1/2, -1)$. The Hessian matrix of f at a point (x,y) is given by

$$\operatorname{Hess} f(x,y) = \begin{pmatrix} 12x^2 & 0 \\ 0 & -6y \end{pmatrix}.$$

With this we analyze the critical points:

- $D(1/2, 1) = -18 < 0$. Saddle point.
- $D(1/2, -1) = 18 > 0$, $f_{xx}(1/2, -1) = 3 > 0$. Local minimum.

Quiz 2

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Write down the algebraic process and *circle the final answers*. No electrical devices and notes can be used in this quiz, Any correct answer without process will get only ONE POINT. You will have 20 minutes for 2 problems.

Problem 1. (5 points) Suppose $f(x,y) = x^2y + 2xy - 3y$ is a function on the region $R = [0,2] \times [-1,1]$.
(a) (2 points) Find critical point(s) of f in R.
(b) (3 points) Find the absolute maximum and minimum values of f on R.

Solution:

(a) Compute $\nabla f(x,y) = (2xy + 2y, x^2 + 2x - 3)$. So

$$\begin{cases} 2y(x+1) = 0 \\ x^2 + 2x - 3 = 0 \end{cases} \implies y = 0 \quad \text{and} \quad x = 1 \text{ or } -3.$$

So the critical point inside R is $(1,0)$ ($(-3,0)$ is outside). Have $f(1,0) = 0$.

(b) Analyze all sides of the square:

- $f(0,y) = -3y$, $-1 \le y \le 1$. Maximum 3 at $(0,-1)$, minimum -3 at $(0,1)$.
- $f(2,y) = 5y$, $-1 \le y \le 1$. Maximum 5 at $(2,1)$, minimum -5 at $(2,-1)$.
- $f(x,1) = x^2 + 2x - 3$, $0 \le x \le 2$. Minimum -3 at $(0,1)$, maximum 5 at $(2,1)$.
- $f(x,-1) = -x^2 - 2x + 3$, $0 \le x \le 2$. By symmetry, no new values.

Conclusion: absolute maximum at $(2,1)$, absolute minimum at $(2,-1)$.

Problem 2. (5 points) Find the absolute maximum and minimum values of the function

$$f(x,y) = x^2 - xy + y^2 - 3x$$

on the region R bounded by the triangle with vertices $(0,0)$, $(0,3)$, $(3,3)$.

Solution: First compute $\nabla f(x,y) = (2x - y - 3, -x + 2y)$. So

$$\begin{cases} 2x - y - 3 = 0 \\ -x + 2y = 0 \end{cases} \implies y = 1 \quad \text{and} \quad x = 2.$$

But $(2,1)$ is not inside R. So no critical points. Now we analyze the sides of the triangle.

- $f(0,y) = y^2$, $0 \le y \le 3$. Minimum 0 at $(0,0)$, maximum 9 at $(0,3)$.
- $f(x,3) = x^2 - 6x + 9$, $0 \le x \le 3$. Minimum 0 at $(3,3)$, maximum 9 at $(0,3)$.
- $f(x,x) = x^2 - 3x$, $0 \le x \le 3$. Minimum $-9/4$ at $(3/2,3/2)$, maximum 0 at $(0,0)$ and $(3,3)$.

Conclusion: absolute maximum 9 at $(0,3)$, absolute minimum at $(3/2,3/2)$.

Quiz 3

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Problem 1. (3 points) Evaluate the iterated integral

$$\int_0^\pi \int_1^2 y\cos(xy)\,\mathrm{d}x\,\mathrm{d}y.$$

Solution: Just compute

$$\begin{aligned}\int_0^\pi \int_1^2 y\cos(xy)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^\pi (\sin(xy))\Big|_1^2\,\mathrm{d}y = \int_0^\pi (\sin(2y)-\sin y)\,\mathrm{d}y \\ &= \left(-\frac{1}{2}\cos(2y)+\cos y\right)\Big|_0^\pi = -\frac{1}{2}-1-\left(-\frac{1}{2}+1\right) \\ &= -2.\end{aligned}$$

Problem 2. (3 points) Find the extreme values of $f(x,y)=x+2y$ on the curve $x^2+2y^2=3$.

Solution: Since we have $\nabla f(x,y)=(1,2)$ and $\nabla g(x,y)=(2x,4y)$, writing that $\nabla f(x,y)=\lambda\nabla g(x,y)$ gives the system

$$\begin{cases}1=2\lambda x\\ 2=4\lambda y\\ x^2+2y^2=3\end{cases}$$

The first two equations immediately give $x=y$. So the constraint equation becomes $3x^2=3$, and $x=\pm1$. Thus

- $f(1,1)=3$, absolute maximum.
- $f(-1,-1)=-3$, absolute minimum.

Problem 3. (4 points) Find the absolute maximum and minimum values of the function

$$f(x,y)=x^4+4y^4$$

on the region $R=\{(x,y)\mid x^2+y^2\leq 5\}$.

Solution: Since $\nabla f(x,y)=(4x^3,16y^3)$ only vanishes for $(x,y)=(0,0)$, which is inside R, we get the candidate $(0,0)$, and $f(0,0)=0$ (which is automatically the absolute minimum, as $f\geq 0$). For the boundary, we use Lagrange multipliers with $\nabla g(x,y)=(2x,2y)$ to get the system

$$\begin{cases}4x^3=2\lambda x\\ 16y^3=2\lambda y\\ x^2+y^2=5\end{cases}$$

Then the first two equations give $x^3y=4xy^3$. If $xy\neq 0$, then $x^2=4y^2$ and the constraint equation becomes just $5y^2=5$, and $y=\pm1$. Then $x^2=4$ implies $x=\pm2$, and we get four more candidates, all achieving the same value

$$f(2,1)=f(-2,1)=f(2,-1)=f(-2,-1)=20.$$

If $xy=0$, we get four more candidates

$$f(\sqrt{5},0)=f(-\sqrt{5},0)=25 \quad\text{and}\quad f(0,\sqrt{5})=f(0,-\sqrt{5})=100.$$

Conclusion: absolute minimum 0 at $(0,0)$, absolute maximum 100 at $(0,\pm\sqrt{5})$.

Quiz 4

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Problem 1. (3 points.) Compute the average of $f(x,y) = x\sin(xy)$ on the region $R = \{(x,y) \mid 0 \le x \le \pi/2, 1 \le y \le 2\}$.

Solution: Straightforward computation:

$$\begin{aligned}
\overline{f} &= \frac{1}{\operatorname{area}(R)} \iint_R x\sin(xy)\,\mathrm{d}A = \frac{1}{(\pi/2)(2-1)} \int_0^{\pi/2} \int_1^2 x\sin(xy)\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi/2} -\cos(xy)\Big|_{y=1}^{y=2} \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi/2} \big(\cos x - \cos(2x)\big)\,\mathrm{d}x \\
&= \frac{2}{\pi} \left(\sin x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi/2} = \frac{2}{\pi}(1 - 0 - (0-0)) \\
&= \frac{2}{\pi}.
\end{aligned}$$

Problem 2. (3 points.) Evaluate the double integral $\iint_R y^2\,\mathrm{d}A$, where R is bounded by $x = -y^2 + 1$ and $x = 0$.

Solution: To avoid dealing with roots, we'll use the following order:

$$\begin{aligned}
\iint_R y^2\,\mathrm{d}A &= \int_{-1}^1 \int_0^{1-y^2} y^2\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^1 y^2 x\Big|_{x=0}^{x=1-y^2} \mathrm{d}y \\
&= \int_{-1}^1 y^2(1-y^2)\,\mathrm{d}y = 2\int_0^1 y^2 - y^4\,\mathrm{d}y \\
&= 2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}.
\end{aligned}$$

Problem 3. (4 points.) Evaluate the following iterated integral by reversing the order of integration:

$$\int_0^2 \int_y^2 x^2\cos(xy)\,\mathrm{d}x\,\mathrm{d}y.$$

Solution: The region of integration is the triangle with vertices in $(0,0)$, $(2,0)$ and $(2,2)$. So:

$$\begin{aligned}
\int_0^2 \int_y^2 x^2\cos(xy)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \int_0^x x^2\cos(xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 x\sin(xy)\Big|_{y=0}^{y=x} \mathrm{d}x \\
&= \int_0^2 x\sin(x^2)\,\mathrm{d}x \overset{(*)}{=} \frac{1}{2}\int_0^4 \sin u\,\mathrm{d}u \\
&= \frac{1}{2}(-\cos u)\Big|_0^4 = \frac{1-\cos 4}{2},
\end{aligned}$$

where in $(*)$ we make $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$ and adjust bounds of integration.

Quiz 5

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Problem 1. (3 points.) Evaluate the following iterated integral using polar coordinates

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x.$$

Solution: First of all, recognize the region of integration as the right half of the unit disk. Not forgetting the correction factor in $\mathrm{d}y\,\mathrm{d}x = r\,\mathrm{d}r\,\mathrm{d}\theta$, compute

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \int_0^1 \sin(r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \overset{(*)}{=} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_0^1 r\sin(r^2)\,\mathrm{d}r$$
$$= \pi\left(-\frac{1}{2}\cos(r^2)\right)\Big|_0^1 = \frac{\pi(1-\cos 1)}{2},$$

where in $(*)$ we use that the integral of a product of functions of separate variables is the product of the integrals.

Problem 2. (3 points.) Find the average distance between points of the annulus $R = \{1 \le x^2+y^2 \le 2\}$ and the origin.

Solution: The distance of a point (x, y) to the origin is $\sqrt{x^2+y^2}$, so we compute

$$\frac{1}{\text{area}(R)} \iint_R \sqrt{x^2+y^2}\,\mathrm{d}A = \frac{1}{\pi}\int_0^{2\pi}\int_1^{\sqrt{2}} r\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\theta\int_1^{\sqrt{2}} r^2\,\mathrm{d}r$$
$$= \frac{1}{\pi}\cdot 2\pi\cdot\frac{r^3}{3}\Big|_1^{\sqrt{2}} = \frac{2}{3}(2\sqrt{2}-1)$$
$$= \frac{4\sqrt{2}-2}{3}.$$

Problem 3. (4 points.) Find the volume of the solid bounded by $z = 12 - 2x^2 - y^2$ and $z = x^2 + 2y^2$.

Solution: To find the region over which we'll integrate the difference of the functions defining the surfaces, we note that $12 - 2x^2 - y^2 = x^2 + 2y^2$ implies $x^2 + y^2 = 4$, meaning that $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4\}$. So

$$\iint_R 12 - 2x^2 - y^2 - (x^2+2y^2)\,\mathrm{d}A = 3\iint_R 4 - x^2 - y^2\,\mathrm{d}A = 3\int_0^{2\pi}\int_0^2 (4-r^2)r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= 3\int_0^{2\pi}\mathrm{d}\theta\int_0^2 4r - r^3\,\mathrm{d}r = 6\pi\left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2$$
$$= 6\pi(8-4) = 24\pi.$$

Quiz 6

MATH 2173, FALL 2019

RECITATION TIME: ———— NAME: SOLUTION KEY

Problem 1. (3 points.) Use cylindrical coordinates to evaluate the following integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Solution: Cylindrical coordinates amount to changing only two of the three variables x, y and z to polar coordinates. And the spatial region D is bounded above by the graph of $z=\sqrt{x^2+y^2}$ and below by the upper semi-circle centered at $(0,0)$ with radius 3. So if $x=r\cos\theta$ and $y=r\sin\theta$, we get

$$\int_0^3\int_0^\pi\int_0^r r\,r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r=\int_0^3\int_0^\pi r^2 z\Big|_{z=0}^{z=r}\mathrm{d}\theta\,\mathrm{d}r=\int_0^3\int_0^\pi r^3\,\mathrm{d}\theta\,\mathrm{d}r=\frac{\pi r^4}{4}\Big|_{r=0}^{r=3}=\frac{81\pi}{4}$$

Problem 2. (3 points.) Evaluate the integral

$$\iiint_D (x^2+y^2+z^2)\,\mathrm{d}V,$$

where $D=\{(x,y,z)|x^2+y^2+z^2\le 4, z\ge 0\}$.

Solution: For spherical coordinates

$$x=\rho\cos\theta\sin\varphi,\quad y=\rho\sin\theta\sin\varphi,\quad z=\rho\cos\varphi,$$

the correction factor (Jacobian determinant) in the change of variables is $J=\rho^2\sin\varphi$. So we have

$$\begin{aligned}\int_0^2\int_0^{2\pi}\int_0^{\pi/2}\rho^2(\rho^2\sin\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\rho &= \int_0^2\int_0^{2\pi}\int_0^{\pi/2}\rho^4\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\rho\\ &=\int_0^2\rho^4\,\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\sin\varphi\,\mathrm{d}\varphi\\ &=\frac{32}{5}\cdot 2\pi\cdot 1\\ &=\frac{64\pi}{5}.\end{aligned}$$

Problem 3. (4 points.) Evaluate the integral

$$\iint_R (x-y)^2(x+y+1)\,\mathrm{d}A,$$

where R is the square with vertices $(0,0),(1,-1),(-1,-1),(0,-2)$.

Solution: Follow your nose and let $u=x-y$ and $v=x+y+1$. The correction factor is found by:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=2\implies\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{2}\implies \mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\mathrm{d}u\,\mathrm{d}v,$$

and since $(x,y) \mapsto (u,v)$ is affine (i.e., the formulas for u and v are first degree polynomials in the variables x and y), the original square will be mapped into a quadrilateral in the plane (u,v). This way, to find the bounds for u and v, we only need to see where the original vertices are mapped to:

$$(0,0) \mapsto (0,1), \quad (1,-1) \mapsto (2,1), \quad (-1,-1) \mapsto (0,-1), \quad (0,-2) \mapsto (2,-1).$$

These are the original vertices, and the new region has edges aligned with the u and v-axes. So

$$\int_{-1}^{1}\int_{0}^{2} u^2 v \left(\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v\right) = \frac{1}{2}\int_{-1}^{1}\int_{0}^{2} u^2 v\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_{-1}^{1} v\,\mathrm{d}v \int_{0}^{2} u^2\,\mathrm{d}u = \frac{1}{2}\cdot 0 \cdot \frac{8}{3} = 0.$$

Quiz 7

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Write down the algebraic process and circle the final answers. No electrical devices and notes can be used in this quiz. Any correct answer without process will get only ONE POINT. You will have 20 minutes for 3 problems.

Problem 1. (3 points.) Let C be the line segment from $(1, 2, -1)$ to $(-5, 4, 2)$. Parametrize C in the form $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, a \le t \le b$.

Solution: Simply do

$$\mathbf{r}(t) = (1-t)(1,2,-1) + t(-5,4,2) = (1-6t, 2+2t, -1+3t),$$

for $0 \le t \le 1$. Note that $\mathbf{r}(\mathbf{0}) = (\mathbf{1}, \mathbf{2}, -\mathbf{1})$ and $\mathbf{r}(1) = (-5, 4, 2)$, as it should.

Problem 2. (3 points.) Evaluate the line integral $\int_C y\,\mathrm{d}x$, where C is the curve given by $\mathbf{r}(t) = \langle t^2, 2t\rangle, 0 \le t \le 1$.

Solution: Use $\mathrm{d}x = x'(t)\,\mathrm{d}t$ with $x(t) = t^2$ to get

$$\int_C y\,\mathrm{d}x = \int_0^1 2t(2t\,\mathrm{d}t) = \int_0^1 4t^2\,\mathrm{d}t = \frac{4}{3}.$$

Problem 3. (4 points.) Evaluate the line integral $\int_C (x^2 - y)\,\mathrm{d}s$, where C is the upper semicircle starting at $(1, 0)$ and ending at $(-1, 0)$.

Solution: Using a parametrization of C by arc-length, we'll have $\mathrm{d}s = \mathrm{d}t$. So let $\mathbf{r}(t) = (\cos t, \sin t)$, for $0 \le t \le \pi$. Then

$$\int_C (x^2 - y)\,\mathrm{d}s = \int_0^\pi (\cos^2 t - \sin t)\,\mathrm{d}t = \frac{\pi}{2} - 2.$$

Quiz 8

MATH 2173, FALL 2019

RECITATION TIME: ———   NAME: SOLUTION KEY

Write down the algebraic process and circle the final answers. No electrical devices and notes can be used in this quiz. Any correct answer without process will get only ONE POINT. You will have 20 minutes for 3 problems. **The last problem is in the back of this page.**

Problem 1. (3 points) Is the vector field $\mathbf{F}(x,y,z) = \langle e^{\sin y}, x(\cos y)e^{\sin y} + z^2, y^2 + z^2\rangle$ conservative?

Solution: No, because

$$\frac{\partial}{\partial z}(x(\cos y)e^{\sin y} + z^2) = 2z \quad \text{and} \quad \frac{\partial}{\partial y}(y^2 + z^2) = 2y$$

are not equal.

Problem 2. (3 points) Let $\mathbf{F}(x,y) = \langle y, -2x\rangle$. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the unit circle arc from $(0,1)$ to $(-1,0)$.

Solution: Take $\mathbf{r}(t) = (\cos t, \sin t)$, with $\pi/2 \le t \le \pi$. Then compute

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{\pi/2}^{\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_{\pi/2}^{\pi} \langle \sin t, -2\cos t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt \\
&= \int_{\pi/2}^{\pi} -\sin^2 t - 2\cos^2 t\,dt = \int_{\pi/2}^{\pi} -1 - \cos^2 t\,dt \\
&= -\frac{\pi}{2} - \int_{\pi/2}^{\pi} \frac{1+\cos(2t)}{2}\,dt = -\frac{\pi}{2} - \frac{\pi}{4} - \frac{\sin(2t)}{4}\Big|_{\pi/2}^{\pi} \\
&= -\frac{3\pi}{4}.
\end{aligned}$$

Problem 3. (4 points) Let $\mathbf{F}(x,y,z) = \langle x+2y, 2x+y, e^z\rangle$.

(a) Is $\mathbf{F}$ conservative?

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = (t, t^3, t^5)$, with $0 \le t \le 1$.

Solution:

(a) Yes. For example, one potential is $\varphi(x,y,z) = \dfrac{x^2+y^2}{2} + 2xy + e^z$.

(b) Forget this curve and apply the Fundamental Theorem of Calculus to get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(\mathbf{r}(1)) - \varphi(\mathbf{r}(0)) = \varphi(1,1,1) - \varphi(0,0,0) = 3 + e - 1 = 2 + e.$$

Quiz 9

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Write down the algebraic process and circle the final answers. No electrical devices and notes can be used in this quiz. Any correct answer without process will get only ONE POINT. You will have 20 minutes for 3 problems. **The last problem is in the back of this page.**

Problem 1. (3 points) Prove that $\sin 2t$ and $\cos 2t$ are solutions to $y'' + 4y = 0$. Write down the general solution for $y'' + 4y = 0$.

Solution: The verification is straightforward. Thus, the general solution has the form $y(t) = a\sin(2t) + b\cos(2t)$, where a and b are real constants (i.e., the general solution is a generic linear combination of two independent particular solutions).

Problem 2. (3 points) Prove that e^{2t} and te^{2t} are solutions to $y'' - 4y' + 4y = 0$, then solve the initial value problem

$$y'' - 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

Solution: Again, the verification that e^{2t} and te^{2t} are solutions is straightforward. Thus the general solution has the form $y(t) = ae^{2t} + bte^{2t}$, where a and b are real constants. We now use the initial conditions given to find a and b. We have that

$$2 = y(0) = a \quad \text{and} \quad 3 = y'(0) = 2a + b,$$

so $a = 2$ and $b = -1$. The solution to the initial value problem is $y(t) = 2e^{2t} - te^{2t}$.

Problem 3. (4 points) Solve the initial value problem

$$y'' - 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

Solution: The characteristic polynomial $\lambda^2 - 3\lambda + 2$ has roots $\lambda = 2$ and $\lambda = 1$, meaning that the general solution of the given differential equation has the form $y(t) = ae^{2t} + be^{t}$, where a and b are real constants. Now we use the initial conditions to find a and b. We have that

$$2 = y(0) = a + b \quad \text{and} \quad -1 = y'(0) = 2a + b,$$

so that $a = -3$ and $b = 5$. So the solution is $y(t) = -3e^{2t} + 5e^{t}$.

Quiz 10

MATH 2173, FALL 2019

RECITATION TIME: ——— NAME: SOLUTION KEY

Problem 1. (3 points.) Solve the following initial value problem:

$$y'' + 3y' + \frac{9}{4}y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

Solution: The characteristic equation is $r^2 + 3r + 9/4 = 0$, which can be factored to $(r + 3/2)^2 = 0$. So $r = -3/2$ is a *double root* and thus the general solution of the differential equation has the form

$$y(t) = c_1 e^{-3t/2} + c_2 t e^{-3t/2},$$

for some constants c_1 and c_2. The initial conditions give

$$3 = y(0) = c_1 \quad \text{and} \quad -2 = y'(0) = -\frac{3c_1}{2} + c_2,$$

so that $y(t) = 3e^{-3t/2} + (5/2)te^{-3t/2}$.

Problem 2. (3 points.) Find the general solution of the following equation:

$$t^2 y'' - 2ty' - 10y = 0, \quad t > 0.$$

Solution: This is a Cauchy-Euler equation, so we try a solution of the form $y(t) = t^p$ to produce the correct characteristic equation. We get

$$t^2 p(p-1)t^{p-2} - 2tpt^{p-1} - 10t^p = 0,$$

and divide everything through by t^p to obtain $p(p-1) - 2p - 10 = 0$. That is, the characteristic equation is $p^2 - 3p - 10 = 0$, whose roots are $p = 5$ and $p = -2$. So the general solution is $y(t) = c_1 t^5 + c_2 t^{-2}$, for some constants c_1 and c_2.

Problem 3. (4 points.) Solve the following initial value problem:

$$y'' + 6y' + 10y = 0, \quad y(0) = 4, \quad y'(0) = -3.$$

Solution: The characteristic equation $r^2 + 6r + 10 = 0$ has solutions $r = -3 \pm i$, so the general solution has the form $y(t) = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t$. The initial conditions become

$$4 = y(0) = c_1 \quad \text{and} \quad -3 = y'(0) = -3c_1 + c_2,$$

so $c_2 = 9$ and thus $y(t) = 4e^{-3t} \cos t + 9e^{-3t} \sin t$.