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Compact locally homogeneous manifolds with parallel Weyl tensor

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Abstract: We construct new examples of compact ECS manifolds, that is, of pseudo-Riemannian manifolds with parallel Weyl tensor that are neither conformally flat nor locally symmetric. Every ECS manifold has rank 1 or 2, the rank being the dimension of a distinguished null parallel distribution discovered by Olszak. Previously known examples of compact ECS manifolds, in every dimension greater than 4, were all of rank 1, geodesically complete, and none of them was locally homogeneous. In contrast, our new examples — all of them geodesically incomplete — realize all odd dimensions starting from 5 and are of rank 2, as well as locally homogeneous.

Keywords: Parallel Weyl tensor, conformally symmetric manifold, compact pseudo-Riemannian manifold.

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Introduction

An essentially conformally symmetric manifold, or *ECS manifold* for short, is a pseudo-Riemannian manifold of dimension $n \geq 4$ which is not locally symmetric and has nonzero parallel Weyl tensor W; see [4]. Its *rank* $d \in \{1, 2\}$ is the dimension of its *Olszak distribution* $\mathcal{D}$, see [19], [5, page 119], which is the null parallel distribution whose sections are the vector fields corresponding via the metric to 1-forms ξ such that $\xi \wedge [W(v, v', \cdot, \cdot)] = 0$ for all vector fields v, v'. (The term 'conformally symmetric' should not be misconstrued as referring to conformal geometry.)

ECS manifolds are of obvious interest due to the naturality and simplicity of the condition $\nabla W = 0$; see [1; 15; 18; 21; 13; 12]. Roter [20, Corollary 3] proved the existence of ECS manifolds in all dimensions $n \geq 4$ and showed in [3, Theorem 2] that their metrics are necessarily indefinite. Locally homogeneous ECS manifolds of either rank exist by [2] for all $n \geq 4$. The local structure of ECS manifolds has been completely described in [5].

Examples of *compact rank-one* ECS manifolds are known by [6; 9] in every dimension $n \geq 5$. They are geodesically complete and not locally homogeneous, which raises three obvious questions: Can a compact ECS manifold have rank two, or be locally homogeneous, or geodesically incomplete? This paper provides affirmative answers to these questions, for every *odd* dimension $n \geq 5$.

Just like in [6; 9], our examples are diffeomorphic to nontrivial torus bundles over the circle, and arise as quotients of certain explicitly described simply connected "model" manifolds $\widehat{M}$ under free and properly discontinuous actions on $\widehat{M}$ of suitable groups Γ of isometries. However, selecting such objects involves two aspects, an analytical aspect for $\widehat{M}$ (the existence of a specific function f of a real variable) and a combinatorial one for Γ, and it is here that our approach fundamentally differs from [6] and [9]. Whereas in those two papers the combinatorial part was trivial, and finding f required extensive work — a messy explicit construction in [6], good only for dimensions n congruent to 2 modulo 3 with $n \geq 5$, and a deformation argument applied to uninteresting constant functions in [9] — the situation here is the exact opposite: f comes from the very simple Formula (4.4), while the groups Γ arise via combinatorial structures ($\mathbb{Z}$-spectral systems), the existence of which we can only establish, with some effort, in Theorem 2.2 for odd dimensions n.

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Every $\mathbb{Z}$-spectral system gives rise to a free Abelian group Σ of isometries in each model manifold of a suitable type, associated with a narrow class of choices for the function f, so that Σ satisfies Conditions (3.9), which in turn allows us to extend Σ to the required group Γ, leading to a compact quotient manifold; see Theorem 5.1. (Our argument used to derive Theorem 5.1 from (3.9) is a modified version of arguments in [6] and [9].) One such choice for f, namely (4.4), makes the resulting compact rank-two ECS manifolds locally homogeneous; see Theorem 6.1. They are also all incomplete, for rather obvious reasons: see Remark 3.4.

The preceding sentence leads to a further question: For a compact ECS manifold, can one have incompleteness without local homogeneity? We answer it in the affirmative: for any f given by (4.4), there is an infinite-dimensional freedom of deforming it, so that Theorem 5.1 still applies, giving rise to compact quotient ECS manifolds which are still incomplete, but not locally homogeneous anymore. They belong to a wider class of compact rank-two ECS manifolds, called *dilational*. Since they are arguably of less interest than the locally homogeneous ones, we relegate their presentation to Appendix B.

In [7, Theorem E] we show that neither local homogeneity nor the dilational property can occur for a compact rank-one ECS manifold which satisfies a natural *genericity condition* imposed on the Weyl tensor. In the case of our simply connected "model" manifolds (Section 3) of dimensions $n \geq 4$, genericity means that $\operatorname{rank} A = n-3$ for a certain nonzero nilpotent endomorphism A of an $(n-2)$-dimensional vector space used in constructing the model; see [7, Formula (6.4)] and [10, Remark 5.4]. The models leading to our rank-two examples, in odd dimensions $n \geq 5$, all have $\operatorname{rank} A = 1$; see Formula (1.3). They thus represent the maximum extent of nongenericity possible in the category of nonzero nilpotent endomorphisms.

We do not know whether locally homogeneous (or dilational) compact ECS manifolds exist in any *even* dimension $n \geq 4$. However, if they do, they cannot be constructed by the same method as our odd-dimensional examples. Namely, as we observe at the end of Section 2, for every $\mathbb{Z}$-spectral system (m, k, E, J) the integer m, corresponding to the dimension $n = m + 2$, is necessarily odd.

1 Preliminaries

Lemma 1.1. *Let $q \in (0, \infty) \setminus \{1\}$ be a real number with $q + q^{-1} \in \mathbb{Z}$. If $\lambda_0, \dots, \lambda_m$ are powers of q with integer exponents, forming pairs of mutual inverses, including the value 1 as its own inverse when m is even, then $\lambda_0, \dots, \lambda_m$ form the spectrum of a matrix in* $\mathrm{GL}(m + 1, \mathbb{Z})$.

Proof. It suffices, cf. [6, page 75], to show that $\lambda_0, \dots, \lambda_m$ are the roots of a degree $m+1$ polynomial with integer coefficients which has the leading coefficient $(-1)^{m+1}$ and the constant term 1. This is immediate if $m = 0$ and $\lambda_0 = 1$, or $m = 1$ and $(\lambda_0, \lambda_1) = (q, q^{-1})$, or $m = 1$ and $(\lambda_0, \lambda_1) = (q^a, q^{-a})$ with any positive integer a. (The last claim is a well-known consequence of the preceding one, since $q^a + q^{-a}$ equals a specific monic degree a polynomial with integer coefficients, evaluated at $q + q^{-1}$.) The required degree $m+1$ polynomial is the product of the quadratic (and possibly linear) ones arising as above when $m = 0$ or $m = 1$. □

Remark 1.2. We call a pseudo-Euclidean inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on an m-dimensional real vector space V *semi-neutral* if its positive and negative indices differ by at most one. Clearly, the matrix representing $\langle\,\cdot\,,\,\cdot\,\rangle$ in a suitable basis $e_1, \dots, e_m$ of V has zero entries except those on the main antidiagonal, all equal to some sign factor $\varepsilon = \pm 1$, which for even m may be assumed equal to 1, but is unique for odd m, as it then equals the difference of the two indices. Equivalently,

$$\langle e_i, e_k \rangle = \varepsilon \delta_{ij} \quad \text{for all } i, j \in \{1, \dots, m\}, \text{ where } k = m + 1 - j, \text{ and } \varepsilon = \pm 1. \tag{1.1}$$

Given V, $\langle\,\cdot\,,\,\cdot\,\rangle$ and $e_1, \dots, e_m$ as above, $q \in (0, \infty)$, and $(a(1), \dots, a(m)) \in \mathbb{R}^m$ with

$$a(1) = 1 \quad \text{and} \quad a(i) + a(j) = 0 \quad \text{whenever } i + j = m + 1, \tag{1.2}$$

we define a nonzero traceless $\langle\,\cdot\,,\,\cdot\,\rangle$-self-adjoint linear endomorphism A of V and a linear $\langle\,\cdot\,,\,\cdot\,\rangle$-isometry $C : V \to V$ such that $CAC^{-1} = q^2 A$ by setting

$$Ae_m = e_1, \quad Ae_i = 0 \ \text{ if } i < m, \quad \text{and} \ \ Ce_i = q^{a(i)} e_i \ \text{ for all } i. \tag{1.3}$$

In fact, as $\langle Ae_m, e_m\rangle$ is the only nonzero entry of the form $\langle Ae_i, e_j\rangle$, the matrix of A in our basis has zeros on the diagonal; $CAC^{-1}e_i$ and q^2Ae_i are both zero if $i < m$ and both equal to q^2e_1 when $i = m$, while the spans of e_i, e_j with $i + j = m + 1$ form an orthogonal decomposition of V into Lorentzian planes (and a line, for odd m and $i = j = (m + 1)/2$), and in each plane C acts as a Lorentzian boost.

We state two more obvious facts as remarks, for easy reference.

Remark 1.3. Every family of eigenvectors of an endomorphism of a vector space, corresponding to mutually distinct eigenvalues, is linearly independent.

Remark 1.4. If the s characteristic roots of an endomorphism Π of an s-dimensional real vector space $\mathcal{Y}$ are all real, distinct, and form the spectrum of a matrix Ξ in $\mathrm{GL}(s,\mathbb{Z})$, then $\Pi(\Sigma) = \Sigma$ for some lattice Σ in $\mathcal{Y}$. (This is true for $\Pi = \Xi$ and $\mathcal{Y} = \mathbb{R}^s$, with $\Sigma = \mathbb{Z}^s$. The general case follows as the algebraic equivalence type of a diagonalizable endomorphism is determined by its spectrum.)

2 $\mathbb{Z}$-spectral systems

By a *$\mathbb{Z}$-spectral system* we mean a quadruple (m, k, E, J) consisting of integers $m, k \geq 2$, an injective function $E : \mathcal{V} \to \mathbb{Z} \setminus \{-1\}$, where $\mathcal{V} = \{1, \dots, 2m\}$, and a function $J : \mathcal{V} \to \{0, 1\}$, satisfying the following conditions for all $i, i' \in \mathcal{V}$.

(a) $k + 1 = 2E(1)$ (and so k must be odd).
(b) $E(i) + E(i') = -1$ and $J(i) \neq J(i')$ whenever $i + i' = 2m + 1$.
(c) $E(i) - E(i') = k$ and $J(i) \neq J(i')$ if $i' = i + 1$ is even.
(d) The set $Y = \{-1\} \cup \{E(i) : i \in \mathcal{V}$ and $J(i) = 1\}$ is symmetric about 0.

In terms of the preimage $S = J^{-1}(1) = \{i \in \mathcal{V} : J(i) = 1\}$, the requirements imposed on J state that S is a simultaneous selector for the two families

$$\{\{i, i'\} \in \mathcal{P}_2(\mathcal{V}) : i + i' = 2m + 1\} \quad \text{and} \quad \{\{i, i'\} \in \mathcal{P}_2(\mathcal{V}) : i' = i + 1 \text{ is even}\} \tag{2.1}$$

of pairwise disjoint 2-element subsets of $\mathcal{V}$, while J equals the characteristic function of S; here $\mathcal{P}_2(\mathcal{V})$ denotes the family of all 2-element subsets of $\mathcal{V}$. Thus, as E was assumed to be injective, with $|\ \ |$ standing for cardinality,

$$|S| = |E(S)| = m, \quad Y = \{-1\} \cup E(S), \quad |Y| = m + 1. \tag{2.2}$$

Remark 2.1. What makes $\mathbb{Z}$-spectral systems relevant for our purposes is the fact that, given any such system (m, k, E, J) and any real number $q \in (0, \infty) \setminus \{1\}$ with $q + q^{-1} \in \mathbb{Z}$, the $(m + 1)$-element set $\{q^a : a \in Y\}$ forms, according to Lemma 1.1, the spectrum of a matrix in $\mathrm{GL}(m + 1, \mathbb{Z})$.

Theorem 2.2. *There exist $\mathbb{Z}$-spectral systems (m, k, E, J) with $k = m + 2$ which realize all odd values of $m \geq 3$.*

Specifically, for $m = 2r - 3$ and $k = 2r - 1$ with any given integer $r \geq 3$, writing $(i, i') = (2j - 1, 2j)$ whenever $i, i' \in \mathcal{V}$ and $i' = i + 1$ is even, we may set

$$(E(2j-1), E(2j)) = \begin{cases} (r, -r+1) & \text{if } j = 1, \\ (j-1, -2r+j) & \text{if } 1 < j < r-1 \text{ and } r \text{ is even}, \\ (2r+j-2, j-1) & \text{if } 1 < j < r-1 \text{ and } r \text{ is odd}, \\ (r-1, -r) & \text{if } j = r-1, \\ (j-2r+2, j-4r+3) & \text{if } r-1 < j < m \text{ and } r \text{ is odd}, \\ (j+1, j-2r+2) & \text{if } r-1 < j < m \text{ and } r \text{ is even}, \\ (r-2, -r-1) & \text{if } j = m \end{cases}$$

and declare $J(i)$ to be 1 *or* 0 *depending on whether $E(i)$ is odd or even, so that Y in* (d) *obviously consists of* -1 *and all odd values of E*:

$$Y = \{-1\} \cup (\mathbb{Z}_{\mathrm{odd}} \cap E(\mathcal{V})), \quad \textit{where } \mathbb{Z}_{\mathrm{odd}} = \mathbb{Z} \setminus 2\mathbb{Z}. \tag{2.3}$$

Also, Y is the intersection of $\mathbb{Z}_{\mathrm{odd}}$ with one or a union of three closed intervals:

$$\begin{aligned} &Y = \mathbb{Z}_{\mathrm{odd}} \cap [-2r+3, 2r-3] \quad \textit{for even } r, \\ &Y = \mathbb{Z}_{\mathrm{odd}} \cap ([-3r+4, -2r-1] \cup [-r, r] \cup [2r+1, 3r-4]) \quad \textit{if } r \textit{ is odd}. \end{aligned} \tag{2.4}$$

Proof. Once we establish (a)–(c) for E, the properties (b) and (c) for J will follow: as $E(i), E(i')$ in (a)–(c) have different parities, $J(i) \neq J(i')$ in $\{0, 1\}$.

Since $k = 2r - 1$, (a) and (c) for E are immediate, with $(i, i') = (2j - 1, 2j)$. To verify (b) for E we display the definition of E in the matrix form

$$\begin{bmatrix} 1 & 2 \\ 2j-1 & 2j \\ 2j-1 & 2j \\ 2r-3 & 2r-2 \\ 2j'-1 & 2j' \\ 2j'-1 & 2j' \\ 2m-1 & 2m \end{bmatrix} \mapsto \begin{bmatrix} E(1) & E(2) \\ E(2j-1) & E(2j) \\ E(2j-1) & E(2j) \\ E(2r-3) & E(2r-2) \\ E(2j'-1) & E(2j') \\ E(2j'-1) & E(2j') \\ E(2m-1) & E(2m) \end{bmatrix} = \begin{bmatrix} r & -r+1 \\ j-1 & -2r+j \\ 2r+j-2 & j-1 \\ r-1 & -r \\ j'-2r+2 & j'-4r+3 \\ j'+1 & j'-2r+2 \\ r-2 & -r-1 \end{bmatrix}$$

where rows 3 and 5 (or 2 and 6) are to be ignored if r is even (or odd), while the ranges of j and j' are $1 < j < r-1$ and $r - 1 < j' < m$.

In the first matrix above two entries have the sum $2m+1 = 4r-5$ if and only if they lie symmetrically about the center of the matrix rectangle, with $j + j' = 2(r - 1)$ (that is, with j and j' lying symmetrically about $r - 1$). The same pairs of entries in the third matrix above have the sum -1, proving (b) for E. Next,

$$\text{the range } E(\mathcal{V}) \text{ contains } \{1, \dots, r\}, \tag{2.5}$$

as the values $r - 2, r - 1, 1$ appear in the first column of the third matrix, and $j - 1$ for $j = 2, \dots, r - 2$ appears in row 2 or 3, depending on the parity of r. Also,

$$E(\mathcal{V}) \text{ includes the } r-3 \text{ values } \begin{cases} r+1, \dots, 2r-3 & \text{if } r \text{ is even}, \\ 2r, 2r+1, \dots, 3r-4 & \text{if } r \text{ is odd}. \end{cases} \tag{2.6}$$

Namely, for even m we get $j' + 1$ (row 6, with $j' = r, \dots, m - 1 = 2r - 4$), and if r is odd then row 3 provides $2r + j - 2$ with $j = 2, \dots, r - 2$. In addition, by (b),

$$E(\mathcal{V}) \text{ is closed under the reflection } i \mapsto -i-1 \text{ about } -1/2. \tag{2.7}$$

Due to (2.5) and (2.6), $E(\mathcal{V})$ contains at least $m = 2r - 3$ positive integers, and by (2.7) at least as many negative ones. Since $\mathcal{V}$ has the cardinality $2m$, injectivity of E follows, and 'at least' in the last sentence amounts to *exactly*. Thus by (2.5)–(2.7) the set $E(\mathcal{V})$ is the intersection of $\mathbb{Z}$ with the union of two or four closed intervals, namely $[-2r+2, -2] \cup [1, 2r-3]$ for even r, and $[-3r+3, -2r-1] \cup [-r-1, -2] \cup [1, r] \cup [2r, 3r-4]$ for odd r. In particular, $-1 \notin E(\mathcal{V})$. Finally, (2.4) is a trivial consequence of this last description of $E(\mathcal{V})$ and (2.3), and it clearly implies symmetry of Y about 0. □

In every $\mathbb{Z}$-spectral system (m, k, E, J), *the integer m must be odd.* Namely, since S has the simultaneous-selector property — see the line preceding (2.1) — if $i \in S$ is odd (or even), then $2m + 1 - i \in \mathcal{V} \setminus S$ will be even (or odd), and so $2m - i \in S$ (or, respectively, $2m + 2 - i \in S$). In other words, if $i \in S$ then $\{i, i'\} \subseteq S$ for the unique $i' \in \mathcal{V}$ with $i \equiv i' \pmod 2$ and $i + i' = 2m + 1 + (-1)^i$. The resulting sets $\{i, i'\}$ form a partition of S, and clearly $i' \neq i$ unless m is odd (with $i' = i$ equal to m or $m + 1$). If m were even, the m-element set S would thus be partitioned into our $m/2$ disjoint 2-element sets $\{i, i'\}$. Oddness of k, due to (a), and (b)–(c) would now give $E(i) \equiv E(i') \pmod 2$ on each such $\{i, i'\}$, making $\sum_{i \in S} E(i)$ even. This contradicts the equality $\sum_{i \in S} E(i) = 1$, which is immediate, since $E : \mathcal{V} \to \mathbb{Z} \setminus \{-1\}$ is injective and $Y = \{-1\} \cup E(S)$ is symmetric about 0, cf. (d) and (2.2).

3 Standard dilational models

We define a *standard dilational ECS model* to be an n-dimensional pseudo-Riemannian manifold

$$(\widehat{M},\widehat{g}) = ((0,\infty)\times\mathbb{R}\times V, \kappa dt^2 + dt ds + \langle\,\cdot\,,\,\cdot\,\rangle), \tag{3.1}$$

built from the data q, n, V, $\langle\,\cdot\,,\,\cdot\,\rangle$, A, C, f consisting of a real number $q\in(0,\infty)\setminus\{1\}$ with $q+q^{-1}\in\mathbb{Z}$, an integer $n\geq 4$, a real vector space V of dimension $n-2$, a pseudo-Euclidean inner product $\langle\cdot,\cdot\rangle$ on V, a nonzero traceless $\langle\,\cdot\,,\,\cdot\,\rangle$-self-adjoint linear operator $A: V\to V$, a linear $\langle\,\cdot\,,\,\cdot\,\rangle$-isometry $C: V\to V$, and a nonconstant C^∞ function $f:(0,\infty)\to\mathbb{R}$, satisfying the conditions

$$\text{a)}\quad CAC^{-1} = q^2 A,\qquad \text{b)}\quad f(t) = q^2 f(qt)\qquad \text{for all } t\in(0,\infty). \tag{3.2}$$

In (3.1) we identify dt, ds and the flat metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on V with their pullbacks to $\widehat{M}$, the function $\kappa:\widehat{M}\to\mathbb{R}$ is defined by $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av,v\rangle$, and (t,s) are the Cartesian coordinates on $(0,\infty)\times\mathbb{R}$.

It is well known, see [5, Theorem 4.1] and [11, Section 1], that (3.1) is an ECS manifold, having rank one if $\operatorname{rank} A>1$, and rank two when $\operatorname{rank} A = 1$.

The following text leading up to Formula (3.6) repeats, almost verbatim, some material from [10, Section 6], albeit in a special case characterized by (3.5); [10, Section 6] also serves as a reference for it, and (3.1) stands, in the rest of this section, for the standard dilational model associated with fixed data q, n, V, $\langle\,\cdot\,,\,\cdot\,\rangle$, A, C, f.

We denote by $\mathcal{W}$ and $\mathcal{E}$ the vector spaces of dimensions 2 and $2(n-2)$ that consist of all C^2 functions $y:(0,\infty)\to\mathbb{R}$, or $u:(0,\infty)\to V$, such that

$$\text{i) } \ddot{y} = fy\quad\text{or}\quad \text{ii) } \ddot{u} = fu + Au,\quad \text{respectively, where the dot stands for } d/dt. \tag{3.3}$$

Let the operator T act on functions $(0,\infty)\ni t\mapsto u(t)$, valued anywhere, by

$$[Tu](t) = u(t/q),\quad \text{so that (3.2-b) reads } Tf = q^2 f. \tag{3.4}$$

Thus, T obviously preserves $\mathcal{W}$. We now *impose on f an additional requirement*:

$$\begin{aligned}&T:\mathcal{W}\to\mathcal{W} \text{ has two distinct eigenvalues } \mu^\pm\in(0,\infty) \text{ with positive}\\ &\text{eigenfunctions } y^+, y^-\in\mathcal{W}, \text{ so that } Ty^\pm = \mu^\pm y^\pm \text{ and } \mu^+\mu^- = q^{-1},\end{aligned} \tag{3.5}$$

the last equality ($\det T = q^{-1}$ in $\mathcal{W}$) being immediate since the formula $\alpha(y^+,y^-) = \dot{y}^+y^- - y^+\dot{y}^-$ (a constant!) defines an area form α on $\mathcal{W}$ and $qT^*\alpha = \alpha$. The space $\mathcal{E}$ is not, in general, preserved either by T or by C acting valuewise via $u\mapsto Cu$, but the composition $CT = TC$ clearly leaves $\mathcal{E}$ invariant, leading to

$$\text{the operator } CT:\mathcal{E}\to\mathcal{E} \text{ given by } [(CT)u](t) = Cu(t/q). \tag{3.6}$$

Next, given $(\hat{r},\hat{u}), (r,u)\in\mathbb{R}\times\mathcal{E}$, we define mappings $\widehat{\gamma},\gamma:\widehat{M}\to\widehat{M}$ by

$$\begin{aligned}\widehat{\gamma}(t,s,v) &= (qt, -\langle\hat{w}(qt), 2Cv + \hat{u}(qt)\rangle + \hat{r} + s/q, Cv + \hat{u}(qt)),\\ \gamma(t,s,v) &= (t, -\langle\dot{u}(t), 2v + u(t)\rangle + r + s, v + u(t))\end{aligned} \tag{3.7}$$

where $\hat{w} = d\hat{u}/dt$. Both $\widehat{\gamma},\gamma$ lie in the isometry group $\operatorname{Iso}(\widehat{M},\widehat{g})$, see [10, Formula (4.7)].

We choose to treat $(\hat{r},\hat{u})\in\mathbb{R}\times\mathcal{E}$ as fixed, and allow (r,u) to range over $\mathbb{R}\times\mathcal{E}$. The set of all γ arising via (3.7) from all $(r,u)\in\mathbb{R}\times\mathcal{E}$ forms a normal subgroup H of $\operatorname{Iso}(\widehat{M},\widehat{g})$, see [10, Formula (4.8)], and, as explained below,

$$\begin{aligned}&\text{i)}\quad (r,u)(r',u') = (\Omega(u',u) + r + r', u + u'),\\ &\text{ii)}\quad \Pi(r,u) = (2\Omega(CTu,\hat{u}) + r/q, CTu), \text{ where}\\ &\text{iii)}\quad \Omega:\mathcal{E}\times\mathcal{E}\to\mathbb{R} \text{ is the symplectic form given by } \Omega(u_1,u_2) = \langle\dot{u}_1,u_2\rangle - \langle u_1,\dot{u}_2\rangle, \text{ and}\\ &\text{iv)}\quad (CT)^*\Omega = q^{-1}\Omega \text{ for the operator } CT:\mathcal{E}\to\mathcal{E} \text{ in (3.6).}\end{aligned} \tag{3.8}$$

Here (3.8-i) describes the group operation of H under the obvious identification $\mathrm{H} = \mathbb{R}\times\mathcal{E}$, cf. [10, (a) in Section 4], the linear operator $\Pi : \mathbb{R}\times\mathcal{E} \to \mathbb{R}\times\mathcal{E}$ in (3.8-ii) equals $\mathrm{H} \ni \gamma \mapsto \widehat{\gamma}\gamma\widehat{\gamma}^{-1} \in \mathrm{H}$, the conjugation by $\widehat{\gamma}$, cf. [10, Remark 4.2], (3.8-iii) is immediate as self-adjointness of A and (3.3-ii) imply constancy of $\Omega(u_1, u_2)$, and (3.8-iv) is a consequence of (3.6).

Consider the following conditions for two objects $\mathcal{L}$ and Σ, with Π as in (3.8-ii) for our fixed $(\hat{r}, \hat{u}) \in \mathbb{R}\times\mathcal{E}$.

$$
\begin{array}{ll}
\text{(A)} & \mathcal{L} \subseteq \mathcal{E} \text{ is a vector subspace of dimension } n-2.\\
\text{(B)} & CT \text{ in (3.6) leaves } \mathcal{L} \text{ invariant.}\\
\text{(C)} & \Sigma \text{ is a (full) lattice in } \mathbb{R}\times\mathcal{L} \text{ and } \Pi(\Sigma) = \Sigma.\\
\text{(D)} & \Omega(u, u') = 0 \text{ whenever } u, u' \in \mathcal{L}, \text{ with } \Omega \text{ as in (3.8-iii).}\\
\text{(E)} & u \mapsto u(t) \text{ is an isomorphism } \mathcal{L} \to V \text{ for every } t \in (0,\infty).
\end{array}
\tag{3.9}
$$

Our choice of symbols has obvious reasons: H is a Heisenberg group, and $\mathcal{L}$ is a Lagrangian subspace of $\mathcal{E}$.

The following remark and lemma use the hypotheses preceding (3.9).

Remark 3.1. As an obvious consequence of (3.8-i), whenever a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ satisfies (3.9-D), then $\mathbb{R}\times\mathcal{L}$ is an Abelian subgroup of $\mathrm{H} = \mathbb{R}\times\mathcal{E} \subseteq \mathrm{Iso}(\widehat{M}, \widehat{g})$ and the group operation in $\mathbb{R}\times\mathcal{L}$ coincides with the vector space addition.

Lemma 3.2. *Condition* (3.9-E) *for a vector subspace* $\mathcal{L} \subseteq \mathcal{E}$ *implies that the assignment* $(t, z, u) \mapsto (t, s, v) = (t, z - \langle \dot{u}(t), u(t)\rangle, u(t))$ *is an* H*-equivariant diffeomorphism* $(0,\infty)\times\mathbb{R}\times\mathcal{L} \to \widehat{M}$.

Proof. This is a special case of [10, Remark 9.1]. □

Remark 3.3. As pointed out in [8, the lines following Formula (7.2)], the coordinate vector field $\partial/\partial s$ in (3.1) is null and parallel. Thus $\partial/\partial s$ spans a one-dimensional null parallel distribution $\mathcal{P}$, contained by [11, Section 1] in the Olszak distribution $\mathcal{D}$, while $\nabla dt = 0$ since $dt = 2g(\partial/\partial s, \cdot)$. The mappings (3.7) multiply t and its gradient $2\partial/\partial s$ by constants, and so $\mathcal{P}$ gives rise to distributions, also denoted by $\mathcal{P}$, on the compact quotients constructed in Sections 5 and 6.

Remark 3.4. A standard dilational model manifold (see Section 3) is never geodesically complete. Namely, we have $\nabla dt = 0$ in Remark 3.3. Thus t restricted to any geodesic is an affine function of its parameter, and so t itself serves as such parameter for a geodesic $t \mapsto x(t)$ through any point x with an initial velocity v at x having $d_v t = 1$. Our claim follows since t ranges over $(0,\infty)$.

4 From $\mathbb{Z}$-spectral systems to conditions (3.9)

Suppose that (m, k, E, J) is a $\mathbb{Z}$-spectral system (Section 2) and that $q \in (0,\infty)\setminus\{1\}$ with $q + q^{-1} \in \mathbb{Z}$, while a C^∞ function $f : (0,\infty) \to \mathbb{R}$ satisfies both (3.2-b) and (3.5) with $\mu^\pm = q^{(-1\pm k)/2}$. We set $n = m + 2$, choose a semi-neutral inner product $\langle\cdot,\cdot\rangle$ on an m-dimensional real vector space V (see Remark 1.2), a basis $e_1, \dots, e_m$ of V satisfying (1.1), and define $A, C : V \to V$ by (1.3) for $a(1), \dots, a(m)$ with

$$a(j) = E(2j-1) + (1-k)/2, \qquad \text{that is, } a(j) = E(2j) + (1+k)/2, \tag{4.1}$$

the equivalence of both descriptions, and (1.2), being due to (a)–(c) in Section 2, which also easily imply that, for our $\mu^\pm = q^{(-1\pm k)/2}$,

$$(\mu^+ q^{a(1)}, \mu^- q^{a(1)}, \dots, \mu^+ q^{a(m)}, \mu^- q^{a(m)}) = (q^{E(1)}, \dots, q^{E(2m)}). \tag{4.2}$$

According to Remark 1.2, these data $q, n, V, \langle\cdot,\cdot\rangle, A, C, f$ have all the properties preceding (3.2). Thus, they lead to a standard dilational model $(\widehat{M}, \widehat{g})$ with (3.1).

Lemma 4.1. *The assumptions just listed have the following consequences.*

(a) *Some ordered basis* $(u_1^+, u_1^-, \dots, u_m^+, u_m^-) = (u_1, \dots, u_{2m})$ *of* $\mathcal{E}$ *consists of eigenvectors of* $CT : \mathcal{E} \to \mathcal{E}$, *cf.* (3.6), *and the respective eigenvalues, equal to* $q^{E(1)}, \dots, q^{E(2m)}$, *are pairwise distinct. With* $y^\pm$ *as in* (3.5) *and suitable functions* $z^{\pm} : (0, \infty) \to \mathbb{R}$, *this basis may be obtained by setting* $u_i^{\pm} = y^{\pm} e_i$ *if* $i < m$ *and* $u_m^{\pm} = y^{\pm} e_m + z^{\pm} e_1$.

(b) $\Omega(u_i, u_j) = 0$ *whenever* $i, j \in \{1, \ldots, 2m\}$ *and* $i + j \neq 2m + 1$, *the basis* $(u_1, \ldots, u_{2m})$ *and* Ω *being as in* (a) *and* (3.8-iii).

Proof. Due to (3.5) and (1.3), the $u_i^{\pm}$ defined in (a) have $CTu_i^{\pm} = \mu^{\pm} q^{a(i)} u_i^{\pm}$ if $i < m$, that is, by (4.2), $CTu_j = q^{E(j)} u_j$ whenever $j \in \{1, \ldots, 2m-2\}$. That $q^{E(1)}, \ldots, q^{E(2m)}$ are distinct follows as $E : \mathcal{V} \to \mathbb{Z} \setminus \{-1\}$ is injective (Section 2).

Given two functions $x^{\pm} : (0, \infty) \to \mathbb{R}$ with $\ddot{x}^{\pm} = f x^{\pm} + y^{\pm}$, the equation (3.3-i) for $y = y^{\pm}$ and (1.3) yield (3.3-ii) for $u^{\pm} = y^{\pm} e_m + x^{\pm} e_1$, so that $u^{\pm} \in \mathcal{E}$ and, again by (3.5), (1.3) and (4.2), $w^{+} = [CT - q^{E(2m-1)}] u^{+}$ and $w^{-} = [CT - q^{E(2m)}] u^{-}$ both lie in the subspace $\mathcal{Z}$ of $\mathcal{E}$ spanned by the eigenvectors u_1, u_2 for the eigenvalues $q^{E(1)}, q^{E(2)}$. (The scalars $q^{E(2m-1)}, q^{E(2m)}$ stand for the corresponding multiples of identity.) Distinctness of the eigenvalues $q^{E(1)}, \ldots, q^{E(2m)}$ implies that $CT - q^{E(2m-1)}$ and $CT - q^{E(2m)}$ map $\mathcal{Z}$ isomorphically onto itself. We may now choose $z^{\pm}$ to be the function such that $CT - q^{E(2m-(1\pm1)/2)}$ sends $(x^{\pm} - z^{\pm}) e_1$ onto $w^{\pm}$, and (a) follows, with linear independence of $u_1, \ldots, u_{2m}$ due to Remark 1.3.

Next, by (3.8-iv) and (a) we have $q^{-1} \Omega(u_i, u_j) = \Omega(CTu_i, CTu_j) = q^{E(i)+E(j)} \Omega(u_i, u_j)$, which, in view of the injectivity of E and (b) in Section 2, yields (b). □

We also fix a pair $(\hat{r}, \hat{u}) \in \mathbb{R} \times \mathcal{E}$, as in the lines following (3.7), and denote by Π the resulting linear operator $\mathbb{R} \times \mathcal{E} \to \mathbb{R} \times \mathcal{E}$ of conjugation by $\widehat{\gamma}$, in (3.8-ii).

Lemma 4.2. *With the data* $q, n, V, \langle \cdot, \cdot \rangle, A, C, f$ *and* $(\hat{r}, \hat{u}), \Pi$ *chosen as above, the conditions* (3.9) *hold for suitable* $\mathcal{L}$ *and* Σ.

Proof. Let $\mathcal{L}$ be the span of $\{u_i : i \in S\}$ for the basis $(u_1^{+}, u_1^{-}, \ldots, u_m^{+}, u_m^{-}) = (u_1, \ldots, u_{2m})$ of $\mathcal{E}$ appearing in Lemma 4.1(a) and the set S associated with our $\mathbb{Z}$-spectral system (m, k, E, J) (see Section 2), that is, $S = \{i \in \mathcal{V} : J(i) = 1\}$. Now (3.9-A) and (3.9-B) follow since $m = n - 2$. As S is a selector for the second family of (2.1), the basis $\{u_i : i \in S\}$ of $\mathcal{L}$ has the form

$$(u_1^{\varepsilon(1)}, \ldots, u_m^{\varepsilon(m)}) \text{ with some signs } \varepsilon(1), \ldots, \varepsilon(m). \tag{4.3}$$

For each fixed $t \in (0, \infty)$, the operator $\mathcal{E} \ni u \mapsto u(t) \in V$ sends $u_i^{\pm}$ to $y^{\pm}(t) e_i$ if $i < m$ and $u_m^{\pm}$ to $y^{\pm}(t) e_m + z^{\pm}(t) e_1$, so that, restricted to $\mathcal{L}$, it is represented in the bases (4.3) and $e_1, \ldots, e_m$ by an upper triangular matrix with all diagonal entries positive in view of (3.5), which proves (3.9-E). Simultaneously, S is a selector for the first family in (2.1), so that $i + j \neq 2m + 1$ if $u_i, u_j \in \mathcal{L}$. Combined with Lemma 4.1(b), this yields (3.9-D). Finally, the existence of a lattice Σ required in (3.9-C) is immediate from Remarks 2.1 and 1.4. □

An example of a C^{∞} function $f : (0, \infty) \to \mathbb{R}$ having both (3.2-b) and (3.5) for $\mu^{\pm} = q^{(-1\pm k)/2}$, as required at the beginning of this section, is provided by

$$f(t) = \frac{k^2 - 1}{4t^2}, \quad \text{with } y^{\pm}(t) = t^{(1\mp k)/2} \text{ in (3.5).} \tag{4.4}$$

For the resulting standard dilational model $(\widehat{M}, \widehat{g})$, cf. the lines following (4.2),

$$(\widehat{M}, \widehat{g}) \text{ is locally homogeneous.} \tag{4.5}$$

Namely, by equation (4.4), the expression (3.1) for g amounts to that for the metric g^P in [2, top of page 170], our coordinate t being denoted there by u^1. Our Remark 1.2 now clearly implies Formula (10) in [2, page 172] which, as stated there, guarantees homogeneity of the metric g^P on $(0, \infty) \times \mathbb{R} \times V$, with $V = \mathbb{R}^{n-2}$.

5 From conditions (3.9) to compact quotients

We now show that conditions (3.9) are sufficient for a standard dilational model to admit compact isometric quotients. Specifically, let $(m, k, E, J), q, f$, along with $n = m+2$ and $V, \langle \cdot, \cdot \rangle, e_1, \ldots, e_m, A, C$, have the properties listed at the beginning of Section 4, so that the data $q, n, V, \langle\cdot,\cdot\rangle, A, C, f$ give rise to a standard dilational model $(\widehat{M}, \widehat{g})$ with (3.1). We denote by $\mathcal{P}$ the one-dimensional null parallel distribution on $(\widehat{M}, \widehat{g})$ defined in Remark 3.3.

Theorem 5.1. *Under these assumptions, we also fix a pair $(\hat{r}, \hat{u}) \in \mathbb{R} \times \mathcal{E}$, cf. the lines following* (3.7)*, and define Π by* (3.8-ii)*. If $\mathcal{L}$ and Σ are any objects satisfying* (3.9)*, then*

$$\text{the group } \Gamma \subseteq \mathrm{Iso}(\widehat{M}, \widehat{g}) \text{ generated by } \widehat{\gamma} \text{ appearing in (3.7) and } \Sigma \tag{5.1}$$

acts on $\widehat{M}$ freely and properly discontinuously with a compact quotient manifold $M = \widehat{M}/\Gamma$. In addition, M is the total space of a torus bundle over the circle, with the leaves of $\mathcal{P}^{\perp}$ serving as the fibres, and its fundamental group Γ has no Abelian subgroup of finite index, so M cannot be diffeomorphic to a torus, nor be covered by a torus.

Proof. By Lemma 3.2, Remark 3.1 and (3.9-C), the action of $\Sigma \subseteq \mathrm{H}$ on each t-level $\{t\} \times \mathbb{R} \times V$ is equivariantly

(i) identified with the additive action of the lattice Σ on $\mathbb{R} \times \mathcal{L}$.

Since Π acts on Σ via conjugation by $\widehat{\gamma}$, cf. the lines following (3.8),

(ii) Σ is an Abelian normal subgroup of Γ,

where we again used Lemma 3.2, Remark 3.1 and (3.9-C). Thus, any element of Γ, being a finite product of factors from the set $\Sigma \cup \{\widehat{\gamma}, \widehat{\gamma}^{-1}\}$, equals $\widehat{\gamma}^r \gamma$ (written multiplicatively) for some $r \in \mathbb{Z}$ and $\gamma \in \Sigma$. For $(t, s, v) \in \widehat{M}$ we obtain from (3.7) that

(iii) $(\widehat{\gamma}^r \gamma)(t, s, v) = (q^r t, s', v')$ for some s', v', which also leads to

(iv) the homomorphism $\Gamma \ni \widehat{\gamma}^r \gamma \mapsto r \in \mathbb{Z}$,

and so Γ acts freely on $\widehat{M}$: if $\widehat{\gamma}^r \gamma$ has a fixed point (t, s, v) then (iii) gives $q^r t = t$. Therefore $r = 0$, and γ, having a fixed point, must equal the identity since by (i) the action of Σ on $\widehat{M}$ is free.

Consider now sequences with the terms $(r, \gamma) \in \mathbb{Z} \times \Sigma$ and $x = (t, s, v) \in \widehat{M}$ such that x and $\widehat{\gamma}^r(\gamma(x))$ both converge. Thus, (iii) implies convergence of the sequence r (and hence its ultimate constancy). For the sequences $\gamma' = \widehat{\gamma}^r \gamma \widehat{\gamma}^{-r}$ in Σ and $x' = \widehat{\gamma}^r(x) \in \widehat{M}$, with this "ultimate constant" r, writing $\gamma' = (r, u)$ and $x' = (t', s', v')$, we obtain convergence of both $\gamma'(x') = \widehat{\gamma}^r(\gamma(x))$ and x', so that (i) implies eventual constancy of γ', and consequently that of $\widehat{\gamma}^r \gamma \in \Gamma$.

The implication established in the last paragraph proves proper discontinuity of the action of Γ on $\widehat{M}$; see [17, Exercise 12-19 on page 337].

Next, $\widehat{M}$ has a compact subset K intersecting every orbit of Γ, which yields compactness of the quotient manifold $M = \widehat{M}/\Gamma$. In fact, we may choose K to be the image, under the H-equivariant diffeomorphism in (a), of $J \times K'$, where $J \subseteq (0, \infty)$ is the closed interval with the endpoints 1, q and K' is a compact set in $\mathbb{R} \times \mathcal{L}$ which intersects all orbits of the lattice Σ acting on $\mathbb{R} \times \mathcal{L}$ by vector space translations. We now modify any $(t, s, v) \in \widehat{M}$ by applying to it elements of Γ twice in a row so as to end up with a point of K. First, $\widehat{\gamma}^r(t, s, v) = (q^r t, s', v')$, cf. (iii), has $q^r t \in J$ for a suitable $r \in \mathbb{Z}$ (as the sum of $\log t$ and some multiple of $\log q$ lies between $\log q$ and 0). We may thus assume that $t \in J$. With this fixed t, (i) allows us to choose $\gamma' \in \Sigma$ sending (t, s, v) into K.

The surjective submersion $\widehat{M} \ni (t, s, v) \mapsto (\log t)/(\log q) \in \mathbb{R}$, being clearly equivariant relative to the homomorphism (iv) along with the obvious actions of Γ on $\widehat{M}$ via (iii) and of $\mathbb{Z}$ on $\mathbb{R}$, descends to a surjective submersion $M \to S^1$ which, due to the compact case of Ehresmann's fibration theorem [14, Corollary 8.5.13], is a bundle projection. This leads, via (i), to the required conclusion about a torus bundle over the circle. The claim about the fibres follows: the leaves of $\mathcal{P}^{\perp}$ in $\widehat{M}$ are the levels of t, since $\mathcal{P}$ is spanned by the parallel gradient of t according to Remark 3.3.

Finally, a finite-index subgroup Γ' of Γ would have a nontrivial image under the homomorphism (iv) (the kernel of which, Σ, has an infinite index in Γ, and hence cannot contain Γ'), and $\Gamma' \cap \Sigma$ would clearly be a finite-index subgroup of the lattice Σ spanning, consequently, the whole space $\mathbb{R} \times \mathcal{L}$. The conjugation by any $\gamma' \in \Gamma' \setminus \Sigma$ would thus lead to the operator (3.8-ii) equal to the identity on $\mathbb{R} \times \mathcal{L}$, and yet having the q-component different from 1. This contradiction proves the final clause of the theorem. □

6 The locally homogeneous case

Constructing compact rank-one ECS manifolds of dimension n via Theorem 5.1 is clearly reduced to finding two objects: a $\mathbb{Z}$-spectral system (m, k, E, J) for $m = n - 2$, and a C^∞ function $f : (0, \infty) \to \mathbb{R}$ with (3.2-b) and (3.5), for $q \in (0, \infty) \setminus \{1\}$ such that $q + q^{-1} \in \mathbb{Z}$ and $\mu^\pm = q^{(-1\pm k)/2}$. One now gets the former from Theorem 2.2, as long as $n \geq 5$ is odd, while an example of the latter is then provided by Formula (4.4).

The resulting existence theorem may be phrased as follows.

Theorem 6.1. *Let $n \geq 5$ be odd. Applying Theorem* (5.1) *to data that include (m, k, E, J) of Theorem* (2.2) *where $m = n - 2$ and f is given by* (4.4)*, we obtain the group Γ in* (5.1) *acting on $\widehat{M}$ freely and properly discontinuously with a locally homogeneous and geodesically incomplete compact quotient rank-two ECS manifold $M = \widehat{M}/\Gamma$ of dimension n, forming the total space of a nontrivial torus bundle over the circle, with the fibres provided by the leaves of $\mathcal{P}^\perp$, while its fundamental group Γ has no finite-index Abelian subgroup.*

In fact, for local homogeneity and incompleteness see (4.5) and Remark 3.4.

Appendix A: Special spectra realized in function spaces

We fix a real number $q \in (0, \infty) \setminus \{1\}$. For a continuous function $f : (0, \infty) \to \mathbb{R}$ satisfying Condition (3.2-b), recall from Section 3 that the two-dimensional space $\mathcal{W}$ of C^2 solutions $y : (0, \infty) \to \mathbb{R}$ to the second-order ordinary differential Equation (3.3-i) is obviously invariant under the *translation operator T* given by

$$[Ty](t) = y(t/q), \quad \text{with } \det T = q^{-1} \text{ in } \mathcal{W}, \tag{A.1}$$

where $\det T = q^{-1}$ as in the line following (3.5). Clearly, (3.2-b) amounts to periodicity, with the period $\log q$, of the function $\mathbb{R} \ni \tau \mapsto e^{2\tau} f(e^\tau)$. Therefore,

$$\begin{aligned}&\text{both the vector space } \mathcal{F} \text{ of continuous functions } f \text{ satisfying (3.2-b)}\\ &\text{and its subspace } \mathcal{F}_0 = \{f \in \mathcal{F} : f(1) = 0\} \text{ are infinite-dimensional.}\end{aligned} \tag{A.2}$$

We will need such f with T having, for some $c \in (0, \infty)$, the spectrum

$$q^{\pm c - 1/2}, \quad \text{that is, positive real eigenvalues and} \quad \operatorname{tr} T = 2q^{-1/2}\cosh(c \log q). \tag{A.3}$$

Examples of real-analytic functions $f \in \mathcal{F}$ with (A.3) are provided by

$$f_c(t) = (c^2 - 1/4)t^{-2}, \quad \text{where } c \in (0, \infty). \tag{A.4}$$

In fact, an obvious basis of $\mathcal{W}$ for $f = f_c$ consists of $y = y_c^\pm$ given by

$$y_c^\pm(t) = t^{\mp c + 1/2}, \quad \text{so that } Ty_c^\pm = q^{\pm c - 1/2} y_c^\pm. \tag{A.5}$$

Theorem A.1. *For any fixed $q \in (0, \infty) \setminus \{1\}$ and $c \in (0, \infty)$ there exists an infinite-dimensional manifold of smooth functions $f : (0, \infty) \to \mathbb{R}$ with* (3.2-b) *such that the corresponding translation operator $T : \mathcal{W} \to \mathcal{W}$ has the eigenvalues $q^{\pm c + 1/2}$, and some basis of $\mathcal{W}$ diagonalizing T consists of positive functions. The same remains true if one replaces 'smooth' by real-analytic.*

More precisely, for any $f_* \in \mathcal{F}_0$ as in (A.2) *sufficiently C^0-close to* 0*, there exists a unique a close to c in $\mathbb{R}$ such that $f = f_* + f_a$ realizes the T-spectrum $\{q^{c-1/2}, q^{-c-1/2}\}$, while the resulting assigment $f_* \mapsto f$ is smooth and injective.*

Proof. Define a mapping $H : \mathcal{F}_0 \times (0, \infty) \to \mathbb{R}$ by $H(f_*, a) = \operatorname{tr} T$ for T arising from $f = f_* + f_a$. Smoothness of H follows since

$$H(f_*, a) = y^+(1/q) + q^{-1}\dot{y}^-(1/q) \tag{A.6}$$

where y^+, y^- are solutions to (3.3) with the initial conditions $(y^+(1), \dot{y}^+(1)) = (1, 0)$ and $(y^-(1), \dot{y}^-(1)) = (0, 1)$. To verify (A.6) note that any $y \in \mathcal{W}$ equals $y(1)y^+ + \dot{y}(1)y^-$. For Ty rather than y this reads, by (A.1), $Ty = y(1/q)y^+ + q^{-1}\dot{y}(1/q)y^-$ which, applied to $y = y^+$ and $y = y^-$, gives

$$(Ty^+, Ty^-) = (y^+(1/q)y^+ + q^{-1}\dot{y}^+(1/q)y^-, y^-(1/q)y^+ + q^{-1}\dot{y}^-(1/q)y^-),$$

showing that the matrix of T in the basis y^+, y^- of $\mathcal{W}$ has the trace claimed in (A.6). Also, as each f_c leads to the spectrum (A.3), $H(0, a) = 2q^{-1/2}\cosh(a \log q)$ for all $a > 0$, including $a = c$. Since $d[H(0, a)]/da \neq 0$ at $a = c$, the implicit mapping theorem [16, page 18] provides neighborhoods of 0 in $\mathcal{F}$ and c in $\mathbb{R}$ with the required smooth mapping $f_* \mapsto a$ sending 0 to c and having $H(f_*, a) = 2q^{-1/2}\cosh(c \log q)$. Injectivity of $f_* \mapsto f_* + f_a$ follows: $f(1) = f_a(1) = a^2 - 1/4$ uniquely determines $a > 0$, and hence f_a and f_* as well.

Finally, positivity of the functions (A.5) on the closed interval with the endpoints $1, q$ yields the same for functions C^0-close to them that diagonalize T for f close to f_c. Being eigenvectors of the translation operator T, they thus remain positive throughout $(0, \infty)$. □

Appendix B: Rank-two ECS manifolds of dilational type

The distribution $\mathcal{P}$ (see Remark 3.3) on every compact rank-two ECS manifold (M, g) arising in Theorem 5.1 is a real line bundle over M with a linear connection induced by the Levi–Civita connection of g. Due to its obvious flatness, the latter connection has a countable holonomy group contained in $\mathbb{R} \setminus \{0\}$.

All our examples (M, g) are *dilational* in the sense that this holonomy group is infinite, which follows since the group Γ in (5.1) contains the element $\widehat{\gamma}$ defined by (3.7) with $q \in (0, \infty) \setminus \{1\}$.

Theorem 6.1 now obviously remains valid if one replaces 'is given by (4.4)' with *arises in Theorem* (A.1) *for* $c = k/2$, and 'locally homogeneous' with *dilational*:

Theorem B.1. *Let* $n \geq 5$ *be odd. Applying Theorem* 5.1 *to* (m, k, E, J) *of Theorem* 2.2 *where* $m = n - 2$ *and* f *arises in Theorem* A.1 *for* $c = k/2$, *we obtain the group* Γ *in* (5.1) *acting on* $\widehat{M}$ *freely and properly discontinuously with a dilational and geodesically incomplete compact quotient rank-two ECS manifold* $M = \widehat{M}/\Gamma$ *of dimension n, forming the total space of a nontrivial torus bundle over the circle, the fibres of which are the leaves of* $\mathcal{P}^\perp$, *and the fundamental group* Γ *of* M *has no finite-index Abelian subgroup.*

Geodesic incompleteness is immediate here by Remark 3.4. Most examples resulting from Theorem B.1 have the dilational property without local homogeneity, which is guaranteed by the infinite-dimensional freedom of choosing f in Theorem A.1: in the locally homogeneous case $|f(t)|^{-1/2}$ must be — according to [9, Formula (3.3)] — an affine function of t, which restricts it to a finite-dimensional moduli space.

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