TIPS ON SIMPLIFYING TRIGONOMETRIC EXPRESSIONS & SOME EXAMPLES

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The most important thing to remember when solving problems involving the simplification of trigonometric functions is the **fundamental identity**:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

This is nothing more than a Pythagorean identity in disguise (think back to the usual triangle inside the unit circle seen in class: the hypothenuse is 1 — as it is a radius of the circle — while the legs measure $\sin\theta$ and $\cos\theta$).

Two more or less important consequences:

- Divide everything by $\sin^2\theta$ to obtain $\boxed{1 + \cot^2\theta = \csc^2\theta}$.
- Divide everything by $\cos^2\theta$ to obtain $\boxed{\tan^2\theta + 1 = \sec^2\theta}$.

Remark. I **do not** expect you to memorize the formulas in blue boxes. It seems too likely to just mix up sec and csc, and tan and cot. I **do** expect you to know by heart the fundamental relation between sin and cos (in the red box), and to derive the formulas in blue boxes if the need arises. Asking you to carry one single step yourselves does not seem unreasonable. If even still, deducing these extra formulas on the spot turns out to be too painful, there is a way around it: write **all** the trigonometric expressions you have in the expression at hand in terms of sin and cos **only**, and then use only the fundamental identity $\sin^2\theta + \cos^2\theta = 1$. Generally, you can expect to be punished by having to do more algebra using this second method, but as a last-ditch attempt, it will get the job done.

And the general strategy is: try to start from one of the sides of the proposed identity, and do algebra to get to the other side. You'll either be successful, or you will start to suspect that what you're trying to prove is false — in this case, you can start to plug values to see whether both sides of the proposed identity give different results. To show that a proposed identity is false, it suffices to exhibit **one** counterexample. On the other hand, showing one numerical case where the identity is true is **not** enough to show that the proposed identity is true in general.

Example. Consider $\sin\theta + \cos\theta = \sec\theta$. If we plug the specific angle $\theta = 0$, then we have $\sin(0) + \cos(0) = 0 + 1 = 1$, and also $\sec(0) = 1/\cos(0) = 1/1 = 1$. We cannot

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conclude from this that the proposed identity is true. It does not matter whether you can give more values for which the thing is true. Say we choose $\theta = \pi/4$ so that

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \quad \text{and} \quad \sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Can we conclude now that $\sin\theta + \cos\theta$ is equal to $\sec\theta$? **No**. Take $\theta = \pi/3$, so that

$$\sin\frac{\pi}{3} + \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1+\sqrt{3}}{2} \quad \text{and} \quad \sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{1/2} = 2.$$

Since $(1+\sqrt{3})/2$ is **not** equal to 2, the proposed identity $\sin\theta + \cos\theta = \sec\theta$ is false.

Let's move on to some exercises taken from the book.

For Exercises 49–54, determine whether the statement is true or false for an acute angle θ by using the fundamental identities. If the statement is false, provide a counterexample by using a special angle: $\frac{\pi}{3}, \frac{\pi}{4}$, or $\frac{\pi}{6}$.

49. $\sin\theta \cdot \tan\theta = 1$

Answers

50. $\cos^2\theta \cdot \tan^2\theta = \sin^2\theta$

51. $\sin^2\theta + \tan^2\theta + \cos^2\theta = \sec^2\theta$

Answers

52. $\csc\theta \cdot \cot\theta = \sec\theta$

53. $\frac{1}{\tan\theta} \cdot \cot\theta + 1 = \csc^2\theta$

Answers

54. $\sin\theta \cdot \cos\theta \cdot \tan\theta + 1 = \cos^2\theta$

Remark. The problem suggests $\pi/3$, $\pi/4$ and $\pi/6$ as possible counterexamples when you suspect something is false. They just suggest these angles because the values of standard trigonometric functions for these angles are something you have to know anyway (from tables or the unit circle) anyway. Choosing 0 and $\pi/2$ is also a possibility, provided all the functions are defined at these angles from the get go. As in, you cannot try $\theta = \pi/2$ as a counterexample for some identity involving $\tan\theta$ anywhere, because $\tan\pi/2$ is undefined.

49: Try to start from one side and get to the other using algebra. In this case

$$\sin\theta \cdot \tan\theta = \sin\theta \cdot \frac{\sin\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}.$$

We do not seem to be getting any closer to obtaining 1 as the result. And indeed, since $\sin^2\theta$ is not equal to $\cos\theta$, this seems to be impossible. The final nail in the coffin will be to give a numerical counterexample[1]: for $\theta = 0$ we have $\sin(0)\tan(0) = 0 \cdot 0 = 0 \neq 1$, so the proposed identity is false.

[1] Use the "KISS principle" and don't try to get overly creative with counterexamples.

50: Let's play the same game: start from one side and try to get to the other using algebra.

$$\cos^2\theta \cdot \tan^2\theta = \cancel{\cos^2\theta} \cdot \frac{\sin^2\theta}{\cancel{\cos^2\theta}} = \sin^2\theta.$$

As we were sucessful this time, the proposed identity is true.

51: There are two ways to go about this. Recall that $1 + \tan^2\theta = \sec^2\theta$ and use this together with the fundamental identity to get

$$\sin^2\theta + \tan^2\theta + \cos^2\theta = 1 + \tan^2\theta = \sec^2\theta.$$

Or write everything in terms of $\sin\theta$ and $\cos\theta$ as

$$\sin^2\theta + \tan^2\theta + \cos^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

In any case, we conclude that the proposed identity is true, but you can note that the second solution was objectively worse than the first one.

52: Try to get from the left hand side to the right hand side again.

$$\csc\theta \cdot \cot\theta = \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin^2\theta}.$$

It does not seem we'll be able to get $\sec\theta$ from the above. Then we try to get a counterexample. You can check that for $\theta = \pi/4$ both sides will give $\sqrt{2}$ as the result, so this particular angle is inconclusive. So let's try $\theta = \pi/3$:

$$\csc\frac{\pi}{3} \cdot \cot\frac{\pi}{3} = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{2}{3} \quad \text{but} \quad \sec\frac{\pi}{3} = 2.$$

Since $2/3 \neq 2$, the proposed identity is false.

53: Recall that $1 + \cot^2\theta = \csc^2\theta$, and use that $\cot\theta = 1/\tan^2\theta$ to write

$$\frac{1}{\tan\theta} \cdot \cot\theta + 1 = \cot\theta \cdot \cot\theta + 1 = \cot^2\theta + 1 = \csc^2\theta.$$

Thus, the proposed identity is true. Again, the alternative (and worse) solution is to write

$$\begin{aligned}
\frac{1}{\tan\theta} \cdot \cot\theta + 1 &= \frac{1}{\sin\theta/\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} + 1 \\
&= \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} + 1 \\
&= \frac{\cos^2\theta}{\sin^2\theta} + 1 \\
&= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta} \\
&= \frac{1}{\sin^2\theta} \\
&= \csc^2\theta.
\end{aligned}$$

54: Let's see how far we can get.

$$\sin\theta\cdot\cos\theta\cdot\tan\theta+1=\sin\theta\cdot\cancel{\cos\theta}\cdot\frac{\sin\theta}{\cancel{\cos\theta}}+1=\sin^2\theta+1.$$

The fundamental identity says that $\cos^2\theta=1-\sin^2\theta$, and not $1+\sin^2\theta$. So we suspect that the proposed identity is false. Let's look for a counterexample. For $\theta=\pi/4$ we have

$$\sin\frac{\pi}{4}\cos\frac{\pi}{4}\tan\frac{\pi}{4}+1=\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot 1+1=\frac{1}{2}+1=\frac{3}{2}\text{ and }\cos^2\frac{\pi}{4}=\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}.$$

Since $3/2\neq 1/2$, we indeed conclude that what we started with is false.

Doing plenty of exercises such as those might give you some intuition for when something like this will be true or false — with enough practice, you might even be able to look at an equality and immediately give a counterexample for it (in case it is false), without trying to prove it first (to then convince yourself it is not going to work out).