Thesis defense: The geometry and structure of compact rank-one ECS manifolds

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These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/thesis_slides.pdf

Part I

Introduction & context

The Weyl curvature tensor

We will start by recalling the definition of the Weyl curvature tensor W of a pseudo-Riemannian manifold (M, g).

The curvature tensor of $\mathbb{S}^n$ equipped with its round metric is given by

$$R(X, Y, Z, V) = \mathrm{g}(Y, Z)\mathrm{g}(X, V) - \mathrm{g}(X, Z)\mathrm{g}(Y, V)$$

$$R(X, Y, Z, V) = \underbrace{\mathrm{g}(Y, Z)\mathrm{g}(X, V) - \mathrm{g}(X, Z)\mathrm{g}(Y, V)}_{(\mathrm{g} \owedge \mathrm{g})(X,Y,Z,V)}$$

This is a quadratic expression in g. Polarize!

$$\begin{aligned} 2(T \owedge S)(X, Y, Z, V) \doteq {} & T(Y, Z)S(X, V) - T(X, Z)S(Y, V) \\ & + S(Y, Z)T(X, V) - S(X, Z)T(Y, V) \end{aligned}$$

The $\owedge$-multiplication between symmetric type $(0, 2)$ tensor fields is always a type $(0, 4)$ tensor field with the "symmetries of a curvature".

In any pseudo-Riemannian manifold (M^n, g), we may $\owedge$-divide R by g:

$$R = \mathrm{g} \owedge P + W, \qquad W = \text{Weyl curvature tensor of } (M, \mathrm{g}).$$

Here are the main facts about W:

- W is the remainder of the $\owedge$-division of R by g.
- W is the "Ricci-traceless" part of R.
- W is the part of R not constrained by Einstein's field equations.
- R has $n^2(n^2-1)/12$ independent components, while Ric has $n(n+1)/2$: the remaining ones all come from W.
- $W = 0$ whenever $\dim M \leq 3$.
- If $\dim M \geq 4$, (M, g) is conformally flat if and only if $W = 0$.

The condition we are interested in is $\nabla W = 0$.

Definition (ECS manifold)

A pseudo-Riemannian manifold (M, g) is called *essentially conformally symmetric* if $\nabla W = 0$ but neither $W = 0$ nor $\nabla R = 0$.

The metric signature

ECS manifolds are objects of strictly indefinite nature:

Theorem (Roter, 1977)

For a Riemannian manifold (M, g)*:* $\nabla W = 0 \iff W = 0$ *or* $\nabla R = 0$.

Roter has also shown that ECS manifolds exist in all dimensions starting from 4, and realizing all possible indefinite metric signatures.

Every ECS manifold carries a distinguished null parallel distribution, which helps control its geometry:

Definition

The Olszak distribution of an ECS manifold (M, g) is $\mathcal{D} \hookrightarrow TM$ given by

$$\mathcal{D}_x = \{v \in T_xM \mid \mathrm{g}_x(v, \cdot) \wedge W_x(v', v'', \cdot, \cdot) = 0, \text{ for all } v', v'' \in T_xM\},$$

for every $x \in M$.

More on the Olszak distribution

The Olszak distribution was originally introduced for the more general study of conformally recurrent manifolds, and in this setting it is already true that $\mathcal{D}$ is indeed smooth, parallel and null.

In the ECS case, the rank of $\mathcal{D}$ is always equal to 1 or 2. For this reason, we speak of rank-one/rank-two ECS manifolds.

Theorem (Derdzinski-Roter, 2009)

Let (M, g) be an ECS manifold, and $\mathcal{D}$ be its Olszak distribution. Then:

1. *The Ricci endomorphism of (M, g) is $\mathcal{D}$-valued.*
2. *The connection induced in the quotient bundle $\mathcal{D}^{\perp}/\mathcal{D}$ over M is flat.*
3. *The connection induced in $\mathcal{D}$ itself is flat when (M, g) is of rank one.*

The local structure of ECS manifolds has been determined by Derdzinski and Roter in 2009.

A rank-one example

Example (Conformally symmetric pp-wave manifolds)

Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space of dimension $n-2 \geq 2$, $A \in \mathfrak{sl}(V)$ be self-adjoint, $I \subseteq \mathbb{R}$ be an open interval and $f \colon I \to \mathbb{R}$ be a smooth function. Consider

$$(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\, \mathrm{d}t^2 + \mathrm{d}t\, \mathrm{d}s + \langle \cdot, \cdot \rangle),$$

where $\kappa \colon \widehat{M} \to \mathbb{R}$ is given by $\kappa(t, s, v) = f(t)\langle v, v\rangle + \langle Av, v\rangle$.

Then $(\widehat{M}, \widehat{\mathrm{g}})$ has $\nabla W = 0$, with:

- $W = 0 \iff A = 0$;
- $\nabla R = 0 \iff f$ is constant.

In the ECS case, the Olszak distribution $\mathcal{D}$ is spanned by the null parallel coordinate vector field ∂_s, and $(V, \langle \cdot, \cdot \rangle)$ is isometrically identified with the vector space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$.

Intuition

We consider such examples because *any point in a rank-one ECS manifold* (M^n, g) *has a neighborhood isometric to an open subset of some* $(\widehat{M}, \widehat{\mathrm{g}})$.

The idea relies on two general facts about rank-one ECS manifolds:

- Ric is $\mathcal{D}$-valued.
- the connections induced on $\mathcal{D}$ and $\mathcal{D}^\perp/\mathcal{D}$ are flat.

Locally, consider: a null parallel vector field w spanning $\mathcal{D}$, and a function t such that $\mathrm{d}t = \mathrm{g}(w, \cdot)$. This way:

- $\mathrm{Ric} = (2-n)f(t)\,\mathrm{d}t \otimes \mathrm{d}t$ for some suitable function f.
- The Weyl tensor acts as a traceless self-adjoint endomorphism A of $V = \mathcal{D}^\perp/\mathcal{D}$ via $A(v + \mathcal{D}) = W(u, v)u + \mathcal{D}$ (where u is any vector field with $\mathrm{g}(u, w) = 1$).

Any null geodesic $t \mapsto x(t)$ with $\mathrm{g}(\dot{x}(t), w_{x(t)}) = 1$ gives rise to a mapping

$$F(t, s, v) = \exp_{x(t)}\left(v_{x(t)} + \frac{s w_{x(t)}}{2}\right), \quad \text{with} \quad F^*\mathrm{g} = \widehat{\mathrm{g}}.$$

Part II

Compact ECS manifolds: existence

About compact ECS manifolds

With the local structure of ECS manifolds being fully understood, the next step is to address global aspects. The first question is whether compact ECS manifolds exist.

Theorem (Derdzinski-Roter, 2010)

In every dimension $n = 3j + 2$, $j = 1, 2, 3, \ldots$, there exists a compact Ricci-recurrent ECS manifold (M, g) of any prescribed indefinite metric signature, which is diffeomorphic to a torus bundle over $\mathbb{S}^1$, but not homeomorphic to (or even covered by) a torus.

These examples are all of the form $M = \widehat{M}/\Gamma$, where Γ is some subgroup of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ acting freely and properly discontinuously on $\widehat{M}$.

The strange dimensions $n = 3j + 2$ were a particularity of their construction, which obtained a 5-dimensional example with $\dim V = 3$, but turned out to be "compatible" with taking cartesian powers of $(V, \langle\cdot,\cdot\rangle)$, leading also to dimensions 8, 11, 14, etc..

The isometry group of $(\widehat{M}, \widehat{\mathrm{g}})$

Again: $(V, \langle\cdot,\cdot\rangle)$ has $\dim V = n-2$, $A \in \mathfrak{sl}(V) \smallsetminus \{0\}$ is self-adjoint, f is nonconstant on an open interval $I \subseteq \mathbb{R}$, and our "rank-one ECS model" is $(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \langle\cdot,\cdot\rangle)$.

1. S is the group of the triples $\sigma = (q, p, C) \in \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V)$ with $CAC^{-1} = q^2 A$ and $q^2 f(qt+p) = f(t)$.
2. $(\mathcal{E}, \Omega)$ is the symplectic vector space of solutions $u\colon I \to V$ of $\ddot{u}(t) = f(t)u(t) + Au(t)$, with $\Omega(u, \hat{u}) = \langle \dot{u}, \hat{u}\rangle - \langle u, \dot{\hat{u}}\rangle$.

Note: $\mathrm{S} \circlearrowright \mathcal{E}, I, \mathbb{R}$ via $(\sigma u)(t) = Cu(q^{-1}(t-p))$, $\sigma t = qt + p$, $\sigma s = q^{-1}s$.

3. The Heisenberg group $\mathrm{H} = \mathbb{R} \times \mathcal{E}$ associated with $(\mathcal{E}, \Omega)$, with operation given by $(r, u)(\widehat{r}, \widehat{u}) = (r + \widehat{r} - \Omega(u, \widehat{u}), u + \widehat{u})$.

Theorem

$\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ *is isomorphic to a semidirect product* $\mathrm{S} \ltimes \mathrm{H}$.

- $(\sigma, r, u)(\widehat{\sigma}, \widehat{r}, \widehat{u}) = (\sigma\widehat{\sigma}, r + q^{-1}\widehat{r} - \Omega(u, \sigma\widehat{u}), u + \sigma\widehat{u})$
- $(\sigma, r, u)(t, s, v) = (\sigma t, -\langle \dot{u}(\sigma t), 2\sigma v + u(\sigma t)\rangle + q^{-1}s + r, \sigma v + u(\sigma t))$

The groups $\mathrm{G}(\sigma)$

S: group of all $\sigma = (q, p, C) \in \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V, \langle\cdot,\cdot\rangle)$ respecting f and A.

As we have seen, the group $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}}) = \mathrm{S} \ltimes \mathrm{H}$ can be difficult to deal with.

We restrict our search for compact-quotient subgroups Γ of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ to specific groups $\mathrm{G}(\sigma)$, with $\sigma \in \mathrm{S}$.

More precisely: $\mathrm{G}(\sigma) = \{(\sigma^k, r, u) \mid k \in \mathbb{Z} \text{ and } (r, u) \in \mathrm{H}\} \cong \mathbb{Z} \ltimes \mathrm{H}$.

The formulas for the group operation in $\mathrm{G}(\sigma)$ and its action on $\widehat{M}$ become simplified versions of what we had in the previous page.

The element $\sigma \in \mathrm{S}$ is always chosen according to two situations:

1. *translational*: $I = \mathbb{R}$ and $\sigma = (1, p, C)$ for some "period" $p > 0$.
2. *dilational*: $I = (0, \infty)$ and $\sigma = (q, 0, C)$ for some $q \in (0, \infty) \smallsetminus \{1\}$.

(In both cases, $C \in \mathrm{O}(V, \langle\cdot,\cdot\rangle)$ has $CAC^{-1} = q^2 A$.)

The translational-dilational dichotomy

The reason for the names "translational" and "dilational" goes beyond the meaning suggested by the actions of the elements $(1, p), (q, 0) \in \mathrm{Aff}(\mathbb{R})$.

In general, we say that an abstract ECS manifold (M, g) is translational or dilational according to whether the holonomy group of the natural flat connection induced in $\mathcal{D}$ is finite or infinite.

If $(\widetilde{M}, \widetilde{\mathrm{g}})$ is the universal covering of (M, g), with $M = \widetilde{M}/\Gamma$ for some $\Gamma \cong \pi_1(M)$, and $t\colon \widetilde{M} \to \mathbb{R}$ is a function whose (parallel) gradient spans $\widetilde{\mathcal{D}}$, then for every $\gamma \in \Gamma$ there is $(q, p) \in \mathrm{Aff}(\mathbb{R})$ such that $t \circ \gamma = qt + p$.

This gives us two homomorphisms

$$\Gamma \ni \gamma \mapsto (q, p) \in \mathrm{Aff}(\mathbb{R}) \quad \text{and} \quad \Gamma \ni \gamma \mapsto q \in \mathbb{R} \smallsetminus \{0\},$$

and it turns out that the holonomy group of the connection induced in $\mathcal{D}$ equals the image of the second homomorphism.

First-order subspaces

Recall: any rank-one ECS model $(\widehat{M}, \widehat{\mathrm{g}})$ gives rise to the symplectic vector space $(\mathcal{E}, \Omega)$ of solutions $u\colon I \to V$ of the ODE $\ddot{u}(t) = f(t)u(t) + Au(t)$.

For each $t \in I$, we have the corresponding evaluation mapping $\delta_t\colon \mathcal{E} \to V$, given by $\delta_t(u) = u(t)$. (They're obviously surjective.)

Definition

A vector subspace $\mathcal{L} \subseteq \mathcal{E}$ is called a first-order subspace of $(\mathcal{E}, \Omega)$ if, for every $t \in I$, the restriction $\delta_t|_{\mathcal{L}}\colon \mathcal{L} \to V$ is an isomorphism.

First-order subspaces of $(\mathcal{E}, \Omega)$ are in one-to-one correspondence with curves $B\colon I \to \mathrm{End}(V)$ satisfying $\dot{B} + B^2 = f + A$, via

$$\mathcal{L} = \{u \in \mathcal{E} \mid \dot{u}(t) = B(t)u(t) \text{ for all } t \in I\}.$$

Here:

1. $\mathcal{L}$ is Lagrangian if and only if each $B(t)$ is self-adjoint.
2. $\mathcal{L}$ is σ-invariant if and only if $B(\sigma t) = q^{-1}CB(t)C^{-1}$.

A criterion for the existence of cocompact subgps. of $\mathrm{G}(\sigma)$

Theorem

For a rank-one ECS model manifold $(\widehat{M}, \widehat{\mathrm{g}})$, and an isometry $\gamma = (\sigma, b, w)$ with $\sigma \in \mathrm{S}$ chosen as before, the following conditions are equivalent:

- (a) *There is a discrete subgroup Γ of $\mathrm{G}(\sigma)$ acting freely and properly discontinuously on $\widehat{M}$ with a compact quotient $M = \widehat{M}/\Gamma$.*
- (b) *There is a σ-invariant first-order subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$, a lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$ with $\mathrm{C}_\gamma[\Sigma] = \Sigma$, and $\theta \geq 0$ such that $\Sigma \cap (\mathbb{R} \times \{0\}) = \mathbb{Z}\theta \times \{0\}$ and $\Omega(u, \hat{u}) \in \mathbb{Z}\theta$ for all $u, \hat{u} \in \Lambda$, where Λ is the image of Σ under the projection $\mathbb{R} \times \mathcal{L} \to \mathcal{L}$.*

If (b) holds, Γ in (a) can be taken to be the group generated by γ and Σ and there is a locally trivial fibration $M \to \mathbb{S}^1$ whose fibers, all diffeomorphic to a torus or to a 2-step nilmanifold according to whether $\mathcal{L}$ is Lagrangian or not, are the leaves of $\mathcal{D}^\perp$. Finally, M equipped with its natural quotient metric is translational and complete, or dilational and incomplete, according to whether $\sigma = (1, p, C)$ or $\sigma = (q, 0, C)$.

Very brief sketch of (b) implies (a)

First, we show that the quotient $N = (\mathbb{R} \times \mathcal{L})/\Sigma$ is compact, where the lattice Σ acts on $\mathbb{R} \times \mathcal{L}$ by *Heisenberg* left-translations.

Then, if ε is 0 or 1 (depending on whether σ is translational or dilational), we let $\widetilde{w} \in \mathcal{L}$ be the unique element with $\widetilde{w}(\sigma\varepsilon) = w(\sigma\varepsilon)$, and let $\widetilde{b}$ be given by $\widetilde{b} = b - \langle \dot{w}(\sigma\varepsilon) - B(\sigma\varepsilon)w(\sigma\varepsilon), w(\sigma\varepsilon)\rangle$.

Then $\phi\colon \mathbb{R} \times \mathcal{L} \to \mathbb{R} \times \mathcal{L}$ given by

$$\phi(r, u) = \big(q^{-1}r + \widetilde{b} - \Omega(\widetilde{w} - 2w, \sigma u), \sigma u + \widetilde{w}\big)$$

is Σ-equivariant, and hence passes to the quotient $\Phi\colon N \to N$.

Finally, we set $M = (I \times N)/\mathbb{Z}$, where $k \cdot (t, \Sigma(r, u)) = (\sigma^k t, \Phi^k \Sigma(r, u))$.

This works.

Theorem (Derdzinski-T., 2022)

There exist compact rank-one translational ECS manifolds of all dimensions $n \geq 5$ and all indefinite metric signatures, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{D}^\perp$, all geodesically complete, and none locally homogeneous. In each fixed dimension and metric signature, there is an infinite-dimensional moduli space of local-isometry types.

Theorem (Derdzinski-T., 2023)

There exist compact rank-one dilational ECS manifolds of all odd dimensions $n \geq 5$ and with semi-neutral metric signature, including locally homogeneous ones, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{D}^\perp$, all of them geodesically incomplete. In each fixed odd dimension, there is an infinite-dimensional moduli space of local-isometry types.

» Discuss translational examples

» Discuss dilational examples

The translational construction

Fix $\sigma = (1, p, \mathrm{Id}_V)$, for $p > 0$.

Based on the criterion for the existence of cocompact subgroups of $\mathrm{G}(\sigma)$, with $\theta = 0$, our goal is to find: a first-order σ-invariant Lagrangian subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$ and a conjugation-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$.

At the same time, we must find a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ and a self-adjoint $A \in \mathfrak{sl}(V) \setminus \{0\}$ with the correct spectral properties to be used as model data.

The key observation for this construction is that any $B\colon \mathbb{R} \to \mathrm{End}(V)$ gives rise to the corresponding f and A, by taking the trace and traceless-part of $\dot{B} + B^2$.

So:

how to find the B making all of it work?

Rephrasing the goal

We take $(V, \langle \cdot, \cdot \rangle)$ to be a standard pseudo-Euclidean $\mathbb{R}^{n-2}$, and restrict our search for $B\colon \mathbb{R} \to \mathrm{End}(\mathbb{R}^{n-2})$ to the ones valued in the hyperplane Δ^{n-2} consisting of diagonal matrices. This makes each $B(t)$ self-adjoint, and the corresponding first-order subspace $\mathcal{L}$ Lagrangian.

As σ-invariance of $\mathcal{L}$ amounts to p-periodicity of B, we may set $\mathbb{R}/p\mathbb{Z} \cong \mathbb{S}^1$ and regard our candidates to B as defined in $\mathbb{S}^1$.

We thus seek $B \in \mathrm{C}^\infty(\mathbb{S}^1, \Delta^{n-2})$ such that:

- i. the trace of $\dot{B} + B^2$ is nonconstant.
- ii. the traceless part of $\dot{B} + B^2$ is a nonzero constant.
- iii. $\sigma|_{\mathcal{L}} = \exp\left(-\int_{\mathbb{S}^1} B\right) \in \mathrm{GL}(n-2, \mathbb{Z})$.

(Condition (iii) ultimately gives the existence of the conjugation-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$.)

The key spectral condition

For $n \geq 3$, we consider the following condition imposed on some tuple $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathbb{R}^{n-2}$:

$$\{\lambda_1, \ldots, \lambda_{n-2}\} \text{ is a subset of } (0, \infty) \smallsetminus \{1\}, \text{ not of the form } \{\lambda\} \text{ or } \{\lambda, \lambda^{-1}\} \text{ with any } \lambda > 0. \qquad (\dagger)$$

Condition (iii) is essentially taken care of by noting that whenever $n \geq 5$, there is a matrix in $\mathrm{GL}(n-2, \mathbb{Z}) \cap \Delta^{n-2}$ whose diagonal entries satisfy $(\dagger)$.

Consider now, for $0 \leq k \leq \infty$:

- $\mathcal{P}\colon \mathrm{C}^{k+1}(\mathbb{S}^1, \Delta^{n-2}) \to \mathrm{C}^k(\mathbb{S}^1, \Delta^{n-2})$ given by $\mathcal{P}(B) = \dot{B} + B^2$.
- $S\colon \mathrm{C}^k(\mathbb{S}^1, \Delta^{n-2}) \to \Delta^{n-2}$ given by $S(B) = \exp\left(-\int_{\mathbb{S}^1} B\right)$.

Theorem

Whenever the entries of $\Theta \in \Delta^{n-2}$ satisfy $(\dagger)$, there is an infinite-dimensional manifold of functions $f \in \mathrm{C}^\infty(\mathbb{S}^1)$ realized as $\operatorname{tr}\mathcal{P}(B)$ for some $B \in \mathrm{C}^\infty(\mathbb{S}^1, \Delta^{n-2})$ having $S(B) = \Theta$ and $\mathcal{P}(B)$ with nonzero constant traceless part.

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Sketch of proof: condition $(\dagger)$ allows us to write $\Theta = S(C)$ for some nonsingular $C \in \Delta^{n-2}$. This makes the derivative $\mathrm{d}\mathcal{P}_C$, explicitly given by $\mathrm{d}\mathcal{P}_C(Y) = \dot{Y} + 2CY$, an isomorphism. We may now apply the Inverse Function Theorem to first deform the constant C into a C^k — and then C^∞ — curve B having $S(B) = \Theta$.

The dilational construction

Fix $\sigma = (q, 0, C)$, with $q \in (0, \infty) \smallsetminus \{1\}$ and C to be chosen later.

Based on the criterion for the existence of cocompact subgroups of $\mathrm{G}(\sigma)$, with $\theta = 0$, our goal is to find: a first-order σ-invariant Lagrangian subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$ and a conjugation-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$.

At the same time, we must find a smooth function $f\colon (0, \infty) \to \mathbb{R}$ and a self-adjoint $A \in \mathfrak{sl}(V) \smallsetminus \{0\}$ with the correct spectral properties to be used as model data.

Obtaining such f and A, in this case, is simple, and it is the existence of $\mathcal{L}$ and Σ which pose a challenge. It ultimately relies on the combinatorial structure we will discuss next.

$\mathbb{Z}$-spectral systems

Definition

A *$\mathbb{Z}$-spectral system* is a quadruple (m, k, E, J) consisting of two integers $m, k \geq 2$, an injective function $E\colon \mathcal{V} \to \mathbb{Z} \smallsetminus \{-1\}$, where $\mathcal{V} = \{1, \ldots, 2m\}$, and a function $J\colon \mathcal{V} \to \{0, 1\}$, satisfying for every $i, i' \in \mathcal{V}$ that:

a. $k + 1 = 2E(1)$ (and so k must be odd).
b. $E(i) + E(i') = -1$ and $J(i) + J(i') = 1$ whenever $i + i' = 2m + 1$.
c. $E(i) - E(i') = k$ and $J(i) + J(i') = 1$ whenever $i' = i + 1$ is even.
d. The set $Y = \{-1\} \cup \{E(i) \mid i \in \mathcal{V} \text{ and } J(i) = 1\}$ is symmetric about zero.

The spectral selector $S = J^{-1}(1)$ is simultaneously a selector for both two-element subset families

$$\{\{i, i'\} \mid i + i' = 2m + 1\} \quad \text{and} \quad \{\{i, i'\} \mid i' = i + 1 \text{ is even}\}.$$

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b. $E(i) + E(i') = -1$ and $J(i) + J(i') = 1$ whenever $i + i' = 2m + 1$.
c. $E(i) - E(i') = k$ and $J(i) + J(i') = 1$ whenever $i' = i + 1$ is even.
d. The set $Y = \{-1\} \cup \{E(i) \mid i \in \mathcal{V} \text{ and } J(i) = 1\}$ is symmetric about zero.

The reason we care about this is that for any $\mathbb{Z}$-spectral system (m, k, E, J) and $q \in (0, \infty) \smallsetminus \{1\}$ such that $q + q^{-1} \in \mathbb{Z}$, the $(m+1)$-element set $\{q^a \mid a \in Y\}$ is the spectrum of some matrix in $\mathrm{GL}(m + 1, \mathbb{Z})$.

"Odd-dimensional" systems...

Example

For every odd integer $m \geq 3$, there is a $\mathbb{Z}$-spectral system $(m, m+2, E, J)$. Writing $m = 2r - 3$ with $r \geq 3$, and $(i, i') = (2j-1, 2j)$ whenever $i, i' \in \mathcal{V}$ and $i' = i + 1$ is even, we define the function E by

$$(E(2j-1), E(2j)) = \begin{cases} (r, -r+1) & \text{if } j = 1, \\ (j-1, -2r+j) & \text{if } 1 < j < r-1 \text{ and } r \text{ is even}, \\ (2r+j-2, j-1) & \text{if } 1 < j < r-1 \text{ and } r \text{ is odd}, \\ (r-1, -r) & \text{if } j = r-1, \\ (j-2r+2, j-4r+3) & \text{if } r-1 < j < m \text{ and } r \text{ is odd}, \\ (j+1, j-2r+2) & \text{if } r-1 < j < m \text{ and } r \text{ is even}, \\ (r-2, -r-1) & \text{if } j = m, \end{cases}$$

and let the function J be given by $J(i) = E(i) \bmod 2$, so that

$$Y = \{-1\} \cup (\mathbb{Z}_{\text{odd}} \cap E[\mathcal{V}]), \quad \text{where } \mathbb{Z}_{\text{odd}} = \mathbb{Z} \smallsetminus 2\mathbb{Z}.$$

... and no "even-dimensional" ones.

Proposition

There are no $\mathbb{Z}$-spectral systems (m, k, E, J) with even m.

Proof idea: Let (m, k, E, J) be a $\mathbb{Z}$-spectral system with even m, written as $m = 2s$ for some $s \in \mathbb{Z}$. The "exponent vector" $\mathbf{E} \in \mathbb{Z}^{4s}$ has the form

$$\mathbf{E} = (a_1, a_1 - k, \ldots, a_s, a_s - k, -1 - a_s + k, -1 - a_s, \ldots, -1 - a_1 + k, -1 - a_1)$$

for some $a_1, \ldots, a_s \in \mathbb{Z}$. Now, let ε_j be 1 or -1 according to whether $\{2j-1, 2m-2j+1\} \subseteq S$ or $\{2j, 2m-2j+2\} \subseteq S$. As the set $Y = \{-1\} \cup E[S]$ is symmetric about zero,

$$1 = \sum_{i \in S} E(i) = \sum_{j=1}^{s} (-1 + \varepsilon_j k),$$

and so $\left(\sum_{j=1}^{s} \varepsilon_j\right) k = s + 1$. For ℓ negative ε_j's, we obtain the relation $(s - 2\ell)k = s + 1$. Both sides have different parities, a contradiction.

Defining dilational ECS data

Let $n \geq 5$ be odd and set $m = n - 2$.

Fix a $\mathbb{Z}$-spectral system (m, n, E, J), a m-dimensional pseudo-Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ with semi-neutral signature, and $q \in (0, \infty) \smallsetminus \{1\}$ with $q + q^{-1} \in \mathbb{Z}$.

Defining A and C: let $(e_1, \ldots, e_m)$ be a basis of $(V, \langle \cdot, \cdot \rangle)$ on which

$$\langle \cdot, \cdot \rangle \sim \begin{bmatrix} 0 & 0 & \ldots & 0 & \varepsilon \\ 0 & 0 & \ldots & \varepsilon & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \varepsilon & \ldots & 0 & 0 \\ \varepsilon & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad \varepsilon \in \{1, -1\},$$

and define

$$a(j) = E(2j - 1) + \frac{1 - n}{2} = E(2j) + \frac{1 + n}{2}, \quad j = 1, \ldots, m.$$

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and define $a(j) = E(2j-1) + \dfrac{1-n}{2} = E(2j) + \dfrac{1+n}{2}, \quad j = 1, \ldots, m.$

Now set:

- $Ae_m = e_1$, and $Ae_j = 0$ for $j = 1, \ldots, m-1$.
- $Ce_j = q^{a(j)} e_j$ for $j = 1, \ldots, m$.

Then $A \in \mathfrak{sl}(V) \setminus \{0\}$ is self-adjoint, $C \in \mathrm{O}(V, \langle\cdot,\cdot\rangle)$, and $CAC^{-1} = q^2 A$.

Defining dilational ECS data

The function f: here, we consider the "scalar version" of $(\mathcal{E}, \Omega)$, that is, the space $\mathcal{W}$ of solutions $y\colon (0,\infty) \to \mathbb{R}$ of $\ddot{y}(t) = f(t)y(t)$.

The operator $T\colon \mathcal{W} \to \mathcal{W}$ given by $(Ty)(t) = y(t/q)$ is indeed $\mathcal{W}$-valued whenever f has the property $q^2 f(qt) = f(t)$.

Its spectrum μ^+, μ^- satisfies $\mu^+\mu^- = q^{-1}$, as $T^*\alpha = \alpha$ for the (symplectic) area form $\alpha(y,z) = \dot{y}(t)z(t) - y(t)\dot{z}(t)$.

The spectrum of $\sigma\colon \mathcal{E} \to \mathcal{E}$ then becomes

$$(\mu^+ q^{a(1)}, \mu^- q^{a(1)}, \ldots, \mu^+ q^{a(m)}, \mu^- q^{a(m)}). \qquad (*)$$

Choosing f so that $\mu^+ = q^{(-1-n)/2}$ and $\mu^- = q^{(-1+n)/2}$, such as

$$f(t) = \frac{n^2-1}{4t^2},$$

the spectrum $(*)$ becomes precisely

$$(q^{E(1)}, q^{E(2)}, \ldots, q^{E(2m-1)}, q^{E(m)}).$$

Defining dilational ECS data

So far: f, A, and $\sigma = (q, 0, C)$ are in place, and the spectrum of $\sigma\colon \mathcal{E} \to \mathcal{E}$ is $(q^{E(1)}, q^{E(2)}, \ldots, q^{E(2m-1)}, q^{E(m)})$, for our spectral system $(n-2, n, E, J)$.

The space $\mathcal{L}$: using more linear algebra, we obtain a basis

$$(u_1, u_2, \ldots, u_{2m-1}, u_{2m}) = (u_1^+, u_1^-, \ldots, u_m^+, u_m^-)$$

of $\mathcal{E}$, of eigenvectors of σ associated with $(q^{E(1)}, q^{E(2)}, \ldots, q^{E(2m-1)}, q^{E(m)})$.

This basis satisfies that $\Omega(u_i, u_j) = 0$, whenever $i, j \in \{1, \ldots, 2m\}$ have $i + j \neq 2m + 1$. Hence, if $S = J^{-1}(1)$ is the spectral selector of the $\mathbb{Z}$-spectral system $(n-2, n, E, J)$,

the direct sum $\mathcal{L} = \bigoplus_{i \in S} \mathbb{R} u_i$ is a first-order σ-invariant Lagrangian subspace of $(\mathcal{E}, \Omega)$.

Defining dilational ECS data

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Now, σ-invariance of $\mathcal{L}$ makes $\mathbb{R} \times \mathcal{L}$ C_γ-invariant for any $\gamma \in \mathrm{G}(\sigma)$.

The spectrum of the restriction $\mathrm{C}_\gamma|_{\mathbb{R} \times \mathcal{L}}$ is given by $\{q^a \mid a \in Y\}$, for $Y = \{-1\} \cup E[S]$ arising from $(n-2, n, E, J)$.

This means that a C_γ-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$ exists.

Part III

The topology of compact rank-one ECS manifolds

The topological structure

Q: What do all known compact rank-one ECS manifolds presented so far have in common?

A: They are all bundles over $\mathbb{S}^1$, and $\mathcal{D}^\perp$ appears as the vertical distribution.

We will see next that this is **not an accident**.

The main result

Theorem (Derdzinski-T., 2022)

Every non-locally-homogeneous compact rank-one ECS manifold (M, g) for which the orthogonal distribution $\mathcal{D}^\perp$ is transversely orientable is the total space of a locally trivial fibration over $\mathbb{S}^1$ whose fibers are the leaves of $\mathcal{D}^\perp$.

The transverse orientability of $\mathcal{D}^\perp$ can be achieved by replacing (M, g) with a suitable isometric double covering, if necessary.

One of the main consequences of the above result is:

Theorem (Derdzinski-T., 2022)

For any compact rank-one ECS manifold (M, g), the leaves of $\widetilde{\mathcal{D}}^\perp$ in the universal covering $(\widetilde{M}, \widetilde{\mathrm{g}})$ are the factor manifolds of a global decomposition of $\widetilde{M}$. More precisely, every leaf L of $\widetilde{\mathcal{D}}^\perp$ in $\widetilde{M}$ is connected and simply connected, and $\widetilde{M}$ is diffeomorphic to $\mathbb{R} \times L$.

The strategy

The central concept used in the proof is what we call the dichotomy property for a codimension-one foliation $\mathcal{V}$ in a smooth manifold M, which has two alternatives (NC) and (AC) imposed on its compact leaves.

The reason why we care about this property is that it turns out that if M is compact, $\mathcal{V}$ is transversely orientable, and some compact leaf of $\mathcal{V}$ satisfies (AC), then there is a locally trivial bundle projection $M \to \mathbb{S}^1$ whose fibers are the leaves of $\mathcal{V}$.

There are two big steps to carry out:

1. Establishing the dichotomy property for $\mathcal{D}^\perp$ (when transversely orientable) in a rank-one ECS manifold (M, g).
2. Showing that some compact leaf of $\mathcal{D}^\perp$ satisfies (AC) when M is compact.

Step (i) does not use compactness of M, and local homogeneity is an obstruction for (ii).

The dichotomy property

Definition

A codimension-one foliation $\mathcal{V}$ in a smooth manifold M has the *dichotomy property* if every compact leaf L of $\mathcal{V}$ has a neighborhood U in M such that the leaves of $\mathcal{V}$ intersecting $U \smallsetminus L$ are either:

NC: all *noncompact*, or

AC: *all compact*, and some neighborhood of L in M saturated by compact leaves of $\mathcal{V}$ may be diffeomorphically identified with the product $\mathbb{R} \times L$ in such a way that $\mathcal{V}$ corresponds to the foliation $\{\{s\} \times L\}_{s \in \mathbb{R}}$.

Example

If both M and $\mathcal{V}$ are real-analytic and $\mathcal{V}$ is transversely orientable, then $\mathcal{V}$ has the dichotomy property. If a compact leaf L of $\mathcal{V}$ does not satisfy (NC), there are compact leaves of $\mathcal{V}$ arbitrarily close to L. Now analyticity implies that L satisfies (AC).

More examples of the dichotomy property

Example

If $\mathcal{V}$ is transversely orientable and has a finite number of compact leaves, then $\mathcal{V}$ clearly has the dichotomy property. Examples of this situation include the Reeb foliation on $\mathbb{S}^3$, and foliations on products $\mathbb{T}^2 \times K$ coming from foliations on $\mathbb{T}^2$ having themselves a finite number of leaves.

Example

Let M be an orientable line bundle over a compact and connected manifold L, equipped with a flat connection ∇, and let $\mathcal{V}$ be the horizontal distribution on M associated with ∇. The compact leaf L (and hence all others) satisfies (NC) or (AC) according to whether the holonomy group $\mathrm{Hol}(\nabla)$ is infinite or trivial.

Establishing the dichotomy property for $\mathcal{D}^\perp$

The last example illuminates the way to proceed:

Theorem

Let (M, g) be a compact rank-one ECS manifold with transversely orientable $\mathcal{D}^\perp$, and let L be a compact leaf of $\mathcal{D}^\perp$. Then, there is some neighborhood U of L in M which can be identified with a neighborhood U' of the zero section $L \hookrightarrow \mathcal{D}^*_L$ as to make the distribution $\mathcal{D}^\perp$ in U correspond in U' to the horizontal distribution of the flat connection in $\mathcal{D}^*_L$.

Sketch of proof: Let $t\colon \widetilde{M} \to \mathbb{R}$ is a function whose parallel gradient $\mathbf{w}$ spans $\widetilde{\mathcal{D}}$, and ϕ be a flow on M which is transverse to $\mathcal{D}^\perp$. Define $U = \phi[(-\varepsilon, \varepsilon) \times L]$ and $\Psi\colon U \to U' = \Psi[U]$ by

$$\Psi(\phi(\tau, x)) = [t(\widetilde{\phi}(\tau, y)) - t(y)]\xi_y \circ (\mathrm{d}\pi_y)^{-1},$$

where $\widetilde{\phi}$ is a lift of ϕ to $\widetilde{M}$, ξ is the parallel section of $\widetilde{\mathcal{D}}^*$ with $\xi(\mathbf{w}) = 1$, and $y \in \pi^{-1}(x)$ is chosen at will. This works.

Establishing the dichotomy property for $\mathcal{D}^\perp$

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So:

Theorem

If (M, g) is a rank-one ECS manifold with transversely orientable $\mathcal{D}^\perp$, then $\mathcal{D}^\perp$ satisfies the dichotomy property. Namely, for a compact leaf L of $\mathcal{D}^\perp$, alternatives (AC) and (NC) correspond to whether the holonomy group of the natural flat connection in the line bundle $\mathcal{D}^*_L$ is finite or infinite.

Towards a compact leaf with (AC): cohomology, $\mathcal{F}$ & P

Our next goal is to show that some compact leaf of $\mathcal{D}^\perp$ in M satisfies alternative (AC) of the dichotomy property.

Closedness of a continuous 1-form ζ means its locally being the differential of a C^1 function. Thus it makes sense to consider a cohomology class $[\zeta] \in H^1_{\mathrm{dR}}(M) \cong \mathrm{Hom}(\pi_1(M), \mathbb{R})$.

We fix again the universal covering $(\widetilde{M}, \widetilde{\mathrm{g}})$ of (M, g), a function $t\colon \widetilde{M} \to \mathbb{R}$ whose parallel gradient spans $\widetilde{\mathcal{D}}$, and express $M = \widetilde{M}/\Gamma$ with $\Gamma \cong \pi_1(M)$.

Considering the space $\mathcal{F}$ of all continuous functions $\chi\colon \widetilde{M} \to \mathbb{R}$ such that $\chi\, \mathrm{d}t$ is closed and Γ-invariant, we may consider the operator

$$P\colon \mathcal{F} \to H^1_{\mathrm{dR}}(M), \qquad \text{given by} \qquad P\chi = [\chi\, \mathrm{d}t].$$

Special functions

Considering the space $\mathcal{F}$ of all continuous functions $\chi\colon \widetilde{M} \to \mathbb{R}$ such that $\chi\,\mathrm{d}t$ is closed and Γ-invariant, we may consider the operator

$$P\colon \mathcal{F} \to H^1_{\mathrm{dR}}(M), \qquad \text{given by} \qquad P\chi = [\chi\,\mathrm{d}t].$$

Theorem

Let (M, g) be a compact rank-one ECS manifold such that $\mathcal{D}^\perp$ is transversely orientable. If (M, g) is not locally homogeneous, then there exists a nonconstant function $\mu \in \mathrm{C}^1(M)$ which is constant along $\mathcal{D}^\perp$.

Sketch of proof: It mainly consists in showing that either

1. $\dim \mathcal{F} < \infty$ and (M, g) is locally homogeneous, or
2. $\dim \mathcal{F} = \infty$ and such μ exists.

In case (i), set-theoretical reasons imply that $f(t) = \varepsilon(t-b)^{-2}$, where $\mathrm{Ric} = (2-n)f(t)\,\mathrm{d}t \otimes \mathrm{d}t$. In case (ii), let $\chi \in \ker P \smallsetminus \{0\}$ and take μ such that $\mathrm{d}\mu$ equals the projected $\chi\,\mathrm{d}t$.

From special functions to compact leaves satisfying (AC)

Let $\mu \in \mathrm{C}^1(M)$ be nonconstant, but constant along $\mathcal{D}^\perp$.

By Sard's theorem, the image of μ in $\mathbb{R}$ contains an open interval of regular values of μ. Any connected component of a level set $\mu^{-1}(c)$, with c in a such open interval, is a compact leaf of $\mathcal{D}^\perp$ with (AC).

Note: Sard's theorem usually applies for a C^k function from an n-manifold into an m-manifold, where $k \geq \max\{n - m + 1, 1\}$. Here, $k = m = 1$ and $n \geq 4$, but compactness of M together with μ being locally a function of t allows us to apply Sard with $n = 1$ instead of $n \geq 4$.

Part IV

ECS genericity and the four-dimensional case

Genericity – the linear-algebraic setting

Finally, we aim to obtain classification results for compact ECS manifolds.

We were able to achieve this under an additional genericity condition.

Definition

Given a pseudo-Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$, a traceless and self-adjoint operator $A\colon V \to V$ is called generic if only finitely many isometries of $(V, \langle\cdot,\cdot\rangle)$ commute with A.

Note: the set of generic operators is open and dense in the space of all traceless and self-adjoint operators.

Example

When $\dim V = 2$, every such nonzero A is generic: only four or two isometries of $(V, \langle\cdot,\cdot\rangle)$ commute with A, according to whether A is diagonalizable or not, respectively.

Genericity – the linear-algebraic setting

Example

If $\dim V = m$ and A has m mutually distinct eigenvalues, then A is generic: there are exactly 2^m isometries of $(V, \langle\cdot,\cdot\rangle)$ commuting with A.

Inspired by the dilational case:

Example

If $\dim V = m$ and A is nilpotent, then A is generic if and only if $\ker A$ is one-dimensional, in which case only $\pm\mathrm{Id}_V$ commutes with A and there is a basis $(e_1, \ldots, e_m)$ of $(V, \langle\cdot,\cdot\rangle)$, unique up to an overall sign change, in which

$$A \sim \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad \langle\cdot,\cdot\rangle \sim \begin{bmatrix} 0 & 0 & \ldots & 0 & \varepsilon \\ 0 & 0 & \ldots & \varepsilon & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \varepsilon & \ldots & 0 & 0 \\ \varepsilon & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Genericity – the ECS setting

Now we move on to the ECS setting.

Recall that for any rank-one ECS manifold (M, g), we have a traceless and self-adjoint operator A of the space $(V, \langle\cdot,\cdot\rangle)$ of parallel sections of the quotient bundle $\mathcal{D}^\perp/\mathcal{D}$ over the universal covering $(\widetilde{M}, \widetilde{\mathrm{g}})$.

It is given by $A(v+\mathcal{D}) = W(u,v)u + \mathcal{D}$ where u is a vector field with $\widetilde{\mathrm{g}}(u,w) = 1$, where w is a null parallel vector field spanning $\mathcal{D}$.

Definition

A rank-one ECS manifold (M, g) is generic if its associated operator A described above is generic.

Based on previous examples:

every four-dimensional rank-one ECS manifold is generic.

A crucial first consequence of genericity

We have seen earlier that every rank-one ECS manifold (M, g) gives rise to a model $(\widehat{M}, \widehat{\mathrm{g}})$. This construction can be done globally in the universal covering $(\widetilde{M}, \widetilde{\mathrm{g}})$ instead of locally in (M, g), and we obtain an isometric mapping $F\colon \widehat{M} \to \widetilde{M}$.

When can we ensure that F is surjective as well?

Theorem (Derdzinski-T., 2023)

The universal covering of a generic compact rank-one ECS manifold (M, g) is globally isometric to a rank-one ECS model $(\widehat{M}, \widehat{\mathrm{g}})$.

The proof of this result uses general facts about complete connections, as well as the notions of maximal completeness and $\mathcal{D}^{\perp}$-completeness.

Classification results in the generic case

Recall that writing $M = \widetilde{M}/\Gamma$, (M, g) is translational or dilational according to whether the image K of $\Gamma \ni \gamma \mapsto q \in \mathbb{R} \smallsetminus \{0\}$ is finite or infinite.

Our first classification result is:

Theorem (Derdzinski-T., 2023)

Any generic compact rank-one ECS manifold is either translational or locally homogeneous.

The proof of this result consists in showing that,

if (M, g) is dilational, then the image K cannot be infinite cyclic.

This is done by using the algebraic structure of a generic nilpotent operator together with the existence of a first-order subspace of $(\mathcal{E}, \Omega)$ to obtain a certain impossible combinatorial structure.

Ruling out the locally homogeneous alternative

As the next step, we refine the previous theorem:

Theorem (Derdzinski-T., 2023)

Any generic compact rank-one ECS manifold is translational.

This time, the argument consists of showing that a generic compact rank-one ECS manifold cannot be locally homogeneous.

Given $q \in (0, \infty) \smallsetminus \{1\}$, there are only two isometries C_q of $(V, \langle\cdot,\cdot\rangle)$ such that $CAC^{-1} = q^2 A$, and a careful analysis of the spectrum of the associated operators $\sigma_q\colon \mathcal{E} \to \mathcal{E}$ yields a bound on the rank of the lattice $\Sigma = \Gamma \cap \mathrm{H}$.

The condition $\operatorname{rank} \Sigma \leq 1$ then implies that Γ is Abelian, and then the transitive commutation property for the identity component of the isometry group of a homogeneous model ultimately implies that Γ cannot act freely and properly discontinuously on $\widehat{M}$.

Goodbye, dimension four!

Theorem (Derdzinski-.T, 2023)

There are no four-dimensional compact rank-one ECS manifolds.

Sketch of proof: If a rank-one ECS manifold (M, g) were compact and four-dimensional, it would be translational and its universal covering would be a model $(\widehat{M}, \widehat{\mathrm{g}})$ with $I = \mathbb{R}$.

Replacing (M, g) with a finite isometric covering if necessary, we may assume that all elements in Γ have trivial $\mathrm{O}(V, \langle \cdot, \cdot \rangle)$-component, i.e., $\gamma = (1, p, \mathrm{Id}_V, r, u)$ with $p \in \mathbb{R}$ and $(r, u) \in \mathrm{H}$.

Then, the image of $\Gamma \ni \gamma \mapsto p \in \mathbb{R}$ must be infinite cyclic:

- if it were trivial, $t\colon \widetilde{M} \to \mathbb{R}$ would be Γ-invariant and survive as a continuous unbounded function $M \to \mathbb{R}$.
- if it were dense, the condition $f(t + p) = f(t)$ (valid for all $\gamma \in \Gamma$) would imply that f is constant and (M, g) is locally symmetric.

Goodbye, dimension four!

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- if it were dense, the condition $f(t+p) = f(t)$ (valid for all $\gamma \in \Gamma$) would imply that f is constant and (M, g) is locally symmetric.

So, denoting again by p the positive generator of the image of the above homomorphism $\Gamma \to \mathbb{R}$, we obtain that $\Gamma \subseteq \mathrm{G}(\sigma)$ for $\sigma = (1, p, \mathrm{Id}_V)$, and so our criterion for the existence of compact quotients can be applied.

The σ-invariant first-order subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$ obtained from Γ gives rise to a curve $B\colon \mathbb{S}^1 \to \mathrm{End}(V)$ which has $\int_{\mathbb{S}^1} \mathrm{tr}\, B = 0$ (as $\det(\sigma|_{\mathcal{L}}) = 1$, since $\sigma|_{\mathcal{L}}$ leaves a lattice invariant), and also $\int_{\mathbb{S}^1} \mathrm{tr}\, B \neq 0$ when $\dim V = 2$ (by a straightforward computation).

This concludes the argument.

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Thank you for your attention!

◂ Return to translational examples

◂ Return to dilational examples

◂ Return to topological structure

◂ Return to ECS genericity and the 4-dim case