Week 2 Discussion Worksheet: Proofs, Sets and Functors

June 29, 2022

Problem 1: Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

(a) the function that assigns to each bit strings the number of ones in the string minues the number of zeros in the string

5 min

(b) the function that assigns to each bit string twice the number of zeros in that string

(c) the function that assigns the number of bits leftover when a bit string is split into bytes (which are blocks of 8 bits)

(d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

(a) $1011 \longmapsto 3-1=2$ Domain: {bit strings}
$0011 \longmapsto 2-2=0$ Range: $\mathbb{Z}$
$n>0$: $\overbrace{1\cdots1}^{n} \longmapsto n$
$n=0$: $10 \longmapsto 1-1=0$
$n<0$: $\overbrace{00\cdots0}^{|n|} \longmapsto 0-|n|=-n$

(b) $1011 \longmapsto 2$
$0011 \longmapsto 4$
all nonneg. even #
Domain: {bit strings} Range: $2\mathbb{Z}_{\geq 0}$ $2\mathbb{N}$

(c) $1011 \longmapsto 4$
$00111100|1011 \longmapsto 8$
Domain: {bit strings} Range: $\{0,1,2,\cdots,8\}$

(d) $5 \longmapsto 2^2=4$
$16 \longmapsto 4^2=16$
$17 \longmapsto 4^2=16$
Domain: $\mathbb{Z}_{>0}$ Range: Perfect □.

Problem 2: Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$. If it is not, state why.

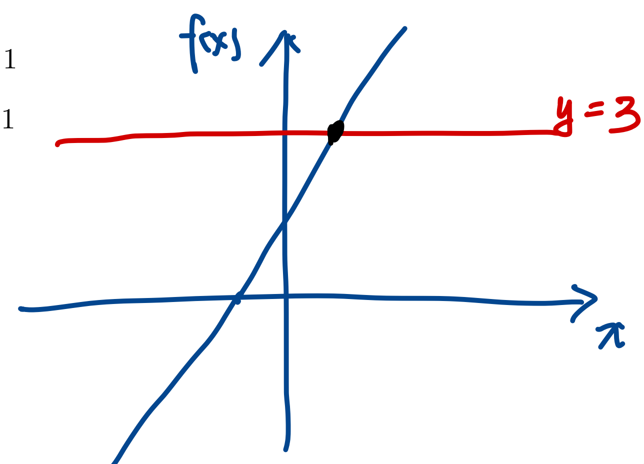
(a) $f(x) = 2x + 1$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^3$

(d) $f(x) = \frac{x^2+1}{x^2+2}$

10 min

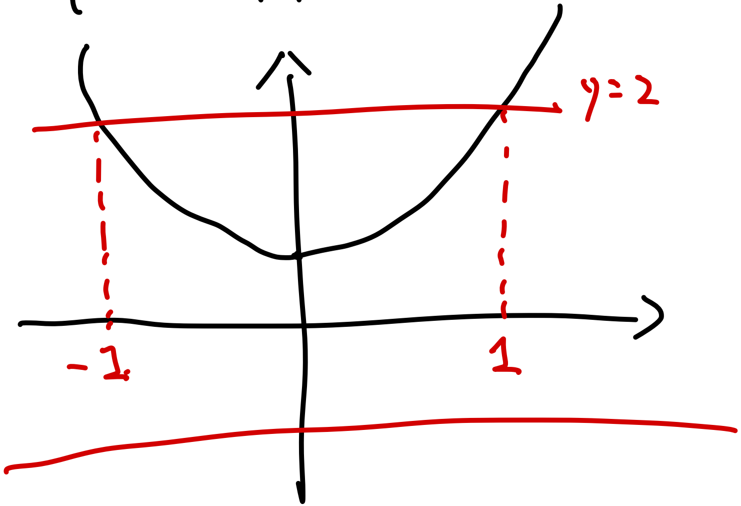


$a = 2x+1$
$x = \frac{a-1}{2}$
$f\left(\frac{a-1}{2}\right) = a.$

$f(x_1) = f(x_2)$
$2x_1+1 = 2x_2+1$
$\Rightarrow x_1 = x_2$

equiv.
$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

(b) $f(x) = x^2 + 1$

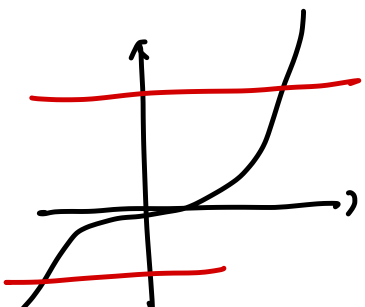


$x^2+1 = 2$

$\Downarrow$

$x^2 = 1$, $x = \pm 1$

(c) $f(x) = x^3$

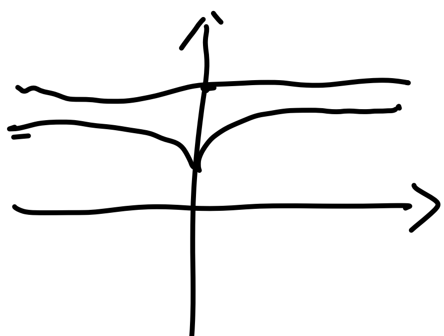


$x^3 = a$.

$\sqrt[3]{a}$

bij.

(d) $f(x) = \dfrac{x^2+1}{x^2+2} = 1 - \dfrac{1}{x^2+2}$.



$(-x)^2 = x^2$

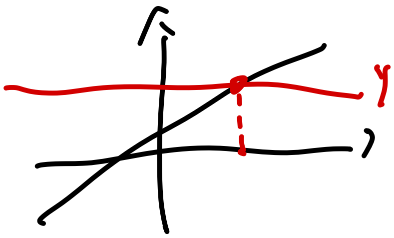
$f(-x) = f(x)$

$f(-1) = f(1)$.

not bij.

Problem 3: Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$, where a and b are constants with $a \neq 0$ is invertible, and find the inverse of f.(Hint: Recall calculus I)

$a \neq 0 \qquad f(x) = ax+b$



$y = ax+b$

$y - b = ax$

$\dfrac{y-b}{a} = x$

$g(y) = \dfrac{y-b}{a}$

$\begin{cases} f(g(y)) = y \\ g(f(x)) = x \end{cases}$

Problem 4: What can you say about the sets A and B if we know that

5 min

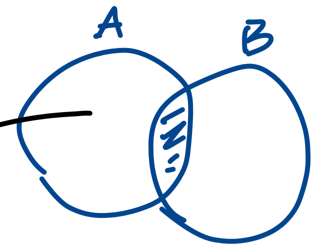
(a) $A \cup B = A$

(b) $A \cap B = A$

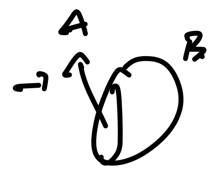
(c) $A - B = A$

Drawing a Venn diagram may help you intuition.

(b)

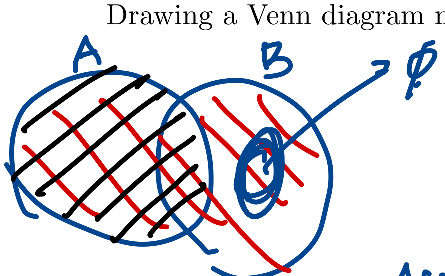


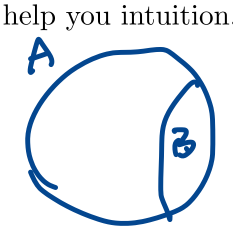
$\emptyset$ c



$A \subseteq B$

(a)





$A \cup B = A \Rightarrow A \supseteq B$

(c)

Problem 5: Find the cardinality, power set and cardinality of the power set for the following sets.

5 min

a. $\{\emptyset, x, y\}$

b. $\{\{x, y\}\}$

Problem 6: Let $A = \{x, y\}$ and $N = \{1, 2, 3, \cdots\}$

(a) Is A a countable set? Is N a countable set?

(b) Describe the elements of $A \times N$.

(c) Is $A \times N$ a countable set?

$A \times N = \{(\alpha, \beta) \mid \alpha \in A, \beta \in N\}$

$(x, 3) \qquad (y, 2)$

1	3	5	
$(x,1)$	$(x,2)$	$(x,3)$	$\cdots$
2	4	6	
$(y,1)$	$(y,2)$	$(y,3)$	$\cdots$