Week 2 Discussion Worksheet: Proofs, Sets and Functors

June 29, 2022

Problem 1: Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

(a) the function that assigns to each bit strings the number of ones in the string minues the number of zeros in the string

(b) the function that assigns to each bit string twice the number of zeros in that string

(c) the function that assigns the number of bits leftover when a bit string is split into bytes (which are blocks of 8 bits)

(d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

(a) Domain: the set of all (finite)bit strings

Range: $\mathbb{N}$

(b) Domain: the set of all (finite)bit strings

Range: $\{2n \mid n \in \mathbb{N}\}$

(c) Domain: the set of all (finite)bit strings

Range: $\{0, 1, 2, 3, 4, 5, 6, 7\}$

(d) Domain: the set of all positive integers

Range: the set of all perfect square integers

Problem 2: Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$. If it is not, state why.

(a) $f(x) = 2x + 1$

Bijection.

(b) $f(x) = x^2 + 1$

Not a bijection since it is neither an injection nor a surjection.

(c) $f(x) = x^3$

Bijection.

(d) $f(x) = \frac{x^2+1}{x^2+2}$

Not a bijection since it is neither an injection nor a surjection.

Problem 3: Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$, where a and b are constants with $a \neq 0$ is invertible, and find the inverse of f.(Hint: Recall calculus I)

$y = f(x)$ is a line in the xy-plane.
$f^{-1}(y) = (y - b)/a$.

Problem 4: What can you say about the sets A and B if we know that

(a) $A \cup B = A$

$B \subseteq A$

(b) $A \cap B = A$

$A \subseteq B$

(c) $A - B = A$

$A \cap B = \emptyset$

Drawing a Venn diagram may help you intuition.

Problem 5: Find the cardinality, power set and cardinality of the power set for the following sets.

a. $\{\emptyset, x, y\}$

cardinality: 3

powerset: $\{\emptyset, \{\emptyset\}, \{x\}, \{y\}, \{\emptyset, x\}, \{\emptyset, y\}, \{x, y\}, \{\emptyset, x, y\}\}$

cardinality of the power set: $8 = 2^3$

b. $\{\{x, y\}\}$

cardinality: 1

powerset: $\{\emptyset, \{x, y\}\}$

cardinality of the power set: $2 = 2^1$

Problem 6: Let $A = \{x, y\}$ and $N = \{1, 2, 3, \cdots\}$

(a) Is A a countable set? Is N a countable set?

Yes both of them are countable set.

(b) Describe the elements of $A \times N$.

$A \times N = \{(m, n) \mid m \in A, n \in N\} = \{(x, 1), (y, 1), (x, 2), (y, 2), \cdots\}$

(c) Is $A \times N$ a countable set?

Yes, we can build an injective function $f : A \times N \to \mathbb{N}$ by the rule that $f((x, n)) = 2n - 1$ and $f((y, n)) = 2n$.