Week 2 Discussion Worksheet: Proofs, Sets and Functors

June 27, 2022

1 Problem 1: (Prove by contrapositive): If $3n + 2$ is even, then n is even.

$3n+2$ is even $\to$ n is even

$\neg(n$ is even$) \to \neg(3n+2)$ is even

n is odd $\to$ $3n+2$ is odd.

$\exists k \in \mathbb{Z}$, $n = 2k+1$, $3n+2 = 3(2k+1)+2$
$= 6k+5$
$= 2(3k+2)+1$

Direct proof
Proof by contrapositive
Proof by contradiction
Proof by cases

15min

Order:
$3 \to 4 \to 2$

Problem 2: Use a direct proof to show that every odd integer is the difference of two squares. (Hint: Find the difference of the squares of $k + 1$ and k, where k is a ~~positive~~ integer.)

$(k+1)^2 - k^2$

$= k^2 + 2k + 1 - k^2 = 2k+1$

If n is odd, then $n = 2k+1$ for some $k \in \mathbb{Z}$, now $n = (k+1)^2 - k^2$
$= 2k+1$

Proof by contradiction:
$n(n+1)$ is even
If it is odd, then both n & $n+1$ are odd
(only odd $\times$ odd $=$ odd)
Contradiction!

Problem 3: Prove that if n is an integer, then n is even if and only if $7n + 4$ is even.

n is even $\to$ $7n+4$ is even

$n = 2k$ $\quad 7n+4 = 14k+4 = 2(7k+2)$

$7n+4$ is even $\to$ n is even

$\sim$ n is odd $\to$ $7n+4$ is odd.

$n = 2k+1$ $\quad 7n+4 = 7(2k+1)+1 = 14k+15$
$= 2(7k+7)+1$

Problem 4: Prove: If x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (Hint: break it into cases.) - Note: if $x = y$, then $\max(x, y) = \min(x, y) = x = y$.

Proof by cases

(1) $x > y$ LHS $= x + y =$ RHS ✓

(2) $x = y$ $\max(x, y) = \min(x, y) = x = y$ ✓

(3) $x < y$ LHS $= y + x = x + y =$ RHS ✓

Problem 5: Express each set using set builder notation. Then if the set is finite, give its cardinality. Otherwise, indicate that the set is infinite.

a. $\{3, 6, 9, 12, \ldots\}$ $= \{3n \mid n \in \mathbb{Z}^+$ or $n \in \mathbb{Z}_{>0}$ or n is positive integer$\}$

infinite.

b. $\{-3, -1, 1, 3, 5, 7, 9\}$

Card = 7.

Problem 6: Determine whether each statement is true or false for any two sets A and B. If the statement is false, explain why.

a. If $A \subseteq B$, ~~then~~ $\Rightarrow$ $A \subset B$. ✗ $(A = B)$

b. If $A \subset B$, then $A \subseteq B$. ✓

c. If $A = B$, then $A \subseteq B$. ✓

d. If $A = B$, then $A \subset B$. ✗

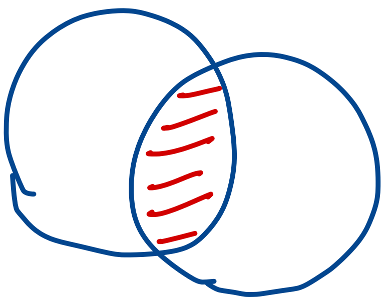
e. If $A \subset B$, then $A \neq B$. ✓

$\subseteq$: subset

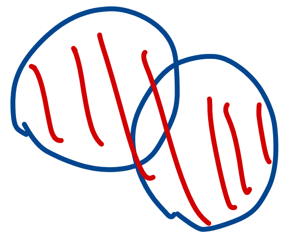
$\subset$: proper subset $(A \neq B)$.

Set Operations

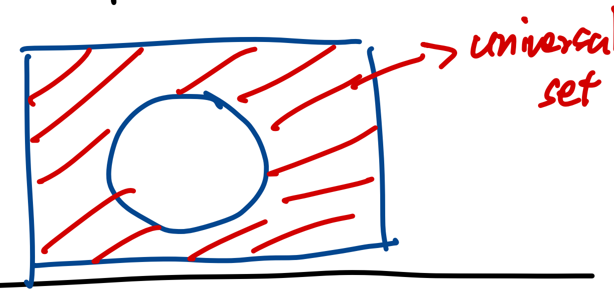
∩ Intersection



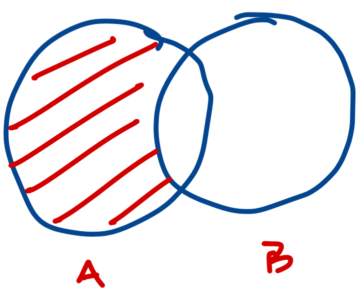
∪ Union



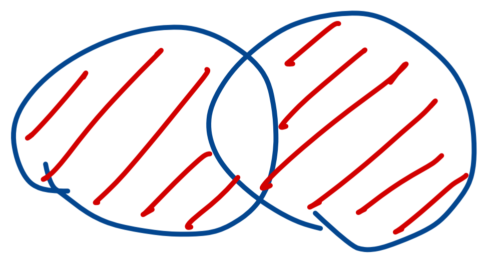
Complement $\overline{A}$



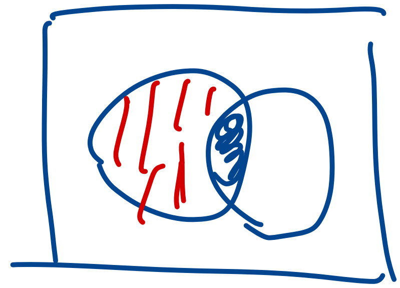
A − B Difference



⊕ Symmetric Difference



10min

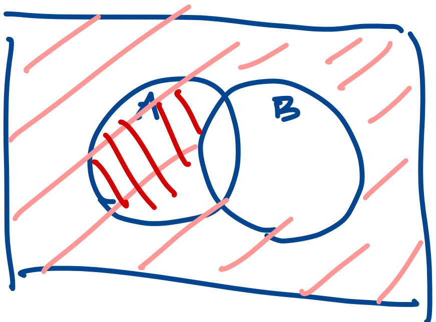


Problem 7: Show that if A and B are sets, then

a. $A - B = A \cap \bar{B}$

b. $(A \cap B) \cup (A \cap \bar{B}) = A$

You can do this by showing that each side is contained in the other, or by using setbuilder notation and logical equivalences. Drawing a Venn diagram may help your intuition, but does NOT constitute a proof.



$A - B \subseteq A \cap \bar{B}$ $\quad$ $A \cap \bar{B} \subseteq A - B$

- $\forall x \in A - B$, $x \in A$, $x \notin B \Rightarrow x \in \bar{B}$

  $x \in A \cap \bar{B}$

  $A - B \subseteq A \cap \bar{B}$

- $\forall x \in A \cap \bar{B}$, $x \in A$, $x \in \bar{B} \Rightarrow x \notin B$

  $x \in A - B$

  $A \cap \bar{B} \subseteq A - B$

Hence $A - B = A \cap \bar{B}$