Week 2 Discussion Worksheet: Proofs, Sets and Functors

June 27, 2022

Problem 1: (Prove by contrapositive): If $3n + 2$ is even, then n is even.

Proof. The contrapositive proposition is: If n is odd then $3n + 2$ is odd. Assuming that n is odd, that is to say, $n = 2k + 1$ for some integer k. We have

$$3n + 2 = 3 * (2k + 1) + 2 = 6k + 5.$$

Since $6k$ is always even, one deduce that $6k + 5$ is odd. □

Problem 2: Use a direct proof to show that every odd integer is the difference of two squares. (Hint: Find the difference of the squares of $k + 1$ and k, where k is a positive integer.)

Proof. For any integer k(not only positive integer!),

$$(k + 1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1.$$

Since every odd integer can be write as $2k + 1$ for some k. □

Problem 3: Prove that if n is an integer, then n is even if and only if $7n + 4$ is even. $\Rightarrow$: Assuming that $7n + 4$ is even, we need to show that n is even. This can be done by prove by contrapositive: suppose n is odd, then $n = 2k + 1$ for some integer k, we have

$$7n + 4 = 14k + 11$$

which is odd.

$\Leftarrow$: Suppose that n is even then one can write $n = 2k$ for some integer k.

$$7n + 4 = 14n + 4$$

is even.

Problem 4: Prove: If x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (Hint: break it into cases.) - Note: if $x = y$, then $\max(x, y) = \min(x, y) = x = y$.

Case 1: $x > y$. $max(x, y) = x, min(x, y) = y$. Therefore

$$max(x, y) + min(x, y) = x + y.$$

- Case 2: $x = y$. $max(x, y) = min(x, y) = x = y$. Therefore

$$max(x, y) + min(x, y) = x + y.$$

Case 3: $x < y$. $max(x, y) = y, min(x, y) = x$. Therefore

$$max(x, y) + min(x, y) = x + y.$$

Problem 5: Express each set using set builder notation. Then if the set is finite, give its cardinality. Otherwise, indicate that the set is infinite.

a. $\{3, 6, 9, 12, \ldots.\}$

b. $\{-3, -1, 1, 3, 5, 7, 9\}$

a. $\{3x \mid x \in \mathbb{Z} \text{ and } x \geq 1\}$. This is an infinite set.
b. $\{2x + 1 \mid x \in \mathbb{Z} \text{ and } -2 \leq x \leq 4\}$. This is a finite set with cardinality equals to 7.

Problem 6: Determine whether each statement is true or false for any two sets A and B. If the statement is false, explain why.

a. If $A \subseteq B$, then $A \subset B$.

b. If $A \subset B$, then $A \subseteq B$.

c. If $A = B$, then $A \subseteq B$.

d. If $A = B$, then $A \subset B$.

e. If $A \subset B$, then $A \neq B$.

a: F. It is possible that A=B.
b: T.
c: T.
d: F: A is not a proper subset of B.
e: T.

Problem 7: Show that if A and B are sets, then

a. $A - B = A \cap \bar{B}$

b. $(A \cap B) \cup (A \cap \bar{B}) = A$

You can do this by showing that each side is contained in the other, or by using setbuilder notation and logical equivalences. Drawing a Venn diagram may help your intuition, but does NOT constitute a proof.

Proof. a. $A - B = \{x \mid (x \in A)^{(}x \notin B)\} = A \cap \bar{B}$.

b. $(A \cap B) \cup (A \cap \bar{B}) = A \cap (B \cup \bar{B}) = A$.

$\square$