Lecture 2: Negative numbers and Powers

In the previous lecture, we introduced natural numbers

$$0, 1, 2, \ldots$$

Now pretend your little brother/sister had no idea about negative numbers and their operations. How do you explain a negative number to them?

1. Negative Numbers

For natural numbers, we can define the subtraction $a - b$ only when $a \geq b$. But how can we extend this to $b - a$?

Definition: For any natural number a, we define $-a$ as the number such that $a + (-a) = 0$ and $(-a) + a = 0$.

We write both conditions because, at this stage, we do not know if addition with negative numbers is commutative.

Now, observe that

$$(b - a) + (a - b) = 0$$

so $b - a = -(a - b)$.

Given $a, b > 0$, the extended addition is defined as:

- $a + b$ is the usual sum.
- $a + (-b) = a + (-b) + b - b = a - b$.
- $(-a) + (-b) = -(a + b)$ because $(-a) + (-b) + b + a = 0$.

For this extended addition, you can prove it is both commutative and associative.

The rule $a - b = a + (-b)$ is very important because it turns a subtraction into an addition. We will revisit this rule later.

Negative Number Multiplication

How do we define multiplication for all integers?

Definition: For any natural number a, $(-1) \cdot a = a \cdot (-1) = -a$.

Now, we can use the distributive law to define multiplication with negatives:

$$(-3) \cdot 8 = (-1-1-1) \cdot 8 = (-1) \cdot 8 + (-1) \cdot 8 + (-1) \cdot 8 = -8-8-8 = -24$$

Or, using the associative law:

$$(-3) \cdot 8 = ((-1) \cdot 3) \cdot 8 = (-1)(3 \cdot 8) = -24$$

Multiplying Two Negative Numbers

To define the product of two negative numbers, we use the distributive law:

$$(-1)(-1) + (1)(-1) = (-1+1) \cdot 1 = 0$$

So $(-1)(-1) + (-1) = (-1)(-1) - 1 = 0$. Thus, $(-1)(-1) = 1$.

Now, for $a, b > 0$, the extended multiplication becomes:

- ab is the usual product.
- $(-a)b = ((-1)(a))(b) = -ab$
- $a(-b) = a((-1)b) = -ab$
- $(-a)(-b) = (-1)a(-b) = (-1)(-ab) = ab$

2. Rational numbers

Similar to introducing negative numbers, division, as the inverse operator of multiplication is not defined in general. So we introduce the set of rational numbers

$$\frac{a}{b}$$

Definition 1: For any nonzero integer a, we define the reciprocical $\frac{1}{a}$ such that $\frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$.

Different than subtraction, we need to consider the case $\frac{1}{a} + \frac{1}{a}$ which is not the reciprocical of an integer. So we further introduce the set of rationals.

Definition 2: For any integer a and any nonzero integer b, we define the fraction $\frac{a}{b} = a \cdot \frac{1}{b} = \frac{1}{b} \cdot a$.

Now for any two integers a, b, the division

$$a/b = a/b \cdot b \cdot \frac{1}{b} = a \cdot \frac{1}{b}$$

So fraction turns division into multiplication.

The expanded multiplication and division is below:

- ab is the usual
- $\frac{a}{b} \cdot \frac{c}{d} = a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d} = \frac{ac}{bd}$
- $\frac{a}{b} / \frac{c}{d} = \frac{a}{b} / \frac{c}{d} \cdot \frac{c}{d} \cdot \frac{d}{c} = \frac{ad}{bc}$

The expanded addition become more complicated now

- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
- $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} = \frac{ad-bc}{bd}$

Exercises: Prove the commutative and associative law for addition of rationals.

3. Powers

Definition: For any number a and a positive integer n, we define a^n as the product of n factors of a, where n is called the exponent.

For example:

- $a^2 = a \cdot a$
- $a^3 = a \cdot a \cdot a$
- and so on.

Exponent Rules

- **Addition of exponents:**
  $a^{m+n} = a^m a^n$ for $m, n > 0$.
  This is a powerful rule and leads to all other exponent rules.
- **Zero exponent:**
  What is a^0?

If we follow the exponent rule:
$a^2 = a^{2+0} = a^2 a^0$
So $a^0 = 1$.

- **Negative exponent:**
  What is a^{-n}?
  If we follow the exponent rule:

$$a^{-n} a^n = a^{-n+n} = a^0 = 1$$

  So $a^{-n} = \frac{1}{a^n}$.

- **Subtraction of exponents:**
  $a^{m-n} = a^{m+(-n)} = a^m a^{-n} = \frac{a^m}{a^n}$
- The above rules expand to all integer exponents.
- **Fractional exponents:**
  How about $a^{\frac{1}{2}}$?
  If we follow the exponent rule:

$$a^{\frac{1}{2}} a^{\frac{1}{2}} = a^1$$

  So $a^{\frac{1}{2}} = \sqrt{a}$.
  This generalizes to:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

- **Exponent of exponent:**

$$(a^2)^3 = a^2 a^2 a^2 = a^{2+2+2} = a^{2 \cdot 3}$$

$$(a^m)^n = a^{mn}$$

- **Product and quotient rules:**

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

  The above rules expand to all rational exponents.

Examples

- The speed of light: 3×10^8 m/s.
- The diameter of the universe: 93 billion light years, which is

$$3 \times 10^8 \times 365 \times 24 \times 3600 \times 93 \times 10^9 < 10^{1+8+3+2+4+2+9} = 10^{29} \text{ m}$$

- The smallest scale in quantum physics: 1.6×10^{-35} m $> 10^{-36}$, called **Planck length**.
- The diameter of an atom: 10^{-11} m.
- The total countable objects in the universe cannot be larger than

$$\left(\frac{10^{29}}{10^{-36}}\right)^3 = (10^{65})^3 < 10^{200}$$

Note:
Physics is about finiteness, while math is about infinity. Are there infinitely many or finitely many primes? It does not matter in physics as long as you find primes less than 10^{200}, but it matters in math.