Lecture 5: Arithmetic and geometric series

Number series

A **number series** is a sequence of numbers.

$$a_1, a_2, a_3, \ldots$$

Example

- the natural number series $0, 1, 2, \ldots$ with the formula $a_n = n - 1$
- the perfect square series $0, 1, 4, 9, \ldots$ with the formula $a_n = (n-1)^2$
- the prime number series $2, 3, 5, 7, \ldots$ with no obvious formula
- Fibonacci series $1, 1, 2, 3, 5, 8, \ldots$ with $a_{n+2} = a_n + a_{n+1}$ when $n \geq 1$
- Harmonic series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ with the formula $a_n = \frac{1}{n}$
- the 9 series $0.9, 0.99, 0.999, 0.9999, 0.99999\ldots.$ and the limit is 1.
- the π series $3, 3.1, 3.14, 3.141, 3.1415, \ldots$
- the series of positive rational numbers $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}, \ldots$ with numerating all nonrepeating fractions with both numerator and denominator up to k
- Check the On-Line Encyclopedia of Integer Sequences (OEIS) (Link) for more awesome series.

Arithmetic progression (series)

An arithmetic progression is a number series with equal distance. It has the formula

$$a_n = a_1 + (n-1)d,$$

where a_1 is the first term and d is the equal distance.

Term counting formula

$$n = \frac{a_n - a_1}{d} + 1$$

Example

- $1, 3, 5, 7, 9, \ldots$ with $a_n = 2n - 1$
- $2, 4, 6, 8, 10, \ldots$ with $a_n = 2n$
- $3, 7, 11, 15, \ldots$ with $a_n = 3 + 4(n-1)$
- $3, 7, 11, \ldots$ what is the 50th term? $a_{50} = 3 + 4 * 49 = 199$
- How many terms of a series starting from 3 and ends at 35 with equal distance 4?

- Term formula gives us total $(35-3)/4+1=9$ terms

The sum formula

The sum of the arithmetic progression has the formula below

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n = (a_1 + a_n) * n/2,$$

where n is the number of terms which can be calculated by

$$n = (a_n - a_1)/d + 1.$$

Example

- $1+2+...+100=(1+100)*100/2=5050$
- $1+3+5+7+...+99=?$
  - the number of terms $n=(99-1)/2+1=50.$
  - the sum $=(1+99)*50/2=50^2=2500$
- $1+3+5+7+...+(2n-1)=(1+2n-1)*n/2=n^2$
  - the famous geometric proof
- $3+7+11+...+103=?$
  - the number of terms $n=(103-3)/4+1=26$
  - the sum $=(3+103)*26/2=106*13=1378$

Geometric series

The geometric series is of the form $a_n = ar^{n-1}$

$$a, ar, ar^2, ar^3, ...$$

The sum formula

Recall the basic algebraic formula below

$$(x-1)(1+x+...+x^n) = x + x^2 + ... + x^{n+1} - 1 - x - x^2 - ... - x^n = x^{n+1} - 1$$

Therefore

$$a + ar + ... + ar^{n-1} = a(1 + r + ... + r^{n-1}) = \frac{a(r^n - 1)}{r - 1}$$

Example $1+2+4+...+2048=?$

- how many terms? $2048 = 2^{11}$ so 11 terms.
- by the formula, we have $r = 2$ and $a = 1$. So the sum equals $2^{12} - 1 = 4096 - 1 = 4095$.
- can you provide a geometric proof?

The infinite sum formula when $|r| < 1$

$$a + ar + ar^2 + ar^3 + ... = a(1 + r + r^2 + r^3 + ...)$$

So it suffices to find the value

$$x = 1 + r + r^2 + r^3 + ...$$

$$rx = r + r^2 + r^3 + ... = x - 1$$

$$1 = (1 - r)x$$

$$x = \frac{1}{1 - r}$$

Thus the sum

$$a + ar + ar^2 + ar^3 + ... = \frac{a}{1 - r}$$

Example

- $1 + \frac{1}{n} + \frac{1}{n^2} + ... = \frac{1}{1 - \frac{1}{n}} = \frac{n}{n-1} = 1 + \frac{1}{n-1}$

Quadratic equation solution example

- $x^2 - x - 1 = 0$
- $x^2 - 3x + 2 = 0$

Fibonacci series

The Fibonacci series is defined with the following recursive formula

$$a_n = a_{n-1} + a_{n-2}$$

with the first few terms:

$$1, 1, 2, 3, 5, 8, ...$$

Why this is related to golden ratio?

$$\frac{a_n}{a_{n-1}} = \frac{a_{n-1} + a_{n-2}}{a_{n-1}} = 1 + \frac{a_{n-2}}{a_{n-1}} = 1 + \frac{1}{\frac{a_{n-1}}{a_{n-2}}}$$

When n is very large, approximately $r = \frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}}$. So we have

$$r = 1 + \frac{1}{r}$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2} = 1.618... \text{ or } -0.618...$$

If we look at the first few terms, the fraction limits to $1.618...$, which is the golden ratio.

Fibonacci series II

For the recursive formula

$$a_n = a_{n-1} + a_{n-2}$$

can we find other solutions?

Let us insert a geometric series $a_n = r^n$ in

$$r^n = r^{n-1} + r^{n-2}$$

$$r^n - r^{n-1} - r^{n-2} = r^{n-2}(r^2 - r - 1) = 0$$

$r = 0$ is a trivial solution so not interested. So the series

$$(\frac{1 \pm \sqrt{5}}{2})^n$$

is a solution for Fibonacci formula and so is

$$T(n) = a(\frac{1 + \sqrt{5}}{2})^{n-1} + b(\frac{1 - \sqrt{5}}{2})^{n-1}$$

for any numbers a, b. This is called the general solution for the Fibonacci recursive formula.

For $n = 1, T(1) = a + b = 1$

For $n = 2, T(2) = a(\frac{1+\sqrt{5}}{2}) + b(\frac{1-\sqrt{5}}{2}) = \frac{a+b}{2} + \frac{a-b}{2}\sqrt{5} = 1$

This implies

$$a - b = \frac{1}{\sqrt{5}}$$

So

$$\begin{aligned}
T(n) &= \frac{1}{2}(1 + \frac{1}{\sqrt{5}})(\frac{1+\sqrt{5}}{2})^{n-1} + \frac{1}{2}(1 - \frac{1}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^{n-1} \\
&= \frac{1}{2}(\frac{1+\sqrt{5}}{\sqrt{5}})(\frac{1+\sqrt{5}}{2})^{n-1} + \frac{1}{2}(\frac{\sqrt{5}-1}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^{n-1} \\
&= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n} - (\frac{1-\sqrt{5}}{2})^{n})
\end{aligned}$$

Exercise Verify $T(3) = 2$ and $T(4) = 3$.