

Lecture 1: Integers and the Binary System

1. Natural Numbers



The sequence of **natural numbers** is:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \dots$$



We define:

- **Addition:** $a + b$ is the union of a apples and b apples.
- **Multiplication:** $a \cdot b$ is the stack of a rows of b apples.

Example:

- Let $a = 2 \rightarrow$ 
- Let $b = 3 \rightarrow$ 

Then:

- $a + b$ is

- $a \cdot b$ is


Note

Here we only define **addition** and **multiplication**.

Subtraction and division are the inverse processes and are omitted.

Basic Laws of Arithmetic



Euler suggested using a, b, c, \dots for specific numbers (and x, y, z, \dots for unknowns). This is the essence of **algebra**.

The natural numbers satisfy the following laws:



- **Commutative Laws**
 - $a + b = b + a$
 - $ab = ba$
- **Associative Laws**
 - $(a + b) + c = a + (b + c)$
 - $(ab)c = a(bc)$
- **Distributive Laws**
 - $a(b + c) = ab + ac$
 - $(a + b)c = ac + bc$

Proof of Commutativity (Intuitive)

Take:

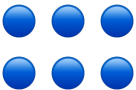
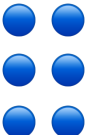
- $a = 2 \rightarrow$ 
- $b = 3 \rightarrow$ 


Then:

- $a + b =$ 
- $b + a =$ 

Clearly, $a + b = b + a$.

For multiplication:

- $a \cdot b =$

- $b \cdot a =$


Thus $ab = ba$. 

Comments

- **Commutative** and **associative** laws mean the order of operations does not matter:

$$(4 \cdot 156) \cdot 25 = 25 \cdot (4 \cdot 156) = (25 \cdot 4) \cdot 156 = 15600$$

- **Distributive law** is helpful for mental arithmetic:

$$8 \cdot 127 = 8 \cdot (125 + 2) = 1000 + 16 = 1016$$

- The second distributive law follows from the first and commutativity:

$$(a + b)c = c(a + b) = ca + cb = ac + bc$$

Decimal System

Using addition and multiplication, we can rigorously define base-10:

$$756 = 7 \cdot 100 + 5 \cdot 10 + 6$$

This makes **vertical calculations** possible.

2. Binary System

We use base 10 because we have ten fingers.

Imagine a world with **two-fingered people** → they would create the **binary system**:

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, \dots$$

This corresponds to natural numbers in base 2.

For example:

$$(1011)_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 11_{10}$$

Conversion from Decimal to Binary

Convert 11_{10} :

- $11 = 2 \cdot 5 + 1$
- $5 = 2 \cdot 2 + 1$
- $2 = 2 \cdot 1 + 0$
- $1 = 1$

Reading remainders upward: 1011_2 .

Binary Arithmetic

- **Addition**
 - $11 + 11 = 110$
 - $111 + 111 = 1110$
- **Multiplication**
 - $11 \cdot 10 = 110$
 - $11 \cdot 111 = 10001$
- **Subtraction**
 - $10000 - 101 = 1011$
- **Division**
 - $1001/11 = 11$ (so 1001 is not prime in binary)
 - $1/11 = 0.010101\dots$

👉 Verify each case against decimal arithmetic.

3. Natural Numbers via Peano Axioms

A formal definition avoids dependence on fingers or aliens.

- 0 is a natural number.
- For every a , $S(a)$ (the **successor**) is a natural number.
- If $S(a) = S(b)$, then $a = b$.
- $S(a) \neq 0$.

Thus:

$$1 = S(0), \quad 2 = S(1) = S(S(0)), \quad \dots$$


Addition is defined recursively:

- $a + 0 = a$
- $a + S(b) = S(a + b)$

Lemma 1. $a + S(0) = S(a)$; in other words, $a + 1 = S(a)$.


Lemma 2. $0 + b = b$

Proof.

- $0 + 0 = 0$ (definition).
- $0 + 1 = 0 + S(0) = S(0 + 0) = S(0) = 1$.
- $0 + 2 = 0 + S(1) = S(0 + 1) = S(1) = 2$.
- Continue forever. 

Lemma 3. $S(a) + b = S(a + b)$

Proof.


- $S(a) + 0 = S(a) = S(a + 0)$.
- $S(a) + 1 = S(a) + S(0) = S(S(a) + 0) = S(S(a)) = S(a + 1)$.
- $S(a) + 2 = S(a) + S(1) = S(S(a) + 1) = S(S(a + 1)) = S(a + 2)$.
- Continue forever. 

Theorem. (Commutativity of Addition)

$$a + b = b + a$$

Proof.

1. **Base case:** $a + 0 = a = 0 + a$ (by Lemma 2).
2. $a + 1 = a + S(0) = S(a + 0) = S(a) = S(0) + a = 1 + a$.
3. $a + 2 = a + S(1) = S(a + 1) = S(1 + a) = S(1) + a = 2 + a$.
4. $a + 3 = a + S(2) = S(a + 2) = S(2 + a) = S(2) + a = 3 + a$.

By continuing, $a + b = b + a$ for all b . 

 **Note:** This is essentially a **proof by induction**:

- Base case: $a + 0 = 0 + a$
- Inductive step: if $a + b = b + a$, then

$$a + S(b) = S(a + b) = S(b + a) = S(b) + a$$

We presented it with small cases for intuition.