

# Lecture 1: Integers and the Binary System



Natural numbers are the sequence of numbers

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

where we can define

- addition  $a + b$  as the union of  $a$  apples and  $b$  apples
- multiplication  $a * b$  as the stack of  $a$  rows of  $b$  apples

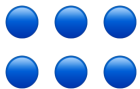
Let's use:

- $a = 2 \rightarrow$  
- $b = 3 \rightarrow$  

Then  $a + b$  is defined as



and  $ab$  is defined as



- Here we only define addition and multiplication for two numbers.
- subtraction and division can be thought as the inverse process of addition and multiplication so we omit here.

## Basic Laws:

Euler told us to use  $a, b, c, \dots$  for a specific number (and  $x, y, z, \dots$  for unknowns). So for a given natural number, we use  $a, b, c, \dots$  to denote a number, which is the essence of the **algebra**.



The addition and multiplication of natural numbers has the following fundamental laws. It is very helpful to use the above naive definition to think geometrically.

- Commutative law
  - $a + b = b + a$
  - $ab = ba$
- Associative law
  - $(a + b) + c = a + (b + c)$

- $(ab)c = a(bc)$  (what is the geometric meaning?)
- Distributive law
  - $a(b + c) = ab + ac$
  - $(a + b)c = ac + bc$

### Proof of commutative laws

Let's use:

- $a = 2 \rightarrow$  
- $b = 3 \rightarrow$  

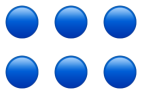
$a + b$  is



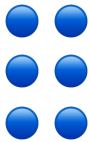
and  $b + a$  is



Similarly,  $a * b$  is



and  $b * a$  is



Thus the commutative laws for addition and multiplication are proved.

### Comments:

- Commutative and associative laws tell us how to combine and rearrange the operation does not matter. That is good for simple calculation like below:

$$(4 * 156) * 25 = 25 * (4 * 156) = (25 * 4) * 156 = 15600$$

- Distributive law can help:

$$8 * 127 = 8 * (125 + 2) = 1016$$

- The last one is derived by commutative and distributive law.

$$(a + b)c = c(a + b) = ca + cb = ac + bc$$

## Decimal system

With addition and multiplication, we can rigorously define the decimal system:

$$756 = 7 * 100 + 5 * 10 + 6$$

which make verticle calculation easy to do.

## Binary system

We use decimal system because we have 10 fingers. What if there is a parallel world with only two-fingers people? What does their math look like? They will only introduce two digits 0 and 1 and create their binary system as below

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...

which corresponds to the natural numbers up to  $(1011)_2 = (11)_{10}$ .

- the expansion of 1011 can be defined similarly:

$$1011 = 1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0 = 11$$

- conversely, any decimal number can be converted to a binary by taking repetive division by 2:

$$11 = 2 * 5 + 1$$

$$5 = 2 * 2 + 1$$

$$2 = 2 * 1 + 0$$

$$1 = 1$$

and taking the remainder inversely.

- all vertical calculation carries the same way by only remembering you can not use 2, 3, ..., 9.

- Addition

- $11 + 11 = 110$

- $111 + 111 = 1110$

- Multiplication

- $11 * 10 = 110$

- $11 * 111 = 10001$

- Substraction

- $10000 - 101 = 1 + 1111 - 101 = 1011$

- Division

- $1001/11 = 11$

- So 1001 is not a prime number in binary case.

- $1/11 = 0.010101....$

- Please verify each step their decimal case.

## Natural number via Peano Axioms

If we would like to define natural number independent of aliens and fingers, Peano Axiom for natural numbers is one solution:

- 0 is a natural number.
- We have the next operator  $S$ . For every natural number  $a$ ,  $S(a)$  is a natural number (the next number).
- If  $S(a) = S(b)$  then  $a = b$ .
- $S(a)$  is never 0.

So  $1 = S(0)$ ,  $2 = S(1) = S(S(0))$ ,....

This does not depend on binary, decimal systems.

- Addition:  $a + 0 = a$  and  $a + S(b) = S(a + b)$

**Lemma 1:**  $a + S(0) = S(a)$ ; in other words,  $a + 1 = S(a)$

**Lemma 2:**  $0 + b = b$

- First we know  $0 + 0 = 0$  by definition.
- Then  $0 + 1 = 0 + S(0) = S(0 + 0) = S(0) = 1$ .
- Then  $0 + 2 = 0 + S(1) = S(0 + 1) = S(1) = 2$ .
- Then repeat on previous steps forever.
- QED

**Lemma 3:**  $S(a) + b = S(a + b)$

- First we know  $S(a) + 0 = S(a) = S(a + 0)$ .
- Then  $S(a) + 1 = S(a) + S(0) = S(S(a) + 0) = S(S(a)) = S(a + 1)$ .
- Then  $S(a) + 2 = S(a) + S(1) = S(S(a) + 1) = S(S(a + 1)) = S(a + 2)$ .
- Then repeat on previous steps forever.
- QED

**Theorem:**  $a + b = b + a$

- First we know  $a + 0 = a = 0 + a$  by Lemma 2.
- Then  $a + 1 = a + S(0) = S(a + 0) = S(a) = S(0) + a = 1 + a$ .
- Then  $a + 2 = a + S(1) = S(a + 1) = S(1 + a) = S(1) + a = 2 + a$ .
- Then repeat on previous steps forever.

- QED