Lecture 7: More on Quadratic Equations and Roots

Quadratic Equation Formula Derivation

Starting from the general quadratic equation:

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

1. Divide by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

2. Complete the square: add $\left(\frac{b}{2a}\right)^2$ to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

3. Recognize the square and simplify the right side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

4. Take square roots and isolate x:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let $\Delta = b^2 - 4ac$ (the discriminant):

- When $\Delta > 0$: two distinct real roots
- When $\Delta = 0$: one double root
- When $\Delta < 0$: no real roots (or two complex conjugate roots)

Example: Analyze the roots of $x^2 + ax + 1 = 0$

- Consider the discriminant $\Delta = a^2 - 4$:
  - When $a > 2$ or $a < -2$: two distinct roots
  - When $a = \pm 2$: one double root
  - When $-2 < a < 2$: no roots

Graphs of Quadratic Functions

The standard form $y = x^2$ gives a parabola opening upward.
For $y = -x^2$, the parabola opens downward.

Key Points:

- For $y = x^2$, the vertex (minimum point) is at $(0, 0)$
- $y = x^2 + c$: shifts the parabola up ($c > 0$) or down ($c < 0$)
- $y = (x + c)^2$: shifts the parabola left ($c > 0$) or right ($c < 0$)
- $y = ax^2$:
  - When $a > 1$: stretches vertically
  - When $0 < a < 1$: compresses vertically
  - When $a < 0$: flips orientation

For any quadratic function $y = ax^2 + bx + c$, we can write:

$$y = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a^2}$$

This form helps us graph using the above transformations.

Examples:

- $y = x^2 - 2x + 1 = (x - 1)^2$: Standard parabola shifted right by 1. Vertex at $(1, 0)$. Intersects x-axis at $x = 1$.
- $y = x^2 - 2x - 3 = (x - 1)^2 - 4$: Shift right by 1, then down by 4. Roots at $x = 3, -1$.
- $y = x^2 - 2x + 2 = (x - 1)^2 + 1$: Shift right by 1, then up by 1. No x-intercepts.

Solving Quadratic Inequalities

Example:

- For $x^2 - 2x - 3 > 0$: Looking at the graph, solutions are $x < -1$ or $x > 3$
- General rule: Solution involves intervals between roots and infinities

Maximum and Minimum of Quadratic Functions

For $y = ax^2 + bx + c$:

- Vertex occurs at $x = -\frac{b}{2a}$

- When $a > 0$: vertex is minimum
- When $a < 0$: vertex is maximum

Example: Maximum Area of Rectangle with Fixed Perimeter
Let $L = a + b$ be half the perimeter. Area is:

$$A = ab = a(L - a) = -a^2 + La = -(a - \frac{L}{2})^2 + \frac{L^2}{4}$$

Maximum occurs at $a = \frac{L}{2}$ (square shape).

Note: For shapes with fixed perimeter, circle maximizes area.

Roots and Powers

Fundamental rules:

- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$

For fractional powers:

- $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Definition: The nth root $\sqrt[n]{a}$ of a nonnegative number a is the nonnegative number whose nth power is a.

Important Notes:

- Odd roots: can handle negative numbers (e.g., $(-1)^3 = -1$)
- Even roots: require non-negative numbers