Lecture 4: Solving Equations

1. Solving Equations in One Variable

Types of Equations

We will use x for the unknown variable and $a, b, c, \ldots$ as fixed numbers.

- **Linear equation:** $ax + b = 0$
- **Quadratic equation:** $ax^2 + bx + c = 0$
- **Cubic equation:** $ax^3 + bx^2 + cx + d = 0$
- **Quartic (degree 4), Quintic (degree 5), Sextic (degree 6):**

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

  The left-hand side is called a **polynomial** of degree n. The solutions (roots of $p(x)$) are values x_0 such that $p(x_0) = 0$.

Solving Linear Equations

For the equation $ax + b = 0$, consider two cases:

- If $a = 0$, the equation is solvable only when $b = 0$, and x can be any number.
- If $a \neq 0$, then $x = -\frac{b}{a}$.

Observation: Every rational number is a solution of a linear equation, and every solution of a linear equation is a rational number.

Solving Quadratic Equations

For the equation $ax^2 + bx + c = 0$, consider two cases:

- If $a = 0$, it reduces to a linear equation (see above). All solutions are rational.
- If $a \neq 0$, divide both sides by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Let $p = \frac{b}{a}$ and $q = \frac{c}{a}$, so the equation becomes:

$$x^2 + px + q = 0$$

Easy and Difficult Example

For the equation $x^2 = 2$, the solution is $x = \pm\sqrt{2}$.

Question: Is $\sqrt{2}$ a rational number? (Recall: linear solutions are always rational.)

First Crisis in Mathematics:
The ancient Greeks believed all numbers were rational because they were "rational"!

Theorem: $\sqrt{2}$ is not a rational number.

Proof:
Suppose $\sqrt{2} = \frac{a}{b}$ for integers a, b with no common factors (in lowest terms).
Then $\frac{a^2}{b^2} = 2$, so $a^2 = 2b^2$.

- a must be even, so $a = 2k$.
- Substitute: $4k^2 = 2b^2 \implies 2k^2 = b^2$, so b must also be even.
- But if both a and b are even, the fraction is not in lowest terms. **Contradiction!**

Aftermath:
Legend says the Greeks could not accept this result, so they threw the mathematician who discovered it into the sea!

2. Solving Quadratic Equations by Factoring and Vieta's Theorem

If the equation $x^2 + px + q = 0$ can be factored as $(x - x_1)(x - x_2)$, then the roots are x_1 and x_2. For other values of x, the right-hand side is not zero.

Examples:

- $x^2 + 2x + 1 = (x + 1)(x + 1) = 0$ so $x = -1$
- $x(x - 2) = 0$ so $x = 0$ or 2
- $x^2 - 3x + 2 = (x - 1)(x - 2) = 0$ so $x = 1$ or 2

Problem: Guessing the factorization can sometimes be difficult.

Hero #1: Vieta, Italian mathematician

Vieta's Theorem:
If $x^2 + px + q = (x - x_1)(x - x_2) = 0$, then

$$p = -(x_1 + x_2)$$

$$q = x_1 x_2$$

Proof:
$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2 = 0$

Corollary:
To factor $x^2 + px + q, findtwonumberswhosesumis$-p$andwhoseproductis$q$.

Examples:

- $x^2 - 2x - 3$: -3 as a product, $(-1) \times 3$ or $(-3) \times 1$, so $(x-3)(x+1)$
- $x^2 + 25x + 84$: $84 = 4 \times 21$, so $(x+4)(x+21)$
- $x^2 - 7x + 10$: $10 = 2 \times 5$, sum is 7, so $(x-2)(x-5)$
- $x^2 - 9x + 9$: perfect square, $(x-3)^2$
- $x^2 - 36$: difference of squares, $(x-6)(x+6)$

Diophantine's Question (2000 years ago):
Find two numbers whose sum is 9 and whose product is 20.

$$a + b = 9$$

$$ab = 20$$

This is equivalent to solving $x^2 - 9x + 20 = 0$ by Vieta's theorem.

Use algebra:

$$(a+b)^2 - 4ab = (a-b)^2$$

So $a - b = \pm\sqrt{81 - 80} = \pm 1$.

Since $a + b = 9$, the two numbers are 4 and 5.

General Solution:

$$a + b = -p$$

$$ab = q$$

So

$$a - b = \pm\sqrt{p^2 - 4q}$$

Therefore,

$$a, b = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Rewrite for Standard Quadratic Equation:

Plug in $p = \frac{b}{a}$ and $q = \frac{c}{a}$, so the solution is

$$\frac{-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - \frac{4c}{a}}}{2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$