Lecture 8: Polynomials in one variable

Definition

We will use x for the unknown variable and $a, b, c, \ldots$ as fixed numbers.

- **Linear:** $ax + b$
- **Quadratic:** $ax^2 + bx + c$
- **Cubic:** $ax^3 + bx^2 + cx + d = 0$
- **In general:**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  is a **polynomial** of degree n. The **roots** of $f(x)$ are values x_0 such that $f(x_0) = 0$. The **coefficients** of $f(x)$ are $a_n, \ldots, a_0$.

Additions and subtractions

We only need to align the degree and make corresponding coefficient addition and subtraction.

Examples

- $(x+1) + (x^6 + 5x^2 - 4x + 4) = x^6 + 5x^2 - 3x + 5$
- $(x^2 - 3x + 1) - (x^2 - 1) = -3x + 2$
- $-(x^2 - x - 1) = -x^2 + x + 1$

Clearly the addition is commutative and associative. The subtraction is the the inverse of the addition.

Multiplication

To define the multiplication, all we need is the distributive rule.

Examples

- $(x+1)(x-1) = x(x-1) + (x-1) = x^2 - x + x - 1 = x^2 - 1$
- Vieta's formula $(x-a)(x-b) = x^2 - (a+b)x + ab$
- $x^6(x^5 + x + 1) = x^{11} + x^7 + x^6$
- $(x+1)(x^2 - x + 1) = x^3 - x^2 + x + x^2 - x + 1 = x^3 + 1$
- The product of a degree n and a degree m polynomial is a degree $n + m$ polynomial.

Division

To define the division of the polynomial, we need to introduce the long division.

Example 1

To divide $x^2 - 1$ by $x + 1$:

$$
\begin{array}{r}
x-1 \\
x+1{\overline{\smash{\big)}\,x^2+0x-1}} \\
\underline{x^2+x} \\
-x-1 \\
\underline{-x-1} \\
0
\end{array}
$$

So, $\frac{x^2-1}{x+1} = x-1$.

Example 2
To divide x^2 by $x+1$:

$$
\begin{array}{r}
x-1 \\
x+1{\overline{\smash{\big)}\,x^2+0x+0}} \\
\underline{x^2+x} \\
-x+0 \\
\underline{-x-1} \\
1
\end{array}
$$

So, $\frac{x^2}{x+1} = x-1+\frac{1}{x+1}$.

Remark: In this example, we see that there is a remainder for polynomial long division just like integer long division. Why? x here is very similar to the roll of 10 in the decimal system. The above two examples is like $100/11$ and $99/11$.

Example 3
Another way of doing long division:

$$\frac{x^3+2x}{x^2-x+1} = \frac{x^3-x^2+x}{x^2-x+1} + \frac{x^2+x}{x^2-x+1} = x + \frac{x^2+x}{x^2-x+1} = x + \frac{x^2-x+1+2x-1}{x^2-x+1} = x+1+\frac{2x-1}{x^2-x+1}$$

The degree of the result polynomial is always the difference between the degree of two polynomials.

Example 4

$$\frac{x^n-1}{x-1} = \frac{x^n-x^{n-1}+x^{n-1}-1}{x-1} = x^{n-1} + \frac{x^{n-1}-1}{x-1} = x^{n-1}+x^{n-2}+\cdots+1$$

Example 5

$$\frac{x^3+1}{x+1} = \frac{x^3+x^2-x^2+1}{x+1} = x^2-x+1$$

$$\frac{x^4+1}{x+1} = \frac{x^4+x^3-x^3+1}{x+1} = x^3 + \frac{-x^3-x^2+x^2+1}{x+1} = x^3-x^2+\frac{x^2+x-x+1}{x+1} = x^3-x^2+x-1+\frac{2}{x+1}$$

In general, no remainder for odd degree polynomial but remainder for even degree polynomial.

Example 6: Remainder of the division by $x-a$

If $f(x) = q(x)(x-a) + \text{remainder}$, then plugging $x=a$, we get the remainder equals $f(a)$.

If a is a root of $f(x)$, i.e., $f(a)=0$, then the remainder is 0. Thus $x-a$ divides $f(x)$ if and only if a is a root of $f(x)$.

Example 7:
Factor x^4+5x-6. Plug in 2 not working. Plug in -2 works.

$$\frac{x^4+5x-6}{x+2}=x^3+\frac{-2x^3+5x-6}{x+2}=x^3-2x^2+\frac{4x^2+5x-6}{x+2}=x^3-2x^3+4x-3$$

Always check $f(1)$ which is the sum of all coefficients and $f(-1)$ which is the alternating sum of all coefficients.