Lecture 1: Integers and the Binary System

1. Natural Numbers

The sequence of **natural numbers** is:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$$

We define:

- **Addition**: $a + b$ is the union of a apples and b apples.
- **Multiplication**: $a \cdot b$ is the stack of a rows of b apples.

Example:

- Let $a = 2 \rightarrow$ 🔵 🔵
- Let $b = 3 \rightarrow$ 🟢 🟢 🟢

Then:

- $a + b$ is
  🔵 🔵 🟢 🟢 🟢
- $a \cdot b$ is
  🔵 🔵 🔵
  🔵 🔵 🔵

Note

Here we only define **addition** and **multiplication**.
Subtraction and division are the inverse processes and are omitted.

Basic Laws of Arithmetic

Euler suggested using $a, b, c, \ldots$ for specific numbers (and $x, y, z, \ldots$ for unknowns). This is the essence of **algebra**.

The natural numbers satisfy the following laws:

- **Commutative Laws**
  - $a + b = b + a$
  - $ab = ba$
- **Associative Laws**
  - $(a + b) + c = a + (b + c)$
  - $(ab)c = a(bc)$
- **Distributive Laws**
  - $a(b + c) = ab + ac$
  - $(a + b)c = ac + bc$

Proof of Commutativity (Intuitive)

Take:

- $a = 2 \rightarrow$ 🔵 🔵
- $b = 3 \rightarrow$ 🟢 🟢 🟢

Then:

- $a + b =$ 🔵 🔵 🟢 🟢 🟢
- $b + a =$ 🟢 🟢 🟢 🔵 🔵

Clearly, $a + b = b + a$.

For multiplication:

- $a \cdot b =$
  🔵 🔵 🔵
  🔵 🔵 🔵
- $b \cdot a =$
  🔵 🔵
  🔵 🔵
  🔵 🔵

Thus $ab = ba$. ✅

Comments

- **Commutative** and **associative** laws mean the order of operations does not matter:

$$(4 \cdot 156) \cdot 25 = 25 \cdot (4 \cdot 156) = (25 \cdot 4) \cdot 156 = 15600$$

- **Distributive law** is helpful for mental arithmetic:

$$8 \cdot 127 = 8 \cdot (125 + 2) = 1000 + 16 = 1016$$

- The second distributive law follows from the first and commutativity:

$$(a + b)c = c(a + b) = ca + cb = ac + bc$$

Decimal System

Using addition and multiplication, we can rigorously define base-10:

$$756 = 7 \cdot 100 + 5 \cdot 10 + 6$$

This makes **vertical calculations** possible.

2. Binary System

We use base 10 because we have ten fingers.
Imagine a world with **two-fingered people** → they would create the **binary system**:

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, \ldots$$

This corresponds to natural numbers in base 2.
For example:

$$(1011)_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 11_{10}$$

Conversion from Decimal to Binary

Convert 11_{10}:

- $11 = 2 \cdot 5 + 1$
- $5 = 2 \cdot 2 + 1$
- $2 = 2 \cdot 1 + 0$
- $1 = 1$

Reading remainders upward: 1011_2.

Binary Arithmetic

- **Addition**
  - $11 + 11 = 110$
  - $111 + 111 = 1110$
- **Multiplication**
  - $11 \cdot 10 = 110$
  - $11 \cdot 111 = 10001$
- **Subtraction**
  - $10000 - 101 = 1011$
- **Division**
  - $1001/11 = 11$ (so 1001 is not prime in binary)
  - $1/11 = 0.010101\ldots$

👉 Verify each case against decimal arithmetic.

3. Natural Numbers via Peano Axioms

A formal definition avoids dependence on fingers or aliens.

- 0 is a natural number.
- For every a, $S(a)$ (the **successor**) is a natural number.
- If $S(a) = S(b)$, then $a = b$.
- $S(a) \neq 0$.

Thus:

$$1 = S(0), \quad 2 = S(1) = S(S(0)), \quad \dots$$

Addition is defined recursively:

- $a + 0 = a$
- $a + S(b) = S(a + b)$

Lemma 1. $a + S(0) = S(a)$; in other words, $a + 1 = S(a)$.

Lemma 2. $0 + b = b$

Proof.

- $0 + 0 = 0$ (definition).
- $0 + 1 = 0 + S(0) = S(0 + 0) = S(0) = 1$.
- $0 + 2 = 0 + S(1) = S(0 + 1) = S(1) = 2$.
- Continue forever. ✅

Lemma 3. $S(a) + b = S(a + b)$

Proof.

- $S(a) + 0 = S(a) = S(a + 0)$.
- $S(a) + 1 = S(a) + S(0) = S(S(a) + 0) = S(S(a)) = S(a + 1)$.
- $S(a) + 2 = S(a) + S(1) = S(S(a) + 1) = S(S(a + 1)) = S(a + 2)$.
- Continue forever. ✅

Theorem. (Commutativity of Addition)

$$a + b = b + a$$

Proof.

1. **Base case**: $a + 0 = a = 0 + a$ (by Lemma 2).
2. $a + 1 = a + S(0) = S(a + 0) = S(a) = S(0) + a = 1 + a$.
3. $a + 2 = a + S(1) = S(a + 1) = S(1 + a) = S(1) + a = 2 + a$.
4. $a + 3 = a + S(2) = S(a + 2) = S(2 + a) = S(2) + a = 3 + a$.

By continuing, $a + b = b + a$ for all b. ✅

> 💡 **Note:** This is essentially a **proof by induction**:
>
> - Base case: $a + 0 = 0 + a$
> - Inductive step: if $a + b = b + a$, then
>
> $$a + S(b) = S(a + b) = S(b + a) = S(b) + a$$
>
> We presented it with small cases for intuition.