# Lecture 8: Polynomials in one variable II

#### Remainder of the division by $x - a$

If $f(x) = q(x)(x - a) + \text{remainder}$, then we first observe that the remainder must be a number (degree $0$). If the remainder degree has larger than $1$, then we can continue dividing $x - a$.

**Example**: Consider $x^2 + 1/(x + 1)$. The long division gives the remainder $2$.

**Theorem**: The remainder equals $f(a)$. In particular, if $f(a) = 0$, then $x - a$ divides $f(x)$.

**Proof**: When we plug $a$ into the long division, we get $f(a) = 0 + \text{remainder}$. So the remainder is $f(a)$.

**Example revisited**: $x + 1 = x - (-1)$. So the remainder is $f(-1) = (-1)^2 + 1 = 2$.

**Example**:

- the remainder of $x^2 - 3x + 2$ divides $x - 2$ equals $f(2) = 4 - 6 + 2 = 0$.
- the remainder of $x^2 - 3x + 2$ divides $x - 1$ equals $f(1) = 1 - 3 + 2 = 0$.
- in other words, $x - 1$ and $x - 2$ both divides $x^2 - 3x + 2$. This is indeed the factorization

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

- Factorization of $x^2 + 17x - 18$. Consider $f(1) = 1 + 17 - 18 = 0$. So $x - 1$ divides $x^2 + 17x - 18$ and the factorization becomes $x^2 + 17x - 18 = (x - 1)(x + 18)$.

#### Applications to factorizations

- $f(1)$ is the sum of all coefficients and $f(-1)$ is the alternating sum of all coefficients of the polynomial. Those are always easy to check first when dealing with complicated factorization problems.
- Factorization of $x^2 + 17x - 18$. Consider $f(1) = 1 + 17 - 18 = 0$. So $x - 1$ divides $x^2 + 17x - 18$ and the factorization becomes $x^2 + 17x - 18 = (x - 1)(x + 18)$.
- Factorization of $x^2 - 17x - 18$. Consider $f(-1) = 1 + 17 - 18 = 0$. So $x + 1$ divides $x^2 - 17x - 18$ and the factorization becomes $x^2 - 17x - 18 = (x + 1)(x + 18)$.
- Factorization of $x^3 - 3x - 2$. Consider $f(-1) = -1 + 3 - 2 = 0$. So $x + 1$ divides $x^3 - 3x - 2 = (x + 1)(x^2 - x - 2) = (x + 1)^2(x - 2)$. We can check $f(2) = 0$ as well.

**Theorem**: a polynomial of degree $d$ can have up to $d$ roots.

**Proof**: If $f(x)$ has a root $a$, then $x - a$ divides $f(x)$. Now $f(x) = (x - a)g(x)$, where $g(x)$ is a polynomial of degree $d - 1$. If $f(x)$ has more than $d$ roots, we do not enough degree to divide all of $x - a_i$.

**Examples**:

- $x^2 - 3x + 2$ has $2$ roots
- $x^2 + 2x + 1$ has one (double) root
- $x^2 + 1$ has no root, but have two distinct complex roots $x^2 + 1 = (x - \sqrt{-1})(x + \sqrt{-1}) = (x - i)(x + i)$.
- $x^2 - 3x - 2$ has one double root and one simple root.
- $(x - 1)^n$ has one root with multiplicity $n$.
- $x^3 + 1 = (x + 1)(x^2 - x + 1)$ has one root and two complex roots.

**The Fundamental Theorem of Algebra (Gauss, the guy who did 1+2+...+100)**: a polynomial of degree $d$ have exactly $d$ roots when counting multiplicities.

**Factorization of a polynomial**

**Example**:

- $x^3+x^2+x+1 = x^2(x+1)+(x+1) = (x^2+1)(x+1)$. You could also use $f(-1)=0$.
- $x^5+x^4+\cdots+x+1 = (x^4+x^2+1)(x+1)$. You could also use $f(-1)=0$.
- $x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-2x^2 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$. No real roots at all.
- $x^5+x+1 = x^5+x^4+x^3-x^4-x^3-x^2+x^2+x+1 = (x^2+x+1)(x^3-x^2+1)$. This is tricky and we do not know if we could factorize $x^3-x^2+1$ any further.
- $x^n-1 = (x-1)(x^{n-1}+x^{n-2}+\ldots+x+1)$

**Examples to more variables**:

- $a^2-b^2 = (a+b)(a-b)$
- $a^3-b^3 = (a+b)(a^2-ab+b^2)$
- $a^3+b^3 = (a+b)(a^2+ab+b^2)$
- $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$. This is really unexpected.

**Rational expressions**

**Example**:

- $\frac{1}{(\frac{1}{a}+\frac{1}{b})/2} = \frac{2ab}{a+b}$. Harmonic mean of two numbers: average speed with half on speed $a$ and half on speed $b$.
- $\frac{1}{1+\frac{1}{x}}$
- $\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$

**Last fun example:**

$$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-a)(x-c)}{(b-a)(b-c)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}$$

$$=\frac{(x-a)(x-b)(a-b)+(x-a)(x-c)(c-a)+(x-b)(x-c)(b-c)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(a-b)x^2-(a^2-b^2)x+ab(a-b)+(c-a)x^2-(c^2-a^2)x+ac(c-a)+(b-c)x^2-(b^2-c^2)x+bc(b-c)}{(a-b)(a-c)(b-c)}$$

$$=\frac{a^2b-ab^2+ac^2-a^2c+b^2c-bc^2}{(a^2-ab-ac+bc)(b-c)}$$

$$=\frac{a^2b-ab^2+ac^2-a^2c+b^2c-bc^2}{(a^2b-ab^2-abc+bc^2-a^2c+abc+ac^2-bc^2)}=1$$