Lecture 6: More on quadratic equations

To find the roots of the following quadratic equation

$$ax^2 + bx + c = 0,$$

there are two main methods: factoization and formulas.

Factorization

If we have a factorization of the form

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0,$$

then there are two solutions

$$x = \alpha \text{ or } \beta.$$

When $a = 1$, using Vieta's formula, we have

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta,$$

so we need to guess $-b = \alpha + \beta$ and $c = \alpha\beta$.

Examples

- $x^2 - 2x + 1 = (x - 1)^2 = 0$. So only one (double) root $x = 1$.
- $x^2 - 3x + 2 = (x - 1)(x - 2) = 0$. So $x = 1, 2$.
- $x^2 - 2x - 143 = (x - 13)(x + 11) = 0$. So $x = 13, -11$.

Using the formula

If there is no easy way to factorize, for example $a \neq 1$, better to use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples

- $x^2 - 2x + 35 = (x-1)^2 + 34 = 0$. So no solution. If we use the formula we will get $b^2 - 4ac = 4 - 4 \cdot 35 < 0$, where we cannot apply the square root.
- $x^2 - 3x + 2 = 0$. So $x = \frac{3 \pm \sqrt{9-8}}{2} = 1, 2$.
- $x^2 - 2x - 143 = 0$. So $x = \frac{2 \pm \sqrt{4 + 4 \cdot 143}}{2} = \frac{2 \pm \sqrt{4 \cdot 144}}{2} = 13, -11$.

the discriminant Δ

We define the discriminant of a quadratic equation

$$\Delta = b^2 - 4ac$$

We observe the following:

- When $\Delta > 0$, there are two distinct roots
- When $\Delta = 0$, there is only one (double) root.
- When $\Delta < 0$, there is no root.

Examples

- $x^2 + 1 = 0$, $\Delta = 0 - 4 = -4$. Still plugging in the formula, we have

$$x = \frac{0 \pm \sqrt{-4}}{2} = \pm\sqrt{-1} = \pm i$$

- In high school, you will learn complex numbers. Quadratic equations are solvable over complex numbers because if $\Delta = -d < 0$ where $d > 0$, then the formula will give us

$$x = \frac{-b \pm \sqrt{-d}}{2a} = \frac{-b \pm i\sqrt{d}}{2a}$$

if we denote $i = \sqrt{-1}$.