Lecture 3: The Binomial Theorem $(a + b)^n$

1. Expanding $(a + b)^2$

Let's expand $(a + b)^2$ step by step:

$$(a+b)^2 = (a+b)(a+b)$$

Apply the distributive law (multiply each term in the first bracket by each term in the second bracket):

$$= a(a+b) + b(a+b) = a \cdot a + a \cdot b + b \cdot a + b \cdot b$$

Apply the commutative law, we get

$$(a+b)^2 = a^2 + 2ab + b^2$$

Visual (Picture) Proof

Imagine a square with side length $(a + b)$.
The area of the square is $(a + b)^2$.

You can split the square into four smaller regions:

- A square of area a^2 (side a)
- A square of area b^2 (side b)
- Two rectangles of area ab each

Here is a diagram:

```
+------------------------+
|         |              |
|   a^2   |    ab        |   a
|         |              |
+------------------------+
|         |              |
|   ab    |    b^2       |   b
|         |              |
+------------------------+
     a          b
```

Exercise Find a geometric proof for $(a+b)(a-b) = a^2 - b^2$

2. Expanding $(a+b)^3$ and $(a+b)^4$

Now let's expand $(a+b)^3$:

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

First, expand $(a+b)(a+b)$ as above:

$$(a+b)(a+b) = a^2 + 2ab + b^2$$

Now multiply this result by $(a+b)$:

$$= (a+b)(a^2 + 2ab + b^2)$$

Apply the distributive law again:

$$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$$

$$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$$

Now apply commutative and associative law, we get

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Exercise Prove this geometrically.

Expanding $(a+b)^4$

$$(a+b)^4 = (a+b)(a^3+3a^2b+3ab^2+b^3)$$

$$= a^4 + a^3b + 3a^2b^2 + ab^3 + ba^3 + 2ba^2b + 2bab^2 + b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

3. Binomial theorem and Pascal's triangle

Pascal's Triangle

Binomial coefficients can be found in **Pascal's Triangle**. Each row of the triangle gives the coefficients for $(a+b)^n$:

```
    1
   1 1
  1 2 1
 1 3 3 1
1 4 6 4 1
```

- The nth row corresponds to the coefficients for $(a+b)^n$. The coefficients in front of each term are called **binomial coefficients**.
- For example, the coefficients for $(a+b)^3$ are $1, 3, 3, 1$.
- the most left and most right numbers are always one.

Theorem: Each number inside Pascal's Triangle is the sum of the two numbers directly above it in the previous row.

Why? to get a^2b, we have two ways $a(ab)$ and $b \cdot a^2$. We have two copies of ab and one copy of a^2 in previous expansion.

Binomial Theorem Formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

This formula uses the coefficients from Pascal's Triangle for each term in the expansion.

Definition:
The binomial coefficient $\binom{n}{k}$ (read as "n choose k") is the number of ways to choose k objects from n objects, without regard to order. It is given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $n!$ means $n \times (n-1) \times \cdots \times 1$.

Another way to understand the binomial coefficient is by counting the number of ways to get k heads when tossing n coins.

Each coin can land as either Heads (H) or Tails (T), so the outcome of tossing one coin is $(H + T)$. For two coins, the possible outcomes are:

$$(H+T)(H+T) = HH + HT + TH + TT$$

- HH (2 heads)
- HT (1 head, 1 tail)
- TH (1 head, 1 tail)
- TT (0 heads)

If we look at the number of ways to get k heads:

- 2 heads: HH (1 way)
- 1 head: HT, TH (2 ways)
- 0 heads: TT (1 way)

These numbers ($1, 2, 1$) match the binomial coefficients for $(a+b)^2$.

For three coins:

$$(H+T)^3 = HHH + HHT + HTH + HTT + THH + THT + TTH + TTT$$

Count the number of ways to get k heads:

- 3 heads: HHH (1 way)
- 2 heads: HHT, HTH, THH (3 ways)
- 1 head: HTT, THT, TTH (3 ways)

- 0 heads: TTT (1 way)

So, the numbers $(1, 3, 3, 1)$ match the binomial coefficients for $(a + b)^3$.

In general:
The binomial coefficient $\binom{n}{k}$ counts the number of ways to get k heads (and $n - k$ tails) when tossing n coins. This is a practical example of combinations and shows how binomial coefficients appear in probability and counting problems.