Lecture 10: Factorization and Roots

Factorization: More Difficult Problems

- $$\begin{aligned}&x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)\\&=xy^2-xz^2+yz^2-yx^2+zx^2-zy^2\\&=x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)\\&=x(y-z)(y+z)+y(z-x)(z+x)+z(x-y)(x+y)\\&=(y-z)\left[x(y+z)-y(z+x)+z(x+y)\right]\\&=(y-z)(y-x)(x-z)\end{aligned}$$
- $$a^{10}+a^5+1=(a^2+a+1)\left(a^8-a^7+a^5-a^4+a^3-a+1\right)$$
- $$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$$
- $$\begin{aligned}(a+b+c)^3-a^3-b^3-c^3&=3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+2abc)\\&=3\left[a^2(b+c)+a(b^2+c^2+2bc)+bc(b+c)\right]\\&=3(b+c)\left[a^2+ab+ac+bc\right]\\&=3(a+b)(a+c)(b+c)\end{aligned}$$
- $$\begin{aligned}&a^3-3a^2b+3ab^2-b^3+b^3-3b^2c+3bc^2-c^3+c^3-3c^2a+3ca^2-a^3\\&=-3a^2b+3ab^2-3b^2c+3bc^2+3a^2c-3ac^2\\&=-3(a^2b-ab^2+b^2c-bc^2-a^2c+ac^2)\\&=-3\left[a^2(b-c)-a(b^2-c^2)+bc(b-c)\right]\\&=-3\left[a^2(b-c)-a(b-c)(b+c)+bc(b-c)\right]\\&=-3(b-c)\left[a^2-ab-ac+bc\right]\\&=-3(b-c)(a-b)(a-c)\end{aligned}$$

Problem 133: If $x+\frac{1}{x}=7$, calculate

- $x^2+\frac{1}{x^2}=$

$$x+\frac{1}{x}=7$$

$$x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} = 49$$

$$x^2 + 2 + \frac{1}{x^2} = 49$$

$$x^2 + \frac{1}{x^2} = 47$$

- $x^3 + \frac{1}{x^3} =$

$$x^3 + 3x^2 \cdot \frac{1}{x} + 3x \cdot \frac{1}{x^2} + \frac{1}{x^3} = 343$$

$$x^3 + 3x + 3\frac{1}{x} + \frac{1}{x^3} = 343$$

$$x^3 + 3(x + \frac{1}{x}) + \frac{1}{x^3} = 343$$

$$x^3 + 21 + \frac{1}{x^3} = 322$$

Roots

Definition The n-th root of a nonnegative number a is the nonnegative number whose n-th power is a, denoted by $\sqrt[n]{a}$.

Example

- $\sqrt{9} = 3$
- $\sqrt[3]{64} = 4$
- we may even define the odd roots for negative numbers, $\sqrt[3]{-64} = -4$. But not for even roots.
- Which one is larger? $\sqrt[3]{3}$ or $\sqrt[4]{4}$. Try taking 12th power.
    - $(\sqrt[3]{3})^{12} = ((\sqrt[3]{3})^3)^4 = 81$
    - $(\sqrt[4]{4})^{12} = ((\sqrt[4]{4})^4)^3 = 64$

- So $\sqrt[3]{3}$ is larger.

Observation: $(a^{\frac{1}{n}})^n = a$. So $\sqrt[n]{a} = a^{\frac{1}{n}}$.

Power rules still hold for fractional numbers m, n:

- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $a^{-n} = \frac{1}{a^n}$

Fraction power: $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

Example

- $a^{\frac{2}{3}} a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}}$

Non-fracitonal powers? How to define $2^{\sqrt{2}}$?
Since $\sqrt{2} = 1.414...$, we can define a series of powers:

- $2^{1.4}$
- $2^{1.41}$
- $2^{1.414}$
- ...
  This series will give us the "limit" number which is $2^{\sqrt{2}}$. This requires calculus to define it.

Awesome proof on $\sqrt{2}^{\sqrt{2}}$
Theorem: There exist an irrational power of an irrational number which is rational.

Proof: If $\sqrt{2}^{\sqrt{2}}$ is rational, then it is proved. If not, then $\sqrt{2}^{\sqrt{2}}$ is irrational. Now consider

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2.$$

So either $\sqrt{2}^{\sqrt{2}}$ is rational or $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is rational.

Theorem is proved without knowing the rationality of $\sqrt{2}^{\sqrt{2}}$!!