Lecture 1: Integers and the Binary System

Natural numbers are the sequence of numbers

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$$

where we can define

- addition $a + b$ as the union of a apples and b apples
- multiplication $a * b$ as the stack of a rows of b apples

Let's use:

- `a = 2` $\rightarrow$ 🔵 🔵
- `b = 3` $\rightarrow$ 🟢 🟢 🟢

Then $a + b$ is defined as

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and ab is defined as
🔵 🔵 🔵
🔵 🔵 🔵

- Here we only define addition and multiplication for two numbers.
- subtraction and division can be thought as the inverse process of additional and multiplication so we omit here.

Basic Laws:
Euler told us to use $a, b, c, \ldots$ for a specific number (and $x, y, z, \ldots$ for unkonwns). So for a given natural number, we use $a, b, c, \ldots$ to denote a number, which is the essence of the **algebra**.

The addition and multiplication of natural numbers has the following fundamental laws. It is very helpful to use the above naive definition to think geometrically.

- Commutative law
  - $a + b = b + a$
  - $ab = ba$
- Associative law
  - $(a + b) + c = a + (b + c)$

- $(ab)c = a(bc)$ (what is the geometric meaning?)
- Distributive law
  - $a(b + c) = ab + ac$
  - $(a + b)c = ac + bc$

Proof of commutative laws

Let's use:

- `a = 2` $\rightarrow$ 🔵 🔵
- `b = 3` $\rightarrow$ 🟢 🟢 🟢

$a + b$ is

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and $b + a$ is

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Similarly, $a * b$ is
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and $b * a$ is
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Thus the commutative laws for addition and multiplication are proved.

Comments:

- Commutative and associative laws tell us how to combine and rearrange the operation does not matter. That is good for simple calculation like below:

$$(4 * 156) * 25 = 25 * (4 * 156) = (25 * 4) * 156 = 15600$$

- Distributive law can help:

$$8 * 127 = 8 * (125 + 2) = 1016$$

- The last one is derived by commutative and distributive law.

$$(a + b)c = c(a + b) = ca + cb = ac + bc$$

Decimal system

With addition and multiplication, we can rigorously define the decimal system:

$$756 = 7 * 100 + 5 * 10 + 6$$

which make verticle calculation easy to do.

Binary system

We use decimal system because we have 10 fingers. What if there is a parallel world with only two-fingers people? What does their math look like? They will only introduce two digits 0 and 1 and create their binary system as below

```
1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...
```

which corresponds to the natural numbers up to $(1011)_2 = (11)_{10}$.

- the expansion of 1011 can be defined similarly:

$$1011 = 1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0 = 11$$

- conversely, any decimal number can be converted to a binary by taking repetivie division by 2:
  $11 = 2 * 5 + 1$
  $5 = 2 * 2 + 1$
  $2 = 2 * 1 + 0$
  $1 = 1$
  and taking the remainder inversely.
- all vertical calculation carries the same way by only remembering you can not use $2, 3, ..., 9$.
- Addition
  - $11 + 11 = 110$
  - $111 + 111 = 1110$
- Multiplication
  - $11 * 10 = 110$
  - $11 * 111 = 10001$
- Substraction
  - $10000 - 101 = 1 + 1111 - 101 = 1011$
- Division
  - $1001/11 = 11$
  - So 1001 is not a prime number in binary case.
  - $1/11 = 0.010101....$

- Please verify each step their decimal case.

Natural number via Peano Axioms

If we would like to define natural number independent of aliens and fingers, Peano Axoim for natural numbers is one solution:

- 0 is a natural number.
- We have the next operator S. For every natural number a, $S(a)$ is a natural number (the next number).
- If $S(a) = S(b)$ then $a = b$.
- $S(a)$ is never 0.

So $1 = S(0)$, $2 = S(1) = S(S(0))$,....

This does not depend on binary, decimal systems.

- Addition: $a + 0 = a$ and $a + S(b) = S(a + b)$

Lemma 1: $a + S(0) = S(a)$; in other words, $a + 1 = S(a)$

Lemma 2: $0 + b = b$

- First we know $0 + 0 = 0$ by definition.
- Then $0 + 1 = 0 + S(0) = S(0 + 0) = S(0) = 1$.
- Then $0 + 2 = 0 + S(1) = S(0 + 1) = S(1) = 2$.
- Then repeat on previous steps forever.
- QED

Lemma 3: $S(a) + b = S(a + b)$

- First we know $S(a) + 0 = S(a) = S(a + 0)$.
- Then $S(a) + 1 = S(a) + S(0) = S(S(a) + 0) = S(S(a)) = S(a + 1)$.
- Then $S(a) + 2 = S(a) + S(1) = S(S(a) + 1) = S(S(a + 1)) = S(a + 2)$.
- Then repeat on previous steps forever.
- QED

Theorem: $a + b = b + a$

- First we know $a + 0 = a = 0 + a$ by Lemma 2.
- Then $a + 1 = a + S(0) = S(a + 0) = S(a) = S(0) + a = 1 + a$.
- Then $a + 2 = a + S(1) = S(a + 1) = S(1 + a) = S(1) + a = 2 + a$.
- Then repeat on previous steps forever.

- QED