Lecture 2: Duality and Desargues' Theorem

The Duality of points and lines

Point–Line Duality in the Projective Plane In the projective plane, there is a near-perfect symmetry between points and lines. We can summarize some basic dual pairs in a table:

Object / Statement	Dual Object / Statement
point	line
line	point
infinitely many points on a line	infinitely many lines through a point
two distinct points determine one line	two distinct lines meet in one point (**except for parallel lines**)
three points are collinear	three lines are concurrent
three non-collinear points	three non-concurrent lines

If you take a statement about points and lines and systematically swap these words using the rule above, you get another correct theorem: the **dual statement**. The following pictures illustrate the duality between points on lines and lines through points.

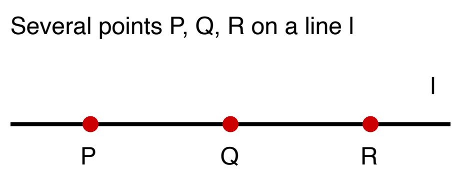

Figure 1: Several points on a line

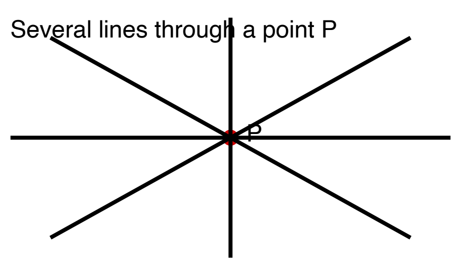

Figure 2: Several lines through a point

Desargues' Triangles and Configuration

Desargues' theorem is one of the most fundamental results in projective geometry, discovered by Girard Desargues in 1639. It establishes a beautiful relationship between two triangles that are in perspective.

Definition: Desargues Triangles Two triangles ABC and $A'B'C'$ are called **Desargues triangles** (or **centrally perspective**) if the lines connecting corresponding vertices are concurrent. That is, the three lines AA', BB', and CC' meet at a single point, often denoted as O.

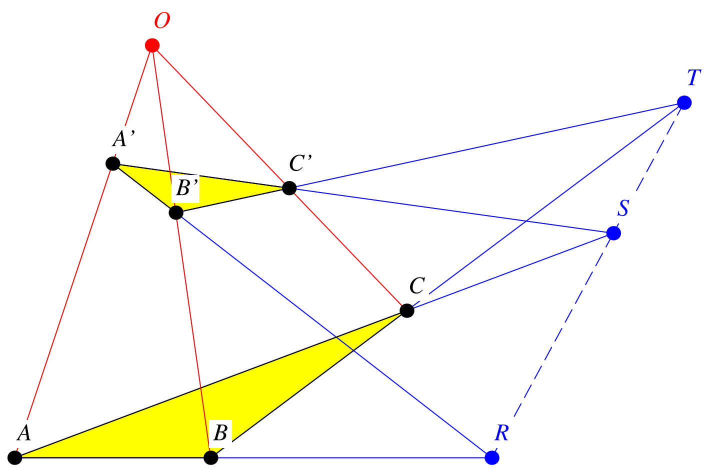

Figure 3: Desargues' configuration

Now we apply the duality principle to define the dual concept for Desargues' configuration.

Object / Statement	Dual Object / Statement
vertices of the triangle	edges (sides) of the triangle
line AA' joining corresponding vertices	intersection point $R = AB \cap A'B'$ of corresponding sides
3 concurrent lines	3 collinear points
Desargues triangles (perspective from a point)	dual Desargues triangles (perspective from a line)

Two triangles are called **dual Desargues triangles** if the three intersection points R, S, T of corresponding sides are collinear, that is, sitting on the same line. Specifically: - Let R be the intersection of lines AB and $A'B'$ - Let S be the intersection of lines BC and $B'C'$ - Let T be the intersection of lines CA and $C'A'$ The common line through R, S, T is called the **perspective axis**, dual to the **perspective center** O.

Desargues' Theorem (Statement)

Theorem (Desargues): Two triangles form Desargues triangles if and only if they are dual Desargues triangles.

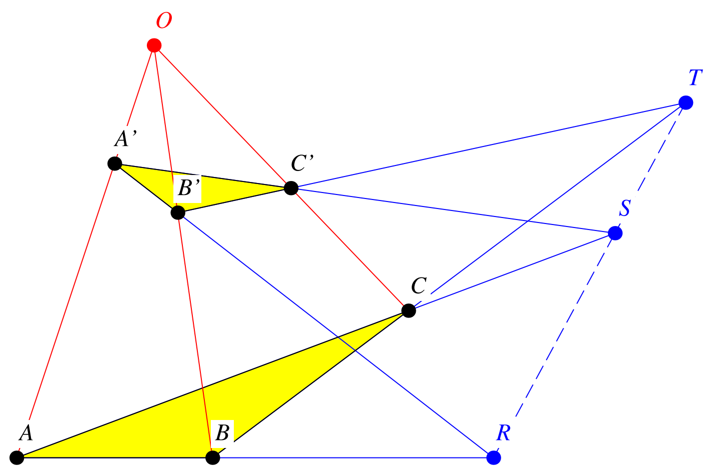

Figure 4: Desargues' configuration

In other words: - If triangles ABC and $A'B'C'$ are Desargues triangles with center O (meaning AA', BB', CC' meet at O), then the three intersection points of corresponding sides lie on a line. - Conversely, if the intersection points of corresponding sides lie on a line, then the lines connecting corresponding vertices meet at a point.

This configuration is called a **Desargues configuration**. It consists of: - 10 points: vertices A, B, C, A', B', C', center O, and three axis points R, S, T - 10 lines: sides of both triangles, the three lines through O, and the perspective axis

Notice the beautiful symmetry: **10 points and 10 lines**!

Proof Sketch (3D Method)

One elegant proof uses three-dimensional geometry: - Imagine the two triangles ABC and $A'B'C'$ lying in different planes in 3D space - If corresponding vertices are connected through a point O outside both planes, the lines AA', BB', CC' naturally meet at O - The sides of the triangles lie in their respective planes - Corresponding sides (like AB and $A'B'$) lie in a plane that contains both lines - The intersection of this plane with both triangle planes gives a line - All three such intersection points lie on the line of intersection of the two triangle planes, which is exactly the perspective axis - This proves that R, S, T are collinear!

Special Cases and Degenerate Configurations

When the two triangles lie in the same plane, Desargues' theorem still holds, but special care must be taken with: - Parallel lines (which meet at "points at infinity" in projective geometry) - Degenerate cases where some points coincide

In Euclidean geometry, if some corresponding sides are parallel, the theorem still applies by considering parallel lines as meeting at infinity.