Lecture 3: Duality Principles and Examples

Review of Duality

From Lecture 2, we learned about the fundamental duality principle in projective geometry: statements about points and lines can be systematically interchanged to produce new valid theorems.

Original Concept	Dual Concept
point	line
line	point
on	through

Duality in Action

Example 0: Two Points on a Line and Two Lines Through a Point The simplest example of duality: consider two distinct points A and B lying on a line ℓ.

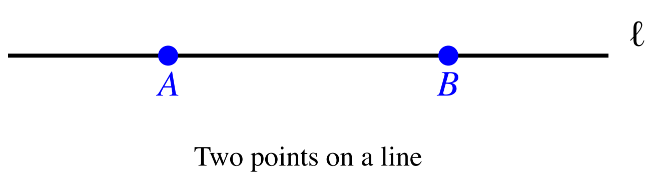

Figure 1: Two points on a line

By duality, we get: two distinct lines a and b passing through a point P.

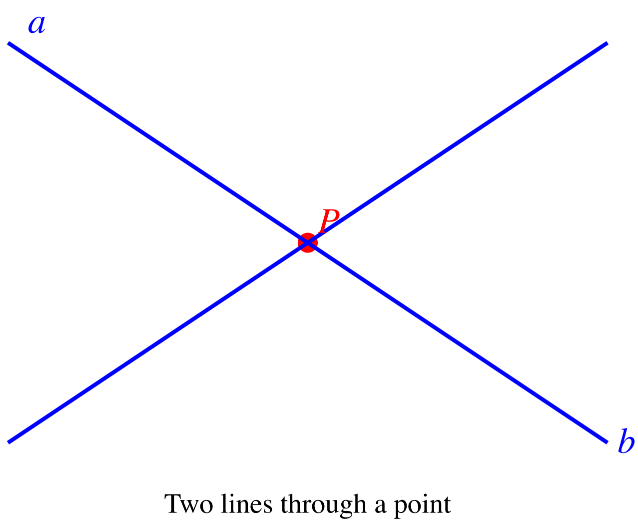

Figure 2: Two lines through a point

This illustrates the fundamental duality: - “Two points determine a line”   “Two lines determine a point”

Example 1: From Three Collinear Points to Three Concurrent Lines Consider three points A, B, C lying on a line ℓ.

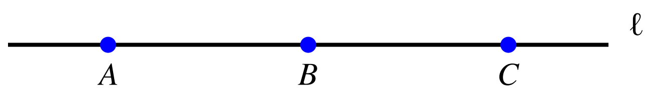

Figure 3: Three collinear points

By duality, we get: three lines a, b, c passing through a point P.

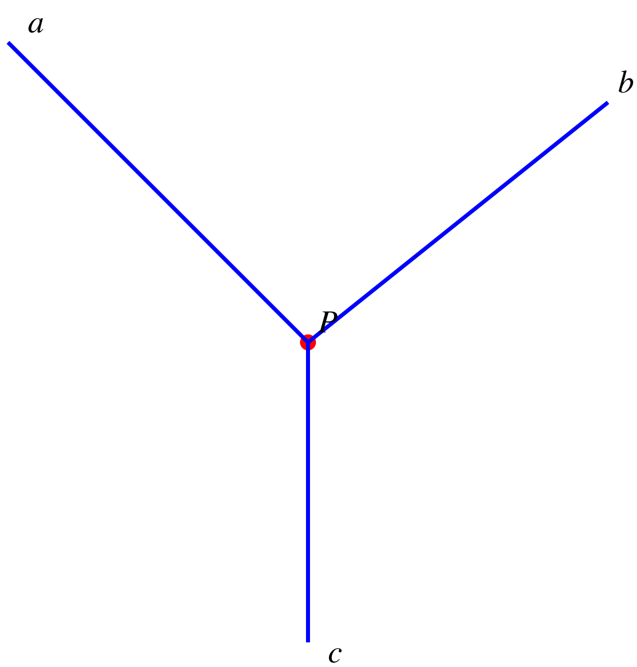

Figure 4: Three concurrent lines

Example 2: The Dual of a Triangle is a Triangle A **triangle** consists of three non-collinear points A, B, C and the three lines d, e, f connecting them.

By duality, we interchange: - The three **points** A, B, C (uppercase) $\to$ three **lines** a, b, c (lowercase, no three concurrent) - The three **lines** d, e, f (lowercase) $\to$ three **points** $D = b \cap c$, $E = b \cap a$, $F = c \cap b$ (uppercase)

The dual configuration is a **trilateral** (triangle formed by three lines).

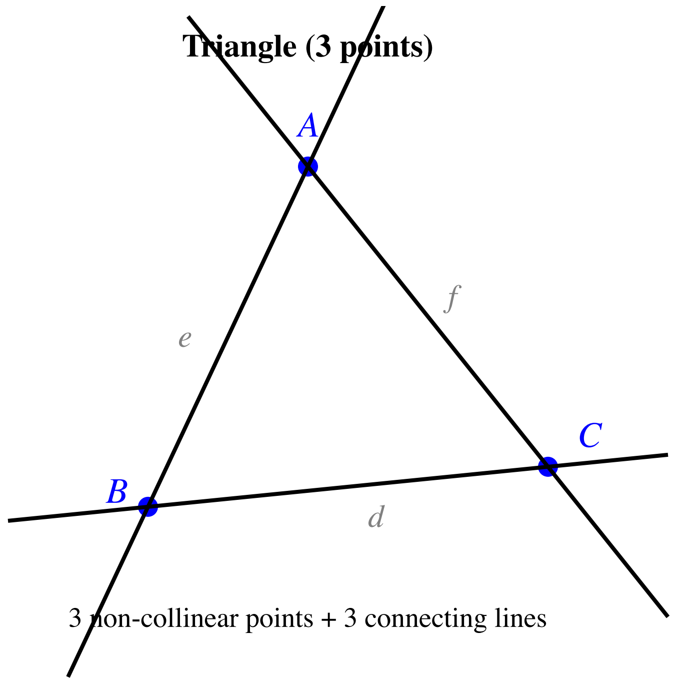

Figure 5: Triangle from three points

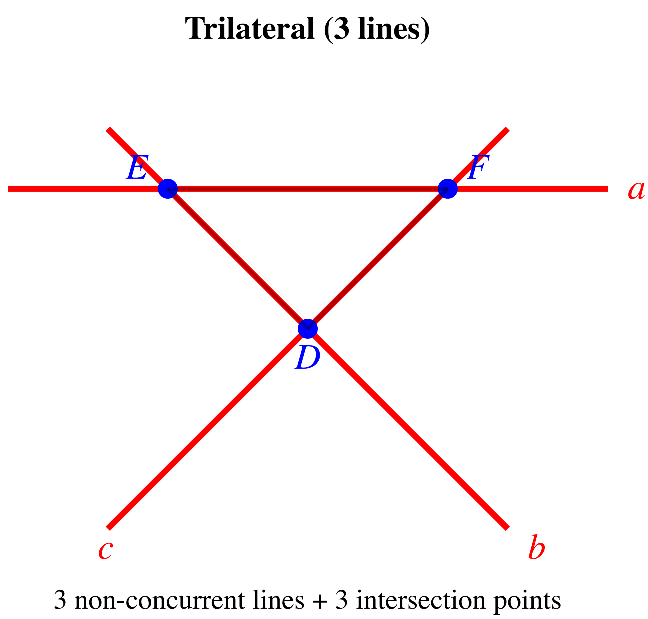

Figure 6: Triangle from three lines (trilateral)

Notice that both configurations represent triangles! This shows that the triangle is a **self-dual** figure in the sense that a triangle of points dualizes to a triangle of lines.

Example 3: Complete Quadrilateral and Complete Quadrangle A **complete quadrilateral** consists of four lines (no three concurrent) and their six intersection points.

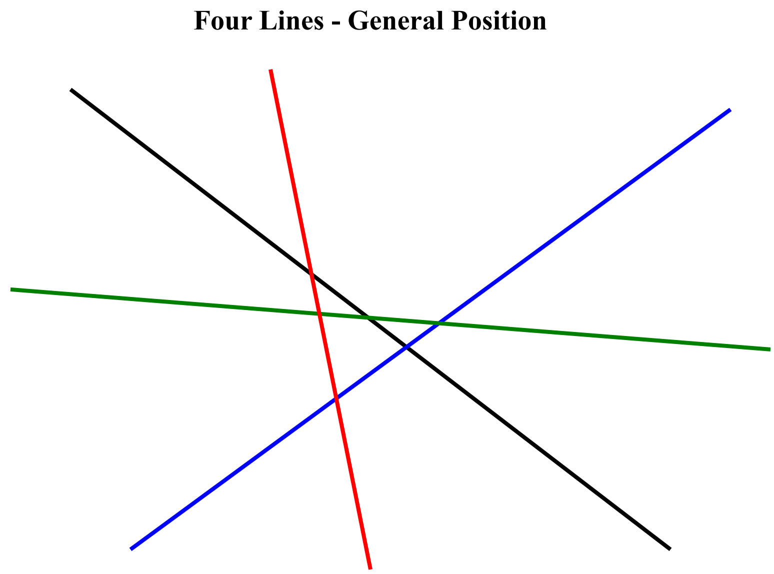

Figure 7: Complete quadrilateral

The dual configuration is a **complete quadrangle**: four points (no three collinear) and their six connecting lines.

The Principle of Duality

Principle of Duality: Given any theorem in projective geometry involving points and lines, we can obtain a dual theorem by systematically interchanging: - "point" "line" - "lie on" "pass through" - "collinear" "concurrent" - "join" "intersect"

Both the original theorem and its dual are simultaneously true in the projective plane.

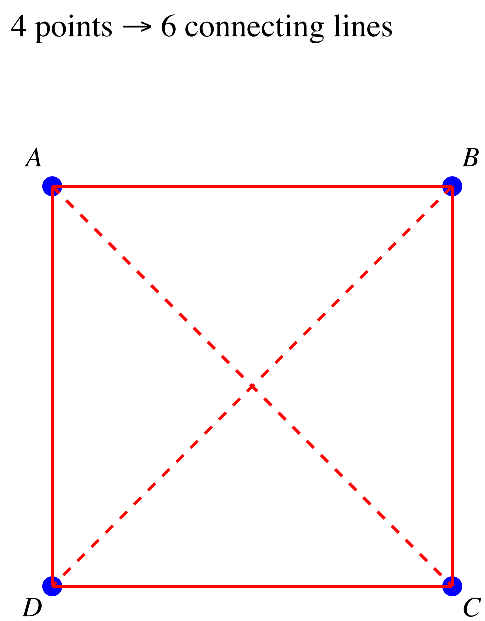

Figure 8: Complete quadrangle