Lecture 8: Form a queue or form a team

There are two basic counting problems that extends the gift swap and handshake problem in general. Given a room of n students, last time, we visited two types of problems

- swap gifts with each other. How many gifts in total? Answer: $n(n-1)$
- the number of handshakes: $\frac{n(n-1)}{2} = 1 + 2 + ... + n - 1$

Here you can also draw a graph with (without) direction to solve the problem. Now let us push the situation in general.

Form a queue

Question 1: Given a room of n students, form a queue of k students.

For simplicity, let us label all students by $1, 2, ..., n$. A queue of k students means an order tuple

$$(a_1, ..., a_k)$$

For a group of 3 students, we have

- all 1-queues correspond to $(1), (2), (3)$
- all 2-queues corresponds to $(12), (13), (21), (23), (31), (32)$. These are also gift swap pairs for 3 students.
- all 3-queues corresponds to all permutations $(123), (132), (213), (231), (312), (321)$.

The general formula is

$$P(n, k) = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!},$$

here we use factorial notation $m! = 1 \cdot 2 \cdots m$. The idea to prove this formula is very easy. We first pull one student as the first place of the queue so we have n choices. When place next student into the queue, we have $n-1$ choices. So go on and on, until we places k-th student in the queue.

In particular, forming n-queue for n students will get $P(n, n) = n!$. n-queue has another name, called *permutations* of n students. Check yourself for $n = 3$ and $n = 4$. One comment here is that $n!$ is a huge number even for $10!$.

A list of properties of $P(n, k)$:

- $k \leq n$;
- $P(n, 1) = n$;
- $P(n, 2) = n(n-1)$;
- $P(n, n) = n!$;
- $P(n, k) \leq P(n, k+1)$

Example

- Take the word "WORD", if we pull k letters and form a new word, how many different outcomes? Answer: $P(4, k)$. In particular, all permuation outcomes are $4! = 24$.
- Take the word "ALGEBRA". How many permutations we get?
  - there are seven letters but two As.
  - let us pretend seven letters are different so we would get $7! = 7 * 6 * 120 = 5040$.
  - notice the symmetry, picking the first A at first place and the second A at the sixth place is the same as picking the first A at the sixth place and picking the second A at the first place.
  - so we have $5040/2 = 2520$.
- All permutations of "ALGEBRA" with AA together.
  - you can consider "AA" as a single letter. So the answer is $6! = 720$.
- All permutations of "WORD" with "O" adjacent to "R".
  - consider "OR" in one letter so we get $3! = 6$.
  - "OR" can be "RO" too. So total 12.

Form a team

Question 2: Given a room of n students, form a team of k students.

For a group of 3 students, we have

- all 1-teams correspond to $(1), (2), (3)$
- all 2-teams corresponds to $(12) = (21), (13) = (31), (23) = (32)$. These are also handshake pairs.
- all 3-team corresponds to one team $(123) = (132) = (213) = (231) = (312) = (321)$.

The general formula is

$$\binom{n}{k} = \frac{P(n, k)}{k!} = \frac{n(n-1)...(n-k+1)}{k!}$$

The reason is that we can numerate all k-queues and then divided by the all k-permutations.

Examples:

- $n = 2$
    - $\binom{2}{1} = 2$, $\binom{2}{2} = 1$
- $n = 4$
    - $\binom{4}{1} = 4$, $\binom{4}{2} = 6, \binom{4}{3} = 4$, $\binom{4}{4} = 1$
- $\binom{n}{1} = n$, $\binom{n}{n-1} = n$, $\binom{n}{n} = 1$
    - picking 1-teams is the same as picking $n-1$-teams
- $\binom{n}{k} = \binom{n}{n-k}$
    - check this is true using the formula as well
    - we define $\binom{n}{0} = \binom{n}{n} = 1$
- Pascal's triangle by putting the binomial coefficient nicely

>```
>          1
>        1   1
>      1   2   1
>    1   3   3   1
>  1   4   6   4   1
>1   5  10    10   5   1
>```

Flipping a coin:

- flipping n times we get 2^n total outcomes
- flipping n times with k heads? Answer: $\binom{n}{k}$
    - if you do not believe, write it down.
    - flipping 3 coins with one head: HTT, THT, TTH
    - idea: picking k slots for heads (team)
- total outcomes are the same as the outcomes with k heads for $k = 0, 1, ..., n$. So we have

$$2^n = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n-1} + \binom{n}{n}$$

Paths on a grid: Moving only upperward or rightward, how many different paths?

Answer: $\binom{7}{3}$

Retangles on grid: $\binom{4}{2} * \binom{5}{2} = 60$