Lecture 13: Algerbra 101 Follow the rules

Basic rules

- Commutative law $a+b=b+a$
- Associative law $a+(b+c)=(a+b)+c$
- Distributive law $a(b+c)=ab+ac$, $(a+b)c=ac+bc$
- Negation law $-a=(-1)a$
- numbers before variables $3x$ not $x3$

Example

- easy ones

$$\begin{aligned} a-(b+c) &= a+(-1)(b+c) \\ &= a+((-1)b+(-1)c) \\ &= a-b-c \end{aligned}$$

$$\begin{aligned} a-(b-c) &= a+(-1)(b-c) \\ &= a+((-1)b-(-1)c) \\ &= a-b+c \end{aligned}$$

$$\begin{aligned} 4a-3-2(a+1) &= 4a-3-2a-2 \\ &= 2a-5 \end{aligned}$$

- more work

$$\begin{aligned} (a+2)(a+1) &= a(a+1)+2(a+1) \\ &= a^2+a+2a+2 \\ &= a^2+3a+2 \end{aligned}$$

$$\begin{aligned} (a+b)(a-b) &= a(a-b)+b(a-b) \\ &= a^2-ab+ba-b^2 \\ &= a^2-b^2 \end{aligned}$$

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= a(a+b)+b(a+b) \\ &= a^2+ab+ba+b^2 \\ &= a^2+2ab+b^2\end{aligned}$$

$$\begin{aligned}(a-b)^2 &= (a+(-b))^2 \\ &= a^2+2a(-b)+(-b)^2 \\ &= a^2-2ab+b^2\end{aligned}$$

$$\begin{aligned}(2a-4)^2 &= (2a)^2-2(2a)(-4)+(-4)^2 \\ &= 4a^2+16a+16\end{aligned}$$

- In general, we have

$$\begin{aligned}(a+b)(c+d) &= a(c+d)+b(c+d) \\ &= ac+ad+bc+bd\end{aligned}$$

- Applications:

$$\begin{aligned}101^2 &= (100+1)^2 \\ &= 10000+200+1=10201\end{aligned}$$

$$\begin{aligned}100^2-99^2 &= (100+99)(100-99) \\ &= 199\end{aligned}$$

- more and more work

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a(a^2+2ab+b^2)+b(a^2+2ab+b^2) \\ &= a^3+2a^2b+ab^2+ba^2+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

$$\begin{aligned}(a+b)^4 &= (a+b)(a^3+3a^2b+3ab^2+b^3) \\ &= a(a^3+3a^2b+3ab^2+b^3)+b(a^3+3a^2b+3ab^2+b^3) \\ &= a^4+3a^3b+3a^2b^2+ab^3+ba^3+3a^2b^2+3ab^3+b^4 \\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

Why? Pascal's triangle?? Pick k of b's out of the product.

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + b^n$$

- more and more work

$$\begin{aligned}(a+b+c)^2 &= ((a+b)+c)^2 \\ &= (a+b)^2 + 2(a+b)c + c^2 \\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\end{aligned}$$

Factorization

A polynomial $p(x)$ in one variable is of the form with x and its powers in such as:

$$x+3, 2x-5, 3x^2+4x+1, x^5-1, \ldots$$

A facorization means the polynomial is expressed as a product of linear terms.

$$x^2 - 4 = (x-2)(x+2)$$

$$x^2 - 2x - 3 = (x-3)(x+1)$$

If you can factor a quadratic polynomial, you can find the solution of $p(x) = 0$. The general quadratic polynomial is of the form

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a})$$

We will start with $a = 1$. Let us pretend

$$\begin{aligned}x^2 + bx + c &= (x - x_1)(x - x_2) \\ &= x^2 - (x_1 + x_2)x + x_1 x_2\end{aligned}$$

In particular, we get a special form of Vieta's formula:

$$c = x_1 x_2$$

$$b = -(x_1 + x_2)$$

Now it is a guess game

Examples:

- $p(x) = x^2 - 2x - 3$, -3 as a product, $(-1) * 3$ or $(-3) * 1$, so must be $(x - 3)(x - (-1))$
- $x^2 + 25x + 84$, factor $84 = 4 * 21$, so we have $(x + 4)(x + 21)$
- $x^2 - 7x + 10$, factor 10 as two and sum up to 7, so $(x - 2)(x - 5)$
- $x^2 - 9x + 9$, perfect square, use $(a - b)^2$, we get $(x - 3)^2$
- $x^2 - 36$, use $a^2 - b^2$, so $(x - 6)(x + 6)$

More difficult examples

Guess the product.

- $9x^2 + 6x + 1 = (3x + 1)(3x + 1)$
- $6x^2 - 23x - 18 = (2x - 9)(3x + 2)$