Lecture 13: Solving equations via algebra

History of Algebra and Number theory

Diophantus of Alexandria was a Greek mathematician around 200-300 A.D. He was the first one who introduce unknowns ($x, y, \ldots$) to solve math problems. To learn more about his life, we start with Diophantus's Riddle, which is a poem that encodes a mathematical problem. In verse, it read as follows:

> 'Here lies Diophantus,' the wonder behold. Through art algebraic, the stone tells how old:
> 'God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife;
> And then yet one-seventh ere marriage begun;
> In five years there came a bouncing new son.
> Alas, the dear child of master and sage After attaining half the measure of his father's life chill fate took him.
> After consoling his fate by the science of numbers for four years, he ended his life.'

To know how long he lived, you need algebra invented by himself. We introduce the unknown x as the number of years Diophantus lived. The poem gives an equation as below:

$$\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x$$

after simplification, we get

$$\frac{75}{84}x + 9 = x$$

So $x = 84$.

In his book "arithmetica", he put together 200 equations to solve. The English translation is here. The first problem is the classical problem everyone learns in middle school

> Problem 1: To split a given number (100) in two parts having a given difference (40).

To solve this, we introduce two unknowns x, y and the problems gives two equations:

$$\begin{aligned} x + y &= 100 \qquad (1) \\ x - y &= 40 \qquad (2) \end{aligned}$$

There are two ways to solve it.

Cancellation trick

Adding two equations together, we get y cancelled and

$$2x = 140$$

So $x = 70$ and $y = 30$.

Substitution trick

From the second equation, we know $x = y + 40$. So we can plug that into first equation and get:

$$2y + 40 = 100$$

which gives the same solution.

Fermat's last theorem

Diophantus's book influences many generations of mathematicians in the next 1800 years. In the same book, Problem 8 of Volume II is stated below:

> Problem II.8: To split a given square (16) into two squares.

Accoding to Wiki, the 1621 edition of Arithmetica by Bachet gained fame after **Pierre de Fermat** wrote his famous **"Last Theorem"** in the margins of his copy:

> If an integer n is greater than 2, then
>
> $$a^n + b^n = c^n$$
>
> has no solutions in non-zero integers a, b, and c. **I have a truly marvelous proof of this proposition which this margin is too narrow to contain.**

If Diophantus is the grandfather of algebra and number theory, then Fermat is the father. His famous quote influences modern algebra and number theory in the next 300 years. Finally, Andrew Wiles closed the long journey on mulitple generations of mathematicians on Fermat's wild guess.

> Theorem (Andrew Wiles): Fermat's last theorem is a theorem!

More equations

Example 1

$$\begin{aligned} 2x + y &= 100 \qquad (1) \\ x - y &= 40 \qquad (2) \end{aligned}$$

Now cancellation trick still works by adding two equations.

Example 2

$$\begin{aligned} 2x + y &= 100 \qquad (1) \\ x - 2y &= 40 \qquad (2) \end{aligned}$$

Now to do cancellation, we need to multiply equation (1) by 2.

$$\begin{aligned} 2x + y &= 100 \qquad (1) \\ 2x - 4y &= 80 \qquad (2) \end{aligned}$$

Now (1)-(2) yields the cancellation.

$$5y = 20$$

So $y = 4$ and $x = 48$.

Example 3

$$\begin{aligned} 2x + 3y &= 100 \qquad (1) \\ 3x - 2y &= 40 \qquad (2) \end{aligned}$$

To make the cancellation work, it takes more effort. We can multiply 3 on (1) and 2 on (2).

$$\begin{aligned} 6x + 9y &= 300 \qquad (1) \\ 6x - 4y &= 80 \qquad (2) \end{aligned}$$

Then cancellation works.

Cramer's rules

Now let me introduce some college math with the complete solutions for above two equations for two unknowns. We use a,b,c,d,e,f below as 6 given numbers.

$$ax + by = e$$
$$cx + dy = f$$

To solve x and y, Cramer's rule has the formula below.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

We introduce the determinant symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The way to remember is to start with the coefficient matrix in the demonimator.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The first column is called x coefficient column and the second is called y coefficient column. The denominator is always the determinant of the coefficient matrix.

If you want to solve for x then swap the x-column with the value column. Similarly if you want to solve for y, swap y column with the value column.

$$\begin{pmatrix} e \\ f \end{pmatrix}$$

Exercise: Use the Cramer's rule to solve previous examples and verify the answers are the same.