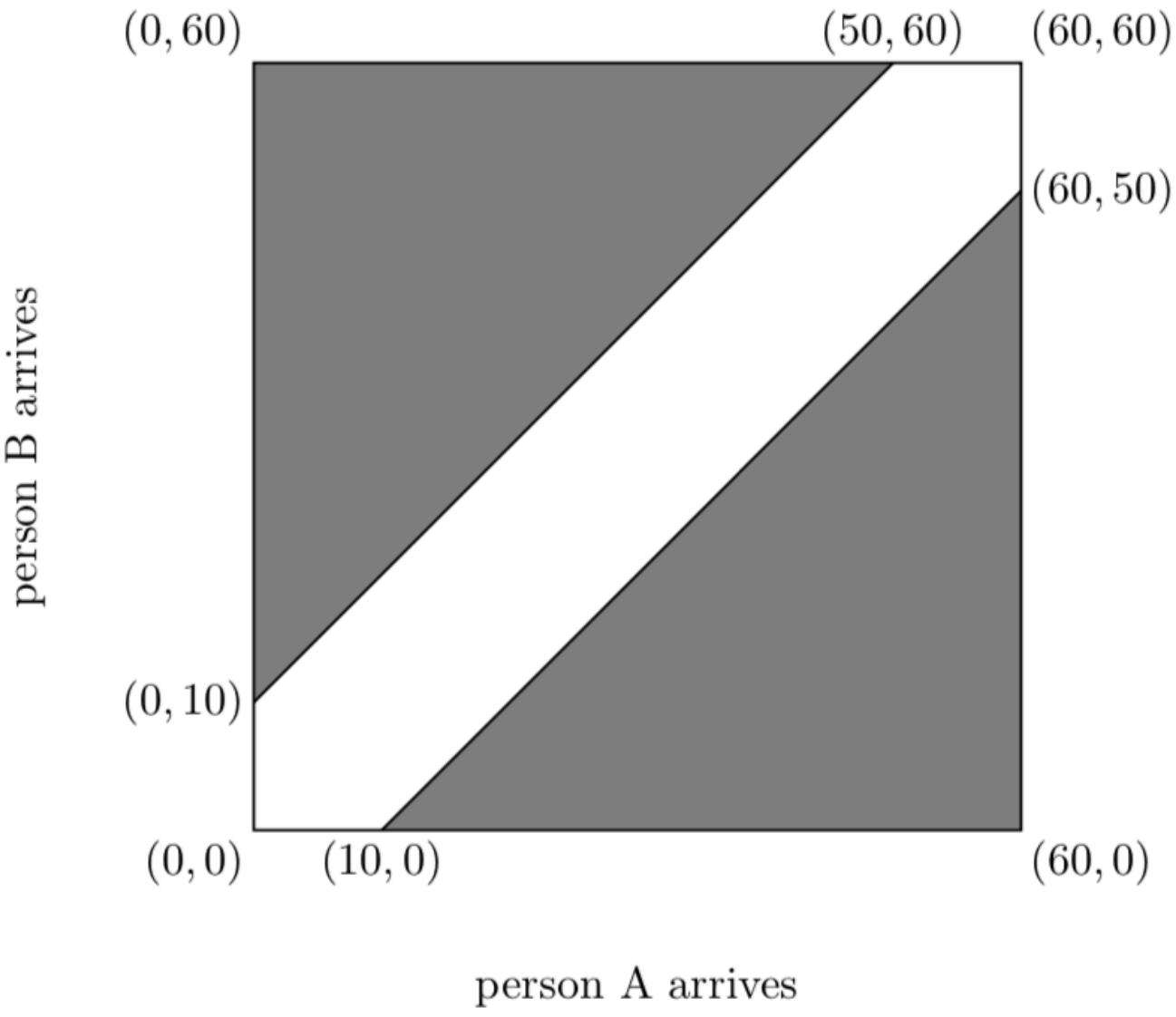
Lecture 11: More on Probability

Waiting problem

A and B agree to meet at a coffeee shop between 1PM and 2PM, and they arrive randomly. The rule is that no one will wait for the other more than 10 mins. What is the probability they will meet?

Draw the time table at a unit square. See the following picture.



So the unshaded area ratio

$$1 - \frac{50 * 50/2}{60 * 60/2} = \frac{11}{36}$$

Flipping a coin until you win

We have a coin with $p(H) = p$ and $p(T) = 1 - p$. Now we gonna threw this coin until we got a head. What is the expectation of steps getting a head?

- 1-toss: getting a head with probability p
- 2-toss: "TH", so the probability is $(1-p)p$
- k-toss: "T...TH", so get $(1-p)^{k-1}p$.
- Expected of tosses

$$E = 1 \cdot p + 2 \cdot (1-p)p + 3 \cdot (1-p)^2 p + 4 \cdot (1-p)^3 p + ...$$

$$\frac{E}{p} = 1 + 2 \cdot (1-p) + 3 \cdot (1-p)^2 + 4 \cdot (1-p)^3 + ...$$

$$\frac{(1-p)E}{p} = 1 \cdot (1-p) + 2 \cdot (1-p)^2 + 3 \cdot (1-p)^3 + 4 \cdot (1-p)^4 + ...$$

$$\frac{E}{p} - \frac{(1-p)E}{p} = 1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + ...$$

$$E = \frac{1}{1-(1-p)} = \frac{1}{p}$$

- So if $p = \frac{1}{2}$, you expect 2 tosses until getting a head.

Happy meals toy collector

You keep buying happy meals until you collected the whole set of toys (a pack of 5-6 toys). How many meals do you expect to buy?

- Buying the first meal, you will get one new toy for sure, with probability 100%.
- Starting from the second meal, getting an additional new toy has probability $\frac{k-1}{k}$. The expected meals you need to buy is the same as "tossing until you win problem", so it is $\frac{k}{k-1}$.
- Once you collected 2 toys, getting an additional new toy has probability $\frac{k-2}{k}$. The expected meals you need to buy is the same as "tossing until you win problem", so it is $\frac{k}{k-2}$.
- So the expected number of meals until you get all k toys:

$$1 + \frac{k}{k-1} + \frac{k}{k-2} + ... + \frac{k}{1} = k(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k})$$

- $k = 2$, you need 3 meals.
- $k = 3$, you need 5.5 meals.
- $k = 5$, you need 11.4 meals.

- $k = 25$, you need 95.4 meals.
- $k = 100$, you need 519 meals.