

Lecture 13: Solving equations via algebra II

Reminder on Cramer's rules

Now let me introduce some college math with the complete solutions for above two equations for two unknowns. We use a, b, c, d, e, f below as 6 given numbers.

$$ax + by = e$$

$$cx + dy = f$$

To solve x and y , Cramer's rule has the formula below.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

We introduce the determinant symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The way to remember is to start with the coefficient matrix in the demonimator.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The first column is called x coefficient column and the second is called y coefficient column. The denominator is always the determinant of the coefficient matrix.

If you want to solve for x then swap the x -column with the value column. Similarly if you want to solve for y , swap y column with the value column.

$$\begin{pmatrix} e \\ f \end{pmatrix}$$

Examples

- (Speed=Distance/Time) Up hills speed 6 miles/hour (mph), Down hills speed 18 miles/hour. If up hill takes 30 mins, what is the average speed?
 - $d = v * t$. So the one way distance is $6 * 0.5 = 3$ miles.

- average speed is the total distance over total time.
 - Uphill takes 0.5 hour
 - Downhill takes $t = \frac{d}{v} = 3/18 = \frac{1}{6}$ hour
 - average speed is $\frac{3+3}{\frac{1}{2} + \frac{1}{6}} = 9$ miles per hour
- (Harmonic mean) Same problem as above but drop 30 mins assumption.
 - let us assume the one way distance is x .
 - average speed is

$$\frac{2x}{\frac{x}{6} + \frac{x}{18}} = \frac{2x}{\frac{2}{9}x} = 9$$

- In general, the formula is

$$\frac{2x}{\frac{x}{a} + \frac{x}{b}} = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b}$$

this is call the harmonic means of two numbers.

- (Tracking the unit) Seven moles take seven minutes to dig seven holes, how long does eight moles dig eight holes?
 - Output is 7 holes. So the efficiency of the hole digging is

$$\frac{7 \text{ holes}}{7 \text{ moles} * 7 \text{ min}} = \frac{1}{7} \text{ holes/mole*min}$$

We can say the digging hole rate is one seventh per mole per min.

- Now the rest is easy.

$$8 \text{ holes} = \frac{1}{7} \text{ holes/mole*min} * 8 \text{ moles} * x \text{ min}$$

All units cancelled out and $x = 7$ min.

- (Two unknowns and two equations) Three chocalate bars and five lollipops cost 22 and four chocalate bars and two lollipops cost 20. How much does each cost?

$$3x + 5y = 22 \quad (1)$$

$$4x + 2y = 20 \quad (2)$$

- cancellation trick $(1) * 4 - (2) * 3$ we get $y = 2$ and $x = 4$.

$$12x + 20y = 88 \quad (1)$$

$$12x + 6y = 60 \quad (2)$$

$$14y = 28$$

- using Cramer's rule

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = 6 - 20 = -14$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = \begin{vmatrix} 22 & 5 \\ 20 & 2 \end{vmatrix} = 44 - 100 = -56$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = \begin{vmatrix} 3 & 22 \\ 4 & 20 \end{vmatrix} = 60 - 88 = -28$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-56}{-14} = 4, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-28}{-14} = 2$$

- (Ratio Rule)

$$\frac{a}{b} = \frac{c}{d}$$

is the same as $ad = bc$

- Jim is twice as his youngest brother's age at 2020. After 20 years, the ratio is $\frac{2}{3}$.

$$\frac{x + 20}{2x + 20} = \frac{2}{3}$$

$$3x + 60 = 2x + 40$$

So $x = 10$ and Jim has age 20 at 2020.