Lecture 6: Arithmetic and geometric series

Number series

A **number series** is a sequence of numbers.

$$a_1, a_2, a_3, \ldots$$

Example

- the natural number series $0, 1, 2, \ldots$ with the formula $a_n = n - 1$
- the perfect square series $0, 1, 4, 9, \ldots$ with the formula $a_n = (n-1)^2$
- the prime number series $2, 3, 5, 7, \ldots$ with no obvious formula
- Fibonacci series $1, 1, 2, 3, 5, 8, \ldots$ with $a_{n+2} = a_n + a_{n+1}$ when $n \geq 1$
- Harmonic series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ with the formula $a_n = \frac{1}{n}$
- the 9 series $0.9, 0.99, 0.999, 0.9999, 0.99999 \ldots.$ and the limit is 1.
- the π series $3, 3.1, 3.14, 3.141, 3.1415, \ldots$
- the series of positive rational numbers $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}, \ldots$ with numerating all nonrepeating fractions with both numerator and denominator up to k
- Check the On-Line Encyclopedia of Integer Sequences (OEIS) (Link) for more awesome series.

Arithmetic progression (series)

An arithmetic progression is a number series with equal distance. It has the formula

$$a_n = a_1 + (n-1)d,$$

where a_1 is the first term and d is the equal distance

Example

- $1, 3, 5, 7, 9, \ldots$ with $a_n = 2n - 1$
- $2, 4, 6, 8, 10, \ldots$ with $a_n = 2n$
- $3, 7, 11, 15, \ldots$ with $a_n = 3 + 4(n-1)$
- in general
- $3, 7, 11, \ldots$ what is the 50th term? $a_{50} = 3 + 4 * 49 = 199$
- How many terms of a series starting from 3 and ends at 35 with equal distance 4?
  - $(35 - 3)/4 + 1 = 9$
  - ends at 11? $(11 - 3)/4 + 1 = 3$

The sum formula

The sum of the arithmetic progression has the formula below

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n = (a_1 + a_n) * n/2,$$

where n is the number of terms which can be calculated by

$$n = (a_n - a_1)/d + 1.$$

Example

- $1 + 2 + ... + 100 = (1 + 100) * 100/2 = 5050$
- $1 + 3 + 5 + 7 + ... + 99 = ?$
  - the number of terms $n = (99 - 1)/2 + 1 = 50.$
  - the sum $= (1 + 99) * 50/2 = 50^2 = 2500$
- $1 + 3 + 5 + 7 + ... + (2n - 1) = (1 + 2n - 1) * n/2 = n^2$
- $3 + 7 + 11 + ... + 103 = ?$
  - the number of terms $n = (103 - 3)/4 + 1 = 26$
  - the sum $= (3 + 103) * 26/2 = 106 * 13 = 1378$

Dirichlet's arithmetic progression theorem

Theorem The arithmetic progression of the form $a + bd$:

$$a, a + d, a + 2d, a + 3d, ...$$

has infinitely many prime when $\gcd(a, d) = 1$.

Comments:

- When $\gcd(a, b)$ is not 1, we have a series of composite numbers such as

$$2, 12, 22, 32, ...$$

so have at most one prime number.

- Series like $4k + 1$ and $4k + 3$ all have infinitely many primes.

Geometric series

The geometric series is of the form $a_n = ar^{n-1}$

$$a, ar, ar^2, ar^3, ...$$

The sum formula when $|r| < 1$

$$a + ar + ar^2 + ar^3 + ... = a(1 + r + r^2 + r^3 + ...)$$

So it suffices to find the value

$$x = 1 + r + r^2 + r^3 + ...$$

$$rx = r + r^2 + r^3 + ... = x - 1$$

$$1 = (1 - r)x$$

$$x = \frac{1}{1 - r}$$

Thus the sum

$$a + ar + ar^2 + ar^3 + ... = \frac{a}{1 - r}$$

Example

- $1 + \frac{1}{n} + \frac{1}{n^2} + ... = \frac{1}{1 - \frac{1}{n}} = \frac{n}{n-1} = 1 + \frac{1}{n-1}$
  - recall in base n, the above is $1.11111..._n = 1 + 0.1111..._n = 1 + \frac{1}{n-1}$