Lecture 4: Decimal vs Binary, and more

A math joke

We have "Thanksgiving = Christmas" because $\mathrm{Dec}\ 25 = \mathrm{Oct}\ 31$.

Decimal or Base-Ten system

Base-Ten system (decimal numeral system) is mostly used because we have ten fingers. All numbers are expressed by $0, 1, \cdots, 9$. The base-ten expression has the following expression.

Example

$$\begin{aligned}12345 &= 10000 + 2000 + 300 + 40 + 5\\ &= 1 * 10^4 + 2 * 10^3 + 3 * 10^3 + 4 * 10^1 + 5 * 10^0\\ &= 12345_{10}\\ &= \mathrm{Dec}\ (12345)\end{aligned}$$

Base-Two (Binary) system

That is the computer numbering system because computer has know 2 digits 0 and 1... You can only use 0 and 1 to label a number $0, 1, 10, 11, 100, 101, 110, 111, 1000 \ldots$.

- $10_2 = 1 * 2^1 + 0 * 2^0 = 2_{10}$
- $111_2 = 1 * 2^2 + 1 * 2^1 + 1 * 2^0 = 7_{10}$
- $10111 = 1 * 2^4 + 111 = 23_{10}$

Base-Three (Ternery) system

You can only use $0, 1, 2$ to label a number $0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, \ldots$

- $122_3 = 1 * 3^2 + 2 * 3^1 + 2 = 17$
- $122_3 = 200_3 - 1_3 = 18 - 1 = 17$

Base-Eight (Octal) system

You can only use $0, \ldots, 7$ to label a number $0, 1, 2, 3, 4, 5, 6, 7, 10, \ldots$.

- $\mathrm{Oct}\ 31 = 31_8 = 3 * 8^1 + 1 = 25_{10} = \mathrm{Dec}\ 31$ (the joke)

Base-Sixteen (Hexadecimal) system

You can only use $0, ..., 9, A, B, C, D, E, F$ to label a number $0, 1, 2, 3, 4, 5, 6, 7, 10, A, B, C, D, E, F, 10, 11, ...$

- $BB = 11 * 16 + 11 = 187_{10}$

Base-n system

With the proper labels, you can design any base-n system. $n < 10$ is easy because we can borrow the ten digits. For n is large, you need to design your own "digit" such as $A, B, C, ...$

Convert a decimal to other base

Use long division.

- Convert 25_{10} to octal
  - $25 = 3 * 8 + 1$
  - So $25_{10} = 31_8$. The joke again.
- Convert 78_{10} to binary
  - **repetitively divides 2 until you cannot. Collect the last divisor and other remaiders in the reserve way**.
    - 78=2*39+0
    - 39=2*19+1
    - 19=2*9+1
    - 9=2*4+1
    - 4=2*2+0
    - 2=2*1+0
  - $78_{10} = 1001110_2 = 2^6 + 2^3 + 2^2 + 2^1 = 64 + 8 + 4 + 2 = 78$
- Convert 78_{10} to base 3.
  - $78 = 3 * 26 + 0$
  - $26 = 3 * 8 + 2$
  - $8 = 3 * 2 + 2$
  - So $78_{10} = 2220_3$

Addition and substraction

Use long addition and substraction!!

- $111_2 + 11_2 = 1010_2$
- the above is the same as $7 + 3 = 10$.
- $10010_2 - 1101_2 = 101_2$

- the above is the same as $18 - 13 = 5$.

Long addition/substraction really works for any base! But need to be careful on add/minus one.

- for octals, $7_8 + 4_8 = 13_8$
- for binary, only need to remember $1_2 + 1_2 = 10_2$ and $10_2 - 1_2 = 1_2$

Multiplication and division

Use long multiplication and long division!!

$9 * 9$ multiplication table for binary

- $0 * 0 = 0$
- $0 * 1 = 1 * 0 = 0$
- $1 * 1 = 1$

Computer likes this because it is really easy!! No need to memorize $9 * 9$ multiplication table. We can do that too but it is too late.... We get so used to decimal system.

Homework Create a base-5 $5 * 5$ multiplication table.

Example

work below out use long multiplication/division

- $111_2 * 11_2 = 7 * 3 = 21$
- $1001110_2/110_2 = 78/6 = 13 = 1101_2$

Bonus

- We know $1/3 = 0.33333....$. What is the division in base three?

$$1_3/10_3 = 0.1_3 = 0.33333....$$

  This extends to all fractions or rational numbers.
- We know $1/2 = 0.5$. What is the division in base two?

$$1/2 = 1_2/10_2 = 0.1$$

- But in base 3

$$1/2 = 0.11111....{}_3 = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...$$

- In fact, for any base n,

$$\frac{1}{n-1} = 0.11111....._n = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + ...$$

- We know in base $3_{10} = 10_3$. But 10_3 is not a prime!! Because 2 can not divide 10_3. Thus a prime number is a prime number for any base.