Lecture 11: Probability and Expectation

More Examples:

- A bag of 10 blocks with 5 red and 5 green. What is the probability of picking 5 with 3 green and 2 red?
  - same as selecting committee problem.
  - total outcome $\binom{10}{5} = \frac{10*9*8*7*6}{120} = 4*9*7 = 252$
  - total possibility $\binom{5}{2} * \binom{5}{3} = 10*10 = 100$
  - probability is $\frac{25}{63}$
- Biased coin flipping: A bad coin gives $2/3$ chance of head. What is the probability of getting 3-heads out of 6 flips?
  - The possible combinations of "HHHTTT" is $\binom{6}{3} = 20$.
  - The probability is actually $20 * (\frac{2}{3})^3 * (\frac{1}{3})^3$.
  - Notice if the coin is unbiased, then the probability is the same old one. $20/2^6$.
- What is the probability of a triangle being isosceles in an octagon?
  - outcomes: $\binom{8}{3} = 56$
  - isosclels outcome: $3*8$
- What is the probability of a group of n student with at least two sharing the same birthday?
  - Use the complementary counting. Check the probability of n students having n different birthdays.
  - we get

$$\frac{365}{365} * \frac{364}{365} * \frac{363}{365} * \frac{362}{365} * \ldots * \frac{365-n+1}{365}$$

  - When $n = 23$, P(at least two sharing same birthday)$> 50\%$.
  - When $n = 70$, P(at least two sharing same birthday)$> 99\%$.
  - counterintuitive, 70 people counts only one fifth of 365 days but sharing same birthday chance is great than 99%.

Geometric probability:

- A line segment AB with mid point C. What is the probability of a point closer to the mid point than to A or B?
  - draw the possibility. You find 0.5.
- A circle embeds into a square. What is the probability in the circle?
  - use area formula
  - If square has length 2, then square area is 4 and circle area is π.
  - the probability is $\frac{\pi}{4}$.
- A square embeds into a circle. What is the probability in the square?
  - use area formula
  - If circle has radius 2, then square area is 2 and circle area is π.
  - the probability is $\frac{2}{\pi}$.
- A target with concentric circles with radius from 1 to 10. What is the probability in the odd rings?
  - total area: 100π.
  - k-th ring: $((k)^2 - (k-1)^2)\pi = (2k-1)\pi$
  - $(1+5+9+13+17)\pi/100\pi = 45\%$

Expected value (Expectation):

- flipping a coin game. You give me one dollar to enter the game. If head, earn 1 dollar, if tail, nothing. Is this a fair game?
  - the expectation is $0.5*1+0.5*0 = 0.5$. That is the expected money you expect to earn.
  - but you have to pay 1 dollar to play. So it is not a fair game.
  - if you only pay less than 0.5\$, you will earn more money.
  - if you only pay more than 0.5\$, you will lose money.
- Rolling a dice to get same amount of dollar. What is the fair game ticket?
  - $(1+2+3+4+5+6)/6 = 3.5$
- Gambling machine, claw machine，powerball are all designed by expectation analysis. You will always lose in the long term. "Stupidity tax": you buy powerball only when you are stupid.
- Expectation cannot explain everything.
  - two options: win 50 or half chance win 100 or nothing. Which to pick?
  - two options: lose 50 or half chance lose 100 or nothing. Which to pick?

Powerball Expectation Analysis: Odds of Winning

The odds of winning each prize are as follows (rounded to the nearest whole number):

- **Jackpot (5 + Powerball)**: 1 in 292,201,338
- **Match 5 (no Powerball)**: 1 in 11,688,053
- **Match 4 + Powerball**: 1 in 913,129
- **Match 4 (no Powerball)**: 1 in 36,525
- **Match 3 + Powerball**: 1 in 14,494
- **Match 3 (no Powerball)**: 1 in 579
- **Match 2 + Powerball**: 1 in 701
- **Match 1 + Powerball**: 1 in 92
- **Powerball only**: 1 in 38

Prize Amounts

Here are the prize amounts for each tier, though the actual amounts can vary based on the drawing:

- **Jackpot (5 + Powerball)**: $400,000,000 (typically varies).
- **Match 5 (no Powerball)**: $1,000,000.
- **Match 4 + Powerball**: $50,000.
- **Match 4 (no Powerball)**: $100.
- **Match 3 + Powerball**: $100.
- **Match 3 (no Powerball)**: $7.
- **Match 2 + Powerball**: $7.
- **Match 1 + Powerball**: $4.
- **Powerball only**: $4.

Calculating Expected Value

To calculate the expected value, we multiply the probability of winning each prize by the value of the prize and sum the results:

$$\text{Expected Value} = \sum \left(\frac{\text{Prize}}{\text{Odds of winning the prize}} \right)$$

Step-by-Step Calculation

We will now compute each term:

$$\text{Expected Value} = \left(\frac{400,000,000}{292,201,338} \right) + \left(\frac{1,000,000}{11,688,053} \right) + \left(\frac{50,000}{913,129} \right) + \left(\frac{100}{36,525} \right) + \left(\frac{100}{14,494} \right) + \left(\frac{7}{579} \right) + \left(\frac{7}{701} \right) + \left(\frac{4}{92} \right) + \left(\frac{4}{38}$$

Results for Each Term

1. $\frac{400{,}000{,}000}{292{,}201{,}338} \approx 1.37$
2. $\frac{1{,}000{,}000}{11{,}688{,}053} \approx 0.0856$
3. $\frac{50{,}000}{913{,}129} \approx 0.0548$
4. $\frac{100}{36{,}525} \approx 0.0027$
5. $\frac{100}{14{,}494} \approx 0.0069$
6. $\frac{7}{579} \approx 0.0121$
7. $\frac{7}{701} \approx 0.0100$
8. $\frac{4}{92} \approx 0.0435$
9. $\frac{4}{38} \approx 0.1053$

Sum of All Terms

$$\text{Expected Value} = 1.37 + 0.0856 + 0.0548 + 0.0027 + 0.0069 + 0.0121 + 0.0100 + 0.0435 + 0.1053 = 1.6899$$

Cost of a Powerball Ticket

A single Powerball ticket costs **$2**.

Interpretation

The expected value is about **1.69** per ticket, meaning that, on average, you can expect to win approximately **1.69** for every $2 ticket you purchase. In other words, you are losing about **0.31** per ticket in the long run.

While the jackpot can be large, the odds are extremely low, so the expected value of purchasing a Powerball ticket is less than the cost of the ticket. This makes the Powerball, like most lotteries, a losing investment in terms of expected value. However, the excitement of the game and the potential for a large win are the main attractions.