Lecture 16: Factorization and quadratic equation formula

Polynomials

A polynomial of degree d is an expression of an unknown x as below

$$p(x) = a_d x^d + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

Here we assume that

- the leading coefficient $a_d \neq 0$, otherwise it is a lower degree polynomial.
- $a_d, ..., a_0$ are called coefficients of the polynomial and they are fixed numbers.
- x is unknown so you can plug in any number into it. For example, $p(0) = a_0$ and

$$p(1) = a_d + a_{d-1} + ... + a_0$$

Examples

- Degree zero polynomials are just constant $p(x) = a_0$.
- Degree one polynomials are called **linear** $p(x) = ax + b$
- Degree two polynomisla are called **quadratics** $p(x) = ax^2 + bx + c$
- Degree three polynomisla are called **cubics** $p(x) = ax^3 + bx^2 + cx + d$
- So can define **quartics**, **qunitcs** and **sextics**.

Roots of a polynomial

A **root** or a solution of a polynomial $p(x)$ is a number x_0 such that $p(x_0) = 0$.

Examples

- for degree one, finding roots means $ax + b = 0$. So there is only one solution $x_0 = -\frac{b}{a}$.
- for degree two, $p(x) = ax^2 + bx + c = 0$ is not trivial.
- for degree three and four, they are solved until 1700.
- for degree five and above, Galois and Abel proved in 1800 that there are not general formula.

Factization

What can help solve the roots for quadratic equation? **Factorization**!!

$$p(x) = x^2 - 4 = (x-2)(x+2) = 0$$

so $x_0 = \pm 2$.

$$p(x) = x^2 - 7x + 10 = (x-2)(x-5) = 0$$

so $x_0 = 2$ or $x_0 = 5$.

A **factorization** of a polynomial $p(x)$ of degree d is a product of linear ones:

$$p(x) = a_d x^d + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = a_d(x - x_1)(x - x_2)...(x - x_d) = 0$$

A degree d polynomial can have up to at most d roots! For degree one, we can always find a solution but for degree two, **FALSE**!!

$$x^2 + 1 = 0$$

Vieta's formula for degree two

$$p(x) = ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a(x - x_1)(x - x_2) = a(x^2 - (x_1 + x_2)x + x_1 x_2)$$

So we get

$$x_1 + x_2 = -\frac{b}{a}$$
$$x_1 x_2 = -\frac{c}{a}$$

When $a = 1$, this is very useful to guess the factorization.

Examples

- For $p(x) = x^2 - 7x + 10$, the sum is 7 and the product is 10. So we have $(x-2)(x-5)$.
- For $p(x) = x^2 + 14x + 49$, the sum is -14 and the product is 49. So we have $(x+7)^2$.
- For $p(x) = x^2 - 2x - 3$, the sum is 2 and the product is -3. So we have $(x-3)(x+1)$.
- For $p(x) = 9x^2 - 12x + 4 = 9(x^2 - \frac{4}{3}x + \frac{4}{9})$, the sum is $\frac{4}{3}$ and the product is $\frac{4}{9}$. So we have $4(x - \frac{2}{3})^2$.

General formula for quadratics

The root formula for $p(x) = ax^2 + bx + c$ is

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So the factorization formula is

$$p(x) = a(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a})(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a})$$

Example

- $x^2 - 7x + 10$
  - $x_0 = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7 \pm 3}{2} = 5, 2$
- two numbers sum to a and product to b.
  - this is the same as $x^2 - ax + b$ by Vieta's formula.
  - so the solution is $x_0 = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$.
- $x^4 - 7x^2 + 10 = 0$
  - let $y = x^2$. So we have $y^2 - 7y + 10 = 0$. So $y^2 = 2, 5$. Thus $y = \pm\sqrt{5}, \pm\sqrt{2}$.
- $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$
  - let the result be x.
  - $x^2 = 1 + x$ or $x^2 - x - 1 = 0$.
  - $x_0 = \frac{1 \pm \sqrt{5}}{2}$. For the plus number, we get $\frac{1 + \sqrt{5}}{2} = 1.618\ldots$ the golden ratio.
- $\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}$
  - let the result be x.
  - $\frac{1}{x} = 1 + x$ or $x^2 + x - 1 = 0$.
  - $x_0 = \frac{-1 \pm \sqrt{5}}{2}$. For the plus number, we get $\frac{-1 + \sqrt{5}}{2} = 0.618\ldots$ the golden ratio.

Why the formula?

Complement square trick

$$\begin{align} x^2 + bx + c &= 0 \tag{1} \\ (x^2 + \frac{b}{2})^2 - \frac{b^2}{4} + c &= 0 \tag{2} \\ (x + \frac{b}{2})^2 &= \frac{b^2}{4} - c \tag{3} \\ x + \frac{b}{2} &= \pm\sqrt{\frac{b^2}{4} - c} \tag{4} \\ x &= -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \tag{5} \end{align}$$