Lecture 5:

Unit Digit

What is the last digit of

- $456 * 789 \rightarrow 2$
- $307 * 188 \rightarrow 6$
- the last digit of a perfect square can only be $0, 1, 4, 9, 6, 5$.
- 3^{2025}
  - $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$, always repeats as $3, 9, 7, 1, 3, ...$
  - $2025 = 4 * k + 1$, so the last digit is 3.
- 2^k: $2 \rightarrow 4 \rightarrow 8 \rightarrow 6 \rightarrow 2 \rightarrow ....$
- 4^k: $4 \rightarrow 6 \rightarrow 4 \rightarrow 6 \rightarrow ...$
- $5^k, 6^k$: always 5 or 6
- 7^k: $7 \rightarrow 9 \rightarrow 3 \rightarrow 1 \rightarrow 7 \rightarrow ...$
- 8^k: $8 \rightarrow 4 \rightarrow 2 \rightarrow 6 \rightarrow 8 \rightarrow ...$
- 9^k: $9 \rightarrow 1 \rightarrow 9 \rightarrow ...$
- the power unit digit repeats itself by a period of $1, 2, 4$.

Lemma a^4 and a have the same unit digit for any number a.

Remainder

The unit digit of a number is the remainder of that number divided by 10. The remainder divided by n is actually the unit digit in base n.

Example
What is the remainder of 2^{2025} divided by 3?

- Find a pattern
  - $2^1 = 2$
  - $2^2 \rightarrow_3 1$
  - $2^3 \rightarrow_3 1 * 2 \rightarrow_3 2$
- order 2!!

Example
What is the remainder of 4^{2025} divided by 5?

- Find a pattern
    - $4^1 = 4$
    - $4^2 \rightarrow_5 1$
    - $4^3 \rightarrow_5 1 * 4 \rightarrow_3 4$
- order 2!!
- Try 2^k divided by 5, $2 \rightarrow 4 \rightarrow 3 \rightarrow 1$,order 4.
- Try 3^k, $3 \rightarrow 4 \rightarrow 2 \rightarrow 1$, order 4.

Example
What is the remainder of 3^{2025} divided by 7?

- Find a pattern
    - $3^1 = 3$
    - $3^2 = 9 \rightarrow_7 2$
    - $3^3 = 27 \rightarrow_7 6$, note that $2 * 3 = 6$ as well
    - $3^4 \rightarrow_7 6 * 3 \rightarrow_7 4$
    - $3^5 \rightarrow_7 4 * 3 \rightarrow_7 5$
    - $3^6 \rightarrow_7 5 * 3 \rightarrow_7 1$
- order 6!!
- 2^k divided by 7, $2 \rightarrow 4 \rightarrow 1$, order 3
- 4^k divided by 7, $4 \rightarrow 2 \rightarrow 1$, order 3
- 5^k divided by 7, $5 \rightarrow 4 \rightarrow 6 \rightarrow 2 \rightarrow 3 \rightarrow 1$, order 6
- 6^k divided by 7, $6 \rightarrow 1 \rightarrow 6$, order 2

Fermart's Little Theorem When p is a prime, the repeating order divides $p - 1$.

This is not true for composite number. 3^k divided by 4, $3 \rightarrow 1$, order 2, not divides $4 - 1 = 3$.

Homework Ask ChatGPT to write a python code to verify Fermat's little theorem for primes up to 100. For each p, what is the number with repeating order exactly $p - 1$? Any pattern?

Fractions

divided by $9, 99, 999, \ldots$

Example

- $\frac{4}{9} = 0.\overline{4}$
- $\frac{41}{99} = 0.\overline{41}$
- $\frac{181}{999} = 0.\overline{181}$

- $\frac{31}{111} = \frac{279}{999} = 0.\overline{279}$

Terminating

A fraction is terminating in decimal expansion if the denominator contains prime factors only 2 and 5. Remember the formula $10^k = 2^k 5^k$.

Example

- $\frac{1}{25} = \frac{4}{100} = 0.04$
- $\frac{1}{8} = \frac{125}{1000} = 0.125$
- $\frac{3}{625} = \frac{3*2^4}{10000} = 0.0048$

Decimals to Fractions

Introduce x.

Example

- $x = 0.1\overline{3}$
  - $10x = 1.\overline{3} = \frac{4}{3}$
  - $x = \frac{4}{30}$
- $x = 0.\overline{37}$
  - $100x = 37 + x$
  - $x = \frac{37}{99}$
- $x = 0.\overline{142857}$
  - $1000000x = 142857 + x$
  - $x = \frac{142857}{999999} = \frac{1}{7}$

A magic number

A decimal that contains all natural numbers, thus every number anywhere, social security number, credit card number, your home address number, etc. Remember that number lies between 0.12 and 0.13, which encodes all numbers...

$$a = 0.12345678910111213...$$