Lecture 9: Applications of permutation and combinatorics

Permutation formula: form a queue

$$P(n,k) = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!},$$

Combinatorics formula: form a team

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n(n-1)...(n-k+1)}{k!}$$

Applications in counting

- flipping n-coins:
  - the number of all possible outcomes of heads and tails: 2^n
  - the number of k heads: $\binom{n}{k}$
  - idea: pick k slots out of n, the order does not matter
  - Fundamental formula

$$2^n = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n-1} + \binom{n}{n}$$

- **Paths on a grid**: Moving only upperward or rightward, how many different paths?

- Answer: $\binom{7}{3}$
  - Denote U as upward and R as rightward, a path is like, UUURRRR. This is the same as flipping coin situation.
- **Retangles on grid**:
  - you need to pick two points on the bottom edge and two points on the vertical edge
  - answer: $\binom{4}{2} * \binom{5}{2} = 60$
- **Triangles on grid**: Consider the following grid of points, how many triangles you can form?

>   o   o   o   o   o
> o   o   o   o   o   o

- Answer: $\binom{5}{2} * 6 + \binom{6}{2} * 5 = 60 + 75 = 135$
- **two sums**: How many sums of positive integers added up to 6?
  - easy: $6 = 1 + 5 = 2 + 4 = 3 + 3 = 4 + 2 = 5 + 1$, so 5.
  - for general n, we have $n - 1$.
- **three sums**: How many 3-sums of positive integers added up to 6?
  - Not so easy: 10

$$\begin{aligned}6 &= 1+1+4 = 1+2+3 = 1+3+2 = 1+4+1 \\ &= 2+1+3 = 2+2+2 = 2+3+1 \\ &= 3+1+2 = 3+2+1 \\ &= 4+1+1\end{aligned}$$

  - for general n, we have $\binom{n-1}{2}$.
  - Consider inserting two "+" into $n - 1$ gaps among the n objects

> ```
> o | o  o | o  o  o
> ```

- **two sums with zero**: How many sums of positive integers added up to 6?
  - easy: $5 + 2 = 7$.
- **three sums with zeros**: How many sums of positive integers added up to 6?
  - not so easy
  - Solution 1: $10 + 3 * 5 = 25$ and 3 left are permutations of $0 + 0 + 6$

> ```
>  o | o  o | o  o  o
> 0| o  | o  o  o  o  o
>  o  |0| o  o  o  o  o
>  o | o  o  o  o  o |0
> ```

- Solution 2: $\binom{8}{2} = 8 * 7/2 = 28$

> ```
> o | o  o | o  o  o
> o || o  o  o  o  o
> ```

- k **sums without zeros adding up to** n: $\binom{n-1}{k-1}$
- k **sums with zeros adding up to** n: $\binom{n+k-1}{k-1}$
- How many ways of 3 sums adding up to 14 with only allowed values $1, ..., 10$?
  - this is without zero sum for sure so at least we have $\binom{13}{2} = 78$.
  - we do not allow

        - $1+1+12$ so 3 of them.
        - $1+2+11$ so 6 of them.
    - so we have $78-3-6=69$.

Case works

- $4*4$ grid points contains how many squares as its vertices?

> ```
> o   o   o   o
> o   o   o   o
> o   o   o   o
> o   o   o   o
> ```

- AMC8 2025 coat hanging problem:
  A classroom has a row of 35 coat hooks. Paulina likes coats to be equally spaced, so that there is the same number of empty hooks before the first coat, after the last coat, and between every coat and the next one. Suppose there is at least 1 coat and at least 1 empty hook. How many different numbers of coats can satisfy Paulina's pattern?